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A weak law of large numbers for realised covariation in a Hilbert space setting

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Abstract

This article generalises the concept of realised covariation to Hilbert-space-valued stochastic processes. More precisely, based on high-frequency functional data, we construct an estimator of the trace-class operator-valued integrated volatility process arising in general mild solutions of Hilbert space-valued stochastic evolution equations in the sense of Da Prato and Zabczyk (2014). We prove a weak law of large numbers for this estimator, where the convergence is uniform on compacts in probability with respect to the Hilbert–Schmidt norm. In addition, we determine convergence rates for common stochastic volatility models in Hilbert spaces.

Keywords: Law of large numbers; High-frequency estimation; Quadratic covariation; Volatility; Hilbert space; Evolution equations

1. Introduction

Stochastic volatility and covariance estimation are of key importance in many fields. Motivated in particular by financial applications, a lot of research has been devoted to constructing suitable (co-) volatility estimators and to deriving their asymptotic limit theory in the setting when discrete, high-frequent observations are available. Initially, the main interest was in (continuous-time) stochastic models based on (Itô) semimartingales, where the so-called realised variance and covariance estimators (and their extensions) proved to be powerful tools. Relevant articles include the works by [3,7–9,32], amongst many others, and the textbooks by [1,33].

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Subsequently, the theory was extended to cover non-semimartingale models, see, for instance, [4,5,21–23] and the survey by [41], where the proofs of the asymptotic theory rely on Malliavin calculus and the famous fourth-moment theorem, see [37]. The multivariate theory has been studied in [30,39].

Common to these earlier lines of investigation is the fact that the stochastic processes considered have finite dimensions. In this article, we extend the concept of realised covariation to an infinite-dimensional framework.

The estimation of covariance operators is elementary in the field of functional data analysis and was elaborated mainly for discrete-time series of functional data (see e.g. [16,27,31,38,42,46]). However, spatio-temporal data that can be considered as functional might also be sampled densely in time, like forward curves for interest rates or commodities and data from geophysical and environmental applications.

In this paper, we consider a separable Hilbert space H and study H-valued stochastic processes Y of the form

$$Y_t = \mathcal{S}(t)Y_0 + \int_0^t \mathcal{S}(t-s)\alpha_s ds + \int_0^t \mathcal{S}(t-s)\sigma_s dW_s, \quad t \in [0,T], \tag{1}$$

for some $T > 0$. Here $(\mathcal{S}(t))_{t\geq 0}$ is a strongly continuous semigroup, $\alpha := (\alpha_t)_{t\in[0,T]}$ a predictable and almost surely integrable H-valued stochastic process, $\sigma := (\sigma_t)_{t\in[0,T]}$ is a predictable operator-valued process, Y_0 with values in H is some initial element and W a so called Q-Wiener process on H (see Section 2 for details).

Our aim is to construct an estimator for the integrated covariance process

$$\left(\int_0^t \sigma_s Q \sigma_s^* ds\right)_{t\in[0,T]}.$$

More precisely, we denote by

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (Y_{t_i} - \mathcal{S}(\Delta_n)Y_{t_{i-1}})^{\otimes 2}, \tag{2}$$

the *semigroup-adjusted realised covariation (SARCV)* for an equally spaced grid $t_i := i\Delta_n$ for $\Delta_n = 1/n$, $i = 1, \ldots, \lfloor t/\Delta_n \rfloor$. We prove uniform convergence in probability (ucp) with respect to the Hilbert–Schmidt norm of the (SARCV) to the integrated covariance process. It is in line with the finite dimensional theory for continuous semimartingales that, apart from the necessary assumptions for stochastic integrability, no assumptions have to be imposed on the stochastic volatility process σ to guarantee the validity of this weak law of large numbers. In that sense, the (SARCV) can be regarded as a natural generalisation of the well known realised quadratic (co-)variation in finite dimensions (which is a special case) to processes of the form (1), which are sometimes coined *mild Itô processes*, c.f. Da Prato et al. [24, section 2].

Nevertheless, our framework certainly differs from common high-frequency settings mainly due to peculiarities that arise from infinite dimensions. Observe that the main motivation to consider processes in this form, is that a vast amount of parabolic stochastic partial differential equations possess only mild (in opposition to analytically strong) solutions, which are of the form (1). That is, Y is (under weak conditions) the mild solution of a stochastic partial differential equation

$$\text{(SPDE)} \quad dX_t = (AX_t + \alpha_t)dt + \sigma_t dW_t, \quad X_0 = Y_0, \quad t \in [0,T],$$

where A is the infinitesimal generator of the semigroup $(\mathcal{S}(t))_{t\geq 0}$ (cf. [25,40] or [35]).

In contrast to finite-dimensional stochastic diffusions, this is a priori not an H-valued semimartingale, but rather an H-valued Volterra process and under certain conditions on the volatility, the rate of convergence can be affected. For instance, in the case of a constant deterministic volatility, the rate is $\mathcal{O}(\Delta_n^{1/2})$ in the semimartingale case, but might be arbitrarily slow in our infinite dimensional mild framework, as it is essentially determined by the continuity of the semigroup on the range of the volatility (see Theorem 3.2 and the subsequent remark). A discussion around different rates of convergence in various cases is included in Section 4 of the paper.

Various recent developments related to statistical inference for (parabolic) SPDEs based on discrete observations in time and space have emerged, see e.g. [15,17,18,20].

To the best of our knowledge, our paper is the first one considering high-frequency estimation of (co-) volatility of infinite-dimensional stochastic evolution equations in an operator setting. This is of interest for various reasons. For instance, a simple and important application might be the parameter estimation for H-valued Ornstein–Uhlenbeck process (that is, $\sigma_s = \sigma$ is a constant operator). Elementary techniques such as functional principal component analysis might then be considered on the level of volatility. In a multivariate setting, dynamical dimension reduction was conducted for instance in [2]. Furthermore, it can be used as a tool for inference of infinite-dimensional stochastic volatility models as in [12] or [14]. In the special case of a semigroup that is continuous with respect to the operator norm, the framework also covers the estimation of volatility for H-valued semimartingales.

We organise the paper as follows: First, we recall the main technical preliminaries of our framework in Section 2. In Section 3, we establish the weak law of large numbers. In Section 4, we study the behaviour of the estimator in special cases of semigroups and volatility. We derive convergence rates for particular examples of semigroups in Section 4.1 and stochastic volatility models in Section 4.2. Section 5 is devoted to the proofs of our main results, while in Section 6 we discuss our results and methods in relation to some existing literature and provide some outlook into further developments.

2. Notation and some preliminary results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ denote a filtered probability space satisfying the usual conditions. Consider two separable Hilbert spaces U, H with scalar products denoted by $\langle \cdot, \cdot \rangle_U$, $\langle \cdot, \cdot \rangle_H$ and norms $\| \cdot \|_U$, $\| \cdot \|_H$, respectively. We denote $L(U, H)$ the space of all linear bounded operators $K : U \to H$, and use the shorthand notation $L(U)$ for $L(U, U)$. Equipped with the operator norm, $L(U, H)$ becomes a Banach space. The adjoint operator of a $K \in L(U, H)$ is denoted by K^*, and is an element on $L(H, U)$.

Following Peszat and Zabczyk [40, Appendix A] we use the following notations: An operator $K \in L(U, H)$ is called *nuclear* or *trace class* if the following representation holds

$$Ku = \sum_k b_k \langle u, a_k \rangle_U, \text{ for } u \in U,$$

where $\{a_k\} \subset U$ and $\{b_k\} \subset H$ such that $\sum_k \|a_k\|_U \|b_k\|_H < \infty$. The space of all nuclear operators is denoted by $L_1(U, H)$; it is a separable Banach space and its norm is denoted by

$$\|K\|_1 := \inf \left\{ \sum_k \|a_k\|_U \|b_k\|_H : Ku = \sum_k b_k \langle u, a_k \rangle_U \right\}.$$

We denote by $L_1^+(U, H)$ the class of all symmetric, non-negative-definite nuclear operators from U to H. We write $L_1(U)$ and $L_1^+(U)$ for $L_1(U, U)$ and $L_1^+(U, U)$, respectively.

For $x \in U$ and $y \in H$, we define the tensor product $x \otimes y$ as the linear operator in $L(U, H)$ defined as $x \otimes y(z) := \langle x, z\rangle_U y$ for $z \in U$. We note that $x \otimes y \in L_1(U, H)$ and $\|x \otimes y\|_1 = \|x\|_U \|y\|_H$, see Peszat and Zabczyk [40, p. 107].

The operator $K \in L(U, H)$ is said to be a *Hilbert–Schmidt operator* if

$$\sum_k \|K e_k\|_H^2 < \infty,$$

for any orthonormal basis (ONB) $(e_k)_{k\in\mathbb{N}}$ of U. The space of all Hilbert–Schmidt operators is denoted by $L_{\text{HS}}(U, H)$. We can introduce an inner product by

$$\langle K, L\rangle_{\text{HS}} := \sum_k \langle K e_k, L e_k\rangle_H, \text{ for } K, L \in L_{\text{HS}}(U, H).$$

The induced norm is denoted $\|\cdot\|_{\text{HS}}$. As usual, we write $L_{\text{HS}}(U)$ in the case $L_{\text{HS}}(U, U)$.

We have the following convenient result for the space of Hilbert–Schmidt operators. Although it is well-known, we include the proof of this result for the convenience of the reader:

Lemma 2.1. *Let U, V, H be separable Hilbert spaces. Then $L_{HS}(U, H)$ is a separable Hilbert space. Moreover, if $K \in L_{HS}(U, V)$, $L \in L_{HS}(V, H)$, then $LK \in L_{HS}(U, H)$ and*

$$\|LK\|_{HS} \leq \|L\|_{op}\|K\|_{HS} \leq \|L\|_{HS}\|K\|_{HS}, \tag{3}$$

where the HS-norms are for the spaces in question.

Proof. It is well-known that $L_{\text{HS}}(U, H)$ is a separable Hilbert space (see e.g. Peszat and Zabczyk [40, Appendix A.2, p. 356]). Indeed, an orthonormal basis is $(e_i \otimes f_j)_{i,j\in\mathbb{N}}$ where $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis for U and $(f_j)_{j\in\mathbb{N}}$ for H. Notice that for any $x \in U$, we have for $L \in L_{\text{HS}}(U, H)$

$$\|Lx\|_H^2 = \sum_{i=1}^{\infty}\langle Lx, e_i\rangle_H^2 = \sum_{i=1}^{\infty}\langle x, L^* e_i\rangle_H^2 \leq \|x\|_H^2 \sum_{i=1}^{\infty}\|L^* e_i\|_H^2 = \|x\|_U^2\|L^*\|_{\text{HS}}^2,$$

where $(e_i)_{i=1}^{\infty}$ is an orthonormal basis in U and we applied the Cauchy–Schwarz inequality. Hence, $\|L\|_{\text{op}} \leq \|L^*\|_{\text{HS}} = \|L\|_{\text{HS}}$. It can be seen directly from the definition of the Hilbert–Schmidt norm that for $L \in L_{\text{HS}}(V, H)$, $K \in L_{\text{HS}}(U, V)$, it holds

$$\|LK\|_{\text{HS}} \leq \|L\|_{\text{op}}\|K\|_{\text{HS}} \leq \|L\|_{\text{HS}}\|K\|_{\text{HS}},$$

and the claimed algebraic structure of Hilbert–Schmidt operators follows. □

2.1. Hilbert-space-valued stochastic integrals

Fix $T > 0$ and assume that $0 \leq t \leq T$. Let H and U be separable Hilbert spaces throughout. Recall that a U-valued random variable X is normal with mean $a \in U$ and covariance operator $Q \in L_1^+(U)$ if $\langle X, f\rangle_U$ is a real-valued normally distributed random variable for each $f \in U$, with mean $\langle a, f\rangle$ and

$$E[\langle X, f\rangle_U\langle X, g\rangle_U] = \langle Qf, g\rangle_U,$$

for all $f, g \in U$.

Definition 2.2. A stochastic process $(W_t)_{t\geq 0}$ with values in U is called a Wiener process with covariance operator $Q \in L_1^+(U)$, if $W_0 = 0$ almost surely, W has independent and stationary increments, and for $0 \leq s \leq t$, we have $W_t - W_s \sim N(0, (t-s)Q)$.

Throughout let W denote a Wiener process taking values in U with covariance operator $Q \in L_1^+(U)$. To this operator we can assign the reproducing kernel Hilbert space $U_0 := Q^{\frac{1}{2}}U$ equipped with the scalar product $\langle h, g\rangle_0 := \langle Q^{-\frac{1}{2}}h, Q^{-\frac{1}{2}}g\rangle_H$, where $Q^{-\frac{1}{2}}$ is the pseudo-inverse of $Q^{\frac{1}{2}}$. The space $(U_0, \langle\cdot,\cdot\rangle_0)$ forms again a separable Hilbert space (c.f. Proposition C.03 in [34]). We define for $T < \infty$ the space $\mathcal{N}_W(0, T; H)$ as the space of all predictable $L_{\mathrm{HS}}(U_0; H)$-valued processes $(\sigma_s)_{s\in[0,T]}$ such that

$$\mathbb{P}\left[\int_0^T \|\sigma_s Q^{1/2}\|_{\mathrm{HS}}^2 ds < \infty\right] = 1. \tag{4}$$

Let $\sigma = (\sigma_t)_{t\geq 0}$ denote a stochastic volatility process where $\sigma \in \mathcal{N}_W(0, T; H)$ for some fixed $T < \infty$. The stochastic integral

$$Y_t := \int_0^t \sigma_s dW_s$$

can then be defined as in [34, Chapter 2] and takes values in the Hilbert space H.

We denote the tensor product of the stochastic integral Y by $(Y_t)^{\otimes 2} = Y_t \otimes Y_t$, and define the corresponding stochastic variance term as the *operator angle bracket* (not to be confused with the inner products introduced above!) given by

$$\langle\langle Y\rangle\rangle_t = \int_0^t \sigma_s Q \sigma_s^* ds = \int_0^t (\sigma_s Q^{1/2})(\sigma_s Q^{1/2})^* ds,$$

see Peszat and Zabczyk [40, Theorem 8.7, p. 114].

Remark 2.3. As in Da Prato and Zabczyk [25, p. 104], we note that $(\sigma_s Q^{1/2}) \in L_{HS}(U, H)$ and $(\sigma_s Q^{1/2})^* \in L_{HS}(H, U)$. Hence the process $(\sigma_s Q^{1/2})(\sigma_s Q^{1/2})^* = \sigma_s Q \sigma_s^*$ for $s \in [0, T]$ takes values in $L_1(H, H)$.

Remark 2.4. The integral $\int_0^t \sigma_s Q \sigma_s^* ds$ is interpreted as a Bochner integral in the space of Hilbert–Schmidt operators $L_{\mathrm{HS}}(H)$. Indeed, $\sigma_s Q \sigma_s^*$ is a linear operator on H, and we have almost surely

$$\begin{aligned}\int_0^t \|\sigma_s Q \sigma_s^*\|_{\mathrm{HS}} ds &= \int_0^t \|\sigma_s Q^{1/2}(\sigma_s Q^{1/2})^*\|_{\mathrm{HS}} ds \\ &\leq \int_0^t \|\sigma_s Q^{1/2}\|_{\mathrm{HS}}^2 ds < \infty,\end{aligned}$$

by appealing to Lemma 2.1 and the assumption that $\sigma \in \mathcal{N}_W(0, T; H)$.

Remark 2.5. From the existence of a localising sequence of stopping times

$$\tau_N := \{t \in [0, T] : \int_0^t \|\sigma_s Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 ds > N\},$$

as described in Liu and Röckner [34, p.36] such that for the stopped process given by $Y_{\min(t,\tau_N)} = \int_0^t \mathbb{I}_{[0,\tau_n]}(s)\sigma_s dW_s$ we have

$$\mathbb{E}\left[\int_0^T \|\mathbb{I}_{[0,\tau_n]}(s)\sigma_s Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 ds\right] < \infty$$

and appealing to Peszat and Zabczyk [40, Theorem 8.2, p. 109] we deduce that the process $(M_t)_{t\geq 0}$ with

$$M_t = (Y_t)^{\otimes 2} - \langle\langle Y\rangle\rangle_t$$

is an $L_1(H)$-valued local martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. Thus, the operator angle bracket process can be called the *quadratic covariation process* of Y_t, which we shall do from now on.

We will need the following result, which is a direct corollary of the Hilbert space version of the Burkholder–Davis–Gundy inequality (c.f. [36]).

Lemma 2.6. *Let $\sigma \in \mathcal{N}_W(0,T;H)$. Then there is a positive constant C_4, independent of σ or t, such that*

$$\mathbb{E}\left[\sup_{s\leq t}\left\|\int_0^t \sigma_s dW_s\right\|_H^4\right] \leq C_4\mathbb{E}\left[\left(\int_0^t \|\sigma_s Q^{\frac{1}{2}}\|_{HS}^2 ds\right)^2\right].$$

This finishes our section with preliminary results.

3. The weak law of large numbers

In this section, we show our main result on the law of large numbers for Volterra-type stochastic integrals in Hilbert space with operator-valued volatility processes. Consider, for some $\mathcal{F}_0$-measurable H-valued Y_0,

$$Y_t := \mathcal{S}(t)Y_0 + \int_0^t \mathcal{S}(t-s)\alpha_s ds + \int_0^t \mathcal{S}(t-s)\sigma_s dW_s, \tag{5}$$

where W is a Q-Wiener process on the separable Hilbert space U, σ is an element of $\mathcal{N}_W(0,T;H)$, $(\mathcal{S}(t))_{t\geq 0}$ is a C_0-semigroup on H and α is an almost surely square integrable (in the Bochner sense) predictable process with values in H. We assume that we observe Y at times $t_i := i\Delta_n$ for $\Delta_n = 1/n$, $i = 1,\ldots,\lfloor t/\Delta_n\rfloor$ and define the semigroup-adjusted increment

$$\tilde{\Delta}_n^i Y := Y_{t_i} - \mathcal{S}(\Delta_n)Y_{t_{i-1}} = \int_{t_{i-1}}^{t_i} \mathcal{S}(t_i-s)\alpha_s ds + \int_{t_{i-1}}^{t_i} \mathcal{S}(t_i-s)\sigma_s dW_s. \tag{6}$$

We define the process of the semigroup-adjusted realised covariation (SARCV) as

$$t \mapsto \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} (\tilde{\Delta}_n^i Y)^{\otimes 2}.$$

The aim is to prove the following weak law of large numbers for the SARCV

$$\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} (\tilde{\Delta}_n^i Y)^{\otimes 2} \stackrel{ucp}{\to} \int_0^t \sigma_s Q\sigma_s^* ds, \qquad \text{as } n\to\infty,$$

in the ucp-topology, that is, for all $\epsilon > 0$ and $T > 0$

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{0\leq t\leq T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} (\tilde{\Delta}_n^i Y)^{\otimes 2} - \int_0^t \sigma_s Q\sigma_s^* ds\right\|_{\text{HS}} > \epsilon\right) = 0. \tag{7}$$

3.1. The main result

As we use the notation quite frequently, we will write $\|\cdot\| := \|\cdot\|_H$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_H$ in what follows. We will first impose a moment condition to hold for the drift and volatility processes, which will later be weakened by localisation:

Assumption 1. Assume that for $T > 0$ the following moment conditions hold:

$$\mathbb{E}\left[\int_0^T \|\alpha_s\|^2 ds\right] + \mathbb{E}\left[\left(\int_0^T \|\sigma_s Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 ds\right)^2\right] \leq C(T), \tag{8}$$

for some constant $C(T) > 0$.

Remark 3.1. Using the Cauchy–Schwarz inequality, we can deduce under Assumption 1

$$\mathbb{E}\left[\int_0^t \|\sigma_s Q^{1/2}\|_{\mathrm{HS}}^2 ds\right] \leq \mathbb{E}\left[\left(\int_0^t \|\sigma_s Q^{1/2}\|_{\mathrm{HS}}^2 ds\right)^2\right]^{\frac{1}{2}} \leq \sqrt{C(T)} < \infty.$$

Thus, the integrability condition on $(\sigma_t)_{t\in[0,T]}$ holds for predictable processes satisfying Assumption 1.

Denote for $t \geq 0$

$$M(t) := \sup_{x\in[0,t]} \|\mathcal{S}(x)\|_{op}, \tag{9}$$

which is finite by the Hille–Yosida bound on the semigroup. In order to prove the ucp-convergence (7) we will first show the following stronger result, which can be used to derive convergence rates under Assumption 1:

Theorem 3.2. *Assume that Assumption 1 holds for some $T > 0$. Then there exist constants $L_1(T), L_2(T), L_3(T) > 0$ such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} (\tilde{\Delta}_n^i Y)^{\otimes 2} - \int_0^t \sigma_s Q \sigma_s^* ds\right\|_{HS}\right] \leq L_1(T)\Delta_n^{\frac{1}{2}} + L_2(T)a_n(T) + L_3(T)b_n(T), \tag{10}$$

where

$$a_n(T) := \mathbb{E}\left[\left(\sup_{i=1,\dots,\lfloor T/\Delta_n\rfloor+1} \int_{t_{i-1}}^{\min(t_i,T)} \left\|\sigma_s Q^{\frac{1}{2}}\right\|_{HS}^2 ds\right)^2\right]^{\frac{1}{4}}, \tag{11}$$

$$b_n(T) := \left(\int_0^T \sup_{x\in[0,\Delta_n]} \mathbb{E}[\|(I-\mathcal{S}(x))\sigma_s Q^{\frac{1}{2}}\|_{op}^2] ds\right)^{\frac{1}{2}}, \tag{12}$$

and

$$L_1(T) := M(\Delta_n)^2\left(\Delta_n^{\frac{1}{2}} C(T) + 2C(T)^{\frac{3}{4}}\right), \tag{13}$$

$$L_2(T) := M(\Delta_n)^2 C(T)^{\frac{1}{4}} (8(1+C_4))^{\frac{1}{2}} + a_n, \tag{14}$$

$$L_3(T) := (1+M(\Delta_n))\, C(T)^{\frac{1}{4}}, \tag{15}$$

where C_4 is the universal constant from Lemma 2.6 and $C(T)$ is the constant from Assumption 1. Moreover, a_n and b_n converge to 0 and we have

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} (\tilde{\Delta}_n^i Y)^{\otimes 2} - \int_0^t \sigma_s Q \sigma_s^* ds\right\|_{HS}\right] = 0.$$

Remark 3.3. The precise forms of $L_1(T)$, $L_2(T)$ and $L_3(T)$ follow by combining Eqs. (37), (39) and (46). One should observe that their magnitude can shrink with larger values of n.

That $(a_n(T))_{n\in\mathbb{N}}$ converges to 0, follows from the integrability condition in Assumption 1 and the implied uniform continuity of the mapping

$$t \mapsto \int_0^t \left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\text{HS}}^2 ds.$$

Observe, that in many cases we may assume the volatility to have integrable fourth moments, i.e.

$$\int_0^T \mathbb{E}\left[\left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\text{HS}}^4\right] ds < \infty. \tag{16}$$

In this case we have $a_n = \mathcal{O}(\Delta_n^{1/4})$, as it is easy to see that

$$a_n(T) \leq \left(\int_0^T \mathbb{E}\left[\left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\text{HS}}^4\right] ds\right)^{\frac{1}{4}} \Delta_n^{\frac{1}{4}}.$$

If we further assume that

$$\mathbb{E}\left[\sup_{s\in[0,T]} \left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\text{HS}}^4\right] < \infty, \tag{17}$$

then we even have $a_n = \mathcal{O}(\Delta_n^{1/2})$ as

$$a_n(T) \leq \left(\mathbb{E}\left[\sup_{s\in[0,T]} \left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\text{HS}}^4\right]\right)^{\frac{1}{4}} \Delta_n^{\frac{1}{2}}.$$

That $(b_n(T))_{n\in\mathbb{N}}$ converges to 0 is an implication of Proposition 5.1. The magnitude of this sequence essentially determines the rate of convergence of the realised covariation by virtue of inequality (10). We will come back to the magnitude of the b_n's in specific cases in Section 4.1.

A localisation argument yields the general law of large numbers

Theorem 3.4. *Assume $\sigma \in \mathcal{N}_W(0,T;H)$, i.e. it is stochastically integrable, and the drift α is almost surely square integrable, i.e.*

$$\mathbb{P}\left[\int_0^T \|\alpha_s\|^2 ds < \infty\right] = 1.$$

Then

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leq s\leq t} \left\| \sum_{i=1}^{\lfloor s/\Delta_n \rfloor} (\tilde{\Delta}_i^n Y)^{\otimes 2} - \int_0^s \sigma_u Q \sigma_u^* du \right\|_{HS} > \epsilon\right) = 0, \tag{18}$$

We also emphasise, that the following holds:

Corollary 3.5. *Let $(\bar{Y}_t)_{t\in[0,T]}$ be another process on another separable Hilbert space $\bar{H}$ of the form*

$$\bar{Y}_t := \bar{\mathcal{S}}(t)\bar{Y}_0 + \int_0^t \bar{\mathcal{S}}(t-s)\bar{\alpha}_s ds + \int_0^t \bar{\mathcal{S}}(t-s)\bar{\sigma}_s dW_s, \tag{19}$$

where $\bar{Y}_0$ is $\mathcal{F}_0$-measurable with values in H, $\bar{\sigma}$ is an element of $\mathcal{N}_W(0, T; \bar{H})$, $(\bar{\mathcal{S}}(t))_{t\geq 0}$ is a C_0-semigroup on $\bar{H}$ and $\bar{\alpha}$ is an almost surely square integrable (in the Bochner sense) predictable process. We have with respect to the Hilbert–Schmidt norm-topology on $L_{HS}(H, \bar{H})$

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (Y_{t_i} - \mathcal{S}(\Delta_n)Y_{t_{i-1}}) \otimes (\bar{Y}_{t_i} - \bar{\mathcal{S}}(\Delta_n)\bar{Y}_{t_{i-1}}) \stackrel{ucp}{\to} \int_0^t \bar{\sigma} Q_s \sigma_s^* ds.$$

Proof. We define the process $\hat{Y} := (Y, \bar{Y})^\top$ on the Hilbert space $H \times \bar{H}$ equipped with the scalar product

$$\langle (h, \bar{h})^\top, (g, \bar{g})^\top \rangle_{H\times\bar{H}} := \langle h, g\rangle_H + \langle \bar{h}, \bar{g}\rangle_{\bar{H}}.$$

Moreover, define the strongly continuous semigroup

$$\hat{\mathcal{S}}(t) := \begin{pmatrix} \mathcal{S}(t-s) & 0 \\ 0 & \bar{\mathcal{S}}(t-s) \end{pmatrix}, \quad t \geq 0$$

on $H \times \bar{H}$. As

$$\hat{Y}_t = \begin{pmatrix} Y_0 \\ \bar{Y}_0 \end{pmatrix} + \int_0^t \hat{\mathcal{S}}(t-s) \begin{pmatrix} \alpha_s \\ \bar{\alpha}_s \end{pmatrix} ds + \int_0^t \hat{\mathcal{S}}(t-s) \begin{pmatrix} \sigma_s & 0 \\ \bar{\sigma}_s & 0 \end{pmatrix} d \begin{pmatrix} W_s \\ 0 \end{pmatrix}. \tag{20}$$

Denote by P_1 the projection from $H \times \bar{H}$ onto the first component given by $P_1(h, \bar{h})^\top = h$ and by P_2 onto $\bar{H}$ given by $P_2(h, \bar{h}) := \bar{h}$. Both P_1 and P_2 are continuous linear projections, and as (20) again is a mild Itô process of the form (1), the law of large numbers in Theorem 3.4 is valid. This is why we obtain

$$\begin{aligned} &\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (Y_{t_i} - \mathcal{S}(\Delta_n)Y_{t_{i-1}}) \otimes (\bar{Y}_{t_i} - \bar{\mathcal{S}}(\Delta_n)\bar{Y}_{t_{i-1}}) - \int_0^t \bar{\sigma}_s Q \sigma_s^* ds \\ =&P_2 \left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\hat{Y}_{t_i} - \hat{\mathcal{S}}(\Delta_n)\hat{Y}_{t_{i-1}})^{\otimes 2} - \int_0^t \begin{pmatrix} \sigma_s & 0 \\ \bar{\sigma}_s & 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_s & 0 \\ \bar{\sigma}_s & 0 \end{pmatrix}^* ds \right) P_1^* \\ &\stackrel{ucp}{\to} 0. \end{aligned}$$

The Corollary follows. □

4. Applications

In this section, we give an overview of potential settings and scenarios for which we can use the techniques described above to infer volatility.

Stochastic integrals of the form (19) arise naturally in correspondence to mild or strong solutions to stochastic partial differential equations. Take as a simple example a process given by

$$\text{(SPDE)} \begin{cases} dY_t = AY_t dt + \sigma_t dW_t, & t \geq 0 \\ Y_0 = h_0 \in H, \end{cases} \tag{21}$$

where A is the generator of a C_0-semigroup $(\mathcal{S}(t))_{t\geq 0}$ on the separable Hilbert space H, W is a Q-Wiener process on a separable Hilbert space U for some positive semidefinite and symmetric trace class operator $Q : U \to U$ and $\sigma \in \mathcal{N}_W(0, T; H)$.

There are three components in this model, which need to be estimated in practice: the covariance operator Q of the Wiener process, the generator A (or the semigroup $(\mathcal{S}(t))_{t\geq 0}$ respectively) and the stochastic volatility process σ.

4.1. Semigroups

The essence of the convergence result in Theorem 3.2 is that we can infer on Q and σ based on observing the path of Y, given that we *know* the semigroup $(\mathcal{S}(t))_{t\geq 0}$. Even more, in this case, Theorem 3.2 allows us to derive rates of convergence, which are specified by the behaviour of the semigroup on the volatility. We outline some examples below.

4.1.1. Martingale case

For $A = 0$ and $S(t) = I$ and for all $t \geq 0$, we have the solution

$$Y_t = \int_0^t \sigma_s dW_s,$$

for the stochastic partial differential equation (21). Clearly in this case we have

$$b_n(T) = 0.$$

4.1.2. Uniformly continuous semigroups

Assume that $(\mathcal{S}(t))_{t\geq 0}$ is continuous with respect to the operator norm. This is equivalent to $A \in L(H)$ and $\mathcal{S}(t) = e^{tA}$.

Lemma 4.1. *Let Assumption 1 hold. If the semigroup $(\mathcal{S}(t))_{t\geq 0}$ is uniformly continuous, we have, for b_n given in (12), that*

$$b_n(T) \leq \Delta_n \|A\|_{op} e^{\|A\|_{op}\Delta_n} C(T)^{\frac{1}{4}}.$$

Proof. Recall the following fundamental equality from semigroup theory (cf. Engel and Nagel [26, Lemma II.1.3]):

$$(\mathcal{S}(x) - I)h = \int_0^x A\mathcal{S}(s)hds, \qquad \forall h \in D(A). \tag{22}$$

Using (22), we get

$$\begin{aligned}\sup_{x\in[0,\Delta_n]} \|(I - \mathcal{S}(x))\|_{\text{op}} &= \sup_{x\in[0,\Delta_n]} \sup_{\|h\|=1} \left\| \int_0^x A\mathcal{S}(s)hds \right\| \\ &\leq \sup_{x\in[0,\Delta_n]} x\|A\|_{\text{op}} e^{\|A\|_{\text{op}} x} = \Delta_n \|A\|_{\text{op}} e^{\|A\|_{\text{op}}\Delta_n}.\end{aligned}$$

It follows that

$$\begin{aligned}b_n^2(T) &= \int_0^T \mathbb{E}[\sup_{x\in[0,\Delta_n]} \|(I - \mathcal{S}(x))\sigma_s Q^{\frac{1}{2}}\|_{\text{op}}^2]ds \\ &\leq \sup_{x\in[0,\Delta_n]} \|(I - \mathcal{S}(x))\|_{\text{op}}^2 \int_0^T \mathbb{E}[\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2]ds \\ &\leq \Delta_n^2 \|A\|_{\text{op}}^2 e^{2\|A\|_{\text{op}}\Delta_n} \mathbb{E}\left[\left(\int_0^T \|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right]^{\frac{1}{2}},\end{aligned}$$

and the claim follows. □

For uniformly continuous semigroups and if (17) holds, we obtain a convergence speed of the order $\Delta_n^{1/2}$ for the convergence of the adjusted realised covariation to the quadratic covariation in Theorem 3.2.

Remark 4.2. Note that, if the semigroup is uniformly continuous and under Assumption 1, we can get back to a case similar to Section 4.1.1: As A is continuous, $(Y_t)_{t\in[0,T]}$ is a strong solution to the SPDE (21) and therefore takes the form

$$Y_t = Y_0 + \int_0^t AY_s ds + \int_0^t \sigma_s dW_s.$$

As the drift process given by $\alpha_s = AY_s$ is square-integrable, we can choose the semigroup equal to the identity and therefore the law of large numbers holds without any the adjustment, i.e. we have the convergence of the (nonadjusted) realised covariation

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(Y_{t_i} - Y_{t_{i-1}}\right)^{\otimes 2} \overset{ucp}{\to} \int_0^t \sigma_s Q \sigma_s^* ds.$$

By definition $b_n(T) = 0$ in this case and if (17) holds, the rate of convergence is $\mathcal{O}(\Delta_n^{1/2})$, similar to the case for the adjusted realised covariation.

Let us turn our attention to a case of practical interest coming from financial mathematics applied to commodity markets.

4.1.3. Forward contracts in commodity and interest rate markets: the Heath–Jarrow–Morton approach

A case of relevance for our analysis is inference on the volatility for forward prices in commodity markets as well as for forward rates in fixed-income markets. The Heath–Jarrow–Morton–Musiela equation (HJMM-equation) describes the term structure dynamics in both of these settings (see [28] for a detailed motivation for the use in interest rate modelling and [11] for its use in commodity markets) and is given by

$$\text{(HJMM)} \begin{cases} dX_t = (\frac{d}{dx}X_t + \alpha_t)dt + \sigma_t dW_t, & t \geq 0 \\ X_0 = h_0 \in H, \end{cases} \tag{23}$$

where H is a Hilbert space of functions $f : \mathbb{R}_+ \to \mathbb{R}$ (the *forward curve space*), $(\alpha_t)_{t\geq 0}$ is a predictable and almost surely locally Bochner-integrable stochastic process and σ and W are as before. Conveniently, the states of this *forward curve dynamics* are realised on the separable Hilbert space

$$H = H_\beta = \left\{h : \mathbb{R}_+ \to \mathbb{R} : h \text{ is absolutely continuous and } \|h\|_\beta < \infty\right\}, \tag{24}$$

for fixed $\beta > 0$, where the inner product is given by

$$\langle h, g\rangle_\beta = h(0)g(0) + \int_0^\infty h'(x)g'(x)\mathrm{e}^{\beta x} dx,$$

and norm $\|h\|_\beta^2 = \langle h, h\rangle_\beta$. This space was introduced and analysed in [28]. As in [28], one may consider more general scaling functions in the inner product than the exponential $\exp(\beta x)$. However, for our purposes here this choice suffices. The suitability of this space is partially due to the following result:

Lemma 4.3. *The differential operator $A = \frac{d}{dx}$ is the generator of the strongly continuous semigroup $(\mathcal{S}(t))_{t\geq 0}$ of shifts on H_β, given by $\mathcal{S}(t)h(x) = h(x + t)$, for $h \in H_\beta$, such that*

$$M(\Delta_n) = \sup_{t\leq \Delta_n} \|\mathcal{S}(t)\|_{op} \leq e^{\Delta_n}. \tag{25}$$

Proof. See for example [28]. For the quasi-contractive property (25) compare [11, Theorem 3.5]. □

The HJMM-equation (23) possesses a mild solution (see e.g. [40])

$$f_t = \mathcal{S}(t) f_0 + \int_0^t \mathcal{S}(t-s)\alpha_s ds + \int_0^t \mathcal{S}(t-s)\sigma_s dW_s. \tag{26}$$

Since forward prices and rates are often modelled under a risk neutral probability measure, the drift has in both cases (commodities and interest rates) a special form. In the case of forward prices in commodity markets, it is zero under the risk neutral probability, whereas in interest rate theory it is completely determined by the volatility via the no-arbitrage drift condition

$$\alpha_t = \sum_{j\in\mathbb{N}} \sigma_t^j \Sigma_t^j, \quad \forall t \in [0,T], \tag{27}$$

where $\sigma_t^j = \sqrt{\lambda_j}\sigma_t(e_j)$ and $\Sigma_t^j = \int_0^t \sigma_s^j ds$ for some eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ and a corresponding basis of eigenvectors $(e_j)_{j\in\mathbb{N}}$ of the covariance operator Q of W (cf. Lemma 4.3.3 in [28]).

Lemma 4.4. *Assume that the volatility process $(\sigma_t)_{t\in[0,T]}$ satisfies (17) and that for each $t \in [0,1]$ the operator σ_t maps into*

$$H_\beta^0 = \{h \in H_\beta : \lim_{x\to\infty} h(x) = 0\}.$$

Then the drift given by (27) has values in H_β, is predictable, and has finite second moments.

Proof. That the drift is well defined follows from Lemma 5.2.1 in [28]. Predictability follows immediately from the predictability of the volatility. We have by Theorem 5.1.1 from [28] that there is a constant K depending only on β such that

$$\|\sigma_t^j \Sigma_t^j\|_\beta \le K \|\sigma_t^j\|_\beta^2.$$

Therefore, we get by the triangle inequality that

$$\|\alpha_t\|_\beta \le K \sum_{j\in\mathbb{N}} \|\sigma_t^j\|_\beta^2 = K\|\sigma_t Q^{\frac{1}{2}}\|_{\text{HS}}^2.$$

We obtain the finite second moment property by (17) as

$$\sup_{t\in[0,T]} \mathbb{E}[\|\alpha_t\|_\beta^2] \le K^2 \sup_{t\in[0,T]} \mathbb{E}[\|\sigma_t Q^{\frac{1}{2}}\|_{\text{HS}}^4].$$

Moreover, the Bochner integrability follows, since we have

$$\mathbb{E}\left[\int_0^T \|\alpha_t\|_\beta dt\right] \le \int_0^T \mathbb{E}[\|\alpha_t\|_\beta^2]^{\frac{1}{2}} dt \le TK \sup_{t\in[0,T]} \mathbb{E}[\|\sigma_t Q^{\frac{1}{2}}\|_{\text{HS}}^4]^{\frac{1}{2}} < \infty.$$

The result follows. □

Remark 4.5. Since we know the exact form of the semigroup $(\mathcal{S}(t))_{t\geq 0}$, we can recover the adjusted increments $\tilde{\Delta}_n^i f$ efficiently from forward curve data by a simple shifting in the spatial (e.g., time-to-maturity) variable of these curves. Theorem 3.2 and more generally Theorem 3.4 can therefore be applied in practice to make inference on σ.

The shift semigroup is strongly, but not uniformly, continuous, leaving us with the question to determine the convergence speed of the estimator established in Eq. (10). We close this subsection by deriving a convergence bound under regularity condition of the volatility in the space variable (that is time to maturity).

Observe that by Theorem 4.11 in [11] we know that for all $r \in [0, T]$ there exist random variables c_r with values in $\mathbb{R}$, f_r, g_r with values in H such that $g_r(0) = 0 = f_r(0)$ and $p_r \in L^2(\mathbb{R}_+^2)$ such that we have

$$\sigma_r Q^{\frac{1}{2}} h(x) = c_r h(0) + \langle g_r, h\rangle_\beta + h(0) f_r(x) + \int_0^\infty q_r(x,z) h'(z) dz,$$

where $q_r(x,z) = \int_0^x p_r(y,z) e^{\frac{\beta}{2}(z-y)} dy$. We denote by $C^{1,\gamma}_{\text{loc}} := C^{1,\gamma}_{\text{loc}}(\mathbb{R}_+)$ the space of continuously differentiable functions with locally γ-Hölder continuous derivative for $\gamma \in (0,1]$. The proof of the following result can be found in Section 5.2.

Theorem 4.6. *Assume that $f_r, q_r(\cdot, z) \in C^{1,\gamma}_{loc}$ for all $z \geq 0$, $r \in [0,T]$ and that for the corresponding local Hölder constants $L^1_r(x)$ of $e^{\frac{\beta}{2}\cdot} f_r'(\cdot)$ and $L^2_r(x,z)$ of p_r, we have that for all $x \in [0,1]$*

$$|e^{\beta(x+y)} f_r'(x+y) - e^{\beta x} f_r'(x)| \leq L^1_r(x) y^\gamma$$

and

$$|p(y+x,z) - p(x,z)| \leq L^2_r(x,z) y^\gamma.$$

Moreover, we assume that L^1_r and L^2_r are square integrable in x and in (x,z) respectively such that for some $\zeta \in (0,T)$

$$\hat{L} := \left(\int_0^T \mathbb{E}\left[\left(|f_r'(\zeta)| + \sqrt{8}(\frac{e^{\frac{\beta+1}{2}}}{\beta})\|f_r\|_\beta + \sqrt{2}\|L^1_r\|_{L^2(\mathbb{R}_+)} + \|L^2_r\|_{L^2(\mathbb{R}_+^2)} + (1+\frac{\beta}{2})\|p_r\|_{L^2(\mathbb{R}_+^2)}\right)^2\right] dr\right)^{\frac{1}{2}} < \infty.$$

Then for $b_n(T)$ as given in (12), we can estimate

$$b_n(T) \leq \hat{L} \Delta_n^{\min(\gamma, \frac{1}{2})}.$$

In the next section, we investigate the asymptotic behaviour for different stochastic volatility models.

4.2. Stochastic volatility models

In this section different models for stochastic volatility in Hilbert spaces are discussed. So far, infinite-dimensional stochastic volatility models are specified by stochastic partial differential equations on the positive cone of Hilbert–Schmidt operators (see [12,14]). We will check therefore, which models satisfy Assumption 1. Throughout this section, we take $H = U$ for simplicity.

4.2.1. Constant volatility

We start with the simple, but important special case of constant volatility, i.e. $\sigma_s = I$ for all $s \in [0, T]$ and we want to make inference on Q. In this case (17) is trivially fulfilled and it is easy to see that $C(T) \leq T^2\mathrm{Tr}(Q)^2$. The convergence rate is thus $\mathcal{O}(\Delta_n^{1/2} + b_n(T))$. The magnitude of $b_n(T)$ is now completely dependent on the range of the square root of the covariance operator $Q^{\frac{1}{2}}$. We define

$$\begin{aligned}\tilde{Z}_n(i) :=&(\tilde{\Delta}_i^n Y)^{\otimes 2} - \int_{t_{i-1}}^{t_i} \mathcal{S}(t_i - s)Q\mathcal{S}(t_i - s)^* ds\\ =&(\tilde{\Delta}_i^n Y)^{\otimes 2} - \int_0^{\Delta_n} \mathcal{S}(\Delta_n - s)Q\mathcal{S}(\Delta_n - s)^* ds.\end{aligned} \tag{28}$$

It is interesting to note the following: As the sequence $(\tilde{Z}_n(i))_{i\in\mathbb{N}}$ is a centred i.i.d. sequence of random variables, we also obtain a convergence result, if $T \to \infty$ and Δ_n is constant. Namely, the classical law of large numbers in Hilbert spaces, see e.g. [16, Theorem 2.4], yields

$$\lim_{T\to\infty} \frac{1}{\lfloor T/\Delta_n \rfloor} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \tilde{Z}_n(i) \xrightarrow{u.c.p.} 0.$$

If the semigroup is the identity, this again yields a consistent way of estimating Q, which is analogous to the finite dimensional case. However, note that, if the semigroup is not equal to the identity, the long time estimator (28) estimates $\int_0^{\Delta_n} \mathcal{S}(\Delta_n - s)Q\mathcal{S}(\Delta_n - s)^* ds$, rather than $\Delta_n Q$.

4.2.2. Barndorff–Nielsen & Shephard (BNS) model

The volatility is oftentimes given as the unique positive square-root of a process Σ_t, e.g.,

$$\sigma_t := \Sigma_t^{\frac{1}{2}}, \tag{29}$$

where Σ takes values in the set of positive Hilbert–Schmidt operators on H. This is for instance the case in the Hilbert space-valued volatility model suggested in [12], extending the BNS-model introduced in [6] to infinite dimensions. There Σ is given by the Ornstein–Uhlenbeck dynamics

$$(BNS)\begin{cases} d\Sigma_t = \mathbb{B}\Sigma_t dt + d\mathcal{L}_t, \\ \Sigma_0 \in L_{\mathrm{HS}}(H), \end{cases}$$

where $\mathbb{B}$ is a positive bounded linear operator on the space of Hilbert–Schmidt operators $L_{\mathrm{HS}}(H)$, $\mathcal{L}$ is a square integrable Lévy subordinator on the same space and Σ_0 is also positive definite. $\mathbb{B}$ is then the generator of the uniformly continuous semigroup given by $\mathbb{S}(t) = \exp(\mathbb{B}t)$ and the equation has a mild solution given by

$$\Sigma_t = \mathbb{S}(t)\Sigma_0 + \int_0^t \mathbb{S}(t-s) d\mathcal{L}_s,$$

which defines a stochastically integrable process in $\mathcal{N}_W(0, T; H)$ (see [12]). We have

$$\sup_{s\in[0,T]} \mathbb{E}[\|\sigma_s\|_{\mathrm{op}}^4] = \sup_{s\in[0,T]} \mathbb{E}[\|\Sigma_s^{\frac{1}{2}}\|_{\mathrm{op}}^4] = \sup_{s\in[0,T]} \mathbb{E}[\|\Sigma_s\|_{\mathrm{HS}}^2].$$

By the Itô isometry, we obtain

$$\begin{aligned}\sup_{t\in[0,T]}\mathbb{E}[\|\Sigma_t\|_{\mathrm{HS}}^2]^{\frac12} &\leq \sup_{t\in[0,T]}\left(\|\mathbb{S}(t)\Sigma_0\|_{\mathrm{HS}}+\mathbb{E}\left[\left\|\int_0^t \mathbb{S}(t-u)d\mathcal{L}_u\right\|_{\mathrm{HS}}^2\right]^{\frac12}\right)\\ &\leq \sup_{t\in[0,T]}\left(\|\mathbb{S}(t)\Sigma_0\|_{\mathrm{HS}}+\left(\int_0^T \|\mathbb{S}(t-u)Q_{\mathcal{L}}^{\frac12}\|_{\mathrm{HS}}^2du\right)^{\frac12}\right)\\ &\leq e^{\|\mathbb{B}\|_{\mathrm{op}}T}\|\Sigma_0\|_{\mathrm{HS}}+e^{\|\mathbb{B}\|_{\mathrm{op}}T}\mathrm{Tr}(Q_{\mathcal{L}})^{\frac12}T^{\frac12},\end{aligned}$$

where $Q_{\mathcal{L}}$ denotes the covariance operator of $\mathcal{L}$. This yields that we can find an upper bound for the constant $C(T)$ from Assumption 1 according to

$$\begin{aligned}C(T) &\leq T^2\sup_{s\in[0,T]}\mathbb{E}\left[\left\|\sigma_s Q^{\frac12}\right\|_{\mathrm{op}}^4\right]\\ &\leq \mathrm{Tr}(Q)^2T^2\left(e^{\|\mathbb{B}\|_{\mathrm{op}}T}\|\Sigma_0\|_{\mathrm{HS}}+e^{\|\mathbb{B}\|_{\mathrm{op}}T}\mathrm{Tr}(Q_{\mathcal{L}})^{\frac12}T^{\frac12}\right)^2.\end{aligned}$$

Moreover, it is easy to see that (16) holds, which is why the rate of convergence in the law of large numbers Theorem 3.2 becomes $\mathcal{O}(b_n(T)+\Delta_n^{1/4})$. Now we can combine this result with the ones from the previous section (for instance for the term structure models) and obtain explicit expressions for the constants $L_1(T)$, $L_2(T)$ and $L_3(T)$ from Theorem 3.2.

It is also possible to derive ucp convergence for rough volatility models, which we present in the following section.

4.2.3. *Rough volatility models*

In [10] pathwise constructions of Volterra processes are established and suggested for the use in stochastic volatility models. In this setting, a process is mostly known to be Hölder continuous almost surely of some particular order.

If H is a Banach algebra (like the forward curve space defined by (24)), we can define the volatility process by

$$\sigma_t h := \exp(\mathcal{Y}_t)h. \tag{30}$$

This is a direct extension of the volatility models proposed in [29], if we define the rough process $\mathfrak{Y}_t$ as follows: For $\rho > 0$ and a locally Bochner integrable function $f:\mathbb{R}_+\to H$ define the fractional integral operator $I^\rho \in L(L^1_{\mathrm{loc}}(\mathbb{R}_+,H))$ as

$$I^\rho(f)(t) := \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}f(s)ds. \tag{31}$$

For the special case of $\rho=0$ we set $I^0 = id_{L^1_{loc}(\mathbb{R}_+;H)}$. Define a noise term $\mathfrak{X}$ as the Gaussian process

$$\mathfrak{X}(t) = \int_0^t (t-s)^{\rho-1}d\mathfrak{W}(s). \tag{32}$$

This integral is well defined pathwise via a Sewing Lemma in Banach spaces (see [10, Prop. 14]), for a process $\mathfrak{W}$ with sample paths in space of γ-Hölder continuous functions $C^\gamma([0,T];H)$ on H, such that $\rho+\gamma-1>0$. For an initial condition $y\in H$, an Ornstein–Uhlenbeck process in this framework is considered to be the solution to the integral equation

$$\mathfrak{Y}_t = y + I^\rho(\mathbb{A}\mathcal{Y})_t + \mathfrak{X}_t \quad t \in [0, T], \tag{33}$$

where $\mathbb{A} \in L(H)$. It was shown in [10, Prop. 26] that this pathwise integral equation possesses a unique solution $\mathfrak{Y}$ with sample paths in $C^\alpha([0, T]; H)$ for $0 < \alpha < \rho + \gamma - 1$. This solution is moreover Gaussian and hence, by virtue of Fernique's theorem, c.f. [40, Theorem 3.31], satisfies (16), which is why the rate of convergence is $\mathcal{O}(b_n(T) + \Delta_n^{1/4})$. More precisely, the cross-covariance structure is characterised by

$$\begin{aligned} Q_{\mathfrak{Y}}(t, t') &:= \mathbb{E}[\mathfrak{Y}_t \otimes \mathfrak{Y}_{t'}] \\ &= \int_0^t \int_0^{t'} (t-r)^{\rho-1} E_{\rho,\rho}(\mathbb{A}(t-r)^\rho) d^2 Q_{\mathfrak{W}}(r, r')(t'-r')^{\rho-1} E_{\rho,\rho}(\mathbb{A}^*(t'-r')^\rho), \end{aligned}$$

where for $\mathbb{B} \in L(H)$

$$E_{\alpha,\beta}(\mathbb{B}) := \sum_{i=1}^\infty \frac{\mathbb{B}^i}{\Gamma(\alpha i + \beta)}$$

is the Mittag-Leffler operator and Γ is the Gamma-function. From these analytic expressions, one can derive again explicit formulas for the constants $L_1(T)$, $L_2(T)$ and $L_3(T)$.

5. Proofs

In this section, we will present the proofs of our previously stated results.

5.1. Proofs of results in Section 3

5.1.1. Uniform continuity of semigroups on compact sets

In order to verify that $b_n(T)$ defined in (12) converges to 0 and to prove Theorem 3.2, we need to establish some convergence properties of semigroups on compacts.

The next proposition follows from Dini's theorem and will be important for our analysis:

Proposition 5.1. *Let U, H be two separable Hilbert spaces. The following holds:*

(i) If σ is an almost surely compact random linear operator with values in $L(U, H)$, we get that

$$\sup_{x \in [0, \Delta_n]} \|(I - \mathcal{S}(x))\sigma\|_{op} \to 0, \qquad \text{as } n \to \infty, \tag{34}$$

where the convergence holds almost surely. If furthermore $\sigma \in L^p(\Omega; L(U, H))$ for some $p \in [1, \infty)$, the convergence holds also in $L^p(\Omega; \mathbb{R})$.

(ii) Assume that $s \mapsto \sigma_s Q^{\frac{1}{2}}$ is a stochastic process of almost surely compact operators, such that

$$\mathbb{P}\left[\int_0^T \|\sigma_s\|_{op}^p \, ds < \infty\right] = 1,$$

for $p \in [1, \infty)$. Then almost surely

$$\int_0^T \sup_{x \in [0, \Delta_n]} \|(I - \mathcal{S}(x))\sigma_s\|_{op}^p ds \to 0, \qquad \text{as } n \to \infty. \tag{35}$$

If $\int_0^T \mathbb{E}\left[\left\|\sigma_s Q^{\frac{1}{2}}\right\|_{op}^p\right] ds < \infty$, *then*

$$\int_0^T \mathbb{E}[\sup_{x\in[0,\Delta_n]} \|(I-\mathcal{S}(x))\sigma_s\|_{op}^p]ds \to 0, \qquad as\ n\to\infty. \tag{36}$$

Proof. Let $B_0(1) := \{h \in H : \|h\| = 1\}$ be the unit sphere in H and fix $\omega \in \Omega$, such that $\sigma(\omega)$ is compact. Since $\sigma(\omega)$ is compact, $\mathcal{C} := \overline{\sigma(\omega)(B_0(1))}$ is compact in H. We define the set $F(\omega)$ of functionals of the form

$$f_n := \sup_{x\in[0,\Delta_n]} \|(I-\mathcal{S}(x))\cdot\| : \mathcal{C} \to \mathbb{R}.$$

The functions in $F(\omega)$ are continuous, as

$$\begin{aligned}|\sup_{x\in[0,\Delta_n]} \|(I-\mathcal{S}(x))h\| - \sup_{x\in[0,\Delta_n]} \|(I-\mathcal{S}(x))g\||\\ &\leq \sup_{x\in[0,\Delta_n]} \|(I-\mathcal{S}(x))(h-g)\|\\ &\leq \sup_{x\in[0,\Delta_1]} \|(I-\mathcal{S}(x))\|_H\|h-g\|,\end{aligned}$$

for all $g, h \in \mathcal{C}$. Hence Dini's theorem (c.f. Theorem 7.13 in [44]) yields (34) in the almost sure sense. Since the sequence is uniformly bounded by $(1 + M(T))\|\sigma\|_{op}$, which has finite pth moment, we obtain $L^p(\Omega;\mathbb{R})$-convergence by the dominated convergence theorem, and therefore (34) holds in the L^p-sense.

The convergences (35) and (36) follow now immediately by appealing to the dominated convergence theorem, as

$$\sup_{x\in[0,\Delta_n]} \|(I-\mathcal{S}(x))\sigma_s\|_{op}^p \leq M(\Delta_n)^p\|\sigma_s\|_{op}^p$$

and $\sup_{x\in[0,\Delta_n]} \|(I-\mathcal{S}(x))\sigma_s\|_{op}^p$, respectively $\mathbb{E}\left[\sup_{x\in[0,\Delta_n]} \|(I-\mathcal{S}(x))\sigma_s\|_{op}^p\right]$, converges to 0 by (34). □

Recall also the following fact:

Lemma 5.2. *The family* $(\mathcal{S}(t)^*)_{t\geq 0}$ *of adjoint operators of the* C_0*-semigroup* $(\mathcal{S}(t))_{t\geq 0}$ *forms again a* C_0*-semigroup on* H.

Proof. See Section 5.14 in [26]. □

Now we can proceed with the proof of our main theorem in the next subsection.

5.1.2. Elimination of the drift

The drift process will not affect the asymptotic behaviour of the realised covariation. This is proved in the next Lemma:

Lemma 5.3. *In order to prove Theorem 3.2, we can without loss of generality assume* $\alpha \equiv 0$ *and* $Y_0 \equiv 0$.

Proof. That we can assume $Y_0 \equiv 0$ can be seen immediately as

$$\widetilde{\Delta}_n^i Y := Y_{t_i} - \mathcal{S}(\Delta_n)Y_{t_{i-1}} = \int_{t_{i-1}}^{t_i} \mathcal{S}(t_i - s)\alpha_s ds + \int_{t_{i-1}}^{t_i} \mathcal{S}(t_i - s)\sigma_s dW_s$$

is not dependent on the initial condition. We can then argue for the drift as follows: We have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}(\tilde{\Delta}_n^i Y)^{\otimes 2}-\int_0^t \sigma_s Q\sigma_s^* ds\right\|_{\text{HS}}\right]\\
&\quad\le \mathbb{E}\left[\sup_{0\le t\le T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left(\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\alpha_s ds\right)^{\otimes 2}\right\|_{\text{HS}}\right]\\
&\qquad+\mathbb{E}\left[\sup_{0\le t\le T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left(\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\alpha_s ds\right)\otimes\left(\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\sigma_s dW_s\right)\right\|_{\text{HS}}\right]\\
&\qquad+\mathbb{E}\left[\sup_{0\le t\le T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left(\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\sigma_s dW_s\right)\otimes\left(\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\alpha_s ds\right)\right\|_{\text{HS}}\right]\\
&\qquad+\mathbb{E}\left[\sup_{0\le t\le T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left(\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\sigma_s dW_s\right)^{\otimes 2}-\int_0^t\sigma_s Q\sigma_s^* ds\right\|_{\text{HS}}\right]\\
&\quad\le \sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\left[\left\|\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\alpha_s ds\right\|^2\right]\\
&\qquad+2\left(\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\left[\left\|\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\sigma_s dW_s\right\|^2\right]\right)^{\frac{1}{2}}\\
&\qquad\qquad\times\left(\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\left[\left\|\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\alpha_s ds\right\|^2\right]\right)^{\frac{1}{2}}\\
&\qquad+\mathbb{E}\left[\sup_{0\le t\le T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left(\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\sigma_s dW_s\right)^{\otimes 2}-\int_0^t\sigma_s Q\sigma_s^* ds\right\|_{\text{HS}}\right]\\
&\quad=(1)+(2)+(3).
\end{aligned}
$$

In order to prove the assertion, we have to show that (1) and (2) converge to 0 as $n\to\infty$. We find by Bochner's inequality

$$
\mathbb{E}\left[\left\|\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\alpha_s ds\right\|^2\right]\le \Delta_n M^2(\Delta_n)\mathbb{E}\left[\int_{t_{i-1}}^{t_i}\|\alpha_s\|^2 ds\right],
$$

and by the Itô isometry

$$
\mathbb{E}\left[\left\|\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\sigma_s dW_s\right\|_H^2\right]\le M^2(\Delta_n)\mathbb{E}\left[\int_{t_{i-1}}^{t_i}\|\sigma_s Q^{\frac{1}{2}}\|_{HS}^2 ds\right],
$$

where we appealed to the bound (9) on the semigroup. Hence, $(1)+(2)=\mathcal{O}(\Delta_n^{\frac{1}{2}})$, so the first two terms will not impact the estimation of the covariation (in the limit). More precisely we have that

$$(1)+(2) \leq \Delta_n M^2(\Delta_n) \int_0^T \mathbb{E}\left[\|\alpha_s\|^2\right] ds \\ + 2\left(\Delta_n M^2(\Delta_n) \int_0^T \mathbb{E}\left[\|\alpha_s\|^2\right] ds\right)^{\frac{1}{2}} \left(M^2(\Delta_n) \int_0^T \mathbb{E}\left[\left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\mathrm{HS}}^2\right] ds\right)^{\frac{1}{2}}$$

and therefore, in view of Assumption 1

$$(1)+(2) \leq \Delta_n^{\frac{1}{2}} M(\Delta_n)^2 \left(\Delta_n^{\frac{1}{2}} C(T) + 2C(T)^{\frac{3}{4}}\right). \tag{37}$$

The Lemma follows. □

5.1.3. *Proof of Theorem 3.2*

In view of Lemma 5.3 we assume in this subsection that the process Y takes the form $Y_t = \int_0^t \mathcal{S}(t-s)\sigma_s dW_s$. The operator bracket process for the semigroup-adjusted increment takes the form

$$\langle\langle \widetilde{\Delta}_n^i Y \rangle\rangle = \int_{t_{i-1}}^{t_i} \mathcal{S}(t_i - s)\sigma_s Q \sigma_s^* \mathcal{S}(t_i - s)^* ds. \tag{38}$$

We have

Proposition 5.4. *Let Assumption 1 hold. Then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\tilde{\Delta}_i^n Y)^{\otimes 2} - \langle\langle \tilde{\Delta}_i^n Y \rangle\rangle \right\|\right] \leq M(\Delta_n)^2 C(T)^{\frac{1}{4}} \left(8(1+C_4)\right)^{\frac{1}{2}} a_n(T). \tag{39}$$

Proof. We define

$$\tilde{Z}_n(i) := (\tilde{\Delta}_i^n Y)^{\otimes 2} - \langle\langle \tilde{\Delta}_i^n Y \rangle\rangle = (\tilde{\Delta}_i^n Y)^{\otimes 2} - \int_{t_{i-1}}^{t_i} \mathcal{S}(t_i - s)\sigma_s Q \sigma_s^* \mathcal{S}(t_i - s)^* ds.$$

First we show that $\sup_{t\in[0,T]} \| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \tilde{Z}_n(i) \|_{\mathrm{HS}}$ has finite second moment. By the triangle inequality and Lemma 2.1

$$\begin{aligned} \sup_{t\in[0,T]} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \tilde{Z}_n(i) \right\|_{\mathrm{HS}} &\leq \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \|\tilde{Z}_n(i)\|_{\mathrm{HS}} \\ &\leq \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \|(\tilde{\Delta}_i^n Y)^{\otimes 2}\|_{\mathrm{HS}} \\ &\quad + \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \left\| \int_{t_{i-1}}^{t_i} \mathcal{S}(t_i - s)\sigma_s Q \sigma_s^* \mathcal{S}(t_i - s)^* ds \right\|_{\mathrm{HS}} \\ &\leq \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \|\tilde{\Delta}_i^n Y\|_H^2 + M(\Delta_n)^2 \int_0^T \left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\mathrm{HS}}^2 ds. \end{aligned}$$

Considering $\mathbb{E}\left[\sup_{t\in[0,T]}\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}_n(i)\|_{\text{HS}}^2\right]$, we get a finite sum of linear combinations of the following terms

$$\mathbb{E}\left[\left(\int_0^T \|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right] = C(T), \tag{40}$$

$$\mathbb{E}\left[\|\tilde{\Delta}_i^n Y\|^2\|\tilde{\Delta}_j^n Y\|^2\right], \tag{41}$$

$$\int_0^T \mathbb{E}\left[\|\tilde{\Delta}_i^n Y\|^2\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2\right] ds. \tag{42}$$

The expression in (40) is finite by the imposed Assumption 1. The term in (41) is finite, since by the Cauchy–Schwarz inequality

$$\begin{aligned}
&\mathbb{E}\left[\|\tilde{\Delta}_i^n Y\|^2\|\tilde{\Delta}_j^n Y\|^2\right]\\
\leq&\mathbb{E}\left[\|\tilde{\Delta}_i^n Y\|^4\right]^{\frac{1}{2}}\mathbb{E}\left[\|\tilde{\Delta}_j^n Y\|^4\right]^{\frac{1}{2}}\\
\leq&C_4\mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i}\|\mathcal{S}(t_i-s)\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left(\int_{t_{j-1}}^{t_j}\|\mathcal{S}(t_j-s)\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right]^{\frac{1}{2}}\\
\leq&M(\Delta_n)^4\mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i}\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left(\int_{t_{j-1}}^{t_j}\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right]^{\frac{1}{2}}\\
\leq&M(\Delta_n)^4 C(T),
\end{aligned} \tag{43}$$

where the second inequality followed from Lemma 2.6. For (42), we apply the Cauchy–Schwarz inequality and argue as for the first two. In conclusion, we obtain a finite second moment as desired.

Now note that $t\mapsto\psi_t=\int_{t_{i-1}}^t\mathcal{S}(t_i-s)\sigma_s dW_s$ is a martingale for $t\in[t_{i-1},t_i]$. From Peszat and Zabczyk [40, Theorem 8.2, p. 109] we deduce that the process $(\zeta_t)_{t\in[t_{i-1},t_i]}$ with

$$\zeta_t=(\psi_t)^{\otimes 2}-\langle\langle\psi\rangle\rangle_t$$

is a centred martingale w.r.t. $(\mathcal{F}_t)_{t\in[t_{i-1},t_i]}$ and hence

$$\mathbb{E}\left[\tilde{Z}_n(i)|\mathcal{F}_{t_{i-1}}\right]=\mathbb{E}\left[\zeta_{t_i}|\mathcal{F}_{t_{i-1}}\right]=0.$$

Also, this shows that $M_m^n:=\sum_{i=1}^m\tilde{Z}_n(i)$ defines a discrete-time martingale in $L_{\text{HS}}(H)$ and therefore $\|M_m^n\|_{\text{HS}}$ a positive real-valued submartingale with respect to $(\mathcal{F}_{t_i})_{i=0,\dots}$. This is why by Doob's martingale inequality Revuz and Yor [43, Corollary (II.1.6)]

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}_n(i)\right\|_{\text{HS}}^2\right]=&\mathbb{E}\left[\max_{m=1,\dots,\lfloor T/\Delta_n\rfloor}\|M_m^n\|_{\text{HS}}^2\right]\\
\leq&4\mathbb{E}\left[\|M_{\lfloor T/\Delta_n\rfloor}^n\|_{\text{HS}}^2\right]\\
=&4\mathbb{E}\left[\left\|\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\tilde{Z}_n(i)\right\|_{\text{HS}}^2\right].
\end{aligned} \tag{44}$$

Moreover, for $j < i$, as each $\tilde{Z}_n(i)$ is $\mathcal{F}_{t_{i-1}}$ measurable and as the conditional expectation commutes with bounded linear operators, and also using the tower property of conditional expectation

$$\begin{aligned}\mathbb{E}\left[\langle \tilde{Z}_n(i), \tilde{Z}_n(j)\rangle_{\text{HS}}\right] =&\mathbb{E}\left[\mathbb{E}\left[\langle \tilde{Z}_n(i), \tilde{Z}_n(j)\rangle_{\text{HS}}|\mathcal{F}_{t_{i-1}}\right]\right]\\ =&\mathbb{E}\left[\langle \mathbb{E}\left[\tilde{Z}_n(i)|\mathcal{F}_{t_{i-1}}\right], \tilde{Z}_n(j)\rangle_{\text{HS}}\right] = 0. \end{aligned} \tag{45}$$

Combining (44) and (45) we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}_n(i)\right\|_{\text{HS}}^2\right] \leq 4\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\left[\|\tilde{Z}_n(i)\|_{\text{HS}}^2\right].$$

Applying the triangle and Bochner inequalities, the basic inequality $(a+b)^2 \leq 2(a^2+b^2)$ and appealing to (43), we find

$$\begin{aligned}\mathbb{E}\left[\|\tilde{Z}_n(i)\|_{\text{HS}}^2\right] &\leq 2\mathbb{E}\left[\|(\tilde{\Delta}_i^n Y)^{\otimes 2}\|_{\text{HS}}^2 + \left(\int_{t_{i-1}}^{t_i}\|\mathcal{S}(t_i-s)\sigma_s Q\sigma_s^*\mathcal{S}(t_i-s)^*\|_{\text{HS}}ds\right)^2\right]\\ &\leq 2\mathbb{E}\left[\|\tilde{\Delta}_i^n Y\|^4 + M(\Delta_n)^4\left(\int_{t_{i-1}}^{t_i}\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right]\\ &\leq 2M(\Delta_n)^4(C_4+1)\mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i}\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right].\end{aligned}$$

Summing up, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}_n(i)\right\|_{\text{HS}}^2\right] \leq 8(1+C_4)M(\Delta_n)^4\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i}\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right]\\ \leq&8(1+C_4)M(\Delta_n)^4\mathbb{E}\left[\int_0^T\left\|\sigma_r Q^{\frac{1}{2}}\right\|_{\text{HS}}^2 dr \sup_{i=1,\dots,\lfloor T/\Delta_n\rfloor}\int_{t_{i-1}}^{t_i}\left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\text{HS}}^2 ds\right]\\ \leq&8(1+C_4)M(\Delta_n)^4\mathbb{E}\left[\left(\int_0^T\left\|\sigma_r Q^{\frac{1}{2}}\right\|_{\text{HS}}^2 dr\right)^2\right]^{\frac{1}{2}}\\ &\times\mathbb{E}\left[\left(\sup_{i=1,\dots,\lfloor T/\Delta_n\rfloor}\int_{t_{i-1}}^{t_i}\left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\text{HS}}^2 ds\right)^2\right]^{\frac{1}{2}}\\ \leq&8(1+C_4)M(\Delta_n)^4 C(T)^{\frac{1}{2}}a_n(T)^2.\end{aligned}$$

Hence, the proposition follows by application of the Cauchy–Schwarz inequality. □

The Law of large numbers, Theorem 3.2, follows now from the following result:

Proposition 5.5. *Suppose that Assumption 1 holds. Then*

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq t\leq T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle\langle\tilde{\Delta}_n^i Y\rangle\rangle - \int_0^t\sigma_s Q\sigma_s^* ds\right\|_{HS}\right]\\ &\leq (1+M(\Delta_n))\, b_n(T)C(T)^{\frac{1}{4}} + a_n(T)^2.\end{aligned} \tag{46}$$

Proof. Recall the expression for $\langle\langle\tilde{\Delta}_n^i Y\rangle\rangle$ in (38). By the triangle and Bochner inequalities, we find,

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left\|\int_0^{\lfloor t/\Delta_n\rfloor\Delta_n}\sigma_s Q\sigma_s^* ds-\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{t_{i-1}}^{t_i}\mathcal{S}(t_i-s)\sigma_s Q\sigma_s^*\mathcal{S}(t_i-s)^* ds\right\|_{\text{HS}}\\
&\leq \sup_{t\in[0,T]}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{t_{i-1}}^{t_i}\|\sigma_s Q\sigma_s^*-\mathcal{S}(t_i-s)\sigma_s Q\sigma_s^*\mathcal{S}(t_i-s)^*\|_{\text{HS}} ds\\
&\leq \sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\int_{t_{i-1}}^{t_i}\|\sigma_s Q\sigma_s^*-\mathcal{S}(t_i-s)\sigma_s Q\sigma_s^*\mathcal{S}(t_i-s)^*\|_{\text{HS}} ds.
\end{aligned}
$$

By Lemma 2.1 and the Cauchy–Schwarz inequality we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle\langle\tilde{\Delta}_n^i Y\rangle\rangle-\int_0^t\sigma_s Q\sigma_s^* ds\right\|_{\text{HS}}\right]\\
&\leq \sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\int_{t_{i-1}}^{t_i}\mathbb{E}[\|(I-\mathcal{S}(t_i-s))\sigma_s Q\sigma_s^*\|_{\text{HS}}]\\
&\quad+\mathbb{E}[\|\mathcal{S}(t_i-s)\sigma_s Q\sigma_s^*(I-\mathcal{S}(t_i-s)^*)\|_{\text{HS}}]ds\\
&\quad+\sup_{t\in[0,T]}\int_{t_n}^t\mathbb{E}[\|\sigma_s Q\sigma_s^*\|_{\text{HS}}]ds\\
&\leq \sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\int_{t_{i-1}}^{t_i}\mathbb{E}[\|(I-\mathcal{S}(t_i-s))\sigma_s Q^{\frac{1}{2}}\|_{\text{op}}\|Q^{\frac{1}{2}}\sigma_s^*\|_{\text{HS}}]\\
&\quad+M(\Delta_n)\mathbb{E}[\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}\|Q^{\frac{1}{2}}\sigma_s^*(I-\mathcal{S}(t_i-s)^*)\|_{\text{op}}]ds\\
&\quad+\sup_{t\in[0,T]}\int_{t_n}^t\mathbb{E}[\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2]ds\\
&\leq (1+M(\Delta_n))\left(\int_0^T\sup_{t\leq\Delta_n}\mathbb{E}\left[\left\|(I-\mathcal{S}(t))\sigma_s Q^{\frac{1}{2}}\right\|_{\text{op}}^2\right]ds\right)^{\frac{1}{2}}\left(\int_0^T\mathbb{E}\left[\left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\text{HS}}^2\right]ds\right)^{\frac{1}{2}}\\
&\qquad+\sup_{t\in[0,T]}\mathbb{E}\left[\left(\int_{t_n}^t\|\sigma_s Q^{\frac{1}{2}}\|_{\text{HS}}^2 ds\right)^2\right]^{\frac{1}{2}}\\
&\leq (1+M(\Delta_n))\, b_n(T)C(T)^{\frac{1}{4}}+a_n(T)^2.
\end{aligned}
$$

This completes the proof. □

5.1.4. *Proof of Theorem 3.4*

Proof of Theorem 3.4. As σ and α are locally square integrable, we can for all $m\in\mathbb{N}$ define the stopping time

$$\tau_m := \inf\left\{t\in[0,T]:\int_0^T\left(\|\alpha_s\|^2+\left\|\sigma_s Q^{\frac{1}{2}}\right\|_{\text{HS}}^2\right)ds>m\right\}.$$

Define with the notations $\sigma_t^{(m)} := \sigma_t \mathbf{1}_{(0,\tau_m]}(t)$ and $\alpha_t^{(m)} := \alpha_t \mathbf{1}_{(0,\tau_m]}(t)$, such that

$$\begin{aligned}Y_t^{(m)} :=& \mathcal{S}(t)Y_0 + \int_0^{\min(t,\tau_m)} \mathcal{S}(t-s)\alpha_s ds + \int_0^{\min(t,\tau_m)} \mathcal{S}(t-s)\sigma_s dW_s\\ =& \mathcal{S}(t)Y_0 + \int_0^t \mathcal{S}(t-s)\alpha_s^{(m)} ds + \int_0^t \mathcal{S}(t-s)\sigma_s^{(m)} dW_s,\end{aligned}$$

where the last equality holds almost surely for all $t \in [0, T]$ (c.f. Lemma 2.3.9 in [34]). We further define

$$\mathcal{Z}_m^n := \sup_{0\leq s\leq t} \left\| \sum_{i=1}^{\lfloor s/\Delta_n \rfloor} (\tilde{\Delta}_n^i Y^{(m)})^{\otimes 2} - \int_0^s \sigma_u^{(m)} Q \sigma_u^{(m)*} du \right\|_{\text{HS}},$$

$$\mathcal{Z}^n := \sup_{0\leq s\leq t} \left\| \sum_{i=1}^{\lfloor s/\Delta_n \rfloor} (\tilde{\Delta}_n^i Y)^{\otimes 2} - \int_0^s \sigma_u Q \sigma_u^* du \right\|_{\text{HS}}.$$

Since $\alpha^{(m)}$ and $\sigma^{(m)}$ satisfy Assumption 1 and thus, the conditions of Theorem 3.2, we obtain that for all $m \in \mathbb{N}$ and $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{Z}_m^n > \epsilon] = 0. \tag{47}$$

We have $\mathcal{Z}_m^n = \mathcal{Z}^n$ on $\Omega_m := \{\tau_m \geq t\}$ and hence

$$\begin{aligned}\mathbb{P}[\mathcal{Z}^n > \epsilon] =& \int_{\Omega_m} \mathbf{1}(\mathcal{Z}^n > \epsilon) d\mathbb{P} + \int_{\Omega_m^c} \mathbf{1}(\mathcal{Z}^n > \epsilon) d\mathbb{P}\\ =& \int_{\Omega_m} \mathbf{1}(\mathcal{Z}_m^n > \epsilon) d\mathbb{P} + \int_{\Omega_m^c} \mathbf{1}(\mathcal{Z}^n > \epsilon) d\mathbb{P}\\ \leq& \mathbb{P}[\mathcal{Z}_m^n > \epsilon] + \mathbb{P}[\Omega_m^c],\end{aligned}$$

which holds for all $n, m \in \mathbb{N}$. Now, by virtue of (47) we obtain for all $m \in \mathbb{N}$ that

$$\limsup_{n\to\infty} \mathbb{P}[\mathcal{Z}^n > \epsilon] \leq \mathbb{P}[\Omega_m^c].$$

As $\Omega_m \uparrow \Omega$ (due to the local integrability of drift and volatility) and by the continuity of $\mathbb{P}$ from below, $\mathbb{P}[\Omega_m^c]$ converges to 0 as $m \to \infty$ and therefore

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{Z}^n > \epsilon] = \limsup_{n\to\infty} \mathbb{P}[\mathcal{Z}^n > \epsilon] = 0. \quad \square$$

5.2. *Proof of Theorem 4.6*

Proof of Theorem 4.6. Since for all $h \in H_\beta$ one has $|h(0)| \leq \|h\|_\beta$, we have for $\|h\|_\beta = 1$ that

$$\begin{aligned}\|(I - \mathcal{S}(x))\sigma_r Q^{\frac{1}{2}} h\|_\beta \leq& \|(I - \mathcal{S}(x)) f_r\|_\beta + \left\| (I - \mathcal{S}(x)) \int_0^\infty q_r(\cdot, z) h'(z) dz \right\|_\beta\\ =& (1) + (2).\end{aligned}$$

The first summand can be estimated as follows: By the mean value theorem, there is a $\zeta \in (0,1)$, such that for $x < 1$ we have

$$
\begin{aligned}
(1) \leq &\left(|f_r(x)|^2 + 2\int_0^\infty ((1-e^{\frac{\beta}{2}x}) f_r'(y+x))^2 e^{\beta y} dy \right. \\
&\left. \qquad +2\int_0^\infty (e^{\frac{\beta}{2}(x+y)} f_r'(y+x) - e^{\frac{\beta}{2}y} f_r'(y))^2 dy \right)^{\frac{1}{2}} \\
&\leq (|f_r'(\zeta)|^2 x^2 + 2\|\mathcal{S}(x) f_r\|_\beta^2 (\frac{2e^{\frac{\beta}{2}}}{\beta})^2 x^2 + 2x^{2\gamma} \|L_r^1\|^2_{L^2(\mathbb{R}_+)})^{\frac{1}{2}} \\
&\leq x^\gamma (|f_r'(\zeta)| + \sqrt{8}(\frac{e^{\frac{\beta+1}{2}}}{\beta}) \|f_r\|_\beta + \sqrt{2}\|L_r^1\|_{L^2(\mathbb{R}_+)}).
\end{aligned} \tag{48}
$$

Here we used in the second inequality $f_r(0) = 0$ and that L_1 is the Hölder constant of $e^{\frac{\beta\cdot}{2}} f'(\cdot)$. In the third inequality we used the subadditivity of the squareroot and that the semigroup is quasi-contractive and satisfies $\|\mathcal{S}(x)\|_{\text{op}} < e^1 = e$ for $x \leq 1$ (c.f. [11, Lemma 3.5]). We can show, using the Hölder inequality, for all $h \in H_\beta$ such that $\|h\|_\beta = 1$, that for some $\zeta' \in (0,1)$

$$
\begin{aligned}
(2) \leq &\left| \int_0^\infty q_r(x,z) h'(z) dz \right| \\
&\qquad + \left(\int_0^\infty \left[\partial_y \int_0^\infty (q_r(y+x,z) - q_r(y,z)) h'(z) dz \right]^2 e^{\beta y} dy \right)^{\frac{1}{2}} \\
= &\left| \int_0^x \int_0^\infty p_r(y,z) e^{\frac{\beta}{2}(z-y)} h'(z) dz dy \right| \\
&\qquad + \left(\int_0^\infty \left[\int_0^\infty \left(e^{-\frac{\beta}{2}x} p_r(y+x,z) - p_r(y,z) \right) e^{\frac{\beta}{2}(z-y)} h'(z) dz \right]^2 e^{\beta y} dy \right)^{\frac{1}{2}} \\
= &\left(\int_0^x \int_0^\infty e^{\beta(z-y)} h'(z)^2 dz dy \right)^{\frac{1}{2}} \left(\int_0^x \int_0^\infty p_r(y,z)^2 dz dy \right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^\infty \left[\int_0^\infty (e^{-\frac{\beta}{2}x} p_r(y+x,z) - p_r(y,z)) e^{\frac{\beta}{2}z} h'(z) dz \right]^2 dy \right)^{\frac{1}{2}} \\
\leq &\left(x \int_0^\infty e^{\beta z} h'(z)^2 dz \right)^{\frac{1}{2}} \|p_r(\cdot,\cdot)\|_{L^2(\mathbb{R}_+^2)} \\
&\qquad + \left(\int_0^\infty \int_0^\infty (e^{-\frac{\beta}{2}x} p_r(y+x,z) - p_r(y,z))^2 dz \|h\|_\beta^2 dy \right)^{\frac{1}{2}} \\
\leq &x^{\frac{1}{2}} \|p_r\|_{L^2(\mathbb{R}_+^2)} + \left(\int_0^\infty \int_0^\infty (e^{-\frac{\beta}{2}x} p_r(y+x,z) - p_r(y,z))^2 dz dy \right)^{\frac{1}{2}}.
\end{aligned}
$$

Now we can estimate, for $x < 1$, using the triangle inequality

$$
\begin{aligned}
(2) \leq &x^{\frac{1}{2}} \|p_r\|_{L^2(\mathbb{R}_+^2)} + \left(\int_0^\infty \int_0^\infty (e^{-\frac{\beta}{2}x} (p_r(y+x,z) - p_r(y,z)))^2 dz dy \right)^{\frac{1}{2}} \\
&+ \left(\int_0^\infty \int_0^\infty (e^{-\frac{\beta}{2}x} - 1)^2 p_r(y,z)^2 dx dz \right)^{\frac{1}{2}} \\
\leq &x^{\frac{1}{2}} \|p_r\|_{L^2(\mathbb{R}_+^2)} + x^\gamma \|L_r^2\|_{L^2(\mathbb{R}_+^2)} + |e^{-\frac{\beta}{2}x} - 1| \|p_r\|_{L^2(\mathbb{R}_+^2)}
\end{aligned}
$$

$$\leq \left(x^{\frac{1}{2}} \|p_r\|_{L^2(\mathbb{R}_+^2)} + x^{\gamma} \|L_r^2\|_{L^2(\mathbb{R}_+^2)} + \frac{\beta}{2} x \|p_r\|_{L^2(\mathbb{R}_+^2)} \right)$$

$$\leq x^{\min(\gamma,\frac{1}{2})} (\|L_r^2\|_{L^2(\mathbb{R}_+^2)} + (1+\frac{\beta}{2})\|p_r\|_{L^2(\mathbb{R}_+^2)}). \tag{49}$$

Combining (48) and (49), we obtain, for $\|h\|_\beta = 1$,

$$\|(I - \mathcal{S}(x))\sigma_r Q^{\frac{1}{2}} h\|_\beta$$

$$\leq x^{\min(\gamma,\frac{1}{2})} \left[|f_r'(\zeta)| + \sqrt{8}(\frac{e^{\frac{\beta+1}{2}}}{\beta})\|f_r\|_\beta + \sqrt{2}\|L_r^1\|_{L^2(\mathbb{R}_+)} + \|L_r^2\|_{L^2(\mathbb{R}_+^2)} + (1+\frac{\beta}{2})\|p_r\|_{L^2(\mathbb{R}_+^2)} \right].$$

Now we can conclude that

$$b_n(T)^2 \leq \int_0^T \mathbb{E}[\sup_{x\in[0,\Delta_n]} \sup_{\|h\|_\beta=1} \|(I - \mathcal{S}(x))\sigma_r Q^{\frac{1}{2}} h\|_\beta^2] dr$$

$$\leq x^{2\min(\gamma,\frac{1}{2})} \int_0^T \mathbb{E}\left[\left(|f_r'(\zeta)| + \sqrt{8}(\frac{e^{\frac{\beta+1}{2}}}{\beta})\|f_r\|_\beta + \sqrt{2}\|L_r^1\|_{L^2(\mathbb{R}_+)} + \|L_r^2\|_{L^2(\mathbb{R}_+^2)} + (1+\frac{\beta}{2})\|p_r\|_{L^2(\mathbb{R}_+^2)} \right)^2 \right] dr.$$

This concludes the proof. □

6. Discussion and outlook

Our paper develops a new asymptotic theory for high-frequency estimation of the volatility of infinite-dimensional stochastic evolution equations in an operator setting. We have defined the so-called semigroup-adjusted realised covariation (SARCV) and derived a weak law of large numbers based on uniform convergence in probability with respect to the Hilbert–Schmidt norm. Moreover, we have presented various examples where our new method is applicable.

Many articles on (high-frequency) estimation for stochastic partial differential equations rely on the so-called spectral approach and assume therefore the applicability of spectral theorems to the generator A (cf. the survey article [19]). This makes it difficult to apply these results on differential operators that do not fall into the symmetric and positive definite scheme, as for instance $A = \frac{d}{dx}$ in the space of forward curves presented in Section 4.1.3, a case of relevance in financial applications that is included in our framework. Moreover, a lot of the related work assumes the volatility as a parameter of estimation to be real-valued (c.f. the setting in [19]). An exception is the spatio-temporal volatility estimation in the recent paper by [17] (see also [18] for limit laws for the power variation of fractional stochastic parabolic equations). Here, the stochastic integrals are considered in the sense of [45] and the generator is the Laplacian. In our analysis, we operate in the general Hilbert space framework in the sense of [25] for stochastic integration and semigroups.

In our framework, we work with high-frequent observations of Hilbert-space valued random elements, hence we have observations, which are discrete in time but not necessarily in space. Recent research on inference for parabolic stochastic partial differential considered observation schemes which allow for discreteness in time and space, cf. [15,17,18,20]. However, as our approach falls conveniently into the realm of functional data analysis, we might reconstruct data in several cases corresponding to well-known techniques for interpolation or smoothing. Indeed, in practice, a typical situation is that the Hilbert space consists of real-valued functions (curves) on $\mathbb{R}^d$ (or some subspace thereof), but we only have access to discrete observations of the curves. We may have data for $Y_{t_i}(x_j)$ at locations x_j, $j = 1, \ldots, m$, or possibly some aggregation of these (or, in more generality, a finite set of linear functionals of Y_{t_i}). For example, in commodity forward markets, we have only a finite number of forward contracts traded at all times, or, like in power forward markets, we have contracts with a delivery period (see e.g. [13]) and hence observations of the average of Y_{t_i} over intervals on $\mathbb{R}_+$. In other applications, like observations of temperature and wind fields in space and time, we may have accessible measurements at geographical locations where meteorological stations are situated, or, from atmospheric reanalysis where we have observations in grid cells regularly distributed in space. From such discrete observations, one must recover the Hilbert-space elements Y_{t_i}. This is a fundamental issue in functional data analysis, and several smoothing techniques have been suggested and studied. We refer to [42] for an extensive discussion of this. However, smoothing introduces another layer of approximation, as we do not recover Y_{t_i} but some approximate version $Y_{t_i}^m$, where the superscript m indicates that we have smoothed based on the m available observations. The construction of a curve from discrete observations is not a unique operation as this is an inverse problem. In future research, it will be interesting to extend our theory to the case when (spatial) smoothing has been applied to the discrete observations.

In addition, there could be cases, in which we do not have knowledge about a closed form of the semigroup, but rather the generator A. One then has to recover the semigroup adjusted increment in some way. Appealing to finite difference schemes like the implicit Euler method could be one way, which nevertheless, approximates the semigroup just strongly. Mathematically, this opens up an interesting numerical problem, which is left for future research.

Interestingly, when we compare our work to recent developments on high-frequency estimation for volatility modulated Gaussian processes in finite dimensions, see e.g. [41] for a survey, it appears that a scaling factor is needed in the realised (co)variation so that an asymptotic theory for Volterra processes can be derived. This scaling factor is given by the variogram of the associated so-called Gaussian core process, and depends on the corresponding kernel function. However, in our case, due to the semigroup property, we are in a better situation than for general Volterra equations, since we actually have (or can reconstruct) the data in order to compute the semigroup-adjusted increments. We can then develop our analysis based on extending the techniques and ideas that are used in the semimartingale case. In this way, the estimator becomes independent of further assumptions on the remaining parameters of the equation. However, the price to pay for this universality is that the convergence speed cannot generally be determined. The semigroup-adjustment of the increments effectively forces the estimator to converge at most at the same rate as the semigroup converges to the identity on the range of the volatility as t goes to 0. At first glance, it seems that the strong continuity of the semigroup suggests that we can obtain convergence just with respect to the strong topology. This would make it significantly harder to apply methods from functional data analysis, even for constant volatility processes. Fortunately, the compactness of the operators $\sigma_t Q^{\frac{1}{2}}$ for $t \in [0, T]$ comes to the rescue and enables us to prove that convergence holds with respect to the Hilbert–Schmidt norm. In this case, we obtain reasonable convergence rates for the estimator.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] Y. Aït-Sahalia, J. Jacod, High-Frequency Financial Econometrics, Princeton University Press, Princeton, New Jersey, 2014.
[2] Y. Aït-Sahalia, D. Xiu, Principal component analysis of high-frequency data, J. Amer. Statist. Assoc. 114 (525) (2019) 287–303.
[3] T.G. Andersen, T. Bollerslev, F.X. Diebold, P. Labys, Modeling and forecasting realized volatility, Econometrica 71 (2) (2003) 579–625.
[4] O.E. Barndorff-Nielsen, J. Corcuera, M. Podolskij, Multipower variation for Brownian semistationary processes, Bernoulli 17 (4) (2011) 1159–1194.
[5] O.E. Barndorff-Nielsen, J. Corcuera, M. Podolskij, Limit theorems for functionals of higher order differences of Brownian semistationary processes, in: A.E. Shiryaev, S.R.S. Varadhan, E. Presman (Eds.), Prokhorov and Contemporary Probability, in: Springer Proceedings in Mathematics and Statistics, vol. 33, 2013, pp. 69–96.
[6] O.E. Barndorff-Nielsen, N. Shephard, Non-Gaussian ornstein-uhlenbeck-based models and some of their uses in economics, J. R. Stat. Soc. Ser. B Stat. Methodol. 63 (2) (2001) (with discussion).
[7] O.E. Barndorff-Nielsen, N. Shephard, Econometric analysis of realized volatility and its use in estimating stochastic volatility models, J. R. Stat. Soc. Ser. B Stat. Methodol. 64 (2) (2002) 253–280.
[8] O.E. Barndorff-Nielsen, N. Shephard, Realized power variation and stochastic volatility models, Bernoulli 9 (2) (2003) 243–265, http://dx.doi.org/10.3150/bj/1068128977.
[9] O.E. Barndorff-Nielsen, N. Shephard, Econometric analysis of realized covariation: High frequency based covariance, regression, and correlation in financial economics, Econometrica 72 (3) (2004) 885–925.
[10] F.E. Benth, F.A. Harang, Infinite dimensional pathwise Volterra processes driven by Gaussian noise–probabilistic properties and applications, Electron. J. Probab. 26 (2021).
[11] F.E. Benth, P. Krühner, Representation of infinite-dimensional forward price models in commodity markets, Commun. Math. Stat. 2 (1) (2014) 47–106.
[12] F.E. Benth, B. Rüdiger, A. Süss, Ornstein–uhlenbeck processes in Hilbert space with non-Gaussian stochastic volatility, Stoch. Proc. Appl. 128 (2) (2018) 461–486.
[13] F. Benth, J. Šaltytė Benth, S. Koekebakker, Stochastic Modeling of Electricity and Related Markets, World Scientific, 2008.
[14] F.E. Benth, I.C. Simonsen, The Heston stochastic volatility model in Hilbert space, Stoch. Anal. Appl. 36 (4) (2018) 733–750.
[15] M. Bibinger, M. Trabs, Volatility estimation for stochastic PDEs using high-frequency observations, Stoch. Proc. Appl. 130 (5) (2020) 3005–3052.
[16] D. Bosq, Linear Processes in Function Space: Theory and Applications, Vol. 149, Springer, 2012.
[17] C. Chong, High-frequency analysis of parabolic stochastic PDEs, Ann. Statist. 48 (2) (2020) 1143–1167.
[18] C. Chong, R.C. Dalang, Power variations in fractional Sobolev spaces for a class of parabolic stochastic PDEs, 2020, E-print arxiv:2006.15817.
[19] I. Cialenco, Statistical inference for SPDEs: an overview, Stat. Inf. Stoch. Proc. 20 (2) (2018) 309–329.
[20] I. Cialenco, Y. Huang, A note on parameter estimation for discretely sampled SPDEs, Stoch. Dyn. 20 (03) (2020).
[21] J.M. Corcuera, E. Hedevang, M.S. Pakkanen, M. Podolskij, Asymptotic theory for Brownian semi-stationary processes with application to turbulence, Stoch. Proc. Appl. 123 (7) (2013) 2552–2574.

[22] J. Corcuera, D. Nualart, M. Podolskij, Asymptotics of weighted random sums, Commun. Appl. Ind. Math. 6 (1) (2014) e–486, 11.
[23] J.M. Corcuera, D. Nualart, J.H. Woerner, Power variation of some integral fractional processes, Bernoulli 12 (4) (2006) 713–735.
[24] G. Da Prato, A. Jentzen, M. Röckner, A mild itô formula for SPDEs, Trans. Amer. Math. Soc. 372 (6) (2019) 3755–3807.
[25] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, second ed., in: Encyclopedia of Mathematics and its Applications, vol. 152, Cambridge University Press, Cambridge, 2014.
[26] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Vol. 194, Springer Science & Business Media, 1999.
[27] F. Ferraty, P. Vieu, Nonparametric Functional Data Analysis: Theory and Practice, Springer, New York, 2006.
[28] D. Filipović, Consistency problems for HJM interest rate models, in: Lecture Notes in Mathematics, vol. 1760, Springer, Berlin, 2001.
[29] J. Gatheral, T. Jaisson, M. Rosenbaum, Volatility is rough, Quant. Finance 18 (6) (2018).
[30] A. Granelli, A.E. Veraart, A central limit theorem for the realised covariation of a bivariate Brownian semistationary process, Bernoulli 25 (3) (2019) 2245–2278.
[31] L. Horváth, P. Kokoszka, Inference for Functional Data with Applications, in: Springer Series in Statistics, Springer, 2012.
[32] J. Jacod, Asymptotic properties of realized power variations and related functionals of semimartingales, Stoch. Proc. Appl. 118 (4) (2008) 517–559.
[33] J. Jacod, P. Protter, Discretization of Processes, in: Stochastic Modelling and Applied Probability, vol. 67, Springer, Heidelberg, 2012.
[34] W. Liu, M. Röckner, Stochastic Partial Differential Equations: An Introduction, in: Universitext, Springer International Publishing, 2015.
[35] V. Mandrekar, L. Gawarecki, Stochastic Differential Equations in Infinite Dimensions, in: Probability and Its Applications, Springer, Berlin, Heidelberg, 2011.
[36] C. Marinelli, M. Röckner, On the maximal inequalities of Burkholder, Davis and Gundy, Expo. Math. 34 (2016) 1–26.
[37] D. Nualart, G. Peccati, Central limit theorems for sequences of multiple stochastic integrals, Ann. Probab. 33 (1) (2005) 177–193.
[38] V.M. Panaretos, S. Tavakoli, Fourier analysis of stationary time series in function space, Ann. Statist. 41 (2) (2013) 568–603.
[39] R. Passeggeri, A. Veraart, Limit theorems for multivariate Brownian semistationary processes and feasible results, Adv. Appl. Probab. 51 (2019) 667–716.
[40] S. Peszat, J. Zabczyk, Stochastic Partial Differential Equations with Lévy Noise, in: Encyclopedia of Mathematics and its Applications, vol. 113, Cambridge University Press, Cambridge, 2007.
[41] M. Podolskij, Ambit fields: survey and new challenges, in: R.H. Mena, J.C. Pardo, V. Rivero, G.U. Bravo (Eds.), XI Symposium of Probability and Stochastic Processes: CIMAT, Mexico, November 18-22, 2013, in: Progress in Probability, vol. 69, Springer, 2015, pp. 241–279.
[42] J. Ramsay, B.W. Silverman, Functional Data Analysis, second ed., in: Springer Series in Statistics, Springer, 2005.
[43] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, in: Grundlehren der mathematischen Wissenschaften, Springer, Berlin, Heidelberg, 1999.
[44] W. Rudin, Principles of Mathematical Analysis, third ed., McGraw-Hill, New York, 1976.
[45] J. Walsh, An introduction to stochastic partial differential equations, in: R. Carmona, H. Kesten, J. Walsh (Eds.), Ecole D'EtÉ de ProbabilitÉs de Saint-Flour XIV (1984), in: Lecture Notes in Mathematics, vol. 1180, Springer, 1986, pp. 265–436.
[46] F. Yao, H.-G. Müller, J.-L. Wang, Functional data analysis for sparse longitudinal data, J. Amer. Statist. Assoc. 100 (470) (2005) 577–590.