$C^\infty-$ REGULARIZATION OF ODES PERTURBED BY NOISE

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Abstract. We study ODEs with vector fields given by general Schwartz distributions, and we show that if we perturb such an equation by adding an "infinitely regularizing" path, then it has a unique solution and it induces an infinitely smooth flow of diffeomorphisms. We also introduce a criterion under which the sample paths of a Gaussian process are infinitely regularizing, and we present two processes which satisfy our criterion. The results are based on the path-wise space-time regularity properties of local times, and solutions are constructed using the approach of Catellier-Gubinelli based on non-linear Young integrals.

1. Introduction and main results

The regularizing effect of adding an irregular stochastic process to an ill-posed ordinary differential equations (ODE) has been extensively studied over the last fifty years. Still it is one of the most surprising results at the intersection of analysis and probability theory. Consider the integral version of an ODE under perturbation of a path $w : [0, T] \to \mathbb{R}^d$ for some $\epsilon \in \mathbb{R}$ given by

$$y_t^x = x + \int_0^t b\,(y_r)\,dr + \epsilon w_t, \qquad x \in \mathbb{R}^d. \tag{1.1}$$

If $\epsilon = 0$, then the classical theory of ODEs would essentially require local Lipschitz continuity of the vector field b to obtain the uniqueness of solutions. However, for $\epsilon \neq 0$ and suitable w one can show the existence of a unique solution under more general assumptions on the vector field b. This has been studied in a number of papers, e.g. [29, 26, 4, 16, 20, 19, 3, 7]. In recent years particular interest has been directed towards the regularizing properties of a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. It has been proven, both by probabilistic means, e.g. in [3], and by path-wise analysis in [7], that the lower we choose H, the more general assumptions we may choose on b. In [7] Catellier and Gubinelli show that uniqueness may hold for Equation 1.1 even if b is only a distribution. More precisely, they show that if w is a fractional Brownian motion with Hurst index H, then for all $b \in B^\alpha_{\infty,\infty}$ with $\alpha > 1 - \frac{1}{2H}$, where $B^\alpha_{\infty,\infty}$ is a Besov space of regularity α, the solution to (1.1) almost surely exists uniquely. The null set outside of which the uniqueness fails depends on the initial condition x, the noise w, and on the vector field $b \in B^\alpha_{\infty,\infty}$. Under the stronger regularity assumption $\alpha > \frac{3}{2} - \frac{1}{2H}$, they show that the flow $x \mapsto y_t^x$ is Lipschitz, but even then the null

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set may depend on b. Catellier and Gubinelli also identify a path-wise condition for w under which uniqueness holds for all sufficiently regular b (measured in terms of "Fourier-Lebesgue regularity" rather than Besov regularity) and all initial conditions x, see [7, Theorem 1.14].

In the more recent work [1] the authors consider an infinite sequence of fractional Brownian motions $\left(\lambda_k w^{H_k}\right)_{k\geq 0}$, where $(\lambda_k)_{k\geq 0}$ and $(H_k)_{k\geq 0}$ are suitable null sequences, and

$$\mathbb{B}_t := \sum_{k\geq 0} \lambda_k w_t^{H_k}. \tag{1.2}$$

Using techniques developed in [3], they show that the equation

$$y_t^x = x + \int_0^t b\left(y_r\right) dr + \mathbb{B}_t$$

has a unique strong solution (in the probabilistic sense) as long as $b \in L^p\left([0,T], L^q\left(\mathbb{R}^d, \mathbb{R}^d\right)\right) \cap L^1\left([0,T], L^\infty\left(\mathbb{R}^d, \mathbb{R}^d\right)\right)$, and furthermore they show that the flow map $x \mapsto y_\cdot^x$ is in C^∞. The techniques used to obtain this results are mainly based on Malliavin calculus and probabilistic methods, and again the null set outside of which the results fail might depend on b.

In the current article we unite the two perspectives of [7] and [1] and we provide a general framework to obtain existence and uniqueness as well as differentiability of the flow map associated to Equation (1.1) for some sufficiently irregular paths w. In contrast to both [7] and [1], our analysis is purely path-wise.

Similarly as in [7] we formulate the equations in the framework of non-linear young theory. But rather than considering directly the regularity of the random map $(t,x) \mapsto \int_0^t b(x+w_s)ds$, which can only be controlled outside of a null set that depends on b, we first control the regularity of the local time L of w, and then write $\int_0^t b(x+w_s)ds = b * (L(-\cdot))$. In this way the null set is independent of b. There is also such a purely pathwise result in [7], but the regularity of L is given in a Fourier-Lebesgue space and therefore in applications also b has to be in a suitable Fourier-Lebesgue space. Here we work with more common function spaces (Sobolev spaces rather than Fourier-Lebesgue spaces), which has the advantage that we get pathwise regularizing effects for b in Hölder spaces or even L^2 Sobolev spaces. A second advantage is that our analysis applies to any regularizing path, and not only stochastic paths. That is, given a path with a sufficiently regular local time, existence and uniqueness of ODEs of the form (1.1) is readily obtained.

To this end, we identify a class of Gaussian processes with exceptional regularizing properties, and we use the space-time regularity of their local times. In fact, if $w : [0,T] \to \mathbb{R}^d$ is a Gaussian process with co-variance function satisfying some simple conditions (translating roughly speaking to sufficient irregularity), then its local time $L : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ almost surely has infinitely many derivatives in its spatial variable for almost all $\omega \in \Omega$. So to analyze the ODE (1.1) we assume that w is a fixed path with smooth local time. By definition of the local time, we have say for bounded measurable b and $x \in \mathbb{R}^d$

$$\int_s^t b\left(x - w_r\left(\omega\right)\right) dr = \left[b * L_{s,t}(\omega)\right](x), \tag{1.3}$$

where $f_{s,t} := f_t - f_s$ for any function f, and $*$ denotes convolution. So for regular $L_{s,t}$ we can make sense of $\int_s^t \nabla b(x - w_r(\omega))dr = [b * \nabla L_{s,t}(\omega)](x)$, even if b is not differentiable. This observation allows us to obtain bounds for integrals appearing in (1.1) which only depend on low regularity norms of b: Consider, for convenience of notation, Equation (1.1) with $\epsilon = -1$. Then $\tilde{y}_t^x = y_t^x + w_t$ solves

$$\tilde{y}_t^x = x + \int_0^t b(\tilde{y}_r^x - w_r)dr, \tag{1.4}$$

and the integral term on the right hand side is very similar to the one on the left hand side of (1.3). In fact, the integral $\int_0^t b(\tilde{y}_r^x - w_r)dr$ can formally be interpreted as

$$\int_0^t b(\tilde{y}_r^x - w_r)dr = \int_0^t [b * L_{dr}]\,(\tilde{y}_r^x). \tag{1.5}$$

For now this expression is purely formal as we need to make sense of the differential L_{dr}, which later we will do via non-linear Young integration (giving a simplified derivation of results from [7]).

We are mainly interested in "infinitely regularizing" paths w. In this case, we will show that the solution y^x to (1.1) exists uniquely (up to a possibly finite explosion time), and the flow $x \mapsto y^x$ is C^∞, only under the assumption that $b \in \mathscr{S}'$ is a Schwartz distribution. Let us first specify what we mean by an infinitely regularizing path:

Definition 1. We say that a continuous path $w : [0,T] \to \mathbb{R}^d$ is *infinitely regularizing* if the local time $L : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, defined in Section 2.1, is in $C_T^\gamma \mathcal{C}^\alpha$ for all $\gamma \in (\frac{1}{2}, 1)$ and $\alpha \in \mathbb{R}$.

In the above definition, and throughout the text, the space $C_T^\gamma \mathcal{C}^\alpha := C^\gamma([0,T], \mathcal{C}^\alpha(\mathbb{R}^d))$ denotes the space of Hölder continuous functions $h : [0,T] \to \mathcal{C}^\alpha(\mathbb{R}^d)$ with values in the Besov space $\mathcal{C}^\alpha(\mathbb{R}^d) := B^\alpha_{\infty,\infty}$. More details on these spaces can be found in Section 2.2.

Our first main result is that existence and uniqueness hold for ODEs perturbed by the path w, with drift coefficients given by general Schwartz distributions in $\mathscr{S}'$. Moreover, the flow mapping $x \mapsto y^x_\cdot$ is infinitely differentiable.

Theorem 2. *Let $b \in \mathscr{S}'$ be a Schwartz distribution, and consider an infinitely regularizing path $w : [0,T] \to \mathbb{R}^d$ as in Definition 1. Then for all $x \in \mathbb{R}^d$ there exists $T^* = T^*(x) \in (0,T] \cup \{\infty\}$ such that there is a unique solution to the equation*

$$y_t^x = x + \int_0^t b\,(y_r^x)\,dr + w_t,$$

in $C\left([0,T^*) \cap [0,T], \mathbb{R}^d\right)$, interpreted in the sense of Definition 31. For $T^*(x) < \infty$ we have $\lim_{t \uparrow T^*(x)} |y_t^x| = \infty$. Moreover, the map $x \mapsto T^*(x)^{-1}$ is locally bounded, and if $\tau < T^*(x)$ for all $x \in U$ for an open set U, then the flow mapping $U \ni x \mapsto y^x_\cdot \in C([0,\tau], \mathbb{R}^d)$ is infinitely Fréchet differentiable.

Proposition 3. *Assume in the setting of Theorem 2 that additionally $b \in B^\alpha_{p,q}$ for some $\alpha \in \mathbb{R}$ and $p, q \in [1,\infty]$. Then $T^*(x) = \infty$ for all $x \in \mathbb{R}^d$.*

It should be noted that all this holds for deterministic paths that are infinitely regularizing. However, the derivation of sharp spatio-temporal regularity results for the local times of deterministic functions (for example the Weierstrass function) is still an open and challenging problem. Therefore, we show that there exist infinitely regularizing stochastic processes. In particular, we will prove the following theorem, outlining sufficient conditions for a Gaussian process to be infinitely regularizing.

Theorem 4. *Let $w : [0,T] \times \Omega \to \mathbb{R}^d$ be a centered Gaussian process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by w. Suppose w satisfies the following local non-determinism condition for any $\zeta \in (0,1)$*

$$\inf_{t\in(0,T]} \inf_{s\in[0,t)} \frac{\mathrm{Var}(w_t|\mathcal{F}_s)}{(t-s)^\zeta} > 0. \tag{1.6}$$

Then for almost all $\omega \in \Omega$ the path $t \mapsto w_t(\omega)$ is infinitely regularizing. These conditions are satisfied by the log-Brownian motion of Definition 20, or the process $\mathbb{B}_t := \sum_k \lambda_k w_t^{H_k}$ in (1.2).

The structure of the paper is as follows:

- In Section 2 we provide some background material on local times and Besov spaces, as well as a statement of the stochastic sewing lemma, recently developed by Lê [17].
- Section 3 is devoted to the proof of Theorem 4.
- In Section 4 we present two infinitely regularizing Gaussian processes. In particular, we will consider a process called $p-$log Brownian motion, and show that is infinitely regularizing. We also show that the processes used in [1] is infinitely regularizing.
- In Section 5 we give a path-wise construction of the so called average operators, in line with what has been done in [7]. We show that when these operators are constructed from an infinitely regularizing path, then they are in $C_T^\gamma \mathcal{C}^\alpha$ for any $\gamma \in (0,1)$, $\alpha > 0$.
- Finally, in Section 6 we prove the existence and uniqueness of solutions to (1.1) and the smoothness of the associated flow.

2. Essentials of local times and Besov spaces

2.1. **Occupation measures and Local times .** The occupation measure of an $\mathbb{R}^d$-valued measurable path $w : [0,T] \to \mathbb{R}^d$ at a time $t \in [0,T]$ is defined by

$$\mu_t(A) := \lambda\{s \in [0,t]\,|w_s \in A\}, \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^d),$$

where λ is the Lebesgue measure. We interpret $\mu_t(A)$ as "the amount of time w spends in A up to time t". Occupation measures have been an important topic in the theory of stochastic processes during the last fifty years. We refer the interested reader to the comprehensive review paper by Geman and Horowitz [11], and the references therein for an introductory account of occupation measures and local times.

Definition 5. Let $w : [0,T] \to \mathbb{R}^d$ be a measurable path. Assume that there exists a measurable function $L : [0,T] \times \mathbb{R}^d \to \mathbb{R}_+$ with $L_0(z) = 0$ for all $z \in \mathbb{R}^d$ and such that

$$\mu_t(A) = \int_A L_t(z)\,dz, \qquad A \in \mathcal{B}(\mathbb{R}^d), t \in [0,T]. \tag{2.1}$$

Then we call L the *local time* of w.

Remark 6. Of course, the local time does not have to exist, and intuitively we interpret its existence and regularity as an irregularity condition for w. Later we will see that for sample paths of Gaussian processes this interpretation is in some sense justified. However, in general it is an open problem to establish a clear link between regularity properties of the local times and irregularity measures such as "true roughness" [14, 9]; see Sections 10 and 11 of [11] for some partial results in that direction.

Note that in $d = 1$ even $w \in C^\infty$ can have a local time: For $a, b \in \mathbb{R}$ with $b > 0$ the path $w_t = a + bt$ has the local time $L_t(z) = b^{-1}\mathbb{1}_{(a,a+bt]}(z)$. However, if a Lipschitz continuous $w \in C([0,T],\mathbb{R})$ has a local time L, then L_t has at least two discontinuities: If $z_0 = w_{t_0} = \max\{w_s : s \in [0,t]\}$, then $L_t(z) = 0$ for $z > z_0$, and with the Lipschitz constant K of w we get $w_s \in [z_0 - \delta, z_0]$ for all $s \in [t_0 - \delta/K, t_0 + \delta/K] \cap [0,t]$. For $\delta > 0$ small enough we must have $t_0 - \delta/K \geq 0$ or $t_0 + \delta/K \leq t$, and therefore

$$\int_{z_0-\delta}^{z_0} L_t(z)dz \geq \frac{\delta}{K}.$$

By the fundamental theorem of calculus this is impossible if L_t is continuous with $L_t(z_0) = 0$. Similarly L_t must have a discontinuity at the minimum of $w|_{[0,t]}$. In other words the local time can only be continuous if w is more irregular than Lipschitz continuous. A similar argument shows that the set of paths with local time $L_T \in C^\alpha(\mathbb{R})$ has empty intersection with $C^\beta([0,T],\mathbb{R})$ for all $\beta > (\alpha+1)^{-1}$.

2.2. **Essentials of Besov spaces.** Here we recall some basic properties of Besov spaces. For a more extensive introduction we refer to [2]. We will denote by $\mathscr{S}$ resp. $\mathscr{S}'$ the space of Schwartz functions on $\mathbb{R}^d$ resp. its dual, the space of tempered distributions. For $f \in \mathscr{S}'$ we denote the Fourier transform by $\hat{f} = \mathscr{F}(f) = \int_{\mathbb{R}^d} e^{-ix\cdot} f(x)dx$, where the integral notation is formal, with inverse $\mathscr{F}^{-1}f = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{iz\cdot}\hat{f}(z)dz$.

Definition 7. Let $\chi, \rho \in C^\infty(\mathbb{R}^d,\mathbb{R})$ be two radial functions such that χ is supported on a ball $\mathcal{B} = \{|x| \leq c\}$ and ρ is supported on an annulus $\mathcal{A} = \{a \leq |x| \leq b\}$ for $a, b, c > 0$, such that

$$\chi + \sum_{j\geq 0} \rho\left(2^{-j}\cdot\right) \equiv 1,$$

$$\operatorname{supp}(\chi) \cap \operatorname{supp}\left(\rho\left(2^{-j}\cdot\right)\right) = \emptyset, \quad \forall j \geq 1,$$

$$\operatorname{supp}\left(\rho\left(2^{-j}\cdot\right)\right) \cap \operatorname{supp}\left(\rho\left(2^{-i}\cdot\right)\right) = \emptyset, \quad \forall |i-j| \geq 1.$$

Then we call the pair (χ, ρ) a *dyadic partition of unity.* Furthermore, we write $\rho_j = \rho(2^{-j}\cdot)$ for $j \geq 0$ and $\rho_{-1} = \chi$, as well as $K_j = \mathscr{F}^{-1}\rho_j$.

The existence of a partition of unity is shown for example in [2]. We fix a partition of unity (χ, ρ) for the rest of the paper.

Definition 8. For $f \in \mathscr{S}'$ we define its Littlewood-Paley blocks by

$$\Delta_j f = \mathscr{F}^{-1}(\rho_j \hat{f}) = K_j * f.$$

It follows that $f = \sum_{j\geq -1} \Delta_j f$ with convergence in $\mathscr{S}'$.

In the following we write

$$\langle x \rangle := (1 + |x|^2)^{1/2}. \tag{2.2}$$

Definition 9. For any $\alpha, \kappa \in \mathbb{R}$ and $p, q \in [1, \infty]$, the *weighted Besov space* $B^\alpha_{p,q}(\langle x\rangle^\kappa)$ is

$$B^\alpha_{p,q}(\langle x\rangle^\kappa) := \left\{ f \in \mathscr{S}' \,\middle|\, \|f\|_{B^\alpha_{p,q}(\langle x\rangle^\kappa)} = \left(\sum_{j\geq -1} \left(2^{j\alpha} \|\langle x\rangle^\kappa \Delta_j f\|_{L^p}\right)^q \right)^{\frac{1}{q}} < \infty \right\},$$

with the usual interpretation as ℓ^∞ norm if $q = \infty$. If $\kappa = 0$, we simply write $B^\alpha_{p,q}$. Furthermore, we denote $\mathcal{C}^\alpha(\langle x\rangle^\kappa) = B^\alpha_{\infty,\infty}(\langle x\rangle^\kappa)$ and $\mathcal{C}^\alpha_p(\langle x\rangle^\kappa) = B^\alpha_{p,\infty}(\langle x\rangle^\kappa)$.

Remark 10. By Theorem 6.5 of [25] we have

$$\|f\|_{B^\alpha_{p,q}(\langle x\rangle^\kappa)} \simeq \|f\langle x\rangle^\kappa\|_{B^\alpha_{p,q}}.$$

Remark 11. For $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ the space $\mathcal{C}^\alpha(\langle x\rangle^\kappa)$ corresponds to a classical weighted Hölder space, see e.g. [18, Lemma 2.1.23]. For all $\alpha \in \mathbb{R}$ the Besov space $B^\alpha_{2,2}$ corresponds to the inhomogeneous Sobolev space H^α defined by

$$H^\alpha := \left\{ f \in \mathscr{S}' \,\middle|\, \|f\|_{H^\alpha} = \|\,(1 + |\cdot|)^\alpha \hat{f}\|_{L^2} < \infty \right\}.$$

Lemma 12 (Besov embedding, see [2], Proposition 2.71)**.** *Let* $1 \leq p_1 \leq p_2 \leq \infty$ *and* $1 \leq q_1 \leq q_2 \leq \infty$, *and let* $\kappa \in \mathbb{R}$. *Then* $B^\kappa_{p_1,q_1}$ *is continuously embedded into* $B^{\kappa - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}_{p_2,q_2}$.

Recall from Definition 1 that a path is infinitely regularizing if its local time is in $C^\gamma_T\mathcal{C}^\alpha$ for all $\gamma \in (0,1)$ and all $\alpha \in \mathbb{R}$. By an interpolation argument this follows from a softer criterion:

Corollary 13. *Let* $w \in C([0,T], \mathbb{R}^d)$ *with associated local time* L *such that*

$$\sup_{t\in[0,T]} \|L_t\|_{H^\alpha} < \infty$$

for any $\alpha > 0$. *Then* w *is infinitely regularizing.*

Proof. We get the necessary time regularity by bounding $L_{s,t}$ in a Besov space that contains measures, and then we use an interpolation argument: For any finite positive measure μ we have by [2, Proposition 2.76]

$$\|\mu\|_{B^0_{1,\infty}} \lesssim \sup_{\substack{\varphi\in\mathscr{S},\\ \|\varphi\|_{B^0_{\infty,1}}\leq 1}} \langle \mu, \varphi\rangle \leq \mu(\mathbb{R}^d) \times \sup_{\substack{\varphi\in\mathscr{S},\\ \|\varphi\|_{B^0_{\infty,1}}\leq 1}} \|\varphi\|_{L^\infty} \lesssim \mu(\mathbb{R}^d).$$

Since the occupation measure is given by $\mu_t(\cdot) = \int_0^t \delta(\cdot - w_r)dr$, we get with the Besov embedding result from Lemma 12 that

$$\|L_{s,t}\|_{\mathcal{C}^{-d}} \lesssim \|L_{s,t}\|_{B^0_{1,\infty}} \leq \int_s^t \|\delta(\cdot - w_r)\|_{B^0_{1,\infty}} dr \lesssim |t-s|,$$

which implies in particular that $L \in C^1_T\mathcal{C}^{-d}(\mathbb{R}^d)$. Now we get for $\alpha > 0$ and $\gamma \in (0,1)$:

$$\begin{aligned}
\|L\|_{C^\gamma_T\mathcal{C}^\alpha} &= \sup_{0\leq s<t\leq T} \sup_{j\geq -1} 2^{j\alpha} \frac{\|\Delta_j L_{s,t}\|_{L^\infty}}{|t-s|^\gamma} \\
&\leq \left(\sup_{s,t,j} 2^{-jd} \frac{\|\Delta_j L_{s,t}\|_{L^\infty}}{|t-s|} \right)^\gamma \left(\sup_{s,t,j} 2^{j\frac{\alpha+\gamma d}{1-\gamma}} \|\Delta_j L_{s,t}\|_{L^\infty} \right)^{1-\gamma} \\
&\leq \|L\|^\gamma_{C^1_T\mathcal{C}^{-d}} \sup_{s,t} \|L_{s,t}\|^{1-\gamma}_{\mathcal{C}^\kappa},
\end{aligned}$$

for $\kappa := \frac{\alpha+\gamma d}{1-\gamma}$. Since $\|L_{s,t}\|_{\mathcal{C}^\kappa} \lesssim \|L_{s,t}\|_{H^{\kappa+d/2}} \lesssim 1$ by assumption, our claim follows. □

2.3. **The stochastic sewing lemma.** To derive the space-time regularity of local times, we will apply the stochastic sewing lemma [17] recently developed by Khoa Lê. We therefore recite here the statement of this lemma, and refer the reader to [17] for the proof and a discussion of this result. We will make use of n- simplices defined by

$$\Delta_n^T := \{(t_1, \dots, t_n) \in [0,T]^n | 0 \leq t_1 \leq \dots \leq t_n \leq T\}. \tag{2.3}$$

Lemma 14 ([17], Theorem 2.1). *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space. Let $p \geq 2$ and let $A : \Delta_2^T \to \mathbb{R}^d$ be a stochastic process such that $A_{s,s} = 0$, $A_{s,t}$ is $\mathcal{F}_t$ measurable, and $(s,t) \mapsto A_{s,t}$ is right-continuous from Δ_2^T into $L^p(\Omega)$. Set $\delta_u A_{s,t} := A_{s,t} - A_{s,u} - A_{u,t}$ for $(s,u,t) \in \Delta_3^T$, and assume that there exists constants $\beta > 1$, $\kappa > \frac{1}{2}$, and $C_1, C_2 > 0$ such that*

$$\begin{aligned} \|\mathbb{E}\left[\delta_u A_{s,t} | \mathcal{F}_s\right]\|_{L^p(\Omega)} &\leq K_1 |t-s|^\beta, \\ \|\delta_u A_{s,t}\|_{L^p(\Omega)} &\leq K_2 |t-s|^\kappa. \end{aligned} \tag{2.4}$$

Then there exists a unique (up to modifications) $\{\mathcal{F}_t\}$-adapted stochastic process $\mathcal{A}$ such that the following properties are satisfied:

(i) *$\mathcal{A} : [0,T] \to L^p(\Omega)$ is right continuous, and $\mathcal{A}_0 = 0$.*
(ii) *There exist constants $C_1, C_2 > 0$ such that for $\mathcal{A}_{s,t} = \mathcal{A}_t - \mathcal{A}_s$:*

$$\begin{aligned} \|\mathcal{A}_{s,t} - A_{s,t}\|_{L^p(\Omega)} &\leq C_1 K_1 |t-s|^\beta + C_2 K_2 |t-s|^\kappa, \\ \|\mathbb{E}\left[\mathcal{A}_{s,t} - A_{s,t} | \mathcal{F}_s\right]\|_{L^p(\Omega)} &\leq C_1 K_1 |t-s|^\beta. \end{aligned} \tag{2.5}$$

Furthermore, for all $(s,t) \in \Delta_2^T$ and for any partition $\mathcal{P}$ of $[s,t]$, define

$$A_{s,t}^{\mathcal{P}} := \sum_{[u,v]\in\mathcal{P}} A_{u,v}. \tag{2.6}$$

Then $A_{s,t}^{\mathcal{P}}$ converge to $\mathcal{A}_{s,t}$ in $L^p(\Omega)$ as the mesh size $|\mathcal{P}| \to 0$.

3. Regularity of local times associated to Gaussian paths

Here we study the space-time regularity of the local times of Gaussian processes. Although there are well known results for the spatial regularity of the local time L_t of Gaussian processes at fixed times (e.g. [11]), it seems more difficult to find results that are uniform in time or that even quantify the time regularity (see however [7] for results about the time regularity of the local time of fractional Brownian motion in certain Fourier-Lebesgue spaces). We therefore present a general criterion for centered Gaussian processes to be infinitely regularizing.

Definition 15. A square-integrable stochastic process $w : \Omega \times [0,T] \to \mathbb{R}^d$ is called *ζ-locally non-deterministic (ζ-LND)* if

$$\inf_{t\in(0,T]} \inf_{s\in[0,t)} \inf_{\substack{z\in\mathbb{R}^d: \\ |z|=1}} \frac{z^T \mathrm{cov}(w_t|\mathcal{F}_s) z}{(t-s)^{2\zeta}} > 0, \tag{3.1}$$

where $\mathrm{cov}(w_t|\mathcal{F}_s) := \mathbb{E}[(w_t - \mathbb{E}[w_t|\mathcal{F}_s])(w_t - \mathbb{E}[w_t|\mathcal{F}_s])^T|\mathcal{F}_s]$.

Remark 16. There exists several different definitions of the concept of local non-determinism of stochastic process, e.g. [5, 22, 28]. The condition in Definition 15 is related to the strong local ϕ non-determinism proposed by Cuzik and DuPerez [8], where $\phi(r) = r^\gamma$; the only (important) difference is that we only condition on the past, while in [8] also information about the future is taken into account. This concept is also discussed in [28].

The next theorem shows that if $\zeta \in [0, \frac{1}{d})$, then ζ-locally non-deterministic centered Gaussian processes have jointly Hölder-Sobolev continuous local times.

Theorem 17. *Let $\zeta \in [0, \frac{1}{d})$ and let $w : \Omega \times [0,T] \to \mathbb{R}^d$ be a continuous centered Gaussian process which is ζ-LND. Then there exists a null set $\mathcal{N} \subset \Omega$ such that for all $\omega \in \mathcal{N}^c$ the function $w(\omega)$ has a local time $L(\omega)$, and for all $\lambda < \frac{1}{2\zeta} - \frac{d}{2}$ and $\gamma \in [0, 1-(\lambda+\frac{d}{2})\zeta)$ we have*

$$\|L_{s,t}(\omega)\|_{H^\lambda} \leq C(\omega)|t-s|^\gamma. \tag{3.2}$$

It follows that $L \in C_T^\gamma H^\lambda$, $\mathbb{P}$-almost surely.

Proof. We control the Sobolev regularity by deriving bounds for the Fourier transform of the occupation measure. By definition, the Fourier transform of $\mu_{s,t}$ is given by $\widehat{\mu_{s,t}}(z) = \int_s^t e^{i\langle z, w_r\rangle} dr$. Consider the process $A_{s,t} := \int_s^t \mathbb{E}[e^{i\langle z, w_r\rangle}|\mathcal{F}_s]dr$ (we will deal with the z dependence of A later, but for now we suppress the notation), where $\{\mathcal{F}_t\}$ is the (completion of the) natural filtration generated by w. We will apply the stochastic sewing lemma to derive bounds for the moments of the limit $\mathcal{A}_{s,t}$ of the Riemann sums $\sum_{[u,v]\in\mathcal{P}} A_{u,v}$. Then we will see that in fact $\mathcal{A}_{s,t} = \widehat{\mu}_{s,t}(z)$.

To apply the stochastic sewing lemma, we need to check that A verifies the necessary conditions. By definition of A, it follows directly that $A_{s,s} = 0$, $A_{s,t}$ is $\mathcal{F}_t$ measurable, and $(s,t) \to A_{s,t}$ is right continuous. Furthermore,

$$\mathbb{E}[\delta_u A_{s,t}|\mathcal{F}_s] = \mathbb{E}\left[\int_s^t \mathbb{E}[e^{i\langle z, w_r\rangle}|\mathcal{F}_s]dr - \int_s^u \mathbb{E}[e^{i\langle z, w_r\rangle}|\mathcal{F}_s]dr - \int_u^t \mathbb{E}[e^{i\langle z, w_r\rangle}|\mathcal{F}_u]dr|\mathcal{F}_s\right] = 0,$$

by the tower property of conditional expectations, and thus the condition $\|\mathbb{E}[\delta_u A_{s,t}|\mathcal{F}_s]\|_{L^p(\Omega)} = 0$ in (2.4) is satisfied. To show the second condition, i.e. $\|\delta_u A_{s,t}\|_{L^p(\Omega)} \leq C_1|t-s|^\kappa$ for some $\kappa > \frac{1}{2}$, let us decompose w into two parts, namely

$$w_r = \mathbb{E}[w_r|\mathcal{F}_s] + (w_r - \mathbb{E}[w_r|\mathcal{F}_s]).$$

For a Gaussian process w, the components $\mathbb{E}[w_r|\mathcal{F}_s]$ and $(w_r - \mathbb{E}[w_r|\mathcal{F}_s])$ are two Gaussian random variables such that $(w_r - \mathbb{E}[w_r|\mathcal{F}_s])$ is independent of $\mathcal{F}_s$, see e.g. [6, Theorem 3.10.1]. This implies that the conditional covariance $\mathrm{cov}\,(w_r|\,\mathcal{F}_s)$ is deterministic, due to the fact that $\mathrm{cov}(w_r|\mathcal{F}_s) = \mathrm{cov}\,(w_r - \mathbb{E}[w_r|\mathcal{F}_s]|\,\mathcal{F}_s)$ and $w_r - \mathbb{E}[w_r|\mathcal{F}_s]$ is independent of $\mathcal{F}_s$. Consequently,

$$\mathbb{E}[e^{i\langle z, w_r\rangle}|\mathcal{F}_s] = \exp\left(i\langle z, \mu_r^{\mathcal{F}_s}\rangle - \frac{1}{2} z^T \Sigma_r^{\mathcal{F}_s} z\right),$$

where $\mu_r^{\mathcal{F}_s} := \mathbb{E}[w_r|\mathcal{F}_s]$ and $\Sigma_r^{\mathcal{F}_s} := \mathrm{cov}(w_r|\mathcal{F}_s)$. This yields

$$\begin{aligned}\|\delta_u A_{s,t}\|_{L^p(\Omega)} &= \left\|\int_u^t \exp\left(i\langle z, \mu_r^{\mathcal{F}_s}\rangle - \frac{1}{2}z^T\Sigma_r^{\mathcal{F}_s}z\right) - \exp\left(i\langle z, \mu_r^{\mathcal{F}_u}\rangle - \frac{1}{2}z^T\Sigma_r^{\mathcal{F}_u}z\right) dr\right\| \\ &\lesssim \int_u^t \left[\exp\left(-\frac{1}{2}z^T\Sigma_r^{\mathcal{F}_s}z\right) + \exp\left(-\frac{1}{2}z^T\Sigma_r^{\mathcal{F}_u}z\right)\right] dr.\end{aligned}$$

By assumption, w is ζ-LND, so denote by M the constant given by the left hand side of (3.1). Since $(r-s)^{2\zeta} \geq (r-u)^{2\zeta}$ for any $(s,u) \in \Delta_2^T$, we observe that

$$\|\delta_u A_{s,t}\|_{L^p(\Omega)} \lesssim \int_u^t \exp\left(-\frac{M}{2}|z|^2(r-u)^{2\zeta}\right) dr.$$

It is readily checked that

$$e^{-\frac{M}{2}(r-u)^{2\zeta}|z|^2} \leq e^{\frac{MT^{2\zeta}}{2}} e^{-\frac{M}{2}(r-u)^{2\zeta}(1+|z|^2)},$$

and that for $\lambda' \geq 0$ we have $e^{-C} \lesssim C^{-\frac{\lambda'}{2}}$, uniformly in $C > 0$. So we get for $\lambda' > 0$ such that $\lambda'\zeta < 1$:

$$\begin{aligned}\|\delta_u A_{s,t}\|_{L^p(\Omega)} &\lesssim \frac{M^{\lambda'}}{2^{\lambda'}} e^{\frac{MT^{2\zeta}}{2}} \int_u^t (1+|z|^2)^{-\frac{\lambda'}{2}}(r-u)^{-\lambda'\zeta} z\, dr \\ &\simeq (1+|z|^2)^{-\frac{\lambda'}{2}}(t-u)^{1-\lambda'\zeta} \\ &= (1+|z|^2)^{-\frac{\lambda'}{2}}(t-u)^{\kappa}.\end{aligned}$$

If $\lambda' < \frac{1}{2\zeta}$, then $1-\lambda'\zeta > \frac{1}{2}$ and we can apply the stochastic sewing lemma, more precisely (2.5) together with Minkowski's inequality, to deduce that the "sewing" $\mathcal{A}_{s,t}$ satisfies

$$\|\mathcal{A}_{s,t}\|_{L^p(\Omega)} \lesssim \|A_{s,t}\|_{L^p(\Omega)} + (1+|z|^2)^{-\frac{\lambda'}{2}}|t-s|^{1-\lambda'\zeta},$$

where we recall that $\|\mathbb{E}[\delta_u A_{s,t}|\mathcal{F}_s]\|_{L^p(\Omega)} = 0$. It is now readily seen, following the lines of the previous analysis, that we also have

$$\|A_{s,t}\|_{L^p(\Omega)} \lesssim (1+|z|^2)^{-\frac{\lambda'}{2}}|t-s|^{1-\lambda'\zeta}.$$

Moreover, we get for $t_k^n = s + (t-s)k/n$

$$\begin{aligned}\|\widehat{\mu_{s,t}}(z) - \mathcal{A}_{s,t}\|_{L^p(\Omega)} &\leq \sum_{k=0}^{n-1} \left\| \int_{t_k^n}^{t_{k+1}^n} (e^{i\langle z, w_r\rangle} - \mathbb{E}[e^{i\langle z,w_r\rangle}|\mathcal{F}_{t_k^n}]) dr \right\|_{L^p(\Omega)} \\ &\leq 2\sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \left\| e^{i\langle z,w_r\rangle} - e^{i\langle z, w_{t_k^n}\rangle} \right\|_{L^p(\Omega)} dr,\end{aligned}$$

and since w is continuous, the dominated convergence theorem shows that the right hand side converges to zero as $n \to \infty$. In conclusion we have shown that

$$\|\widehat{\mu_{s,t}}(z)\|_{L^p(\Omega)} \lesssim (1+|z|^2)^{-\frac{\lambda'}{2}}|t-s|^{1-\lambda'\gamma}.$$

We will now use this moment bound together with Kolmogorov's continuity criterion to derive the claimed regularity of μ. For $p \geq 2$ and $\epsilon \in (0, \lambda' - \frac{d}{2})$ we apply Minkowski's inequality to obtain

$$\begin{aligned}
\mathbb{E}[\|\mu_{s,t}\|^p_{H^{\lambda'-\frac{d}{2}-\epsilon}}]^{\frac{1}{p}} &= \mathbb{E}\left[\left(\int_{\mathbb{R}^d} |\widehat{\mu_{s,t}}(z)|^2(1+|z|^2)^{\lambda'-\frac{d}{2}-\epsilon}dz\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}^d} \||\widehat{\mu_{s,t}}(z)|^2\|_{L^{\frac{p}{2}}(\Omega)}(1+|z|^2)^{\lambda'-\frac{d}{2}-\epsilon}dz\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^d} \|\widehat{\mu_{s,t}}(z)\|^2_{L^p(\Omega)}(1+|z|^2)^{\lambda'-\frac{d}{2}-\epsilon}dz\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{\mathbb{R}^d} \left((1+|z|^2)^{-\frac{\lambda'}{2}}|t-s|^{1-\lambda'\zeta}\right)^2(1+|z|^2)^{\lambda'-\frac{d}{2}-\epsilon}dz\right)^{\frac{1}{2}} \\
&\lesssim |t-s|^{1-\lambda'\zeta}\int_{\mathbb{R}^d}(1+|z|^2)^{-\frac{d}{2}-\epsilon}dz,
\end{aligned}$$

and the integral on the right hand side is finite for any $\epsilon > 0$. Since $p \geq 2$ can be chosen arbitrarily large, it follows from Kolmogorov's continuity theorem that for any $\gamma \in [0, 1-\lambda'\zeta)$ there exists a set $\mathcal{N}^c$ of full measure such that for all $\omega \in \mathcal{N}^c$ and $(s,t) \in \Delta_2^T$ we have

$$\|\mu_{s,t}(\omega)\|_{H^{\lambda'-\frac{d}{2}-\epsilon}} \leq C(\omega)|t-s|^\gamma. \tag{3.3}$$

So with $\lambda = \lambda' - \frac{d}{2} - \epsilon$ and we obtain the claimed result (3.2). Moreover, since $\zeta < \frac{1}{d}$ we can choose $\lambda > 0$ and in particular $\mu_t(\omega) \in L^2$ and the density $L_t(\omega)$ exists. □

Remark 18. Let $b \in C(\mathbb{R}^d)$, and $w : \Omega \times [0,T] \to \mathbb{R}^d$ be a ζ-LND Gaussian process for some $\zeta \in (0, \frac{1}{d})$. Set $T^w_t b(x) := \int_0^t b(x+w_r)dr$, and observe that $T^w_t b = b*(L_t(-\cdot))$ where L_t is the local time of w. Invoking the regularity of the local time obtained in Theorem 17 together with Young's convolution inequality, there exists a null set $\mathcal{N} \subset \Omega$ only depending on w, such that for all $\omega \in \mathcal{N}^c$ and for all $\epsilon > 0$:

$$\|T^{w(\omega)}_t b - T^{w(\omega)}_s b\|_{\mathcal{C}^{\alpha+\frac{1}{2\zeta}-\frac{d}{2}-\epsilon}} \lesssim \|b\|_{H^\alpha}\|L_{s,t}(\omega)\|_{H^{\frac{1}{2\zeta}-\frac{d}{2}-\epsilon}} \lesssim \|b\|_{H^\alpha}|t-s|^\gamma, \tag{3.4}$$

for some $\gamma > \frac{1}{2}$. Compared to Theorem 1.1 of [7] we lose $\frac{d}{2}$ derivatives in our estimate, but we gain integrability. The main difference is that the null set $\mathcal{N} \subset \Omega$ in [7] depends on the function b. It would be possible to directly estimate the regularity of $T^w b$ using similar arguments as in the proof of Theorem 17. Indeed, we observe that the Fourier transform of $T^w b$ is $\hat{b}(z)\int_s^t e^{i\langle z,w_r\rangle}dr$. It is then readily checked that we recover similar regularity results as in [7, Theorem 1.1], although in H^α spaces. But as the main goal of this article is to provide a path-wise analysis of infinitely regularizing paths, we want to avoid the dependence of the null sets on b and therefore we estimate the regularity of L.

Remark 19. At least in the case of a fractional Brownian motion with Hurst parameter ζ we get from [7, Theorem 1.4] a control of the $\frac{1}{2\zeta}-$ regularity of L in a Fourier-Lebesgue space. Implicitly, Conjecture 1.2 of [7] suspects that the loss of $\frac{d}{2}$ derivatives in our result can be avoided and that we should have $L \in C^{\frac{1}{2}+}B^{\frac{1}{2\zeta}-}_{1,1}$. Indeed, if $w \mapsto w * L_t(-\cdot)$ is a bounded operator from $\mathcal{C}^\alpha$ to $\mathcal{C}^{\alpha+\rho}(\langle x\rangle^{-\kappa})$ (see the discussion before Conjecture 1.2), then we obtain for $\alpha = -\rho + \epsilon$:

$$|\langle b, L_t(-\cdot)\rangle| = |(b * L_t(-\cdot))(0)| \lesssim \|b * L_t(-\cdot)\|_{\mathcal{C}^\epsilon(\langle x\rangle^{-\kappa})} \lesssim \|b\|_{\mathcal{C}^{-\rho+\epsilon}},$$

and since $\mathcal{C}^{-\rho+\epsilon} = B^{-\rho+\epsilon}_{\infty,\infty}$ we would get by duality $L_t(-\cdot) \in B^{\rho-\epsilon}_{1,1}$, see Proposition 2.76 in [2]. For the fractional Brownian motion we would be able to take $\rho = \frac{1}{2\zeta}-$, and in particular we would obtain that $L_t \in L^1$ whenever $\frac{1}{2\zeta} > 0$. But of course in general it does not only depend on the Hurst parameter ζ but also on the dimension whether the fractional Brownian motion has an absolutely continuous occupation measure. For example, Xiao [27, Theorem 2.1] shows that if w is a d-dimensional fractional Brownian motion of Hurst index ζ, then the Hausdorff dimension of $(w_t)_{t\in[0,1]}$ is equal to $\min\{d, \frac{1}{\zeta}\}$. If $\frac{1}{\zeta} < d$, the image of $(w_t)_{t\in[0,1]}$ is thus a null set in $\mathbb{R}^d$ and therefore the occupation measure cannot be absolutely continuous. Note that $\frac{1}{\zeta} < d$ is equivalent to $\frac{1}{2\zeta} - \frac{d}{2} < 0$ and that Theorem 17 gives us space regularity $\frac{1}{2\zeta} - \frac{d}{2}-$, i.e. for $\frac{1}{2\zeta} - \frac{d}{2} > 0$ it follows from Theorem 17 that the local time exists (and then immediately has L^2-Sobolev regularity and not just $B_{1,1}$-Besov regularity).

4. Infinitely regularizing stochastic processes

It follows from Theorem 4 together with Corollary 13 that if w is a continuous centered Gaussian process w which is ζ-LND for any $\zeta > 0$, then w is almost surely infinitely regularizing in the sense of Definition 1. Here we present two examples of such processes.

4.1. $p-$log-Brownian motions.

If the conditional variance $\mathrm{Var}(w_{t+h}|\mathcal{F}_t)$ of a continuous centered Gaussian process is bounded below by $\phi(h) := |\ln(1/h)|^{-p}$, for some $p > 0$, then it is ζ-LND for any $\zeta > 0$. Thus our first example has an incremental variance structure resembling ϕ. This is partly inspired by [11], where the authors mention in a remark below Theorem 28.4 that Gaussian processes with incremental variance behaving like the logarithm around the origin, i.e. $\sim |\ln(1/t)|^{-1}$ for $t \to 0$, seem to have local times with exceptional (spatial) regularity. In [21] the authors investigate a Gaussian process they call the log-Brownian motion. The same process has also recently been investigated for the purpose of super rough volatility modelling in [13].

Definition 20. Consider $[0,T] \subset [0,1)$ and a $p > \frac{1}{2}$, and let $B : [0,T] \times \Omega \to \mathbb{R}^d$ be a d-dimensional Brownian motion. We define the *$p-$log Brownian motion* as

$$w^p_t := \int_0^t k(t-s)dB_s, \tag{4.1}$$

where $k(t) := |t\ln(1/t)^{2p}|^{-\frac{1}{2}} \in L^2([0,T])$.

Remark 21. Since for $p > \frac{1}{2}$ the function $t^{-1}\ln(1/t)^{-2p}$ has a non-integrable singularity at $t = 1$, we have to take $T < 1$. For larger T we could rescale the kernel and consider $k_\beta(t) = k(\beta t)$ for $\beta > T$ instead. See also the discussion below Definition 18 of [21] or [13]. To obtain a stationary version we could for example consider $k(t) = (t(|\ln(1/t)^{2p}|\vee 1))^{-\frac{1}{2}} \in L^2(\mathbb{R}_+)$ and then $w^p_t = \int_{-\infty}^t k(t-s)dB_s$ for a two-sided d-dimensional Brownian motion B. For simplicity we do not make these adaptations and we restrict to $T < 1$ for the rest of the subsection.

Proposition 22. *For $p > 1$ there exists a continuous version of the p-log Brownian.*

Proof. See [21], Definition 18 and below, or [13, Remark 2.5]. □

Corollary 23. *For $p>1$ the d-dimensional p-log Brownian motion w^p is $\mathbb{P}-a.s.$ infinitely regularizing.*

Proof. By definition, the d-dimensional p-lBm is a centered Gaussian process, and according to Proposition 22 it is continuous if $p>1$. By Theorem 17 together with Corollary 13 we obtain that if w^p is ζ-LND for any $\zeta>0$, then it is infinitely regularizing. So let us compute the conditional variance for $(s,t)\in\Delta_2$:

$$\mathrm{Var}(w_t^p|\mathcal{F}_s)=\int_s^t k(t-r)^2 dr I_d, \tag{4.2}$$

where $k(t)=|t\ln(1/t)^{2p}|^{-\frac{1}{2}}$ and I_d is the d-dimensional unit matrix. By elementary computations, using that $\frac{d}{dt}\frac{\ln(1/t)^{1-2p}}{2p-1}=k(t)^2$, we obtain that

$$\int_s^t k(t-r)^2 dr=(2p-1)^{-1}\ln(1/(t-s))^{1-2p}. \tag{4.3}$$

Of course $\inf_{t\in(0,T]}\inf_{s\in[0,t)}\frac{|\ln(1/(t-s))|^{1-2p}}{(t-s)^{2\zeta}}>0$, so w^p is ζ-LND for any $\zeta>0$ and therefore infinitely regularizing.

□

4.2. **Infinite series of fractional Brownian motions.** We will here show that also the process considered in [1] is an infinitely regularizing process according to Definition 1.

Proposition 24. *Consider the process $\mathbb{B}:[0,T]\times\Omega\to\mathbb{R}^d$ introduced in* [1] *given by*

$$\mathbb{B}_t:=\sum_{n\geq 0}\lambda_n B_t^{H_n}.$$

Here $(\lambda_n)_{n\geq 0}$ and and $(H_n)_{n\geq 0}\in(0,1)$ are null sequences such that $\lambda_n,H_n>0$ for all $n\geq 0$. Moreover, $(B^{H_n})_{n\geq 0}$ is sequence of independent $\mathbb{R}^d$-valued fractional Brownian motion of Hurst parameter H_n. Additionally, we assume that

$$\sum_{n\geq 0}|\lambda_n|\mathbb{E}\left[\sup_{0\leq s\leq 1}|B_s^{H_n}|\right]<\infty. \tag{4.4}$$

Then there exists a null set $\mathcal{N}\subset\Omega$ such that $\mathbb{B}(\omega)$ is infinitely regularizing for all $\omega\in\mathcal{N}^c$.

Proof. For simplicity we assume that the fractional Brownian motions B^{H_n} are of Riemann Liouville type. However, the argument is readily extendible to other versions of the fractional Brownian motion, at the price of slightly more complicated computations. More precisely, we assume that B^{H_n} is given as the Wiener-Itô integral

$$B_t^{H_n}:=2H_n\int_0^t(t-s)^{H_n-\frac{1}{2}}dB_r^n, \tag{4.5}$$

where $(B^n)_{n\in\mathbb{N}}$ is a sequence of independent $\mathbb{R}^d$-valued Brownian motions and for convenience we chose the normalizing factor $2H_n$ instead of the usual $\Gamma(H_n+\frac{1}{2})^{-1}$. Since $\sum_{n\geq 0}|\lambda_n|\mathbb{E}\left[\sup_{0\leq s\leq 1}|B_s^{H_n}|\right]<\infty$ the process $\mathbb{B}$ is almost surely the uniform limit of continuous functions and therefore continuous itself. So to conclude the proof it suffices to show that $\mathbb{B}$ is ζ-LND for any $\zeta > 0$. The processes B^{H_n} and B^{H_m} are independent for $m \neq n$, and thus the conditional covariance is

$$\operatorname{cov}(\mathbb{B}_t|\mathcal{F}_s) = \sum_{n\in\mathbb{N}} \lambda_n^2 (t-s)^{2H_n} I_d. \tag{4.6}$$

Since $(H_n)_{n\in\mathbb{N}}$ is a null sequence there exists m such that $H_m < \zeta$, and then

$$\frac{\sum_n \lambda_n^2 (t-s)^{2H_n}}{(t-s)^{2\zeta}} \geq \lambda_m^2 T^{2(H_m-\zeta)} > 0,$$

where we used that $\lambda_m > 0$. This concludes the proof. □

5. Path-wise construction of infinitely regularizing averaging operators

In this section we investigate the spatio-temporal regularity of "averaging operators". For a continuous path $w \in C([0,T],\mathbb{R}^d)$ and a measurable function $b : \mathbb{R}^d \to \mathbb{R}^d$, we define the averaging operator T^w as

$$T^w_{s,t} b\,(x) := \int_s^t b\,(x + w_r)\,dr. \tag{5.1}$$

Such operators have previously been studied by Tao and Wright in [24] in the case of deterministic perturbations w, and more recently by Catellier and Gubinelli [7] in their study of the regularizing effect of fractional Brownian motions on ODEs.

Our first result is that if w is infinitely regularizing according to Definition 1, then the averaging operator T^w can be uniquely extended to any $b \in \mathscr{S}'$.

Recall that $\mathcal{C}^\alpha(\langle x\rangle^{-\kappa}))$ denotes the weighted Besov space of Definition 9, with weight $\langle x\rangle^{-\kappa} := (1+|x|^2)^{-\frac{\kappa}{2}}$

Proposition 25. *Let $w \in C([0,T],\mathbb{R}^d)$ be infinitely regularizing and let $b \in \mathscr{S}'$. There exist a $\kappa \in \mathbb{R}$ depending on b and a unique function*

$$T^w b \in \bigcap_{\substack{\gamma\in(0,1),\\ \alpha>0}} C^\gamma([0,T],\mathcal{C}^\alpha(\langle x\rangle^{-\kappa}))$$

such that $T^w_0 b \equiv 0$, and such that for any sequence of continuous functions $(b_n)_{n\in\mathbb{N}} \subset C(\mathbb{R}^d)\cap \mathscr{S}'$ that converges to b in $\mathscr{S}'$ we have

$$\lim_{n\to\infty} \|T^w b - T^w b_n\|_{C^\gamma([0,T],\mathcal{C}^\alpha(\langle x\rangle^{-\lambda}))} = 0$$

for some $\lambda \in \mathbb{R}$ and all $\gamma \in (0,1)$, $\alpha > 0$.

Proof. If b is continuous, then

$$T^w_{s,t} b(x) = \int_s^t b(x+w_r)dr = \int_{\mathbb{R}^d} b(x+z) L_{s,t}(z)dz = \langle b, L_{s,t}(\cdot - x)\rangle,$$

where L is the local time associated to w. Since $L_{s,t}(\cdot - x) \in C^\infty_c(\mathbb{R}^d)$, the right hand side makes sense for all $b \in \mathscr{S}'$ and we take it as the definition of $T^b_{s,t}(x)$. To see the claimed regularity note that for $b \in \mathscr{S}'$ there exist $\kappa \in \mathbb{R}$ and $k \geq 0$ such that $|\langle b, \varphi\rangle| \lesssim \max_{|\alpha|\leq k} \|\langle\cdot\rangle^\kappa \partial^\alpha \varphi\|_\infty$. In particular,

$$\begin{aligned} |T^b_{s,t}(x)| = |\langle b, L_{s,t}(\cdot - x)\rangle| &\lesssim \max_{|\alpha|\leq k} \|\langle\cdot\rangle^\kappa \partial^\alpha L_{s,t}(\cdot - x)\|_\infty \\ &\lesssim \sup_{z\in\mathbb{R}^d} \frac{\langle z\rangle^\kappa}{\langle z-x\rangle^\kappa} |t-s|^\gamma \|L\|_{C^\gamma([0,T],\mathcal{C}^{k+1}(\langle x\rangle^\kappa))} \\ &\lesssim \langle x\rangle^\kappa |t-s|^\gamma, \end{aligned}$$

where we used that $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ and that $\|L\|_{C^\gamma([0,T],\mathcal{C}^{k+1}(\langle x\rangle^\kappa))} \simeq \|L\|_{C^\gamma([0,T],\mathcal{C}^{k+1})}$ as L is compactly supported. To control the derivatives note that $T^w_{s,t}b$ is essentially a convolution, and thus $\partial^\beta T^w_{s,t}b(x) = \langle b, (-1)^\beta(\partial^\beta L_{s,t})(\cdot - x)\rangle$, from where the same arguments as above yield

$$|\partial^\beta T^w_{s,t}b(x)| \lesssim \langle x\rangle^\kappa |t-s|^\gamma, \tag{5.2}$$

and therefore $T^w b \in C^\gamma([0,T],\mathcal{C}^\alpha(\langle x\rangle^{-\kappa}))$ for all $\alpha > 0$ and all $\gamma \in (0,1)$. If a sequence of smooth functions $(b_n)_{n\in\mathbb{N}} \subset \mathscr{S}'$ converges to b in $\mathscr{S}'$, then there exist $\lambda \in \mathbb{R}$ and $\ell \geq 0$ such that $|\langle b_n, \varphi\rangle| \lesssim \max_{|\alpha|\leq\ell} \|\langle\cdot\rangle^\lambda \partial^\alpha\varphi\|_\infty$ uniformly in n, see [23, Theorem V.7]. Therefore, the convergence of $T^w b_n$ to $T^w b$ in $C^\gamma([0,T],\mathcal{C}^\alpha(\langle x\rangle^{-\lambda}))$ follows as above. □

Corollary 26. *If $b \in B^\beta_{p,q}$ for some $\beta \in \mathbb{R}$ and $p, q \in [1,\infty]$, then we have (without weights):*

$$T^w b \in \bigcap_{\substack{\gamma\in(0,1),\\ \alpha>0}} C^\gamma([0,T],\mathcal{C}^\alpha).$$

Proof. If $b \in B^\beta_{p,q}$, then we have with the conjugate exponents p', q' of p, q:

$$|\langle b, L_{s,t}(\cdot - x)\rangle| \lesssim \|b\|_{B^\beta_{p,q}} \|L_{s,t}(\cdot - x)\|_{B^{-\beta}_{p',q'}} = \|b\|_{B^\beta_{p,q}} \|L_{s,t}\|_{B^{-\beta}_{p',q'}} \lesssim \|L_{s,t}\|_{\mathcal{C}^{-\beta+\epsilon}},$$

where in the last step we used that L is compactly supported and therefore we can decrease the integrability index from ∞ to p' while only paying a constant, and that we can replace q' by ∞ if we give up ϵ regularity. This shows that we can take $\kappa = 0$ in the proof (and then in the statement) of Proposition 25. □

6. Existence, uniqueness and flow differentiability of perturbed ODEs

We are now ready to apply the concept of averaging operators to ODEs perturbed by noise. Formally, we will consider the equation

$$\tilde{y}^x_t = x + \int_0^t b\,(\tilde{y}^x_r)\,dr + w_t, \qquad (t,x) \in [0,T]\times\mathbb{R}^d, \tag{6.1}$$

for a Schwartz distribution b and an infinitely regularizing continuous path w. To interpret this equation rigorously, we set $y^x_t := \tilde{y}^x_t - w_t$, and observe that y formally solves

$$y^x_t = x + \int_0^t b\,(y^x_r + w_r)\,dr, \qquad (t,x) \in [0,T]\times\mathbb{R}^d. \tag{6.2}$$

To make sense of the integral on the right hand side we consider a sequence $(b_n)_{n\in\mathbb{N}}$ of continuous functions converging to $b \in \mathscr{S}'$. Then, inspired by the construction of the operator

$T^{w}b$ in Proposition 25, we will show that the following limit exists:

$$\int_0^t b\left(y_r+w_r\right)dr := \lim_{n\to\mathbb{N}}\int_0^t b_n\left(y_r+w_r\right)dr. \tag{6.3}$$

To this end we use the non-linear Young integral of [7], for which we first give a simplified construction.

6.1. **Non-linear Young integration.** Let $\varXi:\Delta_2^T\to\mathbb{R}^d$ and consider the Riemann sum of Ξ over a partition $\mathcal{P}$ of a set $[s,t]\subset[0,T]$:

$$\mathcal{I}_{\mathcal{P}}\left(\varXi\right)_{s,t}=\sum_{[u,v]\in\mathcal{P}}\varXi_{u,v}.$$

The *sewing lemma* ([12, Proposition 1], see also [10, Lemma 4.2]) gives explicit conditions on the function $\varXi$ under which $\lim_{|\mathcal{P}|\to 0}\mathcal{I}_{\mathcal{P}}\left(\varXi\right)$ exists. To state it, we first define the linear functional δ acting on $f:\Delta_2^T\to\mathbb{R}^d$ as

$$\delta_u f_{s,t}=f_{s,t}-f_{s,u}-f_{u,t},\qquad (s,u,t)\in\Delta_3^T. \tag{6.4}$$

Lemma 27 ([10], Lemma 4.2)**.** *Let $\alpha\in(0,1)$ and $\beta\in(1,\infty)$, and let $\varXi:\Delta_2^T\to\mathbb{R}^d$ be such that*

$$\|\delta\varXi\|_\beta:=\sup_{(s,u,t)\in\Delta_3^T}\frac{|\delta_u\varXi_{s,t}|}{|t-s|^\beta}<\infty\qquad \textit{and}\qquad \|\varXi\|_\alpha:=\sup_{(s,t)\in\Delta_2^T}\frac{|\varXi_{s,t}|}{|t-s|^\alpha}<\infty.$$

Then there exists a unique function $\mathcal{I}(\varXi)\in C^\alpha([0,T],\mathbb{R}^d)$ such that $\mathcal{I}(\varXi)_0=0$ and

$$|\mathcal{I}\left(\varXi\right)_{s,t}-\varXi_{s,t}|\le C\|\delta\varXi\|_\beta|t-s|^\beta,$$

where $C>0$ only depends on β and T. Moreover, we have $\mathcal{I}\left(\varXi\right)_{0,t}=\lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}}\varXi_{u,v}$.

Now let us consider again the integral in (6.2). If b is continuous, then

$$\begin{aligned}\int_0^t b\left(y_r+w_r\right)dr &= \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} b\left(y_u+w_u\right)(v-u)=\lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}}\int_u^v b\left(y_u+w_r\right)du\\ &=\lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} T^w_{v,u}b\left(y_u\right),\end{aligned} \tag{6.5}$$

where $T^w b$ is the average operator from (5.1). If w is infinitely regularizing and $\mathcal{P}$ is a fixed partition, then by Proposition 25 the sum on the right hand side is well defined even if only $b\in\mathscr{S}'$. The existence of the limit as $|\mathcal{P}|\to 0$ will follow from the sewing lemma. Note that the limit is not exactly a Young integral, since $T^w b$ is non-linear in its spatial argument:

$$T^w_{s,t}b\left(z+y\right)\neq T^w_{s,t}b\left(z\right)+T^w_{s,t}b\left(y\right).$$

Therefore, we need a non-linear extension of the Young integral, which was introduced by Catellier and Gubinelli in [7] and for which we give a simplified construction.

Proposition 28 (See also [7], Theorem 2.4 or [15], Proposition 2.4)**.** *Let $\beta,\gamma\in(0,1)$ be such that $\beta+\gamma>1$. Let $y\in C_T^\beta:=C_T^\beta\mathbb{R}^d$ and let $Y\in C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ be such that*

$$|\nabla Y_{s,t}(x)|\le F(x)|t-s|^\gamma,\qquad (s,t)\in\Delta_T^2, x\in\mathbb{R}^d,$$

where F is a locally bounded function. Then with $\Xi_{s,t} = Y_{s,t}(y_s)$, the integral

$$\int_s^t Y_{dr}(y_r) := \mathcal{I}(\Xi)_{s,t}, \tag{6.6}$$

is well defined according to Lemma 27.

Proof. Since $Y_{s,t} = Y_t - Y_s$, we have

$$|\delta_u Y_{s,t}(y_s)| = |Y_{u,t}(y_s) - Y_{u,t}(y_u)| \leq \sup_{|x| \leq \|y\|_\infty} F(x)|t-u|^\gamma |y_{s,u}| \leq \sup_{|x| \leq \|y\|_\infty} F(x)\|y\|_{C_T^\beta}|t-s|^{\gamma+\beta}.$$

So the result follows from Lemma 27. □

6.2. **Abstract non-linear Young equations.** Here we use the non-linear Young integral from Proposition 28 to construct solutions to an abstract non-linear integral equation. Later we will apply these abstract results to our equation (6.2).

Proposition 29. *Let $Y \in C^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ be such that for some $\gamma \in (\frac{1}{2}, 1)$ and $\delta > \frac{1}{\gamma}$ the following conditions hold for $s, t \in [0,T]$ and $x, y \in \mathbb{R}^d$:*

$$\begin{aligned} &\text{(i)} && |Y_{s,t}(x)| + |\nabla Y_{s,t}(x)| \leq G(x)|t-s|^\gamma, \\ &\text{(ii)} && |\nabla Y_{s,t}(x) - \nabla Y_{s,t}(y)| \leq F(x,y)|t-s|^\gamma |x-y|^{\delta-1}, \end{aligned}$$

where $G : \mathbb{R}^d \to \mathbb{R}_+$ and $F : \mathbb{R}^{2d} \to \mathbb{R}_+$ are locally bounded functions. Then for all $x \in \mathbb{R}^d$ there is a maximal existence time $T^* = T^*(x) \in (0,T] \cup \{\infty\}$ and a unique solution $y \in C^\gamma([0,T^*) \cap [0,T])$ to

$$y_t = x + \int_0^t Y_{dr}(y_r). \tag{6.7}$$

Here the non-linear Young integral $\int_0^t Y_{dr}(y_r)$ is as in Proposition 28. If $T^* < \infty$, then $\lim_{t \to T^*} |y_t| = \infty$. Moreover, the map $x \mapsto T^*(x)^{-1}$ is locally bounded. If G and F are bounded, then $T^* = \infty$.

Proof. This is quite standard and the result follows from an application of the non-linear sewing lemma, Proposition 28, together with a Picard iteration. For completeness we include the arguments.

Let $\tau \in [0,T]$ and $\gamma' \in (1-\gamma, \gamma)$ be such that $\gamma + \delta(1-\gamma') > 1$ (note that $\gamma + \delta(\gamma - 1) = \delta\gamma > 1$, so this is possible). Let $z \in C_\tau^{\gamma'}$. Define the increment $\Xi_{s,t} := Y_{s,t}(z_s)$. Then we obtain from (i):

$$\begin{aligned} |\Xi_{s,t}| &\leq G(z_s)|t-s|^\gamma, \\ |\delta_u \Xi_{s,t}| &= |Y_{u,t}(z_s) - Y_{u,t}(z_u)| \leq \sup_{|a| \leq \|z\|_\infty} G(a)\|z\|_{C_\tau^{\gamma'}}|t-s|^{\gamma+\gamma'}, \end{aligned} \tag{6.8}$$

where $C_\tau^{\gamma'} = C^{\gamma'}([0,\tau])$. Since $\gamma + \gamma' > 1$ it follows from Lemma 27 that the map

$$\Gamma : \left\{ z \in C^{\gamma'}([0,\tau], \mathbb{R}^d) \middle| z_0 = x \right\} \to \left\{ z \in C^{\gamma'}([0,\tau], \mathbb{R}^d) \middle| z_0 = x \right\},$$

$$\Gamma(z)_t = x + \int_0^t Y_{dr}(z_r)$$

is well defined and satisfies

$$\begin{aligned}|\Gamma(z)_{s,t}| &\leq \left|\int_s^t Y_{dr}\left(z_r\right)-Y_{s,t}\left(z_s\right)\right|+\left|Y_{s,t}\left(z_s\right)\right| \\ &\lesssim |t-s|^{\gamma+\gamma'} \sup_{|a|\leq\|z\|_\infty} G(a)\|z\|_{C_\tau^{\gamma'}}+\sup_{|a|\leq\|z\|_\infty} G(a)|t-s|^\gamma \\ &\lesssim \tau^{\gamma-\gamma'}|t-s|^{\gamma'} \sup_{|a|\leq\|z\|_\infty}\Big(G\left(a\right)\|z\|_{C_\tau^{\gamma'}}+G(a)\Big).\end{aligned}$$

This implies that for sufficiently small $\tau>0$ (depending on $|x|$) the map Γ leaves the ball

$$\mathcal{B}_{2|x|}=\Big\{z\in C_\tau^{\gamma'}\big|z_0=x,\ \|z\|_\infty\vee\|z\|_{C_\tau^{\gamma'}}\leq 2|x|\Big\}$$

invariant. Moreover, for two paths $z,\tilde{z}\in\mathcal{B}_{2|x|}$ we have

$$\left|\Gamma\left(z\right)_{s,t}-\Gamma\left(\tilde{z}\right)_{s,t}\right|\leq\left|Y_{s,t}\left(z_s\right)-Y_{s,t}\left(\tilde{z}_s\right)\right|+\left|\int_s^t\left[Y_{dr}\left(z_r\right)-Y_{dr}\left(\tilde{z}_r\right)\right]-\left[Y_{s,t}\left(z_s\right)-Y_{s,t}\left(\tilde{z}_s\right)\right]\right|.$$

For $u\in[s,t]$ we rewrite

$$\begin{aligned}\delta_u\left[Y_{s,t}\left(z_s\right)-Y_{s,t}\left(\tilde{z}_s\right)\right] &= \left(Y_{u,t}(z_s)-Y_{u,t}(\tilde{z}_s)\right)-\left(Y_{u,t}\left(z_u\right)-Y_{u,t}\left(\tilde{z}_u\right)\right) \\ &= \int_0^1\nabla Y_{u,t}\left(\tilde{z}_s+\lambda\left(z_s-\tilde{z}_s\right)\right)\cdot\left(z_s-\tilde{z}_s\right)d\lambda \\ &\quad-\int_0^1\nabla Y_{u,t}\left(\tilde{z}_u+\lambda\left(z_u-\tilde{z}_u\right)\right)\cdot\left(z_u-\tilde{z}_u\right)d\lambda\end{aligned}\tag{6.9}$$

Invoking condition (ii) on the function Y, we observe that

$$\begin{aligned}&\left|\delta_u\left[Y_{s,t}\left(z_s\right)-Y_{s,t}\left(\tilde{z}_s\right)\right]\right| \\ &\leq\left|\int_0^1\left[\nabla Y_{u,t}\left(\tilde{z}_s+\lambda\left(z_s-\tilde{z}_s\right)\right)-\nabla Y_{u,t}\left(\tilde{z}_u+\lambda\left(z_u-\tilde{z}_u\right)\right)\right]\cdot\left(z_s-\tilde{z}_s\right)d\lambda\right| \\ &\quad+\left|\int_0^1\nabla Y_{u,t}\left(\tilde{z}_s+\lambda\left(z_s-\tilde{z}_s\right)\right)\cdot\left(z_s-\tilde{z}_s-z_u-\tilde{z}_u\right)d\lambda\right| \\ &\leq\int_0^1 F\left(\tilde{z}_s+\lambda\left(z_s-\tilde{z}_s\right),\tilde{z}_u+\lambda\left(z_u-\tilde{z}_u\right)\right)|t-u|^\gamma \\ &\qquad\times\left|\tilde{z}_s+\lambda\left(z_s-\tilde{z}_s\right)-\left(\tilde{z}_u+\lambda\left(z_u-\tilde{z}_u\right)\right)\right|^{\delta-1}d\lambda\|z-\tilde{z}\|_\infty \\ &\quad+\int_0^1 G(\tilde{z}_s+\lambda(z_s-\tilde{z}_s))|t-u|^\gamma d\lambda|t-s|^{\gamma'}\|z-\tilde{z}\|_{C_\tau^{\gamma'}} \\ &\lesssim\|F\|_{\mathcal{B}_{2|x|}}|t-s|^{\gamma+\gamma'(\delta-1)}|2x|^{\delta-1}\|z-\tilde{z}\|_{C_\tau^{\gamma'}}+|t-s|^{\gamma+\gamma'}\|G\|_{\mathcal{B}_{2|x|}}\|z-\tilde{z}\|_{C_\tau^{\gamma'}},\end{aligned}$$

where

$$\|G\|_{\mathcal{B}_{2|x|}}=\sup_{|a|\leq 2|x|}G(a),\qquad\|F\|_{\mathcal{B}_{2|x|}}=\sup_{|a|,|b|\leq 2|x|}F(a,b).$$

Recall that $\gamma+(\delta-1)\gamma'>1$. Furthermore, the bound in (i) gives

$$\left|Y_{s,t}\left(z_s\right)-Y_{s,t}\left(\tilde{z}_s\right)\right|\leq\|G\|_{\mathcal{B}_{2|x|}}|t-s|^\gamma\|z-\tilde{z}\|_\infty\leq\|G\|_{\mathcal{B}_{2|x|}}\tau^{\gamma'}|t-s|^\gamma\|z-\tilde{z}\|_{C_\tau^{\gamma'}},$$

where we used that $z_0 = \tilde{z}_0 = x$, and therefore $\|z - \tilde{z}\|_\infty \le \tau^{\gamma'} \|z - \tilde{z}\|_{C_\tau^{\gamma'}}$. So after possibly further decreasing $\tau > 0$, depending on $|x|$, we get a contraction on $\mathcal{B}_{2|x|}$. Since the maximum possible choice for τ only depends on $|x|$ and it is bounded away from 0 if $|x|$ is bounded, we can choose $\tau(x)$ such that the map $x \mapsto \tau(x)^{-1}$ is locally bounded.

Moreover, it is a simple exercise to check that for $z \in C_\tau^\gamma$ we have $\Gamma(z) \in C_\tau^\gamma$, so the unique fixed point $(y_t)_{t\in[0,T]}$ is even γ-Hölder continuous. Now we can iterate the construction and extend the solution to $[0, \tau + \tau']$ for some $\tau' \le \tau$, etc. We just showed that $\tau(x)$ only depends on the size of the initial condition $|x|$, so if we would have $\sup_{t\in[0,T^*)} |y_t| < \infty$ and $T^* < T$, then we could extend the solution beyond T^* and thus T^* could not have been the maximal time of existence. Since $T^* > \tau$ the local boundedness of $x \mapsto T^*(x)^{-1}$ follows from that of $x \mapsto \tau(x)^{-1}$.

If F and G are bounded, then there exists a fixed $\tau > 0$ such that for any starting point x the map Γ leaves the ball $\mathcal{B}_{2|x|}$ invariant. Therefore, the solution y with initial value x satisfies $\sup_{t\in[0,\sigma]} |y_\sigma| \le 2^{\lceil \frac{\sigma}{\tau} \rceil} |x|$ on any interval $[0, \sigma] \subset [0, T]$, and it does not explode in finite time, i.e. $T^* = \infty$. □

6.3. **Application to perturbed ODEs.** We will now apply the abstract results from the previous section to define solutions to Equation (6.2) and to prove their existence and uniqueness and the smoothness of the associated flow.

Lemma 30. *Let w be infinitely regularizing, let $b \in \mathscr{S}'$, and let $T^w b$ the averaging operator defined in* (5.1)*. Then for all $\epsilon > 0$ and all $y \in C^\epsilon([0,T], \mathbb{R}^d)$ the non-linear Young integral $\int_0^t T^w_{dr} b(y_r)$ is well defined.*

Proof. By Proposition 25 the function $Y_t(x) = T^w_t b(x)$ satisfies $|\nabla Y_{s,t}(x)| \lesssim \langle x \rangle^\kappa |t - s|^\gamma$ for some $\kappa \in \mathbb{R}$ and for all $\gamma < 1$. In particular we can choose $\gamma > 1 - \epsilon$, and then the claim follows from Proposition 28. □

Definition 31. Let w be infinitely regularizing, let $b \in \mathscr{S}'$ and let $\tau \le T$ and $\tilde{y} \in C([0,\tau], \mathbb{R}^d)$. Then we say that $\tilde{y}$ solves the equation

$$\tilde{y}_t = x + \int_0^t b(\tilde{y}_r) dr + w_t, \tag{6.10}$$

if $y = \tilde{y} - w$ is in $C^\epsilon([0,\tau], \mathbb{R}^d)$ for some $\epsilon > 0$ and

$$y_t = x + \int_0^t T^w_{dr} b(y_r), \qquad t \in [0, \tau]. \tag{6.11}$$

Lemma 32. *Let w be infinitely regularizing, let $b \in \mathscr{S}'$ and let $\tau \le T$ and $\tilde{y} \in C([0,\tau], \mathbb{R}^d)$ be such that $y = \tilde{y} - w$ is in $C^\epsilon([0,\tau], \mathbb{R}^d)$ for some $\epsilon > 0$. Then $\tilde{y}$ solves* (6.10) *if and only if for any sequence $(b_n) \subset C(\mathbb{R}^d) \cap \mathscr{S}'$ converging to b in $\mathscr{S}'$ we have*

$$\tilde{y}_t = x + \lim_{n\to\infty} \int_0^t b_n(\tilde{y}_r) dr + w_t, \qquad t \in [0, \tau].$$

Proof. By the convergence result for the average operator in Proposition 25 together with continuity properties of the non-linear Young integral which follow from Lemma 27 we have $\int_0^t T^w_{dr} b(y_r) = \lim_{n\to\infty} \int_0^t T^w_{dr} b_n(y_r)$ for $t \in [0, \tau]$. Therefore, the claim follows from (6.5). □

Lemma 33. *Let w be infinitely regularizing, let $b \in \mathscr{S}'$ and let $\gamma \in (\frac{1}{2}, 1)$. For all $x \in \mathbb{R}^d$ there exists a maximal existence time $T^* = T^*(x) \in (0, T] \cup \{\infty\}$ and a unique solution $y \in C^\gamma([0, T^*) \cap [0, T])$ to*

$$y_t = x + \int_0^t T^w_{dr} b(y_r), \qquad t \in [0, T^*) \cap [0, T]. \tag{6.12}$$

If $T^* < \infty$, then $\lim_{t \to T^*} |y_t| = \infty$. Moreover, the map $x \mapsto T^*(x)^{-1}$ is locally bounded. If $b \in B^\beta_{p,q}$ for some $\beta \in \mathbb{R}$ and $p, q \in [1, \infty]$, then $T^* = \infty$.

Proof. According to Proposition 25 there exists $\kappa \in \mathbb{R}$ such that $Y_t(x) = T^w_t b(x)$ is in $C^\gamma_T \mathcal{C}^\alpha(\langle x \rangle^{-\kappa})$ for all $\gamma < 1$ and $\alpha > 0$. In particular it satisfies the assumptions of Proposition 29. If $b \in B^\beta_{p,q}$, then $Y \in C^\gamma_T \mathcal{C}^\alpha$ (without weight), and therefore the global existence follows from the last part of Proposition 29. □

To complete the proof of Theorem 2 we have to show the differentiability of the flow $x \mapsto y^x$, where y^x solves the equation with $y^x_0 = x$. We will achieve this by solving the equation for $(y^x, \nabla y^x, \ldots, \nabla^k y^x)$, whose explosion time a priori might depend on k. To show that it is independent of k and that the flow exists as long as y^x stays bounded, we introduce an abstract notion:

Definition 34. Let $k \geq 0$ and $d = d_0 + \ldots + d_k$ and let $Y \in C^{0,1}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ be of the form

$$Y_t(z) = [Y^0_t(z^0), Y^1_t(z^{\leq 1}), \ldots, Y^k_t(z^{\leq k})] \tag{6.13}$$

for all $z = (z^0, z^1, \ldots, z^k) \in \mathbb{R}^{d_0 + \ldots + d_k}$, where $z^{\leq \ell} := (z^0, \ldots, z^\ell)$. Let $\gamma \in (\frac{1}{2}, 1)$, $\delta > \frac{1}{\gamma}$ and assume that Y^0 satisfies the condition of Proposition 29, while each of the components Y^ℓ for $\ell \in \{1, \ldots, k\}$ satisfies the following three bounds:

(i) $$|Y^\ell_{s,t}(z^{\leq \ell})| \leq G_\ell(z^{\leq \ell - 1})(1 + |z^\ell|)|t - s|^\gamma,$$

(ii) $$|Y^\ell_{s,t}(z^{\leq \ell}) - Y^\ell_{s,t}(\tilde{z}^{\leq \ell})| \leq |t - s|^\gamma H_\ell(z^{\leq \ell - 1}, \tilde{z}^{\leq \ell - 1}) \times (|z^\ell - \tilde{z}^\ell| + |z^\ell| \times |z^{\leq \ell - 1} - \tilde{z}^{\leq \ell - 1}|),$$

(iii) $$|\nabla Y^\ell_{s,t}(z^{\leq \ell}) - \nabla Y^\ell_{s,t}(\tilde{z}^{\leq \ell})| \leq F_\ell(z^{\leq \ell}, \tilde{z}^{\leq \ell})|t - s|^\gamma |z^{\leq \ell} - \tilde{z}^{\leq \ell}|^{\delta - 1},$$

where the functions G_ℓ, H_ℓ and F_ℓ are positive and locally bounded. Then we say that Y has a *lower triangular structure.*

If Y has a lower triangular structure, then the maximal existence time of $y_t = x + \int_0^t Y_{dr}(y_r)$ is equal to the explosion time of y^0:

Lemma 35. *Assume that Y has a lower triangular structure and let y be the solution to $y_t = x + \int_0^t Y_{dr}(y_r)$, constructed in Proposition 29, with maximal existence time $T^* \in (0, T] \cup \{\infty\}$. If $T^* < \infty$, then $\lim_{t \to T^*} |y^0_t| = \infty$.*

Proof. By definition Y satisfies the conditions of Proposition 29, so y exists. Assume that $T^* < \infty$ and that $\sup_{t < T^*} |y^0_t| = C < \infty$. We claim that then also $\sup_{t < T^*} |y^{\leq \ell}_t| < \infty$ for all $\ell \leq k$, which is a contradiction to the fact that $\sup_{t < T^*} |y_t| = \infty$ by Proposition 29.

Assume that the claim holds for $\ell - 1$ and let us show that then it also holds for ℓ. Because of the lower triangular structure, $y^{\leq \ell - 1}$ solves a non-linear Young equation with non-linearity $Y^{\le \ell-1}$, and since $\sup_{t<T^*}|y_t^{\le\ell-1}|<\infty$ we deduce from Proposition 29 that also $\sup_{t<T^*}\|y^{\le \ell-1}\|_{C_t^\gamma}<\infty$.

To obtain a bound for $|y^\ell|$ let $\Xi^\ell_{s,t}=Y^\ell_{s,t}\left(y_s^{\le \ell}\right)$. Then there exists a constant $C>0$ which depends on $\sup_{t<T^*}|y_t^{\le\ell-1}|$ and $\sup_{t<T^*}\|y^{\le\ell-1}\|_{C_t^\gamma}$ such that

$$|\Xi^\ell_{s,t}|\le G_\ell(y_s^{\le \ell-1})(1+|y_s^\ell|)|t-s|^\gamma\le |t-s|^\gamma C\left(1+|y_0^\ell|+\tau^{\gamma'}\|y^\ell\|_{C_\tau^{\gamma'}}\right), \tag{6.14}$$

$$\begin{aligned}|\delta_u\Xi^\ell_{s,t}|&=|Y_{u,t}(y_u^{\le\ell})-Y_{u,t}(y_s^{\le\ell})|\\&\le |t-s|^\gamma H_\ell(y_s^{\le\ell-1},y_u^{\le \ell-1})(|y_u^\ell-y_s^\ell|+|y_s^\ell|\times|y_u^{\le\ell-1}-y_s^{\le \ell-1}|)\\&\le |t-s|^{\gamma+\gamma'}C\left(\|y^\ell\|_{C_\tau^{\gamma'}}+|y_0^\ell|+\tau^{\gamma'}\|y^\ell\|_{C_\tau^{\gamma'}}\right).\end{aligned} \tag{6.15}$$

With the help of these bounds we obtain from the sewing lemma (Lemma 27):

$$\begin{aligned}|y^\ell_{s,t}|&=\left|\int_s^t Y^\ell_{dr}(y_r^{\le\ell})\right|\\&\lesssim \tau^{\gamma-\gamma'}|t-s|^{\gamma'}C\left(1+|y_0^\ell|+\tau^{\gamma'}\|y^\ell\|_{C_\tau^{\gamma'}}\right)\\&\quad+\tau^\gamma|t-s|^{\gamma'}C\left(\|y^\ell\|_{C_\tau^{\gamma'}}+|y_0^\ell|+\tau^{\gamma'}\|y^\ell\|_{C_\tau^{\gamma'}}\right).\end{aligned}$$

Therefore, there exists $\tau>0$ which only depends on C and γ,γ' such that

$$\|y^\ell\|_{C_\tau^{\gamma'}}\vee\sup_{t\le\tau}|y_t^\ell|\le 2|y_0^\ell|.$$

Since τ is fixed and does not depend on y_0^ℓ we deduce that $\sup_{t<T^*}|y_t^\ell|\le 2^{\left\lceil\frac{T^*}{\tau}\right\rceil}|y_0^\ell|<\infty$ and this concludes the proof. □

Now we are ready to prove Theorem 2:

Theorem (Theorem 2)**.** *Let $b\in\mathscr{S}'$ be a Schwartz distribution, and consider an infinitely regularizing path $w:[0,T]\to\mathbb{R}^d$ as in Definition 1. Then for all $x\in\mathbb{R}^d$ there exists $T^*=T^*(x)\in(0,T]\cup\{\infty\}$ such that there is a unique solution to the equation*

$$y_t^x=x+\int_0^t b\left(y_r^x\right)dr+w_t,$$

in $C\left([0,T^*)\cap[0,T],\mathbb{R}^d\right)$. For $T^*(x)<\infty$ we have $\lim_{t\uparrow T^*(x)}|y_t^x|=\infty$. Moreover, the map $x\mapsto T^*(x)^{-1}$ is locally bounded, and if $\tau<T^*(x)$ for all $x\in U$ with an open set U, then the flow mapping $U\ni x\mapsto y_\cdot^x\in C([0,\tau],\mathbb{R}^d)$ is infinitely Fréchet differentiable.

Proof. It remains to prove the smoothness of the flow. Let $k\in\mathbb{N}$. We define

$$Y^\ell_{s,t}(z^{\le\ell})=\sum_{j=1}^{\ell}\sum_{i_1+\ldots+i_j=\ell}\nabla^jT^w_{s,t}b(z^0)\left(z^{i_1}\otimes\ldots\otimes z^{i_j}\right),\qquad \textit{for}\qquad 0\le\ell\le k.$$

Since Y^ℓ is an affine function of z^ℓ it is not hard to see that $Y=(Y^1,\ldots,Y^k)$ has a lower triangular structure. Let now $x\in\mathbb{R}^d$, let U be an open neighborhood of x and let $\tau\in[0,T]$

be such that $T^*(z) > \tau$ for all $x' \in U$. For $x' \in U$ assume that $z^{x'} = (z^{0,x'}, \dots, z^{k,x'})$ solves

$$z_t^{x'} = \chi + \int_0^t Y_{dr}(z_r^{x'})$$

on the maximum existence interval, where

$$\chi = (x', I_d, 0, \dots, 0)$$

for the d-dimensional unit matrix I_d. Then $z^{0,x'} = y^{x'}$ by definition of Y, and therefore Lemma 35 shows that $z^{x'}$ exists on $[0,\tau]$. We claim that $z^{\ell,x} = \nabla^\ell y^x$, as a Fréchet derivative in $C([0,\tau],\mathbb{R}^d)$ equipped with the uniform norm. Below we prove this for $\ell = 1$, the general case is similar but the notation becomes more involved.

Before we prove the first order differentiability we first show local Lipschitz continuity. So let $x' \in U$ and define the integrand $\Xi_{s,t} = T^w_{s,t}b(y_s^{x'}) - T^w_{s,t}b(y_s^x)$. Then $y^x - y^{x'}$ is the sewing of Ξ. There exists a constant C that depends on y^x and $y^{x'}$ such that for $0 \le s \le t \le \sigma \le \tau$:

$$\begin{aligned}
|\Xi_{s,t}| &\le C|t-s|^\gamma \sup_{t\le\sigma} |y_t^{x'} - y_t^x| \le C|t-s|^\gamma (|x'-x| + \sigma^{\gamma'} \|y^{x'} - y^x\|_{C_\sigma^{\gamma'}}) \\
|\delta_u \Xi_{s,t}| &= |(T^w_{u,t}b(y_s^{x'}) - T^w_{u,t}b(y_u^{x'})) - (T^w_{u,t}b(y_s^x) - T^w_{u,t}b(y_u^x))| \\
&\le C|t-s|^{\gamma+\gamma'} \|y^{x'} - y^x\|_{C_\sigma^{\gamma'}}.
\end{aligned}$$

So by the sewing lemma (Lemma 27) we get for a new $C > 0$:

$$|y_{s,t}^x - y_{s,t}^{x'}| \le C|t-s|^{\gamma'} \sigma^{\gamma-\gamma'} (|x'-x| + \|y^{x'} - y^x\|_{C_\sigma^{\gamma'}}),$$

and therefore we have for sufficiently small σ (depending only on C):

$$\|y^{x'} - y^x\|_{C_\sigma^{\gamma'}} \vee \sup_{t\le\sigma} |y_t^{x'} - y_t^x| \le 2|x'-x|,$$

and then iteratively

$$\|y^{x'} - y^x\|_{C_\tau^{\gamma'}} \vee \|y^{x'} - y^x\|_\infty \lesssim |x'-x|,$$

which proves the local Lipschitz continuity.

Next we want to show that $z^{1,x}$ is the Fréchet derivative of y^x in x. For that purpose we define the new integrand

$$\Xi_{s,t} = T^w_{s,t}b(y_s^{x'}) - T^w_{s,t}b(y_s^x) - \nabla T^w_{s,t}(y_s^x) z_s^{1,x}(x'-x).$$

Then $y^x - y^{x'} - z^{1,x}(x'-x)$ is the sewing of Ξ. There exists a $C > 0$ that depends on $y^x, y^{x'}, z^{1,x}$, such that for $0 \le s \le t \le \sigma \le \tau$:

$$\begin{aligned}|\Xi_{s,t}| &\leq C|t-s|^\gamma \left(\|y^{x'}-y^x\|_\infty^2 + \sup_{r\leq\sigma}|y_r^{x'}-y_r^x - z_r^{1,x}(x'-x)|\right)\\ &\lesssim C|t-s|^\gamma\left(|x'-x|^2+\sigma^{\gamma'}\|y^{x'}-y^x-z^{1,x}(x'-x)\|_{C_\sigma^{\gamma'}}\right)\\ |\delta_u\Xi_{s,t}| &= \Big|(T_{u,t}^w b(y_s^{x'})-T_{u,t}^w b(y_u^{x'}))-(T_{u,t}^w b(y_s^x)-T_{u,t}^w b(y_u^x))\\ &\qquad -(\nabla T_{u,t}^w(y_s^x)z_s^{1,x}(x'-x)-\nabla T_{u,t}^w(y_u^x)z_u^{1,x}(x'-x))\Big|\\ &\leq C|t-s|^{\gamma+\gamma'}\left(|x-x|^2+\|y^{x'}-y^x-z^{1,x}(x'-x)\|_{C_\sigma^{\gamma'}}\right).\end{aligned}$$

From here we obtain as before that

$$\|y^{x'}-y^x-z^{1,x}(x'-x)\|_{C_\sigma^{\gamma'}} \vee \|y^{x'}-y^x-z^{1,x}(x'-x)\|_\infty \lesssim |x-x'|^2,$$

and therefore $z^{1,x}$ is indeed the Fréchet derivative of y^x.

So far we showed that y^x is k times Fréchet differentiable, but since k was arbitrary y^x is infinitely Fréchet differentiable as claimed. □

References

[1] Oussama Amine, David Banos, and Frank Proske. C-infinity-regularization by noise of singular ode's, 2017.

[2] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[3] David Banos, Torstein Kastberg Nilssen, and Frank Proske. *Strong existence and higher order Frechet differentiability of stochastic flows of fractional brownian motion driven SDEs with singular drift.* Arxiv 1511 02717v1, Nov-2015.

[4] Richard F. Bass and Zhen-Qing Chen. Stochastic differential equations for Dirichlet processes. *Probab. Theory Related Fields*, 121(3):422–446, 2001.

[5] Simeon M. Berman. Local times and sample function properties of stationary Gaussian processes. *Trans. Amer. Math. Soc.*, 137:277–299, 1969.

[6] Vladimir I. Bogachev. *Gaussian measures*, volume 62 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[7] Rémi Catellier and Massimiliano Gubinelli. Averaging along irregular curves and regularisation of ODEs. *Stochastic Process. Appl.*, 126(8):2323–2366, 2016.

[8] Jack Cuzick and Johannes P. DuPreez. Joint continuity of Gaussian local times. *Ann. Probab.*, 10(3):810–817, 1982.

[9] Peter Friz and Atul Shekhar. Doob-Meyer for rough paths. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 8(1):73–84, 2013.

[10] Peter K. Friz and Martin Hairer. *A course on rough paths.* Universitext. Springer, Cham, 2014. With an introduction to regularity structures.

[11] Donald Geman and Joseph Horowitz. Occupation densities. *Ann. Probab.*, 8(1):1–67, 02 1980.

[12] Massimiliano Gubinelli. Controlling rough paths. *J. Funct. Anal.*, 216(1):86–140, nov 2004.

[13] Archil Gulisashvili. Time-inhomogeneous gaussian stochastic volatility models: Large deviations and super roughness, 2020.

[14] Martin Hairer and Natesh S. Pillai. Regularity of laws and ergodicity of hypoelliptic SDEs driven by rough paths. *Ann. Probab.*, 41(4):2544–2598, 2013.

[15] Yaozhong Hu and Khoa Lê. Nonlinear Young integrals and differential systems in Hölder media. *Trans. Amer. Math. Soc.*, 369(3):1935–2002, 2017.

[16] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005.
[17] Khoa Lê. A stochastic sewing lemma and applications. *Arxiv: 1810.10500*, 2018.
[18] Jörg Martin. *Refinements of the Solution Theory for Singular SPDEs*. PhD thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät, 2018.
[19] Olivier Menoukeu-Pamen, Thilo Meyer-Brandis, Torstein Nilssen, Frank Proske, and Tusheng Zhang. A variational approach to the construction and Malliavin differentiability of strong solutions of SDE's. *Math. Ann.*, 357(2):761–799, 2013.
[20] Thilo Meyer-Brandis and Frank Proske. Construction of strong solutions of SDE's via Malliavin calculus. *J. Funct. Anal.*, 258(11):3922–3953, 2010.
[21] Oana Mocioalca and Frederi Viens. Skorohod integration and stochastic calculus beyond the fractional brownian scale. *Journal of Functional Analysis*, 222(2):385 – 434, 2005.
[22] Loren D. Pitt. Local times for Gaussian vector fields. *Indiana Univ. Math. J.*, 27(2):309–330, 1978.
[23] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.
[24] Terence Tao and James Wright. L^p improving bounds for averages along curves. *J. Amer. Math. Soc.*, 16(3):605–638, 2003.
[25] Hans Triebel. *Theory of function spaces. III*, volume 100 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006.
[26] Alexander Yu.. Veretennikov. On strong solution and explicit formulas for solutions of stochastic integral equations. *Math. USSR Sb.*, 39:387–403, 1981.
[27] , Yi Min Xiao. Dimension results for Gaussian vector fields and index-α stable fields. *Ann. Probab.*, 23(1):273–291, 1995.
[28] Yimin Xiao. Properties of local-nondeterminism of Gaussian and stable random fields and their applications. *Ann. Fac. Sci. Toulouse Math. (6)*, 15(1):157–193, 2006.
[29] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974.

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