Letter

New nonlinear instability for scalar fields

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In this Letter, we introduce the nonlinear partial differential equation $\partial_\tau^2\pi \propto (\vec{\nabla}\pi)^2$ showing a new type of instability. Such equations appear in the effective field theory (EFT) of dark energy for the k-essence model as well as in many other theories based on the EFT formalism. We demonstrate the occurrence of instability in the cosmological context using a relativistic N-body code, and we study it mathematically in $3+1$ dimensions within spherical symmetry. We show that this term dominates for the low speed of sound limit where some important linear terms are suppressed.

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I. INTRODUCTION

One of the main goals of the upcoming large cosmological surveys [1–4] is to understand the physical mechanism behind the mysterious late-time accelerating expansion of the Universe [5–7]. Accurate modeling of the current viable dark energy (DE) and modified gravity (MG) candidates over all scales of interest is critical for the highly precise datasets that these surveys will deliver over the coming decade.

To study many possible models that include a DE component, or where the theory of gravity is altered, the effective field theory (EFT) framework has been suggested [8–11]. In the EFT scheme, a general form of the action is considered up to a certain energy scale, and the idea is that only some degrees of freedom are relevant below that scale, while those degrees of freedom that describe properties of a system at higher energy scales can be integrated out [12,13]. The EFT of DE is particularly useful for cosmologists, as one can map most of the interesting MG/DE theories to this language by choosing the set of free parameters appropriately. The EFT of DE thus provides a framework for a generic study of DE/MG theories [14–16].

As a first step toward implementing the EFT of DE in an N-body simulation, we have developed the k-evolution code [17] based on *gevolution* [18,19]. k evolution is able to simulate nonlinear structure formation with k-essence dark energy [20,21].

Our extensive numerical studies using k evolution have led us to the discovery of a new type of nonlinear instability that appears naturally in such EFT expansions and is not limited to the k-essence type of theories. This instability is not in the form of rapid growth of the scalar field but rather is an instability in the mathematical sense, in which the scalar field solution ceases to exist at a finite "blowup" time, leading to the breakdown of the EFT framework.

II. EFT EQUATIONS OF MOTION FOR k-ESSENCE

The action for a general scalar field theory constructed from the scalar field ϕ and the kinetic term $X \doteq g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ can be written as

$$S = \int d^4x\sqrt{-g}P(X,\phi), \tag{1}$$

where P is in general an arbitrary scalar function of its arguments, g is the determinant of the metric, and the integral is taken over the four-dimensional space-time. This class of theories is known as k-essence [20,21]. In the EFT of DE framework, assuming small scalar field fluctuations, a $3+1$ split of space-time can be defined by using the scalar field as a "clock" to define constant-time hypersurfaces. Writing the action as an expansion in terms of geometric scalars, we obtain [11,22]

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2\right], \tag{2}$$

where $M_{\rm pl}$ is the Planck mass; R is the four-dimensional Ricci scalar; $\Lambda(t)$, $c(t)$, and $M_2^4(t)$ are time-dependent functions; and δg^{00} is the perturbation of g^{00} around its

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background value. We have ignored terms that are of higher order in the fluctuations δg^{00} because these terms are negligible in the weak-field expansion relevant for cosmology. The scalar field and its perturbation π can be reintroduced, as usual in this framework, with the Stückelberg trick. See Ref. [23] for more details. Our starting point for studying the phenomenology of the k-essence scalar field is the EFT of DE action Eq. (2). As mentioned earlier, the action Eq. (1) of the full theory can give rise to any number of additional parameters that are relevant in the ultraviolet limit of the particular theory. The utility of the EFT approach stems from the fact that we do not need to specify what these are.

The variation of the action (2) with respect to the metric results in the gravitational field equations [18], while the variation with respect to the scalar field perturbation π results in a nonlinear partial differential equation (PDE) for the k-essence scalar field,

$$\begin{aligned}&\partial_\tau^2\pi + \mathcal{H}(1-3w)\partial_\tau\pi + (\partial_\tau\mathcal{H} - 3w\mathcal{H}^2 + 3c_s^2(\mathcal{H}^2 - \partial_\tau\mathcal{H}))\pi \\ &\quad - \partial_\tau\Psi + 3\mathcal{H}(w - c_s^2)\Psi - 3c_s^2\partial_\tau\Phi - c_s^2\nabla^2\pi \\ &\quad = \mathcal{N}(\pi, \partial_\tau\pi, \vec{\nabla}\pi, \vec{\nabla}\partial_\tau\pi, \nabla^2\pi),\end{aligned} \tag{3}$$

where $\mathcal{N}(\pi, \partial_\tau\pi, \vec{\nabla}\pi, \vec{\nabla}\partial_\tau\pi, \nabla^2\pi)$ includes all the nonlinear terms in the equation,

$$\begin{aligned}&\mathcal{N}(\pi, \partial_\tau\pi, \vec{\nabla}\pi, \vec{\nabla}\partial_\tau\pi, \nabla^2\pi) \\ &\quad = -\frac{\mathcal{H}}{2}(5c_s^2 + 3w - 2)\boxed{(\vec{\nabla}\pi)^2} \\ &\qquad + 2(1-c_s^2)\vec{\nabla}\pi\cdot\vec{\nabla}\partial_\tau\pi - [(c_s^2-1)(\partial_\tau\pi + \mathcal{H}\pi - \Psi) \\ &\qquad + c_s^2(\Phi - \Psi) + 3\mathcal{H}c_s^2(1+w)\pi]\nabla^2\pi + (2c_s^2-1)\vec{\nabla}\Psi\cdot\vec{\nabla}\pi \\ &\qquad - c_s^2\vec{\nabla}\Phi\cdot\vec{\nabla}\pi + \frac{3(c_s^2-1)}{2}\partial_i(\partial_i\pi(\vec{\nabla}\pi)^2).\end{aligned} \tag{4}$$

We have parametrized the model with an equation of state w and a speed of sound c_s which, respectively, relate to the EFT parameters through

$$w = \frac{c - \Lambda}{c + \Lambda}, \qquad c_s^2 = \frac{c}{c + 2M_2^4}. \tag{5}$$

In these expressions, $\mathcal{H} > 0$ is the (conformal) Hubble parameter that gives the expansion rate of the Universe, and Φ, Ψ are the two gravitational potentials in longitudinal gauge. The equation of state parameter w is close to -1 if the k-essence field is to play the role of the dark energy.

The equation of motion as well as the stress-energy tensor are discussed in detail in Ref. [23]. However, in this letter, we only focus on the PDE and the instability which is caused by the first term on the right-hand side of Eq. (4).

III. NEW INSTABILITY

Numerical simulations using k-evolution empirically show that the evolution under Eq. (3) is unstable for small values of the speed of sound c_s. We also find that there is a critical value c_s^* such that for a speed of sound lower than c_s^* the evolution becomes singular at a finite time, before the Universe reaches its present age. On the other hand, for speeds of sound much larger than the critical value, the singularity is avoided altogether. The instability forms in the regions with the highest curvature of the gravitational potential (center of halos), and the blowup time depends on the initial curvature of the potential wells.

In Fig. 1, we show the evolution of the scalar field perturbation π times the Hubble parameter $\mathcal{H}$ on a two-dimensional (2D) slice taken from a cosmological simulation. As we can see, the instability is formed suddenly at a certain redshift, and the scalar field solution ceases to exist at this time.

By considering subsets of the terms in $\mathcal{N}$, we find that the nonlinear instability is generated by the term quadratic in the gradient, $(\vec{\nabla}\pi)^2$ in Eq. (3). However, for large speed of sound, Eq. (3) is dominated by the linear terms rather than the nonlinear ones. This is expected because in this limit the sound horizon of the scalar field is of order of the Hubble scale and the scalar field perturbations decay inside the sound horizon. Thus, we do not expect to have nonlinear effects at small scales. On the other hand, in

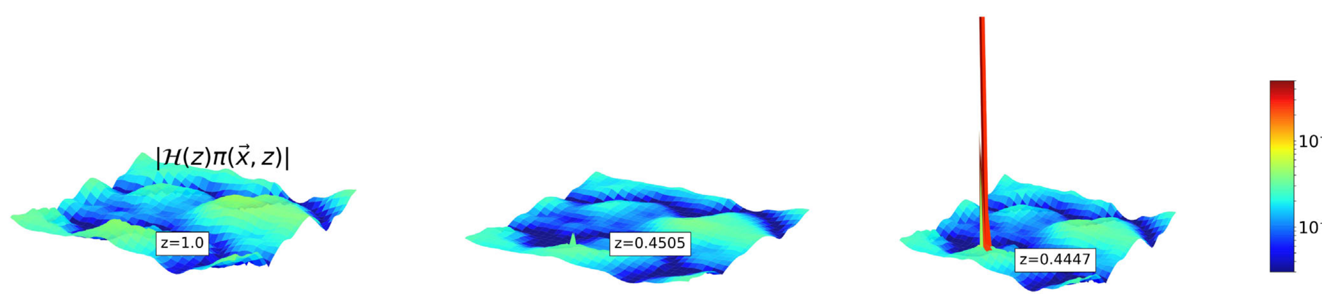

FIG. 1. From left to right: the evolution of the absolute value of the scalar field perturbation times the Hubble parameter, $\mathcal{H}\pi$, from k-evolution in time is shown. At the time corresponding to $z \approx 0.45$, the system blows up due to the nonlinear instability. This instability acts very quickly and leads to a divergence of π in finite time. Note that for the sake of illustration we plot the absolute value of the scalar field and that the blowup occurs at a local minimum.

the low speed of sound limit, the corresponding sound horizon is small, and in Eq. (3), the linear restoring force, $c_s^2\vec{\nabla}^2\pi$, is suppressed. As a result, the nonlinear terms become important, and the instability forms.

IV. MATHEMATICAL RESULTS

In this section, we are going to mathematically show that the problematic nonlinear term in Eq. (4) will inevitably give rise to a singularity in finite time once it dominates the dynamics. We only focus on the single problematic term rather than the whole PDE. In other words, we consider the simple nonlinear equation in $3+1$ dimensions,

$$\partial_\tau^2\pi = \alpha\vec{\nabla}\pi\cdot\vec{\nabla}\pi, \tag{6}$$

where α is the coefficient of the problematic term.[1] Around the local extrema x_* where $\vec{\nabla}\pi|_{x_*}=0$, we choose spherical coordinates to study the behavior of the solution. This is a reasonable choice as, according to Eq. (6), such a point remains an extremum at all times.[2] Moreover, the spherical symmetry is preserved under time evolution. Thus, for such points and for spherically symmetric initial conditions, we have the following PDE:

$$\partial_\tau^2\pi(\tau,r) = \alpha[\partial_r\pi(\tau,r)]^2. \tag{7}$$

This PDE is unstable independently of the sign of α; in the case of positive α, the singularity occurs at local minima of the scalar field, whereas for negative α, it occurs at local maxima. For the EFT of k-essence, $\alpha>0$, and the instability is formed in the minima which generally coincide with the centers of halos.

We may choose units such that $\alpha=1$, and we solve this equation for the initial conditions $\pi(\tau_0,r)$ and $\partial_\tau\pi(\tau_0,r)$. It is worth noting that, even if we assume $\pi(\tau_0,r)=0$ and $\partial_\tau\pi(\tau_0,r)=0$ in a cosmological scheme, the gravitational potential Ψ would eventually source the scalar field, as is evident from Eq. (3). Here, we instead consider a general initial condition for the scalar field and do not keep the gravitational source term. One particular solution to the nonlinear PDE (7) is given by

$$\pi_s(\tau,r) = \kappa(\tau)r^2, \tag{8}$$

where $2\kappa(\tau)$ represents the curvature of the scalar field $\pi_s(\tau,r)$ in time and $\kappa(\tau)$ is a solution to the ordinary differential equation (ODE)

$$\partial_\tau^2\kappa(\tau) = 4[\kappa(\tau)]^2. \tag{9}$$

The initial conditions $\kappa(\tau_0)$ and $\partial_\tau\kappa(\tau_0)$ can be obtained based on the assumed initial condition for $\pi(\tau_0,r)$ and $\partial_\tau\pi(\tau_0,r)$. We can think of this ODE as Newton's second law with the force $F(x)=4x^2$, which corresponds to a potential $V(x)=-\frac{4}{3}x^3$. No matter what the initial conditions for $x(0)$ and $\frac{dx}{d\tau}(0)$ or equivalently $\kappa(0)$ and $\frac{d\kappa}{d\tau}(0)$ [except $\kappa(0)=\frac{d\kappa}{d\tau}(0)=0$] are, a particle on this potential rolls to $+\infty$ eventually. Here, we are going to show that in fact the particle (in our case the curvature of the scalar field) goes to infinity in a finite time τ_b. To solve Eq. (9), we multiply both sides by $\kappa'=\frac{d\kappa}{d\tau}$,

$$\frac{1}{2}\frac{d(\kappa'(\tau)^2)}{d\tau} = \frac{4}{3}\frac{d(\kappa(\tau)^3)}{d\tau}. \tag{10}$$

Integrating results in the following expression:

$$\kappa'(\tau)^2 = \kappa'(0)^2 + \frac{8}{3}\kappa(\tau)^3 - \frac{8}{3}\kappa(0)^3. \tag{11}$$

Integrating once more, we obtain

$$\int_{\kappa(0)}^{\kappa(\tau)} \frac{d\kappa}{\sqrt{\kappa'(0)^2+\frac{8}{3}\kappa^3-\frac{8}{3}\kappa(0)^3}} = \int_0^\tau d\tau' = \tau. \tag{12}$$

Changing the integration variable from κ to s for $s^3=\frac{8}{3}\frac{\kappa^3}{C}$ and $C=\kappa'(0)^2-\frac{8}{3}\kappa(0)^3$, we find that τ is bounded by

$$\tau_b = \left(\frac{3}{8}\right)^{\frac{1}{3}}\left(\frac{1}{C}\right)^{\frac{1}{6}}\int_{s(\kappa_0)}^{\infty}\frac{ds}{\sqrt{1+s^3}}; \tag{13}$$

i.e., the solution blows up in finite time. In the cosmological context, we can set $\kappa(0)=0$ so that $C>0$ and $s(\kappa_0)=0$, giving us a blowup time of

$$\tau_b = \left(\frac{3}{8}\right)^{\frac{1}{3}}\left(\frac{1}{\kappa'(0)}\right)^{\frac{1}{3}}\frac{2\Gamma[\frac{1}{3}]\Gamma[\frac{7}{6}]}{\sqrt{\pi}}. \tag{14}$$

We can also see that a solution of the nonlinear ODE corresponding to a specific choice of initial condition for $\kappa(\tau_0)$ and $\kappa'(\tau_0)$ is

$$\kappa(\tau) = \frac{3}{2(\tau-\tau_b)^2}, \tag{15}$$

which is the blowup behavior for all solutions $\kappa(\tau)$ near the blowup time τ_b; it is characterized by a critical exponent of 2.

We summarize our observations about the PDE in a cosmological framework as follows, where some of them are discussed extensively in a mathematical study being carried out by some of us [24–26]:

[1] In a general EFT of DE theory, α depends on the EFT parameters. For the EFT of k-essence, we have $\alpha=-\frac{\mathcal{H}}{2}(5c_s^2+3w-2)$.

[2] In addition to $\vec{\nabla}\pi|_{x_*}=0$, we also need to have $\vec{\nabla}\frac{\partial\pi}{\partial\tau}|_{x_*}=0$ at initial time, which is a reasonable assumption. In our numerical studies, we consider the scalar field and its time derivative to be zero at initial time and the scalar field being generated solely by the gravitational coupling to the matter perturbations.

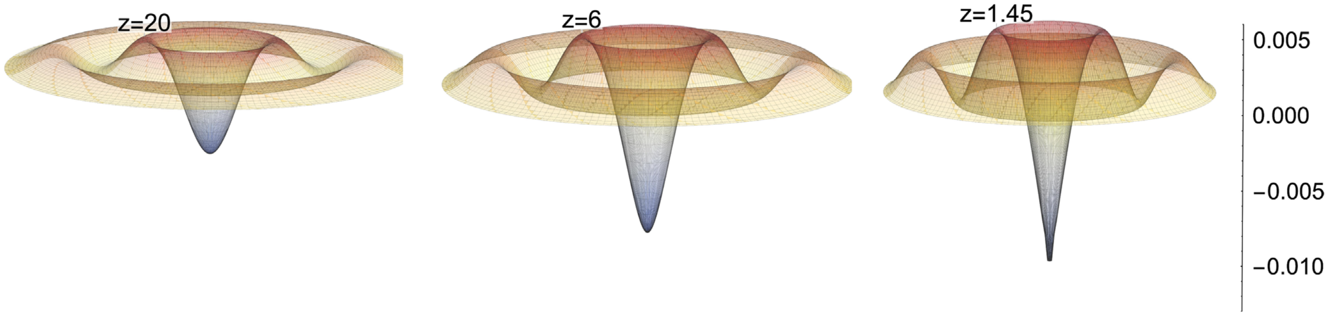

FIG. 2. From left to right: the evolution of the scalar field π for a spherically symmetric scenario when only the $(\partial_x \pi)^2$ is considered as a nonlinear term and all linear terms in Eq. (3) are considered. According to the figure, we see similar behavior compared to the blowup we see in the cosmological N-body simulations.

(i) The equation $\partial_\tau^2 \pi = (\partial_r \pi)^2$ is unstable and blows up at time τ_b given by Eq. (14). For certain initial conditions which are relevant in cosmology, i.e., when the scalar field and its time derivative vanish initially, this is a local phenomenon, in the sense that the blowup point (at a minimum) does not move during its evolution.

(ii) Assuming a small initial value for the scalar field (as a result, small $|\kappa(0)|$), we can see that $|\kappa'(0)|$ is sourced by the gravitational potential and the blowup time depends on $\sim|\kappa'(0)|^{-1/3} \sim |\partial_r^2 \Phi(\tau_{\rm ini})|^{-1/3}$; a higher curvature of the initial gravitational potential (or equivalently a higher density) leads to a faster instability of the system.

(iii) Based on the solution, we expect that the minima become more curved in time and finally at $\tau = \tau_b$ the curvature becomes infinite. It is important to note that the mathematical discussion here was based on considering the particular solution (8) that is quadratic in r. However, in our mathematical papers, we also study this PDE for a more general gravitational potential form [e.g., $\Psi(r) = 1 - \cos(r) = \frac{1}{2}\pi^2 r^2 - \frac{1}{24}\pi^4 r^4 + \cdots$], where corrections of higher order than r^2 contribute,

$$\pi(\tau, r) = b(\tau)\frac{r^2}{2} + d(\tau)\frac{r^4}{4!} + \ldots \tag{16}$$

In that case, we find a leading-order blowup behavior $b(\tau) = \frac{3/2}{(\tau - \tau_b)^2}$ as discussed above, as well as

$$d(\tau) = \frac{\text{const}}{(\tau_b - \tau)^{2\beta - 2}} + \cdots, \tag{17}$$

where $\beta = -1.25 + \sqrt{97}/4 = 1.212\ldots$ is a new critical exponent.

Equation (17) implies that, even for nonquadratic initial conditions, the instability exists. In Fig. 2, we show the numerical solution of Eq. (7) in a $3+1$-dimensional $(3+1\text{D})$ spherically symmetric setup. As we can see, the minimum of the scalar field becomes sharper in time and develops the instability in a finite time (here at redshift $z = 1.45$). Resemblance between our numerical $3+1\text{D}$ cosmological results with the simplified PDE in Eq. (7) suggests that we have correctly identified the source of instability in the full PDE in Eq. (3).

Except for being second order in time, the PDE is similar to the Hamilton-Jacobi equation, $\partial_\tau \pi = (\vec{\nabla}\pi)^2$, and even though this does not *a priori* imply that there should be any relation between the solutions of the two PDEs in our mathematical studies, we show that certain aspects of the time evolution of the problem in fact do reflect this analogy.

V. CONCLUSIONS

This Letter presents a new instability appearing in nonlinear PDEs that arise naturally in EFT descriptions of physical problems. We discovered the instability while studying the equations for k-essence dark energy in the EFT framework with $3+1\text{D}$ cosmological N-body simulations. A mathematical study shows that such nonlinear PDEs are unstable and blow up in finite time. This PDE is rich and interesting from a mathematical point of view, as it does not seem to fit into any mathematical scheme developed so far.

The potential presence of this instability in the EFT of DE framework for cosmology seems unavoidable, as the relevant term, $(\vec{\nabla}\pi)^2$, appears generically for models beyond the standard model of cosmology ΛCDM. This is almost independent of the physics that the EFT approach is applied to. For example, in the EFT of inflation [22], similarly to the EFT of DE, this term appears in the second-order equations. Moreover, in the EFT of gravity [27], we also expect to have such a term in the equations of motion beyond linear order. Whenever this term is present, and is not balanced by a pressure term $\propto \vec{\nabla}^2 \pi$ with sufficiently large coefficient, then we expect that the solutions cease to exist at finite time, effectively signaling the breakdown of the whole EFT scheme.

In Refs. [28–31], it has been found that some covariant k-essence theories with Lagrangian $P(X)$ form instabilities in a finite time. In our approach, we study the scalar field equations rather than the fluid equations (usually being studied in the covariant approaches), where we obtain a second-order (rather than a first-order) partial differential equation. As a result, as shown in Figs. 1 and 2, this instability is of a different type than what has been found in the covariant approaches (caustic formation). Furthermore, the instability we found is formed in a realistic cosmological setup where the weak-field approximation is expected to be valid. We have demonstrated that this is a relevant instability for cosmology, as its characteristic timescale can be much shorter than the age of the Universe.

Whether the breakdown of the EFT scheme is a sign that such methods cannot be applied to these problems or whether it points to a fundamental issue with the physical models that it describes [29,32–35] is not yet clear. In the latter case, large classes of models, including low speed of sound k-essence, become unviable and would effectively be ruled out. This could, for example, be due to shell crossing in a scalar theory that leads to divergences in the field and the stress-energy tensor. If it is only the EFT that fails, then it might be a hint that strong-field effects become important. In this case, and in the cosmological context, it could be that black holes are formed that screen or modify the divergent dynamics. That would be an extremely interesting result, as it could help to explain the presence of supermassive black holes in the centers of galaxies. These questions are the subject of ongoing work. What we can say is that our numerical and analytic studies show that this instability is formed first in the regions with highest density and that the blowup can happen at early times ($z \sim 30$) depending on the density of the center of halos. Moreover, based on numerical studies, this phenomenon is localized, meaning that it does not affect regions that are located well away from the blowup point.

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