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Diagonal sums of doubly stochastic matrices

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ABSTRACT

Let Ω_n denote the class of $n \times n$ doubly stochastic matrices (each such matrix is entrywise nonnegative and every row and column sum is 1). We study the diagonals of matrices in Ω_n . The main question is which $A \in \Omega_n$ are such that the diagonals in A that avoid the zeros of A all have the same sum of their entries. We give a characterization of such matrices and establish several classes of patterns of such matrices.

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1. Introduction

Let M_n denote the (vector) space of real $n \times n$ matrices and on this space we consider the usual scalar product $A \cdot B = \sum_{i,j} a_{ij}b_{ij}$ for $A, B \in M_n$, $A = [a_{ij}]$, $B = [b_{ij}]$.

A permutation $\sigma = (k_1, k_2, \dots, k_n)$ of $\{1, 2, \dots, n\}$ can be identified with an $n \times n$ permutation matrix $P = P_\sigma = [p_{ij}]$ by defining $p_{ij} = 1$, if $j = k_i$, and $p_{ij} = 0$, otherwise ($i \leq n$). If $X = [x_{ij}]$ is an $n \times n$ matrix, the entries of X in the positions of X in which P has a 1 is the *diagonal* D_σ of X corresponding to σ and P , and their sum

$$d_P(X) = \sum_{i=1}^n x_{i,k_i}$$

is a *diagonal sum* of X . Sometimes we refer to the set of positions as a diagonal of X . Permutations $\sigma_1, \sigma_2, \dots, \sigma_k$ of $\{1, 2, \dots, n\}$, and their corresponding permutation matrices, are *pairwise disjoint* provided no two of them agree in any position; equivalently, no two of their corresponding permutation matrices have a 1 in the same position. We also then say that the associated diagonals are pairwise disjoint. A *zero diagonal* of X is a diagonal of X with 0's in all its positions. Without some restriction on the entries of X , diagonal sums can be quite arbitrary.

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Let $X = [x_{ij}]$ be a *doubly stochastic matrix*. Thus, X is a square nonnegative real matrix with all row and column sums equal to 1:

$$x_{ij} \geq 0 \ (i, j = 1, 2, \dots, n) \quad \text{and} \quad \sum_{i=1}^n x_{ij} = \sum_{i=1}^n x_{ji} = 1 \ (j = 1, 2, \dots, n).$$

The set of $n \times n$ doubly stochastic matrices is denoted by Ω_n . Ω_n is a polytope of dimension $(n-1)^2$ in the space M_n of real matrices of order n with the standard inner product.

A $n \times n$ matrix A is *partly decomposable* if suitable permutations of its rows and columns give a matrix

$$\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix},$$

where A_1 and A_3 are square, nonempty matrices. If A is not partly decomposable, it is called *fully indecomposable*. If A is fully indecomposable, then there exists a doubly stochastic matrix with the same pattern as A . If a doubly stochastic matrix is partly decomposable, then, after suitable row and column permutations, it is a direct sum of fully indecomposable doubly stochastic matrices. Thus in studying properties of doubly stochastic matrices, one usually assumes that they are fully indecomposable.

Sinkhorn [1] and Balasubramanian [2] independently proved the following theorem.

Theorem 1.1: *Let $X \in \Omega_n$, and let $\mathcal{D} = \{D_{\sigma_1}, D_{\sigma_2}, \dots, D_{\sigma_k}\}$ be a set of k pairwise disjoint zero diagonals of X . Assume that every diagonal of X that is disjoint with the diagonals in \mathcal{D} has a constant diagonal sum. Then all entries of X not on any of the diagonals in \mathcal{D} equal $\frac{1}{n-k}$.*

For a matrix X , let $\xi(X)$ be the set of positions in which X has 0's. Generalizing Theorem 1.1, Achilles [3] proved the following theorem.

Theorem 1.2: *Let $X, Y \in \Omega_n$ and let $Z_X \subseteq \xi(X)$ and $Z_Y \subseteq \xi(Y)$. Assume that all diagonals of X disjoint from Z_X have diagonal sum equal to α and all diagonals of Y disjoint from Z_Y have diagonal sum equal to β . The following hold:*

- (i) *If $Z_X \subseteq Z_Y$, then $\alpha \leq \beta$.*
- (ii) *If $Z_X = Z_Y$, then $\alpha = \beta$ and $X = Y$.*

Theorem 1.1 follows from this result by letting Y be the doubly stochastic matrix with the zeros as prescribed by the set $\mathcal{D} = \{D_{\sigma_1}, D_{\sigma_2}, \dots, D_{\sigma_k}\}$ of zero diagonals and with all other elements equal to $\frac{1}{n-k}$; see [3] for details.

Corollary 1.3: *An $n \times n$ doubly stochastic matrix X with a specified set Z of zeros all of whose diagonal sums avoiding Z are equal is uniquely determined.*

Since their publication more than 40 years ago, the three papers [1–3] seem not to have received any attention in the literature. In fact, Theorem 1.2 is true under more general circumstances as shown next.

Lemma 1.4: Let Γ be a polytope (in an inner product space) with set of extreme points $W = \{w_1, w_2, \dots, w_p\}$. Let $u, v \in \Gamma$ and let $X, Y \subseteq W$ such that

$$u = \sum_{x \in X} c_x x, \quad c_x \geq 0 \ (x \in X), \quad \sum_{x \in X} c_x = 1$$

and

$$v = \sum_{y \in Y} c_y y, \quad c_y \geq 0 \ (y \in Y), \quad \sum_{y \in Y} c_y = 1.$$

Assume that $u \cdot x = a$ for all $x \in X$ and $v \cdot y = b$ for all $y \in Y$. If $Y \subseteq X$, then $a \leq b$. If $Y = X$, then $a = b$ and $u = v$.

Proof: The proof follows the proof in [3]:

$$u \cdot u = u \cdot \left(\sum_{x \in X} c_x x \right) = \sum_{x \in X} c_x (u \cdot x) = a \sum_{x \in X} c_x = a(1) = a.$$

Similarly, $v \cdot v = b$. Now suppose that $Y \subseteq X$. Then a similar computation shows that $u \cdot v = a$ and thus

$$0 \leq (u - v) \cdot (u - v) = u \cdot u - 2u \cdot v + v \cdot v = a - 2a + b = b - a$$

and hence $a \leq b$. If $Y = X$, then we also have that $b \leq a$, and hence $a = b$ and $u = v$. ■

Let A be an $n \times n$ $(0, 1)$ -matrix which, without loss of generality, is assumed to be fully indecomposable. The matrix A defines a face $\mathcal{F}(A)$ of Ω_n consisting of all doubly stochastic matrices X with $\xi(A) \subseteq \xi(X)$ (see [4]). In general, $\mathcal{F}(A)$ contains matrices X where $\xi(A)$ is a proper subset of $\xi(X)$ such that all the diagonals of X not containing any positions in $\xi(A)$ have equal diagonal sums. The following example [3] is instructive.

Example 1.5: Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Consider a doubly stochastic matrix $X \in \mathcal{F}(A)$. Then X is of the form

$$X = \begin{bmatrix} a + b + c - 2 & 1 - c & 1 - b & 1 - a \\ 1 - c & c & 0 & 0 \\ 1 - b & 0 & b & 0 \\ 1 - a & 0 & 0 & a \end{bmatrix} \quad (0 \leq a, b, c \leq 1, a + b + c - 2 \geq 0).$$

In order for X to have the same set of 0's as A , we must have $0 < a, b, c < 1$ and $a + b + c - 2 > 0$. There are four diagonals of X that avoid the displayed 0's. A simple computation shows that if the corresponding four diagonal sums are equal, then $a = b = c$ and

$a + b + c - 2 = 0$ and hence

$$X = \frac{1}{3} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \tag{1}$$

We conclude that there does not exist a doubly stochastic matrix with 0's exactly where A has 0's and where all of the diagonals avoiding the positions of the 0's of A have the same sum. Thus not every zero pattern of a fully indecomposable $(0, 1)$ -matrix is realizable as the zero pattern of a doubly stochastic matrix whose diagonal sums avoiding the 0's are constant. Let the $(0, 1)$ -matrix A' be obtained from A by replacing the 1 in position $(1, 1)$ with a 0. Then $A' \leq A$ and A' is realizable as the zero pattern of a doubly stochastic matrix (namely X as in (1)) whose diagonal sums avoiding the 0's have constant value 2.

For later reference, we note that the matrix X in (1) and its zero pattern can be permuted to obtain

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array} \right] \rightarrow \frac{1}{3} \left[\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ \hline 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 \end{array} \right].$$

A motivation for considering our investigations is the following: Given an $n \times n$ matrix $X = [x_{ij}]$, the optimal assignment problem (OAP) asks for a permutation (i_1, i_2, \dots, i_n) of $\{1, 2, \dots, n\}$ such that the corresponding diagonal sum in X is maximum. Thus, x_{ij} is regarded as representing the value a person (corresponding to row i) brings to a job (corresponding to column j). An assignment of people $1, 2, \dots, n$ to jobs $1, 2, \dots, n$ is denoted by a permutation of $\{1, 2, \dots, n\}$, equivalently, an $n \times n$ permutation matrix. Let $A = [a_{ij}]$ be an $n \times n$ fully indecomposable $(0, 1)$ -matrix corresponding to people and jobs as above where an entry $a_{ij} = 0$ is interpreted as person i is not qualified for job j , and an entry $a_{ij} = 1$ is interpreted as person i is qualified for job j . Thus, the only allowable assignments are those avoiding the 0's in A . Assume that $\xi(X) = \xi(A)$. The largest allowable diagonal sum of X solves the OAP, under the restrictions imposed by A . Since A is fully indecomposable, so is X and as is well known, there exist diagonal matrices D_1 and D_2 with entries on the diagonal positive, such that $D_1 X D_2$ is doubly stochastic. We now assume that X is doubly stochastic, that is, we replace X with $D_1 X D_2$. Thus the values x_{ij} have been normalized so that the total value each person brings to the jobs and the total value of each job equals 1. If all diagonal sums of X avoiding the 0's of A are equal, then any permissible assignment solves the OAP.

Let A be an $n \times n$ fully indecomposable $(0, 1)$ -matrix such that there exists an $n \times n$ doubly stochastic matrix X with $\xi(X) = \xi(A)$, where all diagonals of X disjoint from $\xi(X)$ have equal sum. Call the matrix X a *restricted constant diagonal sum* (abbreviated to *RCDS*) doubly stochastic matrix determined by A , and call A the *pattern of an RCDS doubly stochastic matrix*. Note that if A is the pattern of an RCDS doubly stochastic matrix, so is PAQ for permutation matrices P and Q . An analogous assertion holds for X . Our goal is to investigate and give methods of construction of RCDS doubly stochastic matrices and their patterns, and some generalizations as discussed above. Note that if $A = J_n$ (the

$n \times n$ matrix of all 1's) so that $\xi(A) = \emptyset$, then $\frac{1}{n}J_n$ is an RCDS doubly stochastic matrix determined by A .

Example 1.6: Let A be a $(0, 1)$ -matrix with k 1's in each row and column. Define the $n \times n$ matrix X so that $\xi(X) = \xi(A)$ and every nonzero entry in X is $1/k$, i.e. $X = (1/k)A$. Then $X \in \Omega_n$ and every diagonal disjoint from $\xi(X)$ consists of only entries being $1/k$, so that the diagonal sum is n/k . Therefore, X is the RCDS doubly stochastic matrix determined by A . It is well known that when A has this form, A contains k pairwise disjoint diagonals, so this example is of the type considered in Theorem 1.1. Note that all nonzero entries in X are equal. Clearly, every matrix with this property must have the form of this example. Below is a specific example with $n = 4$ and $k = 2$, where we indicate a diagonal in boldface

$$A = \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 \\ 1 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 1 & \mathbf{1} \end{bmatrix}, \quad X = (1/2)A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

We also note that one can determine in polynomial time if a given matrix X is an RCDS doubly stochastic matrix. First, one checks if X is doubly stochastic (trivial), and then one solves two optimal assignment problems, namely

$$\max_{P \leq X} P \cdot X \quad \text{and} \quad \min_{P \leq X} P \cdot X,$$

where P ranges through permutation matrices P satisfying $P \leq X$. We then check if these two optimal values coincide.

The remaining part of the paper is organized as follows. In the next section, we give a characterization of RCDS doubly stochastic matrices and a method for their construction. A discussion of a strengthening of the RCDS property is given next. In the two sections that follow, we develop certain classes of RCDS doubly stochastic matrices. In the final section, we briefly consider the difference of diagonal sums of a doubly stochastic matrix.

Notation: A is a *nonnegative matrix* (resp., *positive matrix*), and we write $A \geq O$ (resp., $A > 0$), if each entry in A is nonnegative (resp., positive).

2. Characterization of RCDS matrices

In this section, we use the duality theorem of linear programming to give a characterization of RCDS doubly stochastic matrices, which affords a means to construct them.

Theorem 2.1: Let $A = [a_{ij}]$ be a fully indecomposable $(0, 1)$ -matrix of size $n \times n$ and let $R = (r_1, r_2, \dots, r_n)$ and $S = (s_1, s_2, \dots, s_n)$ be the row and column sum vectors of A .

- (i) Let $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ be real vectors. Define $Y = Y(u, v) = [y_{ij}] \in M_n$ by $y_{ij} = u_i + v_j$ whenever $a_{ij} = 1$, and $y_{ij} = 0$ otherwise. Assume that $y_{ij} > 0$ whenever $a_{ij} = 1$ and that all row and column sums of Y are equal to some positive

number α , i.e.

$$\begin{aligned} u_i r_i + \sum_{j:a_{ij}=1} v_j &= \alpha \quad (i \leq n), \\ v_j s_j + \sum_{i:a_{ij}=1} u_i &= \alpha \quad (j \leq n). \end{aligned} \tag{2}$$

Then $X = (1/\alpha)Y(u, v)$ is an RCDS doubly stochastic matrix of A .

(ii) Conversely, assume X is an RCDS doubly stochastic matrix of A . Then $X = (1/\alpha)Y(u, v)$, as in (i), for some vectors u and v , and α as the common line sum of $Y(u, v)$.

Proof: Since all row and column sums of $Y = Y(u, v)$ are equal to α , and $y_{ij} \geq 0$ ($i, j \leq n$), $X := (1/\alpha)Y$ is a doubly stochastic matrix. Clearly, $\xi(X) = \xi(Y) = \xi(A)$. Consider a nonzero diagonal in Y corresponding to the permutation $\sigma = (k_1, k_2, \dots, k_n)$. The associated diagonal sum in Y is

$$d_\sigma^Y = \sum_{i=1}^n y_{i,k_i} = \sum_{i=1}^n (u_i + v_{k_i}) = \sum_{i=1}^n u_i + \sum_{i=1}^n v_{k_i} = \sum_{i=1}^n u_i + \sum_{i=1}^n v_i$$

which is independent of σ . Thus, all diagonal sums in Y , and therefore in X , are equal, and X is an RCDS doubly stochastic matrix of A . This proves (i).

To prove (ii), for the given RCDS matrix $X = [x_{ij}]$ consider the linear optimization problem

$$\begin{aligned} &\text{minimize} && \sum_{i,j} x_{ij} y_{ij} \\ &\text{subject to} && \sum_j y_{ij} = 1 \quad (i \leq n) \\ &&& \sum_i y_{ij} = 1 \quad (j \leq n) \\ &&& y_{ij} \geq 0 \quad (i, j \leq n). \end{aligned} \tag{3}$$

Here, we use variables y_{ij} only for those (i, j) such that $x_{ij} \neq 0$; the other y_{ij} can be assumed to be 0. It is well-known that the coefficient matrix in (3) is totally unimodular (see [5]), so there is an optimal solution which is integral; therefore, $Y = [y_{ij}]$ is a permutation matrix. Thus, the optimal value γ in (3) is the minimum diagonal sum in the matrix X ; in fact, all diagonal sums in X equal γ , by assumption, and this optimal assignment problem is solved. By the duality theorem of linear optimization, γ is also equal to the maximum value in the dual problem

$$\begin{aligned} &\text{maximize} && \sum_i u_i + \sum_j v_j \\ &\text{subject to} && u_i + v_j \leq x_{ij} \quad (i, j \leq n). \end{aligned} \tag{4}$$

Here, the constraints are present only for those (i, j) such that $x_{ij} \neq 0$. So, there exists $u_i^*, v_j^* (i \leq n)$ such that

$$u_i^* + v_j^* \leq x_{ij} \quad (i, j \leq n) \quad \text{and} \quad \sum_i u_i^* + \sum_j v_j^* = \gamma.$$

To examine the duality relation closer note that if $Y = [y_{ij}] \in \Omega_n$ (so it satisfies the constraints in (3)), then

$$\sum_{i,j} x_{ij} y_{ij} \geq \sum_{i,j} (u_i^* + v_j^*) y_{ij} = \sum_i u_i^* \sum_j y_{ij} + \sum_j v_j^* \sum_i y_{ij} = \sum_i u_i^* + \sum_j v_j^* = \gamma. \tag{5}$$

If Y is a permutation matrix, then the left-hand side here is also equal to γ , and therefore the inequality holds with equality. This means that if $y_{ij} = 1$, then $x_{ij} = u_i^* + v_j^* (i, j \leq n)$. Since A is fully indecomposable, for every (i, j) outside $\xi(A)$, there exists a permutation matrix with a 1 in that position. Therefore, $x_{ij} = u_i^* + v_j^* (i, j \leq n)$ as desired. ■

We remark that the assumption that A is fully indecomposable is only used in part (ii) of the theorem. Moreover, Theorem 2.1 is closely related to Theorems 2.6.3 and 2.6.4 in [6] where one studies products of diagonals of doubly stochastic matrices. Our result may be obtained by taking the logarithm of these products. The proofs are quite similar in using integrality of the polytope of doubly stochastic matrices and LP duality. However, our proof is shorter for the main part (ii) due to our analysis of equality in (5).

The construction given in the theorem is illustrated in the next example.

Example 2.2: Let A be the $(0, 1)$ -matrix

$$\begin{bmatrix} & 1 & 1 & 1 & & & \\ 1 & & & & 1 & 1 & \\ 1 & & 1 & & & & \\ 1 & & & 1 & & & \\ & 1 & & & 1 & & \\ & 1 & & & & & 1 \end{bmatrix}.$$

Choose vectors u and v as indicated below and let $y_{ij} = u_i + v_j$ for each $(i, j) \notin \xi(A)$

| $u \setminus v$ | 0 | 0 | 1 | 1 | 1 | 1 |
|-----------------|---|---|---|---|---|---|
| 1 | | 1 | 2 | 2 | | |
| 1 | 1 | | | | 2 | 2 |
| 2 | 2 | | 3 | | | |
| 2 | 2 | | | 3 | | |
| 2 | | 2 | | | 3 | |
| 2 | | 2 | | | | 3 |

where every line sum is 5. Therefore, by Theorem 2.1, the diagonal sums (avoiding zeros of A) are equal, in fact equal to 14. The corresponding RCDS doubly stochastic matrix is

$$X = \frac{1}{5} \begin{bmatrix} & 1 & 2 & 2 & & \\ 1 & & & & 2 & 2 \\ 2 & & 3 & & & \\ 2 & & & 3 & & \\ & 2 & & & 3 & \\ & 2 & & & & 3 \end{bmatrix}.$$

Example 2.3: Consider the following RCDS doubly stochastic matrix and its construction from vectors u and v :

$$\frac{1}{4} \begin{bmatrix} 1 & & & 1 & 2 & & \\ & & & & 2 & 2 & \\ & & & & & 2 & 2 \\ 1 & & & 1 & & & 2 \\ \hline 2 & 2 & & & & & \\ & 2 & 2 & & & & \\ & & 2 & 2 & & & \end{bmatrix}, \quad \begin{array}{c|cccc|ccc} u \setminus v & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ \hline 2 & 1 & & & 1 & 2 & & \\ 2 & & & & & 2 & 2 & \\ 2 & & & & & & 2 & 2 \\ 2 & 1 & & & 1 & & & 2 \\ \hline 3 & 2 & 2 & & & & & \\ 3 & & 2 & 2 & & & & \\ 3 & & & 2 & 2 & & & \end{array}$$

Note that here v has some negative components. By adding 1 to the components of u and subtracting 1 from the components of v , we obtain nonnegative u and v .

Theorem 2.1 may be used to construct classes of RCDS doubly stochastic matrices in the following way:

- (1) Let $n \geq 1$, and let $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ be real vectors. Define the matrix $Y = Y(u, v) = [y_{ij}] \in M_n$ by $y_{ij} = u_i + v_j$ and assume that u and v are chosen so that Y is nonnegative (see remark below).
- (2) Choose an $n \times n$ $(0, 1)$ -matrix S such that the *Hadamard product* (that is, entrywise product) $Y \circ S$ has constant positive line (row and column) sums, and let α denote this sum.
- (3) Then $V = (1/\alpha)Y \circ S$ is an RCDS doubly stochastic matrix.

Clearly, the nontrivial part is step 2 which is to select an appropriate S , that is, to select entries from Y such that all line sums for the selected entries are constant. Then the fact that $V = (1/a)Y \circ S$ is an RCDS doubly stochastic matrix follows from Theorem 2.1. Moreover, *any* such RCDS doubly stochastic matrix may be constructed in this way. Those entries of Y which are not selected could have been negative without altering conclusion 3 above.

We give an example of this procedure. Let $k, t \geq 1$ be integers and consider an $n \times n$ matrix

$$V = \frac{1}{tp} \begin{bmatrix} tI_k & tI_k & \cdots & tI_k \\ \hline & A & & \end{bmatrix}, \tag{6}$$

where the block tI_k occurs p times, $n = kp$, and A is an $(n - k) \times n$ $(0, 1)$ -matrix.

We denote the column vector of k 1's by e_k .

Theorem 2.4: Let $k, t, p \geq 1$ be integers with $t \leq k$, and let $n = kp$. Define constant vectors $R = tpe_{n-k}$ and $S = t(p-1)e_n$, and let A be a $(0, 1)$ -matrix with row sum vector R and column sum vector S . Then V in (6) is an RCDS doubly stochastic matrix.

Proof: Let $u = (t, t, \dots, t, 1, 1, \dots, 1) \in \mathbb{R}^n$ where the first k components are t . Let v be the zero vector of length n . Then the matrix $Y = Y(u, v)$ (defined above) has n columns, each equal to u . So every entry in the first k rows of Y is t and all other entries are 1. Thus, $V = (1/\alpha)Y \circ S$ where $\alpha = tp$ and S is the $(0, 1)$ -matrix with ones in the positions of the nonzeros of V . By the procedure above V is an RCDS doubly stochastic matrix. It only remains to show that a matrix $A \in \mathcal{A}(R, S)$ exists. Let $R = (r_1, r_2, \dots, r_n)$ and $S = (s_1, s_2, \dots, s_n)$. Note that $t(p-1) \leq n-k$ as $n-k = kp-k = (p-1)k$ and $t \leq k$. Thus, $r_i \leq n$ and $s_j \leq n-k$ for each i and j . Moreover, $\sum_i r_i = (n-k)tp = ntp - ktp$ and $\sum_j s_j = n(p-1)t$. But then the class $\mathcal{A}(R, S)$ is nonempty as both R and S are constant vectors. This may be verified from the Gale-Ryser theorem (see e.g. [7]) as S is majorized by the conjugate R^* of R . ■

Example 2.5: Let $k = 3, t = 2, p = 2$ and $n = kp = 6$. Then the following matrix:

$$V = (1/4) \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ \hline 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

is an RCDS doubly stochastic matrix with a constant diagonal sum $9/4$.

Theorem 2.4 shows that RCDS patterns may be complicated, and at least as complicated as the pattern of $(0, 1)$ -matrices with constant row sums and constant column sums.

We return to the characterization in Theorem 2.1. Let A be a given $n \times n$ fully indecomposable $(0, 1)$ -matrix. Let $R(A) = (r_1, r_2, \dots, r_n)$ and $S(A) = (s_1, s_2, \dots, s_n)$ be the row sum and column sum vectors of A . Let D_R and D_S be the diagonal matrices with main diagonals $R(A)$ and $S(A)$, respectively. Define

$$R_i(A) = \{j : a_{ij} = 1\}, \quad (i \leq n) \quad \text{and} \quad C_j(A) = \{i : a_{ij} = 1\} \quad (j \leq n).$$

Thus, $r_i = |R_i(A)|$ and $s_j = |C_j(A)|$ for each i and j . In Equation (2) in Theorem 2.1, we may assume $\alpha = 1$. This gives

$$\begin{aligned} r_i u_i + \sum_{j \in R_i(A)} v_j &= 1 \quad (i \leq n), \\ s_j v_j + \sum_{i \in C_j(A)} u_i &= 1 \quad (j \leq n), \end{aligned} \tag{7}$$

which is a linear system of equations with $2n$ variables u_i, v_j ($i, j \leq n$) and $2n$ constraints. Rewriting this system in matrix form gives

$$Hx = e, \quad \text{where } H = \begin{bmatrix} D_R & A \\ A^T & D_S \end{bmatrix} \text{ and } x = \begin{bmatrix} u \\ v \end{bmatrix}. \quad (8)$$

Here, e is the all ones vector (of suitable dimension).

The unexpected observation is that the matrix H is equal to the signless Laplacian matrix of the bipartite graph $BG(A)$ whose biadjacency matrix is A . Therefore, a lot is known on H in terms of spectral properties, e.g. H is positive semidefinite and singular. The vector $w = (e, -e)$ lies in the null space of H . Thus, the system (8) has H as the coefficient matrix, the right-hand side is all ones, and we look for a solution with a certain nonnegativity property. We may solve this system using the block structure:

$$D_R u + Av = e, \quad A^T u + D_S v = e$$

which gives $u = D_R^{-1}(e - Av)$, so $A^T D_R^{-1}(e - Av) + D_S v = e$, i.e.

$$(A^T D_R^{-1} A - D_S) v = A^T D_R^{-1} e - e.$$

We next discuss whether this system has a solution or, equivalently, if $Hx = e$ has a solution.

The following is the main result on RCDS patterns, based on the discussion above.

Theorem 2.6: *Let A be an $n \times n$ fully indecomposable $(0, 1)$ -matrix. Then the following holds:*

- (i) *There exists $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ such that (u, v) is a solution of the system (8). The solution is unique up to adding a constant to each component in u and subtracting the same constant from each component in v .*
- (ii) *A is the pattern of an RCDS doubly stochastic if and only if $u_i + v_j > 0$ for all (i, j) with $a_{ij} = 1$, where (u, v) is an arbitrary solution of (8).*

Proof: First, as A is fully indecomposable, the bipartite graph $BG(A)$ is connected. Since H is the signless Laplacian of $BG(A)$, and this graph is connected, it is a known fact that 0 is a simple eigenvalue of H . So H has rank $2n - 1$. $Hx = e$ has a solution provided that e lies in the range (column space) of H , so we need to show this. Let L denote the null space of H , so $L = \text{span}\{w\}$ where $w = (e, -e)$. Then L is the orthogonal complement of the row space of H . The row space and the column space are equal, as H is symmetric. But

$$e \cdot w = n - n = 0.$$

Thus e is in L^\perp , and it follows that e lies in the range of H . Hence $Hx = e$ has a solution, and a general solution is obtained by adding some multiple of the vector w . This shows (i).

Next, assume A is a pattern of an RCDS. Then, by Theorem 2.1 and the discussion above there exists a solution $x = (u, v)$ of $Hx = e$. By (i) in this theorem, the solution is unique up to adding a multiple of $w = (e, -e)$ (w spans the null space of H), but this does not change the value of $x_{ij} = u_i + v_j$. Thus, we must have that $x_{ij} > 0$ (from the initial assumption). The converse implication follows directly from Theorem 2.1, and the proof is complete. ■

Corollary 2.7: *Let A be an $n \times n$ fully indecomposable, symmetric $(0, 1)$ -matrix which is the pattern of an RCDS doubly stochastic matrix. Then there exists a symmetric, RCDS doubly stochastic matrix X with pattern A , and a vector $w = (w_1, w_2, \dots, w_n)$ such that $X = A \circ W$ where $W = [w_{ij}]$ with $w_{ij} = w_i + w_j$ ($1 \leq i, j \leq n$).*

Proof: Let Z be an RCSD doubly stochastic matrix with pattern A . Since A is symmetric, Z^T is also an RCSD doubly stochastic matrix with pattern A . Hence $X = Z + Z^T$ is a symmetric, RCSD doubly stochastic matrix with pattern A . By Theorem 2.6, there exists $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ such that, with the matrix $Y(u, v) = [y_{ij}]$ where $y_{ij} = u_i + v_j$ (for all i, j), $X = A \circ Y(u, v)$. But then we also have $X = A \circ Y(\frac{u+v}{2}, \frac{u+v}{2})$. ■

If A is fully indecomposable, we have a polynomial-time algorithm for deciding if A is an RCDS pattern: One first finds a (near-unique) solution $x = (u, v)$ of $Hx = e$. An efficient way of finding x is by solving

$$(A^T D_R^{-1} A - D_S) v = A^T D_R^{-1} e - e$$

and defining $u = D_R^{-1}(e - Av)$. Next, we simply check if $x_{ij} = u_i + v_j > 0$ for (i, j) with $a_{ij} = 1$. If these strict inequalities hold, then A is an RCDS pattern; otherwise, it is not.

We remark that this algorithm may also be used to find some ‘random’ RCDS patterns. One generates a random $(0, 1)$ -matrix A and runs the algorithm above. Then, even if A is not an RCDS pattern it may happen that the resulting matrix X is doubly stochastic, but its support is *contained* in the support of A . This procedure has been used in the example below. Finally, we note that the rank of H satisfies $n \leq \text{rank}(H) \leq 2n - 1$ where the lower bound is obtained when A is a permutation matrix.

Example 2.8: Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Using the procedure above we compute v and u : $v = (0, 0.3, 0.1, 0, 0.25)$ and $u = (0.25, 0.2, 0.25, 0.2, 0.3)$. From this, we obtain

$$X = \begin{bmatrix} 0.25 & 0 & 0 & 0.25 & 0.5 \\ 0 & 0.5 & 0.3 & 0.2 & 0 \\ 0.25 & 0 & 0 & 0.25 & 0.5 \\ 0.2 & 0.5 & 0.3 & 0 & 0 \\ 0.3 & 0 & 0.4 & 0.3 & 0 \end{bmatrix}$$

which is an RCDS doubly stochastic matrix corresponding to the pattern A .

Example 2.9: A_1 and A_2 below are some RCDS patterns found by the procedure above.

$$A_1 = \begin{bmatrix} 1 & 1 & & 1 & 1 \\ 1 & & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & 1 & 1 & & \\ 1 & 1 & & & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} & & 1 & 1 & \\ 1 & & 1 & 1 & 1 \\ & 1 & & 1 & 1 \\ & 1 & & & 1 \\ 1 & 1 & & 1 & 1 \end{bmatrix}.$$

Next, recall the fully indecomposable matrix A in Example 1.5; we showed that it is not an RCDS pattern. When the procedure above is applied to the matrix A , we obtain the doubly stochastic matrix X below. Since entry $(1, 1)$ is zero in X , this proves that A is not an RCDS pattern, see Theorem 2.6. We can also conclude that the pattern of X is RCDS, and this is the matrix A' mentioned in Example 1.5.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} & 1/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & & \\ 1/3 & & 2/3 & \\ 1/3 & & & 2/3 \end{bmatrix}.$$

3. Compatible classes of permutation matrices

In view of our earlier discussion, we now consider, primarily through examples, the following general problem (cf. Lemma 1.4). Let A be an $n \times n$ fully indecomposable $(0, 1)$ -matrix and let $\mathcal{P}(A)$ be the set of permutation matrices $P \leq A$. Thus, $\mathcal{F}(A) = \{X : X \leq A, X \in \Omega_n\}$ is a face of Ω_n whose set of extreme points is $\mathcal{P}(A)$. The cardinality of $\mathcal{P}(A)$, the number of extreme points of $\mathcal{P}(A)$, equals the permanent, $\text{per}(A)$, of A .

The set $\mathcal{P}(A)$ is a *compatible class* of permutation matrices provided that the scalar product $Q \cdot \sum_{P \in \mathcal{P}(A)} P$ is a constant $\gamma(A)$ for all $Q \in \mathcal{P}(A)$. We also describe this by saying that A has *compatible permutation support* (abbreviated to *CPS*). If A has CPS, then the matrix

$$\widehat{A} = \frac{1}{\text{per } A} \sum \{P : P \in \mathcal{P}(A)\} \tag{9}$$

is a doubly stochastic matrix with constant diagonal sums avoiding the zero set $\xi(A)$ of A . Thus, the CPS property determines a subclass of RCDS doubly stochastic matrices and provides another possible way to construct such matrices.

In our discussion that follows, it is usually more convenient to drop the normalizing factor $\frac{1}{\text{per}(A)}$ and to use instead of (9) the matrix

$$\widehat{A} = \sum \{P : P \in \mathcal{P}(A)\}. \tag{10}$$

In order that $\mathcal{P}(A)$ be a compatible class of permutations, it is necessary and sufficient that the sum of the permanental minors of A corresponding to the 1's of each permutation matrix $P \leq A$ equals a constant $\gamma(A)$ independent of the permutation matrix P . This is because the permanental minor of a 1 in A counts the number of permutation matrices $Q \leq A$ which use that 1. Hence if A has CPS, then for each permutation $\sigma = (j_1, j_2, \dots, j_n)$

of $\{1, 2, \dots, n\}$ with corresponding permutation matrix $P_\sigma \leq A$,

$$\gamma(A) = \sum_{i=1}^n \text{per } A(i | j_i), \tag{11}$$

where $\text{per } A(i | j_i)$ is the permanent of the matrix obtained from A by deleting row i and column j_i . Thus, the matrix in (9) is an RCDS doubly stochastic matrix if and only if the sum of the permanental minors of the 1's corresponding to each permutation matrix $P \leq A$ is constant. We remark here that Bapat [8] proved that if an $n \times n$ fully indecomposable $(0, 1)$ -matrix A satisfies that the permanental minors of all entries (both 0's and 1's) are constant, then $A = J_n$ or (after row and column permutations) $A = I_n + P_n$, where P_n is the permutation matrix corresponding to the permutation $(2, 3, \dots, n, 1)$.

Example 3.1: The following two examples of RCDS doubly stochastic matrices come from simplex faces of the polytope Ω_7 using the construction given in (9):

$$\frac{1}{4} \begin{bmatrix} 1 & & & 1 & 2 & & \\ & & & 2 & 2 & & \\ & & & & & 2 & 2 \\ 1 & & & 1 & & & 2 \\ \hline 2 & 2 & & & & & \\ & 2 & 2 & & & & \\ & & 2 & 2 & & & \end{bmatrix} \quad \text{and} \quad \frac{1}{9} \begin{bmatrix} & & & & 3 & 3 & 3 \\ & 1 & 1 & 1 & 6 & & \\ & 1 & 1 & 1 & & 6 & \\ & 1 & 1 & 1 & & & 6 \\ \hline 3 & 6 & & & & & \\ 3 & & 6 & & & & \\ 3 & & & 6 & & & \end{bmatrix}.$$

The first has the constant diagonal sum $\frac{13}{4}$; the second has the constant diagonal sum $\frac{7}{3}$. Both matrices are symmetric. For instance, for the first matrix $u = (1, 1, 1, 1, 2, 2, 2)$ and $v = (0, 0, 0, 0, 2, 2, 2)$ work as in Theorem 2.6.

Example 3.2: We consider again the matrix A in Example 2.8, repeated below. The permanental minors of its 1's are given in the matrix M where * corresponds to a permanental minor of a 0 and so is not included in our calculations:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 3 & * & * & 3 & 6 \\ * & 6 & 4 & 2 & * \\ 3 & * & * & 3 & 6 \\ 2 & 6 & 4 & * & * \\ 4 & 8 & 4 & 4 & * \end{bmatrix}.$$

M has diagonal sums of $3 + 4 + 6 + 6 + 4 = 23$ and $6 + 6 + 3 + 2 + 4 = 21$. Hence M does not have equal diagonal sums avoiding the 0's of A . While A is the nonzero pattern of an RCDS doubly stochastic matrix, A does not have CPS. We conclude that CPS is a stronger property than RCDS.

There is a cubic bipartite graph G of order 54 contained in the complete bipartite graph $K_{27,27}$, called the *Gray graph*, whose automorphism group of cardinality 1296 acts transitively on each set of the bipartition (but not on the complete vertex set) and transitively on the edge set (an *edge-transitive graph* but not a *vertex-transitive graph*). The Gray graph is the smallest cubic edge-transitive graph which is not vertex-transitive [9].

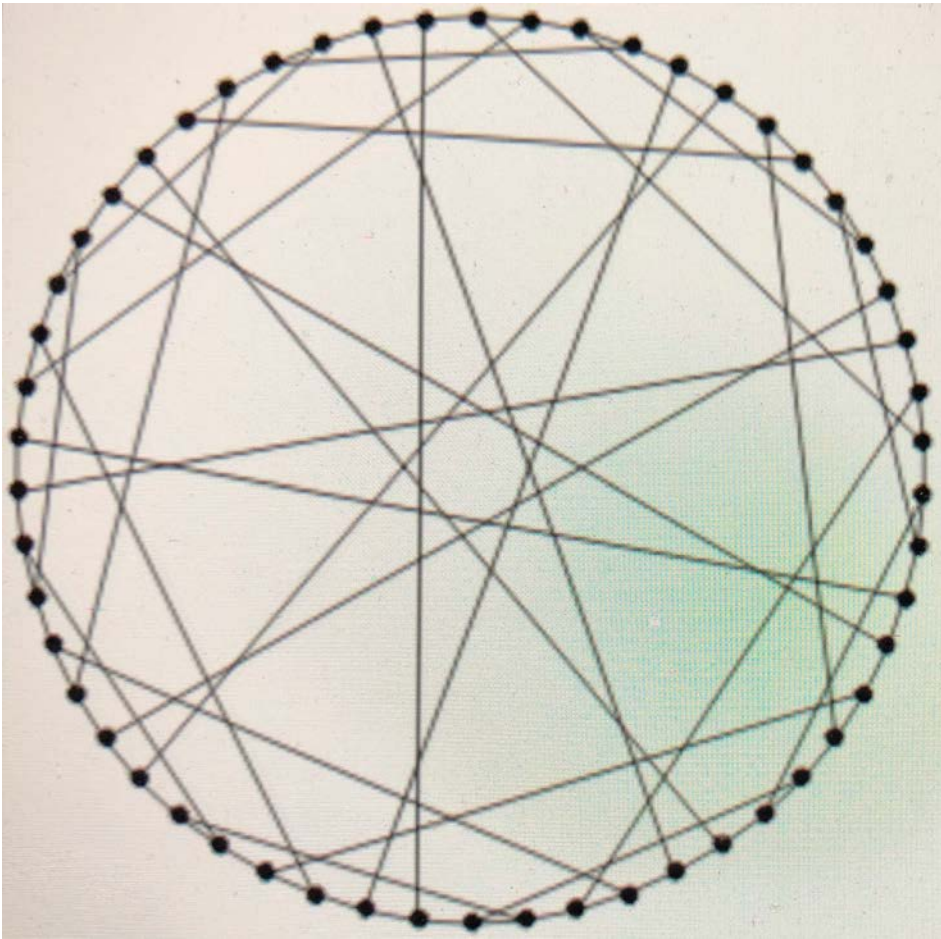


Figure 1. The Gray Graph G (from Wikipedia, en.wikipedia.org).

Let A be the 27×27 adjacency matrix of G with suitable labelling of its vertices. Using Figure 1, we have constructed A as shown in Figure 2.

Since the automorphism group of G is edge-transitive, every edge must be in the same number of perfect matchings since perfect matchings are preserved under automorphisms. Since the perfect matchings containing an edge correspond to permutation matrices $P \leq A_G$ containing the 1 corresponding to the edge, each 1 is in the same number of permutation matrices $P \leq A_G$. This implies that the permanent minors of the 1's of A_G are all equal. Thus, A_G has CPS and, in particular, \hat{A}_G (see (9)) is a RDCS doubly stochastic matrix, and indeed has a much stronger property.

There are other classes of $n \times n$ $(0,1)$ -matrices whose 1's have constant permanent minors. The matrices J_n and $I_n + P_n$ as previously discussed (all of whose entries, not just the entries equal to 1) have constant permanent minors. The $n \times n$ $(0,1)$ -matrix $J_n - I_n$ of all 1's except for 0's on the main diagonal has all the permanent minors of its 1's equal to the permanent of an $(n - 1) \times (n - 1)$ $(0, 1)$ -matrix with exactly $(n - 2)$ 0's where no two of these 0's belong to the same row or column. The permanent minors of the 0's are also

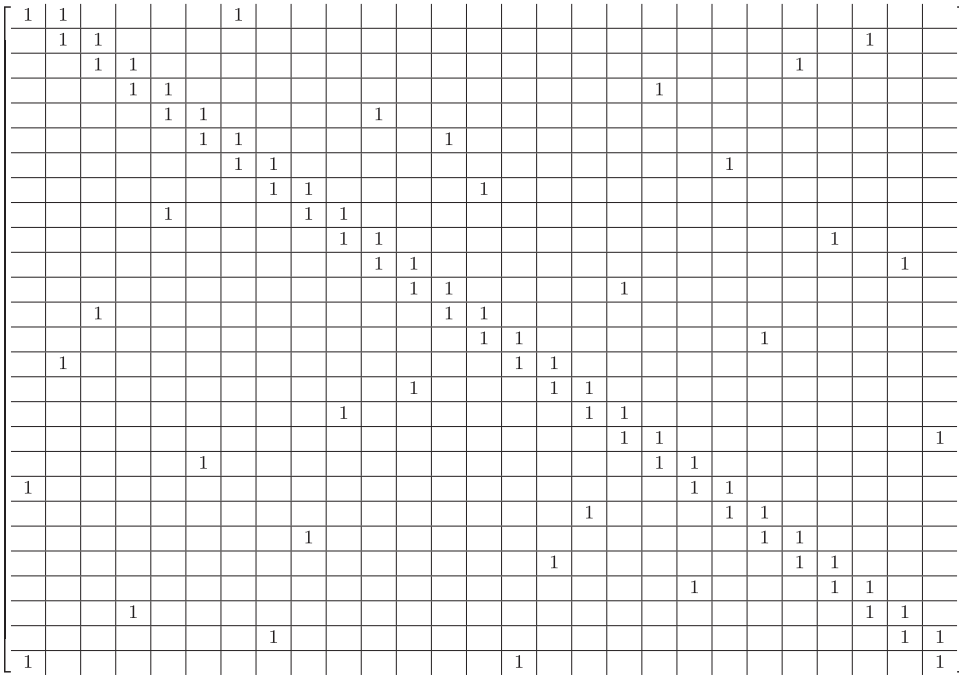


Figure 2. Adjacency matrix A_G of the Gray graph G .

constant but a different constant if $n \geq 4$. Thus, $\frac{1}{D_n}(J_n - I_n)$ is a RCDS doubly stochastic matrix where D_n is the permanent of $J_n - I_n$ (the n th derangement number).

In general, simplex faces of the polytope Ω_n correspond to $n \times n$ fully indecomposable $(0,1)$ -matrices which, after permutations of rows and columns, have the form

$$A = \left[\begin{array}{c|c} A_3 & A_1 \\ \hline A_2 & O \end{array} \right], \tag{12}$$

where, for some p with $0 \leq p \leq n - 2$, A_3 is an $(n - p) \times (p + 1)$ nonzero matrix, and A_1 and A_2^T are vertex-edge incidence matrices of trees T_1 and T_2 and so have exactly two 1's in each column (see Theorem 3.5 of [4]). Note that for Example 3.1, the 1's in A_3 are in the rows and columns determined by the pendent vertices of T_1 and T_2 , respectively. Each such row and column of A_3 must contain at least one 1 in order that A be fully indecomposable. The number of permutations $P \leq A$ equals the number of 1's in A_3 with each such 1 on exactly one such permutation matrix P . Let there be k 1's in A_3 so that there are exactly k permutation matrices P_1, P_2, \dots, P_k with the $P_i \leq A$. Then $X = P_1 + P_2 + \dots + P_k$ is a nonnegative integral matrix with pattern A and all its row and column sums equal k . Thus $\frac{1}{k}X$ is a doubly stochastic matrix.

Consider a simplex face given by (12) in which A_3 has at least one 1 in those rows corresponding to the pendent vertices of T_1 and in those columns corresponding to the pendent vertices of T_2 , and in no other positions. Then A is fully indecomposable. Let A_1^* be the matrix obtained from A_1 by including as a new first column, the column vector which has 1's exactly in those rows in which A_3 has a 1. Let A_2^* be defined in a similar way using the columns of A_3 . Then A_1^* and A_2^{*T} are square, vertex-edge incidence matrices of loopy

trees with loops on exactly the pendent vertices of T_1 and T_2 , respectively. Then A has CPS if and only if the permanent minors of the new 1's in A_1^* are constant and similarly for A_2^* .

4. Tridiagonal RCDS patterns and trees

Let A be an $n \times n$ $(0, 1)$ -matrix. Recall that $\mathcal{F}(A) = \{X \in \Omega_n : X \leq A\}$ is the face of the polytope Ω_n determined by A and consists of all those doubly stochastic $n \times n$ matrices that have zeros wherever A has. If A is fully indecomposable, then (see e.g. [10]) the dimension of this face is $\sigma(A) - 2n + 1$ where $\sigma(A)$ is the number of ones in A . For a matrix A , we let $\rho(A)$ denote its term rank.

Lemma 4.1: *Let X be an RCDS in Ω_n . Let X' be a 2×2 submatrix such that $X' > 0$ (entrywise) and its complementary submatrix X'' satisfies $\rho(X'') = n - 2$. Then the two diagonals in X' have the same sum.*

Proof: This follows from the fact that a diagonal in X'' can be combined with any of the two diagonals in X' to get a diagonal of X . ■

Lemma 4.1 gives a necessary condition for a matrix to be an RCDS doubly stochastic matrix. In a certain situation, this condition is also sufficient, as we now discuss.

Tridiagonal doubly stochastic matrices were studied in [11], and a number of results on this special face Ω_n^t of Ω_n were established. Any matrix $X = [x_{ij}] \in \Omega_n^t$ is symmetric and determined by the entries on the superdiagonal, $x_i = x_{i,i+1}$ ($i = 1, 2, \dots, n - 1$). We write $X = X(x)$ to indicate this dependency. For instance, for $n = 5$, the general form of a tridiagonal matrix doubly stochastic matrix is

$$X(x) = \begin{bmatrix} 1 - x_1 & x_1 & & & \\ x_1 & 1 - x_1 - x_2 & x_2 & & \\ & x_2 & 1 - x_2 - x_3 & x_3 & \\ & & x_3 & 1 - x_3 - x_4 & x_4 \\ & & & x_4 & 1 - x_4 \end{bmatrix},$$

where the remaining entries are zero.

Theorem 4.2: *Let $A = [a_{ij}]$ be the tridiagonal $(0, 1)$ -matrix of order n with $a_{ij} = 1$ whenever $|i - j| \leq 1$, and $a_{ij} = 0$ otherwise. Then A is the pattern of an RCDS doubly stochastic matrix X , and $X = X(x)$ where x is uniquely determined by*

$$x_{i-1} + 4x_i + x_{i+1} = 2 \quad (i \leq n) \tag{13}$$

and $x_0 = x_{n+1} = 0$. The solution x is such that $X(x)$ is doubly stochastic.

Proof: Let $X = [x_{ij}] = X(x) \in \Omega_n^t$ and assume that X is an RCDS of A . Then each entry x_{ij} where $|i - j| \leq 1$ is positive, or equivalently,

$$x_i > 0 \quad (i \leq n - 1), \quad x_{i-1} + x_i < 1 \quad (i \leq n), \tag{14}$$

where $x_0 = x_n = 0$. The only positive 2×2 submatrices in X are those containing row and column i and $i + 1$ ($i = 1, 2, \dots, n - 1$). For each of these submatrices, the complementary

submatrix has full term rank (as the main diagonal contains nonzeros). Therefore, as X is an RCDS matrix, by Lemma 4.1, the following equations must hold:

$$x_i + x_i = (1 - x_{i-1} - x_i) + (1 - x_i - x_{i+1}), \quad (i \leq n) \quad (15)$$

which gives (13). The coefficient matrix C corresponding to the linear Equations (13) is strictly diagonally dominant, so the system has a unique solution x . Below, we show that x also satisfies (14). Moreover, $X = X(x)$ is in fact an RCDS of A which is seen as follows. Consider the identity matrix and let α be its sum in X . Any other permutation matrix P with $\xi(P) \subseteq \xi(A)$ is obtained by replacing some 2×2 submatrices being I_2 by L_2 , see [11]. However, each such interchange does not change the diagonal sum due to (15). Thus all nonzero diagonals in X have the same sum.

It only remains to show that x satisfies (14). This is done by solving (13), using Gaussian elimination. Let $C = [c_{ij}]$ be the coefficient matrix, i.e. $c_{ii} = 4$ ($i \leq n$), $c_{i,i+1} = c_{i+1,i} = 1$ ($i = 1, 2, \dots, n-1$), and $c_{ij} = 0$ otherwise. Also let b be the n -vector with all components being 2. The algorithm is: start with the augmented matrix $[C \ b]$, and for $i = 1, 2, \dots, n$, (a) multiply row i by the inverse of the (current) entry (i, i) , and then, if $i < n$, (b) subtract that row from the next. Let $F = [f_{ij}]$ be the resulting $n \times (n+1)$ matrix. Then $f_{ii} = 1$, and the remaining nonzeros are in positions $(i, i+1)$ ($i < n$) and in the final column. Let h_i be the value in position (i, i) of E right after row $i-1$ has been subtracted from row i . We then get successively $h_1 = 4$, $f_{1,n+1} = 1/2$, and

$$\begin{aligned} h_i &= 4 - 1/h_{i-1} \quad (i = 2, 3, \dots, n), \\ f_{i,i+1} &= 1/h_i \quad (i = 1, 2, \dots, n-1), \\ f_{i,n+1} &= (2 - f_{i-1,n})/h(i) \quad (i = 2, 3, \dots, n). \end{aligned} \quad (16)$$

Moreover, the solution x is found by back-substitution

$$\begin{aligned} x_i &= f_{i,n+1} - f_{i,i+1}x_{i+1} \\ &= (2 - f_{i-1,n+1} - x_{i+1})/h(i) \quad (i = n, n-1, \dots, 1), \end{aligned} \quad (17)$$

where $x_{n+1} = 0$.

Claim 1: $3.7 < h_i < 3.75$ ($i = 2, 3, \dots, n$). Proof of Claim 1: The function $g(h) = 4 - 1/h$ is strictly increasing for $h > 0$ and a computation shows $3.7 < g(3.7) < g(3.75) < 3.75$. The claim then follows by induction.

Claim 2: $0.4 \leq f_{i,n+1} \leq 0.5$ and $0.2 < x_i < 0.5$ ($i = 1, 2, \dots, n$). Proof of Claim 2: Assume $0.4 \leq f_{i,n+1} < 0.5$ for some i . From (16) and Claim 1

$$f_{i,n+1} = (2 - f_{i-1,n})/h(i) \leq (2 - 0.4)/3.7 = 0.4324 < 0.5$$

and

$$f_{i,n+1} = (2 - f_{i-1,n})/h(i) \geq (2 - 0.5)/3.75 = 0.4.$$

Thus, by induction, $0.4 \leq f_{i,n+1} \leq 0.5$ ($i = 1, 2, \dots, n$). Assume $0.2 < x_{i+1} < 0.5$ for some i , then

$$x_i = (2 - f_{i-1,n+1} - x_{i+1})/h(i) \leq (2 - 0.4 - 0.2)/3.7 = 0.3784 < 0.5$$

and

$$x_i = (2 - f_{i-1,n+1} - x_{i+1})/h(i) \geq (2 - 0.5 - 0.5)/3.75 = 0.2667 > 0.2.$$

By induction (backward on i), $0.2 < x_i < 0.5$ ($i = 1, 2, \dots, n$), and Claim 2 is proved. Finally, Claim 2 shows that x satisfies (14), so the matrix $X(x)$ is doubly stochastic, as desired. ■

One can show further properties of the solution x of the linear system (13), but this is not done here. In fact, the exact solution may be found by solving this as a linear second-order difference equation. Note, however, that the modified system where the first and last component on the right-hand side is changed to $5/6$, has the solution $x' = (1/3, 1/3, \dots, 1/3)$. The solution x of (13) is close to x' (except near ‘the two ends’).

In an attempt to generalize the above result for tridiagonal matrices, we introduce the following notion. Let T be a tree on vertices $1, 2, \dots, n$ and let T^* be a *loopy tree* obtained from T by putting a loop at each pendent vertex and possibly also at other vertices. Let $A = A(T^*) = [a_{ij}]$ be the $n \times n$ adjacency matrix of T^* with 1's on the main diagonal corresponding to the loops. Note that A is a symmetric (0,1)-matrix. When T is a path on n vertices we obtain A as in Theorem 4.2 by putting a loop at each vertex. With loops allowed, a *perfect matching* of T^* is a collection of edges (including loops) that are vertex disjoint and meet all vertices. Perfect matchings of T^* are in one-to-one correspondence with the permutation matrices $P \leq A$ and thus their number is the permanent of A .

We have the following lemma.

Lemma 4.3: *Let A be an $n \times n$ symmetric (0, 1)-matrix, and let G be the loopy graph whose adjacency matrix is A . Then every permutation matrix $P \leq A$ is symmetric, that is, corresponds to a perfect matching of G , if and only if G does not have any cycles of length $k \geq 3$.*

Proof: If there are no permutation matrices $P \leq A$, then the lemma is vacuously true. Assume that there is a permutation matrix $P \leq A$. If P is the identity matrix I_n , then P is symmetric. Now assume that $P \neq I_n$. Let the set of vertices of G be $\{1, 2, \dots, n\}$. Then P determines a bijection $f: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ such that (a) $\{i, f(i)\}$ is an edge of G for all i and (b) there exists a k such that $f(k) \neq k$. If P is not symmetric, there exists $i_1 \neq i_2$ such that $f(i_1) = i_2$ and $f(i_2) \neq i_1, i_2$. Since f is a bijection, there exists $i_3 \neq i_1, i_2$ such that $f(i_2) = i_3$. Continuing like this we see that, since there are only finitely many vertices, there exists distinct i_1, i_2, \dots, i_k with $k \geq 3$ such that $f(i_j) = i_{j+1}$ for $1 \leq j \leq k-1$ and $f(i_k) = i_1$. This implies that G contains a cycle of length $k \geq 3$ that gives a permutation cycle of length $k \geq 3$ of P and hence P is not symmetric. The converse is clear since if P has a cycle of length $k \geq 3$, then P is not symmetric. ■

It follows from Lemma 4.3 that if X is an $n \times n$ RCDS matrix and T^* is a loopy tree with $\xi(X) = \xi(A(T^*))$, then all diagonals of X avoiding its 0's are symmetric. But not all such X constructed as in (9) are RCDS matrices.

Example 4.4: Let T be the tree with vertices $1, 2, \dots, 8$ and edges

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{4, 7\}, \{4, 8\}\}.$$

Let T^* be the loopy tree obtained by putting loops only at the pendent vertices $3, 5, 6, 7, 8$, and let A be the adjacency matrix of T^* . Let X be the sum of the permutation matrices $P \leq A$. Then

$$X = \begin{bmatrix} & 2 & 4 & 2 & & & & \\ 2 & & & & 3 & 3 & & \\ 4 & & 4 & & & & & \\ 2 & & & & & & 3 & 3 \\ & 3 & & & 5 & & & \\ & 3 & & & & 5 & & \\ & & & 3 & & & 5 & \\ & & & 3 & & & & 5 \end{bmatrix}.$$

Then X has two diagonals neither of which contain any 0's and with different sums

$$5 + 5 + 2 + 2 + 4 + 3 + 3 + 5 = 29 \quad \text{and} \quad 5 + 3 + 3 + 4 + 4 + 3 + 3 + 5 = 30.$$

Now, let A be the adjacency matrix

$$\begin{bmatrix} & 1 & 1 & 1 & & & \\ 1 & & & & 1 & 1 & \\ 1 & & 1 & & & & \\ 1 & & & 1 & & & \\ & 1 & & & & 1 & \\ & 1 & & & & & 1 \end{bmatrix}$$

of a loopy tree T^* with five perfect matchings. Then

$$X = \frac{1}{5} \begin{bmatrix} & 1 & 2 & 2 & & & \\ 1 & & & & 2 & 2 & \\ 2 & & 3 & & & & \\ 2 & & & 3 & & & \\ & 2 & & & 3 & & \\ & 2 & & & & & 3 \end{bmatrix}$$

is an RCDS doubly stochastic with restricted diagonal sums equal to $\frac{14}{5}$.

Example 4.5: Let $n \geq 2$ and let $x_i \geq 0$ ($2 \leq i \leq n$). Define the symmetric $n \times n$ matrix $V_n = [v_{ij}]$ by $v_{1i} = v_{i1} = x_i$ and $v_{ii} = 1 - x_i$ ($2 \leq i \leq n$), and $x_{11} = 1 - \sum_{i=2}^n x_i$, while all

other entries are zero. For instance, when $n = 5$ the matrix is

$$V_5 = \begin{bmatrix} 1 - \sum_{i=2}^5 x_i & x_2 & x_3 & x_4 & x_5 \\ x_2 & 1 - x_2 & & & \\ x_3 & & 1 - x_3 & & \\ x_4 & & & 1 - x_4 & \\ x_5 & & & & 1 - x_5 \end{bmatrix}.$$

V_n is a doubly stochastic matrix when $x_i \geq 0$ ($2 \leq i \leq n$) and $\sum_{i=2}^n x_i \leq 1$. If all these inequalities are strict, the graph of the matrix is a star with edges $\{1, i\}$ ($2 \leq i \leq n$) and a loop in every vertex i . Assume that X is an RCDS matrix. By Lemma 4.1, the following equations must hold:

$$x_i + x_i = (1 - x_i) + \left(1 - \sum_{k=2}^n x_k\right), \quad (2 \leq i \leq n)$$

or equivalently

$$4x_i + \sum_{k \neq i} x_k = 2, \quad (2 \leq i \leq n).$$

The unique solution is $x_i = 2/(n + 2)$ ($2 \leq i \leq n$). However, $\sum_{i=2}^n x_i = 2(n - 1)/(n + 2)$ which is ≤ 1 if and only if $n \leq 4$. Thus, when $n \geq 5$, the corresponding matrix is not doubly stochastic (the entry in position $(1, 1)$ is negative) and this proves that there is no RCDS doubly stochastic matrix with this pattern. The remaining cases $n \leq 4$ correspond to the following matrices:

$$V_2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1/5 & 2/5 & 2/5 \\ 2/5 & 3/5 & \\ 2/5 & & 3/5 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & & \\ 1/3 & & 2/3 & \\ 1/3 & & & 2/3 \end{bmatrix}$$

and it is easy to check that each of these is an RCDS doubly stochastic matrix.

To summarize, we have shown that when the (loopy) tree T is a path, the corresponding adjacency matrix A is the pattern of a RCDS doubly stochastic, while when T is a star, A is not an RCDS pattern for $n \geq 5$. Moreover, we gave two other trees, in either of these two categories. Based on this, one might guess that the loopy trees that correspond to RCDS patterns must satisfy some constraint on the maximum degree. It is an open question to characterize the loopy trees that correspond to RCDS patterns.

5. Further classes of RCDS matrices

From Theorem 1.2, we immediately obtain the following result.

Corollary 5.1: *The only RCDS doubly stochastic matrix without any zeros is $(1/n)J_n$.*

Proof: We give an alternative proof of this corollary without applying Theorem 1.2. Let $A = [a_{ij}]$ be a RCDS doubly stochastic matrix with no zeros. Consider a 2×2 submatrix B of A . By Lemma 4.1, and as A has no zeros, the two diagonals in B have the same sum. Let $i \leq n$ and assume $a_{ij} = a_{ik}$ for some $j \neq k$. Then column j and column k must be equal. Assume $A \neq (1/n)J_n$. Since A is doubly stochastic, at most $n-2$ columns are equal. Consider two unequal columns, say columns j and j' . Then $a_{ij} \neq a_{ij'}$ ($i \leq n$). Then there are i, i' such that $a_{ij} > a_{ij'}$ and $a_{i'j} < a_{i'j'}$. So, in the submatrix of A consisting of rows i, i' and columns j, j' , we have

$$a_{ij} + a_{i'j'} > a_{ij'} + a_{i'j},$$

a contradiction. This shows that the only possibility is $A = (1/n)J_n$. ■

We next identify another class of RCDS doubly stochastic matrices. Let r, s and n be positive integers satisfying $s < r < n$, and define the matrix $X = X^{(r,s,n)} = [x_{ij}] \in M_n$ by

$$x_{ij} = \begin{cases} 1/r & (i \leq r, j \leq s), \\ (r-s)/(r(n-s)) & (i \leq r, s < j \leq n), \\ 0 & (r < i \leq n, j \leq s), \\ 1/(n-s) & (r < i \leq n, s < j \leq n). \end{cases} \quad (18)$$

Proposition 5.2: $X^{(r,s,n)}$ is an RCDS doubly stochastic matrix for each $s < r < n$.

Proof: All entries of $X = X^{(r,s,n)}$ are nonnegative. For each $j \leq s$ the j 'th column sum is $r \cdot (1/r) = 1$, and for each $s < j \leq n$ the j 'th column sum is

$$r \cdot (r-s)/(r(n-s)) + (n-r) \cdot 1/(n-s) = (r-s+n-r)/(n-s) = 1.$$

Next, for each $i \leq r$ the i 'th row sum is

$$s \cdot (1/r) + (n-s) \cdot (r-s)/(r(n-s)) = (s+r-s)/r = 1.$$

while for $r < i \leq n$ the i 'th row sum is $(n-s) \cdot (1/(n-s)) = 1$. Therefore, X is doubly stochastic.

Consider a diagonal D in X avoiding zeros. From the first s columns it contains s times the entry $1/r$. Next D contains additionally $r-s$ entries in the r first rows and each such entry must be in the last $(n-s)$ columns and is therefore equal to $(r-s)/(r(n-s))$. Finally, D contains $n-r$ additional entries in the last $n-s$ columns but from the last $n-r$ rows. Each such entry is $1/(n-s)$. Thus, the diagonal D contains

$$1/r \text{ (} s \text{ times), and } (r-s)/(r(n-s)) \text{ (} (r-s) \text{ times), and } 1/(n-s) \text{ (} (n-r) \text{ times).}$$

So, all diagonals contain exactly the same numbers, and their sums are equal. This shows that X is an RCDS doubly stochastic matrix. ■

Proposition 5.2 provides a construction of a fully indecomposable $n \times n$ RCDS matrix whose zeros form a $p \times q$ submatrix for any positive p and q with $p+q \leq n-1$.

Example 5.3: Consider the matrix $X = X^{(3,2,6)} \in \Omega_6$ given by

$$X = \begin{bmatrix} 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 1/3 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

Any diagonal avoiding zeros must contain the entries $1/3, 1/3$ (from the first two columns), $1/12$ (from one of the first three rows), and $1/4, 1/4$ and $1/4$ (from three of the last columns).

The class discussed in the previous proposition can be extended to matrices with staircase pattern. A matrix is *constant* if all entries are equal (so it is a multiple of the all ones matrix). Let $k \geq 3$. Let X be a $n \times n$ matrix of the form

$$X = \begin{bmatrix} X_1 & X_2 & & & & \\ & X_3 & X_4 & & & \\ & & X_5 & X_6 & & \\ & & & \ddots & \ddots & \\ & & & & X_k & X_{k+1} \end{bmatrix}, \tag{19}$$

where X_i ($i \leq k + 1$) are constant matrices and open space indicates a zero matrix. We permit X_{k+1} to be void. The constant associated with X_i is denoted by $c(X_i)$ ($i \leq k + 1$), so $X_i = c(X_i)J$. We call X a *zig-zag matrix*. Let X_i have dimension $r_i \times s_i$ ($i \leq k + 1$). Consider the following conditions on the dimensions:

$$\sum_{i=1}^t s_i < \sum_{i=1}^t r_i < \sum_{i=1}^{t+1} s_i \quad (t \leq k). \tag{20}$$

Example 5.4: The following matrix X is a zig-zag doubly stochastic matrix:

$$X = \begin{bmatrix} X_1 & X_2 & & & & \\ & X_3 & X_4 & & & \\ & & X_5 & X_6 & & \\ & & & & & \\ & & & & & \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 & 1/4 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1/4 & 1/4 & 1/2 \end{bmatrix}.$$

Theorem 5.5: Let X be a doubly stochastic zig-zag matrix satisfying (19) and (20). Then X is an RCDS matrix.

Proof: Consider a diagonal D in X with no zeros. Since $r_1 > s_1$, we get $c(X_1) = 1/r_1$ and D contains one entry from each of the first s_1 columns of X . So, D has s_1 entries from X_1 . Moreover, as $s_1 < r_1 < s_1 + s_2$, D contains the remaining $r_1 - s_1$ entries in the r_1 first rows in the last $(n - s_1)$ columns, i.e. D has $r_1 - s_1$ entries in X_2 , all equal to $c(X_2)$. The

remaining $s_2 - (r_1 - s_1) = s_1 + s_2 - r_1 > 0$ entries that D contains in the columns in X corresponding to X_2 must all lie in X_3 . This again implies that D has a fixed number of entries in X_4 , etc. So, continuing like this, any diagonal contains a fixed number of entries in X_i ($i \leq k$). Therefore each such diagonal contains the same numbers, and then clearly their sums are equal. ■

We proceed and identify a large class of RCDS matrices. They are constructed inspired by the procedure after Theorem 2.1. Let $\mathcal{A}_{k,n}$ be the class of $n \times n$ $(0, 1)$ -matrices with k ones in every row and column. Consider a $n \times n$ $(0, 1)$ -matrix

$$A = [a_{ij}] = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array} \right] \quad (21)$$

where each of the blocks A_i has size $p \times p$ ($i \leq 4$) so $n = 2p$. Assume that $A_i \in \mathcal{A}_{k_i,p}$ where $k_i \leq p$ ($i \leq 4$).

Now, let $u = (u_i) = (a_1, \dots, a_1, a_2, \dots, a_2)$ and $v = (v_j) = (b_1, \dots, b_1, b_2, \dots, b_2)$ where the first p components are equal, in each of these vectors. Let $X' = [x'_{ij}]$ be the $n \times n$ matrix where $x'_{ij} = u_i + v_j$ when $a_{ij} = 1$, and $x'_{ij} = 0$ otherwise ($i, j \leq n$). The i 'th row sum in X' is

$$r_i(X') = \begin{cases} k_1(a_1 + b_1) + k_2(a_1 + b_2) & (i \leq p), \\ k_3(a_2 + b_1) + k_4(a_2 + b_2) & (i > p) \end{cases}$$

and the j 'th column sum in X' is

$$s_j(X') = \begin{cases} k_1(a_1 + b_1) + k_3(a_2 + b_1), & (j \leq p), \\ k_2(a_1 + b_2) + k_4(a_2 + b_2) & (j > p). \end{cases}$$

We want all these four sums to be equal; this gives three equations where one is redundant. By some simplification, we get the equivalent system

$$\begin{aligned} k_1(a_1 + b_1) &= k_4(a_2 + b_2), \\ k_2(a_1 + b_2) &= k_3(a_2 + b_1). \end{aligned} \quad (22)$$

The next theorem is obtained by finding a solution to these two equations.

Theorem 5.6: Let $A = [a_{ij}]$ be as in (21) where $A_i \in \mathcal{A}_{k_i,p}$ and $k_i \leq p$ ($i \leq 4$) satisfy $k_1 + k_4 = k_2 + k_3$. Let $X' = [x'_{ij}]$ be the $n \times n$ matrix where $x'_{ij} > 0$ precisely when $a_{ij} = 1$ and

$$x'_{ij} = \begin{cases} k_4, & (i, j \leq p), \\ k_3, & (i \leq p, j > p), \\ k_2, & (i > p, j \leq p), \\ k_1, & (i > p, j > p). \end{cases}$$

Define $X = (1/(k_1k_4 + k_2k_3)) \cdot X'$. Then X is an RCDS doubly stochastic matrix and A is the corresponding RCDS pattern.

Proof: Equation (22) is a linear system of two equations in four unknowns a_1, a_2, b_1, b_2 . One way to find a solution is to solve (if possible)

$$a_1 + b_1 = k_4, \quad a_2 + b_2 = k_1, \quad a_1 + b_2 = k_3, \quad a_2 + b_1 = k_2$$

or

$$a_1 = k_4 - b_1, \quad a_2 = k_1 - b_2, \quad (k_4 - b_1) + b_2 = k_3, \quad (k_1 - b_2) + b_1 = k_2.$$

The last two equations give

$$b_2 - b_1 = k_3 - k_4, \quad b_2 - b_1 = k_1 - k_2$$

which is consistent as $k_1 + k_4 = k_2 + k_3$. Choose

$$b_1 = 0, \quad b_2 = k_3 - k_4, \quad a_1 = k_4, \quad a_2 = k_2.$$

This is a solution of (22). The corresponding matrix X' is then as described in the theorem, and, by the discussion before the theorem, all row and column sums in X' are equal to some number α . We compute $\alpha = k_1k_4 + k_2k_3$. Therefore, $X = (1/\alpha)X'$ is doubly stochastic and has the same pattern as X . Finally, by Theorem 2.1, X has all diagonals sums that avoid zeros in A equal, so the conclusion of the theorem follows. ■

Note that it should not be difficult to find *all* solutions of the (22), and possible more RCDS patterns.

Example 5.7: Let $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, p = 5$ and $n = 2p = 10$. Then, by Theorem 5.6, the following matrix is an RCDS doubly stochastic matrix

$$X = (1/10) \begin{bmatrix} & & & 4 & & & 3 & 3 & & \\ & & & & 4 & & 3 & & & 3 \\ & 4 & & & & & 3 & & & 3 \\ & & 4 & & & & & & & 3 & 3 \\ 4 & & & & & & 3 & & 3 & & \\ \hline & 2 & 2 & 2 & & & 1 & 1 & & 1 & 1 \\ 2 & 2 & & & 2 & & & 1 & 1 & 1 & 1 \\ 2 & & 2 & & 2 & & 1 & 1 & 1 & & 1 \\ & & 2 & 2 & 2 & & 1 & & 1 & 1 & 1 \\ 2 & 2 & & 2 & & & 1 & 1 & 1 & 1 & \end{bmatrix}.$$

6. The diagonal width

To conclude, we mention a generalization of the concept of having equal diagonal sums. Let $X \in \Omega_n$. Define the *diagonal width* $DW(X)$ of X by

$$DW(X) = \max\{d_p^X : P \in \mathcal{P}_n, \xi(X) \subseteq \xi(P)\} - \min\{d_Q^X : Q \in \mathcal{P}_n, \xi(X) \subseteq \xi(Q)\}. \quad (23)$$

This is the maximum difference between two diagonal sums in X , where the diagonals avoid the zeros of X . Thus, X is an RCDS (doubly stochastic matrix) if and only if $DW(X) = 0$.

The following result characterizes $DW(X)$, and may be used to analyse the diagonal width. The proof uses the duality ideas in the proof of Theorem 2.1.

Theorem 6.1: *Let $X \in \Omega_n$. Then*

$$DW(X) = \theta^*(X) - \theta_*(X), \quad (24)$$

where

$$\theta^*(X) = \min \left\{ \sum_i u_i + \sum_j v_j : u_i + v_j \geq x_{ij} \ ((i,j) \notin \xi(X)) \right\},$$

$$\theta_*(X) = \max \left\{ \sum_i u_i + \sum_j v_j : u_i + v_j \leq x_{ij} \ ((i,j) \notin \xi(X)) \right\}.$$

Proof: Recall from the proof of Theorem 2.1 that the linear optimization problem

$$\begin{aligned} & \text{minimize} && \sum_{i,j} x_{ij} y_{ij} \\ & \text{subject to} && \sum_j y_{ij} = 1 \quad (i \leq n) \\ & && \sum_i y_{ij} = 1 \quad (j \leq n) \\ & && y_{ij} \geq 0 \quad ((i,j) \notin \xi(X)) \end{aligned} \quad (25)$$

has optimal value $\min\{d_P^X : P \in \mathcal{P}_n, \xi(X) \subseteq \xi(P)\}$, the minimum diagonal sum in X avoiding the zeros of X . Here, there is a variable y_{ij} for each $(i,j) \notin \xi(X)$, $i, j \leq n$. Moreover, by the duality theorem

$$\min\{d_P^X : P \in \mathcal{P}_n\} = \theta_*(X),$$

where $\theta_*(X)$ is the optimal value of the dual problem

$$\begin{aligned} & \text{maximize} && \sum_i u_i + \sum_j v_j \\ & \text{subject to} && u_i + v_j \leq x_{ij} \quad ((i,j) \notin \xi(X)). \end{aligned} \quad (26)$$

We use this duality relation to find an expression for $\max\{d_P^X : P \in \mathcal{P}_n\}$:

$$\begin{aligned} & \max\{d_P^X : P \in \mathcal{P}_n\} \\ & = \max\{X \cdot P : X \in \Omega_n\} \\ & = -\min\{(-X) \cdot P : X \in \Omega_n\} \\ & = -\max \left\{ \sum_i u_i + \sum_j v_j : u_i + v_j \leq -x_{ij} \ ((i,j) \notin \xi(X)) \right\} \end{aligned}$$

$$\begin{aligned}
 &= -\max \left\{ -\sum_i (-u_i) + \sum_j (-v_j) : (-u_i) + (-v_j) \geq x_{ij} \ ((i, j) \notin \xi(X)) \right\} \\
 &= \min \left\{ \sum_i u_i + \sum_j v_j : u_i + v_j \geq x_{ij} \ ((i, j) \notin \xi(X)) \right\} \\
 &= \theta^*(X)
 \end{aligned}$$

by a change of variables. This gives (24). ■

Example 6.2: Let $X \in \Omega_2$ so

$$X = \begin{bmatrix} 1 - \alpha & \alpha \\ \alpha & 1 - \alpha \end{bmatrix}.$$

Assume $\alpha \geq 1/2$ (by symmetry). If $\alpha = 1$, clearly $DW(X) = 0$, so assume $\alpha < 1$. Then there are two diagonals and their sums are $2(1 - \alpha)$ and 2α . Then $DW(X) = 2\alpha - 2(1 - \alpha) = 4\alpha - 2$.

The previous theorem may be used to find upper bounds on $DW(X)$ for a given $X \in \Omega_n$, as the following corollary shows.

Corollary 6.3: Let $X \in \Omega_n$. Let u_i, v_i and u'_i, v'_i be such that

$$u_i + v_j \leq x_{ij} \leq u'_i + v'_j \quad ((i, j) \notin \xi(X)).$$

Then

$$DW(X) \leq \sum_i (u_i - u'_i) + \sum_j (v_j - v'_j).$$

Proof: This follows directly from Theorem 6.1. ■

We refer to [12] and [13] for some other results on diagonal sums of doubly stochastic matrices.

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