Non-standard models for MINPAR

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Abstract

In the following essay we shall describe models for the theory MINPAR, the theory for finite binary trees, described in Jervell [2003]. In particular we shall investigate non-standard models simpler than the one presented by Roger, denoted $\mathscr{M}_\star$. These include the models for both right and left oriented ω-ladders and the zik-zak model. We believe that $\mathscr{M}_\star$ is the maximal non-standard model for MINPAR, and that it is constructively inaccessible from any simpler model, not only those presented in the paper.

1 The theory of binary trees

A theory for binary trees has been presented by Jones [1997] and axiomatized in Jervell [2003]. In the following essay we shall investigate non-intended modells for this theory. We know that the theory is strong enough to be incomplete in the sense of Gödel [1931]. But, since the theory is quite weak we can easily find concrete sentences that ought to be provable, which are not. The first example is taken from number theory, where we give a direct translation of $\neg sx = x$ into MINPAR. This will result in an exciting study of various non-standard models for binary trees. We always have the term modell in mind, when constructing a modell. Expressions in the language are underlined, meaning $\underline{\mathrm{nil}}$ is the linguistic expression denoting nil.

We define the language $\mathscr{L}$ for MINPAR by recursion as follows

Definition 1 *The terms of $\mathscr{L}$ is defined by*

1. $\underline{\mathrm{nil}}$ is a label in $\mathscr{L}$, and therefore also a term in $\mathscr{L}$.
2. The variables x, y, z, $\dots$ are terms in $\mathscr{L}$.
3. If t_1 and t_2 are terms in $\mathscr{L}$, then $(t_1.t_2)$ is a term in $\mathscr{L}$.

Definition 2 *The sentences in $\mathscr{L}$, is defined as follows;*

1. t_1 and t_2 are terms in $\mathscr{L}$ then $t_1 = t_2$ (equality of trees), $t_1 < t_2$ (containment of trees) are the atomic sentences in $\mathscr{L}$.

2. If A and B are sentences in $\mathscr{L}$, then $\neg A$, $A \to B$, $\forall x < y\, A(x)$, $\forall x\, A(x)$, are all sentences in $\mathscr{L}$.

Then $\wedge$, $\vee$, $\leftrightarrow$, $\exists x < y\, A(x)$ and $\exists x\, A(x)$ are defined in the obvious way. The theory of finite binary trees might be extended, and we consider extentions of the label set denoted $\mathbb{L}$. In $\mathscr{L}$, the label set is a singleton set; $\mathbb{L} = \{\underline{\text{nil}}\}$.

2 The axioms of MINPAR

We consider Jervell's theory MINPAR, denoted T, over the language $\mathscr{L}$ with the standard axioms for equality;

$\vdash x = x$ $\quad (\mathrm{E_R})$
$\vdash x = y \to y = x$ $\quad (\mathrm{E_S})$
$\vdash x = y \wedge y = z \to x = z$ $\quad (\mathrm{E_T})$
$\vdash x = u \wedge y = v \to (x.y) = (u.v)$ $\quad (\mathrm{E_F})$
$\vdash (x.y) = (u.v) \to x = u \wedge y = v$ $\quad (\mathrm{E_I})$

$(\mathrm{E_R})$, $(\mathrm{E_S})$, $(\mathrm{E_T})$ says that equality is respectively reflexive, symmetric and transitive. $(\mathrm{E_F})$ states that the pairing operator is a function, while $(\mathrm{E_I})$ requires that this function is injective. Next we give axioms specific for the theory MINPAR:

$\vdash \neg x < \underline{\text{nil}}$ $\quad (\mathrm{A}^1_<)$
$\vdash x < (y.z) \leftrightarrow x = y \vee x < y \vee x = z \vee x < z$ $\quad (\mathrm{A}^2_<)$
$\vdash \neg \underline{\text{nil}} = (x.y)$ $\quad (\mathrm{A}^3_=)$
$\vdash x = \underline{\text{nil}} \vee \exists y \exists z (x = (y.z))$ $\quad (\mathrm{A}^4_=)$

$(\mathrm{A}^1_<)$ states that $\underline{\text{nil}}$ is the minimal element, while $(\mathrm{A}^2_<)$ defines the containment relation, that the tree x is contained in the tree $(y.z)$. $(\mathrm{A}^3_=)$ states that $\underline{\text{nil}}$ is not a pair (analogous to $(\mathrm{A}^1_<)$), while $(\mathrm{A}^4_=)$ is the weak induction axiom defining the totality of elements in MINPAR.

In addition all true Δ^0_0 sentences are axioms of T.

3 The intended model

The standard model for MINPAR, denoted $\mathscr{M}$, is the one capturing finite binary trees, with nil in the leaf nodes, where we can decide whether a tree t_1 is either equal to ar contained in t_2. We also require that $<$ is monotone and transitive captured respectively by $(\mathrm{A}^5_<)$ and $(\mathrm{A}^6_<)$:

$$\vdash x < y \land z < w \rightarrow (x.z) < (y.w) \quad (\mathrm{A}^5_<)$$
$$\vdash x < y \land y < z \rightarrow x < z \quad (\mathrm{A}^6_<)$$

Inside $\mathscr{M}$ we can represent the numerals, and even sequences of numerals, hence $\mathscr{M}$ is a model for elementary arithmetic: By interpreting the successor symbol for arithmetic as $\underline{\mathrm{nil}}$, we define the successor of x as the function $(\underline{\mathrm{nil}}.x)$. Given the functions head and tail, denoted $\mathrm{hd}(x)$, $\mathrm{tl}(x)$; Then internally we have all the natural numbers represented by numerals of the form:

$$\mathrm{NAT}(x) \leftrightarrow x = \underline{\mathrm{nil}} \lor \forall y < x(\mathrm{hd}(y) = \underline{\mathrm{nil}} \lor y = \underline{\mathrm{nil}})$$

If $+$ and $\times$ are introduced as new symbols, then we can interpret the axioms of for instance Robinson Arithmetic inside MINPAR.

$$\vdash s(x) = s(y) \rightarrow x = y \quad (\Re_1)$$
$$\vdash x = 0 \lor \exists y(x = s(y)) \quad (\Re_2)$$
$$\vdash \neg 0 = s(x) \quad (\Re_3)$$
$$\vdash x + 0 = x \quad (\Re_4)$$
$$\vdash x + s(y) = s(x + y) \quad (\Re_5)$$
$$\vdash x \times 0 = 0 \quad (\Re_6)$$
$$\vdash x \times s(y) = x \times y + x \quad (\Re_7)$$

The following theorems are not provable in Robinson arithmetic:

$$\vdash \neg\, s(x) = x \quad (\Re_8)$$
$$\vdash 0 + x = x \quad (\Re_9)$$
$$\vdash s(y) + x = s(x + y) \quad (\Re_{10})$$
$$\vdash 0 \times x = 0 \quad (\Re_{11})$$
$$\vdash s(y) \times x = x \times y + x \quad (\Re_{12})$$
$$\vdash x + y = y + x \quad (\Re_{13})$$
$$\vdash x + (y + z) = (x + y) + z \quad (\Re_{14})$$
$$\vdash x \times s(0) = x \quad (\Re_{15})$$
$$\vdash x \times y = y \times x \quad (\Re_{16})$$
$$\vdash x \times (y \times z) = (x \times y) \times z \quad (\Re_{17})$$

This shows how weak Robinson arithmetic is, almost nothing is provable in it.

Observation 1 *There is a modell for Robinson arithmetic evaluating* $\Re_8$ - $\Re_{17}$ *to false.*

Proof: We consider the model $\mathcal{N}_\infty$ by interpreting $+$ and $\times$. The proof is easy once an interpretation is settled. By the modell proposed by Jervell [2003] due to ?.

s	
n	$n+1$
∞_0	∞_0
∞_1	∞_1

$+$	m	∞_0	∞_1
n	$n+m$	∞_1	∞_0
∞_0	∞_0	∞_1	∞_0
∞_1	∞_1	∞_1	∞_0

$\times$	0	m	∞_0	∞_1
n	0	$n+m$	∞_0	∞_1
∞_0	0	∞_1	∞_1	∞_1
∞_1	0	∞_0	∞_0	∞_0

We present the proofs of $\mathcal{N}_\infty \not\models \Re_{10}$, $\mathcal{N}_\infty \not\models \Re_{12}$ and $\mathcal{N}_\infty \not\models \Re_{14}$ as one-liner rewrites of equations and leave the rest to the reader.

$\Re_{10}$: $s(\infty_1) + \infty_0 = \infty_1 + \infty_0 = \infty_1 \neq \infty_0 = s(\infty_0) = s(\infty_0 + \infty_1)$

$\Re_{12}$: $s(\infty_1) \times \infty_0 = \infty_1 \times \infty_0 = \infty_0 \neq \infty_1 = \infty_1 + \infty_0 = \infty_0 \times \infty_1 + \infty_0$

$\Re_{14}$: $\infty_0 + (\infty_1 + \infty_0) = \infty_0 + \infty_1 = \infty_0 \neq \infty_1 = \infty_0 + \infty_0 = (\infty_0 + \infty_1) + \infty_1$

$\square$

4 Non-standard models for MINPAR

Let now $\mathscr{M}_R$ be a non-standard model for MINPAR defined as follows: First we extend the language by a new constant $\mathscr{L}_L = \mathscr{L} \cup \{\underline{\infty}\}$. Then we obtain a new model $\mathscr{M}_R$, where ∞ in the modell can be seen as a new label in the trees in addition to nil. The interpretation is obvious; $\underline{\infty}^{\mathscr{M}_R} = \infty$ and $\underline{\text{nil}}^{\mathscr{M}_R} = \text{nil}$. The label ∞ is its own successor, that is

$$\infty = (\text{nil}.\infty) \quad (A^R_\infty)$$

Hence our theory is extended with to a new theory $\mathrm{T}_R = \mathrm{T} \cup \{\underline{\infty} = (\underline{\text{nil}}.\underline{\infty})\}$ Then we see that the theorem of arithmetic

$$\forall x \neg s(x) = x \quad \Re_8,$$

translated into T as

$$\neg(\text{nil}.x) = x \quad \Re^{\mathrm{L}}_8$$

does not hold in $\mathscr{M}_R$, although it holds in the standard model $\mathscr{M}$ for T.

Proposition 1 *Let $\mathscr{L}_R$ and T_R be as described above. Then $\mathscr{M}_R$ is a non-standard model for* MINPAR*, that evaluates $(\Re_8)$ to false.*

Proof: We have already seen that $\Re^{\mathrm{L}}_8$ (and hence $\Re_8$) does not hold in $\mathscr{M}_R$, although it holds in the standard (arithmetical) model $\mathscr{M}$ for T. But this is not enough. We need to argue that the other axioms of T are true in $\mathscr{M}_R$. Define the following four sentences to be true inside $\mathscr{M}_R$:

$$\underline{\text{nil}} < \underline{\infty}\ (1) \quad \neg\underline{\infty} < \underline{\text{nil}}\ (2) \quad \neg\underline{\text{nil}} = \underline{\infty}\ (3).$$

We observe first that the standard equality axioms, $(\mathrm{E_R})$, $(\mathrm{E_S})$, $(\mathrm{E_T})$ holds for every element in $\mathscr{M}_R$ and hence $\underline{\infty} = \underline{\infty}$ $(\dagger)$. Composition of trees is still a function and introducing the non-standard element does not violate the injectivity of pairing, hence $(\mathrm{E_F})$ and $(\mathrm{E_I})$ is also true in $\mathscr{M}_R$. In addition $(\mathrm{A}^1_<)$ holds in case $x = \underline{\infty}$, because of (2).

The containment axiom, $(\mathrm{A}^2_<)$, is a bit more delicate; the truth value of a sentence $x < (y.z)$ is evaluated by recursively nesting $(y.z)$ and x if it can not be decided directly. But this is possible. A case analysis shows that tree containment is interpreted in $\mathscr{M}_R$. Consider therefore

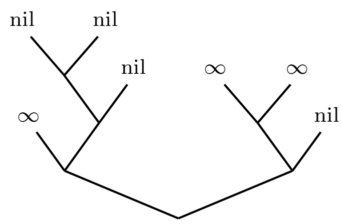
$$\infty < \mathrm{nil}\ (a) \quad \mathrm{nil} < \infty\ (b) \quad \infty < \infty\ (c) \quad \infty < (\infty.\mathrm{nil})\ (d) \quad \infty < (\infty.\infty)\ (e)$$

(a) is false according to (2), while (b) is true because of (1).
Consider (c): Since $\infty = (\mathrm{nil}.\infty)$ it is sufficient to treat $\infty < (\mathrm{nil}.\infty)$. Writing out the axiom $(\mathrm{A}^2_<)$ gives $\mathrm{nil} = \infty \vee \mathrm{nil} < \infty \vee \infty = \infty \vee \infty < \infty$, and by $(\dagger)$, reflexivity of ∞, the third disjunt makes (c) true in $\mathscr{M}_R$.
Consider (d), $\infty < (\infty.\mathrm{nil})$. Since $\infty = \infty \vee \infty < \infty \vee \infty = \mathrm{nil} \vee \infty < \mathrm{nil}$ is true by $(\dagger)$ or (c), we conclude that (d) is also true.
Consider (e), $\infty < (\infty.\infty)$: Plugging the instance into the axiom gives the formula $\infty = \infty \vee \infty < \infty \vee \infty = \infty \vee \infty < \infty$, where each of the disjuncts are true according to either $(\dagger)$ or (c), hence $\mathscr{M}_R \models \infty < (\infty.\infty)$.

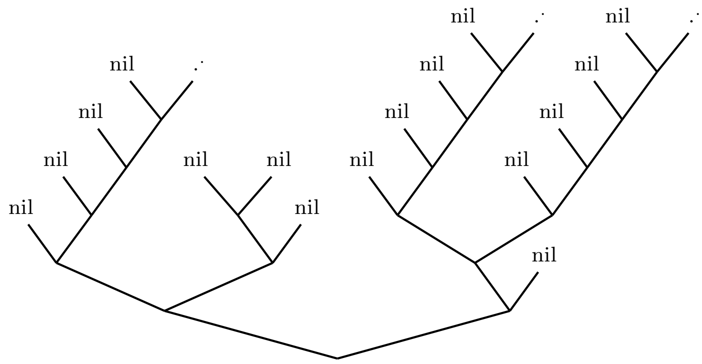
The truth of $(\mathrm{A}^3_=)$ is not violated by introducing more labels since nil is still a minimal element in the extended theory according to (3).

$(\mathrm{A}^4_=)$ holds in the extended model. Since the non-standard trees introduce even more elements $(\mathrm{nil}.\infty)$, the non-standard labels are making the second disjunct true; that is each label is of the form $x = \mathrm{nil} \vee x = \infty$, i.e. $x = \mathrm{nil} \vee x = (\mathrm{nil}.\infty)$, hence $x = \mathrm{nil} \vee \exists y \exists z (x = (y.z))$. $\square$

Consider the non-standard $\mathscr{M}_R$ tree, $((\infty.((\mathrm{nil}.\mathrm{nil}).\mathrm{nil})).((\infty.\infty).\mathrm{nil}))$, denoted π. For non-standard trees, there is a finite representation. The tree π, can be represented by π_c:



An infinite representation of π, can be obtained by expanding the tree by $3 \times \omega$ applications of the equation (A^R_∞), giving the tree π_e;



where each occurence of ∞ by (A^R_∞) corresponds to what we call a *right oriented ω-ladder*. We say that the tree π_e is the *complete expansion* of π, written $\mathrm{ex}(\pi)$, if each non-standard label in π is replaced by ω times applications of its defining equation. Conversely, π_c is the *complete contraction* of a tree π, written $\mathrm{co}(\pi)$, if every ω-ladder is replaced by the non-standard object defining it.

Observation 2

1. $\mathrm{co}(\mathrm{ex}(\pi)) = \mathrm{co}(\pi) = \mathrm{co}(\mathrm{co}(\pi))$.
2. $\mathrm{ex}(\mathrm{ex}(\pi)) = \mathrm{ex}(\pi) = \mathrm{ex}(\mathrm{co}(\pi))$.

Observation 3 *Every non-standard infinite tree in $\mathscr{M}_R$ can be collapsed to a finite non-standard tree with $\mathbb{L}_R = \{\underline{\mathrm{nil}}, \underline{\infty}\}$ and vice verca.*

Observation 4 *Every tree in $\mathscr{M}_R$ has cardinality $\leq \omega$.*

Proposition 2 *There is an interpretation of the Robinson axioms in* MINPAR*, in the right oriented modell $\mathscr{M}_R$, where $\Re_1$ - $\Re_7$ are true but $\Re_8$ is false.*

Proof: We already know now that $\Re_8$ is false in $\mathscr{M}_R$, so our goal is to explore the modell $\mathscr{M}_R$. This is done by investigating the right oriented numerals, denoted ron:

$$\mathrm{nil}^0_R := \mathrm{nil} \quad \mathrm{nil}^{n+1}_R := (\mathrm{nil}.\mathrm{nil}^n_R)$$

Then we define the interpretation of the function symbols s, $+$ and $\times$ as follows.

s	
nil^n_R	$(\mathrm{nil}.\mathrm{nil}^n_R)$
∞_R	$(\mathrm{nil}.\infty_R)$

$+$	nil^m_R	∞_R
nil^n_R	nil^{n+m}_R	∞_R
∞_R	∞_R	∞_R

$\times$	nil	nil^m_R	∞_R
nil	nil	nil	nil
nil^n_R	nil	$\mathrm{nil}^{n\times m}_R$	∞_R
∞_R	nil	∞_R	∞_R

$\Re_1$ is a special case of the injectivity axiom for MINPAR. $\Re_2$ holds by the definition the right oriented numerals. $\Re_2$ is true since $\mathrm{nil} \neq (\mathrm{nil.nil})$. A case analysis of ron gives the result. We prove that the recursion equations for addition is satisfied in the modell, $\mathscr{M}_R \models \Re_4$ and $\mathscr{M}_R \models \Re_5$, and leave the rest as exercises.

$$(\Re_4)\ \ \mathrm{nil}_R^n + \mathrm{nil} = \mathrm{nil}_R^{n+0} = \mathrm{nil}_R^n \quad \infty_R + \mathrm{nil} = \infty_R + \mathrm{nil}_R^0 = \infty_R$$

Then consider the axiom $\Re_5$, $x + s(y) = s(x + y)$. There are four cases; either

$$\begin{array}{llllll} x = \mathrm{nil}_R^n & \text{and} & y = \mathrm{nil}_R^m & (a), & \text{or} \\ x = \mathrm{nil}_R^n & \text{and} & y = \infty_R & (b), & \text{or} \\ x = \infty_R & \text{and} & y = \mathrm{nil}_R^m & (c), & \text{or} \\ x = \infty_R & \text{and} & y = \infty_R & (d). & \end{array}$$

We prove (b) and leave the other as exercises.

$$(\Re_5)\ \ \mathrm{nil}_R^n + (\mathrm{nil}.\infty_R) =^1 \mathrm{nil}_R^n + \infty_R =^2 \infty_R.$$

$=^1$ follows by A_∞^R and $=^2$ by the definition of $+$. $\Box$

By using ron from the previous proof, the concept of 'right oriented ω-ladder' can be defined precise:

$$\mathrm{ex}(\infty_R) = \lim_{n\to\infty} \mathrm{nil}_R^n$$

The previous definition is not a formal definition in the logical sense of the word since it is not formulated in the language of logic. A formal definition, that is an arithmetization of the infinite object $\mathrm{ex}(\infty_R)$, is given by the formula

$$\mathrm{ex}(\infty_R) \quad \leftrightarrow \quad \exists x\, \forall y < x\, (\mathrm{hd}(y) = \underline{\mathrm{nil}} \wedge \exists z (y < z \wedge z < x))$$

Since the containment relation collaps in case of non-standard elements, a more explicit statement of containment must be made in order to compare standard elements with non-standard.

Lemma 1 $\forall n\ (\mathrm{nil}_R^n < \infty_R \wedge \mathrm{nil}_R^n \neq \infty_R)$

Proof: By induction over n. Basis is observed to be correct by recapitulating the construction of $\mathscr{M}_R$; $\mathrm{nil}_R^1 = \mathrm{nil} < \infty_R$ and $\mathrm{nil}_R^1 = \mathrm{nil} \neq \infty_R$.

Suppose therefore as the lemma holds for n:

$$\mathrm{nil}_R^n < \infty_R \wedge \mathrm{nil}_R^n \neq \infty_R$$

Then by axiom $(\mathrm{A}_<^5)$, we have the instance;

$$\mathrm{nil}_R^n < \infty_R \rightarrow (\mathrm{nil.nil}_R^n) < (\mathrm{nil}.\infty_R)$$

which by the definition of ron and A^R_∞ is the same as;

$$\mathrm{nil}^n_R < \infty_R \to \mathrm{nil}^{n+1}_R < \infty_R$$

Furthermore by contraposition of the axiom of injectivity $(\mathrm{E_I})$

$$\mathrm{nil}^n_R \neq \infty_R \to (\mathrm{nil}.\mathrm{nil}^n_R) \neq (\mathrm{nil}.\infty_R)$$

which by a similar argument gives;

$$\mathrm{nil}^n_R \neq \infty_R \to \mathrm{nil}^{n+1}_R \neq \infty_R$$

□

The construction of right oriented ω-ladders was a bit arbitrary, since the representation of numbers could equally have been represented by the equations:

$$\underline{0} := \underline{\mathrm{nil}} \quad \underline{n+1} := (\underline{n}.\underline{\mathrm{nil}})$$

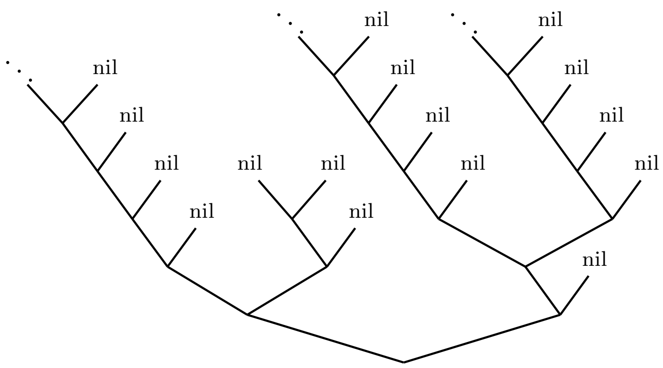
Hence the corresponding collapsing equation for this representation is;

$$\infty_L = (\infty_L.\mathrm{nil}) \quad (A^L_\infty)$$

Proposition 3 *Let* $\mathscr{L}_L = \mathscr{L} \cup \{\underline{\infty_L}\}$ *and* $\mathrm{T}_L = \mathrm{T} \cup \{\underline{\infty_L} = (\underline{\infty_L}.\underline{\mathrm{nil}})\}$. *Then* $\mathscr{M}_L$ *is a non-standard model for* MINPAR *making* $(\Re_8)$ *false.*

Proposition 4 *There is an interpretation of the Robinson axioms in* MINPAR*, in the right oriented modell* $\mathscr{M}_L$*, where* $\Re_1$ *-* $\Re_7$ *are true but* $\Re_8$ *is false.*

The tree π presented previously, relabelled from ∞ to ∞_L and completely expanded yields:



4.1 The limits of complete expansions

To avoid confushion let $\underline{\infty_R}$ denote the right oriented element $\underline{\infty}$ introduced before. In taking the limit of big trees we are only interested in its infinite behaviour. This is captured by the purely non-standard trees. A tree π is *purely non-standard* if its complete contraction $\mathrm{co}(\pi)$ contains only non-standard labels. Then we can define two limit trees for each of the nonstandard models $\mathscr{M}_L$ and $\mathscr{M}_R$. Hence by recursion on the natural numbers we define two specific classes of purely non-standard trees.

Definition 3 *The purely non-standard trees* ∞_L^n *and* ∞_R^n *are defined as follows;*

$$\begin{array}{ll} \infty_L^1 = \infty_L & \infty_R^1 = \infty_R \\ \infty_L^{n+1} = (\infty_L.\infty_L^n) & \infty_R^{n+1} = (\infty_R^n.\infty_R) \end{array}$$

Both ∞_L^n and ∞_R^n defines an infinite sequence of purely non-standard trees. Although it seems at face value that the trees are not grounded, one can consider the constructor as "pushing" the tree upwards. The limit is then given by:

$$\infty_L^\omega \ = \ \lim_{n\to\infty} \infty_L^n \qquad \infty_R^\omega \ = \ \lim_{n\to\infty} \infty_R^n$$

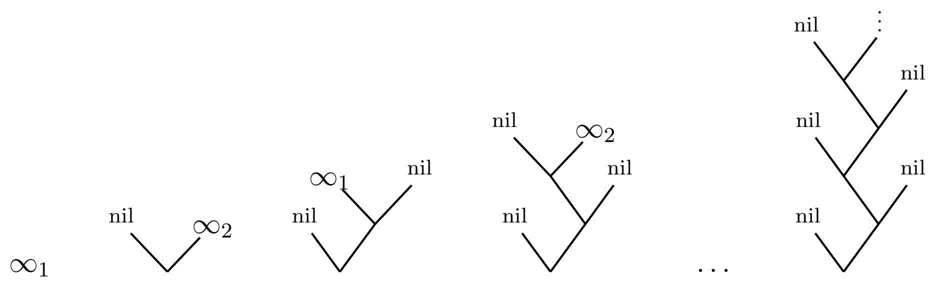
Observation 5 *Neither* $\mathrm{ex}(\infty_L^\omega)$ *nor* $\mathrm{ex}(\infty_R^\omega)$ *are balanced trees.*

4.2 Mutually recursively defined labels

Non-standard elements can be defined by mutual recursion. Consider for instance the extension of T with two labels ∞_1 and ∞_2:

$$\infty_1 = (\mathrm{nil}.\infty_2) \quad \infty_2 = (\infty_1.\mathrm{nil})$$

Consider the tree ∞_1. The complete expansion of ∞_1 is the infinite tree depicted to the right



Is the zik-zak model $\mathscr{M}_z$ generated by extending the language and the theory;

$$\mathscr{L}_z = \mathscr{L} \cup \{\underline{\infty_1}, \underline{\infty_2}\} \quad \mathrm{T}_z = \mathrm{T} \cup \{\infty_1 = (\mathrm{nil}.\infty_2), \infty_2 = (\infty_1.\mathrm{nil})\}$$

a non-standard modell? The answer is yes:

Proposition 5 *Let $\mathscr{L}_z$ and T_z be as defined above. Then there is a non-standard zik-zak model $\mathscr{M}_z$ for* MINPAR *that evaluates* $(\Re_8)$ *to false.*

Proof: First observe we need only consider finite trees, the complete contraction, and expand the trees finitely if necessary (no complete expansion will be made). We define $\mathscr{M}_z$ as follows:

First we give another interpretation of the numerals to ensure that $(\Re_8)$ is false in $\mathscr{M}_z$. Let $s(x)$ be interpreted in $\mathscr{M}_z$ as the function giving the *slalom numerals* $(\mathrm{nil}.(x.\mathrm{nil}))$. The slalom numerals are defined inside zik-zak models by the Δ^0_0 formula:

$$\mathrm{NAT}(x) \leftrightarrow x = \underline{\mathrm{nil}} \vee \forall y < x\,(\mathrm{hd}(y) = \underline{\mathrm{nil}} \wedge \mathrm{tl}(\mathrm{tl}(y)) = \mathrm{nil} \vee y = \underline{\mathrm{nil}}).$$

By a two-step expansion of ∞_2, we can make $(\Re_8)$ false by considering the equation $\infty_1 = (\mathrm{nil}.\infty_2) = (\mathrm{nil}.(\infty_1.\mathrm{nil}))$, hence $\infty_1 = s(\infty_1)$.

Then we declare that $\infty_1 = \infty_1$ (1) and $\infty_2 = \infty_2$ (2). This makes the reflexivity axiom $(\mathrm{E_R})$ true. But then symmetry and transitivity also holds, i.e. $(\mathrm{E_S})$ and $(\mathrm{E_T})$ are both true.

Moreover, let both $\mathscr{M}_z \models \neg\underline{\infty_1} < \underline{\mathrm{nil}}$ (3) and $\mathscr{M}_z \models \neg\underline{\infty_2} < \underline{\mathrm{nil}}$ (4). This makes $(\mathrm{A}^1_<)$ true. Since a one step expansion of both ∞_1 and ∞_2 results in a composite tree, we also require that $\mathrm{nil} \neq \infty_1$ (5) and $\mathrm{nil} \neq \infty_2$ (6). Hence $(\mathrm{A}^3_=)$ is true since both $\mathscr{M}_z \models \neg\underline{\mathrm{nil}} = (\underline{\mathrm{nil}}.\underline{\infty_2})$ and $\mathscr{M}_z \models \neg\underline{\mathrm{nil}} = (\underline{\infty_1}.\underline{\mathrm{nil}})$.

Pairing is still a function, hence $\mathscr{M}_z \models \mathrm{E_F}$. To guarantee that pairing is injective we require $\infty_1 \neq \infty_2$. If not we would have by one-steps expansions on each side of the equation $(\mathrm{nil}.\infty_2) = (\infty_1.\mathrm{nil})$, and then by $(\mathrm{E_I})$ we would have $\mathrm{nil} = \infty_1$ and $\infty_1 = \mathrm{nil}$ which are false by (5) and (6).

The axiom $(\mathrm{A}^3_=)$ is true since by (5) and (6), $\mathscr{M}_z \models \underline{\mathrm{nil}} \neq \underline{\infty_1} \wedge \underline{\mathrm{nil}} \neq \underline{\infty_2}$, and therefore also $\mathscr{M}_z \models \underline{\mathrm{nil}} \neq (\underline{\mathrm{nil}}.\underline{\infty_2}) \wedge \underline{\mathrm{nil}} \neq (\underline{\infty_1}.\underline{\mathrm{nil}})$.

The Robinson axiom $(\mathrm{A}^4_=)$ holds in $\mathscr{M}_z$, since ∞_1 and ∞_2 by a one step expansion yields a composite tree: The labels in a contracted zik-zak tree are of three kinds, $\mathscr{M}_z \models x = \underline{\mathrm{nil}} \vee x = \underline{\infty_1} \vee x = \underline{\infty_2}$, hence the one step expansion gives $\mathscr{M}_z \models x = \underline{\mathrm{nil}} \vee x = (\underline{\mathrm{nil}}.\underline{\infty_2}) \vee x = (\underline{\infty_1}.\underline{\mathrm{nil}})$, which yields $\mathscr{M}_z \models x = \underline{\mathrm{nil}} \vee \exists y \exists z\,(x = (y.z))$.

Finally we consider containment with respect to the non-standard labels, the axiom $(\mathrm{A}^2_<)$. There are four cases:

$$\infty_1 < \infty_2\,(a) \quad \infty_1 < \infty_1\,(b) \quad \infty_2 < \infty_1\,(c) \quad \infty_2 < \infty_2\,(d)$$

Consider (a): Expanding $\infty_1 < \infty_2$ gives $\infty_1 < (\infty_1.\mathrm{nil})$, and since $\infty_1 = \infty_1$ is true by (1), the first disjunct in $(\mathrm{A}^2_<)$ gives $\mathscr{M}_z \models \underline{\infty_1} < \underline{\infty_2}$.
Consider (b): Expanding $\infty_1 < \infty_1$ gives $\infty_1 < (\mathrm{nil}.\infty_2)$ and since $\infty_1 < \infty_2$ is

true by (a), (b) is also true by the fourth disjunct in $(\mathrm{A}^2_<)$, i.e. $\mathscr{M}_z \models \underline{\infty_1} < \underline{\infty_1}$.
Consider (c): Expanding $\infty_2 < \infty_1$ gives $\infty_2 < (\mathrm{nil}.\infty_2)$. But since $\infty_1 = \infty_1$ by (2) we get from the third disjunct in $(\mathrm{A}^2_<)$, $\mathscr{M}_z \models \underline{\infty_2} < \underline{\infty_1}$.
Consider (d): Expanding $\infty_2 < \infty_2$ gives $\infty_2 < (\infty_1.\mathrm{nil})$. Similarly by (c) the second disjunct in $(\mathrm{A}^2_<)$ gives $\mathscr{M}_z \models \underline{\infty_2} < \underline{\infty_2}$.
This case analysis is exhaustive since any contracted tree in the zik-zak model can be compared with respect to the labels. □

4.3 The Roger label

Let $\star$ denote the *Roger label*. Extend the language $\mathscr{L}$ with the non-standard Roger label, $\mathscr{L}_\star = \mathscr{L} \cup \{\underline{\star}\}$ and augement the theory with equations saying

$$\star = (\star.\star) \quad (A_\star)$$

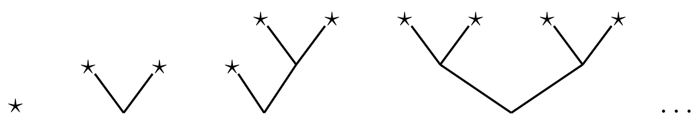
In other words, $\mathrm{T}_\star = \mathrm{T} \cup \{\underline{\star} = (\underline{\star}.\underline{\star})\}$.

Observation 6 *The expansion of the single Roger tree $\star$ gives a infinite balanced binary tree without labels.*

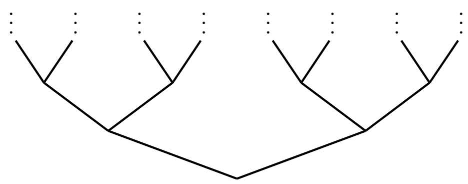
Proposition 6 *$\mathscr{M}_\star$ is a model for* MINPAR.

Proof: See forthcomming essay by Roger Antonsen. □

A step wise expansion of $\star$ gives for instance the following *expansion sequence*:



The complete expansion of $\star$ is the infinite tree depicted below:



Proposition 7 *The Roger tree $\star$ can not be reached constructively from any combination of $\mathscr{M}_L$, $\mathscr{M}_R$ or $\mathscr{M}_z$. $\star$ can not even be reached from any transfinite closure of the models presented in the paper.*

Proof: The first part is obvious. That no transifinite closure can ever reach $\star$ is left as a problem to the reader. $\square$

Conjecture 1 *The Roger tree $\star$ is constructively inaccessible from any non-standard modell simpler than $\mathscr{M}_\star$.*

The problem for deciding this one is to give precise meaning to 'simple' and 'constructively'. This is left as an open problem.

5 Conclusion

The non-standard models presented in the paper are all making the theorem $\Re_8$ of arithmetic false except the $\mathscr{M}_\star$. But more general we observe that what makes them non-standard is the fact that the containment relation collaps, i.e. there are objects x that are self identical, $x = x$, yet contained in themselves, $x < x$. The obvious theorem in MINPAR making this false $\neg x < x$, which is not provable. The reason why the models are so simple is that MINPAR has no power of comparing every element. The elements are not totally ordered, as in number theory given by the theorem $x = y \vee x < y \vee y < x$.

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Exercises

At the end we present some easy and some difficult exercises to the reader who wants to go deeper into this subject.

Exercise 1 *(Easy) Suppose that everything you know is $\mathscr{M}_R$. Are the following true? Give a formal proof of your answer.*

1. $(\infty.((\mathrm{nil}.\mathrm{nil}).\mathrm{nil})) < ((\infty.((\mathrm{nil}.\mathrm{nil}).\mathrm{nil})).((\infty.\infty).\mathrm{nil}))$
2. $((\infty.\infty).\mathrm{nil}) < ((\infty.((\mathrm{nil}.\mathrm{nil}).\mathrm{nil})).((\infty.\infty).\mathrm{nil}))$
3. $((\infty.((\infty.\infty).\mathrm{nil})).((\infty.\infty).\mathrm{nil})) < ((\infty.((\mathrm{nil}.\mathrm{nil}).\mathrm{nil})).((\infty.\infty).\mathrm{nil}))$
4. $(\infty.((\mathrm{nil}.\mathrm{nil}).\mathrm{nil})) < ((\infty.\infty).\mathrm{nil})$

5. $((\infty.\infty).\text{nil}) < (\infty.((\text{nil}.\text{nil}).\text{nil}))$

6. $((\text{nil}.(\text{nil}.(\text{nil}.\infty))).((\text{nil}.\text{nil}).\text{nil})) < ((\infty.((\text{nil}.\text{nil}).\text{nil})).((\infty.\infty).\text{nil}))$

7. $((\infty.((\text{nil}.\text{nil}).\text{nil})).((\infty.\infty).\text{nil})) < ((\infty.((\text{nil}.\text{nil}).\text{nil})).((\infty.\infty).\text{nil}))$

Exercise 2 *(Easy) Now jump over to the left oriented modell $\mathscr{M}_L$ and interpret the label ∞ in the previous exercise as ∞_L instead. Decide which of the sentences in the previous exercise that are true or false.*

Exercise 3 *(Intermediate) Prove formally that $\mathscr{M}_L$ is a model for* MINPAR. *This means prove Proposition 3.*

Exercise 4 *(Easy) Complete the proof of Observation 1.*

Exercise 5 *(Easy) Complete the proof of Proposition 2.*

Exercise 6 *(Intermediate) Define left orient numerals,* lon. *Prove formally that $\mathscr{M}_L$ is a model for the Robinson axioms. This means prove Proposition 4.*

Exercise 7 *(Easy) Consider the zik-zak tree π given as follows:*

$$((((\text{nil}.((\text{nil}.\infty_2).\text{nil})).\infty_1).((\text{nil}.((\text{nil}.(\infty_1.\text{nil})).\text{nil})).\text{nil})).(\text{nil}.(\infty_1.\infty_2)))$$

How many 1-step contractions can be performed to get the normal form of the previous zik-zak tree? Draw both trees. (This means that you shall compute $\mathrm{co}(\pi)$ by counting each rewrite step.)

Exercise 8 *(Intermediate) Show that to be a left oriented ω-ladder is definable using the language $\mathscr{L}$. This means by using only* $(.)$, $\underline{\text{nil}}$ *and logic. (You can of course also use* $\mathrm{hd}()$ *and* $\mathrm{tl}()$.*) Show that you can not define it with a Δ_0^0 sentence.*

Exercise 9 *(Hard) Decide how to solve the conjecture and do it.*

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