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Chern numbers of matroids

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Matroids are combinatorial objects that abstract the notion of independence in mathematics. Motivated by the Chern classes of manifolds and non-singular varieties, we define *Chern numbers* of an arbitrary matroid, even when the matroid is not necessarily representable. These numbers are obtained when intersecting appropriate matroid Chern-Schwartz-MacPherson cycles as defined in [MRS20].

To a matroid M of rank 3, we associate two Chern numbers, namely $\overline{c}_2(M)$, and $\overline{c}_1^2(M)$. We prove that both Chern numbers of matroids of rank 3 on a ground set of n elements are positive, and that their ratio is bounded: $(2n-6)/(n-2) \leq \overline{c}_1^2(M)/\overline{c}_2(M) \leq 3$. If the matroid is orientable, the ratio is bounded above by $5/2$. Moreover, we perform computations for the Chern numbers of matroids of rank 4. Finally, we give a formula for the Chern numbers of the uniform matroid of any rank.

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Contents

List of Figures

List of Tables

CHAPTER 1

Introduction

Matroids are combinatorial objects that abstract the notion of independence throughout several branches of mathematics. The Japanese mathematician Takeo Nakasawa and the American mathematician Hassler Whitney independently developed the theory of matroids in the 1930s. While Nakasawa's work was forgotten for a long time, Whitney's work was the first to be acknowledged in the mathematical world [NK09].

Our take on matroid theory will be from an algebraic geometry perspective. More specifically, given a hyperplane arrangement $\mathcal{A} \subseteq \mathbb{P}_K^{d+1}$, there is an associated matroid $M_{\mathcal{A}}$ which encodes the combinatorial properties of the linear subspaces of $\mathcal{A}$. Matroids that are constructed in this way are called *representable over a field* K. Note that representable matroids constitute about 0% of all matroids [Nel16]. A natural question is: can we define invariants of a general matroid, i.e., a matroid that is not necessarily representable, but that also have a geometric meaning when the matroid is in fact representable?

In this thesis we introduce a new invariant of matroids, namely the *Chern numbers* of a matroid, which are some weights that we obtain when intersecting appropriate weighted polyhedral fans that are given by the matroid.

1.1 Background and contribution

We will now give a brief overview of the necessary prerequisites to give a formal definition of the Chern numbers of matroids, as well as a motivation for introducing this new invariant.

There are at least eight equivalent definitions for a matroid. The definition that is the most appropriate for our purposes is in terms of a rank function. A matroid on a finite set E of rank $d+1$ is a function $\mathrm{r} : 2^E \to \mathbb{Z}_{\geq 0}$ satisfying:

1. $0 \leq \mathrm{r}(S) \leq |S|$,

2. $S \subseteq U$ implies $\mathrm{r}(S) \leq \mathrm{r}(U)$,

3. $\mathrm{r}(S \cup U) + \mathrm{r}(S \cap U) \leq \mathrm{r}(S) + \mathrm{r}(U)$.

Moreover, a *flat* F of a matroid M is a subset of E such that $\mathrm{r}(F) < \mathrm{r}(F \cup \{i\})$ for any element $i \in E \setminus F$. When the matroid arises from a hyperplane arrangement, then the flats F are in one to one correspondence with the linear subspaces $L_F \subseteq \mathcal{A}$, and the value $\mathrm{r}(F)$ corresponds to the codimension of L_F in $\mathbb{P}^d_K$. Moreover, to a matroid of rank $d+1$ we can associate a rational polyhedral fan Σ_M called the *Bergman fan* of the matroid, living in $\mathbb{R}^{|E|-1}$, having a cone for each chain of flats of the matroid.

The *Chow ring* of a variety is a ring generated by the cycles of that variety, quotient by rational equivalence. For matroid M, the Chow ring $A^*(M)$ is a polynomial ring given by

$$A^*(M) = \frac{\mathbb{Z}[x_F : \ F \text{ is a flat of } M]}{\mathcal{I}_M + \mathcal{J}_M},$$

where $\mathcal{I}_M$, and $\mathcal{J}_M$ are ideals encoding the inclusion properties of the flats. The Chow ring of a matroid has some important properties. First of all, Feichter and Yuzvinsky proved in [FY04] that when the matroid arises from a hyperplane arrangement, then the Chow ring the matroid is isomorphic to the Chow ring of the de Concini and Procesi *wonderful compactification of the complement of the arrangement* $W_{\mathcal{A}}$ [DP95]. In this case, if we let $L_F \subsetneq \mathcal{A}$ be the linear subspace corresponding to a flat F, then the generator $x_F \in A^*(M_{\mathcal{A}})$ corresponds to the strict transform $\hat{L}_F \subseteq W_{\mathcal{A}}$ of the linear subspace $L_F \subseteq \mathcal{A}$. Moreover, Adiprasito, Huh, and Katz proved that the Chow ring of a matroid satisfies Poincaré duality and linear duality [AHK18], which give us the following isomorphism

$$A^*(M) \cong MW_*(\Sigma_M),$$

where $MW_*(\Sigma_M)$ is the ring of *Minkowski weights*, i.e., the ring of weighted $k-$skeletons of the Bergman fan Σ_M satisfying a certain balancing condition. The ring structure on $MW_*(\Sigma_M)$ is given by a certain cap-product, moreover there exists an isomorphism $\mathrm{wt}_0 : MW_0(\Sigma_M) \to \mathbb{Z}$, naturally defined by the weight assigned to the vertex of the fan.

López de Medrano, Rincón, and Shaw introduce in [MRS20] the *Chern-Schwartz-MacPherson (CSM) cycles* of a matroid. A k-dimensional CSM cycle $\mathrm{csm}_k(M) \in MW_k(M)$ is a Minkowski weight, where the weight is given according to some specific combinatorial rule, see Chapter 5. They prove that the CSM cycles of a matroid are related to the CSM classes in algebraic geometry. These classes are a generalization of the Chern class of the tangent bundle over a non-singular compact variety, see Section 5.3. When the matroid $M_{\mathcal{A}}$ arises from an arrangement $\mathcal{A}$, the authors in [MRS20] prove that

$$\mathrm{CSM}(\mathbb{1}_{C(\mathcal{A})}) = \sum_{k=0}^{d} \mathrm{csm}_k(M_{\mathcal{A}}) \in A_*(W_{\mathcal{A}}) \cong MW_*(\Sigma_{M_{\mathcal{A}}}),$$

where $\mathrm{CSM}(\mathbb{1}_{C(\mathcal{A})})$ is the CSM class of the group of constructible function on the complement of the arrangement $C(\mathcal{A})$.

The relation between CSM classes and representable matroids inspired us to define a new invariant of matroids:

Definition 1.1.1. (Definition 5.4.1) Let M be a rank $d+1$ matroid. We define the *Chern numbers of a matroid* $\bar{c}_1^{k_1}\bar{c}_2^{k_2}\cdots\bar{c}_d^{k_d}(M)$ to be

$$\bar{c}_1^{k_1}\cdots\bar{c}_d^{k_d}(M) = \mathrm{wt}_0(\mathrm{csm}_{d-1}^{k_1}(M)\mathrm{csm}_{d-2}^{k_2}(M)\cdots\mathrm{csm}_0^{k_d}(M)),$$

where $\sum_{i=0}^{d} i \cdot k_i = d$, and the map wt_0 is the map in Equation (3.3).

We want to emphasize that the distribution of Chern numbers of algebraic manifolds of general type is already an established field of study, called the *geography of manifolds*, see for example [Hun89]. This field deals with problems related to finding bounds, and possible values of Chern numbers. In this thesis we will see that results in the field of geography of manifolds generalize to Chern numbers of matroids.

Line arrangements $\mathcal{A} \subseteq \mathbb{P}^2$, not all intersecting in the same point, give rise to matroids of rank 3, see Example 2.2.6. In Chapter 6, we examine properties of the Chern numbers of rank 3. Among other things, we prove that results about Chern numbers of line arrangements defined in [EFU18], generalize to hold for any matroid of rank 3.

Proposition 1.1.2 (Proposition 6.4.1)**.** *Let M be a simple matroid of rank 3 on the ground set $E = \{1, \ldots, n\}$, and let t_m be the number of flats of rank 2 of size m. If M has $t_n = t_{n-1} = 0$, then its Chern numbers are positive.*

Theorem 1.1.3 (Theorem 6.4.4)**.** *Let M be a simple matroid of rank 3 on $E = \{1, \ldots, n\}$, such that $t_n = t_{n-1}$. Then,*

$$\frac{2n-6}{n-2} \leq \frac{\overline{c}_1^2(M)}{\overline{c}_2(M)} \leq 3$$

Left inequality holds if and only if M is the uniform matroid $U_{3,n}$, and right equality holds if and only if M is the matroid of a finite projective plane.

In Chapter 7, we do computations for the Chern numbers of matroids of rank 4. Among other things, we give a formula for the Chern numbers of the uniform matroid of rank 4. Note that the *uniform matroid* $U_{d+1,n+1}$, which is a central matroid throughout the whole thesis, is defined as follows, for $S \subseteq E$

$$\mathrm{r}(S) = \begin{cases} |S| & \text{if} |S| < d+1 \\ d+1 & \text{otherwise.} \end{cases}$$

Then, the Chern numbers of $U_{4,n+1}$ are given by

$$\begin{aligned} \overline{c}_1^3(U_{4,n+1}) &= (3-n)^3, \\ \overline{c}_1\overline{c}_2(U_{4,n+1}) &= -\frac{1}{2}(n-3)^2(n-2), \\ \overline{c}_3(U_{4,n+1}) &= -\binom{n-1}{3}, \end{aligned}$$

see Example 7.0.5. In fact, we have also found a formula for the Chern numbers of the uniform matroid of arbitrary rank, see Proposition 5.4.5. Moreover, after deriving a formula for the Chern numbers of matroids arising from finite projective 3-dimensional spaces, see Example 7.0.6, we prove the following corollary.

Corollary 1.1.4 (Proposition 7.0.7)**.** *Let M be a matroid arising from the finite projective 3-space $PG(3, q)$ for a prime power q. Then*

$$\frac{\overline{c}_1^3(M)}{\overline{c}_3(M)} = 16.$$

Finally, we have written a script in Macaulay2 to compute the Chern numbers of matroids of both rank 3 and 4. This is particularly useful for computing the Chern numbers of matroids without as nice combinatorial properties as the uniform matroid and the finite projective space matroid.

1.2 Outline

The rest of the text is organized as follows:

Chapter 2 introduces matroids and objects related to them. We begin by introducing the combinatorics of hyperplane arrangements, then we give a rigorous definition of a matroid in terms of a rank function. Moreover we introduce operations on matroids such as deletion and contraction.

Chapter 3 after a brief introduction to polyhedral geometry, we give the definition of the Bergman fan a matroid. Then we introduce the de Concini-Procesi wonderful compactification of the complement of a hyperplane arrangement.

Chapter 4 introduces the Chow ring of a matroid $A^*(M)$, a quotient polynomial ring generated by variables corresponding to the flats of a matroid. Moreover, we introduce the ring of Minkowski weights $MW_*(\Sigma_M)$, which is the ring of balanced weighted k-skeletons of the Bergman fan. Finally, we show that $A^*(M)$ satisfies linear and Poincaré duality, which induces the isomorphism $A^*(M) \cong MW_*(\Sigma_M)$.

Chapter 5 introduces the Chern-Schwartz-MacPherson cycles (CSM cycles) of a matroid, which are Minkowski weights given by the Beta invariant. Then we show that the CSM cycles of a representable matroid $\mathcal{A}$ are related to CSM classes. Finally we define a new invariant of matroids, namely Chern numbers of matroids.

Chapter 6 presents computations and results on Chern numbers of matroids of rank 3. We prove for example that the Chern numbers of matroids of rank 3 are positive, and that their ratio is bounded by 3.

Chapter 7 presents computations and results on Chern numbers of matroids of rank 4. Among other things, we compute the Chern numbers of the uniform matroid of rank 4 of arbitrary size, and of the finite projective 3 space $PG(3, q)$ for arbitrary q's.

Chapter 8 presents questions for further investigation related to the geography of Chern numbers of matroids.

PART I

Matroid Theory

CHAPTER 2

Matroid Theory

In this chapter we introduce the background needed for the rest of the thesis. There exist several definitions for a matroid; for our purpose, defining a matroid in terms of a rank function will be the most convenient. Before introducing the rigorous definition of a matroid, we begin by introducing our motivation. That is the study of the combinatorics of hyperplane arrangements, or dually, of vector configurations, with the help of a rank function.

2.1 Hyperplane Arrangements

In this section we follow tightly Section 2 of [Kat14], and Chapter 4 of [MS15].

Let K be a field. Given some vectors $v_0, \dots, v_n$ in K^{d+1} spanning a subspace $V \subseteq K^{d+1}$, we investigate the combinatorics of the span of the subsets of these vectors. We call $E = \{0, , \dots, n\}$ the *ground set*, and we define a rank function $\mathrm{r}_{\mathrm{vec}} : 2^E \to \mathbb{Z}$, by for $S \subseteq E$:

$$\mathrm{r}_{\mathrm{vec}}(S) = \dim(\mathrm{Span}(\{v_i \mid i \in S\})).$$

Dually, each vector defines a hyperplane in $\mathbb{P}^d_K$ by setting

$$H_i = \{z \in \mathbb{P}^d_K : v_i \cdot z = 0\} = V(f_i = \sum_{j=0}^{d} v_{ij} x_j),$$

where i indexes the vector, and j the entry. The set $\mathcal{A} = \{H_i : 0 \leq i \leq n\}$ is called an *arrangement of* $n+1$ *hyperplanes* in $\mathbb{P}^d_K$. We can define another rank function $\mathrm{r}_{\mathrm{arr}} : 2^E \to \mathbb{Z}$, by, for $S \subseteq E$,

$$\mathrm{r}_{\mathrm{arr}}(S) = \mathrm{codim}(\cap_{i \in S} H_i).$$

We use as a convention that $\dim(\emptyset) = -1$, and hence that $\mathrm{codim}(\emptyset) = d+1$. Note that r_{arr} and $\mathrm{r}_{\mathrm{vec}}$ define the same function on the subsets of E. Given a subset $S \subseteq E$, let A_S be the matrix whose rows are the vectors $\{v_i \mid i \in S\}$, then $\bigcap_{i \in S} H_i = \ker A_S$. Moreover, the equality

$$d + 1 = \mathrm{rank}(A_S) + \mathrm{nullity}(A_S)$$

implies that the following equality

$$\mathrm{r}_{\mathrm{arr}}(S) = d + 1 - \mathrm{nullity}(A_S) = \mathrm{r}_{\mathrm{vec}}(S)$$

holds. For example, if we let $S = \emptyset$ we obtain the following value

$$\mathrm{r_{arr}}(\emptyset) = \mathrm{codim}(\cap_{i\in\emptyset} H_i) = \mathrm{codim}(\mathbb{P}^d_K) = 0,$$

or dually

$$\mathrm{r_{vec}}(\emptyset) = \dim(\mathrm{Span}(\{v_i \mid i \in \emptyset\})) = \dim(0) = 0.$$

Where the 0 in $\dim(0)$, is the 0-vector in K^{d+1}.

Example 2.1.1. Let

$$v_0 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ v_1 = \begin{bmatrix}0\\1\\0\end{bmatrix},\ v_2 = \begin{bmatrix}0\\0\\1\end{bmatrix},\ \text{and } v_3 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

be vectors in $\mathbb{C}^3$, and let $E = \{0, 1, 2, 3\}$ be the corresponding ground set. The map $\mathrm{r_{vec}} : 2^{|E|} \to \mathbb{Z}$ takes the following values on different subsets $S \subseteq E$:

$$\begin{aligned}
&\mathrm{r_{vec}}(\emptyset) = 0,\\
&\mathrm{r_{vec}}(\{i\}) = \dim(\mathrm{span}(v_i)) = 1 \quad \text{for } i \in E\\
&\mathrm{r_{vec}}(\{i,j\}) = \dim(\mathrm{span}(v_i, v_j)) = 2 \quad \text{for } i \neq j \in E\\
&\mathrm{r_{vec}}(\{0,1,2\}) = \dim(\mathrm{span}(v_1, v_2, v_3)) = 3.
\end{aligned}$$

Dually, each vector defines a hyperplane in $\mathbb{P}^d$. Hence we get the hyperplane

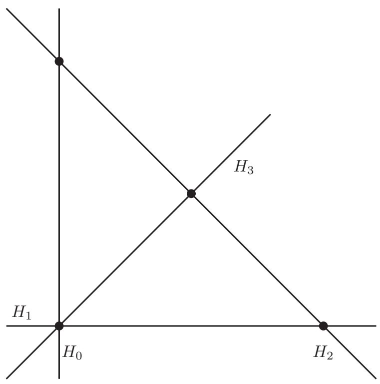

Figure 2.1: The arrangement of four lines in $\mathbb{P}^2$.

arrangement $\mathcal{A} \subset \mathbb{P}^d$ consisting of the following hyperplanes

$$H_0 = V(x_0),\ H_1 = V(x_1),\ H_2 = V(x_2),\ \text{and } H_3 = V(x_0 - x_1),$$

see Figure 2.1. Let us check that the map $\mathrm{r_{arr}} : 2^{|E|} \to \mathbb{Z}$ takes the same values as $\mathrm{r_{vec}}$ on different subsets $S \subseteq E$:

$$\begin{aligned}
&\mathrm{r_{arr}}(\emptyset) = \mathrm{codim}(\mathbb{P}^d) = 0,\\
&\mathrm{r_{arr}}(\{i\}) = \mathrm{codim}(H_i) = 1 \quad \text{for } i \in E\\
&\mathrm{r_{arr}}(\{i,j\}) = \mathrm{codmi}(H_i \cap H_j) = 2 \quad \text{for } i \neq j \in E\\
&\mathrm{r_{arr}}(\{0,1,2\}) = \mathrm{codmi}(H_0 \cap H_1 \cap H_2) = \mathrm{codim}(\emptyset) = 3.
\end{aligned}$$

It is easy to check that the maps $\mathrm{r}_{\mathrm{vec}}$ and $\mathrm{r}_{\mathrm{arr}}$ take the same values on the remaining subsets of E, and hence are the same rank function on the ground set E. ∎

Now, we show that we can identify hyperplane arrangements not necessarily intersecting in a single point of size $n+1$ with linear subspaces of $\mathbb{P}^n_K$, we follow the same construction as in [MS15, Chapter 4.1]. Assume that the vectors $\{v_i \mid i \in E\}$ span K^{d+1}, so the following equality

$$\mathrm{r}_{\mathrm{vec}}(E) = d + 1 = \mathrm{r}_{\mathrm{arr}}(E)$$

holds. Hence, the codimension of the intersection $\cap_{i\in E} H_i \subseteq \mathbb{P}^d_K$ of the corresponding hyperplanes is $d+1$, which implies that the intersection of the hyperplanes is empty. Now, let $\iota : \mathbb{P}^d_K \to \mathbb{P}^n_K$ be the inclusion map given by

$$z \to [f_0(z) : \cdots : f_n(z)] = [v_0 \cdot z : \cdots : v_n \cdot z]. \tag{2.1}$$

The map in Equation (2.1) is injective, since the vectors v_i span K^{d+1}, the 0-vector is the only element in the kernel. Hence, the image $L = \iota(\mathbb{P}^d_K)$ is a d-dimensional linear subspace of $\mathbb{P}^n_K$ [MS15]. Moreover, the image of a hyperplane $\overline{H_i} := \iota(H_i)$ is a new hyperplane in the linear subspace L. Let $[x_0 : \cdots : x_n]$ be the coordinates of $\mathbb{P}^n_K$, then the hyperplane $\overline{H_i}$ is given by $\overline{H_i} = L \cap (x_i = 0) \subseteq L$ for $0 \le i \le n$. So the arrangement $\mathcal{A}$ in $\mathbb{P}^d_K$ maps to the arrangement $\overline{\mathcal{A}} = \{\overline{H_i} : 0 \le i \le n\}$ in L, and the two arrangements have equivalent combinatorial properties. Finally, we can describe the linear subspace L as the vanishing of the ideal I constructed in the following way. As they do in Chapter 4.1 in [MS15], let A be the $(d+1)\times(n+1)$ matrix with the vectors v_i as column vectors, and let $B = (b_{ij})$ be the matrix whose column vectors are the basis of the null space of A. Then, the vanishing of the ideal

$$I = (f_j = \sum_{i=0}^{n} b_{ij} x_i \mid \text{for } 0 \le j \le n-d-1)$$

corresponds to the linear subspace $L \subseteq \mathbb{P}^n_K$.

Example 2.1.2. We continue with the previous example. Let the matrix A be as follows

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and as before let $\mathcal{A} \subseteq \mathbb{P}^2$ be the hyperplane arrangement given by the column vectors of A. Now, let $\iota : \mathbb{P}^2 \to \mathbb{P}^3$ be the embedding given by

$$[x_0 : x_1 : x_2] \to [x_0 : x_1 : x_2 : x_0 - x_1].$$

The null space of A is spanned by the single column vector in the matrix

$$B = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then, the vanishing of the ideal $I = (-x_0+x_1+x_3)$ is the image $\iota(\mathbb{P}^2)$. Moreover, the image of the hyperplane arrangement $\iota(\mathcal{A})$ is isomorphic to the arrangement $\overline{\mathcal{A}} \subseteq \mathbb{P}^3$ given by

$$\overline{\mathcal{A}} = \{H_i = V(x_i) \cap L \mid 0 \leq i \leq 3\} \subseteq L.$$

■

Hence hyperplane arrangements $\mathcal{A}$ in $\mathbb{P}^d_K$ are in one to one correspondence with linear subspaces L of $\mathbb{P}^n_K$ not contained in any hyperplane. In the next example we encounter a well known line arrangement called *the Braid arrangement*.

Example 2.1.3. Consider the homogeneous ideal

$$I = (x_0 - x_1 - x_3, x_0 - x_2 - x_4, x_1 - x_2 - x_5)$$

in $K[x_0, \dots, x_5]$, a linear ideal which defines the plane $L = V(I)$ in $\mathbb{P}^5$. And let $\mathcal{A} = \{H_i : 0 \leq i \leq 5\}$ be the arrangement of 6 lines in the plane given by $H_i = L \cap V(x_i)$, see Figure 2.2.

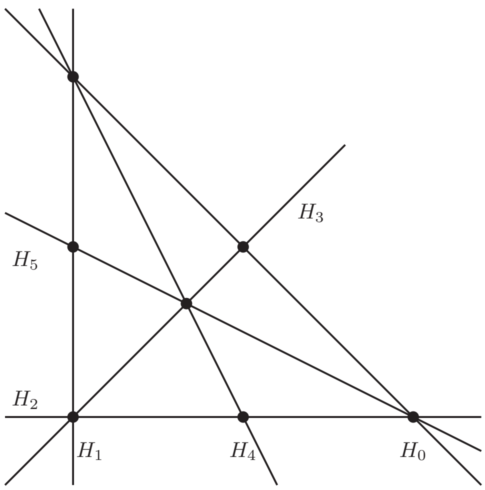

Figure 2.2: The Braid arrangement.

Let $E = \{0, \dots, 5\}$ be the ground set. Then, the rank of any one-element set is one, i.e., $\mathrm{r}(\{i\}) = \mathrm{codim}(H_i) = 1$ for any i in E, since the codimension of a line in a plane is one. The rank of any two-element set is two, i.e., $\mathrm{r}(\{i,j\}) = \mathrm{codim}(H_i \cap H_j) = 2$ for $i \neq j$ in E, since any two lines in a projective plane intersect in a point, and the codimension of a point in a plane is two. The rank of any $n-$set for $n \geq 3$, depends on whether the corresponding lines in the set intersect. For example, $\mathrm{r}(\{0,1,3\}) = \mathrm{codim}(H_0 \cap H_1 \cap H_3) = 2$, whereas $\mathrm{r}(\{0,1,2\}) = \mathrm{codim}(H_0 \cap H_1 \cap H_2) = \mathrm{codim}(\emptyset) = 3$. ■

2.2 Rank function of a matroid

Finally, with the above discussion in mind, we are ready to define our main object of study, namely *matroids*.

Definition 2.2.1 ([Kat14, Definition 3.1])**.** A *matroid* on a finite set E of rank $d+1$ is a function

$$2^E \to \mathbb{Z}$$

satisfying

1. $0 \leq \mathrm{r}(S) \leq |S|$,
2. $S \subseteq U$ implies $\mathrm{r}(S) \leq \mathrm{r}(U)$,
3. $\mathrm{r}(S \cup U) + \mathrm{r}(S \cap U) \leq \mathrm{r}(S) + \mathrm{r}(U)$ and
4. $\mathrm{r}(\{0, \ldots, n\}) = d + 1$.

Proposition 2.2.2. *The rank functions* r_{vec}*, and therefore* r_{arr}*, as defined in the previous section, are matroids.*

Proof. It is easy to check that $\mathrm{r}_{\mathrm{vec}}$, and therefore $\mathrm{r}_{\mathrm{arr}}$, as defined in the previous section, satisfy the first two matroid axioms. For the last axiom, as noted in [Kat14], if for $U \subset E$ we let

$$V_U = \mathrm{span}(v_i | i \in S),$$

then showing the third axiom is equivalent to showing

$$\mathrm{r}_{\mathrm{vec}}(V_{U \cap S}) \leq \mathrm{r}_{\mathrm{vec}}(V_U \cap V_S).$$

The inequality holds since $V_{U \cap S} \subseteq V_U \cap V_S$, moreover the inclusion can be strict as there are not necessarily vectors in the set $U \cap S$ spanning the subspace $V_U \cap V_S$. Hence, the function $\mathrm{r}_{\mathrm{vec}}$ together with a finite set of vectors, or equivalently, the function $\mathrm{r}_{\mathrm{arr}}$ together with a finite set of hyperplanes, are examples of a matroid. ■

We distinguish between *representable* and *non-representable* matroids.

Definition 2.2.3 ([Kat14, Definition 3.3])**.** A matroid is said to be *representable over a field* K if it is isomorphic to a matroid arising from a vector configuration in a vector space over K. A matroid is said to be *representable* if it is representable over some field. A matroid is said to be *regular* if it is representable over every field.

The rank function in Example 2.1.1 is an example of a matroid coming from a hyperplane arrangement, or dually from a vector configuration, hence it is representable. In fact, most matroids that we can think of, do come from a vector configuration. However, almost no matroids actually do. In 2016, Nelson proved that as n tends to infinity, the proportion of matroids on an n-element set that are representable tends to zero [Nel16].
We will explore more this notion of representable and non-representable matroids in Section 2.3 through some examples. We will see examples of matroids that are representable over only some given field, and an example of a matroid which is not representable over any field.

In the rest of the thesis, we often assume that the matroid is *simple*.

Definition 2.2.4 ([Kat14, Definition 3.8])**.** A loop of a matroid is an element $i \in E$ with $\mathrm{r}(i) = 0$. A pair of parallel points (i, j) of a matroid are elements $i, j \in E$ such that $\mathrm{r}(i) = \mathrm{r}(j) = \mathrm{r}(i, j) = 1$. A matroid is said to be *simple* if it has neither loops nor parallel points.

For hyperplane arrangements, a loop corresponds to a hyperplane $V(f_i = 0)$. Moreover, two distinct hyperplanes in a projective space always intersect, and hence for $i, j \in E$ such that $H_i \neq H_j$, the rank $r(\{i, j\}) = 2$. Hence, a pair of parallel points is a pair of identical hyperplanes. In the rest of the thesis, when we talk about a hyperplane arrangement $\mathcal{A}$, we mean a hyperplane arrangement with no repetition, and no degenerate hyperplanes. Moreover, hyperplane arrangements $\mathcal{A} = \{H_i \mid i \in E\} \subseteq \mathbb{P}^d$ such that the intersection $\cap_{i \in E} H_i = \emptyset$ are called *essential hyperplane arrangements.* Both in Example 2.1.1 and in Example 2.1.3, we encountered essential line arrangements in $\mathbb{P}^2$ giving rise to simple matroids of rank 3. We can generalize this result.

Proposition 2.2.5. *An essential hyperplane arrangement $\mathcal{A} \subseteq \mathbb{P}^d$ give rise to a simple matroid of rank $d + 1$.*

Example 2.2.6. Essential line arrangements $\mathcal{A} \subseteq \mathbb{P}^2$ give rise to simple matroids of rank 3. In fact, line arrangements and simple matroids of rank 3 share similar properties. For example, we know that two lines L_1 and L_2 in $\mathbb{P}^2$ intersect in exactly one point. The matroid equivalent statement is that for a simple matroid M of rank 3 on a ground set E, for any two flats $\{i\} \neq \{j\} \subseteq E$ of rank 1, there exists exactly one flat F of rank 2 containing both $\{i\}$ and $\{j\}$. First of all, there must exist such a flat F since $1 \leq r(\{i, j\}) \leq 2$ by the first two axioms in Definition 2.2.1. And, since we have assumed no parallel points, then $r(\{i, j\}) = 2$. Moreover, assume by contradiction that there exists a flat F' of rank 2 such that $F' \neq F$ and such that F' contains both $\{i\}$ and $\{j\}$. Then, by the second axiom of Definition 2.2.9 $(F' \cap F)$ is a flat. And since $(F' \cap F)$ contains $\{i, j\}$ it is of rank 2, and since, by definition, there can not be a strict inclusion of flats of rank 2, the flats F', and F must be the same. ■

At this point, it is maybe not clear yet why matroids are an abstraction of the notion of independence. But, in fact, one of the equivalent definitions for a matroid is in terms of independent subsets, i.e., subsets of E corresponding to linearly independent subsets. These are subsets $I \subset E$ such that $r(I) = |I|$.

Definition 2.2.7 ([Kat14, Definition 3.7])**.** A matroid is a collection of subsets $\mathcal{I}$ of E such that

1. $\mathcal{I}$ is nonempty,
2. Every subset of a member of $\mathcal{I}$ is a member of $\mathcal{I}$, and
3. If X and Y are in $\mathcal{I}$ and $|X| = |Y| + 1$, then there is an element $x \in X \setminus Y$ such that $Y \cup \{x\}$ is in $\mathcal{I}$.

The simplest example of a collection as defined in Definition 2.2.7 is given a finite set of vectors V the collection $\mathcal{I}_V$ of subsets consisting of linearly independent vectors. Proving that the collection $\mathcal{I}_V$ satisfies axiom one and two is quite straight forward, whereas proving that the collection $\mathcal{I}_V$ satisfies the last axiom requires a little bit more of work.

There are at least five other equivalent definitions for a matroid. Depending on the the context, or the purpose, other definitions can be better to apply. However, what is common for all definitions, is that there are a couple of straightforward axioms, and one which is harder to prove or to grasp. And, as Katz puts it in [Kat14], it is this last axiom which ads the flavor to the subject.

We end this section by presenting another equivalent way of defining a matroid, which is in terms of *flats*.

Definition 2.2.8 ([Kat14, Definition 3.4])**.** A flat of r is a subset $S \subseteq E$ such that for any $j \in E$, $j \notin S$, $\mathrm{r}(S \cup \{j\}) > \mathrm{r}(S)$.

If the matroid M is representable, then the flats are in bijection with the linear subspaces of the corresponding hyperplane arrangement $M_{\mathcal{A}}$.

Definition 2.2.9 ([Kat14, Definition 3.5])**.** A matroid is a collection of subsets $\mathcal{F}$ of a set E that satisfies the following conditions

1. $E \in \mathcal{F}$,
2. $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$, and
3. if $F \in \mathcal{F}$ and $\{F_1, F_2, \ldots, F_k\}$ is the set of minimal members of $\mathcal{F}$ properly containing F then the sets $F_1 \setminus F, F_2, \ldots, F_k \setminus F$ partition $E \setminus F$.

The set of flats form a partially ordered set by inclusion, the *lattice of flats* that we denote by $\mathcal{L}_M$. We denote $\hat{0}$ the minimal flat, if the matroid has no coloops, then $\hat{0} = \emptyset$. Moreover, a *flag of flats* is a chain of flats $F_1, \ F_2 \ldots, F_k \in \mathcal{F}$ on the form $\hat{0} \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k \subseteq E$. If the matroid is representable, then an inclusion of flats $F_i \subseteq F_j$ corresponds to the reversed inclusion of the corresponding subspaces $L_{F_j} \subseteq L_{F_i}$.

Example 2.2.10. The lattice of flats of the matroid in Example 2.1.3 is given in Figure 2.3. The edges in the graph represent inclusion of flats. Moreover, since the matroid is representable, the flats are in bijection with the linear subspaces of the Braid arrangement $\mathcal{A}$. Hence an inclusion of flats corresponds to an inclusion of the corresponding subspaces. For example the flat inclusion $\{0\} \subseteq \{0, 3\}$ corresponds to the inclusion of the point $H_0 \cap H_3$ on the line H_0 in the arrangement.

We end this section by giving two central definitions related to matroids that will only be mentioned briefly in the rest of this thesis. These definitions are also found in Chapter 3 in [Kat14].

Definition 2.2.11. Let M be a matroid of rank $d+1$ on a ground set E, then we call a subset $B \subseteq E$ a *basis* if $|B| = d+1$ and $\mathrm{r}(B) = d+1$.

Definition 2.2.12. Let M be a matroid of rank $d+1$ on a ground set E, then we call a subset $C \subseteq E$ a *circuit* if C is a minimal subset of E that is not contained in a basis.

2.3 Examples

In this section we introduce examples of matroids that will appear later in the thesis.

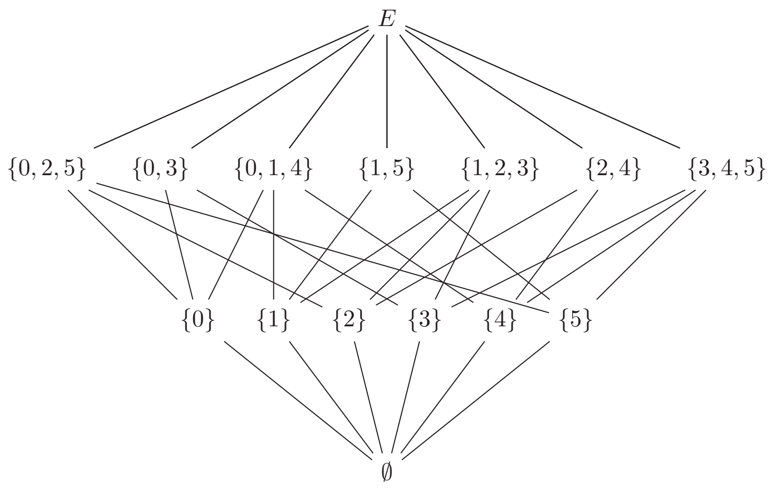

Figure 2.3: Lattice of flats of the Braid arrangement from Figure 2.2.

Example 2.3.1. The *uniform matroid* $U_{d+1,n+1}$ of rank $d+1$ on the ground set E of size $n+1$ is defined to be the rank function r for $S \subseteq E$

$$\mathrm{r}(S) := \begin{cases} |S| & \text{if } |S| < d+1 \\ d+1 & \text{otherwise.} \end{cases} \tag{2.2}$$

It is clear from the definition that any subset S of size $|S| \leq d$ is a flat. Hence, there are $\binom{n}{i}$ number of flats for each rank $1 \leq i \leq d$.

Note that the uniform matroid is not necessarily representable over any field k. It might be that the field is too small. For example the matroid $U_{2,4}$ is not representable over the field $\mathbb{F}_2$. That is because $U_{2,4}$ is the matroid of rank 2 on the set of 4 elements, where each element is of rank 1. On the other hand, the vector space $V_{\mathbb{F}}^2$ is only composed of the vectors

$$V_{\mathbb{F}_2}^2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

Hence there are not enough non-zero vectors $v \in V_{\mathbb{F}_2}^2$ to represent the matroid. In general, the uniform matroid $U_{d+1,n+1}$ is representable over a field k if either k is infinite, or if the vector space $|k|^{d+1} \setminus \{0\}$ has at least $n+1$ linearly independent vectors. ■

Definition 2.3.2 ([Sta+04, Section 1.1.]). A hyperplane arrangement $\mathcal{A} = \{H_0, \ldots, H_d\} \subseteq \mathbb{P}^d$ is in *general position* if

$$\begin{aligned} \{H_{i_0}, \ldots, H_{i_p}\} \subseteq \mathcal{A},\ p \leq d \quad &\Rightarrow \quad \operatorname{codim}(H_{i_0} \cap \ldots \cap H_{i_d}) = p \\ \{H_{i_0}, \ldots, H_{i_p}\} \subseteq \mathcal{A},\ p > d \quad &\Rightarrow \quad H_{i_0} \cap \ldots \cap H_{i_p} = \emptyset. \end{aligned}$$

Definition 2.3.2 looks suspiciously similar to the definition of the uniform matroid. In fact, the uniform matroid $U_{n+1,d+1}$ arises from a hyperplane arrangement in $\mathbb{P}^d$ in general position.

We have already seen that essential hyperplane arrangements $\mathcal{A} \subseteq \mathbb{P}^d$ give rise to simple matroids of rank d, see Proposition 2.2.5. Now, we will explore more hyperplane arrangements arising from finite projective spaces, as these arrangements have nice combinatorial properties. We follow tightly section 6.1. in [Oxl06].

Let $V = \mathbb{F}_q^{n+1}$ be a vector space over the finite field $\mathbb{F}_q$, where q is a prime power. The finite projective geometry $PG(n, q)$ associated to the vector space V is the usual projective space $\mathbb{P}^n_{\mathbb{F}_q}$ arising from identifying points on lines through the origin in $V \setminus 0$. Since a finite projective space $PG(n, q)$ consists of a finite set of points, the span of the vectors corresponding to these points give rise to a finite set of hyperplanes in V. Hence, by Proposition 2.2.5, finite projective spaces give rise to simple matroids of rank $n + 1$. From now on, we denote by $PG(n, q)$ the matroid arising from the finite projective space $PG(n, q)$, and so the flats of the matroid $PG(n, q)$ correspond to the linear subspace of the finite projective space $PG(n, q)$. If $n = 2$, we call the finite projective space $PG(2, q)$ a finite projective plane.

Example 2.3.3. The finite projective plane $PG(2, 2)$ is called the *Fano plane*, and it can be represented as the point configuration in Figure 2.4.

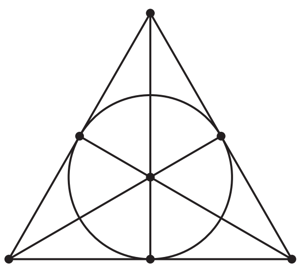

Figure 2.4: Fano matroid.

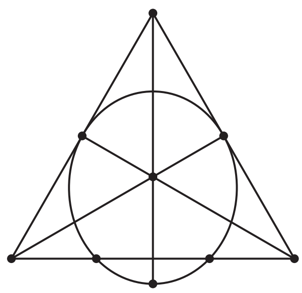

Figure 2.5: non-Fano matroid.

The points corresponds to the seven non-zero vectors in

$$\mathbb{F}_2^3 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix} \right\},$$

and the seven lines correspond to the $\binom{7}{2} = 21$ planes through the origin spanned by pairwise non-zero vectors of $V^3_{\mathbb{F}_2}$. Note that as every triple of vectors on the form $(x, y, x + y)$ in $\mathbb{F}_2^3$ lie on the same plane, we are counting each plane $\binom{3}{2} = 3$ times, hence the number of distinct planes in $V^3_{\mathbb{F}_2}$, and hence of distinct lines in $PG(2, 2)$, is $21/3 = 7$. Hence, the Fano matroid is a simple matroid of rank 3 consisting of 7 flats of rank 1, and 7 flats of rank 2. A related matroid to the Fano matroid is the *non-Fano* matroid. It arises from the configuration in Figure 2.5, where three of the points in the Fano plane are no longer collinear. The non-Fano plane consists then of 6 flats of rank 2 of size 3, and 3 of size 2, and it is representable exactly over all fields whose characteristic is different from 2 [Kat14]. ■

In the next example, we need the *Gaussian coefficient*:

$$\begin{bmatrix} r \\ k \end{bmatrix}_q = \frac{(q^r - 1)(q^r - q)\cdots(q^r - q^{k-1})}{(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})}$$

which is defined for all integers r and k with $0 \leq k \leq r$ see Section 6.1 in [Oxl06].

Example 2.3.4. The smallest three-dimensional finite projective space is $PG(3,2)$. The matroid $PG(3,2)$ is a simple matroid of rank 4, consisting of

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}_2 = 15 \text{ rank 1 flats,} \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_2 = 35 \text{ rank 2 flats,} \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix}_2 = 15 \text{ rank 3 flats.}$$

Moreover, every single plane in $PG(3,2)$ is isomorphic to the Fano plane $PG(2,2)$. ■

The following theorem is a central theorem in projective geometry.

Theorem 2.3.5 ([Oxl06, Theorem 6.1.11])**.** *(Pappus's Theorem) Let* $\{1,2,3\}$ *and* $\{4,5,6\}$ *be triples of distinct points that lie on the lines* L *and* L'*, respectively, of* $\mathbb{P}^2_K$ *such that none of these six points is on both* L *and* L'*. Let* $7, 8,$ *and* 9 *be the points on intersection of the pairs of lines,* 15 *and* 24*,* 16 *and* 34*, and* 26 *and* 35*, respectively. Then,* $7, 8,$ *and* 9 *are collinear.*

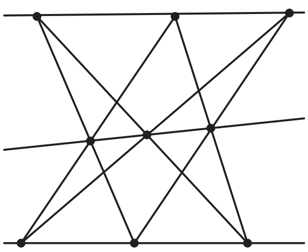

Figure 2.6: Pappus.

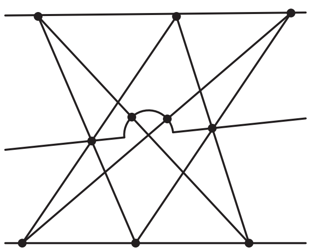

Figure 2.7: non-Pappus.

In the next example, we present an example of a matroid which is not representable over any field.

Example 2.3.6. The simple matroid of rank 3 arising from the line arrangement given in Figure 2.6 is called the *Pappus matroid.* It consist of nine flats of rank 1 (the nine lines), and 18 flats of rank 2 (the intersection points), there are nine flats of rank 2 that are of size 2, and nine that are of size 3. If we call the points on the top line 1, 2, and 3, and the points on the bottom line 4, 5, and 6, it is easy to check that the line arrangement satisfies Pappus's Theorem. The *non-Pappus matroid*, see Figure 2.7, is obtained from the Pappus pseudoline arrangement by bending the line in the middle such that the three points in the middle are no longer collinear. Hence, the non-Pappus matroid still consists of 9 flats of rank 1, whereas the number of flats of rank 2 has increased to 20, of which 12 are of size 2, and 8 of size 3. Note, that the non-Pappus matroid violets Pappus's Theorem, hence it is not representable over any field. ■

We end this section by introducing another fundamental class of matroids, namely the *graphic matroids*. Up until now, we have mostly seen matroids arising from hyperplane arrangements, or dually from vector spaces. Thus, the notion of independence that we have in mind is that of linearly independent vectors. However, graph theory was also fundamental for Whitney's development of matroid theory. As mentioned in the introduction, a graph also defines a matroid. We follow Section 1 in [Oxl06].

Proposition 2.3.7 ([Oxl06, Proposition 1.1.7])**.** *Let E be the set of edges of a graph G and let I be the set of edge sets that do not contain the edge set of a cycle of G. Then I is the set of independent sets of a matroid on E.*

A matroid arising from a graph G in the way explained in Proposition 2.3.7 is denoted by $M(G)$. Equivalently, a matroid $M(G)$ arising from a graph G is the rank function r on the ground set E, returning the size of the spanning forest of a subgraph of G. A matroid that is isomorphic to a matroid $M(G)$ for some graph G is called *graphic*, see the following definition.

Definition 2.3.8 ([Oxl06, Example 1.1.8])**.** Two matroids M_1 and M_2 are isomorphic, written $M_1 = M_2$, if there is a bijection ψ from the underlying set $E(M_1)$ to the underlying set $E(M_2)$ such that, for all $X \subseteq E(M_1)$, the set $\psi(X)$ is independent in M_2 if and only if X is independent in M_1. We call such a bijection ψ an isomorphism from M_1 to M_2.

Example 2.3.9. Let G be the graph in Figure 2.8 with edges $E = \{e_0, e_1, e_2, e_3\}$, and let $M(G)$ be the corresponding matroid. Then, the independent sets of

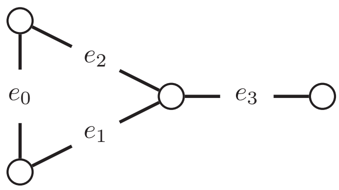

Figure 2.8: The graph G.

$M(G)$ are

$$I = \{\{e_i\}, \{e_i, e_j\}, \{e_0, e_1, e_3\}, \{e_0, e_2, e_3\}, \{e_1, e_2, e_3\} \mid \text{for } i \neq j \in [0, 1, 2, 3]\}.$$

Moreover, note that if we let V be the set consisting of the column vectors of the matrix A over $\mathbb{R}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by $V = \{v_0, v_1, v_2, v_3\}$, and let $M(A)$ be the matroid arising from the matrix A, it is easy to check that the map $\psi : V \to E$ defined by $\psi(v_i) = e_i$ defines an isomorphism of matroids. Note that the independent sets in $M(A)$ are exactly

$$I = \{\{v_i\}, \{v_i, v_j\}, \{v_0, v_1, v_3\}, \{v_0, v_2, v_3\}, \{v_1, v_2, v_3\} \mid \text{for } i \neq j \in [0, 1, 2, 3]\}.$$

Hence $M(G)$ is representable over $\mathbb{R}$. ■

In Example 2.3.9 we saw that the matroid $M(G)$ is representable over $\mathbb{R}$. In fact, we can say something even stronger.

Proposition 2.3.10 ([Oxl06, Proposition 6.1.4 (i)]). *If G is a graph, then $M(G)$ is representable over any field.*

The proof follows from the fact that from a graph G, we can always construct a directed graph $D(G)$ by assigning a direction to the edges of the graph. Moreover, we can describe the directed graph $D(G)$ with the help of an incidence matrix $A_{D(G)}$. Finally, the matrix $A_{D(G)}$ is in fact a representation of the the graph G, see [Oxl06] for details. We briefly review how the incidence matrix can be constructed, we use the same notation as in [Oxl06]. Let V be the set of vertices and E the set of edges of a graph G, then the incidence matrix $A_{D(G)} = |V| \times |E|$ of a directed graph $D(G)$ is constructed in the following way:

$$a_{ij} = \begin{cases} 1, & \text{if vertex } i \text{ is tail of edge } j \\ -1, & \text{if vertex } i \text{ is head of edge } j \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.3.11. Let us see how we recovered the matrix A in Example 2.3.9. Let $D(G)$ be the directed graph of the graph G in Example 2.3.9 given in Figure 2.9. Then the incidence matrix $A_{D(G)}$ is given by

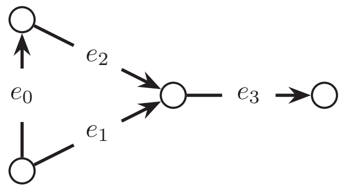

Figure 2.9: The directed graph $D(G)$.

$$A_{D(G)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A.$$

■

We will not explore graphic matroids much in depth, as our take on matroids is mostly from an algebraic geometry point of view. The idea behind presenting the above example is mostly to illustrate the diversity of matroids. But also, to keep in mind, that a lot of the inspiration behind results in matroid theory comes from graph theory. For example in Section 5.1 we will see that the *characteristic polynomial* of matroids is a generalization of the chromatic polynomial of graphs.

2.4 Operations on Matroids

In this section we introduce three operations on matroids, namely *deletion*, *restriction* and *contraction*. These operations will be useful to compute some central invariants of the matroid that will be introduced later in this thesis. We follow the conventions of Section 5 in [Kat14]. Let M be a matroid of rank $d+1$ given by a rank function r on a finite ground set E.

Definition 2.4.1 ([Kat14, Section 5.]). For a subset $X \subseteq E$, the *deletion* $M \setminus X$ is defined to be the matroid on the ground set $E \setminus X$ with rank function given by: for $S \subseteq E \setminus X$

$$\mathrm{r}_{M\setminus X}(S) = \mathrm{r}(S).$$

If the matroid is representable, i.e., arising from a hyperplane arrangement $\mathcal{A} = \{H_i \mid i \in E\}$, then the deletion $M \setminus X$ corresponds to the matroid arising from the hyperplane arrangement

$$\mathcal{A}_{M\setminus X} = \{H_i \mid i \in E \setminus X\}.$$

Moreover, the lattice of flats $\mathcal{L}_{M\setminus X}$ of the matroid $M \setminus X$ can be obtained from the lattice of flats $\mathcal{L}_M$ of M by removing the set X. Then a flat $F \in \mathcal{L}_M$ is either disjoint from X, so that $F \in \mathcal{L}_{M\setminus X}$. Or, the flat $F \in \mathcal{L}_M$ intersects X, then either $\mathrm{r}(F \setminus X) = \mathrm{r}(F)$, so $F \setminus X \in \mathcal{L}_{M\setminus X}$, or $\mathrm{r}(F \setminus X) < \mathrm{r}(F)$, and $F \setminus X$ is no longer a flat of $\mathcal{L}_{M\setminus X}$.

Example 2.4.2. Let M be the matroid arising from the line arrangement in Figure 2.10. The lattice of flats of the matroid M is illustrated in Figure 2.11. Now, the deletion matroid $M \setminus \{0\}$ is the matroid on the ground set $\{1, 2, 3\}$ with rank function $\mathrm{r}_{M\setminus\{0\}}$ for $S \subseteq E \setminus \{0\}$ given by

$$\mathrm{r}_{M\setminus\{0\}}(S) = \mathrm{r}(S). \tag{2.3}$$

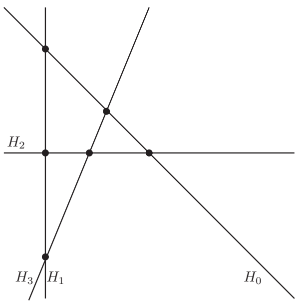

Figure 2.10: $\mathcal{A}$.

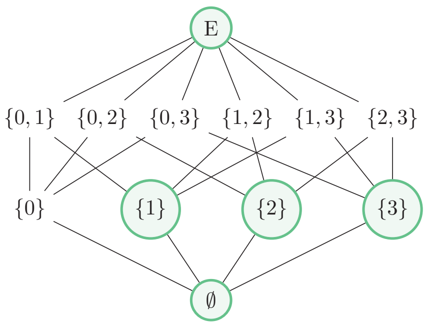

Figure 2.11: Lattice of flats of M.

By applying the rank function $\mathrm{r}_{M\setminus\{0\}}(S)$ given in Equation (2.3), it is easy to check that the lattice of flats of $M \setminus \{0\}$ corresponds to the the flats in Figure 2.13. Geometrically, the matroid $M \setminus \{0\}$ corresponds to the matroid arising from the hyperplane arrangement

$$\mathcal{A} \setminus H_0 = \{H_1, H_2, H_3\} \subseteq \mathbb{P}^2,$$

see Figure 2.12. ■

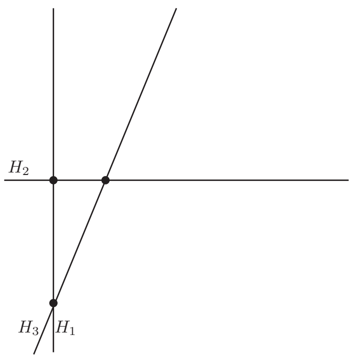

Figure 2.12: Arrangement $\mathcal{A} \setminus H_0$.

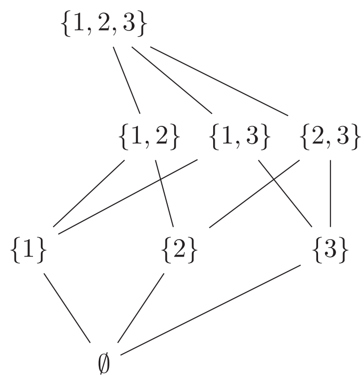

Figure 2.13: Lattice of flats of $M \setminus \{0\}$.

As before, let M be a matroid of rank $d+1$ given by a rank function r on a ground set E.

Definition 2.4.3 ([Kat14, Section 5.]). For a subset $T \subseteq E$ the *restriction* $M|_T$ is defined as the matroid

$$M \setminus (E \setminus T).$$

If $F \subseteq E$ is a flat, then the lattice of flats $\mathcal{L}_{M|F}$ is $[\hat{0}, F]$, i.e., flats between $\hat{0}$ and F. If the matroid has no loops, then $\hat{0} = \emptyset$.

Definition 2.4.4 ([Kat14, Section 5.]). Let $X \subseteq E$, the *contraction* X/E is defined to be the matroid on the ground set $E \setminus X$ with rank function $r_{M/X}$ for $S \subseteq E \setminus X$

$$\mathrm{r}_{M/X}(S) = \mathrm{r}(S \cup X) - \mathrm{r}(X).$$

From Definition 2.4.4 we easily deduce that the matroid M/X is of rank $d + 1 - r(X)$. If F is a flat, then the lattice of flats $\mathcal{L}_{M/F}$ is isomorphic to $[F, E]$.

Moreover, if the original matroid M arises from a hyperplane arrangement $\mathcal{A}$, then, if we denote by $H_{\bar{S}}$ the subspace $\cap_{i \in \bar{S}} H_i$, the matroid M/S corresponds to the matroid arising from the hyperplane arrangement

$$\mathcal{A}_{M/S} = \{H_{\bar{S}} \cap H_j \mid j \in E \setminus \bar{S}\} \subseteq H_{\bar{S}}.$$

Note that when defining the arrangement $\mathcal{A}_{M/S}$ we have taken the closure of S because if j is an element of the set $\bar{S} \setminus S$, then $H_{\bar{S}} \cap H_j = H_{\bar{S}}$, and hence not a hyperplane in $H_{\bar{S}}$.

Example 2.4.5. Let M be the matroid given in Example 2.4.2. The contraction matroid $M/\{0\}$ is the rank 2 matroid on the ground set $\{1, 2, 3\}$ with rank function $r_{(M/\{0\})}$ for $S \subseteq \{1, 2, 3\}$

$$\mathrm{r}_{(M/\{0\})}(S) = \mathrm{r}(S \cup \{0\}) - \mathrm{r}(\{0\}). \tag{2.4}$$

By applying the rank function $\mathrm{r}_{(M/\{0\})}$ given in Equation (2.4), it is easy to check that the lattice of flats of $M/\{0\}$ corresponds to the flats highlighted in green in Figure 2.11. Geometrically, the matroid $M/\{0\}$ corresponds to the matroid arising from the point arrangement

$$\mathcal{A}_{H_0} = \{H_{01} = H_0 \cap H_1,\ H_{02} = H_0 \cap H_2,\ H_{03} = H_0 \cap H_3\} \subseteq H_0.$$

See Figure 2.14 for the point arrangement $\mathcal{A}_{M/\{0\}}$, and Figure 2.15 for the lattice of flats of the matroid arising from the the point arrangement $\mathcal{A}_{M/\{0\}}$. As expected, the lattice of flats given by the highlighted flats in Figure 2.11 and in Figure 2.15 are isomorphic. ■

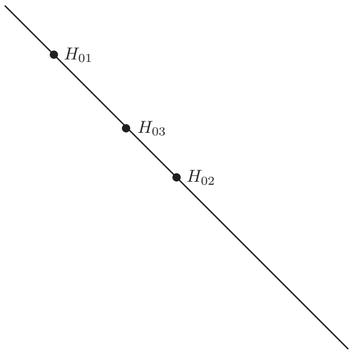

Figure 2.14: Arrangement $\mathcal{A}_{H_0}$.

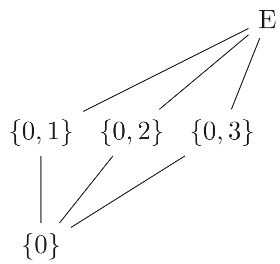

Figure 2.15: Lattice of flats of $M/\{0\}$.

Note that deletion and contraction commutes.

Definition 2.4.6 ([Kat14, Definition 5.1])**.** A matroid M' is said to be a minor of M if it is obtained by deleting and contracting elements of the ground set of M.

Note that both in Example 2.4.2, and in Example 2.4.5 the original matroid is the uniform matroid $U_{3,4}$. Moreover, the deletion matroid $U_{3,4}\backslash\{0\}$ resulted in the uniform matroid $U_{3,3}$. Whereas, the contraction matroid $U_{3,4}/\{0\}$ resulted in the uniform matroid $U_{2,3}$. We can generalize the results in Example 2.4.2, and in Example 2.4.5.

Example 2.4.7. Let $U_{d+1,n+1}$ be the uniform matroid on the ground set E, and let $i \in E$. Then

$$\begin{aligned} &U_{d+1,n+1} \setminus \{i\} = U_{d+1,n}, \text{ and} \\ &U_{d+1,n+1} \,/\, \{i\} = U_{d,n}. \end{aligned}$$

First of all, both the deletion matroid and the restriction matroid are matroids on a ground set with an element less. Moreover, the rank function of the deletion matroid is still given by for $S \subseteq E \setminus \{i\}$

$$\mathrm{r}_{U_{d+1,n+1}\setminus\{i\}}(S) = \begin{cases} |S| & \text{for } |S| \leq d+1 \\ d+1 & \text{otherwise.} \end{cases}$$

Whereas, the rank function of the contraction matroid is given by for $S \subseteq E\backslash\{i\}$

$$\mathrm{r}_{U_{d+1,n+1}\,/\,\{i\}}(S) = \begin{cases} \mathrm{r}(S \cup \{0\}) - \mathrm{r}(\{0\}) = |S| & \text{for } |S| \leq d \\ \mathrm{r}(S \cup \{0\}) - \mathrm{r}(\{0\}) = d & \text{otherwise.} \end{cases}$$

■

We end this section by briefly mentioning that a matroid can be decomposed into *connected* components.

Definition 2.4.8 ([Kat14, Definition 5.2])**.** A matroid is *connected* if for every $i, j \in E$, there exists a circuit containing i and j.

We will not explore this decomposition any further, as we only need Definition 2.4.8 in the rest of the thesis, for more details see Section 5.2 in [Kat14].

CHAPTER 3

The Bergman fan of matroids

A matroid defines a rational polyhedral fan called the *Bergman fan* of the matroid. After giving a brief introduction to polyhedral geometry we give a rigorous definition of the Bergman fan. We will also see that there is a correspondence between polyhedral geometry, and toric varieties, and that the Bergman fan of a matroid is in fact a toric variety. We end this chapter by introducing the de Concini and Procesi *Wonderful compactification* of the complement of a hyperplane arrangement $C(\mathcal{A}) \subsetneq L$, which is constructed by a series of blow-ups on L.

3.1 Fans and toric geometry

In this section we give a brief introduction to toric varieties. The reason is that, as we will see in Section 3.2, matroids define a toric variety, more specifically a polyhedral fan constructed according to some rules. This section is purely preparatory. Toric varieties are varieties having numerous nice properties. For instance, there is a correspondence between toric varieties and the polyhedral geometry of cones and polytopes, which makes computations far easier. We begin by reviewing polyhedral geometry, and then give the correspondence to toric geometry, which can be skipped. We follow the conventions of Chapter 1, and Chapter 3 of [CLS11].

Let N and M be dual lattices with associated vector spaces $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Definition 3.1.1 ([CLS11, Definition 1.2.1.])**.** A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set on the form

$$\sigma = \mathrm{Cone}(S) = \Big\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \Big\} \subseteq N_{\mathbb{R}},$$

where $S \subseteq N_{\mathbb{R}}$ is finite. We say that σ is *generated* by S. Also let $\mathrm{Cone}(\emptyset) = \{0\}$.

An example is the following cone

$$\sigma = \mathrm{Cone}(e_2, e_1 - e_2) \subseteq \mathbb{R}^2 \simeq \mathbb{Z}^2 \otimes \mathbb{R}. \tag{3.1}$$

Definition 3.1.2 ([CLS11, Definition 1.2.3.])**.** Given a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, its *dual cone* is defined by

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

The dual cone of the cone given in Equation (3.1) is the cone given by

$$\sigma^{\vee} = \mathrm{Cone}(e_1, e_1 + e_2). \tag{3.2}$$

Definition 3.1.3 ([CLS11, Definition 1.2.5.])**.** A *face of a cone* of the polyhedral cone σ is $\tau = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \cap \sigma$ for some $m \in \sigma^{\vee}$, written $\tau \preceq \sigma$. Using $m = 0$ shows that σ is a face of itself, i.e., $\sigma \preceq \sigma$. Faces $\tau \neq \sigma$ are called *proper faces*, written $\tau \preceq \sigma$.

The proper faces of the cone in Equation (3.2) are the rays e_1, and $e_1 + e_2$, and the origin $\{0\}$.

Proposition 3.1.4 ([CLS11, Proposition 1.2.12.])**.** *Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone. Then σ is strongly convex if and only if $\{0\}$ is a face of σ.*

Since $\{0\}$ is a proper face of the cone in Equation (3.2), the cone is strongly convex.

Definition 3.1.5 ([CLS11, Definition 1.2.14.])**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is *rational* if $\sigma = \mathrm{Cone}(S)$ for some finite set $S \subseteq N$.

So, since the cone in Equation (3.2) is generated by two rays it is rational.

Definition 3.1.6 ([CLS11, Definition 3.1.2])**.** A *fan* Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that:

1) Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

2) For all $\sigma \in \Sigma$, each face of σ is also in Σ.

3) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each (hence also Σ).

Furthermore, if Σ is a fan, then Σ_r is the set of $r-$dimensional cones of Σ which we call the *r-skeleton*.

Definition 3.1.7 ([CLS11, Definition 3.1.18])**.** Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan.

1) Σ is *unimodular*, if for every cone σ in Σ, the minimal generators of σ form part of a $\mathbb{Z}$-basis of N,

2) Σ is *simplicial* if for every cone σ in Σ the minimal generators of σ are linearly independent over $\mathbb{R}$.

Next, we briefly give the correspondence of polyhedral geometry and toric geometry.

Definition 3.1.8 ([CLS11, Definition 1.2.])**.** A *complex toric variety* is an irreducible variety X containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on V.

Given a rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, the lattice points

$$S_\sigma = \sigma^\vee \cap M \subseteq M$$

form a semigroup [CLS11]. For example the semigroup of the cone σ in Equation (3.1) is $S_\sigma \simeq \mathbb{N}^2$ since it is generated by the linearly independent lattice points $e_1 \cap M$, and $(e_1 + e_2) \cap M$.

Theorem 3.1.9 ([CLS11, Theorem 1.2.18.]). *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a rational polyhedral cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then*

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$$

is an affine toric variety. Furthermore,

$$\dim U_\sigma = n \iff \textit{the torus of } U_\sigma \textit{ is } T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \iff \sigma \textit{ is strongly convex.}$$

The affine toric variety associated to the cone in Equation (3.1) is then $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) \simeq \mathrm{Spec}(\mathbb{C}[\mathbb{N}^2]) \simeq \mathrm{Spec}(\mathbb{C}[x, y]) = \mathbb{A}^2$. We already know that σ is strongly convex, and this fits well with the fact that $\dim(U_\sigma) = \dim(\mathbb{A}^2) = 2$.

We are finally ready to define the *toric variety* X_Σ of a fan Σ.

Definition 3.1.10. The *abstract toric variety* X_Σ associated to the fan Σ is obtained by gluing the affine varieties U_σ for $\sigma \in \Sigma$, see Chapter 3 in ([CLS11]) for technical details.

Theorem 3.1.11 ([CLS11, Theorem 3.2.6]). *(Orbit-Cone Correspondence) Let X_Σ be the toric variety of the fan Σ in $N_{\mathbb{R}}$. Then there is a bijective correspondence*

$$\begin{aligned} \{\textit{cones } \sigma \textit{ in } \Sigma\} &\longleftrightarrow \{T_N\textit{-orbits in } X_\Sigma\} \\ \sigma &\longleftrightarrow O(\sigma). \end{aligned}$$

3.2 Bergman fans and tropicalization

Matroids are highly geometric objects. This is because the inspiration for matroid theory often comes from algebraic geometry. But also because, even when the matroid is not representable, i.e., the matroid does not come from anything geometric, it still defines a toric variety. Specifically, the lattice of flats of a matroid M can be represented by a polyhedral fan called its *Bergman fan*, which was first introduced in [AK06]. We follow the notation of [Eur20]:

- $\{e_i \mid i \in E\}$ the standard basis of $\mathbb{Z}^E$ and $\langle \cdot, \cdot \rangle$ the standard dot-product on $\mathbb{Z}^E$
- $N := \mathbb{Z}^E / \mathbb{Z}\mathbf{1}$ be a lattice where $\mathbf{1}$ denotes the vector $\sum_{i \in E} e_i \in \mathbb{Z}^E$,
- u_i the image of e_i in N for $i \in E$,
- $u_S = \sum_{i \in S} u_i$ for a subset $S \subset E$.

Definition 3.2.1 ([Eur20, Definition 2.2.]). Let M be a loopless matroid of rank $r = d + 1$ on a ground set E. With the notations as above, the Bergman fan

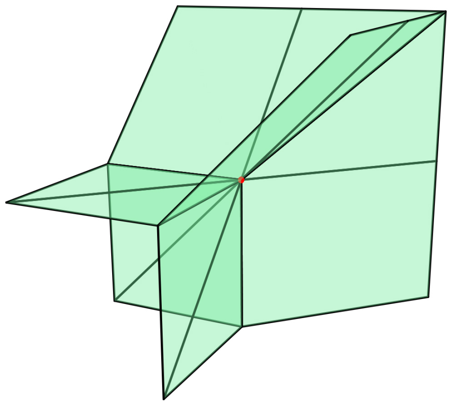

Figure 3.1: The Bergman fan of $U_{3,4}$

$\sum_M$ is the pure d-dimensional polyhedral fan in $N_\mathbb{R} := N \otimes \mathbb{R}$ that is comprised of cones

$$\sigma_\mathcal{F} := \mathrm{Cone}(u_{F_1}, u_{F_2}, \cdots, u_{F_k}) \subset N_\mathbb{R}$$

for each chain of flats $\mathcal{F} : \emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$ in $\mathcal{L}_M$.

Note that, each k-dimensional cone $\sigma_\mathcal{F}$ is generated by k-linearly independent vectors over $\mathbb{R}$, hence the Bergman fan Σ_M of a matroid M is simplicial. In fact, some computations can show that the Bergman fan of a matroid is even unimodular.

Example 3.2.2. The Bergman fan of the uniform matroid $U_{3,4}$ is a 2-dimensional polyhedral fan living in $\mathbb{Z}^4/\mathbb{Z}\mathbf{1} \otimes \mathbb{R}$, see Figure 3.1. Let $E = \{0, 1, 2, 3\}$ be the ground set of $U_{3,4}$. Then, the fan consists of 12 cones, corresponding to the flags $\emptyset \subsetneq \{i\} \subsetneq \{i, j\} \subsetneq E$, for each pair $i \neq j \in E$. The following script in [polymake] generates the fan in Figure 3.1.

```
application "fan";

$fan = new fan::PolyhedralFan(INPUT_RAYS=>[
[1,0,0,0],
[0,1,0,0],
[0,0,1,0],
[0,0,0,1],
[0,-1,-1,-1],
[0,1,1,0],
[0,1,0,1],
[0,0,-1,-1],
[0,0,1,1],
[0,-1,0,-1],
[0,-1,-1,0]
],
INPUT_CONES => [[0,1,5],[0,1,6],[0,1,7],[0,2,5],
[0,2,8],[0,2,9],[0,3,6],[0,3,8],[0,3,10],[0,4,7],[0,4,9],[0,4,10]]);

$complex = new fan::PolyhedralComplex($fan);
$complex -> VISUAL;
```

■

Example 3.2.3. The Bergman fan of the matroid in Example 2.1.3 is a 2-dimensional polyhedral fan living in $\mathbb{Z}^6/\mathbb{Z}\mathbf{1} \otimes \mathbb{R}$. We can not imagine a 2-dimensional fan living in a 5-dimensional space. However, we can imagine, and

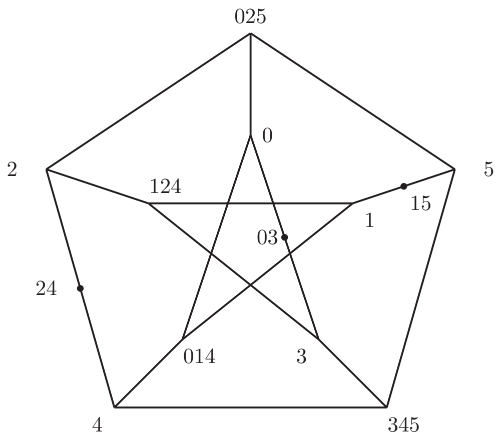

Figure 3.2: The Petersen graph

even draw, the intersection of the fan with the surface of a sphere containing the fan. In this specific example, the graph given by the intersection of the fan with the surface of the sphere is isomorphic to the Petersen graph see Figure 3.2. The intersection points, labeled by the proper flats of the matroid, correspond to the one dimensional rays in the Bergman fan. The edges correspond to the two dimensional cones, which are spanned by two rays corresponding to proper flats of which one is properly contained in the other. In general, the intersection of the Bergman fan in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ with a sphere is called is called the *Bergman Complex of the matroid*, and there exist explicit formulas for computing it, see [AK06]. ■

When the matroid is representable, i.e., arising from some linear space, the Bergman fan of the matroid can be achieved through something called *tropicalization*. The field of *tropical geometry* aims at turning problems related to algebraic varieties into problems related to polyhedral complexes. The process of transforming an algebraic variety into a polyhedral complex is called *tropicalization*, see for example [Bru+15] for a brief introduction to tropical geometry. In 2002, Sturmfels proved that the Bergman fan of the matroid arising from a linear space L corresponds to the tropicalization of L, see [Stu02] for details.

3.3 The wonderful compactification

We will now introduce De Concini-Procesi wonderful compactification $W_{\mathcal{A}}$ of the complement of an arrangement $\mathcal{A}$. The reason is that in the next chapter we will see that to a matroid M we can associate a polynomial ring, namely the *Chow ring* $A^*(M)$. When the matroid M is representable, the ring $A^*(M)$ is in fact the Chow ring $A^*(W_{\mathcal{A}})$.

In this section $L \subseteq \mathbb{P}^n$ is a d-dimensional linear subspace, and

$$\mathcal{A} = \{H_i = \mathrm{V}(x_i) \cap L \mid 0 \leq i \leq n\} \subseteq L$$

is the corresponding hyperplane arrangement. Let $C(\mathcal{A}) \subseteq L$ be the complement of the hyperplane arrangement. Moreover, recall that a k-dimensional subspace L_F of $\mathcal{A}$ corresponds to a flat $F \in \mathcal{L}_{M_{\mathcal{A}}}$ of rank $d-k$ of the matroid $M_{\mathcal{A}}$ defined by the arrangement. The following definition was first presented in [DP95], but we choose to use [Eur20] conventions.

Definition 3.3.1 ([Eur20, Definition 2.3.])**.** The *wonderful compactification* $W_{\mathcal{A}}$ of the complement $C(\mathcal{A})$ is obtained by a series of blow-ups on L in the following way: First blow-up the points $\{L_F\}_{r(F)=d}$, then blow-up the strict transforms of the lines $\{L_F\}_{r(F)=d-1}$, and continue until having blown-up strict transforms of the hyperplanes $\{L_F\}_{r(F)=1}$.

We now review an explicit map for De Concini-Procesi wonderful compactification stated in Definition 3.3.1. In the original paper [DP95], the construction is stated in terms of building set G. Building sets were first introduced in [FS04], and are a generalization of the lattice of flats. For our purpose, we only need the case where the building set is the lattice of flats, therefore, we state the definition in terms of the latter. We follow the conventions of Section 5.4 in [Den14].

With the notation as above, for each flat $F \in \mathcal{L}_{M_{\mathcal{A}}}$, the coordinate projection $\mathbb{C}^{n+1} \to \mathbb{C}^F$ induces a rational map $p_F : \mathbb{P}^n \dashrightarrow \mathbb{P}^{|F|-1}$, and let

$$p : \mathbb{P}^n \dashrightarrow \prod_{F \in \mathcal{L}_{M_{\mathcal{A}}}} \mathbb{P}^{|F|-1} \tag{3.3}$$

be the map whose Fth coordinate is p_F.

Let L_F be the image of the linear subspace L under the map p_F. The following definition is equivalent to Definition 3.3.1. Note that the map p is regular on $C(A) \subseteq L$, hence the following definition makes sense.

Definition 3.3.2 ([Den14, Definition 5.10.])**.** The De Concini-Procesi wonderful compactifiaction (Definition 3.3.1) of $\mathcal{A}$ with lattice of flats $\mathcal{L}_{\mathcal{A}}$ is

$$W_{\mathcal{A}} = \overline{p(C(\mathcal{A}))} \subset \prod_{F \in \mathcal{L}_{M_{\mathcal{A}}}} L_F.$$

Note that since the ground set E is a flat, the image $p_E(C(\mathcal{A})) = C(\mathcal{A})$ is a component of the product. Moreover, let L_F be a linear subspace corresponding to a flat $|F| = k$, i.e., the linear subspace L_F is the locus $V(x_i, \ldots, x_j) \subseteq L$, for k distinct coordinates. And let T_{p_F} denote the graph of p_F, i.e., the graph

$$T_{p_F} = \{(x, p_F(x)) \mid x \in C(\mathcal{A})\} \subseteq C(\mathcal{A}) \times \mathbb{P}^{|F|-1}.$$

Then the closure of T_{p_F} in $C(\mathcal{A}) \times \mathbb{P}^{|F|-1}$ is the actual blow-up of L along the linear space L_F. And since the graph

$$T_{p_F} \subseteq C(\mathcal{A}) \times \mathbb{P}^{k-1}$$

is a component of the image $p(C(\mathcal{A}))$ for every flat $F \in \mathcal{L}_{M_\mathcal{A}}$, we can conclude that the map p defines the blow-ups in Definition 3.3.1 all performed at once.

Example 3.3.3. Let L be the linear space $V(-x_0 + x_1 + x_3)$, and $\mathcal{A}$ be the hyperplane arrangement from Example 2.1.1, and Example 2.1.2. Recall that $\mathcal{A}$ consists of the four lines

$$\mathcal{A} = \{H_i = V(x_i) \cap L \mid 0 \leq i \leq 3\} \subseteq L.$$

As before, we denote the corresponding matroid $M_\mathcal{A}$ with lattice of flats $\mathcal{L}_{M_\mathcal{A}}$. The map p given in Equation (3.3) is composed of the coordinate maps p_F for every flat $F \in \mathcal{L}_{M_\mathcal{A}}$. For example, the component $p_{\{0,2\}} : C(\mathcal{A}) \to \mathbb{P}^1$ given by the flat $\{0, 2\}$ is the projection:

$$[x_0 : x_1 : x_2 : x_0 - x_1] \to [x_0 : x_2],$$

so the image $p_{\{02\}}(C(\mathcal{A})) = L_{\{02\}} = [x_0 : x_2]$ for x_0, and x_2 different from 0. On the other hand, the component $p_{\{0,1,3\}} : C(\mathcal{A}) \to \mathbb{P}^2$ is the projection:

$$[x_0 : x_1 : x_2 : x_0 - x_1] \to [x_0 : x_1 : x_0 - x_1] \simeq [x_0 : x_1],$$

so the image $p_{\{013\}}(C(\mathcal{A})) = L_{\{013\}} = [x_0 : x_1]$, again for x_0, and x_1 different from 0. By similar computations, we get that the image $p(C(\mathcal{A}))$ is the product

$$L_{\{02\}} \times L_{\{12\}} \times L_{\{23\}} \times L_{\{013\}} \times L_{\{E\}} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3,$$

given by the coordinates

$$([x_0 : x_2], [x_1 : x_2], [x_2 : x_1 - x_0], [x_0 : x_1], [x_0 : x_1 : x_2 : x_1 - x_2]).$$

And the wonderful compactification $W_\mathcal{A}$ of the complement of the hyperplanes $C(\mathcal{A})$ is the closure

$$\overline{L_{\{02\}} \times L_{\{12\}} \times L_{\{23\}} \times L_{\{013\}} \times V_{\{E\}}} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3.$$

This is in fact $\mathbb{P}^2$ blown-up at the 4 points $\{02\}$, $\{12\}$, $\{23\}$, and $\{013\}$ ∎

CHAPTER 4

Chow ring of a matroid

In this chapter we introduce the *Chow ring* of a matroid $A^*(M)$. In algebraic geometry, the Chow ring of a variety is the ring of equivalence classes of cycles up to rational equivalence. The Chow ring of a matroid is given by a quotient of a polynomial ring generated by variables corresponding to the flats of the matroid, and the quotient is by two ideals that encode the inclusion properties of the flats. We will see that the Chow ring of a matroid has in fact a geometric meaning when the matroid is representable. Moreover, we will see that the Chow ring of a matroid satisfies linear and Poincaré duality, which give us the isomorphism $A^*(M) \cong MW_*(M)$, where the latter is the ring of balanced weighted k-skeletons of the Bergman fan [AHK18].

4.1 Chow Ring of a matroid

As a rough simplification, the *Chow ring* of a variety encodes the intersection properties of the cycles of that variety. We begin this section by reviewing the general definition of the Chow ring, and then we review the analogy for the Chow ring of a matroid.

The following definitions are taken from Chapter 1 in [EH16]. Let X be an algebraic variety, the *group of cycles* on X, denoted by

$$Z(X) = \bigoplus_k Z_k(X),$$

is the free abelian group generated by the set of subvarieties of X, graded by dimension. Two cycles $Y_1, Y_2 \in Z(X)$ are *rationally equivalent* if there exists a cycle of $\mathbb{P}^1 \times X$ such that the restriction to two fibres $\{t_0\} \times X$, and $\{t_1\} \times X$ are Y_1, and Y_2. For example any two hyperplanes $H_0 = V(f_0)$, and $H_1 = V(f_1)$, given by two linear polynomials f_1 and f_2, in $\mathbb{P}^n$ are rationally equivalent. Let $H \subseteq \mathbb{P}^1 \times \mathbb{P}^n$ be the cycle given by $H = V(t_0 f_0 + t_1 f_1)$, then the restriction to the two fibres $\{[0:1]\} \times \mathbb{P}^n$, and $\{[1:0]\} \times \mathbb{P}^n$ are the two hyperplanes H_0, and H_1.

Definition 4.1.1 ([EH16, Definition 1.3.])**.** The *Chow group* of X is the quotient

$$A(X) = \bigoplus_k Z_k(X)/\mathrm{Rat}(X),$$

the *group of rational equivalence classes of cycles on* X. The equivalence class of a subvariety $Y \in Z(X)$ is denoted by $[Y] \in A(X)$.

Let Y_1 and Y_2 be two subvarieties of X, such that all three are smooth at a point p. Denote by $T_{Y_1,p}$ and $T_{Y_2,p}$ the tangent spaces of Y_1, and Y_2 respectively at p. Then, if the span of $T_{Y_1,p}$ and $T_{Y_2,p}$ is the tangent space of X at p, then the subvarieties Y_1, Y_2 of a variety X intersect *transversely* at the point p ([EH16]). Moreover, if Y_1, and Y_2 intersect transversely at each point $p \in Y_1 \cap Y_2$, then the varieties Y_1, and Y_2 are said to be *genererically transverse* ([EH16]).

Theorem 4.1.2 ([EH16, Theorem-Definition 1.5.]). *If X is a smooth quasi-projective variety, then there is a unique product structure on $A(X)$ satisfying the condition: If two subvarieties Y_1, Y_2 of X are generically transverse, then*

$$[Y_1][Y_2] = [Y_1 \cap Y_2].$$

This structure makes

$$A(X) = \bigoplus_{c=0}^{\dim X} A^c(X)$$

into an associative, commutative ring, graded by codimension, called the Chow ring *of X.*

The above definition makes sense because of the *moving lemma*, which says that if two subvarieties Y_1, and Y_2 are not generically transverse, then there exists a rational equivalent subvariety $Y_1' \in [Y_1]$ such that Y_1' and Y_2 are generically transverse. Note also that since for a smooth quasiprojective variety X of dimension n, the only cycle of dimension n is the variety X itself, the n-component $A^n(X) \simeq \mathbb{Z}$.

Theorem 4.1.3 ([EH16, Theorem 2.1.]). *The Chow ring of $\mathbb{P}^n$ is*

$$A^*(\mathbb{P}^n) = \mathbb{Z}[H]/(H^{n+1}),$$

where $H \in A^1(\mathbb{P}^n)$ is the rational equivalence class of a hyperplane; more generally, the class of a variety of codimension k and degree d is dH^k.

The Chow ring of a matroid was first defined in [FY04], inspired by, and based on the work of [DP95] who first introduced the combinatorics of the wonderful compactification of the complement of a hyperplane arrangement, see Section 3.3. The original definition of the Chow ring of a matroid is stated in more general terms, namely in terms of any building set, see Definition 3 in [FY04]. We choose to state the definition in terms of the lattice of flats.

Let M be a (loopless) matroid on a ground set E, and let $\bar{\mathcal{L}}_M$ be the reduced lattice of flats of the matroid M.

Definition 4.1.4. The *Chow ring* $A^*(M)$ of M is the quotient ring

$$A^*(M) := \frac{\mathbb{Z}[x_F : F \in \bar{\mathcal{L}}_M]}{\mathcal{I}_M + \mathcal{J}_M}$$

where the ideal $\mathcal{I}_M$ is generated by

$$\mathcal{I}_M := (x_F x_{F'} \mid F, F' \text{ not comparable})$$

and the ideal $\mathcal{J}_M$ is generated by

$$\mathcal{J}_M := (\sum_{F \ni i} x_F - \sum_{G \ni j} x_G \mid i, j \in E).$$

Example 4.1.5. In this example we compute the Chow ring of the uniform matroid of rank 3 on 3 elements $U_{3,3}$. The lattice of flats of the matroid $U_{3,3}$ is given in Figure 4.1. Hence, the Chow ring of the matroid is given by

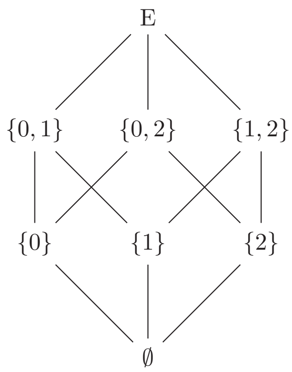

Figure 4.1: The lattice of flats of $U_{3,3}$.

$$A^*(U_{3,3}) = \frac{\mathbb{Z}[x_0, x_1, x_2, x_{01}, x_{02}, x_{12}]}{I_M + J_M},$$

where the ideals I_M, and J_M are given by:

$$\mathcal{I}_M = (x_0x_1, x_0x_2, x_1x_2, x_0x_{12}, x_1x_{02}, x_2x_{01}, x_{01}x_{02}, x_{01}x_{12}, x_{02}x_{12}),$$
$$\mathcal{J}_M = (x_0 + x_{02} - x_1 - x_{12}, x_0 + x_{01} - x_2 - x_{12}, x_1 + x_{01} - x_2 - x_{02}).$$

The relations $\mathcal{I}_M$, and $\mathcal{J}_M$ give us that the Chow ring $A^*(U_{3,3})$ is the direct sum of the following groups:

$$A^0(U_{3,3}) \cong \mathbb{Z}$$
$$A^1(U_{3,3}) \cong \mathbb{Z}\langle x_0, x_{01}, x_{02}, x_{12}\rangle$$
$$A^2(U_{3,3}) \cong \mathbb{Z}\langle x_0x_{01}\rangle.$$

For example the linear monomial x_1 is given by

$$x_1 = -x_{12} - x_{01} + x_0 + x_{01} + x_{02}$$
$$= -x_{12} + x_0 + x_{02},$$

whereas the squared monomials x_0^2 and x_{01}^2 are given by

$$x_0^2 = x_0(-x_{01} - x_{02} + x_2 + x_{02} + x_{12})$$
$$= -x_0x_{01},$$
$$x_{01}^2 = x_{01}(-x_0 - x_{02} + x_2 + x_{02} + x_{12})$$
$$= -x_0x_{01}.$$

■

Note that for harder computations than the one given in Example 4.1.5, we have used a Macaulay2 package which, among other things, gives a script that computes the Chow ring of a matroid, see [Che].

The Chow ring of a matroid in Definition 4.1.4 is in fact a special case of the more general definition of the Chow ring $A^*(\Sigma)$ of any unimodular fan Σ in $N_{\mathbb{R}}$, see Definition 5.4 in [AHK18], or [Dan78] for the original construction. Now, we find a set of generators for the Chow ring $A^*(\Sigma)$. We now use the same construction as the one given in section 5.1 of [AHK18]. Let Σ be a unimodular fan, and let P_Σ be the set of primitive ray generators of Σ. Let S_Σ be the polynomial ring over $\mathbb{Z}$ with variables indexed by P_Σ:

$$S_\Sigma := \mathbb{Z}[x_e]_{e\in P_\Sigma}.$$

For each $k-$dimensional cone σ in Σ, we associate a degree k square-free monomial

$$x_\sigma := \prod_{e\in\sigma} x_e.$$

The subgroup of S_Σ generated by all such monomials x_σ will be denoted

$$Z^k(\Sigma) := \bigoplus_{\sigma\in\Sigma_k} \mathbb{Z}_{x_\sigma}.$$

Let $Z^*(\Sigma)$ be the sum of $Z^k(\Sigma)$ over all nonnegative integers k.

Proposition 4.1.6 ([AHK18, Proposition 5.5])**.** *Let Σ be a unimodular fan. The group $A^k(\Sigma)$ is generated by $Z^k(\Sigma)$ for each nonnegative integer k.*

In the case when the fan of interest is the Bergman fan Σ_M of some matroid M, the polynomial ring S_Σ corresponds to the usual polynomial ring $\mathbb{Z}[x_F : F \in \bar{\mathcal{L}}_F]$ in Definition 4.1.4. Moreover, each k-dimensional cone σ of Σ_M corresponds to some flag $\mathcal{F}_\sigma = \emptyset \subset F_1 \subseteq \ldots \subseteq F_k \subseteq E$ of length $k+2$. Hence, the square-free monomial $x_\sigma \in Z^k(\Sigma)$ corresponds to

$$x_\sigma = x_{F_1} \cdots x_{F_k} \in A^k(M).$$

So, Proposition 4.1.6 says that the Chow group $A^k(M)$ is generated by $Z^k(\Sigma_M)$. For example, recall that in Example 4.1.5 we saw that the monomials $x_0^2 = -x_0x_{01}$ and $x_{01}^2 = -x_0x_{01}$ in the ring $A^*(U_{3,3})$.

When a matroid is representable over some field k, the Chow ring of the matroid has a specific geometric interpretation. Let $M_{\mathcal{A}}$ be the matroid arising from some hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^d_k$, and let $W_{\mathcal{A}}$ be the wonderful compactification of the complement as defined in Definition 3.3.1, then we have the isomorphism $A^*(M_{\mathcal{A}}) \cong A^*(W_{\mathcal{A}})$ [FY04]. Moreover, if we let $L_F \subsetneq \mathcal{A}$ be the linear subspace corresponding to a flat F, then the generators $x_F \in A^*(M_{\mathcal{A}})$ correspond to the strict transforms $\hat{L}_F \subseteq W_{\mathcal{A}}$ of the linear subspaces $L_F \subseteq \mathcal{A}$. Finally, products of generators in $A^*(M_{\mathcal{A}})$ correspond to intersections of the corresponding strict transforms in $W_{\mathcal{A}}$.

Example 4.1.7. Note that a realization of the uniform matroid $U_{3,3}$ is an arrangement $\mathcal{A} \subseteq \mathbb{P}^2$ consisting of three lines not intersecting in the same point. Hence, the Chow ring of the uniform matroid $A^*(U_{3,3})$ computed in Example 4.1.5 is isomorphic to the Chow ring of the wonderful compactification $W_{\mathcal{A}}$ of the arrangement, i.e., of the blow-up of $\mathbb{P}^2$ in the 3 intersection points of the arrangement $\mathcal{A}$. Then, if we let $\hat{H}$ be the pullback of the hyperplane class $H \subseteq \mathbb{P}^2$ in $W_{\mathcal{A}}$, and $\hat{L}_{01}$, $\hat{L}_{02}$, and $\hat{L}_{12}$ be the strict transforms of the intersection points of the arrangement $\mathcal{A} \subseteq \mathbb{P}^2$, then

$$\mathbb{Z}\langle x_0, x_{01}, x_{02}, x_{12}\rangle = A^1(U_{3,3}) \simeq A^1(W_{\mathcal{A}}) = \mathbb{Z}\langle \hat{H}, \hat{L}_{01}, \hat{L}_{02}, \hat{L}_{12}\rangle.$$

■

The autors of [FY04] presented another geometric interpretation for the Chow ring $A^*(M)$ of a loopless matroid M, even when the matroid is not representable. The following theorem is a special case of Theorem 3 in [FY04].

Theorem 4.1.8. *Let Σ_M be the Bergman fan of a loopless matroid M, and let X_Σ be the toric variety associated the fan Σ_M. Then, the assignment x_σ to the orbit closure $\overline{O(\sigma)}$ in X_Σ, extends to an isomorphism*

$$A^*(M) \simeq A^*(X_\Sigma).$$

4.2 Minkowski Weights

In this section we introduce the ring of Minkowski weights of the Bergman fan Σ_M, and we see how the ring of Minkowski weights is related to the Chow ring of the corresponding matroid M. Again, we use the same construction and notation as in section 5.1 of [AHK18].

Denote by Σ a simplicial fan living in $N_\mathbb{R}$, a $n-$dimensional latticed vector space. As before, the group Σ_k consists of the $k-$dimensional cones in Σ. If $\tau \subsetneq \sigma$ is of codimension 1, and σ is a simplicial cone, we write

$e_{\sigma/\tau} :=$ the primitive generator of the unique 1-dimensional face of σ not in τ.

Recall that a k-dimensional cone σ in a simplicial fan is generated by k linearly independent vectors over $\mathbb{R}$. Hence, the above definition makes sense. The group of Minkowski weights on a simplicial fan was originally defined in [FS97] in the study of intersection theory on toric varieties. Following we state the definition reported in [AHK18], in order to be consistent with the conventions.

Definition 4.2.1 ([AHK18, Definition 5.1])**.** A k-dimensional Minkowski weight on Σ is a function

$$\omega : \Sigma_k \to \mathbb{Z}$$

which satisfies the *balancing condition*: For every $(k-1)$-dimensional cone τ in Σ,

$$\sum_{\tau \subset \sigma} \omega(\sigma) e_{\sigma/\tau} \text{ is contained in the subspace generated by } \tau.$$

The *group of Minkowski weights* on Σ is the group

$$MW_*(\Sigma) := \bigoplus_{k \in \mathbb{Z}} MW_k(\Sigma),$$

where $MW_k(\Sigma) := \{k - \text{dimensional Minkowski weights on } \Sigma\} \subset \mathbb{Z}^{\Sigma_k}$.

It is clear from the definition that:

1. There is a natural isomorphism

$$\mathrm{wt}_0 : MW_0(\Sigma) \to \mathbb{Z}. \tag{4.1}$$

2. If $k > \dim(MW(\Sigma))$, then the group $MW_k(\Sigma) = 0$.

When the fan we study is the Bergman fan of some matroid M, we denote the fan by Σ_M. Recall that every cone $\sigma \in \Sigma_M$ corresponds to a flag of flats of length k:

$$\mathcal{F} = F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k.$$

We can exploit the correspondence between cones and flags to rephrase the sum in the balancing condition in Definition 4.2.1. Let $\tau \in \Sigma_M$ be a cone of dimension $(k-1)$ and let $\mathcal{F} = F_1 \subsetneq \ldots \subsetneq F_{k-1}$ be the corresponding flag. Then, every cone $\sigma \supset \tau$ of dimension k, corresponds to a flag composed of the flag $\mathcal{F}$ extended with a flat F_k which fits into the flag $\mathcal{F}$. We denote σ_{F_k} the cone corresponding to the flag $\mathcal{F}$ extended with F_k. Then, the sum in Definition 4.2.1 corresponds to sum over all flats F_k that can fit into the flag for τ:

$$\sum_{F_k \text{ fits into } \mathcal{F}} \omega(\sigma_{F_k})_{F_k} \text{ is contained in the subspace } \langle u_{F_1}, \ldots u_{F_{k-1}} \rangle.$$

where u_{F_k} is the ray corresponding to the flat F_k in the Bergman fan Σ_M. Moreover, as in Section 3.1, we denote by $\{0\}$ the vertex of the fan.

Hopefully, the following example will demystify the definition of a Minkowski weight. The example is similar to [KV19, Example 6.2]

Example 4.2.2. The Bergman fan Σ_M of the uniform matroid $U_{3,3}$ is a 2-dimensional fan living in $N_{\mathbb{R}} = \mathbb{Z}^3/(e_0 + e_1 + e_2) \otimes \mathbb{R}$, where e_0, e_1 and e_3 are the standard unit vectors of $\mathbb{Z}^3$. Moreover, recall that for a flat F the vector $e_F = \sum_{i \in F} e_i$, and the vector u_F is the image of e_F in $N_{\mathbb{R}}$. The Bergman fan Σ_M is then composed of the following rays:

$$\begin{aligned} u_0 &= e_0, & u_1 &= e_1, & u_2 &= -e_0 - e_1, \\ u_{01} &= e_0 + e_1, & u_{02} &= -e_1, & u_{12} &= -e_0, \end{aligned}$$

and of the following cones:

$$\begin{aligned} \sigma_{0,01} &= \mathrm{cone}(u_0, u_{01}), & \sigma_{0,02} &= \mathrm{cone}(u_0, u_{02}), & \sigma_{1,01} &= \mathrm{cone}(u_1, u_{01}), \\ \sigma_{1,12} &= \mathrm{cone}(u_1, u_{12}), & \sigma_{2,02} &= \mathrm{cone}(u_2, u_{02}), & \sigma_{2,12} &= \mathrm{cone}(u_2, u_{12}). \end{aligned}$$

A 1-dimensional Minkowski weight on $\Sigma_{U_{3,3}}$ is a function

$$\omega : \Sigma_1 \to \mathbb{Z},$$

which has to satisfy the balancing condition. As the only 0-dimensional cone τ in Σ_M is the origin, we get the following condition:

$$\omega_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \omega_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + w_2 \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \omega_{01} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \omega_{02} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \omega_{12} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0. \tag{4.2}$$

The condition Equation (4.2) gives us the following relations:

$$\omega_0 - \omega_2 + \omega_{01} - \omega_{12} = 0,$$
$$\omega_1 - \omega_2 + \omega_{01} - \omega_{02} = 0.$$

The weights ω_0, ω_1, ω_2, and ω_{01} determine the remaining two weights ω_{12}, and ω_{02}, hence $MW_1(\Sigma_{U_{3,3}}) \cong \mathbb{Z}^4$.

A 2-dimensional Minkowski weight on Σ_M, is a function

$$\omega : \Sigma_2 \to \mathbb{Z},$$

which has to satisfy the balancing condition. The cones containing u_0, are σ_{01}, and σ_{02}, and the primitive generators are given by

$$e_{\sigma_{0,01}/u_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad e_{\sigma_{0,02}/u_0} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Hence we get the following condition:

$$\omega_{0,01} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \omega_{0,02} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0.$$

Hence, the weight function must satisfy $\omega_{0,01} = \omega_{0,02}$. Similar computations give us

$$\omega_{0,01} = \omega_{0,02} = \omega_{1,01} = \omega_{1,12} = \omega_{2,02} = \omega_{2,12}.$$

It follows that the group of 2-dimensional Minkowski weights $MW_2(\Sigma_{U_{3,3}}) \cong \mathbb{Z}$. Note that the group of Minkowski weights $MW(\Sigma_{U_{3,3}})$ is isomorphic to the Chow group $A^*(U_{3,3})$ calculated in Example 4.1.5. We will explore this relation in more depth in the next section. ■

The fact that $MW_2(\Sigma_M) \cong \mathbb{Z}$, for a 2-dimensional Bergman fan is not a coincidence. Let Σ_M be the d-dimensional Bergman fan of some matroid of rank $d + 1$.

Proposition 4.2.3 ([AHK18, Proposition 5.2])**.** *A d-dimensional weight on d-dimensional Bergman fan Σ_M satisfies the balancing condition if and only if it is constant.*

It follows that there is a canonical isomorphism given by:

$$\text{weight}_d : MW_d(\Sigma_M) \to \mathbb{Z} \tag{4.3}$$
$$\omega \mapsto \omega(\sigma). \tag{4.4}$$

In the next section, we show that the isomorphism in Equation (4.3) implies $A^d(\Sigma_M) \simeq \mathbb{Z}$. But before, we need to introduce some preliminaries.

4.3 Linear duality and Poincaré duality

As noted in [AHK18], the group of $k-$dimensional Minkowski weights $MW_k(\Sigma)$ on Σ can be identified with the dual of $Z^k(\Sigma)$ in the following way

$$t_\Sigma : MW_k(\Sigma) \to \text{Hom}_{\mathbb{Z}}(Z^k(\Sigma), \mathbb{Z}), \quad \omega \to (x_\sigma \to \omega(\sigma)).$$

By Proposition 4.1.6, the image of t_Σ contains $\text{Hom}_{\mathbb{Z}}(A^k(\Sigma), \mathbb{Z})$.

Proposition 4.3.1 ([AHK18, Proposition 5.6.]). *The isomorphism t_Σ restricts to the bijection between the subgroups*

$$MW_k(\Sigma) \to Hom_{\mathbb{Z}}(A^k(\Sigma), \mathbb{Z}).$$

In [AHK18] the bijection in Proposition 4.3.1 is used to define the *cap product*

$$A^l(\Sigma) \times MW_k(\Sigma) \to MW_{k-l}(\Sigma), \quad \xi \cap \omega(\sigma) := t_\Sigma \omega(\xi \cdot x_\sigma).$$

It is worth lingering on the cap product defined in Proposition 4.3.1. Remark that by Proposition 4.1.6, we can write elements in $A^k(\Sigma)$ as elements of $Z^k(\Sigma)$, i.e., as the sum of square free monomials of degree k. Hence for $\xi \in A^l(\Sigma)$, and for $\sigma \in \Sigma_{k-l}$, the polynomial $\xi \cdot x_\sigma$ is an element of $Z^k(\Sigma)$, i.e., of the form

$$\xi \cdot x_\sigma = \sum_{\sigma' \in \Sigma_k} c_{\sigma'} x_{\sigma'},$$

for some $c_{\sigma'} \in \mathbb{Z}$. Then we get

$$\begin{aligned} t_\Sigma \omega(\xi \cdot x_\sigma) &= t_\Sigma \omega\Big(\sum_{\sigma' \in \Sigma_k} c_{\sigma'} x_{\sigma'} \Big) \\ &= \sum_{\sigma' \in \Sigma_k} c_{\sigma'} \omega(\sigma'). \end{aligned}$$

The bijection in Proposition 4.3.1 makes the group $MW_*(\Sigma)$ into a graded module over the Chow ring $A^*(\Sigma)$ [AHK18]. Note that because of the relations in the Chow ring, the choice of representative for ξ is not unique. However, the cap product is still well defined, since the Minkowski weight function is constructed such that it is indifferent to the choice of representative for ξ. Let us look at an example to demonstrate how the cap product works.

Example 4.3.2. We continue with Example 4.2.2, so Σ_M is the Bergman fan of the uniform matroid $U_{3,3}$. Let us calculate the cap product

$$A^1(\Sigma_M) \times MW_1(\Sigma) \to MW_0(\Sigma), \quad \xi \cap \omega(\sigma) := t_\Sigma \omega(\xi \cdot x_\sigma),$$

for $\xi = x_0 + x_{01} + x_{02} \in A^1(\Sigma_M)$, and $\omega \in MW_1(\Sigma)$ taking the values:

$$\omega(u_i) = 1, \quad \omega(u_{ij}) = 0.$$

Then, the the cap product is an element of $MW_0(\Sigma_M)$:

$$\xi \times \omega \to \hat{\omega} = \xi \cap \omega \in MW_0(\Sigma_M),$$

so $\hat{\omega}$ is just a function on the origin $\{0\}$ of the fan Σ_M, which is given by

$$\begin{aligned} \hat{\omega}(\{0\}) &= t_\Sigma \omega(x_0 + x_{02} + x_{02}) \\ &= \omega(u_0) + \omega(u_{02}) + \omega(u_{02}) = 1. \end{aligned}$$

Now, let us calculate the cap product

$$A^1(\Sigma_M) \times MW_2(\Sigma) \to MW_1(\Sigma), \quad \xi \cap \omega(\sigma) := t_\Sigma \omega(\xi \cdot x_\sigma),$$

for $\xi = x_0 + x_{01} \in A^1(\Sigma_M)$, and for the constant function $\omega \in MW_2(\Sigma_M)$ taking the value 1 on all the top dimensional cones. Then the cap product is an element of $MW_1(\Sigma_M)$:

$$\xi \times \omega \to \hat{\omega} = \xi \cap \omega \in MW_1(\Sigma_M),$$

so $\hat{\omega} \in MW_1(\Sigma_M)$ is a function on the one dimensional rays of the fan. Let us check that $\hat{\omega} \in MW_1(\Sigma_M)$ actually satisfies the balancing condition:

$$\begin{aligned}&\hat{\omega}(u_0) + \hat{\omega}(u_1) + \hat{\omega}(u_2) + \hat{\omega}(u_{01}) + \hat{\omega}(u_{02}) + \hat{\omega}(u_{12}) = \\ &t_\Sigma\omega(x_0^2 + x_0x_{01}) + t_\Sigma\omega(x_1x_{01}) + t_\Sigma\omega(x_{01}^2).\end{aligned}$$

We apply Proposition 4.1.6, and by Example 4.1.5, we rewrite the last line:

$$\begin{aligned}&t_\Sigma\omega(-x_0x_{02} + x_0x_{01}) + t_\Sigma\omega(x_1x_{01}) + t_\Sigma\omega(-x_0x_{01}) = \\ &-\omega(\sigma_{0,02}) + \omega(\sigma_{0,01}) + \omega(\sigma_{1,01}) - \omega(\sigma_{0,01}) = 0.\end{aligned}$$

Hence, $\hat{\omega} \in MW_1(\Sigma_M)$ satisfies the balancing condition. ■

Definition 4.3.3 ([AHK18, Definition 5.7.]). To an element i of the ground set E of matroid M, we associate the linear form

$$\alpha_{M,i} := \sum_{i \in F} x_F \in A^1(M)$$

Recall that when a matroid of rank $d+1$ is representable over some field k, the matroid $M_\mathcal{A}$ arises from some hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^d_k$. Moreover, the generators $x_F \in A^*(M_\mathcal{A})$, correspond to the strict transforms $\hat{L}_F$ of the linear subspaces $L_F \subset \mathcal{A}$ in $W_\mathcal{A}$. Note that the polynomial $\alpha_{M,i}$ is the sum of the monomials x_F, for $F \subseteq H_i$. It follows that the polynomial $\alpha_{M,i}$ corresponds to the strict transform of H_i plus the union of the strict transform of all its subspaces $L_F \subset H_i$, which is the total transform $\pi^{-1}(H_i) \subset W_\mathcal{A}$ of the hyperplane $H_i \subset \mathcal{A}$.

The divisor of $\alpha_{M,i}$ is independent of the choice of i, hence it will be denoted by α_M. In [AHK18], they show that $A^d(\Sigma_M)$ is generated by the element α_M^d, where d is the dimension of Σ_M.

Proposition 4.3.4 ([AHK18, Proposition 5.8]). *Let $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ be any flag of nonempty proper flats of M.*

(1) *If the rank of F_m is not m for some $m \leq k$, then*

$$x_{F_1}x_{F_2}\cdots x_{F_k}\alpha_M^{d-k} = 0 \in A^d(\Sigma_M).$$

(2) *if the rank of F_m is m for all $m \leq k$, then*

$$x_{F_1}x_{F_2}\cdots x_{F_k}\alpha_M^{d-k} = \alpha_M^d \in A^d(\Sigma_M)$$

In particular, for any two maximal flags of nonempty proper flats $\mathscr{F}_1$ and $\mathscr{F}_2$ of M,

$$x_{\mathscr{F}_1} = x_{\mathscr{F}_2} \in A^d(\Sigma_M).$$

There is a geometric intuition behind the first result in Proposition 4.3.4. It suffices to consider the case where the flag is of length 1, i.e., to consider the monomial

$$x_F \alpha_M^{d-1} \in A^d(\Sigma_M),$$

to grasp the idea. So, let us understand why $\mathrm{r}(F) > 1$ implies $x_F \alpha_M^{d-1} = 0$. We begin by assuming that the matroid $M_{\mathcal{A}}$ arises from an essential hyperplane arrangement $\mathcal{A} \subset \mathbb{P}_k^d$. And note that if the intersection $L_F \cap H = \emptyset$ in $\mathcal{A}$, it implies that $\hat{L}_F \cap \pi^{-1}(H) = \emptyset$ in $W_{\mathcal{A}}$. Moreover, the polynomial α_M^{d-1} corresponds to the total transform of H^{d-1}, i.e., of some hyperplane H intersected with itself $d-1$ times. Recall that α is independent of the choice of H, hence we can choose $d-1$ different hyperplanes H, and as these intersect transversely, the codimension of H^{d-1} is $d-1$. Moreover, if $\dim(L_F) < d-1$, we can always choose a hyperplane H such that $H \cap L_F = \emptyset$, which implies

$$H^{d-1} \cap L_F = \emptyset.$$

It follows that $x_F \alpha_M^{d-1} = 0$. By a similar argument, we can show that

$$x_F \alpha_M^l = 0 \text{ in } A^{l+1}(\Sigma_M) \quad \text{ if } \mathrm{r}(F) + l > d,$$

or equivalently that

$$H^l \cap L_F = \emptyset \quad \text{ if } \operatorname{codim}(L_F) + l > d.$$

Definition 4.3.5 ([AHK18, Proposition 5.9])**.** The *degree map* of M is the homomorphism obtained by taking the cap product

$$\deg : A^d(\Sigma_M) \to \mathbb{Z}, \quad \xi \longmapsto \xi \cap 1_M,$$

where $1_M = 1$ is the constant d-dimensional Minkowski weight on Σ_M.

By Proposition 4.1.6, the homomorphism deg is uniquely determined by the fact that $\deg(x_\sigma) = 1$ for all monomials x_σ corresponding to a d-dimensional cone in Σ_M.

Proposition 4.3.6 ([AHK18, Proposition 5.9])**.** *The degree map of M is an isomorphism.*

In 2020, Eur wrote an explicit formula for the degree map in Definition 4.3.5 applied to monomials of degree d, which will be of interest in later sections.

Theorem 4.3.7 ([Eur20, Theorem 3.2.])**.** *Let M be a matroid of rank $d+1$ on a ground set E. Let $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E$ be a chain of flats in $\mathcal{L}_M$ with ranks $r_i := \mathrm{r}(F_i)$, and let $d_1, \ldots, d_k$ be positive integers such that $\sum_i d_i = d$. Denote by $\tilde{d}_i := \sum_{j=1}^{i} d_j$. Then*

$$\deg(x_{F_1}^{d_1} \cdots x_{F_k}^{d_k}) = (-1)^{d-k} \prod_{i=1}^{k} \binom{d_i - 1}{\tilde{d}_i - r_i} \mu^{\tilde{d}_i - r_i}(M|F_{i+1}/F_i)$$

where $\mu^i(M')$ denotes the i-th unsigned coefficient of the reduced characteristic polynomial $\chi_{M'}(t) = \mu^0(M')t^{\mathrm{r}(M')-1} - \mu^1(M')t^{\mathrm{r}(M')-2} + \cdots + (-1)^{\mathrm{r}(M')-1}\mu^{\mathrm{r}(M')-1}(M')$ of a matroid M'.

The above degree map will be useful when computing the Chern numbers of a matroid.

Theorem 4.3.8 ([AHK18, Theorem 6.19])**.** *(Poincaré Duality). For any integer $k \leq r$, the multiplication map*

$$A^k(M) \times A^{r-k}(M) \to A^r(M)$$

defines an isomorphism between groups

$$A^{r-k}(M) \cong Hom_{\mathbb{Z}}(A^k(M), A^r(M)).$$

In particular, the groups $A^q(M)$ are torsion free.

Let M be a matroid of rank $d+1$, then, by Proposition 4.3.6, the Chow group $A^d(M) \cong \mathbb{Z}$. So for $r = d$, the isomorphism in Theorem 4.3.8 is given by:

$$A^{d-k}(M) \cong \mathrm{Hom}_{\mathbb{Z}}(A^k(M), \mathbb{Z}).$$

Since the Chow group $A^k(M)$ is a torsion free finitely generated $\mathbb{Z}$-module, we get that $A^k(M)$ is isomorphic to its dual, hence

$$\dim(A_k(M)) = \dim(A^{d-k}(M)) = \dim(A^k(M)),$$

where dim is the dimension of the Chow group as a $\mathbb{Z}$-module. Moreover, let Σ_M be the Bergman fan of M, then by the linear duality given in Proposition 4.3.1, and the Poincaré duality given in Theorem 4.3.8 we get the following isomorphism:

$$MW_k(\Sigma_M) \cong \mathrm{Hom}_{\mathbb{Z}}(A^k(M), \mathbb{Z}) \cong A^{d-k}(M). \tag{4.5}$$

The isomorphism in 4.5 gives a ring structure to the groups of Minkowski weights $MW_*(\Sigma_M)$. The product on the Chow ring induces a well defined product on

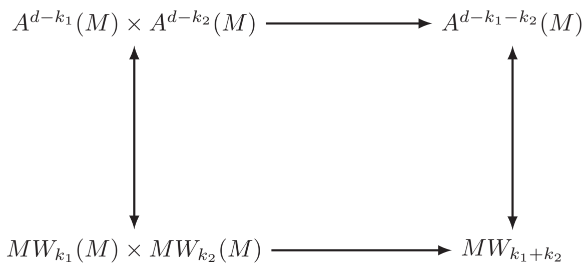

Figure 4.2: Product structure on $MW_*(\Sigma_M)$.

the Minkowski weights $MW_*(\Sigma_M)$, see Figure 4.3. It is worth remarking that, when the fan Σ is complete, a ring structure of $MW_*(\Sigma)$ is given in [FS97]. Hence, we have the following ring isomorphism

$$MW_*(\Sigma_M) \cong A^*(M). \tag{4.6}$$

For later purposes, we denote the isomorphism in 4.6 by

$$\psi : MW_*(\Sigma_M) \to A^*(M). \tag{4.7}$$

In particular, we denote the restriction of the isomorphism 4.6 on the 0-dimensional Minkowski weights Σ_M by

$$\psi_0 : MW_0(\Sigma_M) \to A^d(M). \tag{4.8}$$

Moreover, the restriction map ψ_0 in Equation (4.8), the degree map deg in Definition 4.3.5, and the map wt_0 in Equation (4.1), give us the commutative diagram in Section 4.3.

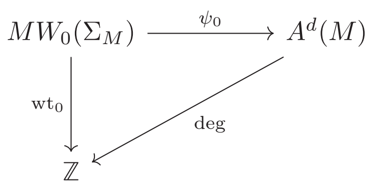

Figure 4.3: Commutative diagram.

PART II

Chern numbers of matroids

CHAPTER 5

The Chern–Schwartz–MacPherson cycles of matroids

In this chapter we give an overview of the *Chern-Schwartz-MacPherson cycles* (CSM cycles) of the matroid. The weights of the CSM cycles are assigned to the underlying fan of the Chern-Schwartz-MacPherson cycles by the *Beta invariant*, which we introduce in the first section. Next we review that, when the matroid arises from a hyperplane arrangement, the CSM cycles are related to the *Chern-Schwartz-MacPherson (CSM) class* of the complement of the arrangement. Two different approaches to the CSM classes were developed by M.H. Schwartz [Sch65] and McPherson [Mac74] independently, and proven to coincide in [Bra81]. Finally, we define the *Chern numbers* of a matroid.

5.1 The beta invariant

The *characteristic polynomial* of a matroid is a generalization of the chromatic polynomial of a graph. The chromatic polynomial $\chi_G(\lambda)$, first introduced in [Bir12], is a function counting the number of ways of coloring the vertices a graph G with $\lambda \in \mathbb{Z}_{\geq 1}$ different colors, given the condition that two adjacent vertices are not to be given the same color. Note that a graph with a loop has no coloring, and in fact, also the characteristic polynomial for a matroid with loops is defined to be 0.

Let M be a loopless matroid of rank $d+1$ on a ground set $E = \{0, 1, \ldots, n\}$. The *characteristic polynomial* of M is

$$\chi_M(\lambda) = \sum_{S \subseteq E} (-1)^{|S|} \lambda^{\mathrm{crk}(S)}, \tag{5.1}$$

where the sum is over all subsets $S \subset E$, and $\mathrm{crk(S)} = d+1-\mathrm{r}(S)$. Furthermore, we will see that the characteristic polynomial can also be defined in terms of flats. First, we need to define the Möbius function.

Definition 5.1.1 ([MRS20, Definition 4])**.** Let $\mathcal{L}_M$ be the lattice of flats F of a matroid M. The *Möbius function* of $\mathcal{L}_M$ is the function $\mu : \mathcal{L}_M \times \mathcal{L}_M \to \mathbb{Z}$

defined recursively by

$$\mu(F,G) := \begin{cases} 0 & \text{if } F \nsubseteq G, \\ 1 & \text{if } F = G, \\ -\sum_{F\subseteq G' \subsetneq G} \mu(F,G') & \text{if } F \subsetneq G. \end{cases}$$

Then, the *characteristic function* is

$$\chi_M(\lambda) = \sum_F \mu(\emptyset, F)\lambda^{\mathrm{crk}(F)}.$$

A fundamental property of the characteristic polynomial is the deletion/contraction property: If i is not a loop or a coloop of M then

$$\chi_M(\lambda) = \chi_{M\setminus i}(\lambda) - \chi_{M/i}(\lambda). \tag{5.2}$$

To get an intuition for the deletion/contraction property, it is worth examining the chromatic polynomial analogy. In graph theory, deleting an edge $G \setminus e$ corresponds to removing the edge e from the graph. Then, the two adjacent vertices v_1, v_2 of e are no longer subject to the condition of having different colors. Whereas, contracting an edge G/e, corresponds to identifying the two adjacent vertices to e. Hence, the coloring of G/e corresponds to the coloring of G with the additional condition on v_1 and v_2 to have the same color. In particular, if e is not a loop or a coloop of G:

$$\chi_G(\lambda) = \chi_{G\setminus e}(\lambda) - \chi_{G/e}(\lambda).$$

Example 5.1.2. Let G be as in Example 2.3.9. The graphs $G \setminus e_0$, and G/e_0 are given in Figure 5.2.

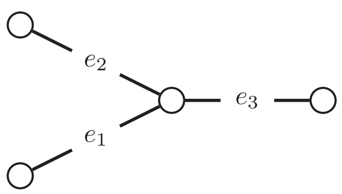

Figure 5.1: The graph $G \setminus e_0$.

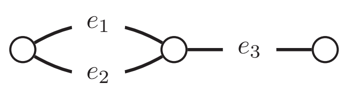

Figure 5.2: The graph G/e_0.

The chromatic polynomial $\chi_{G\setminus e_0}$ is given by

$$\chi_{G\setminus e_0}(\lambda) = \lambda(\lambda-1)^3,$$

since for the first central node there are λ different possible coloring, whereas for the three adjacent vertices there are $(\lambda-1)$ possible coloring. The chromatic polynomial χ_{G/e_0} is given by

$$\chi_{G/e_0}(\lambda) = \lambda(\lambda-1)^2.$$

since for the first central node there are λ different possible coloring, whereas for the two adjacent vertices there are $(\lambda - 1)$ possible coloring. Hence, the chromatic polynomial of G is given by

$$\begin{aligned}\chi_G(\lambda)(G) =& \lambda(\lambda-1)(\lambda-1)(\lambda-2) - \lambda(\lambda-1)(\lambda-1)\\ =& \lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda.\end{aligned}$$

■

We continue with some invariants of the matroid related to the characteristic polynomial. The *reduced characteristic polynomial* of M is the polynomial

$$\bar{\chi}_M(q) = \chi_M(q)/(q-1).$$

In [Cra67], Crapo proves that the *beta invariant* can be computed directly from the lattice of flats, instead of over all subsets of E, see definition below.

Definition 5.1.3. Let M be a loopless matroid, then the *beta invariant* of M is

$$\begin{aligned}\beta(M) &= (-1)^{d+1} \sum_{F \in \mathcal{L}_M} \mu(\emptyset, F) r(F)\\ &= (-1)^d \overline{\chi}_M(1).\end{aligned}$$

If M has a loop then $\beta(M)$ is defined to be zero.

Theorem 5.1.4 ([FLL87, Theorem 7.3.2. 4])**.** *The beta invariant of a matroid M satisfies:*

1. $\beta(M) \geq 0$.
2. $\beta(M) > 0$ *if and only if M is connected and is not a loop*

Example 5.1.5. Let us compute the characteristic polynomial $\chi_{U_{3,3}}$ for the uniform matroid $U_{3,3}$. By applying the formula in Equation (5.1) we compute the characteristic polynomial.

$$\begin{aligned}\chi_{U_{3,3}}(\lambda) =& (-1)^{|\emptyset|}\lambda^{\mathrm{crk}(\emptyset)} + 3(-1)^{|\{1\}|}\lambda^{\mathrm{crk}(\{1\})}\\ &+ 3(-1)^{|\{1,2\}|}\lambda^{\mathrm{crk}(\{1,2\})} + (-1)^{|E|}\lambda^{\mathrm{crk}(E)}\\ =& \lambda^3 - 3\lambda^2 + 3\lambda - 1\end{aligned}$$

Equivalently, we can compute the characteristic polynomial by applying Equation (5.2). Note that in this particular example, the subsets of E and the flats of M coincide, but keep in mind that this is absolutely not the case in general. First, we calculate the values for the Möbius function

$$\begin{aligned}&\mu(\emptyset, \emptyset) = 1,\\ &\mu(\emptyset, \{0\}) = -1,\\ &\mu(\emptyset, \{0,1\}) = -(1-2) = -1,\\ &\mu(\emptyset, E) = -(1-3+3) = -1.\end{aligned}$$

Remark that, by symmetry, $\mu(\emptyset, \{0\}) = \mu(\emptyset, \{1\}) = \mu(\emptyset, \{2\})$, and that $\mu(\emptyset, \{0,1\}) = \mu(\emptyset, \{0,2\}) = \mu(\emptyset, \{1,2\})$. It follows that the characteristic polynomial is

$$\chi_{U_{3,3}}(\lambda) = \mu(\emptyset, \emptyset)\lambda^{\mathrm{crk}(\emptyset)} + 3\mu(\emptyset, \{0\})\lambda^{\mathrm{crk}(\{0\})}$$

$$+ 3\mu(\emptyset, \{0,1\})\lambda^{\mathrm{crk}(\{0,1\})} + \mu(\emptyset, E)\lambda^{\mathrm{crk}(E)}$$
$$= \lambda^3 - 3\lambda^2 + 3\lambda - 1,$$

as expected. Moreover, the reduced characteristic polynomial is

$$\overline{\chi}(\lambda) = \lambda^2 - 2\lambda + 1.$$

And since $\overline{\chi}(1) = 0$, the beta invariant is $\beta(M) = 0$. ■

In the next example we write an explicit formula for the beta invariant of an arbitrary uniform matroid. We follow tightly Example 2 in [MRS20].

Example 5.1.6. Let F be a flat of the uniform matroid $U_{d+1,n+1}$. Then, the values of the Möbius function are given by

$$\mu(F) = \sum_{i=0}^{\mathrm{r}(F)-1} (-1)^{i+1}\binom{|F|}{i} = \begin{cases} (-1)^{|F|} & \text{for } \mathrm{r}(F) < d+1 \\ \sum_{i=0}^{d}(-1)^{i+1}\binom{n+1}{i} & \text{for } \mathrm{r}(F) = d+1. \end{cases}$$

Since there are $\binom{n+1}{i}$ number of flats of rank i for $0 \leq i \leq d$, the characteristic polynomial of $U_{d+1,n+1}$ is given by

$$\begin{aligned} \chi_{U_{d+1,n+1}}(\lambda) = & \sum_{F \in \mathcal{L}_{U_{d+1,n+1}} \setminus E} (-1)^{|F|}\lambda^{d+1-|F|} + \sum_{i=0}^{d}(-1)^{i+1}\binom{n+1}{i} \\ = & \sum_{i=0}^{d}\binom{n+1}{i}(-1)^i\lambda^{d+1-i} + (-1)^{i+1}\binom{n+1}{d} \\ = & \sum_{i=0}^{d}\binom{n+1}{i}(-1)^i(\lambda^{d+1-i} - 1). \end{aligned}$$

Moreover, by using that $(\lambda^{d+1-i} - 1)\backslash(\lambda - 1) = \sum_{i=0}^{d-i}\lambda^i$, we get that the reduced characteristic polynomial is given by

$$\overline{\chi}_{U_{d+1,n+1}}(\lambda) = \sum_{i=0}^{d}\binom{n}{i}(-1)^i\lambda^{d-i}.$$

Finally, by setting $\lambda = 1$, and by multiplying with $(-1)^d$ we get that the beta invariant is

$$\beta(U_{d+1,n+1}) = \binom{n-1}{d}.$$

■

With the help of the package [Che], we have implemented a function to compute the beta invariant of a matroid in Macaulay2 [GS].

```
loadPackage "Matroids"

betaInvariant = Matroid -> (
    d = rank(Matroid);
    T:= characteristicPolynomial Matroid;
    R = ring T;
    Q = frac R;
```

```
    lift (T, R);
    g = T/(R_0 - 1);
    beta = (-1)^(d-1)*sub(g, Q_0=>1);
    return beta
)

U_33 = uniformMatroid(3,3);
betaInvariant(U_33)

FanoM = specificMatroid "fano";
betaInvariant(FanoM)

U_45 = uniformMatroid(4,5)
betaInvariant(U_45)
```

This generates the output:

```
ii22 : U_33 = uniformMatroid(3,3);
ii23 : betaInvariant(U_33)
oo23 = 0


ii26 : FanoM = specificMatroid "fano";
ii27 : betaInvariant(FanoM)
oo27 = 3


ii31 : U_45 = uniformMatroid(4,5);
ii32 : betaInvariant(U_45)
oo32 = 1
```

5.2 Chern-Schwarts-MacPherson cycles of matroids

The *Chern-Schwarts-MacPherson cycles* (CSM-cycles) of a matroid M were first defined in [MRS20] as a collection of weighted rational polyhedral fans. In Theorem 2.3 they prove that the CSM-cycles are balanced fans according to Definition 4.2.1. We choose to state their definition directly in terms of Minkowski weights.

Definition 5.2.1 ([MRS20, Definition 2.8.])**.** Suppose M is a matroid of rank $d+1$ on $n+1$ elements. For $0 \leq k \leq d$, the k-dimensional Chern-Schwartz-MacPherson (CSM) cycle $\mathrm{csm}_k(M)$ of M is a k-dimensional Minkowski weight ω. If M is a loopless matroid, the weight of the cone $\sigma_{\mathcal{F}}$ corresponding to a flag of flats $\mathcal{F} := \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = \{0, ..., n\}\}$ is:

$$\omega(\sigma_{\mathcal{F}}) := (-1)^{d-k} \prod_{i=0}^{k} \beta(M|F_{i+1}/F_i),$$

where $M|F_{i+1}/F_i$ denotes the minor of M obtained by restricting to F_{i+1} and contracting to F_i. If M has a loop then we define $\mathrm{csm}_k(M) := \emptyset$ for all k.

Example 5.2.2. Let M be a loopless matroid of rank $d+1$. The 0-dimensional Chern-Schwartz-MacPherson cycle $\mathrm{csm}_0(M)$ of M is the origin $\{0\}$ of Σ_M equipped with a weight equal to

$$\omega(\{0\}) = (-1)^d \beta(M).$$

■

Example 5.2.3. Let $F_i \subsetneq F_{i+1}$ be two flats of the uniform matroid, and note that

$$U_{d+1,n+1}|F_{i+1}/F_i = U_{\mathrm{r}(F_{i+1})-\mathrm{r}(F),|F_{i+1}|-|F_i|},$$

by a similar argument as in Example 2.4.7. Then, the weight of a cone $\sigma_{\mathcal{F}}$ corresponding to a flag of flats

$$\mathcal{F} := \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = \{0, ..., n\}\}$$

is:

$$\begin{aligned}\omega(\sigma_{\mathcal{F}}) &= (-1)^{d-k} \prod_{i=0}^{k} \beta(U_{\mathrm{r}(F_{i+1})-\mathrm{r}(F),|F_{i+1}|-|F_i|}) \\ &= (-1)^{d-k} \prod_{i=0}^{k} \binom{|F_{i+1}| - |F_i| - 2}{\mathrm{r}(F_{i+1}) - \mathrm{r}(F_i) - 1},\end{aligned}$$

by Example 5.1.6. Note that if $\mathrm{r}(F_{i+1}) - \mathrm{r}(F_i) \geq 1$ for a $0 \leq i \leq k$, then $\omega(\sigma_{\mathcal{F}}) = 0$, otherwise we get that the weight is given by

$$\omega(\sigma_{\mathcal{F}}) = (-1)^{d-k} \binom{n-k-1}{d-k}.$$

For example, for a top dimensional cone σ of $\Sigma_{U_{d+1,n+1}}$ the weight is given by:

$$\omega(\sigma_{\mathcal{F}}) = 1.$$

■

Recall that, by linear and Poincaré duality, we have the following isomorphism

$$\psi : MW_k(\Sigma_M) \to A^{d-k}(M),$$

see Equation (4.5). Hence, the k-dimensional CSM cycle $\mathrm{csm}_k(M)$ of a matroid M lives naturally in the Chow ring $A^*(M)$ of the matroid. Tara Fife and Felipe Rincón have conjectured an explicit formula for the CSM cycles in the Chow ring. Let $x_E = -\sum_{F \ni i} x_F$, where we are summing over all flats containing i, and note that the monomial x_E is indifferent of the choice of $i \in E$. Moreover, let

$$ch_k(M) = \psi(\mathrm{csm}_{d-k}(M))$$

denote the degree k polynomial in $A^k(M)$ corresponding to the $(d-k)$-dimensional CSM cycle.

Conjecture 5.2.4 ([FR22])**.** *Let M be a matroid of rank $d+1$, then*

$$ch_k(M) = (-1)^k \sum_{F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k \subseteq E} C_{\mathrm{r}(F_1),\mathrm{r}(F_2),\ldots,r(F_k)} x_{F_1} x_{F_2} \ldots x_{F_k},$$

where $S = \{s_1 \leq s_2 \leq \cdots \leq s_k\} = \{\mathrm{r}(F_1) \leq \mathrm{r}(F_2) \leq \cdots \leq \mathrm{r}(F_k)\}$ is a multisubset of the set $\{1, \ldots, d+1\}$. Denote by $m_S(i)$ the multiplicity of the number i in S (equal to zero if i is not in S), then the coefficients are given by

$$C_S = \frac{(s_1)(s_2-1)(s_3-2)\ldots(s_k-k+1)}{m_S(1)!m_S(2)!\ldots m_S(d+1)!}.$$

Theorem 5.2.5 ([FR22]). *Conjecture 5.2.4 holds for $ch_1(M)$, and $ch_d(M)$.*

Example 5.2.6. Let M be a matroid of rank $d+1$ with lattice of flats $\mathcal{L}_M$, and reduced lattice of flats $\hat{\mathcal{L}}_M$. Then, by Conjecture 5.2.4

$$ch_1(M) = -\sum_{F\in\mathcal{L}_M} \mathrm{r}(F)x_F. \tag{5.3}$$

We can use the identity $x_E = -\sum_{F\ni i} x_F$, to remove the flats of rank 1 from the sum in Equation (5.3). Recall that, by the linear relation in the Chow ring, the monomial x_i is given by for $F \in \hat{L}_M$:

$$x_i = -x_E - \sum_{F\ni i,\mathrm{r}(F)\geq 2} x_F.$$

Hence, we can rewrite $ch_1(M)$ as

$$\begin{aligned} ch_1(M) &= -\Bigg(\sum_{i\in E}\Bigg(-x_E - \sum_{F\in\hat{\mathcal{L}}_M, F\ni i, \mathrm{r}(F)\geq 2} x_F\Bigg) + \sum_{F \;\; \mathcal{L}_M F\ni i,\mathrm{r}(F)\geq 2} \mathrm{r}(F)x_F\Bigg) \\ &= -\sum_{\mathrm{r}(F)\geq 2} (\mathrm{r}(F) - |F|)x_F, \end{aligned}$$

where the last sum is over all flats F in $\mathcal{L}_M$. For example, for the uniform matroid $U_{d+1,n+1}$, we have the following equality

$$ch_1(U_{d+1,n+1}) = -(d+1-|E|)x_E.$$

■

Example 5.2.7. Let M be a matroid of rank $d+1$, then if Conjecture 5.2.4 is true, we have the following equality

$$ch_2(M) = (-1)^k \sum_{F_1\subseteq F_2} C_{\{F_1,F_2\}} x_{F_1} x_{F_2},$$

where the coefficients are given by

$$C_{\{F_1,F_2\}} = \frac{\mathrm{r}(F_1)(\mathrm{r}(F_2)-1)}{m_{\{F_1,F_2\}}(\mathrm{r}(F_1))! m_{\{F_1,F_2\}}(\mathrm{r}(F_2))!}.$$

■

5.3 CSM classes

In [MRS20], the authors relate the CSM-cycles of matroids $M_{\mathcal{A}}$ arising from a hyperplane arrangement $\mathcal{A}$ to a well known geometric invariant, namely to the CSM class of the complement of the arrangement $C(\mathcal{A})$. CSM classes are a generalization of Chern classes of a manifold. Usually, Chern classes of a variety are defined for the tangent bundle over that variety, and hence can only be defined over a non-singular variety. Whereas CSM classes are also defined for singular and non-compact varieties. In this section we review [MRS20] results. But first we review some definitions. We follow tightly section 3 of [MRS20].

For an algebraic variety X over $\mathbb{C}$, let $C(X)$ denote the additive group

$$\mathbb{Z}\langle \mathbb{1}_Y \rangle,$$

which is generated by the indicator functions $\mathbb{1}_Y$, for all subvarieties $Y \subset X$. The group $C(X)$ is called the group of *constructible functions* on X. In fact C defines a functor from the category of algebraic varieties over $\mathbb{C}$ to the category of abelian groups. Moreover, the Chow group $A(X)$ as defined in Definition 4.1.1, also lives in the category of abelian groups. Let A define the functor from the category of algebraic varieties over $\mathbb{C}$ to the category of abelian groups, which sends X to $A(X)$. Note that the functor A is only defined for proper morphisms. Now, for a complete and non-singular variety X, let $c(\mathcal{T}_X) \in A^*(X)$ denote the Chern class of the tangent bundle over X. Robert MacPherson proved that there is a natural transformation from the functor C to the functor A called the *Chern-Schwartz-MacPherson (CSM) class*, which satisfies

$$\mathrm{CSM}(\mathbb{1}_X) = c(\mathcal{T}_X) \cap [X],$$

when the variety X is smooth and complete, see [Mac74].

There are two important properties of the CSM classes that we will mention. First of all, the degree zero component of $\mathrm{CSM}(\mathbb{1}_X)$ corresponds to the usual topological Euler characteristic $\chi_{\mathrm{top}}(X)$. In fact, for every subvariety $Y \subseteq X$, and CSM class $\mathrm{CSM}(\mathbb{1}_Y) \in A(X)$, the zero component $\mathrm{CSM}_0(\mathbb{1}_Y) = \chi_{\mathrm{top}}(Y)$. Moreover, the CSM classes obey the inclusion-exclusion property, namely that for two varieties $Y_1, Y_2 \subseteq X$, we have that

$$\mathrm{CSM}(\mathbb{1}_{Y_1 \cup Y_2}) = \mathrm{CSM}(\mathbb{1}_{Y_1}) + \mathrm{CSM}(\mathbb{1}_{Y_2}) - \mathrm{CSM}(\mathbb{1}_{Y_1 \cap Y_2}) \in A(X).$$

The above property is a powerful tool for computations. For example, if we want to compute the CSM class of the constructible function $\mathbb{1}_{C(\mathcal{A})}$ of the complement of a hyperplane arrangement $\mathcal{A} \subsetneq \mathbb{P}^d_{\mathbb{C}}$, it is enough to know $\mathrm{CSM}(\mathbb{1}_{\mathbb{P}^n})$ for $0 \leq n \leq d$, which is a well known invariant. Let $\zeta \in A^1(\mathbb{P}^n)$ be the hyperplane class, then

$$\mathrm{CSM}(\mathbb{1}_{\mathbb{P}^n}) = (1+\zeta)^{n+1} \cap [\mathbb{P}^n],$$

see Section 5.7.1 in [EH16]. Note that the cap product with $[\mathbb{P}^n]$ gives the following isomorphism:

$$\begin{aligned} A^*(\mathbb{P}^n) &\to A_*(\mathbb{P}^n) \\ \zeta &\to [\mathbb{P}^{n-1}], \end{aligned}$$

and hence for every $0 \leq k \leq n$

$$\zeta^k \to [\mathbb{P}^{n-k}].$$

Example 5.3.1. Let $\mathcal{A} \subsetneq \mathbb{P}^2$ be an essential arrangement of 3 lines. Then, since

$$\begin{aligned} \mathrm{CSM}(\mathbb{1}_{\mathbb{P}^2}) &= 3[\mathbb{P}^0] + 3[\mathbb{P}^1] + [\mathbb{P}^2], \\ \mathrm{CSM}(\mathbb{1}_{\mathbb{P}^1}) &= 2[\mathbb{P}^0] + [\mathbb{P}^1], \\ \mathrm{CSM}(\mathbb{1}_{\mathbb{P}^0}) &= [\mathbb{P}^0], \end{aligned}$$

then by using the inclusion-exclusion property we get that

$$\begin{aligned}\mathrm{CSM}(\mathbb{1}_{C(\mathcal{A})}) &= 3[\mathbb{P}^0] + 3[\mathbb{P}^1] + [\mathbb{P}^2] - 3(2[\mathbb{P}^0] + [\mathbb{P}^1]) + 3[\mathbb{P}^0] \\ &= [\mathbb{P}^2].\end{aligned}$$

■

Now we can finally state the results in [MRS20]. Recall first that, by linear and Poincaré duality, we have the isomorphism $MW_*(\Sigma_M) \cong A_*(M)$, see Section 4.3. Moreover, let $M_{\mathcal{A}}$ be a matroid arising from a hyperplane arrangement $\mathcal{A}$, and let $W_{\mathcal{A}}$ be the wonderful compactification of the complement of the arrangement $\mathcal{A}$, see Definition 3.3.1, then $A_*(M_{\mathcal{A}}) \cong A_*(W_{\mathcal{A}})$, see [FY04].

Theorem 5.3.2 ([MRS20, Theorem 3.1.]). *Let $W_{\mathcal{A}}$ be the maximal wonderful compactification of the complement of an arrangement of hyperplanes $\mathcal{A}$ in $\mathbb{P}^d_{\mathbb{C}}$. Then*

$$\mathrm{CSM}(\mathbb{1}_{C(\mathcal{A})}) = \sum_{k=0}^{d} \mathrm{csm}_k(M_{\mathcal{A}}) \in A_*(W_{\mathcal{A}}) \simeq MW_*(\Sigma_{M_{\mathcal{A}}}).$$

The above theorem shows that the CSM cycles of a matroid representable over $\mathbb{C}$, which are constructed completely combinatorially, in fact have a deeper geometric meaning. An important ingredient of the proof is the following lemma, see Section 1.4.5. in [Coh+09] for original reference.

Lemma 5.3.3. *Let $\mathcal{A} \subseteq \mathbb{P}^{d+1}_{\mathcal{A}}$ be an essential arrangement, and let $M_{\mathcal{A}}$ be the corresponding matroid. Then the Euler characteristic of the complement of the arrangement is given by*

$$\chi_{\mathrm{top}}(C(\mathcal{A})) = (-1)^d \beta(M_{\mathcal{A}}),$$

where $\beta(M_{\mathcal{A}})$ is the beta invariant of the corresponding matroid as defined in Definition 5.1.3.

5.4 Chern numbers of matroids

We are finally ready to define *Chern numbers of a matroid*, a new geometric invariant of a matroid. We define Chern numbers of a matroid to be multiplicities associated to the vertex of the fan when intersecting appropriate CSM cycles of a fixed matroid, in order to get a zero dimensional intersection.

Definition 5.4.1. Let M be a rank $d+1$ matroid. We define the *Chern numbers of a matroid* $\bar{c}_1^{k_1}\bar{c}_2^{k_2}\cdots\bar{c}_d^{k_d}(M)$ to be

$$\bar{c}_1^{k_1}\cdots\bar{c}_d^{k_d}(M) = \mathrm{wt}_0(\mathrm{csm}_{d-1}^{k_1}(M)\mathrm{csm}_{d-2}^{k_2}(M)\cdots\mathrm{csm}_0^{k_d}(M)),$$

where $\sum_{i=0}^{d} i \cdot k_i = d$, and the map wt_0 is the map in Equation (3.3).

Note that the Chern numbers are indexed by codimension, while the csm cycles by dimension. Moreover, note that the number of Chern numbers associated to a given a matroid of rank $d+1$, is the number of partitions of d.

Example 5.4.2. Let M be a matroid of rank $d+1=3$, then as $d=2$ partition into 2, and 1+1, we get two Chern numbers associated to the matroid, namely $\bar{c}_2$ and $\bar{c}_1^2$:

$$\bar{c}_2(M) = \mathrm{wt}_0(\mathrm{csm}_0(M)),$$
$$\bar{c}_1^2(M) = \mathrm{wt}_0(\mathrm{csm}_1^2(M)).$$

■

Example 5.4.3. Let M be a matroid of rank $d+1=4$, then as $d=3$ partition into 3, 1+1+1, and 1+2, we get three Chern numbers associated to the matroid, namely $\bar{c}_3$, $\bar{c}_1\bar{c}_2$ and $\bar{c}_1^3$:

$$\bar{c}_3(M) = \mathrm{wt}_0(\mathrm{csm}_0(M)),$$
$$\bar{c}_1\bar{c}_2(M) = \mathrm{wt}_0(\mathrm{csm}_2(M)\mathrm{csm}_1(M)),$$
$$\bar{c}_1^3(M) = \mathrm{wt}_0(\mathrm{csm}_2^3(M)).$$

■

The following result follows immediately from the definition.

Proposition 5.4.4. *Let M be a rank $d+1$ matroid. The Chern number*

$$\bar{c}_d(M) = (-1)^d\beta(M).$$

Proof. See Example 5.2.2. ■

Note that, by the commutative diagram in Section 4.3, we have that

$$\mathrm{weight}_0(\mathrm{csm}_{d-1}^{k_1}(M)\cdots\mathrm{csm}_0^{k_d}(M)) = \deg(\psi(\mathrm{csm}_{d-1}^{k_1}(M)\cdots\mathrm{csm}_0^{k_d}(M))). \tag{5.4}$$

Note also that, as ψ is an isomorphism, we have that

$$\psi(\mathrm{csm}_{d-1}^{k_1}(M)\cdots\mathrm{csm}_0^{k_d}(M)) = ch_1^{k_1}\cdots ch_0^{k_d}. \tag{5.5}$$

The equalities Equation (5.4) and Equation (5.5) are useful when computing the Chern numbers, as computation is usually easier in the Chow ring. However, in some cases, for example for the uniform matroid, it is easier to compute the Chern numbers by directly intersecting the CSM cycles, see the proposition below.

Proposition 5.4.5. *The Chern numbers of the uniform matroid $U_{d+1,n+1}$ are given by*

$$\bar{c}_1^{k_1}\cdots\bar{c}_d^{k_d}(U_{d+1,n+1}) = (-1)^d\prod_{i=1}^{d}\binom{n-(d-i)-1}{i}^{k_i}.$$

Proof. Let the fan $C\in MW_*(\Sigma_{U_{d,n+1}})$ be the Bergman fan of $U_{d,n+1}$ with weight 1 on all its top dimensional cones. Then, by Example 15 in [MRS20], the fan $C^k \in MW_*(\Sigma_{U_{d-k+1,n+1}})$ is the Bergman fan of $U_{d-k+1,n+1}$ with weight 1 on all its top dimensional cones. Moreover, by Example 5.2.3, we have that:

$$\mathrm{csm}_k(U_{d+1,n+1}) = (-1)^{d-k}\binom{n-k-1}{d-k}C^{d-k}.$$

Then, by Definition 5.4.1. we get that the Chern number of $U_{d+1,n+1}$ are given by

$$\begin{aligned}\bar{c}_1^{k_1}\cdots\bar{c}_d^{k_d}(U_{d+1,n+1}) =& \mathrm{wt}_0\left(\prod_{i=1}^{d}(-1)^{k_i(d-(d-i))}\binom{n-(d-i)-1}{d-(d-i)}^{k_i}C^{ik_i}\right)\\ =&(-1)^d\prod_{i=1}^{d}\binom{n-(d-i)-1}{i}^{k_i},\end{aligned}$$

where the last equality follows from the fact that the fan C^d is just the vertex of the fan with weight 1, and from the fact that $\sum_{i=1}^d ik_i = d$. ■

In [Eur20], Eur defines an invariant of the matroid which is closely related to the divisor $ch_1(M) \in A^1(M)$.

Definition 5.4.6 ([Eur20, Definition 5.1.]). For a matroid M with reduced lattice of flats $\overline{\mathcal{L}_M}$, define its *shifted rank divisor* D_M to be

$$D_M := \sum_{F\in\overline{\mathcal{L}_M}} \mathrm{r}(F)x_F,$$

and define the *shifted rank volume of a matroid* M to be the volume of its shifted rank divisor:

$$\mathrm{shRVol}(M) := \deg\Big(\sum_{F\in\overline{\mathcal{L}_M}} \mathrm{r}(F)x_F\Big)^{\mathrm{r}(M)-1}.$$

For representable matroids, the volume measures how general the associated hyperplane arrangement is [Eur20].

Theorem 5.4.7 ([Eur20, Theorem 5.5.]). *Let M be a representable matroid of rank $d+1$ on $n+1$ elements. Then*

$$\mathit{shRVol}(M) \leq \mathit{shRVol}(U_{d+1,n+1}) = n^{r-1} \quad \text{with equality iff } M = U_{d+1,n+1}.$$

Proposition 5.4.8. *Let M be a matroid of rank $d+1$ on a ground set E, then*

$$ch_1(M) = D_M + (d+1)x_E.$$

In Section 6.4, we will see that Theorem 5.4.7 can apply for the Chern number $\bar{c}_1^2(M)$, when M is a matroid of rank 3.

CHAPTER 6

Chern numbers of matroids of rank 3

In this chapter we investigate properties of the Chern numbers of matroids of rank 3.

6.1 The Chern numbers of matroids of rank 3

Recall, by Example 5.4.2, that to a matroid M of rank 3 we can associate two Chern numbers, namely $\overline{c}_1^2(M)$ and $\overline{c}_2(M)$. Now, let us write an explicit formula for the Chern numbers $\overline{c}_1^2(M)$ and $\overline{c}_2(M)$ in terms of the size, and the number of flats of rank 2.

Proposition 6.1.1. *Let M be a simple matroid of rank 3 on a ground set E. If we let $|E| = n$, and let t_m be the number of flats F of rank 2 of size m, we get that the Chern numbers of M are*

$$\overline{c}_1^2(M) = (3-n)^2 - \sum_{m\geq 2}(2-m)^2 t_m,$$
$$\overline{c}_2(M) = 3 - 2n + \sum_{m\geq 2}(m-1)t_m.$$

We use the following results to prove Proposition 6.1.1. As before, Let M denote a simple matroid of rank 3.

Lemma 6.1.2. *Let x_i be the monomial corresponding to the rank 1 flat $\{i\} \subseteq E$ for $0 \leq i \leq |E|$, and let x_F be the monomial corresponding to the rank 2 flats $F \subseteq E$, then the linear monomials in $A^1(M)$ are given by*

$$x_i = -\sum_{F\ni i, F\neq i} x_F + x_j + \sum_{F\ni j} x_F, \text{ and}$$
$$x_{F'} = -\left(x_i + \sum_{F\ni i, F\neq F'} x_F\right) + x_j + \sum_{F\ni j} x_F,$$

for an $i \in F'$ and a $j \notin F'$.

Proof. Follows from the linear relation in the Chow ring $A^*(M)$. ■

Recall, be Definition 4.3.5, and Proposition 4.3.6, that for a matroid M of rank 3 we have the following isomorphism

$$\deg : A^2(M) \to \mathbb{Z}, \tag{6.1}$$

uniquely determined by the property that $\deg(x_i x_F) = 1$, for a flat F of rank 2 containing i. In order to compute the Chern numbers, we need to know the degrees of the quadric monomials, see proposition below.

Proposition 6.1.3. *Let M be a simple matroid of rank 3 on a ground set E, and let F denote a flat of rank 2 containing an $i \in E$. Moreover let $k_i = |\{F : F \ni i\}$. Then, the degrees of the quadric monomials are given in Table 6.1.*

Monomials	$x_i x_F$	x_i^2	x_F^2	$x_i x_E$	$x_F x_E$	x_E^2
Degrees	1	$1 - k_i$	-1	-1	0	1

Table 6.1: Degrees of the quadric monomials.

Proof. We prove the degrees of the monomials x_i^2 and x_F^2 in Table 6.1, by applying Eur's formula given in Theorem 4.3.7. For the monomial x_i^2, we need the following the following values for computations:

$$\mathrm{r}_1 = 1,\ d_1 = 2,\ \tilde{d}_1 = 2,\ k = 1.$$

Then, by inserting the above values in the formula given in Theorem 4.3.7, we get the following equality

$$\deg(x_i^2) = -\mu^1(M/i).$$

Recall, by Section 2.4 that the matroid M/i is the matroid arising from the lattice of flats $[i, E]$, which is the rank 2 matroid consisting of k_i flats of rank 1. Note that k_i is the number of flats of rank 2 of the original matroid M containing i. Then, the reduced characteristic polynomial $\overline{\chi}_{M/i}(\lambda) = \lambda + 1 - k_i$, and the unsigned coefficient $\mu^1(M/i) = -(k_i - 1)$. It follows that $\deg(x_i^2) = (1 - k_i)$.

For the monomial x_F^2 we need the following values

$$\mathrm{r}_1 = 2,\ d_1 = 2,\ \tilde{d}_1 = 2,\ k = 1.$$

Then, by inserting the above values in the formula given in Theorem 4.3.7, we get the following equality

$$\deg(x_F^2) = -\mu^0(M/F).$$

Recall, by Section 2.4 that the matroid M/F is just the one element matroid of rank 1. Hence the reduced characteristic polynomial is given by $\overline{\chi}_{M/F} = 1$, and so the unsigned coefficient $\mu^0(M/F) = 1$. It follows, that $\deg(x_F^2) = -1$.

For the monomial x_E^2, we apply directly the relations in the Chow ring. Let $i \neq j \in E$, and denote by F_{ij} the unique rank 2 flat containing i, and j. Then we have the following equality

$$x_E^2 = (-\sum_{F \ni i} x_F)(-\sum_{F \ni j} x_F)$$

$$= x_i x_{F_{ij}} + x_j x_{F_{ij}} + x_{F_{ij}}^2.$$

Hence $\deg(x_E^2) = 1$. Finally, we find the degrees of the monomials $x_i x_E$ and $x_F x_E$ by applying Proposition 4.3.4. ∎

Now, we are finally ready to prove Proposition 6.1.1.

Proof of Proposition 6.1.1. We compute the Chern numbers $\overline{c}_1^2(M)$ and $\overline{c}_2(M)$ by applying the formula in Conjecture 5.2.4. Recall by Theorem 5.2.5, that the conjecture is proven for these Chern numbers. We begin with the first statement. By Example 5.2.6, the Chern number $\overline{c}_1^2(M)$ is given by

$$\begin{aligned}\overline{c}_1^2(M) =& \Big((-1) \sum_{\mathrm{r}(F)\geq 2} (\mathrm{r}(F) - |F|) x_F\Big)^2 \\ =& \sum_{\mathrm{r}(F)=2} (2-|F|)^2 x_F^2 + \sum_{\mathrm{r}(F)=2} 2(2-|F|)(3-|E|) x_F x_E \\ &+ (3-|E|)^2 x_E^2.\end{aligned}$$

Then, by the degrees of the quadric monomials given in Table 6.1, the Chern number $\overline{c}_1^2(M)$ is given by

$$\overline{c}_1^2(M) = (3-|E|)^2 - \sum_{\mathrm{r}(F)=2} (2-|F|)^2.$$

Finally, by letting $|E| = n$, and t_m be the number of flats F of rank 2 of size m, we get

$$\overline{c}_1^2(M) = (3-n)^2 - \sum_{m\geq 2} (2-m)^2 t_m. \tag{6.2}$$

Now we prove the second statement. By Example 5.2.7 and Definition 5.4.1, we have the following equality

$$\overline{c}_2(M) = \sum_{F_1 \subseteq F_2} ch_S x_{F_1} x_{F_2},$$

where the sum is over all chains $F_1 \subseteq F_2$, for flats F_1, and F_2 of rank at least 1. Recall that the coefficients in the sum are given by

$$ch_S = \frac{\mathrm{r}(F_1)(\mathrm{r}(F_2)-1)}{m_S(1)! m_S(2)! \dots m_S(d+1)!},$$

where the multisubset $S = \{r(F_1), r(F_2)\}$. Hence, if we denote by F the flats of rank 2, the Chern number $\overline{c}_2(M)$ is given by

$$\overline{c}_2(M) = \sum_{i\in F} x_i x_F + \sum_{i\in E} 2 x_i x_E + \sum_F x_F^2 + \sum_{F\subsetneq E} 4 x_F x_E + 3x_E^2.$$

By applying the degrees of the quadric monomials in Table 6.1, we get the following equality

$$\overline{c}_2(M) = \sum_F |F| - \sum_{i\in E} 2 - \sum_F 1 + 3.$$

Finally, by letting $|E| = n$, and t_m be the number of flats F of rank 2 of size m, we get the following equality

$$\begin{aligned}\overline{c}_2(M) =& \sum_{m\geq 2} m t_m - 2n - \sum_{m\geq 2} t_m + 3\\ =& 3 - 2n + \sum_{m\geq 2}(m-1)t_m.\end{aligned}$$

■

Now, we want to point out that there is another way of computing the Chern number $\overline{c}_2(M)$. Recall, by Proposition 5.4.4, that if M is a loopless matroid of rank 3, we have the following equality

$$\overline{c}_2(M) = \beta(M).$$

Let us check that the above equality agrees with the findings in Proposition 6.1.1.

Lemma 6.1.4. *Let M be a loopless matroid of rank* 3*, then*

$$\beta(M) = 3 - 2n + \sum_{m\geq 2}(m-1)t_m.$$

Proof. Recall that the Beta invariant of a loopless matroid M is

$$\beta(M) = (-1)^{r(M)-1}\overline{\chi}_M(1).$$

To find the characteristic polynomial χ_M, we use the Möbius function, see Definition 5.1.1. Particularly, we use that for flats $F \in \mathcal{L}_M$ of rank 2

$$\mu(F, \emptyset) = -(1 - |F|),$$

hence, if we as usual let t_m be the number of flats of size m, we get that

$$\sum_{F\in\mathcal{L}_M} \mu(\emptyset, F) = \sum_{m\geq 2}(m-1)t_m.$$

Then, the characteristic polynomial is given by

$$\chi_M(\lambda) = \lambda^3 - n\lambda^2 + \sum_{m\geq 2}(m-1)t_m\lambda - (1 - n + \sum_{m\geq 2}(m-1)t_m).$$

Finally, the beta invariant is given by

$$\overline{\chi_M}(1) = 3 - 2n + \sum_{m\geq 2}(m-1)t_m,$$

which equals the Chern number $\overline{c}_2(M)$.

■

6.2 Chern numbers of line arrangemnts

Recall that essential line arrangements give rise to simple matroids of rank 3, see Proposition 2.2.5. In this section we see that the *Chern numbers of a line arrangement* $\mathcal{A}$ as defined in [EFU18] are related to the Chern numbers of the corresponding simple matroid $M_{\mathcal{A}}$ of rank 3.

Definition 6.2.1. [EFU18] Let $\mathcal{A}$ be an arrangement of n lines, and let t_m be the number of *m-points* of $\mathcal{A}$, i.e., a point which belongs to exactly m lines. We define the integers

$$\bar{c}_1^2(\mathcal{A}) = 9 - 5n + \sum_{m\geq 2}(3m-4)t_m \text{ and} \tag{6.3}$$

$$\bar{c}_2(\mathcal{A}) = 3 - 2n + \sum_{m\geq 2}(m-1)t_m. \tag{6.4}$$

They are called the *Chern numbers* of $\mathcal{A}$.

Example 6.2.2. Let $\mathcal{A}$ be a *trivial* arrangement, i.e., an arrangement consisting of n lines intersecting at the same point. Then, the number of n-points $t_n = 1$, whereas the number of k-points $t_k = 0$ for $k \neq n$. Moreover, the Chern numbers are given by

$$\bar{c}_1^2(\mathcal{A}) = -2n + 5, \text{ and } \bar{c}_2(\mathcal{A}) = -n + 2.$$

Now, let $\mathcal{A}$ be a *quasi-trivial* arrangement, i.e., an arrangement consisting of $n-1$ lines intersecting at the same point, and one other line intersecting the first $n-1$ lines in $n-1$ distinct points, one for each line. Then, the number of 2-points $t_2 = n-1$, and the number of $(n-1)$-point $t_{n-1} = 1$, whereas the number of k-points $t_k = 0$ for all other k's. Moreover, we get that the Chern numbers of the quasi-trivial arrangement are

$$\bar{c}_1^2(\mathcal{A}) = 0, \text{ and } \bar{c}_2(\mathcal{A}) = 0.$$

■

Proposition 6.2.3. *Let $\mathcal{A}$ be a essential arrangement of lines, and let $M_{\mathcal{A}}$ be the corresponding simple matroid of rank* 3*. Then*

$$\bar{c}_1^2(\mathcal{A}) = \bar{c}_1^2(M_{\mathcal{A}}),$$
$$\bar{c}_2(\mathcal{A}) = \bar{c}_2(M_{\mathcal{A}}),$$

where $\bar{c}_2(M_{\mathcal{A}})$ and $\bar{c}_1^2(M_{\mathcal{A}})$ are the Chern numbers defined in Definition 6.2.1.

Proof. The second statement follows directly from the definition of the Chern number $\bar{c}_2(\mathcal{A})$, see Definition 6.2.1, and from Proposition 6.1.1.

For the first statement, we use that

$$\bar{c}_1^2(M_{\mathcal{A}}) - \bar{c}_1^2(\mathcal{A}) = n^2 - n - \sum_{m\geq 2}(m^2-m)t_m,$$

and prove by induction on n the following equality

$$n^2 - n = \sum_{m\geq 2}(m^2-m)t_m. \tag{6.5}$$

For $n = 3$, the arrangement must consist of three lines not all intersecting in the same point, hence the corresponding matroid has three flats of rank 2 of size 2, so the equality holds.

Now, let $\mathcal{A}$ be an essential line arrangement of size n, and let $M_{\mathcal{A}}$ be the corresponding matroid. We want to show that adding a line l_i to the arrangement $\mathcal{A}$, or equivalently a flat $\{i\}$ of rank 1 to the corresponding matroid $M_{\mathcal{A}}$, changes the left and right hand side of Equation (6.5) of the same quantity. First of all, we easily compute that the change in the left hand side is $2n$. For the right hand side, we first denote by $\mathcal{A}'$ the arrangement $\mathcal{A}$ with one line added, and by $M_{\mathcal{A}'}$ the corresponding matroid.

Note that the rank 2 flats of $M_{\mathcal{A}'}$ consist of those not containing i, which are also rank 2 flats of $M_{\mathcal{A}}$, and those containing i. Now, let $\mathcal{F}_i$ be the set of rank 2 flats of $M_{\mathcal{A}'}$ containing i. The set $\mathcal{F}_i$ equals the disjoint union $\mathcal{F}_i = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1$ consists of the flats $F \in \mathcal{F}_i$ such that $F \setminus \{i\}$ is a rank 2 flat of $M_{\mathcal{A}}$, and $\mathcal{F}_2$ consists of the flats $F \in \mathcal{F}_2$ such that $F \setminus \{i\}$ is a flat of rank 1, hence F is of size 2. Remark that, by the covering axiom of flats, the size of $\mathcal{F}$ is $|\mathcal{F}_2| = n - \sum_{F \in \mathcal{F}_1}(|F|-1)$. Then, adding a flat $\{i\}$ to the matroid $M_{\mathcal{A}}$ changes the right hand side of Equation (6.5) by

$$\begin{aligned}&\sum_{F\in\mathcal{F}_1}(|F|^2-|F|) - \sum_{F\in\mathcal{F}_1}((|F|-1)^2-(|F|-1)) + 2(n - \sum_{F\in\mathcal{F}_1}(|F|-1))\\ &= 2n.\end{aligned}$$

This proves Equation (6.5), hence the first statement in Proposition 6.2.3. ■

6.3 Examples

In this section we calculate some examples for the Chern numbers of simple matroids of rank 3.

Example 6.3.1. We compute the Chern numbers of the uniform matroid $U_{3,n+1}$, which has $n(n+1)/2$ number of flats of rank 2 of size 2. Hence the multiplicities are given by $t_2 = n(n+1)/2$, and $t_k = 0$ for all other k's. It follows that the Chern numbers are given by

$$\begin{aligned}\bar{c}_1^2(U_{3,n+1}) &= (n-2)^2,\\ \bar{c}_2(U_{3,n+1}) &= \frac{1}{2}(n-2)(n-1),\end{aligned}$$

by applying the formulas given in Proposition 6.1.1. Note that the above results are consistent with the findings in Proposition 5.4.5, namely with the formula for the Chern numbers of the uniform matroid of arbitrary rank. In Table 6.2 we have listed the Chern numbers of $U_{3,n+1}$ for some $n \in \mathbb{N}$. ■

As mentioned in Section 2.3, hyperplane arrangements arising from finite projective spaces $PG(n,q)$ have nice combinatorial properties. In particular, the finite projective plane $PG(2,q)$ gives rise to a hyperplane arrangement consisting of q^2+q+1 lines, and q^2+q+1 intersection points. Moreover, each intersection point is on $q+1$ distinct lines, see Section 6.1 in [Oxl06].

M	$\overline{c}_1^2(M)$	$\overline{c}_2(M)$
$U_{3,3}$	0	0
$U_{3,4}$	1	1
$U_{3,5}$	4	3
$U_{3,7}$	16	10
$U_{3,9}$	36	21

Table 6.2: Chern numbers of the uniform matroid $U_{3,n+1}$.

Example 6.3.2. The finite projective plane matroid $PG(2,q)$ is a matroid on a ground set E of size q^2+q+1 having q^2+q+1 flats of rank 2 of size $q+1$. Hence, its Chern numbers are given by

$$\begin{aligned}
\overline{c}_1^2(PG(2,q)) &= (2-(q^2+q))^2 - (q^2+q+1)(2-(q+1))^2 \\
&= 3(q^3-q^2-q+1), \\
\overline{c}_2(PG(2,q)) &= 1-2(q^2+q)+q(q^2+q+1) \\
&= q^3-q^2-q+1.
\end{aligned}$$

For example, the Fano plane $PG(2,2)$ introduced in Example 2.3.3 has Chern numbers

$$\begin{aligned}
\overline{c}_1^2(PG(2,2)) &= 9, \\
\overline{c}_2(PG(2,2)) &= 3.
\end{aligned}$$

In Table 6.3 we have listed the Chern numbers of the projective plane matroid $PG(2,q)$ for different $q \in \mathbb{Z}_{\geq 0}$. ■

M	$\overline{c}_1^2(M)$	$\overline{c}_2(M)$
$PG(2,2)$	9	3
$PG(2,4)$	135	45
$PG(2,8)$	1323	441
$PG(2,9)$	1920	640

Table 6.3: Chern numbers of the finite projective plane matroid $PG(2,q)$.

We want to point out that there exists finite projective planes that are not representable over a field. See definition below for the general definition of a projective plane, we have used [Lam91] notation.

Definition 6.3.3. A *finite projective plane of order* q, with $q>0$, is a collection of q^2+q^1+1 lines and q^2+q+1 points such that

1. every line contains $q+1$ points,
2. every point is on $q+1$ lines,
3. any two distinct lines intersect at exactly one point, and
4. any two distinct points lie on exactly one line.

There exists finite projective planes of order 9 that are not representable over any field ([HSK59]). But whether there exists projective planes of order not a prime power is still an open question.

Proposition 6.3.4. *Let M be the finite projective plane matroid $PG(2,q)$, not necessarily representable over a field, then the following equality holds*

$$\frac{\overline{c}_1^2(M)}{\overline{c}_2(M)} = 3.$$

Proof. See Example 6.3.2, and Definition 6.3.3. ■

Example 6.3.5. In this example we compute the Chern numbers of the Pappus matroid M_P, and of the non-Pappus matroid M_{NP}, see Section 2.3. Recall that the Pappus matroid M_P is a matroid on a ground set of size 9 having 9 flats of rank 2 of size 2, and 9 flats of rank 2 of size 3. Hence the Chern numbers are given by

$$\overline{c}_1^2(M_P) = 27,$$
$$\overline{c}_2(M_P) = 12.$$

Whereas, the non-Pappus matroid is a matroid on a ground set of size 9 having 8 flats of rank 2 of size 3, and 12 flats of rank 2 of size 2. Hence its Chern numbers are

$$\overline{c}_1^2(M_{NP}) = 28,$$
$$\overline{c}_2(M_{NP}) = 13.$$

■

6.4 The geography of rank 3 matroids

In this section we show that there are some bounds on the Chern numbers of matroids of rank 3. The inspiration comes from results in the study of geography of manifolds, which deals with the possible values of Chern numbers of algebraic manifolds of general type [Hun89]. Specifically, we begin by generalizing two propositions of [EFU18] on the Chern numbers of line arrangements, to hold for Chern numbers of simple matroids of rank 3. The proofs follow the same lines as in [EFU18], but are stated in terms of Chern numbers of matroids rather than Chern numbers of line arrangements.

Proposition 6.4.1. *Let M be a simple matroid of rank 3 on the ground set $E = \{1, \dots, n\}$, and let t_m be the number of flats of rank 2 of size m. If M has $t_n = t_{n-1} = 0$, then its Chern numbers are positive.*

Proof. Recall that the Chern number $\overline{c}_2(M) = \beta(M)$, moreover recall by Theorem 5.1.4 that $\beta(M)$ is non-negative, and that $\beta(M) = 0$ if and only if M is disconnected of a loop. We have already assumed that M is not a loop. Moreover, note that M disconnected implies that $t_{n-1} = 1$, which we have assumed to be 0. Hence, the Chern number $\overline{c}_2(M)$ is positive.

Now, we prove by induction on n, that $\overline{c}_1^2(M)$ is positive. Assume that

$n = 4$, then $t_n = t_{n-1} = 0$ implies $t_2 = 6$; so the Chern number of the matroid is $\overline{c}_1^2(M) = 1$. Assume now that M is a matroid on a ground set $|E| = n + 1 \geq 5$, and that $i \in E$ is such that it is contained in $t \geq 3$ rank 2 flats. Such an element must exist by the assumption that $t_n = t_{n-1} = 0$. Let $\mathcal{F}_i = \{F_1 \dots F_t\}$ be the rank 2 flats of M containing i, and denote by $M \setminus i$ the deletion matroid. We partition $\mathcal{F}_i$ as in the proof of Proposition 6.2.3, i.e., we let $\mathcal{F}_i = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1$ consists of flats F such that $F \setminus i$ is a flat of rank 2 of $M \setminus i$, and $\mathcal{F}_2$ consists of flats F of rank 2 such that $F \setminus i$ is a flat of rank 1 of $M \setminus i$. Recall also that the size of $\mathcal{F}_2$ is $|\mathcal{F}_2| = n - \sum_{F \in \mathcal{F}_1}(|F| - 1)$. Then, by Definition 6.2.1, and Proposition 6.2.3 the Chern number $\overline{c}_1^2(M)$ is given by

$$
\begin{aligned}
\overline{c}_1^2(M) =& \overline{c}_1^2(M \setminus i) - 5 - \sum_{F \in \mathcal{F}_1} (3(|F| - 1) - 4) + \sum_{F \in \mathcal{F}_1} (3|F| - 4) \\
& + 2(n - \sum_{F \in \mathcal{F}_1} (|F| - 1)) \\
=& \overline{c}_1^2(M \setminus i) - 5 + \sum_{F \in \mathcal{F}_1} 1 + 2(\sum_{F \in \mathcal{F}_1} 1 + n - \sum_{F \in \mathcal{F}_1} (|F| - 1)) \\
\geq& \overline{c}_1^2(M \setminus i) - 5 + 2t \\
\geq& \overline{c}_1^2(M \setminus i) + 1 \\
\geq& 1.
\end{aligned}
$$

The last inequality follows from the induction hypothesis, i.e., that the Chern number $\overline{c}_1^2(M \setminus i) \geq 0$ of the deletion matroid $M \setminus i$ is positive, hence $\overline{c}_1^2(M)$ is positive as well. ■

In the next lemma we want to relate the Chern number $\overline{c}_1^2(M)$ to the shifted rank volume of a matroid shRVol(M) defined in [Eur20].

Lemma 6.4.2. *Let M be a simple matroid of rank 3 on a ground set E of size $n + 1$, then the Chern number*

$$\overline{c}_1^2(M) = \mathrm{shRVol}(M) + 3 - 6n.$$

Proof. Denote by $\mathcal{L}_M$, and by $\hat{\mathcal{L}}_M$ the lattice of flats of M, and the reduced lattice of flats of M respectively. Recall, by Definition 5.4.1 that the Chern number $\overline{c}_1^2(M)$ is given by

$$
\begin{aligned}
\overline{c}_1^2(M) &= \deg\Big(\sum_{F \in \mathcal{L}_M \setminus \emptyset} \mathrm{r}(F) x_F\Big)^2 \\
&= \deg\Big(\sum_{F \in \hat{\mathcal{L}}_M} \mathrm{r}(F) x_F + 3x_E\Big)^2 \\
&= \deg\Big(\sum_{F \in \hat{\mathcal{L}}_M} \mathrm{r}(F) x_F\Big)^2 + \deg(9x_E^2) + 6\deg \sum_{F \in \hat{\mathcal{L}}_M} \mathrm{r}(F) x_F x_E \\
&= \mathrm{shRVol}(M) + 3 - 6n.
\end{aligned}
$$

Where the last equality follows from the degree values of the monomials listed in Table 6.1. ■

Hence, for a given matroid M, the Chern number $ch_1^2(M)$ and the shifted rank volume value shRVol(M) differ only by a constant. Hence, the Chern number $\overline{c}_1^2(M_{\mathcal{A}})$ of a representable matroids $M_{\mathcal{A}}$ can also be understood as a measure of how general the underlying line arrangement is. In fact, for simple matroids of rank 3, we can even generalize Theorem 5.4.7, which only holds for representable matroids, to hold for all simple matroids of rank 3.

Proposition 6.4.3. *Let M be a simple matroid of rank 3 on a ground set E of size $n+1$. If M has $t_{n+1} = t_n = 0$, then the Chern numbers $\overline{c}_1^2(M)$, and $\overline{c}_2(M)$ are bounded by*

$$\begin{aligned}\overline{c}_1^2(M) &\leq \overline{c}_1^2(U_{3,n+1}), \text{ and}\\ \overline{c}_2(M) &\leq \overline{c}_2(U_{3,n+1}).\end{aligned}$$

Proof. We begin with the first statement. Recall that the Chern number $\overline{c}_1^2(M)$ is given by

$$\overline{c}_1^2(M) = (|E| - \mathrm{r}(E))^2 - \sum_{\mathrm{r}(F)=2} (|F| - \mathrm{r}(F))^2,$$

and recall that for the uniform matroid $U_{3,n+1}$, the Chern number $\overline{c}_1^2(U_{3,n+1})$ is given by

$$\overline{c}_1^2(U_{3,n+1}) = (|E| - \mathrm{r}(E))^2$$

Then, since both the Chern number $\overline{c}_1^2(M)$ and the sum $\sum_{\mathrm{r}(F)=2}(|F| - \mathrm{r}(F))^2$ are positive, we get the following inequality

$$\overline{c}_1^2(M) \leq (|E| - \mathrm{r}(E))^2 = \overline{c}_1^2(U_{3,n+1}).$$

Now, let $f_m(M) = \sum_{m\geq 2} t_m(m-1)$. Then, for the second statement, we need to prove that

$$f_m(M) \leq f_m(U_{3,n+1}). \tag{6.6}$$

We begin by noting that

$$f_m(M) = \sum t_m(m-1) = \sum t_m m - \sum t_m,$$

which equals the number of edges minus the number of flats of rank 2 in the reduced lattice of flats $\hat{\mathcal{L}}_M$ of M. We also note that we can always achieve the lattice of flats of the uniform matroid $U_{3,n+1}$ from the lattice of flats of an arbitrary matroid M of rank 3, by relaxing the flats of rank 2 of size $k \geq 3$ into flats of rank 2 of size 2. Every time we relax a flat of size k, we get $\binom{k}{2}$ new flats of size 2, and one less flat of size k. Moreover, the number of edges increases by $2\binom{k}{2} - k$. If we denote by M' the matroid arising from relaxing a flat of size k, we get that

$$\begin{aligned}f_m(M') - f_m(M) =& 2\binom{k}{2} - k - \left(\binom{k}{2} - 1\right)\\ =& \binom{k}{2} - k + 1 \geq 1,\end{aligned}$$

which proves the Equation (6.6), and hence the right inequality of the second statement. ∎

Following, we generalize Proposition 3.4 in [EFU18].

Theorem 6.4.4. *Let M be a simple matroid of rank 3 on $E = \{1, \dots, n\}$, such that $t_n = t_{n-1} = 0$. Then,*

$$\frac{2n-6}{n-2} \leq \frac{\overline{c}_1^2(M)}{\overline{c}_2(M)} \leq 3$$

Left equality holds if and only if M is the uniform matroid $U_{3,n}$, and right equality holds if and only if M is the matroid of a finite projective plane.

Proof. Proving left inequality is equivalent to showing the inequality below

$$0 \leq (n-2)\overline{c}_1^2(M) - (2n-6)\overline{c}_2(M). \tag{6.7}$$

Recall from the proof of Proposition 6.2.3 that the following equality

$$-n^2 + n = \sum_{m \geq 2} (m - m^2) t_m$$

holds, when keeping the notation as in Proposition 6.2.3. Then, by inserting for $\overline{c}_1^2(M)$, and $\overline{c}_2(M)$ in Equation (6.7) we get the following equality

$$\begin{aligned}(n-2)\overline{c}_1^2(M) - (2n-6)\overline{c}_2(M) &= -n^2 + n + \sum_{m \geq 2} (mn - 2n + 2) t_m \\ &= \sum_{m \geq 2} (-m^2 + m) t_m + \sum_{m \geq 2} (mn - 2n + 2) t_m \\ &= \sum_{m \geq 2} (-m^2 + m(1+n) + (2-2n)) t_m.\end{aligned}$$

Moreover, the inequality $-m^2 + m(1+n) + (2-2n) \geq 0$ holds for $2 \leq m \leq n-1$ and the inequality $-m^2 + m(1+n) + (2-2n)) > 0$ holds for all $3 \leq m \leq n-2$. And since m is in fact less then $n-1$ by our assumption, the inequality in Equation (6.7) holds. Moreover, since the uniform matroid $U_{3,n}$ has $n(n-1)/2$ flats of rank 2 of size 2, the left equality in the theorem holds for the uniform matroid.

Proving the right inequality is equivalent to showing the following inequality

$$\overline{c}_1^2(M) - 3\overline{c}_2(M) = n - \sum_{m \geq 2} t_m \leq 0.$$

The inequality follows from inserting $p = 1$, and $r = 3$, in Conjecture 1 in [HW17], see [DW75] for original result. Moreover, recall from Example 6.3.2, that for a finite projective plane $PG(2, q)$ both the number of elements n and the number of flats of rank 2 is $q^2 + q + 1$, hence right equality holds. ∎

Next, we generalize a theorem in [EFU18] on line arrangements in the real projective plane to hold for *pseudoline arrangements*. See for example Figure 2.7 for a pseudoline arrangement. First we need to introduce some definitions.

Definition 6.4.5. [Bjö+99, Definition 6.2.2.] A simple closed curve L in $\mathbb{P}^2_{\mathbb{R}}$ is called a *pseudoline* if $\mathbb{P}^2 \setminus L$ has one connected component. A collection of pseudolines $\mathcal{A} = (L_e)_{e \in E}$ is called an *arrangement of pseudolines* if $\bigcap \mathcal{A} = \emptyset$ and every pair of pseudolines L_e and L_f in $\mathcal{A}$, for $e \neq f$, intersect in exactly one point.

A pseudoline arrangement decomposes the projective plane $\mathbb{P}^2_{\mathbb{R}}$ into 0-cells, 1-cells and 2-cells, which will be called *vertices*, *edges*, and *polygons* respectively, as they do in [Bjö+99]. Moreover, we call *r-gons* the polygons bounded by r edges. If every 2-cell is a 3-gone, the arrangement is called a *simplicial pseudoline arrangement.*

Pseudoline arrangements are in fact related to a spacial type of matroids which have some extra structure to them, namely *oriented matroids*, see for example [Bjö+99] for an introduction to oriented matroids. The following result is a special case of results due to Folkman, and Lawrence, see [FL78] for the original reference.

Theorem 6.4.6. *[Bjö+99, Section 1.3]] There is a one-to-one correspondence between arrangements of pseudolines and simple rank 3 oriented matroids.*

The following theorem is stated in terms of pseudoline arrangements, but keep in mind the correspondence to oriented matroids. The next theorem is a generalization of Theorem 3.5 in [EFU18].

Theorem 6.4.7. *Let M be simple matroid of rank 3 on $E = \{1, \dots, n\}$ with $t_n = t_{n-1} = 0$. If M arises from some pseudoline arrangement $\mathcal{A} = (L_e)_{e\in E}$, then*

$$\overline{c}_1^2(M) \leq \frac{5}{2}\overline{c}_2(M).$$

Equality is achieved if and only if the pseudoline arrangements of M are simplicial.

Proof. We follow the conventions of Section 1.1 in [Hir83], and the proof of [EFU18]. Note first that

$$5\overline{c}_2(M) - 2\overline{c}_1^2(M) = -3 - \sum_{m\geq 2}(m-3)t_m.$$

Let p_m be the number of m-gons, and let f_0, f_1 and f_2 be the number of vertices, edges and 2-cells respectively. Then $f_0 = \sum_{m\geq 2} t_m$, $f_2 = \sum_{m\geq 3} p_m$, and note that

$$f_1 = \sum_{m\geq 2} mt_m = \frac{1}{2}\sum_{m\geq 2} mp_m.$$

Then by the Euler characteristic formula, and by using that the Euler characteristic of $\mathbb{P}^2_{\mathbb{R}}$ is 1, we get that

$$\begin{aligned}
3f_0 - 3f_1 + 3f_2 =& 3\sum_{m\geq 2} t_m - (\sum_{m\geq 2} m\cdot t_m + \sum_{m\geq 2} m\cdot p_m) + 3\sum_{m\geq 2} p_m \\
=& -\sum_{m\geq 2}(m-3)\cdot t_m - \sum_{m\geq 2}(m-3)\cdot p_m \\
=& 3.
\end{aligned}$$

Finally we get the following equality

$$\sum_{m\geq 2}(m-3)p_m = -3 - \sum_{m\geq 2}(m-3)t_m.$$

Since every pair of pseudolines intersect in exactly one point the multiplicity $p_2 = 0$, hence the sum $\sum_{m\geq 2}(m-3)p_m \geq 0$. And the original inequality holds. Moreover, if the pseudoline arrangement $\mathcal{A}$ is simplicial, then then multiplicity $p_k = 0$ for all $k \neq 3$. Hence, equality is achieved if and only if the matroid M arises from a simplicial pseudoline arrangement. ■

Corollary 6.4.8. *If M is a simple orientable matroid of rank 3 with $t_n = t_{n-1} = 0$, then*

$$\frac{\overline{c}_1^2(M)}{\overline{c}_2(M)} \leq \frac{5}{2}.$$

Proof. Follows from Theorem 6.4.6, and Theorem 6.4.7. ■

Finally, we end this section by computing the ratio $\overline{c}_1^2(M)/\overline{c}_2(M)$ for some specific matroids, see Table 6.4. Note that we have also included the matroid arising from the Braid arrangement, see Example 2.1.3. Moreover, we have plotted the Chern number pairs $(\overline{c}_2(M), \overline{c}_1^2(M))$ in Figure 6.1 for some specific matroids of rank 3, see Table 6.4. The red line has slope 3, and the orange line has slope 2.5.

M	$\overline{c}_1^2(M)$	$\overline{c}_2(M)$	$\overline{c}_1^2/\overline{c}_2(M)$	
$U_{3,3}$	0	0	-	
$U_{3,4}$	1	1	1	A
$U_{3,5}$	4	3	≈ 1.33	B
$U_{3,7}$	16	10	1.6	C
$U_{3,9}$	36	21	≈ 1.71	
$PG(2,2)$	9	3	3	D
$PG(2,4)$	135	45	3	
$PG(2,8)$	1323	441	3	
$PG(2,9)$	1920	640	3	
non-Fano	10	4	2.5	E
Pappus	27	12	2.25	F
non-Pappus	28	13	≈ 2.15	G
Braid	5	2	2.5	H

Table 6.4: Chern numbers of matroids of rank 3.

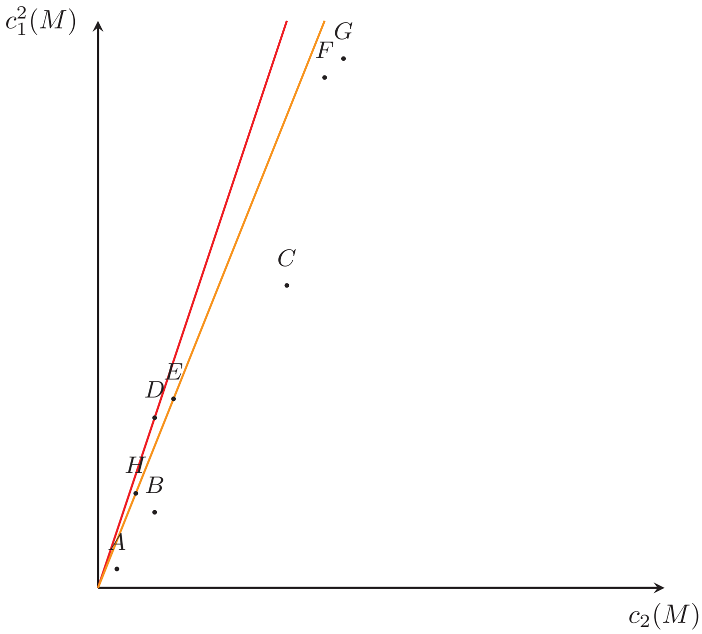

Figure 6.1: Chern numbers of matroids of rank 3.

CHAPTER 7

Chern numbers of matroids of rank 4

In this chapter we investigate the Chern numbers of a simple matroid of rank 4. Recall that by Example 5.4.3 we can associate three Chern numbers to a simple matroid M of rank 4, namely the Chern numbers $\overline{c}_3(M)$, $\overline{c}_1\overline{c}_2(M)$, and $\overline{c}_1^3(M)$, which are given by

$$\begin{aligned} \overline{c}_1^3(M) &= \deg(ch_1^3(M)), \\ \overline{c}_1\overline{c}_2(M) &= \deg(ch_1 ch_2(M)), \text{ and} \\ \overline{c}_3(M) &= \deg(ch_3(M)). \end{aligned}$$

We begin by calculating the polynomials $ch_1(M) \in A^1(M)$, $ch_2(M) \in A^2(M)$, and $ch_3(M) \in A^3(M)$ by applying Conjecture 5.2.4. It is important to keep in mind that the conjecture is proven for $ch_1(M)$, and $ch_3(M)$, but not for ch_2. Hence, the following corollary is true for $ch_2(M)$ only if Conjecture 5.2.4 is true.

Corollary 7.0.1. *Let M be a simple matroid of rank 4 on a ground set E with lattice of flats $\mathcal{L}_M$. Moreover, denote by $\mathcal{F}$ the set of flats of rank 2, and by $\mathcal{G}$ the set of flats of rank 3, then assuming Conjecture 5.2.4 is true the polynomials $ch_1(M)$, $ch_2(M)$, and $ch_3(M)$ are given by*

$$\begin{aligned} ch_1(M) = &- \sum_{F\in\mathcal{F}} (2-|F|)x_F - \sum_{G\in\mathcal{G}} (3-|G|)x_G - (4-|E|)x_E, \\ ch_2(M) = &\sum_{i\in E, F\in\mathcal{F}} x_i x_F + \sum_{i\in E, G\in\mathcal{G}} 2x_i x_G + \sum_{i\in E} 3x_i x_E + \sum_{F\in\mathcal{F}} x_F^2 \\ &+ \sum_{F\in\mathcal{F}, G\in\mathcal{G}} 4x_F x_G + \sum_{F\in\mathcal{F}} 6x_F x_E + \sum_{G\in\mathcal{G}} 9x_G x_E + 6x_E^2, \\ ch_3(M) = &\sum_{i\in E, F\in\mathcal{F}, G\in\mathcal{G}} x_i x_F x_G + \sum_{i\in E, G\in\mathcal{G}} x_i x_G^2 + \sum_{i\in E, F\in\mathcal{F}} 2x_i x_F x_E \\ &+ \sum_{i\in E, G\in\mathcal{G}} 4x_i x_G x_E + \sum_{i\in E} 3x_i x_E^2 + \sum_{F\in\mathcal{F}, G\in\mathcal{G}} x_{F_2}^2 x_{F_3} + \sum_{F\in\mathcal{F}} 2x_F^2 x_E \\ &+ \sum_{F\in\mathcal{F}, G\in\mathcal{G}} 2x_F x_G^2 + \sum_{F\in\mathcal{F}} 6x_F x_E^2 + \sum_{F\in\mathcal{F}, G\in\mathcal{G}} 8x_F x_G x_E \\ &+ \sum_{G\in\mathcal{G}} 6x_G^2 x_E + \sum_{G\in\mathcal{G}} 9x_G x_E^2 + \sum_{G\in\mathcal{G}} x_G^3 + 4x_E^3, \end{aligned}$$

where we only sum over flags.

Proof. The proof follows from Conjecture 5.2.4, and Example 5.2.6. ■

In order to compute the Chern numbers, we need to compute the values of the degrees of the cubic monomials, as we did for matroids of rank 3. And, in order to do so, we need the following result. We use the same notation as in Corollary 7.0.1.

Lemma 7.0.2. *The linear monomials in $A^1(M)$ are given by*

$$x_i = -\sum_{F\in\mathcal{F},F\ni i} x_F - \sum_{G\in\mathcal{G},G\ni i} x_G + x_j + \sum_{F\in\mathcal{F},F\ni j} x_F + \sum_{G\in\mathcal{G},G\ni j} x_G,$$

$$x_{F'} = -\left(x_i + \sum_{F\in\mathcal{F},F\ni i,F\neq F'} x_F + \sum_{G\in\mathcal{G},G\ni i} x_G\right) + x_j + \sum_{F\in\mathcal{F},F\ni j} x_F + \sum_{G\in\mathcal{G},G\ni j} x_G,$$

$$x_{G'} = -\left(x_k + \sum_{F\in\mathcal{F},F\ni k} x_F + \sum_{G\in\mathcal{G},G\ni k,G\neq G} x_G\right) + x_j + \sum_{F\in\mathcal{F},F\ni j} x_F + \sum_{G\in\mathcal{G},G\ni j} x_G.$$

Proof. The proof follows from the linear relations in the Chow ring $A^*(M)$. ■

Recall, by Definition 4.3.5, and by Proposition 4.3.6, that there is an isomorphism

$$\deg : A^3(M) \to \mathbb{Z}$$

uniquely determined by the property that $\deg(x_i x_F x_G) = 1$, for a chain $\{i\} \subsetneq F \subsetneq G$. In the following proposition we compute the degree values for the remaining cubic monomials.

Proposition 7.0.3. *Let M be a simple matroid of rank 4, and $\mathcal{F}$ and $\mathcal{G}$ denote the set of flats of rank 2, and 3 respectively. Moreover let the multiplicity $k_F = |\{G \in \mathcal{G} : G \supsetneq F\}|$ for a flat $F \in \mathcal{F}$. Then, for some flats $\emptyset \subsetneq \{i\} \subsetneq F \subsetneq G \subsetneq E$, the degrees of the cubic monomials are given in Table 7.1.*

Monomials	$x_i x_G^2$	$x_F^2 x_G$	$x_F x_G^2$	$x_i x_F^2$	x_F^3	x_G^3	
Degrees	-1	-1	0	$1-k_F$	$2k_F-2$	1	
Monomials	$x_i x_E^2$	$x_i x_F x_E$	$x_i x_G x_E$	$x_F x_E^2$	$x_F x_G x_E$	$x_G x_E^2$	x_E^3
Degrees	1	-1	0	0	0	0	-1
Monomials	$x_F^2 x_E$	$x_G^2 x_E$					
Degrees	1	0					

Table 7.1: Degrees of the cubic monomials

Proof. For the monomials in the first row of the table we have applied Eur's formula given in Theorem 4.3.7. We only show the computations for the monomial $x_i x_F^2$, since the other monomials are computed in a similar way. First of all, recall by Section 2.4 that for two flats $F_i \subsetneq F_{i+1}$ the matroid $M|F_{i+1}/F_i$ is the matroid arising from the lattice of flats $[F_i,\ F_{i+1}]$. We keep the notations as in Theorem 4.3.7. Then for the monomial $x_i x_F^2$, we need the following values for the computation:

$$\begin{aligned} &\mathrm{r}_1 = \mathrm{r}(\{i\}) = 1, && d_1 = 1, && \tilde{d}_1 = 1, \\ &\mathrm{r}_2 = \mathrm{r}(F) = 2, && d_2 = 2, && \tilde{d}_2 = 3. \end{aligned}$$

Then, since $d - k = 1$, and by inserting the above values in the formula given in Theorem 4.3.7, we get that

$$\deg(x_i x_F^2) = -\mu^0(M|F/\{i\})\mu^1(M/F).$$

The matroid $M|F/\{i\}$ is just the matroid arising from the lattice of flats $[\{i\},\ F]$, hence a matroid of rank 1 consisting of one element. Note that the reduced characteristic polynomial of a one element matroid is $\overline{\chi}_{M|F/\{i\}}(\lambda) = 1$. Hence, the unsigned coefficient $\mu^0(M|F/\{i\}) = 1$. The matroid M/F is the matroid arising from the lattice of flats $[F, E]$, hence it is a rank 2 matroid, where the rank 1 flats are the rank 3 flats of the original matroid M containing F. Recall that we have denoted by $k_F = |\{G : G \supsetneq F\}|$, then the reduced characteristic polynomial $\overline{\chi}_{M/F} = \lambda + 1 - k_F$, and hence the unsigned coefficient $\mu^1(M/F) = k_F - 1$. Finally we get that $\deg(x_i x_F^2) = 1 - k_F$.

For the monomials in the third row we need to apply the relations in the Chow ring, and the results in Lemma 7.0.2. We denote by L a flat in the reduced lattice of flats. Moreover, it follows from the flat axioms that for a simple matroid M of rank 4:

1. For every pair of elements $i, j \in E$ there exists a unique flat $F_{i,j}$ of rank 2 containing both i, and j.

2. For an element $k \in E$, and a flat F of rank 2 such that $k \notin F$, there exists a unique flat $G_{\{F,k\}}$ of rank 3 containing both k and F.

Now, choose $i \notin F$, and let $G_{\{F,i\}}$ be the unique rank 3 flat containing F and i, then the monomial $x_F^2 x_E \in A^3(M)$ is given by

$$\begin{aligned} x_F^2 x_E &= x_F^2\big(-\sum_{L \ni i} x_L\big) \\ &= -x_F^2 x_{G_{\{F,i\}}}. \end{aligned}$$

Hence, we get $\deg(x_F^2 x_E) = 1$. Now, choose $i \notin G$, then the monomial $x_G^2 x_E \in A^3(M)$ is given by

$$x_G^2 x_E = x_G^2\big(\sum_{L \ni i} x_L\big) = 0.$$

Finally for $i \neq j \neq k$, such that $\mathrm{r}(\{i,j,k\}) = 3$, we have that

$$\begin{aligned} x_E^3 &= -\sum_{L \ni i} x_L \sum_{L \ni j} x_L \sum_{L \ni k} x_L \\ &= -x_i x_{F_{ij}} x_{G_{ijk}}, \end{aligned}$$

where F_{ij} is the unique flat of rank 2 containing i and j, and G_{ijk} is the unique rank 3 flat containing i, j, and k. Hence $\deg(x_E^3) = -1$.

For the monomials in the second row of the table we have applied Proposition 4.3.4, and that $\deg(x_E^3) = -1$. ■

We are finally ready to express the Chern numbers of a simple matroid M of rank 4 in terms of the size and number of its flats.

Proposition 7.0.4. *Let M be a simple matroid of rank 4 on a ground set E. Let $\mathcal{F}$ and $\mathcal{G}$ be the sets of flats of rank 2, and 3 respectively, and let $k_F = |\{G \in \mathcal{G} : G \supsetneq F\}|$. Then, assuming Conjecture 5.2.4 is true, the Chern numbers $\overline{c}_1^3(M)$, and $\overline{c}_3(M)$ of M are given by*

$$\begin{aligned} \overline{c}_1^3(M) =& -\Bigg((|E|-4)^3 + \sum_{F \in \mathcal{F}} (|F|-2)^2 \Big(2(k_F-1)(|F|-2) + 3(|E|-4)\Big) \\ & - 3\sum_{F \subsetneq G} (|F|-2)^2(|G|-3) + \sum_{G \in \mathcal{G}} (|G|-3)^3\Bigg), \\ \overline{c}_3(M) =& -\Bigg(3|E| - 4 + \sum_{F \in \mathcal{F}} ((|F|-1)(k_F-2)) - \sum_{G \in \mathcal{G}} (|G|-1)\Bigg), \\ \overline{c}_1\overline{c}_2(M) =& -\Bigg(-6(4-|E|) + \sum_{F \in \mathcal{F}} \Big(|F|\Big((2-|F|)((1-k_F)-3) - (4-|E|)\Big) \\ & + (2-|F|)(2k_F-2) + 6(2-|F|) + (4-|E|)\Big) \\ & + \sum_{F \in \mathcal{F}, G \in \mathcal{G}} |F|\Big((2(2-|F|) + (3-|G|)\Big) \\ & - (4(2-|F|) + (3-|G|))\Big) + \sum_{G \in \mathcal{G}} 3(3-|G|) \\ & + \sum_{i \in E, G \in \mathcal{G}} -2(3-|G|) + \sum_{i \in E} 3(4-|E|)\Bigg) \end{aligned}$$

Proof. We begin with the first statement. We compute the cube of the polynomial $ch_1(M)$ as given in Corollary 7.0.1:

$$\begin{aligned} ch_1^3(M) =& (-1)\Bigg(\sum_{F \in \mathcal{F}} (|F|-2)^3 x_F^3 + 3\sum_{F \subsetneq G} (|F|-2)^2(|G|-3) x_F^2 x_G \\ & + 3\sum_{F \in \mathcal{F}} (|F|-2)^2(|E|-4) x_F^2 x_E + \sum_{G \in \mathcal{G}} (|G|-3)^3 x_G^3 + (|E|-4)^3\Bigg), \end{aligned}$$

then by applying the results in Proposition 7.0.3, we get the wanted result.

For the second statement, we apply the results in Proposition 7.0.3 to the formula given for ch_3 in Corollary 7.0.1.

For the last statement, we need to assume that Conjecture 5.2.4 is true. We begin by computing the polynomial $ch_1(M)ch_2(M)$ with the results given in Corollary 7.0.1.

$$\begin{aligned}
ch_1(M)ch_2(M) = -\Bigg(& \sum_{i\in E, F\in\mathcal{F}} ((2-|F|)x_i x_F^2) + 3(2-|F|)x_i x_F x_E \\
& + (4-|E|)x_i x_F x_E) + \sum_{i\in E, F\in\mathcal{F}, G\in\mathcal{G}} (2(2-|F|)x_i x_F x_G \\
& + (3-|G|)x_i x_F x_G) + \sum_{F\in\mathcal{F}} ((2-|F|)x_F^3 + (4-|E|)x_F^2 x_E) \\
& + \sum_{F\in\mathcal{F}, G\in\mathcal{G}} 4(2-|F|)x_F^2 x_G + \sum_{F\in\mathcal{F}} 6(2-|F|)x_F^2 x_E \\
& + \sum_{F\in\mathcal{F}} 6(2-|F|)x_F x_E^2 + \sum_{i\in E, G\in\mathcal{G}} 2(3-|G|)x_i x_G^2 \\
& + \sum_{F\in\mathcal{F}, G\in\mathcal{G}} (3-|G|)x_F^2 x_G + \sum_{G\in\mathcal{G}} 3(3-|G|)x_G^3 \\
& + \sum_{i\in E} 3(4-|E|)x_i x_E^2 + 6(4-|E|)x_E^3 \Bigg)
\end{aligned}$$

Moreover, we get the formula for the Chern number $\overline{c}_1\overline{c}_2(M)$ by computing the degree $\deg(ch_1 ch_2(M))$ with the help of the results in Proposition 7.0.3. ■

Example 7.0.5. Recall that, by Proposition 5.4.5, we already have a formula for computing the Chern numbers of the uniform matroid. Let M be the uniform matroid of rank 4 on $n+1$, then the Chern numbers of M are given by:

$$\begin{aligned}
\overline{c}_1^3(M) &= (3-n)^3, \\
\overline{c}_1\overline{c}_2(M) &= -\frac{1}{2}(n-3)^2(n-2), \\
\overline{c}_3(M) &= -\binom{n-1}{3}.
\end{aligned}$$

Note that, if we denote by $\mathcal{F}$ the set of flats of rank 2, and by $\mathcal{G}$ the set of flats of rank 3, then we have the following values for $U_{4,n+1}$:

$$|\mathcal{F}| = \binom{n+1}{2}, \quad |\mathcal{G}| = \binom{n+1}{3},$$

and

$$|F| = 2 \text{ for } F \in \mathcal{F}, \quad |G| = 3 \text{ for } G \in \mathcal{G}.$$

Moreover, let k_F denote the number of flats of rank 3 containing $F \in \mathcal{G}$, then $k_F = n-1$, since, for every flat $F \in \mathcal{F}$, the set $F \cup \{i\}$, for an $i \in E \setminus F$, is a flats of rank 3. If we insert the values above in the formulas given for the Chern numbers for matroids of rank 4 in Proposition 7.0.4, we get the exact same values for the Chern numbers as in Proposition 5.4.5.
In Table 7.2 we have listed the Chern numbers of $U_{4,n+1}$ for some specific values for n. ■

M	$\overline{c}_1^3(M)$	$\overline{c}_1\overline{c}_2(M)$	$\overline{c}_3(M)$
$U_{4,4}$	0	0	0
$U_{4,5}$	-1	-1	-1
$U_{4,6}$	-8	-6	-4
$U_{4,8}$	-64	-40	-20
$U_{4,15}$	-1331	-936	-286

Table 7.2: Chern numbers of the uniform matroid $U_{4,n+1}$

What is particular interesting about Example 7.0.5, is that for the uniform matroid $U_{4,n+1}$, we know what the Chern number $\overline{c}_1\overline{c}_2(U_{4,n+1})$ should be for all $n \geq 4$, see Proposition 5.4.5. And this coincides with the results when computing $\overline{c}_1\overline{c}_2(U_{4,n+1})$ by applying Conjecture 5.2.4. Hence, the above example strengthen the likelihood of the conjecture to be true!

Example 7.0.6. In this example we compute the Chern numbers of the matroid arising from the finite projective 3-space $PG(3, q)$ for a prime power q. The ground set E corresponds to the set of planes of $PG(3, q)$, the set of rank 2 flats $\mathcal{F}$ corresponds to the set of lines, and the set of rank 3 flats $\mathcal{G}$ corresponds to the set of points. As before, denote by $k_F = |\{G : G \supseteq F\}|$, then the matroid $PG(3, q)$ has

$$\begin{aligned}
|E| &= q^3 + q^2 + q + 1 \text{ number of elements},\\
|\mathcal{F}| &= (q^2 + 1)(q^2 + q + 1) \text{ number of flats of rank 2},\\
|\mathcal{G}| &= q^3 + q^2 + q + 1 \text{ number of flats of rank 3},\\
|F| &= q + 1 \text{ sized flats } F \in \mathcal{F},\\
|G| &= q^2 + q + 1 \text{ sized flats } G \in \mathcal{G},
\end{aligned}$$

Moreover, the multiplicity $k_F = q + 1$ for all $F \in \mathcal{F}$. For details on the above results, see for example Section 6.1. in [Oxl06]. And keep in mind that the size $|F|$ corresponds to the number of planes through a line, the size $|G|$ corresponds to the number of lines through a point, and the multiplicity k_F corresponds to the number of points on a line.

Then, by inserting the above values in the formulas given in Proposition 7.0.4, we get that the Chern numbers of the matroid arising from $PG(3, q)$ are given by

$$\begin{aligned}
\overline{c}_1^3(PG(3, q)) &= -16(q^6 - q^5 - q^4 + q^2 + q - 1),\\
\overline{c}_2\overline{c}_1(PG(3, q)) &= -6(q^6 - q^5 - q^4 + q^2 + q - 1),\\
\overline{c}_3(PG(3, q)) &= -(q^6 - q^5 - q^4 + q^2 + q - 1).
\end{aligned}$$

In Table 7.3 we have listed some values for the Chern numbers for different prime powers q.

■

M	$\overline{c}_1^3(M)$	$\overline{c}_1\overline{c}_2(M)$	$\overline{c}_3(M)$
$PG(3,2)$	-336	-126	-21
$PG(3,3)$	-6656	-2496	-416
$PG(3,4)$	-45360	-17010	-2835

Table 7.3: Chern numbers of the finite projective 3-space matroid $PG(3,q)$.

Proposition 7.0.7. *Let M be a matroid arising from the finite projective 3-space $PG(3,q)$ for a prime power q. Then the following equalities*

$$\frac{\overline{c}_1^3(M)}{\overline{c}_3(M)} = 16, \text{ and } \frac{\overline{c}_1^3(M)}{\overline{c}_1\overline{c}_2(M)} = \frac{8}{3}$$

hold.

Proof. See Example 7.0.6. ∎

For an arbitrary matroid M of rank 4 we have implemented a function in Macaulay2 in order to compute the Chern number $\overline{c}_1^3(M)$ with the help of the Package [Che], see script below. In addition of having a test example for a known value, we have included both the *Vamos matroid* V_8, and the related matroid V_8^+. The Vamos matroid is a non-representable matroid of rank 4 on a ground set of 8 elements, whereas the matroid V_8^+ is a matroid obtained from V_8 by relaxing a flat, see Example 2.1.25 in [Oxl06] for the construction of these matroids.

```
loadPackage "Matroids";

c1_cubed = Matroid -> (
    M = Matroid; Flats = flats M;
    I = idealChowRing M;   R = ring I;  A = R/I; l = #Flats;
    AVars = hashTable apply(gens ambient A, i -> (set last baseName i, sub(i,A)));
    ch_1 = 0;
    XE = 0;
    -- Choose the generator for A^3(M)
    x_sigma = AVars#(set {0});
    n=#M.groundSet;
    m=2;
    while m < 4 do(
        if rank(M, Flats#n)== m  then if  member(0, Flats#n) then (
            x_sigma=x_sigma*AVars#(Flats#n);
            m = 3;
            F = Flats#n;
        );
        n = n+1;
        if rank(M, Flats#n)== 3  then if  isSubset(F, Flats#n) then (
            x_sigma=x_sigma*AVars#(Flats#n);
            m = 4;
        );
    );
    -- Compute ch_1 and XE
    j = 1;
    while j < #Flats - 1 do(
        ch_1 = ch_1 + rank(M, Flats#j)*AVars#(Flats#j);
        if member(0, Flats#j) then XE = XE - AVars#(Flats#j);
        j = j+1;
```

```
    );
    ch_1 = -(ch_1 + rank(M, Flats#(l-1))*XE); ch1_3 = ch_1^3;
    c1_3 = coefficient(x_sigma, ch_1^3);
    return c1_3;
)

PG32 = projectiveGeometry(3,2)
c1_cubed(PG32)

vamos = specificMatroid "vamos"
c1_cubed(vamos)

V8_plus = specificMatroid "V8+"
c1_cubed(V8_plus)
```

This generates the output:

```
i17 : c1_cubed(ProjMat2)

o17 = -336

i23 : c1_cubed(V8_plus)

o23 = -58

i25 : c1_cubed(vamos)

o25 = -59
```

Hence, we get that the Chern number $\overline{c}_1^3(PG(3,2)) = 336$ for the finite projective 3-space $PG(3,2)$ as expected. Moreover we get that the Chern numbers $\overline{c}_1^3(V8) = 59$, and $\overline{c}_1^3(V8+) = 58$ for the Vamos matroid, and the $V8+$ matroid.

Moreover, recall by Proposition 5.4.4 that the Chern number $\overline{c}_3(M)$ is given by $(-1)^3\beta(M)$ for a simple matroid M of rank 4. Hence we can use the script in Chapter 5 in order to compute $\overline{c}_3(M)$. The following code

```
vamos = specificMatroid "vamos";
betaInvariant(vamos)

V8 = specificMatroid "V8+";
betaInvariant(V8)
```

generates the following output

```
i7 : betaInvariant(vamos)

o7 = 15

i9 : betaInvariant(V8)

o9 = 14
```

Hence, the Chern numbers $\overline{c}_3(V8) = -15$, and $\overline{c}_3(V8+) = -14$ respectively.

CHAPTER 8

Further research

In this chapter we present some questions for further investigation. In Section 8.1, and Section 8.2 we concentrate on matroids of rank 3 and 4 respectively. Finally, in Section 8.3 we address general matroids.

8.1 Open questions for matroids of rank 3

In Chapter 6 we showed that both Chern numbers $\overline{c}_1^2(M)$ and $\overline{c}_2(M)$ of a simple matroid of rank 3, as well as the ratio $c_1^2(M)/c_2(M)$, are bounded. For simple matroids of rank 3, we pose the following questions:

Question 8.1.1 (Geography of Chern number pairs in rank 3)**.** *For which pairs* $(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ *does there exists a matroid such that* $(\overline{c}_1^2(M), \overline{c}_2(M)) = (a, b)$*?*

Question 8.1.1 is related to the study of the geography of Chern numbers of surfaces in complex algebraic geometry, where more about these numbers is known, see for example [Spr] for a brief overview, or [Hun89].

Question 8.1.2 (Representability of Chern numbers in rank 3)**.** *Given any simple matroid* M *of rank 3 with Chern numbers* $(\overline{c}_1^2(M), \overline{c}_2(M))$*, does there exist a representable matroid* $M_{\mathcal{A}}$ *of rank* 3 *such that*

$$(\overline{c}_1^2(M_{\mathcal{A}}), \overline{c}_2(M_{\mathcal{A}})) = (\overline{c}_1^2(M), \overline{c}_2(M))?$$

For instance, consider the non-Pappus matroid.

Example 8.1.3. Recall that the non-Pappus matroid is the matroid arising from the pseudoline arrangement given in Section 2.3. It is a simple matroid of rank 3 consisting of 9 flats of rank 1, and 12 flats of rank 2 of size 2, and 8 flats of rank 2 of size 3. Recall also from Example 6.3.5 that the Chern numbers of the non-Pappus matroid are $(28, 13)$. Finding a representable matroid $M_{\mathcal{A}}$ of rank 3 such that $(\overline{c}_1^2(M_{\mathcal{A}}), \overline{c}_2(M_{\mathcal{A}})) = (28, 13)$ corresponds to finding a line arrangement $\mathcal{A}$ consisting of 9 lines intersecting in 12 points with multiplicity 2, and in 8 points with multiplicity 3. Such an arrangement exists, see Figure 8.1. ■

Finally, we want to point out the possible extent of Question 8.1.1 and Question 8.1.2. Recall, from Theorem 6.4.4, that for a simple matroid of rank 3 $\overline{c}_1^2(M)/\overline{c}_2(M) = 3$ if and only if M is the matroid of a finite projective plane.

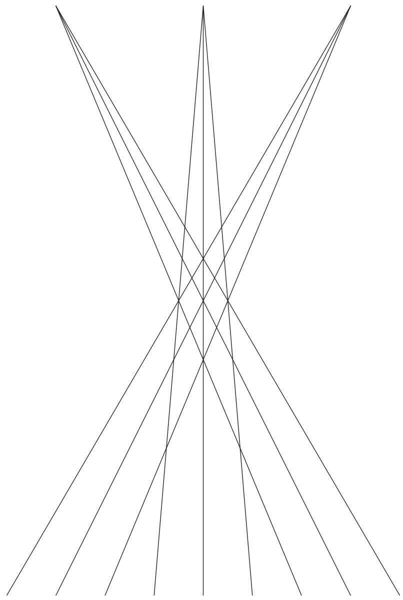

Figure 8.1: Line arrangement $\mathcal{A}$.

Recall also that the finite projective plane matroid of order q has Chern numbers

$$\begin{aligned}\overline{c}_1^2(PG(2,q)) =& 3(q^3 - q^2 - q + 1),\\ \overline{c}_2(PG(2,q)) =& q^3 - q^2 - q + 1.\end{aligned}$$

Hence, finding out for which values of q there exists a matroid M having Chern numbers $(\overline{c}_1^2(M), \overline{c}_2(M)) = (3(q^3 - q^2 - q + 1), q^3 - q^2 - q + 1)$, and if so if the matroid is representable over a field, would in fact solve the prime power conjecture.

8.2 Open questions for matroids of rank 4

We saw in Chapter 6 that the Chern numbers of matroids of rank 3 are positive. From our calculations in Chapter 7 it seems plausible that the Chern numbers of matroids of rank 4 are either zero, or negative. Let M be a matroid of rank 4, then by Proposition 5.4.4, the equality

$$\overline{c}_3(M) = -\beta(M)$$

holds, hence $\overline{c}_3(M)$ is always zero, or negative.

Conjecture 8.2.1. *Let M be a matroid of rank 4, then the Chern numbers $\overline{c}_1^3(M)$, and $\overline{c}_1\overline{c}_2(M)$ are either zero, or negative.*

If Conjecture 8.2.1 were to be true, it would be a generalization of a result on Chern numbers of 3-folds under some assumptions, see [Hun89].

Moreover, it would be interesting to have some bounds for matroids of rank 4 as we did for matroids of rank 3. Given some constrains on the size of the flats, it may be possible to prove the following conjecture.

Conjecture 8.2.2. *Let M be a simple matroid of rank 4 on $E = \{0, 1, \ldots, n\}$, then we have the following inequalities*

$$\frac{6(n-4)^2}{(n-3)(n-2)} \leq \left|\frac{\overline{c}_1^3(M)}{\overline{c}_3(M)}\right| \leq 16, \text{ and} \tag{8.1}$$

$$\left|\frac{\overline{c}_1^3(M)}{\overline{c}_1(M)\overline{c}_2(M)}\right| \leq \frac{8}{3}. \tag{8.2}$$

For the inequality (8.1), note first that $6(n-4)^2/(n-3)(n-2) \leq 16$, as long as $n \geq 4$, but this is always the case for matroids of rank 4. Moreover, left equality is achieved for the uniform matroid $U_{4,d+1}$, see Example 7.0.5, and right equality is achieved for the finite projective 3-space matroids, see Proposition 7.0.7. For the inequality (8.2), right equality is achieved for the finite projective 3-space, see Example 7.0.5. It is also worth mentioning that if the second inequality actually holds, it would also be a generalization of a result on Chern numbers of 3-folds under some assumptions, see [Hun89].

Finally, for simple matroids of rank 4, we pose the following questions.

Question 8.2.3 (Geography of Chern number triples in rank 4)**.** *For what triples $(a, b, c) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ does there exist a matroid such that $(\overline{c}_1^3(M), \overline{c}_1\overline{c}_2(M), \overline{c}_3(M)) = (a, b, c)$?*

Question 8.2.3 is related to the study of the geography of Chern numbers of threefolds of general type, see for example [Liu97].

Question 8.2.4 (Representability of Chern numbers in rank 4)**.** *Given any simple matroid M of rank 4 having Chern numbers $(\overline{c}_1^3(M), \overline{c}_1\overline{c}_2(M), \overline{c}_3(M))$, does there exist a representable matroid $M_{\mathcal{A}}$ of rank 4 such that $(\overline{c}_1^3(M_{\mathcal{A}}), \overline{c}_1\overline{c}_2(M_{\mathcal{A}}), \overline{c}_3(M_{\mathcal{A}})) = (\overline{c}_1^3(M), \overline{c}_1\overline{c}_2(M), \overline{c}_3(M))$?*

8.3 Open questions for matroids of any rank

Overall, it would be interesting to examine Chern numbers of matroids of any rank. Specifically, we pose the following question.

Recall from Proposition 6.3.4, that for the finite projective plane matroid $M = PG(2, q)$, the equality $\overline{c}_1^2/\overline{c}_2(M) = 3$ holds, and from Proposition 7.0.7 that for the finite projective 3-space matroid $M = PG(3, q)$ the equality $\overline{c}_1^3 M/\overline{c}_3(M) = 16$ holds. This leads us to pose the following question.

Question 8.3.1. *Does the equality*

$$\frac{\overline{c}_1^d(M)}{\overline{c}_d(M)} = (d+1)^{d-1}$$

hold for a matroid M arising from a finite projective d-dimensional space?

Finally, geography problems like the ones addressed in Question 8.1.1, and Question 8.1.2 for matroids of rank 3, and in Question 8.2.3, and Question 8.2.4

for matroids of rank 4, regarding the geography and representability of the Chern numbers, are interesting to consider for matroids of any rank, although quite difficult.

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