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Logarithmic Motives with Compact Support

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Building upon recent work by Binda, Park, and Østvær we construct a theory of motives with compact support in the setting of logarithmic algebraic geometry.

Starting from the notion of finite logarithmic correspondences with compact support we define the logarithmic motive with compact support analogous to the classical case. After establishing a Gysin sequence, we prove a Künneth formula, which as a special case, proves homotopy invariance of the logarithmic motive with compact support. This presents an important distinction from the theory of motives with compact support which is not homotopy invariant. Relating our theory to the classical theory we provide an affirmative answer to a question raised in Binda–Park–Østvær concerning the theory's relation to the classical theory. We then prove an analogue of the classical duality theorem, which together with a calculation of the logarithmic motive with compact support of the affine line, culminates in a proof of a cancellation theorem for logarithmic schemes. Moreover, we provide a new homology and cohomology theory for logarithmic schemes, and give a new homotopy invariant generalization of Bloch's higher Chow groups to logarithmic smooth fs logarithmic schemes.

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CHAPTER 0

Introduction

0.1 Overview

Logarithmic geometry can in full generality be regarded as an enlargement of algebraic geometry from commutative rings to commutative monoids. It originally sprang out from arithmetic geometry in the 80's by work of K. Kato to study log crystalline cohomology, log poles, and semi-stable degenerations ([Kat89]), and was further developed by Fontaine–Illusie, Deligne–Faltings, Tsuji, and Ogus, to name a few ([Ogu18]). Logarithmic geometry today impacts various areas, in particular moduli theory, deformation theory and p-adic Hodge theory. For the purpose of algebraic geometry, logarithmic algebraic geometry provides a convenient language to describe schemes with boundary ("open schemes") which gives a natural framework to describe compactifications and degenerations.

Recent work by Binda, Park and Østvær initiates a new theory on motivic homotopy theory in the setting of logarithmic algebraic geometry ([BPØ20], to appear in Astérisque). In their fundamental work they construct the category $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$ of derived motives of logarithmic schemes as an enlargement of Voevodsky's category of derived motives $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$ of schemes ([Voe00]). In doing so, they generalize many classical results to logarithmic algebraic geometry, like Gysin distinguished triangles, the Gysin isomorphism, blow-up distinguished triangles, a projective bundle theorem, a Thom isomorphism, representability of cohomology theories, and many more.

Their motivation come from the fact that there are many invariants that are not $\mathbb{A}^1$-invariant, and thus cannot be studied in the classical $\mathbb{A}^1$-homotopy theory of Voevodsky. Examples include

(i) Algebraic K-theory, $K_n(X)$ if X is a non-regular scheme,

(ii) p-adic cohomology, $H^n_{\text{ét}}(X, \mathbb{Z}/p)$, if p is not invertible in $\mathcal{O}_X$,

(iii) Hodge cohomology, $H^n_{\mathrm{Zar}}(X, \Omega^j)$,

(iv) Cyclic homology, $HC_n(X)$,

(v) Hochschild homology, $HH_n(X)$,

(vi) Topological Hochschild homology, $THH_n(X)$,

(vii) Topological cyclic homology, $TC_n(X)$.

However, generalizing their definition to log schemes it is believed that all these examples are insensitive to a compactification of $\mathbb{A}^1$, which is the projective line $\mathbb{P}^1$ pointed at infinity. This object we denote by

$$\overline{\square} := (\mathbb{P}^1, \infty) \in lSm/k.$$

One of the main result of [BPØ20] is that Hodge cohomology, and thus cyclic homology, is $\overline{\square}$-invariant and satisfies dividing descent and strict Nisnevich descent. Hence they are representable in $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$. Since these cohomology theories are not $\mathbb{A}^1$-invariant, and thus not representable in $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$, this provides a first example that $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$ differs from $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$. Embedding $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$ fully faithfully in $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$, they thus construct an enlargement of Voevodsky's classical theory.

Taking $\overline{\square}$ instead of $\mathbb{A}^1$ as the algebraic replacement of the unit interval $[0, 1] \subset \mathbb{R}$ has the advantage of being both contractible and compact (in contrast to $\mathbb{A}^1$ which is not compact). However, to describe $\overline{\square}$ appropriately we are forced to consider pairs $(X, \partial X)$ where X is a scheme and ∂X acts as a sort of infinitesimal boundary on X. Here the theory of logarithmic geometry provides a natural framework, but there are also alternatives given by *motives with modulus* as presented in [KSY19].

This thesis aims at creating a new theory of motives with compact support in the setting of logarithmic algebraic geometry. In the classical setting, motives with compact support has important consequences on the general motivic theory by providing duality (Theorem 8.2 in [FV00]), cancellation (Theorem 4.3.1 in [Voe00]), new homological invariants on schemes, and representability of Bloch's higher Chow groups for non-smooth schemes. We intend to set up the theory of logarithmic motives with compact support and explore its properties in order to reveal similar impacts on the general logarithmic motivic theory. By generalizing the theory of motives with compact support to logarithmic geometry we get new results for log schemes, which by specializing to the classical theory of schemes recover classical results.

We begin the theory similarly to Chapter 16 in [MVW11] by defining the category of *logarithmic* correspondences with compact support lCor^c/k for a field k. For any fs log scheme X we go on to define the corresponding strict Nisnevich presheaf with log transfers on lSm/k as

$$Y \mapsto \Lambda^c_{\mathrm{ltr}}(X)(Y) := \mathrm{lCor}^c(Y, X) \otimes_{\mathbb{Z}} \Lambda \in \mathbf{Psh}^{\mathrm{ltr}}(k, \Lambda),$$

whose image in $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$ we call the *logarithmic motive* of X *with compact support*. This is done in Section 3.2 after having established the necessary properties in Section 3.1.

We then proceed with investigating various properties of logarithmic motives with compact support.

In the classical case, the fact that the motive with compact support distributes over products is a corollary of the localization theorem. However, as we do not have such an analogue of the localization theorem for log schemes we must prove it directly. Its proof makes use of a generalization of the classical Gysin sequence, i.e., a distinguished triangle

$$M^c(X, Z) \to M^c(X) \to M^c(Z)(1)[2] \to M^c(X, Z)[1],$$

for a smooth log smooth fs log scheme $Y = (X, Z)$. This we state as Theorem 3.2.9, and the combined proof of this statement and the logarithmic version of the Künneth formula occupy the majority of Section 3.2. We state the Künneth formula as follows:

Theorem 0.1.1 (Kunneth formula, Theorem 3.2.12). *Assume that k admits resolution of singularities. Let X and Y be log smooth fs log schemes over k. Then there is an isomorphism*

$$M^c(X \times Y) \simeq M^c(X) \otimes M^c(Y).$$

We then use the Gysin sequence, strict Nisnevich descent, and dividing descent to prove the isomorphisms

$$M^c(\mathbb{A}^1) \simeq \mathbb{Z}(1)[2]$$

and

$$M^c(\mathbb{A}_{\mathbb{N}}) \simeq \mathbb{Z}(1)[1]$$

in $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$.

The Künneth formula presents an important property of the theory, namely that the logarithmic motive with compact support is $\overline{\square}$-invariant, that is that

$$M^c(X \times \overline{\square}) \simeq M^c(X) \otimes M^c(\overline{\square}) \simeq M^c(X) \otimes M(\overline{\square}) \simeq M^c(X)$$

holds for all log smooth fs log schemes X. This is in contrast to the classical case where taking the product with the classical homotopy interval $\mathbb{A}^1$ gives a shift, i.e.,

$$M^c(X \times \mathbb{A}^1) \simeq M^c(X)(1)[2]$$

in $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$. It is also a simple corollary of the Künneth theorem that M^c is $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariant.

When generalizing a theory it is of particular interest to examine how it relates to the original theory. It was shown in [BPØ20] that if k admits resolution of singularities there is a pair of adjoint functors

$$\omega_\sharp : \mathbf{logDM}^{\mathrm{eff}}(k, \Lambda) \rightleftarrows \mathbf{DM}^{\mathrm{eff}}(k, \Lambda) : R\omega^*,$$

producing an equivalence of triangulated categories

$$\mathbf{logDM}^{\mathrm{eff}}_{\mathrm{prop}}(k, \Lambda) \simeq \mathbf{DM}^{\mathrm{eff}}(k, \Lambda), \tag{1}$$

where $\mathbf{logDM}^{\mathrm{eff}}_{\mathrm{prop}}(k, \Lambda)$ is the smallest subcategory of $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$ that is closed under small sums and shift, and generated by all $M(X)$, where $X \in lSm/k$ and the underlying scheme $\underline{X}$ is proper over k (Theorem 8.2.17 in [BPØ20]). We also approach the same problem, in which case our main result is the following generalization of Proposition 8.2.6 in [BPØ20]:

Theorem 0.1.2 (Theorem 3.2.14). *Assume that k admits resolution of singularities. Let X be a smooth scheme over k and Y an log smooth fs log scheme over k. Then for every integer $i \in \mathbb{Z}$ there is an isomorphism*

$$\mathrm{Hom}_{\boldsymbol{logDM}^{\mathrm{eff}}(k,\Lambda)}(M(Y)[i], M^c(X)) \simeq \mathrm{Hom}_{\boldsymbol{DM}^{\mathrm{eff}}(k,\Lambda)}(M(Y - \partial Y)[i], M^c(X)).$$

By calculating the logarithmic motive with compact support of the affine line, this result provides an affirmative answer to the question raised in Remark 8.2.7 of [BPØ20], which is that

$$\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M(Y)[i], M(X)) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M(Y-\partial Y)[i], M(X))$$

does not in general hold for smooth non-proper schemes X since we in general do not have the equivalence $M^c(X) \simeq M(X)$. Moreover, this result enables us to prove that the unit of the above adjunction induces an isomorphism

$$M^c(X) \to R\omega^*\omega_\sharp M^c(X)$$

where $X \in Sm/k$ is a scheme with trivial log structure. This is furthermore used to prove a generalization of the classical duality theorem ([MVW11, Theorem 16.24]) in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$.

Theorem 0.1.3 (Duality, Theorem 3.2.17)**.** *Assume that k admits resolution of singularities. If $T \in lSm/k$ is of pure dimension d over k, $X \in Sm/k$, and $Y \in lSm/k$, then there are isomorphisms*

$$\mathrm{Hom}(M(Y\times T)[n], M^c(X)) \simeq \mathrm{Hom}(M(Y)(d)[2d+n], M^c(X \times (T-\partial T)))$$

in $\boldsymbol{logDM}^{\mathit{eff}}(k,\Lambda)$ for every $n \in \mathbb{Z}$.

Using the majority of our established results, our theory culminates in a generalization of the cancellation theorem (Theorem 16.25 in [MVW11]) to the setting of logarithmic motives.

Theorem 0.1.4 (Cancellation, Theorem 3.2.18). *Assume that k admits resolution of singularities. For X and Y in lSm/k there is an isomorphism*

$$\mathrm{Hom}_{\boldsymbol{logDM}^{\mathit{eff}}(k,\Lambda)}(M(Y), M(X)) \simeq \mathrm{Hom}_{\boldsymbol{logDM}^{\mathit{eff}}(k,\Lambda)}(M(Y)(1), M(X)(1)).$$

In Section 3.3 we introduce a new homology and cohomology theory for logarithmic schemes, namely *motivic cohomology with compact support*

$$H^{n,i}_{lc}(X,\Lambda) := \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M^c(X), \Lambda(i)[n]),$$

and *(Borel–Moore) motivic homology with compact support*

$$H^{\mathrm{lBM}}_{n,i}(X,\Lambda) := \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(\Lambda(i)[n], M^c(X)).$$

We then define a logarithmic analogue of Borel–Moore fundamental classes. Motivated by the equivalence between Borel–Moore homology and Bloch's higher Chow groups, we give a new definition for logarithmic schemes

$$\mathrm{lCH}_i(X,m) := \mathrm{CH}_i(X-\partial X, m),$$

and establish some of its basic properties. In contrast to previous generalization of Chow groups to logarithmic geometry ([Bar20]), this definition is $\overline{\square}$-invariant.

We conclude by giving some open problems and speculations. Many of these questions originates in our driving belief that classical results should have logarithmic analogues, but we have yet to find the correct statements for several such results. For the sake of simplicity we have not carried out the theory in its fullest generality, but this section discusses possible developments in that regard.

0.2 Outline

The thesis is organized into three chapters and one appendix which concern the following:

Chapter 1 introduces Voevodsky's category of derived motives $\mathbf{DM}^{\text{eff}}(k, \Lambda)$ and the motive with compact support $M^c(X)$. Most of this chapter is concerned with examining results about the motive with compact support which we will discuss in the setting of Chapter 2 in Chapter 3.

Chapter 2 reviews the theory of logarithmic motives as developed in [BPØ20]. It is within this framework that we develop a theory of motives with compact support in Chapter 3. In this chapter we study the properties of $\mathbf{logDM}^{\text{eff}}(k, \Lambda)$ for later reference and as a way of introducing notation. As [BPØ20] sets the scene, we heavily rely on this paper and its techniques. Indeed, many of our results, and proofs, are generalizations of the results presented in this work.

Chapter 3 explores the theory of logarithmic motives with compact support, and this chapter thus presents our contribution.

The chapter begins similar to Chapter 16 in [MVW11] by introducing finite logarithmic correspondences with compact support. After having settled basic properties, we use them to define the logarithmic motive with compact support. This construction is our main focus, and the rest of this chapter is devoted to study its properties. Having the logarithmic motive with compact support at hand we define two new cohomology theories for log schemes: motivic cohomology with compact support and Borel–Moore motivic homology for log schemes, and provide a new $\overline{\square}$-invariant definition of Bloch's higher Chow groups for log schemes. We have strived to seek logarithmic analogues of the results presented in Chapter 1, but for several important results these questions remain open. We therefore conclude this chapter with a discussion of open problems and further developments.

Appendix A provides a brief introduction to logarithmic algebraic geometry. Since logarithmic geometry is not in the standard curriculum, our goal has been to make it possible for a reader familiar with algebraic geometry to follow our arguments by looking up definitions and preliminary results when necessary.

0.3 Notation and Terminology

Fix a perfect field k, Λ a unital commutative ring, we then let Sm/k denote the category of smooth and separated schemes of finite type over k and $\mathbf{Psh}(Sm/k, \Lambda)$ the category of presheaves of Λ-modules on Sm/k with coefficients in Λ. By convention, all log schemes (Definition A.2.4) are separated and of finite type over $\operatorname{Spec} k$, where $\operatorname{Spec} k$ is the point equipped with the trivial log structure. The category of all fs (Definition A.1.7 and Definition A.1.7) log schemes we denote by $lSch/k$, while the category of all fs log smooth (Definition A.2.18) log schemes over k we denote by lSm/k. For a log scheme X we let $\underline{X}$ denote the underlying scheme, and $SmlSm/k$ denote the full subcategory of lSm/k whose underlying scheme $\underline{X}$ is smooth over k. If $(X, \mathcal{O}_X)$ is a scheme, and we refer to $\mathcal{O}_X$ as a sheaf of monoids, we will use its multiplicative structure. For a log scheme X we let ∂X denote the points of X with non-trivial log structure. If X is an fs log scheme, the complement of the log structure $\underline{X} - \partial X$ will be an open subset of $\underline{X}$, and there is a canonical open immersion $\underline{X} - \partial X \to \underline{X}$.

We use the notation

$$\overline{\square} := (\mathbb{P}^1, \infty),$$

which we call "box", and let

$$\overline{\square}^n := ((\mathbb{P}^1)^n, \infty \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 + \mathbb{P}^1 \times \infty \times \cdots \times \mathbb{P}^1 + \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times \infty)$$

and $\mathbb{A}_{\mathbb{N}} := (\mathbb{A}^1, 0)$. Considering the pair of projective spaces $(\mathbb{P}^n, \mathbb{P}^{n-1})$ we will always consider $\mathbb{P}^{n-1}$ as a hyperplane in $\mathbb{P}^n$. For convenience, we will write $(X, Z_1 + \cdots + Z_r)$ for $((X, Z_{s+1} + \cdots + Z_r), Z_1 + \cdots + Z_s)$ when adding the divisor $Z_1 + \cdots Z_s$ to the log structure.

We usually refer to the *classical* theory as the theory of motivic homotopy theory on schemes, while we reserve the term *general* theory for motivic homotopy theory on log schemes.

CHAPTER 1

Motivic Homotopy Theory

Motivic homotopy theory began as an attempt by Alexander Grothendieck in the 60's to unify the various *Weil cohomology theories* on smooth projective algebraic varieties in what should be the category $\mathcal{M}_k$ of the so called *pure motives*. This attempt remains largely conjectural even to this day, as Grothendieck proved its existence equivalent to the unsolved standard conjecture on algebraic cycles.

> *In order to express this kinship of these different cohomological theories, I formulated the notion of "motive" associated to an algebraic variety. By this term I want to suggest that it is the "common motive" (or "common reason") behind this multitude of cohomological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possible.*
>
> – A. Grothendieck 1986, Récoltes et Semailles (english translation).

In the late 90's there were given several constructions by Hanamura, Levine, and Voevodsky, as to what should be the *derived* category of the category of motives. They were all found to be equivalent ([VSF00]), and the field of motivic homotopy theory usually refers to the study of these constructions, their properties, and related areas. However, Voevodsky's construction remains the most central for our purposes. Its construction and properties is therefore the main subject of this chapter. For this work, and its applications to his proof of the Milnor conjecture, Voevodsky was awarded the Fields medal in 2002 [1]. Later, Voevodsky and his collaborators generalized the theory and utilized it in the spectacular proof of the Bloch-Kato conjecture (also known as the norm residue isomorphism theorem) ([Voe11]).

In general, Motivic homotopy theory attempts to apply algebraic topological methods in algebraic geometry, creating a homotopy theory on the category of (smooth) schemes over k. Since there is no a priori unit interval in Sm/k, we first have to find a suitable replacement. Here Voevodsky chooses the affine line $\mathbb{A}^1$, and initiates the study of $\mathbb{A}^1$-homotopy theory. As the affine line is not compact, we will in Chapter 2 instead choose a suitable compactification of $\mathbb{A}^1$. However this forces us into the realm of logarithmic geometry, a generalization of algebraic geometry.

[1] The original citation reads: "He defined and developed motivic cohomology and the $\mathbb{A}^1$-homotopy theory, provided a framework for describing many new cohomology theories for algebraic varieties. He proved the Milnor conjectures on the K-theory of fields."

In this chapter we review the basic theory of motivic homotopy theory, with a special focus on motives with compact support. The basic theory will be generalized to the logarithmic setting in Chapter 2, and we will generalize the theory of motives with compact support to the logarithmic motivic setting in Chapter 3.

1.1 Construction of DM$^{\text{eff}}(k, \Lambda)$

We here give a brief description of the construction of Voevodsky's category of derived motives $\mathbf{DM}^{\text{eff}}(k, \Lambda)$.

The category of smooth schemes does not have all the necessary categorical properties in order to do homotopy theory on it. Especially, it does not have all colimits.

Example 1.1.1. The colimit (or pushout) of the diagram

$$* \longleftarrow \{0, 1\} \longrightarrow \mathbb{A}^1$$

is isomorphic to the node $V(y^2 - x^2(x-1)) \subset \mathbb{A}^2$ which is not smooth at $(0, 0)$, hence it is not in Sm/k.

We must therefore find another category of "spaces" with good categorical properties into which the category of smooth schemes embeds. In particular, we want our category of spaces to have all small limits and colimits and internal Hom-objects. This is analogous to the case in algebraic topology where one restricts the theory to weak Hausdorff compactly generated spaces since the category of topological spaces is not Cartesian closed.

Grothendieck found a way to formally add all small limits and colimits, namely passing to presheaves and embed the original category by the Yoneda-embedding.

Definition 1.1.2 (The Yoneda embedding, 1954)**.** Let $\mathcal{C}$ be a small category, $\mathbf{Ab}$ be the category of abelian groups, and let $\mathbf{Psh}(C) := [\mathcal{C}^{\text{op}}, \mathbf{Ab}]$ (or into Set) denote the functor category of contravariant functors from $\mathcal{C}$ to the category $\mathbf{Ab}$. An object in $\mathbf{Psh}(C)$ is called a *presheaf* on $\mathcal{C}$ with values in $\mathbf{Ab}$. An object $X \in \mathcal{C}$ defines a presheaf

$$R_X : \mathcal{C}^{\text{op}} \to \mathbf{Ab}$$

defined by $Y \mapsto \text{Hom}_{\mathcal{C}}(X, Y)$.

The *Yoneda embedding* is the functor $\mathcal{C} \to \mathbf{Psh}(C)$ that associates to any object $X \in \text{ob}(\mathcal{C})$ the representable presheaf R_X.

The important thing is that the Yoneda embedding is full and faithful (Theorem 1.7.4 in [Yek19]) and that $\mathbf{Psh}(\mathcal{C})$, being a locally small category, has all small limits and colimits (Corollary 2.4.3 in [KS06]).

Choosing the category $\mathbf{Psh}(Sm/k)$ as the category of spaces has a disadvantage since the Yoneda embedding does not preserve pushout squares. That is, if $X = U \cup V$ is a Zariski covering of a scheme X by two Zariski open subsets

U and V, then there is a pullback square in Sm/k:

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X. \end{array}$$

However, the corresponding square of representable presheaves

$$\begin{array}{ccc} R_{U\cap V} & \longrightarrow & R_U \\ \downarrow & & \downarrow \\ R_V & \longrightarrow & R_X. \end{array}$$

is not a pushout square if X is not equal to U or V. We therefore introduce the Nisnevich topology as having the primary purpose of forcing such squares to be pushout squares. The definition of the Nisnevich topology is originally due to Nisnevich in [Nis89].

Definition 1.1.3 ([Voe98, Definition 2.1])**.** An *elementary distinguished (Nisnevich) square* in Sm/k is a pullback diagram of the form

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

such that p is an étale morphism, i is a Zariski open embedding and

$$p^{-1}(X - U) \to X - U$$

is an isomorphism (where $X - U$ is equipped with the reduced induced scheme structure).

Example 1.1.4 ([AE17, Example 3.41])**.** If k is a field of characteristic different from 2 and $a \in k$ a non-zero element, then the square

$$\begin{array}{ccc} p^{-1}(\mathbb{A}^1 - \{a\}) & \longrightarrow & \mathbb{A}^1 - \{0\} \\ \downarrow & & \downarrow p \\ \mathbb{A}^1 - \{a\} & \xhookrightarrow{i} & \mathbb{A}^1, \end{array}$$

where p is the étale map given by $x \mapsto x^2$ and i is the inclusion, is an elementary distinguished square if and only if a is a square in k.

Definition 1.1.5. The *Nisnevich topology* on Sm/k is the Grothendieck topology generated by all elementary distinguished squares.

Every Zariski open covering $X = U \cup V$ gives rise to an elementary distinguished square, hence the Nisnevich topology is finer then the Zariski topology. From Example 1.1.4 we see that not all étale coverings give rise to elementary distinguished squares, hence the Nisnevich topology is coarser than the étale topology. Concluding, we have

$$\text{Zariski topology} \subset \text{Nisnevich topology} \subset \text{Étale topology}.$$

Definition 1.1.6 ([Voe98, Definition 2.2])**.** A presheaf F of sets on Sm/k is called a *sheaf* in the Nisnevich topology if the following two conditions are satisfied:

(i) $F(\emptyset) = \mathrm{pt}$

(ii) For any elementary distinguished square, the square of sets

$$\begin{array}{ccc} F(X) & \longrightarrow & F(V) \\ \downarrow & & \downarrow p \\ F(U) & \xrightarrow{i} & F(p^{-1}(U)) \end{array}$$

is a pushout square, i.e., $F(X) = F(U) \times_{F(p^{-1}(U))} F(V)$.

Let $\mathbf{Shv}(Sm/k)$ denote the full subcategory of Nisnevich sheaves in $\mathbf{Psh}(Sm/k)$. This category has all small limits and colimits (Theorem 2.3 in [Voe98]), and internal Hom-objects. It is possible to take this as the category of spaces, which (forcing étale descent instead of Nisnevich descent) leads to Ayoub's construction of $\mathbf{DA}_{\text{ét}}(k,\Lambda)$ in [Ayo07] (or in [Ayo14] for an English version). However, we will follow Voevodsky and introduce finite correspondences which enrich the category of smooth schemes with an extra structure. This extra structure will be particularly useful in computing motivic cohomology, and provides important results in the category of the so called "derived motives" $\mathbf{DM}^{\text{eff}}(k,\Lambda)$. However, we note that the category $\mathbf{DA}_{\text{ét}}(k,\Lambda)$ is simpler than $\mathbf{DM}^{\text{eff}}(k,\Lambda)$, and that this simplification has its own advantages and disadvantages.

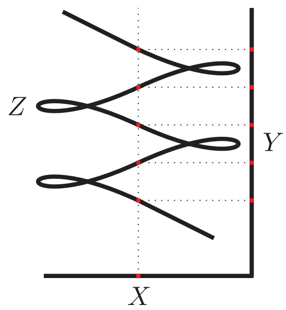


Figure 1.1: An (complicated) elementary correspondence Z from X to Y.

The motivation for considering correspondences (apart from the fact that many cohomology theories have transfers) is that we want to have a way of describing a homotopy between maps similar the case in topology. Taking $\mathbb{A}^1$ as our unit interval, we have the "naive" definition:

Definition 1.1.7. Two maps $f, g : X \to Y$ of (smooth) schemes are called *elementary* $\mathbb{A}^1$*-homotopic* if there exists a map

$$H : X \times \mathbb{A}^1 \to Y$$

such that $H \circ i_0 = f$ and $H \circ i_1 = g$, where i_0 (resp. i_1) is the inclusion of $\{0\}$ (resp. $\{1\}$) in $\mathbb{A}^1$.

In contrast to the case in topology, this is not an equivalence relation as it is not transitive. However, enriching the category Sm/k with correspondences acting as multivalued functions into a new category Cor/k, this relation becomes transitive. Hence it is preferable to include correspondences as morphisms in order to get a well behaved homotopy theory on schemes.

Definition 1.1.8 ([MVW11, Definition 1.1]). Let X be a smooth scheme over k and Y any separated scheme over k. An *elementary correspondence* from X to Y is a closed irreducible subset $Z \subset X \times Y$ that is finite and surjective over a component of X.

Let $\mathrm{Cor}(X, Y)$ be the free abelian group generated by elementary correspondences. An element of $\mathrm{Cor}(X, Y)$ is called a *finite correspondence.*

For $X \in Sm/k$, let $\Lambda_{\mathrm{tr}}(X)$ denote the representable presheaf with transfer that sends $U \in Sm/k$ to the group $\mathrm{Cor}(U, X)$. It is a Nisnevich sheaf with transfers by Lemma 6.2 in [MVW11].

Example 1.1.9. For every morphism $f : X \to Y$, the graph Γ_f is an elementary correspondence from X to Y. We may think of correspondences as multivalued functions or "wrong way" maps. See Figure 1.1

Definition 1.1.10. We let Cor/k be the additive category whose objects are smooth separated schemes over k and whose morphisms are finite correspondences.

Composition of finite correspondences is constructed such that Cor/k contains Sm/k (see Chapter 1 in [MVW11]).

Definition 1.1.11. A *presheaf with transfers F* is a contravariant additive functor

$$\mathrm{Cor}/k^{\mathrm{op}} \to \mathbf{Ab}\,.$$

We say that F is a *Nisnevich sheaf with transfers* if the underlying presheaf on Sm/k is a sheaf in the Nisnevich topology. Let $\mathbf{Shv}(\mathrm{Cor}/k, \Lambda)$ denote the category of Nisnevich sheaves of Λ-modules with transfers. It is this category that we take as "Spaces".

The construction of $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$ is straightforward at this point. We let $C_*(\mathbf{Shv}(k, \Lambda))$ be the category complexes of Nisnevich sheaves of Λ-modules with transfers. Then $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$ is the homotopy category of $C_*(\mathbf{Shv}(k, \Lambda))$ with respect to the $\mathbb{A}^1$-local descent model structure ([BPØ20, p. 80]).

An alternative construction is as follows ([MVW11, p. 109]): Let

$$D(\mathbf{Shv}(k, \Lambda))$$

be the derived category of Nisnevich sheaves of Λ-modules with transfers, and let $\tau_{\mathbb{A}^1}$ be the smallest thick subcategory in $D(\mathbf{Shv}(k, \Lambda))$ that contains the morphisms $\Lambda_{\mathrm{tr}}(X \times \mathbb{A}^1) \to \Lambda_{\mathrm{tr}}(X)$ (induced by projections $X \times \mathbb{A}^1 \to X$) and is closed under direct sums. We then define $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$ as the Verdier quotient ([Ver96]) of $D(\mathbf{Shv}(k, \Lambda))$ by $\tau_{\mathbb{A}}$. That is, $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$ is the localization $\mathbf{D}(\mathbf{Shv}(k, \Lambda))[W_{\mathbb{A}}^{-1}]$, where $W_{\mathbb{A}}^{-1}$ is the class of maps in $D(\mathbf{Shv}(k, \Lambda))$ whose cone is in $\tau_{\mathbb{A}}$.

Definition 1.1.12. For $X \in Sm/k$ we define the *motive* of X, denoted $M(X)$, as the image of X in $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$. By abuse of notation, we let Λ denote $M(\mathrm{Spec}(k))$, which is the unit for the tensor-structure $\otimes_L^{\mathrm{tr}}$ on $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$. If $f : X \to Y$ is a morphism in Sm/k, we define $M(Y \xrightarrow{f} X)$, or simply $M(f)$, as the cone in $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$ associated to the map of complexes $\Lambda^{\mathrm{tr}}(Y) \to \Lambda^{\mathrm{tr}}(X)$ in $C_*(\mathbf{Shv}(k, \Lambda))$.

We then have the following maps of categories

$$M(-): Sm/k \longrightarrow \mathbf{Shv}(k, \Lambda) \longrightarrow C_*(\mathbf{Shv}(k, \Lambda)) \longrightarrow \mathbf{DM}^{\text{eff}}(k, \Lambda).$$

Remark 1.1.13. We note that the construction of Voevodsky was initially carried out for bounded above chain complexes (See Chapter 14 in [MVW11]). The generalization to unbounded chain complexes was carried out in [CD19].

Remark 1.1.14. We present the construction of $\mathbf{DM}^{\text{eff}}(k, \Lambda)$ in the Nisnevich topology. The same argument works for the étale topology as well, but for our purposes the construction given by the Nisnevich topology is more central. It is in a similar topology we present the generalization of this construction to logarithmic geometry (although a generalization for the étale topology to logarithmic geometry exists as well).

Remark 1.1.15. The construction presented here can be generalized to work for many base schemes S, especially Noetherian schemes. The category $\mathbf{DA}^{\text{ét}}(S, \Lambda)$ becomes equivalent to $\mathbf{DM}^{\text{ét}}(S, \Lambda)$, when S has dimension greater than or equal to 1 (Theorem 4.4 in [Ayo14]).

Voevodsky goes on to define various versions of $\mathbf{DM}^{\text{eff}}(k, \Lambda)$, such as $\mathbf{DM}(k, \Lambda)$ (by inverting the Tate-motive $\Lambda(1)$), $\mathbf{DM}^{\text{eff}}_{\text{gm}}(k, \Lambda)$ and $\mathbf{DM}_{\text{gm}}(k, \Lambda)$. However these are not central to the thesis and are therefore left out. We instead refer the reader to [MVW11].

Properties of $\mathbf{DM}^{\text{eff}}(k, \Lambda)$

For the convenience of the reader, and for future reference, we briefly survey some of the properties of $\mathbf{DM}^{\text{eff}}(k, \Lambda)$. Many of these properties have a logarithmic analog. See Section 2.4.

The following list of properties is due to Properties 14.15 in [MVW11].

(i) (Monoidal structure) For every X and Y in Sm/k we have an isomorphism

$$M(X \times Y) \simeq M(Y) \otimes M(Y). \tag{1.1}$$

(ii) (Mayer-Vietoris) For every Zariski open cover U, V of a smooth scheme X we have a triangle

$$M(U \cap V) \longrightarrow M(U) \otimes M(V) \longrightarrow M(X) \longrightarrow M(U \cap V)[1] \tag{1.2}$$

in $\mathbf{DM}^{\text{eff}}(k, \Lambda)$.

(iii) (Vector bundles) If $E \to X$ is a vector bundle, there is an isomorphism

$$M(E) \xrightarrow{\simeq} M(X). \tag{1.3}$$

(iv) (Blow-up triangle) Let $X' \to X$ be a blow up along a smooth center Z. Then there is a triangle

$$M(Z \times_X X') \longrightarrow M(X') \longrightarrow M(X) \longrightarrow M(Z \times_X X')[1]. \tag{1.4}$$

(v) (Gysin triangle) If X and Z are smooth schemes and Z has codimension c in X, then there is a distinguished triangle

$$M(X - Z) \longrightarrow M(X) \longrightarrow M(Z)(c)[2c] \longrightarrow M(X - Z)[1]. \tag{1.5}$$

(vi) (Cancellation) (Under the assumption of resolution of singularities) For every $M, N \in \mathbf{DM}^{\text{eff}}(k, \Lambda)$ there is an isomorphism

$$\text{Hom}_{\mathbf{DM}^{\text{eff}}(k,\Lambda)}(M(1), N(1)) \xrightarrow{\simeq} \text{Hom}_{\mathbf{DM}^{\text{eff}}(k,\Lambda)}(M, N). \tag{1.6}$$

(vii) (Chow motives) Grothendieck's category of effective Chow motives embeds fully faithfully in $\mathbf{DM}^{\text{eff}}(k, \Lambda)$, i.e., if X and Y are smooth projective schemes then there is an isomorphism

$$\text{Hom}_{\mathbf{DM}^{\text{eff}}(k,\Lambda)}(M(X), M(Y)) \simeq \text{Hom}_{\mathsf{Chow}}(X, Y) = \text{CH}^{\dim X}(X \times Y).$$

1.2 Motivic Homotopy Theory with Compact Support

In this section we introduce the motive with compact support. We will carry out a similar approach in Chapter 3 where we generalize the theory presented here to the logarithmic setting of Chapter 2. After presenting the definition, we survey the properties of motives with compact support and how these effect the general theory of motives presented above.

Definition 1.2.1 ([MVW11, Definition 16.1])**.** Let Z be a scheme of finite type over S such that Z dominates a component of S. We call Z *equidimensional of relative dimension* m if for every point $s \in S$, the fibre Z_s is either empty or each of its components has dimension m.

If Y is a scheme of finite type over k and $r \geq 0$ an integer, we let $\Lambda^c_{\text{tr}}(Y, r)$ denote the presheaf with transfers defined as follows. For each smooth scheme X, we let $\Lambda^c_{\text{tr}}(Y, r)(X)$ denote the free abelian group generated by closed and irreducible subschemes Z of $X \times Y$ that are dominant and equidimensional of relative dimension r over a component of X.

Given a map $S' \to S$, the pullback of relative cycles gives a natural map

$$\Lambda^c_{\text{tr}}(Y, r)(S) \longrightarrow \Lambda^c_{\text{tr}}(Y, r)(S').$$

The presheaf $\Lambda^c_{\text{tr}}(Y, r)$ is an étale sheaf, and we may construct transfer maps to make it a étale sheaf with transfers. It is covariant for proper maps and contravariant for flat maps, if we adjust the dimension index r appropriately. See Chapter 16 in [MVW11] for details.

The case where $r = 0$ will be of special interest, and we will denote $\Lambda^c_{\text{tr}}(Y, 0)$ by $\Lambda^c_{\text{tr}}(Y)$. From Definition 1.2.1 we then have that $\Lambda^c_{\text{tr}}(Y)(X)$ is the free abelian group generated by closed irreducible subschemes $Z \subset X \times Y$ that are dominant and quasi-finite over a component of X. We call Z a *finite correspondence with compact support* from X to Y. Let Cor^c/k denote the category with the same objects as Sm/k and morphisms finite correspondences with compact support.

Proposition 1.2.2 ([MVW11, Example 16.2])**.** *If X is a smooth proper scheme over k, then*

$$\Lambda^c_{tr}(X) \simeq \Lambda_{tr}(X).$$

Proof. By [Har77, Ex. III.11.2], every closed subscheme $Z \subset U \times X$ is proper over U. Hence it is quasi-finite over U if and only if it is finite over U. □

We remind the reader of the following useful description of quasi-finite morphisms.

Theorem 1.2.3 (Zariski's Main Theorem, [Sta21, Lemma 37.39.3])**.** *A quasi-finite morphism between noetherian schemes factors as an open immersion followed by a finite morphism, that is, if $f : Y \to X$ is a quasi finite morphism, then there a scheme X', an open immersion g and a finite morphism h such that*

$$Y \xrightarrow{g} X' \xrightarrow{h} X, \qquad (f : Y \to X).$$

and $f = h \circ g$.

Definition 1.2.4 ([MVW11, Definition 16.13])**.** For any scheme X we define the *motive* of X *with compact support*, denoted $M^c(X)$ as the image of $\Lambda^c_{\mathrm{tr}}(X)$ in $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$.

If X is proper, then there is an equivalence

$$M^c(X) \simeq M(X) \tag{1.7}$$

by Proposition 1.2.2. In general, the inclusion $\Lambda_{\mathrm{tr}}(X) \subset \Lambda^c_{\mathrm{tr}}(X)$ induces a canonical morphism

$$M(X) \longrightarrow M^c(X).$$

Example 1.2.5. We have the following identities,

$$M^c(\mathbb{A}^n) \simeq \Lambda(n)[2n]$$

by Corollary 4.1.8 in [Voe00], and

$$M^c(\mathbb{P}^n) \simeq M(\mathbb{P}^n) \simeq \bigoplus_{i=0}^{n} \mathbb{Z}(i)[2i]$$

by Corollary 15.5 in [MVW11].

Most of the following results assume that the ground field k admits resolution of singularities. We therefore provide the definition.

Definition 1.2.6 ([BPØ20, Definition 7.6.3])**.** The field k admits resolution of singularities if the following conditions hold:

(i) For every integral scheme X of finite type over k, there is a proper birational morphism $Y \to X$ of schemes over k such that Y is smooth.

(ii) Let $f : Y \to X$ be a proper birational morphism of integral schemes over k such that X is smooth, and let $Z_1, \ldots, Z_r$ be smooth divisors forming a strict normal crossing divisor on X. Suppose that

$$f^{-1}(X - (Z_1 \cup \cdots \cup Z_r)) \to X - (Z_1 \cup \cdots \cup Z_r)$$

is an isomorphism. Then there is a sequence of blow-ups

$$X_n \overset{f_{n-1}}{\to} X_n \overset{f_{n-2}}{\to} \cdots \overset{f_0}{\to} X_0 = X$$

along smooth centers $W_i \subset X_i$ such that

(i) the composition $X_n \to X$ factors through f.

(ii) the W_i are contained in the preimage of $Z_1 \cup \cdots \cup Z_r$ in X_i.

(iii) the W_i have strict normal crossings with the sum of the strict transforms of

$$Z_1, \ldots, Z_r, f_0^{-1}(W_0), \ldots, f_{i-1}^{-1}(W_{i-1})$$

in X_i.

Condition (i) is satisfied if k is perfect. If k has characteristic 0, then condition (i) is satisfied by Main Theorem I in [Hir64] and condition (ii) is satisfied by Main theorem II in [Hir64].

One of the most important theorems regarding the motive with compact support is the localization theorem stated below. As a corollary we get a Künneth formula and a Mayer-Vietoris sequence, and it is used in proving important theorems such as Duality and Cancellation.

Theorem 1.2.7 (Localization, [MVW11, Theorem 16.15])**.** *Assume that k admits resolution of singularities. Assume that $i : Z \to X$ is a closed immersion with open complement $j : U \to X$. Then there is a distinguished triangle*

$$M^c(Z) \xrightarrow{i_*} M^c(X) \xrightarrow{j^*} M^c(U) \longrightarrow M^c(Z)[1]. \tag{1.8}$$

As a consequence we get a Kunneth formula:

Theorem 1.2.8 (Künneth formula, [MVW11, Corollary 16.16])**.** *Assume that k admits resolution of singularities. For every scheme X and Y there is a natural isomorphism*

$$M^c(X \times Y) \simeq M^c(X) \otimes M^c(Y).$$

The logarithmic analogue of this result (Theorem 3.2.12) is an important result of this thesis. Since we did not find a complete proof of this classical result in the literature we have provided one below.

Proof. When X and Y are smooth and proper this is simply (1.1).

For the case of X being proper (and not smooth) and Y smooth and proper, we blow up the singular locus Z of X and Z' of $X \times Y$. We then, by Theorem 13.26 in [MVW11], get the blow-up triangle

$$M(Z \times_X X') \to M(Z) \oplus M(X') \to M(X) \to M(Z \times_X X')[1] \tag{1.9}$$

where $X' = \mathrm{Bl}_Z X$ is the blow up of X with center Z with the exceptional divisor $Z' = Z \times_X X'$. Moreover, we also have a blow-up triangle

$$M(E) \longrightarrow M(Z \times Y) \oplus M(B) \longrightarrow M(X \times Y) \longrightarrow M(E)[1],$$

where $B = \mathrm{Bl}_{Z\times Y}(X \times Y)$ is the blow up of $X \times Y$ along $Z \times Y$ with exceptional divisor $E = (Z \times Y) \times_{X\times Y} B$. The axioms of the tensor triangulated structure ensures that tensoring (1.9) with $M(Y)$ is still a distinguished triangle, and we get a diagram

$$\begin{array}{ccccccc} M(Z\times_X X')\otimes M(Y) & \longrightarrow & (M(Z)\oplus M(X'))\otimes M(Y) & \longrightarrow & M(X)\otimes M(Y) & \longrightarrow & (M(Z\times_X X')[1])\otimes M(Y) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M(E) & \longrightarrow & M(Z\times Y)\oplus M(B) & \longrightarrow & M(X\times Y) & \longrightarrow & M(E)[1]. \end{array}$$

We now apply (1.1) four times in order to conclude that

$$M(X) \otimes M(Y) \xrightarrow{\simeq} M(X \times Y)$$

is an isomorphism by the five-lemma. Using that X and Y are proper we have obtain the desired isomorphism $M^c(X) \otimes M^c(Y) \simeq M^c(X \times Y)$. Applying the argument once more we prove the statement when Y is not necessarily smooth.

If X is any scheme and Y is proper, we compactify $X \hookrightarrow \overline{X}$, apply the localization sequence, and use the five-lemma on the diagram

$$\begin{array}{ccccccc} M^c(\partial X)\otimes M^c(Y) & \longrightarrow & M(\overline{X})\otimes M^c(Y) & \longrightarrow & M^c(X)\otimes M^c(Y) & \longrightarrow & (M^c(\partial X))[1]\otimes M^c(Y) \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow & & \downarrow\simeq \\ M^c(\partial X\times Y) & \longrightarrow & M^c(\overline{X}\times Y) & \longrightarrow & M^c(X\times Y) & \longrightarrow & M^c(\partial X\times Y)[1]. \end{array}$$

For the general case we apply the above argument to Y as well. □

The motive with compact support relates importantly with the general motive in the following theorem.

Theorem 1.2.9 (Duality, [FV00, Theorem 8.2])**.** *Assume that k admits resolution of singularities. Let T be a smooth scheme of dimension d. Then for every X and Y in Sch/k there are canonical isomorphisms between*

$$\mathrm{Hom}_{\boldsymbol{DM}^{\mathrm{eff}}(k,\Lambda)}(M(X \times T), M^c(Y))$$

and

$$\mathrm{Hom}_{\boldsymbol{DM}^{\mathrm{eff}}(k,\Lambda)}(M(X)(d)(2d + n), M^c(T \times Y))$$

in $\boldsymbol{DM}^{\mathrm{eff}}(k, \Lambda)$ for every $n \geq 0$.

One important consequence of Theorem 1.2.9 is the following.

Theorem 1.2.10 (Cancellation, [Voe10])**.** *Let k be a perfect field and M, N two objects in $\boldsymbol{DM}^{\mathrm{eff}}(k, \Lambda)$. Then tensoring with $\mathbb{Z}(1)$ induces an isomorphism*

$$\mathrm{Hom}_{\boldsymbol{DM}^{\mathrm{eff}}(k,\Lambda)}(M, N) \xrightarrow{\simeq} \mathrm{Hom}_{\boldsymbol{DM}^{\mathrm{eff}}(k,\Lambda)}(M(1), N(1)).$$

In Theorem 3.2.18 we will prove a generalization of this theorem. Since it is interesting to see how the two proofs relate, we have included the original proof of Theorem 16.25 in [MVW11] below.

Proof. Let $M^c(X)$ and $M^c(Y)$ be the motives of two smooth and proper schemes X and Y. We have isomorphisms

$$\mathrm{Hom}(M(X)[n], M(Y)) \overset{(1.7)}{\simeq} \mathrm{Hom}(M(X)[n], M^c(Y))$$

$$\begin{aligned}
&\overset{1.1}{\simeq} \mathrm{Hom}(M(X \times \mathbb{A}^1)[n], M^c(Y)) \\
&\overset{1.2.9}{\simeq} \mathrm{Hom}(M(X)(1)[2+n], M^c(Y \ \times \mathbb{A}^1)) \\
&\overset{1.2.5}{\simeq} \mathrm{Hom}(M(X)(1)[2], M^c(Y)(1)[2]) \\
&\overset{1.1}{\simeq} \mathrm{Hom}(M(X)(1)[2], M(Y)(1)[2]),
\end{aligned}$$

where all Hom groups are taken in $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$. Now removing the shifts yields the desired isomorphism, and an examination of the involved isomorphisms shows that the isomorphism is induced by tensoring with $\mathbb{Z}(1)$. Since these motives generate $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \Lambda)$, this argument shows that the statement is true for all $M, N \in \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \Lambda)$. Furthermore, since

$$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M, \oplus_\alpha N_\alpha) \simeq \oplus_\alpha \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M, N_\alpha)$$

and

$$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(\oplus_\alpha M_\alpha, N) \simeq \oplus_\alpha \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M_\alpha, N),$$

and using that $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$ is generated from $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \Lambda)$ by shifts and direct sums, this allows us to conclude for all objects of $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$. □

The motive with compact support provides another characterization of the classical motive as a sort of dualizing object.

Proposition 1.2.11 ([MVW11, Example 20.11]). *Let $X \in Sm/k$ of dimension d. Then there is an isomorphism*

$$M^c(X) \simeq M(X)^*(d)[2d] := \underline{RHom}(M(X), \mathbb{Z}(d))[2d].$$

We may use the motive with compact support to define new representable homology and cohomology theories in a similar way as we define motivic homology and cohomology (Definition 14.17 in [MVW11]).

Definition 1.2.12 ([MVW11, Definition 16.20]). For every scheme X of finite type over k, we define *motivic cohomology with compact support* as

$$H^{n,i}_c(X, \Lambda) := \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M^c(X), \Lambda(i)[n]).$$

Similarly, we define *(Borel–Moore) motivic homology with compact support* as

$$H^{\mathrm{BM}}_{n,i}(X, \Lambda) := \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(\Lambda(i)[n], M^c(X)).$$

Borel–Moore motivic cohomology has the important property of describing higher Chow groups for non-smooth schemes (Proposition 3.3.6).

An often useful result is the following induced morphism.

Proposition 1.2.13 ([Voe00, Corollary 4.2.4]). *If $f : Y \to X$ is a flat equidimensional morphism of relative dimension n, and if k admits resolution of singularities, there is a canonical morphism*

$$f^* : M^c(X)(n)[2n] \longrightarrow M^c(Y).$$

Using this result we can construct fundamental classes in Borel–Moore homology.

Construction 1.2.14 ([Nie06, p. 720])**.** If X is a scheme of finite type over k, the structure morphism $p: X \to \operatorname{Spec} k$ is a flat morphism of relative dimension equal to the dimensional of X. Using Proposition 1.2.13 we get a morphism

$$cl_X := p^* : \mathbb{Z}(n)[2n] \to M^c(X)$$

which we call the *fundamental class of* X, which in view of Definition 1.2.12 defines a homology class in $H^{\mathrm{BM}}_{2n,n}(X)$. Moreover, if $j : Y \to X$ is a closed subscheme of dimension m, the composition

$$\mathbb{Z}(n)[2n] \stackrel{cl_Y}{\to} M^c(Y) \stackrel{j^*}{\to} M^c(X)$$

represents the *Borel–Moore fundamental class* of Y in X.

CHAPTER 2

Logarithmic Motivic Homotopy Theory

In their fundamental work on motives of logarithmic schemes Binda, Park, and Østvær initiate in [BPØ20] a theory of *logarithmic* motivic homotopy theory. Their motivation originates in the observation that there are many "phenomena" that are not $\mathbb{A}^1$-invariant, and hence is not captured by the classical theory. Examples of such "phenomenas" include Hodge cohomology and cyclic homology.

The transition from algebraic geometry to logarithmic algebraic geometry comes naturally in this setting from the need to describe a compactification of $\mathbb{A}^1$ while at the same time remembering the extra structure of the boundary point. By compactifying $\mathbb{A}^1$ we get a homotopy theory similar to the ordinary case of algebraic topology in which case the object that parametrized homotopies, namely the unit interval, is both contractible and compact.

This chapter introduces logarithmic motivic homotopy theory on which we will build our theory of logarithmic motives with compact support in Chapter 3. As the fundamental work of [BPØ20] sets the scene, we heavily rely on this paper, and therefore cite it extensively.

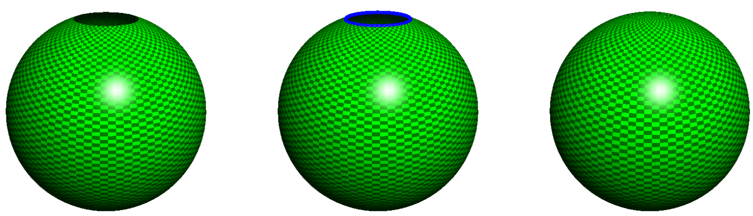


Figure 2.1: Topological realization of $\mathbb{A}^1$, $\overline{\square}$, and $\mathbb{P}^1$. We view $\mathbb{A}^1$ as a (real) sphere punctured at infinity, and $\overline{\square}$ as a punctured sphere with an infinitesimal boundary at infinity, and $\mathbb{P}^1$ corresponding to the sphere S^2.

2.1 Finite logarithmic correspondences

We start with an analogue of finite correspondences (Definition 1.1 in [MVW11]) in the logarithmic setting.

Definition 2.1.1 ([BPØ20, Definition 2.1.1])**.** For X and Y in lSm/k, an *elementary log correspondence* Z from X to Y consists of

(i) an integral closed subscheme $\underline{Z}$ of $\underline{X} \times \underline{Y}$ that is finite and surjective over a connected component of $\underline{X}$, and

(ii) a morphism $Z^N \to Y$ of fs log schemes (A.2.4), where Z^N denotes the fs log scheme whose underlying scheme is the normalization of Z and whose log structure is induced by the one on X. More precisely, if $p : \underline{Z^N} \to \underline{X}$ denotes the induced scheme morphism, then the log structure $\mathcal{M}_{Z^N}$ is given as $p^*_{\log} \mathcal{M}_X$.

A *finite log correspondence* from X to Y is a formal sum $\sum n_i Z_i$ of elementary log correspondences from X to Y. We let $\mathrm{lCor}(X, Y)$ denote the free abelian group generated by finite log correspondences. Let $lCor/k$ denote the category with the same objects as lSm/k and morphisms finite log correspondences.

Composition of finite log correspondences $\alpha \in \mathrm{lCor}(X, Y)$ and $\beta \in \mathrm{lCor}(Y, Z)$ is given by first defining the underlying scheme (similar to the case of finite correspondences), and then equipping it with a fitting log structure. Making the log structure compatible is the main reason for considering the second condition in the definition of elementary log correspondences. Indeed, this condition is the minimal condition we can impose on the log structure to make the projection $Z \to X$ a strict morphism. This construction of the composition is non-trivial, but described in detail in the proof of Lemma 2.3.3 in [BPØ20]. We note that if the log schemes have trivial log structure, then the composition agrees with the case of finite correspondences.

When X is log smooth, and ∂X denotes the set of points og X with non-trivial log structure, the complement $X - \partial X$ is smooth and open and there is an open immersion $X - \partial X \to X$. Note that we have a faithful functor

$$\gamma : \, lSm/k \longrightarrow \mathrm{lCor}/k \tag{2.1}$$

that sits in the commutative diagram

$$\begin{array}{ccc} lSm/k & \xrightarrow{\gamma} & \mathrm{lCor}/k \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ Sm/k & \xrightarrow{\gamma} & \mathrm{Cor}/k. \end{array} \tag{2.2}$$

where γ is the functor $X \mapsto X$ and $(f : X \to Y) \mapsto \Gamma_f$, and ω is the functor $X \to X - \partial X$ and $f \mapsto \underline{f}$.

2.2 Topologies on fs log schemes

Definition 2.2.1 ([Par19, Definition 7.2])**.** A Cartesian square of fs log schemes

$$Q = \begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

is a

(i) *Zariski distinguished square* if f and q are open immersions.

(ii) *Strict Nisnevich distinguished square* if f is strict (A.2.5) étale, g is an open immersion, and f induces an isomorphism

$$f^{-1}(\underline{X} - g(\underline{X'})) \xrightarrow{\sim} \underline{X} - g(\underline{X'})$$

with respect to the reduced scheme structure.

(iii) *Dividing distinguished square* if $Y' = X' = \emptyset$ and f is a surjective proper log étale monomorphism, i.e., a log modification (A.3.5).

Associated to the Zariski distinguished squares (resp. strict Nisnevich distinguished squares) we have the corresponding *Zariski cd-structure* (resp. *Nisnevich cd-structure*) which gives rise to the *Zariski topology* (resp. *strict Nisnevich topology*). We let Zar (resp. sNis) be shorthand for the Zariski topology (resp. strict Nisnevich topology).

The *dividing Zariski cd-structure* (resp. *dividing Nisnevich cd-structure*) is the union of the Zariski (resp. strict Nisnevich) topology and the dividing cd-structures. We refer to the associated topology as the *dividing Zariski topology* (resp. *dividing Nisnevich topology*). We let $dZar$ (resp. $dNis$) be shorthand for the dividing Zariski topology (resp. dividing Nisnevich topology).

Remark 2.2.2. Every distinguished Nisnevich square of schemes is a strict Nisnevich distinguished square. Moreover, for every strict Nisnevich distinguished square Q the induced square $\underline{Q}$ of the underlying schemes

$$\begin{array}{ccc} \underline{Y'} & \xrightarrow{\underline{g'}} & \underline{Y} \\ \big\downarrow{\scriptstyle \underline{f'}} & & \big\downarrow{\scriptstyle \underline{f}} \\ \underline{X'} & \xrightarrow{\underline{g}} & \underline{X} \end{array}$$

is a distinguished Nisnevich square of schemes since f and g are strict.

Remark 2.2.3. By the above remark we see that the strict Nisnevich topology on log schemes generalizes the Nisnevich topology on schemes. However, in the log setting we also want to consider the dividing topology, and create the dividing Nisnevich topology. The reason for this is that we want to replace the classical Tate twist $\mathbb{Z}(1) := M(\operatorname{Spec} k \to \mathbb{G}_m)[-1]$ with $M(\operatorname{Spec} k \to (\mathbb{P}^1, 0+\infty))[-1]$, where $(\mathbb{P}^1, 0+\infty)$ is the compactification of $\mathbb{G}_m$. The problem then is that the ordinary multiplication morphism

$$m : \mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathbb{G}_m$$

does not extend to a morphism

$$\overline{m} : (\mathbb{P}^1, 0+\infty) \times (\mathbb{P}^1, 0+\infty) \longrightarrow (\mathbb{P}^1, 0+\infty).$$

To remedy this, we can instead consider the blow-up of $(\mathbb{P}^1, 0+\infty) \times (\mathbb{P}^1, 0+\infty)$ at $(0,\infty) + (\infty + 0)$ to get a map

$$\overline{m'} \ : (Bl_{(0,\infty)+(\infty+0)}(\mathbb{P}^1 \times \mathbb{P}^1), H_1 + H_2 + H_3 + H_4 + E_1 + E_2) \longrightarrow (\mathbb{P}^1, 0+\infty),$$

and use the dividing topology to identify it with $(\mathbb{P}^1, 0+\infty) \times (\mathbb{P}^1, 0+\infty)$.

Example 2.2.4. The cartesian squares

$$\begin{array}{ccc} \mathbb{A}^n - 0 & \longrightarrow & (Bl_0\, \mathbb{A}^n, E) \\ \downarrow & & \downarrow \\ \mathbb{P}^n - 0 & \longrightarrow & (Bl_0 \mathbb{P}^n, E) \end{array} \tag{2.3}$$

and

$$\begin{array}{ccc} (Bl_0\, \mathbb{A}^n, E) & \longrightarrow & (Bl_0 \mathbb{P}^n, E) \\ \downarrow & & \downarrow \\ \mathbb{A}^n & \longrightarrow & \mathbb{P}^n \end{array} \tag{2.4}$$

where E denotes the exceptional divisor, are examples of strict Nisnevich distinguished squares. The second one corresponds to the classical Nisnevich distinguished square

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1, \end{array}$$

which is responsible for the $\mathbb{A}^1$-weak equivalence

$$\mathbb{P}^1 \simeq \mathbb{A}^1 / (\mathbb{A}^1 - 0)$$

in $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$.

2.3 Sheaves with logarithmic transfers

Definition 2.3.1 ([BPØ20, Definition 4.1.1])**.** A *presheaf of* Λ*-modules with log transfers* is an additive presheaf of Λ-modules on the category of finite log correspondences lCor/k. We denote the category of all presheaves of Λ-modules with log transfers by $\mathbf{Psh}^{\mathrm{ltr}}(k, \Lambda)$. For $X \in lSm/k$, we define $\Lambda_{\mathrm{ltr}}(X)$ as the representable presheaf of Λ-modules with log transfers given by

$$Y \mapsto \Lambda_{\mathrm{ltr}}(X)(Y) := \mathrm{lCor}(Y, X) \otimes \Lambda.$$

Let $\mathbf{Psh}^{\log}(k, \Lambda)$ be the category of presheaves of Λ-modules on lSm/k. There is a pair of adjoint functors

$$\omega : lSm/k \underset{\longrightarrow}{\longleftarrow} Sm/k : \lambda, \tag{2.5}$$

where ω is the fully faithful functor that assigns the trivial log structure and λ is the forgetful functor $X \mapsto X - \partial X$. This gives rise to associated adjoint functors in the sense of [AGV72]

$$\mathbf{Psh}(k,\Lambda) \underset{\underset{\omega_*}{\longleftarrow}}{\overset{\overset{\lambda_\sharp}{\longrightarrow}}{\underset{— \lambda_* \simeq \omega^* \rightarrow}{\leftarrow \lambda^* \simeq \omega_\sharp —}}} \mathbf{Psh}^{\log}(k,\Lambda).$$

We will use the same notation for the functors between the corresponding chain complexes

$$C_*(\mathbf{Psh}(k,\Lambda)) \underset{\underset{\omega_*}{\longleftarrow}}{\overset{\overset{\lambda_\sharp}{\longrightarrow}}{\underset{— \lambda_* \simeq \omega^* \rightarrow}{\leftarrow \lambda^* \simeq \omega_\sharp —}}} C_*(\mathbf{Psh}^{\log}(k,\Lambda)).$$

Definition 2.3.2 ([BPØ20, Definition 4.1.6])**.** The tensor product of two representable presheaves with log transfers on lSm/k is defined by

$$\Lambda_{\mathrm{ltr}}(X) \otimes \Lambda_{\mathrm{ltr}}(Y) := \Lambda_{\mathrm{ltr}}(X \times Y).$$

More generally, we define the tensor product of $F, G \in \mathbf{Psh}^{\mathrm{ltr}}(k,\Lambda)$ by

$$F \otimes G := \varinjlim_{X,Y} \Lambda_{\mathrm{ltr}}(X) \otimes \Lambda_{\mathrm{ltr}}(Y),$$

where F and G are colimits of representable sheaves

$$F \simeq \varinjlim_{X} \Lambda_{\mathrm{ltr}}(X) \text{ and } F \simeq \varinjlim_{Y} \Lambda_{\mathrm{ltr}}(G).$$

This gives a symmetric monoidal structure on $\mathbf{Psh}^{\mathrm{ltr}}(k,\Lambda)$.

Proposition 2.3.3. *For every fs log scheme X smooth log smooth over k, the sheaf with log transfers $\Lambda_{ltr}(X)$ is a strict Nisnevich sheaf (or even a strict étale sheaf).*

Proof. See the proof of Lemma 4.4.3 in [BPØ20]. The origin of this argument dates back to [Voe00]. □

We have a dNis-sheafication functor a^*_{dNis} and a forgetful functor $a_{\mathrm{dNis}\,*}$ forming an adjoint pair

$$a^*_{\mathrm{dNis}} : \mathbf{Psh}^{\log}(k,\Lambda) \rightleftarrows \mathbf{Shv}^{\log}(k,\Lambda) : a_{\mathrm{dNis}\,*}.$$

By abuse of notation, we let

$$a_{\mathrm{dNis}\,*} : \mathbf{Shv}^{\mathrm{ltr}}(k,\Lambda) \longrightarrow \mathbf{Psh}^{\mathrm{ltr}}(k,\Lambda)$$

be the inclusion functor, and

$$\gamma^* : \mathbf{Shv}^{\mathrm{ltr}}(k,\Lambda) \longrightarrow \mathbf{Shv}^{\log}(k,\Lambda)$$

the restriction of the functor $\gamma^* : \mathbf{Psh}^{\mathrm{ltr}}(k,\Lambda) \longrightarrow \mathbf{Psh}^{\log}(k,\Lambda)$. We note that there is an equivalence

$$\gamma^* a_{\mathrm{dNis}\,*} \simeq a_{\mathrm{dNis}\,*}\gamma^*.$$

We would have hoped at this point that the dNis-sheafication of $\Lambda_{\mathrm{ltr}}(X)$ would be a sheaf with transfers in the dividing Nisnevich topology, and that we could associate $lCor(X, Y)$ with

$$\mathrm{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}(k,\Lambda)}\left(a^*_{\mathrm{dNis}}\Lambda_{\mathrm{ltr}}(Y), a^*_{\mathrm{dNis}}\Lambda_{\mathrm{ltr}}(X)\right).$$

However this is not the case as it is not a dividing Nisnevich sheaf ([BPØ20, p. 72]).

Nonetheless, it turns out that we ca describe $a^*_{\mathrm{dNis}}\Lambda_{\mathrm{ltr}}(Y)(X)$ with algebraic cycles by introducing *dividing log correspondences*.

Definition 2.3.4 ([BPØ20, Definition 4.6.1]). For X and Y in lSm/k we define an *elementary dividing log correspondence* Z from X to Y to be a closed irreducible subscheme such that $\underline{Z} \subset \underline{X} \times \underline{Y}$ is finite and surjective over an irreducible component of $\underline{X}$ together with a morphism

$$u' : Z' \to X' \times Y$$

subject to the following set of conditions

(i) The image of the composition $\underline{Z'} \xrightarrow{u'} \underline{X'} \times \underline{Y} \to \underline{X} \times \underline{Y}$ is $\underline{Z}$.

(ii) $\underline{Z'^N}$ is the normalization of $\underline{Z}$.

(iii) The composition $Z'^N \xrightarrow{u'} X' \times Y \to X'$ is strict.

A *dividing log correspondence* from X to Y is a formal sum of elementary dividing log correspondences, and we let $\mathrm{lCor}^{\mathrm{div}}(X, Y)$ be the free abelian group of dividing log correspondences from X to Y.

We define composition of log correspondences

$$\circ : \mathrm{lCor}^{\mathrm{div}}(X, Y) \times \mathrm{lCor}^{\mathrm{div}}(X, Y) \to \mathrm{lCor}^{\mathrm{div}}(X, Y)$$

from the identification

$$\mathrm{lCor}(X, Y) \simeq \mathrm{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}(k,\Lambda)}\left(a^*_{\mathrm{dNis}}\Lambda_{\mathrm{ltr}}(Y), a^*_{\mathrm{dNis}}\Lambda_{\mathrm{ltr}}(X)\right),$$

and using the composition in $\mathbf{Shv}^{\mathrm{ltr}}(k, \Lambda)$.

Remark 2.3.5. We can consider a dividing log correspondence $Z \in \mathrm{lCor}_{\mathrm{div}}(X, Y)$ a log correspondence after replacing X by a log modification. See Remark 4.6.2 in [BPØ20].

The reason why we care about dividing log correspondences is due to the following result.

Proposition 2.3.6 ([BPØ20, Proposition 4.6.3]). *For every X and Y in lSm/k there is an isomorphism*

$$\mathrm{lCor}^{div}(X, Y) \otimes \Lambda \simeq a^*_{dNis}\Lambda_{ltr}(Y)(X).$$

Proof. We can associate the group $\mathrm{lCor}^{\mathrm{div}}(X, Y)$ with $\mathrm{lCor}(X, Y)$ after replacing X by a log modification $Y \to X$. By use of the identification (Lemma 4.4.2 in [BPØ20]) we have

$$a^*_{\mathrm{dNis}}\Lambda_{\mathrm{ltr}}(X) \simeq \operatorname*{colim}_{Y \to X} \Lambda_{\mathrm{ltr}}(Y),$$

where $Y \to X$ is a log modification of X. This concludes the proof. □

Let $\mathbf{Psh}^{\mathrm{dltr}}(k,\Lambda)$ be the category of presheaves of Λ-modules with log transfers on $\mathrm{lCor}^{\mathrm{div}}$ and identify $\mathbf{Shv}^{\mathrm{dltr}}(k,\Lambda)$ as the full subcategory of $\mathbf{Psh}^{\mathrm{dltr}}(k,\Lambda)$ consisting of the presheaves that are sheaves with log transfers in the dividing Nisnevich topology. Furthermore, let $\mathbf{Psh}^{\mathrm{dltr}}(SmlSm/k,\Lambda)$ denote the category of presheaves of Λ-modules with log transfers on $\mathrm{lCor}^{\mathrm{div}}_{SmlSm/k}$ and identify $\mathbf{Shv}^{\mathrm{dltr}}(SmlSm/k,\Lambda)$ as the full subcategory of $\mathbf{Psh}^{\mathrm{dltr}}(SmlSm/k,\Lambda)$ consisting of presheaves that are sheaves with log transfers in the dividing Nisnevich topology.

Proposition 2.3.7. *There are equivalences of categories*

$$\begin{aligned}\boldsymbol{Shv}^{dltr}(k,\Lambda) &\overset{(4.6.6)}{\simeq} \boldsymbol{Shv}^{dltr}(SmlSm/k,\Lambda)\\ &\overset{(4.7.4)}{\simeq} \boldsymbol{Shv}^{ltr}(SmlSm/k,\Lambda)\\ &\overset{(4.7.5)}{\simeq} \boldsymbol{Shv}^{ltr}(k,\Lambda).\end{aligned}$$

Proof. The result is a combination of prop. 4.6.6, lemma 4.7.4 and prop 4.7.5 in [BPØ20]. The proof of these results covers the majority of the sections 4.6 and 4.7 in [BPØ20], so we have not included them here. □

The following theorem ensures that the sheafication functor a^*_{dNis} respects log transfers.

Theorem 2.3.8 ([BPØ20, Theorem 4.5.7])**.** *The dividing Nisnevich topology on lSm/k is compatible with log transfers.*

2.4 The category of logarithmic motives

The construction of $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ is straightforward at this point. By abusing notation we also write $\overline{\square}$ for the projection $X \times \overline{\square} \to X$ onto the first factor.

Definition 2.4.1 ([BPØ20, Definition 5.2.1])**.** The *derived category of effective log motives* $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ is the homotopy category of $C_*(\mathbf{Shv}^{\mathrm{ltr}}(k,\Lambda))$ with respect to the $\overline{\square}$-local descent model structure (See appendix B in [BPØ20]).

For X in lSm/k, we define the *motive of* X, denoted $M(X)$ as the image of $a^*_{\mathrm{dNis}}\Lambda_{\mathrm{ltr}}(X)$ in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$.

Remark 2.4.2. In constructing $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$, Proposition 2.3.7 shows that it is enough to consider smooth fs log schemes log smooth over k as there is an equivalence

$$\mathbf{logDM}^{\mathrm{eff}}(SmlSm/k,\Lambda) \simeq \mathbf{logDM}^{\mathrm{eff}}(k,\Lambda),$$

where $\mathbf{logDM}^{\mathrm{eff}}(SmlSm/k,\Lambda)$ is the homotopy category of

$$C_*(\mathbf{Shv}(SmlSm/k,\Lambda))$$

with respect to the $\overline{\square}$-local descent model structure.

The following properties follows from the construction.

(i) (Monoidal structure) For every X and Y in lSm/k there is a naturally induced isomorphism of log motives

$$M(X \times Y) \simeq M(X) \otimes M(Y). \tag{2.6}$$

Especially, for every X in lSm/k there is a naturally induced isomorphism of log motives

$$M(X) \simeq M(X \times \overline{\square}). \tag{2.7}$$

(ii) (Mayer-Vietoris) For every strict Nisnevich distinguished square in lSm/k

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array} \tag{2.8}$$

there is a naturally induced homotopy cartesian diagram square of log motives

$$\begin{array}{ccc} M(Y') & \longrightarrow & M(Y) \\ \downarrow & & \downarrow \\ M(X') & \longrightarrow & M(X). \end{array} \tag{2.9}$$

(iii) (Log modification) Every log modification $f: Y \to X$ of fs log schemes log smooth over k induces an isomorphism of log motives

$$M(f): \; M(Y) \to M(X). \tag{2.10}$$

Relating the classical theory to this setting we find:

Proposition 2.4.3 ([BPØ20, Proposition 8.2.2])**.** *Assume that k admits resolution of singularities. For every smooth and proper scheme X over k there is an equivalence*

$$a^*_{dNis}\Lambda_{ltr}(X) \simeq \omega^*\Lambda_{tr}(X)$$

in $\boldsymbol{logDM}^{\mathit{eff}}(k, \Lambda)$.

Let $\mathbf{logDM}^{\mathrm{eff}}_{\mathrm{prop}}(k, \Lambda)$ be the smallest triangulated subcategory of

$$\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$$

that is closed under small sums and shifts and is generated by all $M(X)$ for $X \in lSm/k$, where $\underline{X}$ is proper over k (Definition 5.2.8 in [BPØ20]).

Theorem 2.4.4 ([BPØ20, Theorem 8.2.17])**.** *Assume that k satisfies resolution of singularities. Then the functor*

$$R\omega^* \; \boldsymbol{DM}^{\mathit{eff}}(k, \Lambda) \; \longrightarrow \boldsymbol{logDM}^{\mathit{eff}}(k, \Lambda)$$

is fully faithful, and whose essential image we can identify with

$$\boldsymbol{logDM}^{\mathit{eff}}_{\mathit{prop}}(k, \Lambda),$$

i.e., there is an equivalence of triangulated categories

$$\boldsymbol{logDM}^{\mathit{eff}}_{\mathit{prop}}(k, \Lambda) \simeq \boldsymbol{DM}^{\mathit{eff}}(k, \Lambda).$$

We briefly mention the recent paper [BM21] by Merici and Binda where they show that $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$ admits a homotopy t-structure, which induces the classical homotopy t-structure on $\mathbf{DM}^{\mathrm{eff}}(k, \Lambda)$ under the equivalence of Theorem 2.4.4.

Definition 2.4.5 ([BPØ20, Definition 7.0.1])**.** The *Tate object* in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ is

$$\Lambda(1) := M(\operatorname{Spec} k \to \mathbb{P}^1)[-2],$$

were $\operatorname{Spec} k \to \mathbb{P}^1$ denotes the 0-section. For $n \geq 0$ we Let $\Lambda(n)$ denote the n-fold product $\Lambda(1) \otimes \cdots \otimes \Lambda(1)$. We define the n-th *Tate twist* of $M \in \mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ by

$$M(n) := M \otimes \Lambda(n).$$

We are now in position to generalize the definition of motivic homology and cohomology (Definition 14.17 in [MVW11]) to the logarithmic setting.

Definition 2.4.6. We define *logarithmic motivic cohomology* as

$$H^{p,q}(X,\Lambda) := \operatorname{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(M(X), \Lambda(i)[n]),$$

and similarity *logarithmic motivic homology* as

$$H_{p,q}(X,\Lambda) := \operatorname{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(\Lambda(i)[n], M(X)).$$

In general, for a strictly $\overline{\square}$-invariant complex $\mathcal{F}$ of dividing Nisnevich sheaves with log transfers, i.e., a complex satisfying

$$\mathbf{H}^i_{\mathrm{dNis}}(X \times \overline{\square}, \mathcal{F}) \simeq \mathbf{H}^i_{\mathrm{dNis}}(X, \mathcal{F})$$

for all $X \in lSm/k$, Proposition 5.2.3 in [BPØ20] provides the following identification

$$\operatorname{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(M(X), \mathcal{F}(i)[n]) \simeq \mathbf{H}^i_{\mathrm{dNis}}(X, \mathcal{F}).$$

We also like to mention the following technique to prove representability of a cohomology theory of schemes or log schemes in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$. We begin by the (usually easy) generalization of the theory to log schemes in $SmlSm/k$, and then extend it further to all log schemes in lSm/k by the following construction.

Construction 2.4.7 ([BPØ20, Construction 5.4.3])**.** If a complex $\mathcal{F}$ of dividing Nisnevich sheaf is only defined on $SmlSm/k$, we can naturally extend $\mathcal{F}$ to a complex $\iota_\sharp \mathcal{F}$ on lSm/k by defining

$$\iota_\sharp \mathcal{F} := \operatorname*{colim}_{Y \in X^{Sm}_{div}} \mathcal{F}(Y),$$

where X^{Sm}_{div} denotes the category of log modifications $Y \to X$ where $Y \in SmlSm/k$. Then using Proposition 2.3.7 we conclude that

$$\mathbf{H}^i_{\mathrm{dNis}}(X, \mathcal{F}) \simeq \mathbf{H}^i_{\mathrm{dNis}}(X, \iota_\sharp \mathcal{F})$$

for $X \in SmlSm/k$.

If $\mathcal{F}$ is strictly $\overline{\square}$-invariant, $\iota_\sharp \mathcal{F}$ is too, and we further have the identification

$$\operatorname{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(M(X), \iota_\sharp \mathcal{F}(i)[n]) \simeq \mathbf{H}^i_{\mathrm{dNis}}(X, \mathcal{F}).$$

Then proving $\overline{\square}$-invariance proves representability. However, we can alternatively make use of the following theorem.

Theorem 2.4.8 ([BPØ20, Proposition 7.3.1 and Construction 7.8.4])**.** *There is an equivalence of $\overline{\square}$-invariant sheaves satisfying dividing Nisnevich descent in lSm/k and $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariance sheaves satisfying strict Nisnevich descent in $SmlSm/k$.*

CHAPTER 3

Logarithmic Motives with Compact Support

In this chapter we develop a theory of logarithmic motives with compact support, i.e., we generalize the theory of Chapter 1 to the setting of Chapter 2.

In generalizing a theory it becomes of primary interest to develop its properties and its relation to the original theory. Thus after having established basic definitions and results, this will be our main focus. In particular, we will seek logarithmic analogues of the theorems presented in Chapter 1. However, our theory differs from the original theory in an important way: Using Theorem 3.2.12 we show that the logarithmic motive with compact support is homotopy invariant ($\overline{\square}$-invariant), whereas the classical motive with compact support is not homotopy invariant ($\mathbb{A}^1$-invariant).

This chapter presents our contribution. We provide many new definitions, and in establishing their properties we indeed get many new results. However, we rely heavily on the setting of [BPØ20], and many of our arguments are adaptations or generalizations of the arguments presented there, especially in Section 3.1. We therefore cite *loc. cit.* extensively. Several questions remain unanswered which we have gathered in the end of this chapter. We hope to continue working on these problems in the future.

3.1 Finite logarithmic correspondences with compact support

We begin similarly to Chapter 16 in [MVW11] by defining an analogue finite correspondences with compact support which we use to define the logarithmic motive with compact support.

Definition 3.1.1. Let Y be any fs log scheme over k and $r \geq 0$ an integer. We define the presheaf $\Lambda^c_{\mathrm{ltr}}(Y,r)$ on lSm/k as follows: For any $X \in lSm/k$ we let $\Lambda^c_{\mathrm{ltr}}(Y,r)(X)$ be the free abelian group with coefficients in Λ generated by integrally closed subschemes $\underline{Z} \subset \underline{X} \times \underline{Y}$ such that

(i) $\underline{Z}$ is dominant and equidimensional of relative dimension r over a component of $\underline{X}$.

(ii) a morphism $Z^N \to Y$ of fs log schemes, where Z^N denotes the fs log scheme whose underlying scheme is the normalization of Z and whose log structure is induced by the one on X. More precisely, if $p: \underline{Z^N} \to \underline{X}$ denotes the induced scheme morphism, then the log structure $\mathcal{M}_{Z^N}$ is given as the pullback log structure $p^*_{\log}\mathcal{M}_X$, where $p^*_{\log}\mathcal{M}_X$ is the log structure induced by the prelog structure

$$f^{-1}(\mathcal{M}_X) \to f^{-1}(\mathcal{O}_{Z^N}) \to \mathcal{O}_{Z^N}.$$

Remark 3.1.2. This definition is a rephrasing of the definition of correspondences with compact support given in Example 16.2 in [MVW11] adapted to logarithmic algebraic geometry. The first condition is a direct translation of the standard definition, while the second definition is necessary to construct composition of log correspondences with compact support. We note that the second condition is equivalent to the second condition in Definition 2.1.1.

Remark 3.1.3. Given a morphism $S' \to S$, the pullback of cycles induces a natural map

$$\Lambda^c_{\mathrm{ltr}}(X, r)(S) \to \Lambda^c_{\mathrm{ltr}}(X, r)(S').$$

Moreover, given a flat morphism $f: U \to X$ of relative dimension r, the pullback of cycles gives a morphism

$$f^*: \Lambda^c_{\mathrm{ltr}}(U, n) \to \Lambda^c_{\mathrm{ltr}}(X, n + r)$$

by Proposition 3.6.2 in [SV00]. Similarly, if $g: Y \to X$ is a proper map of relative dimension r, the the pushforward of cycles gives a morphism

$$g_*: \Lambda^c_{\mathrm{ltr}}(Y, m) \to \Lambda^c_{\mathrm{ltr}}(X, m + r)$$

by Proposition 3.6.4 in [SV00].

We will mainly be concerned with the case $r = 0$, for which we denote $\Lambda^c_{\mathrm{ltr}}(Y, 0)$ by $\Lambda^c_{\mathrm{ltr}}(Y)$. The conditions of Definition 3.1.1 then reads that $\Lambda^c_{\mathrm{ltr}}(Y)(X)$ is the free abelian group with coefficients in Λ generated by integral closed subschemes $\underline{Z} \subset \underline{X} \times \underline{Y}$ which are dominant and quasi-finite over an connected component of $\underline{X}$, and satisfying Definition 3.1.1.(ii). We call elements of $\Lambda^c_{\mathrm{ltr}}(Y)(X)$ *finite log correspondences with compact support* from X to Y.

Example 3.1.4. If $p: \underline{Z^N} \to \underline{X}$ and $q: \underline{Z^N} \to \underline{Y}$ denotes the underlying scheme morphisms, then referring to Definition III.1.1.5 in [Ogu18] the second condition of Definition 3.1.1 is equivalent to giving a morphism

$$q^*_{\log}\mathcal{M}_Y \to p^*_{\log}\mathcal{M}_X.$$

If we consider the presheaf with log transfers $\Lambda^c_{\mathrm{ltr}}(\mathrm{pt}_{\mathbb{N}})$, where $\mathrm{pt}_{\mathbb{N}}$ is a point with a non-trivial log structure, it is clear that this is the empty presheaf since there can be no such morphism q.

Remark 3.1.5. The representable sheaf $\Lambda_{\mathrm{ltr}}(X)$ is a subsheaf of $\Lambda^c_{\mathrm{ltr}}(X)$ since the structure morphism associated to $U \to V$ are compatible, i.e., the induced map

$$\Lambda^c_{\mathrm{ltr}}(X)(U) \to \Lambda^c_{\mathrm{ltr}}(X)(V)$$

is also the pullback of relative cycles.

Some properties are immediate:

- Since every finite and surjective closed subset $\underline{Z} \subset \underline{X} \times \underline{Y}$ is trivially dominant and quasi-finite, we have $\mathrm{lCor}_k(X, Y) \otimes \Lambda \subset \Lambda^c_{\mathrm{ltr}}(Y)(X)$.
- If Y has trivial log structure, then the condition on the log structure is automatically satisfied, and $\Lambda^c_{\mathrm{ltr}}(Y)(X) = \Lambda^c_{\mathrm{tr}}(Y)(\underline{X})$, where the sheaf with transfers $\Lambda^c_{\mathrm{tr}}(Y)$ were defined in Definition 1.2.1.
- If X has trivial log structure, then Z^N has trivial log structure, and $Z^N \to Y$ factors through $Y - \partial Y$. Hence $\Lambda^c_{\mathrm{ltr}}(Y)(X) = \Lambda^c_{\mathrm{tr}}(X)(Y - \partial Y)$.
- If X and Y have trivial log structure, then combining the two last properties gives
$$\Lambda^c_{\mathrm{ltr}}(Y)(X) = \Lambda^c_{\mathrm{tr}}(X)(Y).$$

We note that all projective schemes are proper (or complete) (Definition A.2.15). In particular we have that $\overline{\square}$ and $\mathbb{P}^n$ are proper. This proposition justifies the name "compact support".

Proposition 3.1.6. *Let X be an fs log scheme. If X is proper then $\Lambda^c_{ltr}(X) = \Lambda_{ltr}(X)$.*

Proof. A morphism is finite if and only if it is both quasi-finite and proper ([Har77, Exercise III.11.2]).

Since X is proper (A.2.15), the valuative criterion of properness of schemes ([Gro61, p. 7.3.8]) makes it clear that the underlying scheme $\underline{X}$ is proper. Thus every closed subset $\underline{Z} \subset \underline{U} \times \underline{X}$ is proper over $\underline{U}$, so from the statement above we have that $\underline{Z}$ is finite, and hence lies in $\Lambda_{\mathrm{ltr}}(X)(U)$. □

Properties of logarithmic sheaves

The following series of lemmas are taken from [BPØ20], to which case we have generalized them slightly to our setting. They are used to prove Proposition 3.1.11, which says that $a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(X)$ is a dividing Nisnevich sheaf with log transfers.

We refer to Definition A.2.7 for the definition of a *solid* log scheme.

Lemma 3.1.7. *Let $f : X \to Y$ be a quasi-finite and dominant morphism of fs log schemes from X to Y. If Y is smooth over k, then X is solid.*

Proof. Owing to Lemma 2.2.2 and Lemma 2.2.7 in [BPØ20], it suffices to show that f is an open morphism. Since Y is normal by Lemma 2.2.8 in [BPØ20], we conclude that f is universally open by Lemma 37.67.2 in [Sta21]. □

Lemma 3.1.8. *Let X and Y be fs log schemes log smooth over k, and suppose that $\underline{Z}$ is a closed subscheme of $\underline{X} \times \underline{Y}$. Then there exists at most one elementary log correspondence with compact support whose underlying scheme is $\underline{Z}$.*

Proof. Let Z^N be the fs log scheme whose underlying scheme is the normalization $\underline{Z^N}$ of $\underline{Z}$. We equip it with the log structure induced from the canonical map $\underline{Z^N} \to \underline{Z} \to \underline{X}$. Letting $q : \underline{Z^N} \to \underline{Y}$ be the induced morphism, we need to show that there is at most one morphism $r : Z^N \to Y$ of ls log schemes such that $\underline{r} = q$. By Lemma 2.2.7 in [BPØ20] we have that Y is solid. Since

Z^N is quasi-finite over X, it is solid by Lemma 3.1.7. Hence the log structure $\mathcal{M}_{Z^N} \to \mathcal{O}_{Z^N}$ is injective by Lemma 2.2.5 in [BPØ20]. We are thus in a position in which we can apply Lemma 2.2.3 in [BPØ20] which shows that there can be at most one such morphism r. □

Lemma 3.1.9. *Let X and Y be fs log schemes that are log smooth over k. Then the homomorphism of abelian groups*

$$\Lambda^c_{ltr}(Y)(X) \longrightarrow \Lambda^c_{tr}(Y - \partial Y)(X - \partial X)$$

mapping V to $V - \partial V$ is injective.

Proof. We may assume that X and Y are integral and that

$$V = n_1 V_1 + \ldots + n_r V_r$$

is a finite log correspondence with compact support with $r > 0$, $n_i \neq 0$, and that the V_i are all distinct. If $V - \partial V = 0$, then $V_i - \partial V_i = V_j - \partial V_j$ for some $i \neq j$. The closure of $V_i - \partial V_i$ and $V_j - \partial V_j$ in $\underline{X} \times \underline{Y}$ are equal to $\underline{V_i}$ and $\underline{V_j}$ respectively, hence $\underline{V_i}$ equals $\underline{V_j}$. Applying Lemma 2.3.1 in[BPØ20] this contradicts the fact that for every closed subscheme Z of $\underline{X} \times \underline{Y}$ there is at most one elementary log correspondence with compact support from X to Y whose underlying scheme is $\underline{Z}$. Hence we must have $V - \partial V \neq 0$ which completes the proof. □

Proposition 3.1.10. *Let $X \in lSm/k$. Then $\Lambda^c_{ltr}(X)$ is a strict étale sheaf.*

Proof. We argue similarly as the proof of Proposition 4.5.1 in [BPØ20].

Letting Y_1 and Y_2 be fs log schemes log smooth over k, we have

$$\Lambda^c_{\mathrm{ltr}}(X)(Y_1 \amalg Y_2) = \Lambda^c_{\mathrm{ltr}}(X)(Y_1) \oplus \Lambda^c_{\mathrm{ltr}}(X)(Y_2).$$

It therefore suffices to show that the sequence

$$0 \to \Lambda^c_{\mathrm{ltr}}(X)(Y) \to \Lambda^c_{\mathrm{ltr}}(X)(U) \xrightarrow{(+,-)} \Lambda^c_{\mathrm{ltr}}(X)(U \times_Y U) \tag{3.1}$$

is exact for every étale covering $p: U \to Y$.

There is an induced square of Λ-modules

$$\begin{array}{ccc} \Lambda^c_{\mathrm{ltr}}(X)(Y) & \longrightarrow & \Lambda^c_{\mathrm{ltr}}(X)(U) \\ \downarrow & & \downarrow \\ \Lambda^c_{\mathrm{ltr}}(X - \partial X)(Y - \partial Y) & \longrightarrow & \Lambda^c_{\mathrm{ltr}}(X - \partial X)(U - \partial U). \end{array}$$

The lower horizontal map is injective, as $\Lambda^c_{\mathrm{ltr}}(X - \partial X) = \Lambda^c_{\mathrm{tr}}(X - \partial X)$ is an étale sheaf by [MVW11, p. 125]. By Lemma 3.1.9, the vertical maps are injective as well, hence the upper horizontal map is injective.

It remains to show that (3.1) is exact at $\Lambda^c_{\mathrm{ltr}}(X)(U)$. Consider the cartesian diagram of fs log schemes

$$\begin{array}{ccc} (U - \partial U) \times (X \times \partial X) & \longrightarrow & U \times X \\ \downarrow & & \downarrow \\ (Y - \partial Y) \times (X \times \partial X) & \longrightarrow & Y \times X. \end{array}$$

Suppose that $W \in \Lambda^c_{\mathrm{ltr}}(X)(U)$ is a log correspondence with compact support with trivial image in $\Lambda^c_{\mathrm{ltr}}(X)(U \times_Y U)$, and consider the finite correspondence with compact support

$$W - \partial W \in \mathrm{Cor}^c(U - \partial U, X - \partial X).$$

Since $\Lambda^c_{\mathrm{tr}}(X)$ is an étale sheaf, we may find a finite correspondence

$$V' \in \mathrm{Cor}^c(Y - \partial Y, X - \partial X)$$

that maps to $W - \partial W$. Define $\underline{V}$ as the closure of V' in $\underline{Y} \times \underline{X}$.

Let $u : \underline{W} \to \underline{V} \times_{\underline{Y}} \underline{U}$ be the induced map between closed subschemes of $\underline{U} \times \underline{X}$. The pullback to $(U - \partial U) \times (X - \partial X)$ is by construction the isomorphism

$$W - \partial W \to (V - \partial V) \times_{(Y - \partial Y)} (X - \partial X).$$

Since $\underline{p} : \underline{U} \to \underline{Y}$ is an étale covering, and $V - \partial V$ is dense in $\underline{Y}$, we have that

$$(V - \partial V) \times_{(Y - \partial Y)} (U - \partial U)$$

is dense in $\underline{V} \times_{\underline{Y}} \underline{U}$. Moreover, $W - \partial W$ is dense in $\underline{W}$ since u is a closed immersion. Note that $\underline{V}$ is quasi-finite over $\underline{Y}$ by Proposition IV.2.7.1(ii) in [GJ66], since $\underline{V} \times_{\underline{Y}} \underline{U}$ is quasi-finite over $\underline{U}$, and $\underline{p} : \underline{U} \to \underline{Y}$ is an étale cover. The natural induced morphism $v : \underline{W} \to \underline{V}$ is an étale morphism, since u is an isomorphism and v is the pullback of p.

Let V^N be the fs log scheme whose underlying scheme is $\underline{V}$, equipped with the induced log structure by Y. By Proposition II.11.3.13(ii) in EGA the underlying product $\underline{V^N} \times_{\underline{V}} \underline{W}$ is normal as v is étale. Thus there is a cartesian square of schemes

$$\begin{array}{ccc} \underline{W^N} & \longrightarrow & \underline{W} \\ \downarrow & & \downarrow \\ \underline{V^N} & \longrightarrow & \underline{V}. \end{array}$$

The log transfer structure gives a morphism $r : W^N \to X$ of fs log schemes over k. By assumption, the two composite arrows in the diagram

$$W^N \times_U (U \times_Y U) \rightrightarrows W^N \xrightarrow{r} X$$

coincide. By Corollary III.1.4.5 in [Ogu18], there is a morphism $V^N \to X$ of fs log schemes over k such that the composition $W^N \to V^N \to X$ is equal to X, as p is strict étale morphism. The pair $(\underline{V}, V^N) \to X$ now gives a finite log correspondence with compact support from X to Y as $\underline{V}$ is quasi-finite over Y, and the pullback of V to U is W. □

Proposition 3.1.11. *For every $X \in lSm/k$ the sheaf $a^*_{dNis}\Lambda^c_{ltr}(X)$ is a dividing Nisnevich sheaf with log transfers.*

Proof. Since it is a strict Nisnevich sheaf by Proposition 3.1.10, the result follows from the fact that the dividing Nisnevich topology is compatible with log transfers (Theorem 4.5.7 in [BPØ20]). □

Dividing logarithmic correspondences with compact support

In general we cannot associate the sheaf $\Lambda^c_{\mathrm{ltr}}(Y)(X)$ with

$$\mathrm{Hom}_{Shv^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)}(a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(Y), a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(X))$$

as it is not a dividing Nisnevich sheaf.

It turns out that we can describe it using algebraic cycles, hence we take what seemingly is a detour in order to describe it by *dividing* logarithmic correspondences with compact support.

Definition 3.1.12. For X and Y in lSm/k, we define an elementary *dividing* log correspondence with compact support Z from X to Y as a closed irreducible subscheme $\underline{Z} \subset \underline{X} \times \underline{Y}$ that is quasi-finite and dominant over a component of $\underline{X}$, together with a log modification $X' \to X$ and a morphism

$$u' : Z'^N \to X' \times Y$$

subject to the following set of conditions:

(i) The image of the composition $\underline{Z'} \overset{u'}{\to} \underline{X'} \times \underline{Y} \to \underline{X} \times \underline{Y}$ is $\underline{Z}$.

(ii) $\underline{Z'^N}$ is the normalization of $\underline{Z} \times_{\underline{X}} X'$.

(iii) The composition $Z'^N \overset{u'}{\to} X' \times Y \to X'$ is strict.

A *dividing log correspondence with compact support* from X to Y is a formal sum of elementary dividing log correspondences with compact support, and let $\mathrm{lCor}^c_{\mathrm{div}}(X, Y)$ be the free abelian group of dividing log correspondences with compact support from X to Y.

Remark 3.1.13. As remarked in Remark 4.6.2 in [BPØ20], due to the fact that a finite log correspondence with compact support $Z \in \Lambda^c_{\mathrm{ltr}}(Y)(X)$ is determined by $Z - \partial Z \in \Lambda^c_{\mathrm{ltr}}(Y - \partial Y)(X - \partial X)$ (Lemma 2.3.2 in [BPØ20]), there is an alternative definition of a finite log correspondence with compact support:

An elementary log correspondence with compact support Z from X to Y is an integral closed subscheme $\underline{Z} \subset \underline{X} \times \underline{Y}$ that is quasi-finite and dominant over X, together with a morphism

$$u : Z^N \to X \times Y$$

subject to the following conditions:

(i) The image of $\underline{Z}^N \to \underline{X} \times \underline{Y}$ is $\underline{Z}$.

(ii) $\underline{Z^N}$ is the normalization of $\underline{Z}$.

(iii) The composition $Z^N \overset{u}{\to} X \times Y \to Y$ is strict.

This makes it possible to consider a dividing log correspondence with compact support $Z \in \mathrm{lCor}^c_{\mathrm{div}}(X, Y)$ as a log correspondence with compact support after replacing X by a log modification of X.

Thus we arrive at the reason for considering dividing log correspondence with compact support.

Proposition 3.1.14. *For every X and Y in lSm/k there is an isomorphism*

$$\mathrm{lCor}^c_{div}(X,Y) \otimes \Lambda \simeq a^*_{dNis}\Lambda^c_{ltr}(Y)(X).$$

Proof. We can associate the group $\mathrm{lCor}^c_{\mathrm{div}}(X,Y)$ with $\Lambda^c_{\mathrm{ltr}}(Y)(X)$ after replacing X by a log modification $Y \to X$. By use of a similar identification as in Lemma 4.4.3 in [BPØ20] we have

$$a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(X) \simeq \operatorname*{colim}_{Y \to X} \Lambda^c_{\mathrm{ltr}}(Y),$$

where $Y \to X$ is a log modification of X. □

Admissible blow-ups

Definition 3.1.15 ([BPØ20, Definition 7.6.1])**.** An *admissible blow-up* is a proper birational morphism

$$X' \to X$$

of fs log schemes log smooth over k such that the induced morphism

$$X' - \partial X' \to X - \partial X$$

is an isomorphism. We let $\mathcal{AB}l/k$ denote the class of admissible blow-ups and $(\mathcal{AB}l/k) \downarrow Y$ the class of admissible blow-ups over an fs log scheme Y.

Assuming resolution of singularities, the class of admissible blow-ups $\mathcal{AB}l/k$ admits a calculus of right fractions (Proposition 7.6.6 in [BPØ20]) in the sense of the dual of Definition I.2.2 in [GZ67]. Thus, according to Proposition 7.6.7 in [BPØ20], log motives are invariant to admissible blow-ups (Proposition 7.6.7 in [BPØ20]), i.e., any morphism $f : Y \to X$ in $(\mathcal{AB}l/k)$ induces an isomorphism

$$M(Y) \xrightarrow{\simeq} M(X).$$

We are interested in admissible blow-ups because of the next proposition. It relates the correspondences between X and $Y - \partial Y$ with correspondences between X and admissible blow-ups of Y. Because $(\mathcal{AB}l/k)$ admits a calculus of right fractions, the class $(\mathcal{AB}l/k) \downarrow Y$ of admissible blow-ups of Y is cofiltered, and we may take the colimit.

Proposition 3.1.16. *Assume that k admits resolution of singularities. Let X be a smooth scheme over k and Y an fs log scheme log smooth over k. Then there is a naturally induced isomorphism:*

$$\operatorname*{colim}_{Y' \in (\mathcal{AB}l/k)\downarrow Y} \Lambda^c_{ltr}(X)(Y') \simeq \Lambda^c_{ltr}(X)(Y - \partial Y).$$

Proof. Following the proof of Proposition 8.2.1 in [BPØ20] we let $Y' \to Y$ be an admissible blow-up. The induced morphism $Y' - \partial Y' \to Y - \partial Y$ allows us to form a homomorphism

$$\varphi_{Y'} : \Lambda^c_{\mathrm{ltr}}(X)(Y') \to \Lambda^c_{\mathrm{ltr}}(X)(Y' - \partial Y') \simeq \Lambda^c_{\mathrm{ltr}}(X)(Y - \partial Y).$$

Gathering the $\varphi(Y')$'s we get a morphism

$$\varphi : \operatorname*{colim}_{Y' \in (\mathcal{AB}l/k)\downarrow Y} \Lambda^c_{\mathrm{ltr}}(X)(Y') \longrightarrow \Lambda^c_{\mathrm{ltr}}(X)(Y - \partial Y).$$

Let $W \in \Lambda^c_{\mathrm{ltr}}(X, Y')$ be a log elementary correspondence with compact support. The map $\varphi_{Y'}$ is injective since $\varphi_{Y'}(W) = 0$, implies that $W = 0$, since it is the closure of $\varphi_{Y'}$ in $Y' \times X$.

For the surjectivity, let $Z \in \Lambda^c_{\mathrm{ltr}}(X)(Y - \partial Y)$. From platification (Theorem 5.7.9 in [RG71] or equivalently Theorem 2.2.2 in [FV00]) there is a scheme $\underline{Y'}$ and a morphism $f : \underline{Y'} \to \underline{Y}$ such that f is an isomorphism on $Y - \partial Y$ and the closure $\underline{Z'}$ of $\underline{Z}$ in $Y' \times X$ is flat over $\underline{Y'}$. The subscheme $\underline{Z}$ is a correspondence with compact support from $\underline{Y'}$ to X.

By resolution of singularities, there is a blow-up $g : \underline{Y''} \to Y'$ such that Y'' is smooth over k, and the complement of $(f \circ g)^{-1}(Y - \partial Y)$ in $\underline{Y''}$ consists of strict normal crossing divisors $Z'_1, \ldots, Z'_r$. Letting Y'' be the log scheme corresponding to $(\underline{Y''}, Z'_1 + \ldots + Z'_r)$, the induced morphism $Y'' \to Y$ is a admissible blow-up. The closure $\underline{Z''}$ of $\underline{Z}$ in $\underline{Y''} \times X$ is a closed subscheme of $W' \times_Y Y'$, which preserved under base change, is quasi-finite. It follows that W can be extended to a correspondence with compact support from $\underline{Y''}$ to X. Since X has trivial log structure, this gives a log correspondence with compact support from Y'' to X. □

The following proposition generalizes Proposition 8.2.2 in [BPØ20], but its proof is very similar.

Proposition 3.1.17. *Assume that k admits resolution of singularities. For every smooth scheme X over k, there is an isomorphism*

$$a^*_{dNis}\Lambda^c_{ltr}(X) \simeq \omega^*\Lambda^c_{tr}(X).$$

Proof. Using Lemma 4.4.3 in [BPØ20] there is an isomorphism

$$a^*_{\mathrm{dNis}} v^* v_\sharp \Lambda^c_{\mathrm{ltr}}(X) \simeq v^* v_\sharp a^*_{\mathrm{dNis}} \Lambda^c_{\mathrm{ltr}}(X).$$

Applying Proposition 3.1.16 we get the isomorphism

$$a^*_{\mathrm{dNis}} v^* v_\sharp \Lambda^c_{\mathrm{ltr}}(X) \simeq a^*_{\mathrm{dNis}} \omega^* \Lambda^c_{\mathrm{tr}}(X),$$

and using Remark 7.6.8 in [BPØ20] we moreover have

$$v^* v_\sharp a^*_{\mathrm{dNis}} \Lambda^c_{\mathrm{ltr}}(X) \simeq a^*_{\mathrm{dNis}} \Lambda^c_{\mathrm{ltr}}(X).$$

We conclude by the fact that $\omega^*\Lambda^c_{\mathrm{tr}}(X)$ is a dividing Nisnevich sheaf, since the functor

$$\eta_\sharp : \mathbf{Shv}(k_{\mathrm{\acute{e}t}}, \Lambda) \to \mathbf{Shv}^{\log}_{l\acute{e}t}(k, \Lambda)$$

is fully faithful (Lemma 8.5.2 in [BPØ20]), where $\eta : k_{\mathrm{\acute{e}t}} \to lSm/k$ is the inclusion functor. □

Lemma 3.1.18. *Assume that k admits resolution of singularities. Let X be a smooth scheme over k. Then there are isomorphisms*

$$\omega^*\Lambda^c_{tr}(X) \simeq C_*\omega^*\Lambda^c_{tr}(X) \simeq \omega^* C^{\mathbb{A}^1}_* \Lambda^c_{tr}(X)$$

in $\boldsymbol{logDM}^{\mathit{eff}}(k, \Lambda)$.

Proof. The first isomorphism is comes from Proposition 6.2.9(4) and Remark 7.6.8 in [BPØ20]. The second isomorphism comes from applying Proposition 8.2.3 in [BPØ20]. □

Lemma 3.1.19. *If $f : X' \to X$ is a log modification in lSm/k and $V \in \Lambda^c_{dNis}(X)(Y)$, there exists a dividing Zariski cover $g' : Y' \to Y$ and $W \in \Lambda^c_{dNis}(X')(Y')$ sitting in a commutative diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{W} & X' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{V} & X. \end{array}$$

Proof. The proof of Lemma 4.5.5 in [BPØ20] works in this setting so we copy its proof: Since the question is Zariski local on Y we can assume that Y has an fs chart P. Letting $P: V^N \times_X X' \to V^N$ be the projection and $q: V^N \to V$ the structure morphism there is a subdivison of fans $M \to \operatorname{Spec} P$ by [BPØ20, A.11.5] such that the projection $V^N \times_{\mathbb{A}_P} \mathbb{A}_M \to V^N$ admits a factorization

$$V^N \times_{\mathbb{A}_P} \mathbb{A}_M \to V^N \times_X X' \to V^N.$$

Letting $Y' := Y \times_{\mathbb{A}^P} \mathbb{A}_M$ and $V' = V \circ u$, where $u: Y' \to Y$ is the projection. The structure morphism $v: V'^N \to X$ factors through $V^N \times_Y Y' \simeq V^N \times_{\mathbb{A}_P} \mathbb{A}_M$, hence it also factors through $V^N \times_X X'$. It follows that it also factors through X'.

Replacing Y by Y' and V by V' we can assume that the structure morphism $V^N \to X$ factors through X'. Thus there is a commutative diagram of fs log schemes

$$\begin{array}{ccccc} V^N & \xrightarrow{w} & Y \times X' & \longrightarrow & X' \\ & & \big\downarrow & & \big\downarrow \\ & & Y \times X & \longrightarrow & X. \end{array}$$

Since X' is proper over X, the morphism w is proper. Let $\underline{W}$ be the image of $\underline{V^N} \to \underline{Y} \times \underline{X'}$, which we consider as the closed subscheme of $\underline{Y} \times \underline{X'}$ with reduced scheme structure. Then $\underline{V^N}$ is the normalization of $\underline{W}$ since $\underline{V^N}$ is the normalization of the image of $\underline{V^N} \to \underline{Y} \times \underline{X}$. Thus w provides a correspondence W from Y to X' with image V in $\Lambda^c_{\mathrm{ltr}}(X)(Y')$. □

The inclusion functor $\gamma: lSm/k \to \mathrm{lCor}/k$ induces a functor

$$\gamma^*: \mathbf{Psh}^{\mathrm{ltr}}(k, \Lambda) \longrightarrow \mathbf{Psh}^{\log}(k, \Lambda)$$

and a functor between the corresponding chain complexes

$$\gamma^*: C_*(\mathbf{Psh}^{\mathrm{ltr}})(k, \Lambda) \longrightarrow C_*(\mathbf{Psh}^{\log}(k, \Lambda)).$$

By abuse of notation we also write

$$\gamma^*: \mathbf{Shv}^{\mathrm{ltr}}(k, \Lambda) \longrightarrow \mathbf{Shv}^{\log}(k, \Lambda)$$

for the restriction of γ^*. Moreover, the sheafication functor a^*_{dNis} and the forgetful functor a_{dNis^*} gives an adjoint functor pair

$$a^*_{\mathrm{dNis}}: \mathbf{Psh}^{\log}(k, \Lambda) \rightleftarrows \mathbf{Shv}^{\log}(k, \Lambda): a_{\mathrm{dNis}^*},$$

and we have that

$$\gamma^* a_{\mathrm{dNis}^*} \simeq a_{\mathrm{dNis}^*} \gamma^*.$$

Proposition 3.1.20. *If $f : Y \to X$ is a log modification we have an induced isomorphism*

$$a^*_{dNis}\gamma^*\Lambda^c_{ltr}(Y) \to a^*_{dNis}\gamma^*\Lambda^c_{ltr}(X) \tag{3.2}$$

of dividing Nisnevich sheaves.

Proof. Arguing similarly to Lemma 4.5.6 in [BPØ20], for any fs log scheme T there is a commutative diagram

$$\begin{array}{ccc} \Lambda^c_{\mathrm{ltr}}(Y)(T) & \longrightarrow & \Lambda^c_{\mathrm{ltr}}(X)(T) \\ \downarrow & & \downarrow \\ \Lambda^c_{\mathrm{ltr}}(Y-\partial Y)(T-\partial T) & \longrightarrow & \Lambda^c_{\mathrm{ltr}}(X-\partial X)(T-\partial T). \end{array}$$

The vertical morphisms are injections by Lemma 3.1.9, and since the lower morphism is an isomorphism, the top arrow is an isomorphism as well. Thus $\Lambda^c_{\mathrm{ltr}}(Y) \to \Lambda^c_{\mathrm{ltr}}(X)$ is a monomorphism, which implies that

$$a^*_{\mathrm{dNis}}\gamma^*\Lambda^c_{\mathrm{ltr}}(Y) \to a^*_{\mathrm{dNis}}\gamma^*\Lambda^c_{\mathrm{ltr}}(X)$$

is a monomorphism since a^*_{dNis} and γ^* are exact.

Due to Lemma 3.1.19 there exists a dividing Nisnevich cover $g : T' \to T$ and a finite log correspondence with compact support $W \in \Lambda^c_{\mathrm{dNis}}(Y)(T')$ such that $f \circ W = g \circ V$ for every $V \in \Lambda^c_{\mathrm{dNis}}(Y)(T)$. This proves that (3.2) is an epimorphism, finishing the proof. □

3.2 Logarithmic motives with compact support

We are now in a position to define the main subject of this thesis: the logarithmic motive with compact support.

Definition 3.2.1. Let X be an fs log scheme log smooth over k. The *logarithmic motive with compact support* of X, denoted $M^c(X)$, is the image of $a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(X)$ in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$. We will often abbreviate it as the *compact log motive* of X. Given a morphism $f : Y \to X$ in lSm/k we define

$$M^c(Y \xrightarrow{f} X)$$

as the cone in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ associated to the complex induced by f

$$a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(Y) \longrightarrow a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(X)$$

in $C_*(\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda))$. We will also denote it by $M^c(Y \to X)$ if the morphism is understood, or sometimes simply $M^c(f)$.

The inclusion $\Lambda_{\mathrm{ltr}}(X) \subset \Lambda^c_{\mathrm{ltr}}(X)$ induces a canonical morphism $M(X) \to M^c(X)$. If X is proper over k, Proposition 3.1.6 implies that

$$M^c(X) = M(X). \tag{3.3}$$

Using the covariance and contravariance of $\Lambda^c_{\mathrm{ltr}}(-)$ from Remark 3.1.3 we see that the compact log motive $M^c(X)$ is contravariant in X for open immersions, and covariant in X for closed immersions.

The following two propositions show that the compact log motive satisfies strict Nisnevich descent and dividing descent.

Proposition 3.2.2. *Suppose that*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

is a strict Nisnevich distinguished square. Then there is an isomorphism

$$M^c(Y \to X) \xrightarrow{\simeq} M^c(Y' \to X')$$

in $\boldsymbol{logDM}^{\mathit{eff}}(k, \Lambda)$*.*

Proof. We have to show that for every strictly $\overline{\square}$-invariant complex of dividing Nisnevich sheaves $\mathcal{F}$ with log transfers there is an isomorphism

$$\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M^c(Y \to X), \mathcal{F}) \simeq \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M^c(Y' \to X'), \mathcal{F}).$$

Ignoring transfers, it suffices to prove that this holds in $\mathbf{logDA}^{\mathrm{eff}}(k, \Lambda)$ (Definition 5.2.1 in [BPØ20]). Arguing similarly as Proposition 5.2.3 in [BPØ20] we may assume that $\mathcal{F}$ is a fibrant object in $C(\mathbf{Shv}^{\log}_{\mathrm{dNis}}(k, \Lambda))$ with regard to the $\overline{\square}$-local descent structure. We then have the equivalence

$$\mathrm{Hom}_{\mathbf{logDA}^{\mathrm{eff}}(k,\Lambda)}(M^c(X), \mathcal{F}) \simeq \mathrm{Hom}_{D(\mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda))}(a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(X), \mathcal{F}).$$

Using the exactness of

$$0 \to a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(Y) \to a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(Y') \oplus a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(X) \to a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(X') \to 0,$$

if follows that there is an equivalence between

$$\mathrm{Hom}_{D(\mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda))}(a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(Y \to X), \mathcal{F})$$

and

$$\mathrm{Hom}_{D(\mathbf{Shv}^{\log}_{\mathrm{dNis}}(k,\Lambda))}(a^*_{\mathrm{dNis}}\Lambda^c_{\mathrm{ltr}}(Y' \to X'), \mathcal{F}).$$

□

Remark 3.2.3. With a formal argument (Proposition C.1.7 in [BPØ20]), this also shows that there are isomorphisms

$$M^c(Y \to Y') \xrightarrow{\simeq} M^c(X \to X').$$

Proposition 3.2.4. *For any log modification* $f : Y \to X$ *of fs log schemes there is an induced isomorphism*

$$M^c(Y) \to M^c(X)$$

in $\boldsymbol{logDM}^{\mathit{eff}}(k, \Lambda)$*.*

Proof. This is a consequence of Proposition 3.1.20. □

Remark 3.2.5. Proposition 3.2.4 can also be proved similarly as Proposition 3.2.2 by using a dividing distinguished square instead of a strict Nisnevich distinguished square.

Properties of logarithmic motives with compact support

In this section we explore properties of logarithmic motives with compact support. We begin by proving a Gysin sequence which we use to prove an analogue of the Künneth formula, that is, establishing an isomorphism

$$M^c(X \times Y) \simeq M^c(X) \otimes M^c(Y)$$

for log smooth fs log schemes X and Y. Assuming resolution on singularities we prove this as Theorem 3.2.12, and as a corollary we prove $\overline{\square}$-invariance and $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariance.

From the lack of a localization sequence for motives of log schemes, we cannot directly generalize the proof of Theorem 1.2.8. Instead our proof will consist of four steps, proving it for

(i) $X, Y \in Sm/k$, and when X, Y are proper fs log schemes log smooth over k,

(ii) $X = (\underline{X}, Z)$, with Z a smooth irreducible divisor on $\underline{X}$ and $Y \in Sm/k$,

(iii) $X \in lSm/k, Y \in Sm/k$,

(iv) $X, Y \in lSm/k$.

Step 1 is straightforward, and **step 3** follows from **step 2** by dividing descent. We emphasize that **step 2** is an important step as it proves $\overline{\square}$-invariance of the compact log motive, i.e.,

$$M^c(X \times \overline{\square}) \simeq M^c(X)$$

which we state as Corollary 3.2.13. Establishing a Gysin sequence in Theorem 3.2.9, **step 2** follows easily, and **Step 4** reduces to **step 3**. The hardest step is establishing Theorem 3.2.9 where we apply a technique called *deformation to the normal cone* originating from the proof of [MV99, Theorem 2.23] and generalized to log schemes in Theorem 7.5.4 of [BPØ20]. In order to introduce this technique we need some preliminary definitions:

Definition 3.2.6 ([BPØ20, Definition 7.4.1])**.** The *deformation of the pair of spaces* (X, Z), where $X, Z \in Sm/k$, is

$$D_Z X := Bl_{Z\times 0}(X \times \overline{\square}) - Bl_Z X.$$

For a pair of spaces (X, Z) and a smooth log smooth scheme $Y = (X, Z_1 + \ldots + Z_n)$, where $Z_1, \ldots, Z_n$ form a strict normal crossing divisor, we consider the normal bundle $p: N_Z X \to X$ and define the normal bundle of Y with respect to Z as

$$N_Z Y := (N_Z X, p^{-1}(Z_1) + \ldots + p^{-1}(Z_n)).$$

Similarly, we consider the blow-up $Bl_Z X \to X$ and define

$$Bl_Z Y := (Bl_Z X, W_1 + \ldots + W_n),$$

where W_i are the strict transform of Z_i in $Bl_Z X$. We then define the blow-up of Y with respect to Z as $(Bl_Z Y, E)$, where E is the exceptional divisor of $Bl_Z X$.

Definition 3.2.7 ([BPØ20, Definition 7.5.1]). For $X \in Sm/k$ and Z a smooth closed subscheme of X we have a commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Z \times \overline{\square} & \longleftarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & D_Z X & \longleftarrow & N_Z X \\ \downarrow & & \downarrow & & \downarrow \\ \{1\} & \xrightarrow{i_1} & \overline{\square} & \xleftarrow{i_0} & \{0\} \end{array}$$

called the *deformation to the normal cone*, where

(i) each square is cartesian,

(ii) i_0 is the zero section and i_1 is the 1-section,

(iii) $Z \to N_Z X$ is the 0-section into the normal bundle.

(iv) $D_Z X \to \overline{\square}$ is the composition

$$D_Z X \to Bl_Z(X \times \overline{\square}) \to X \times \overline{\square} \to \overline{\square},$$

where the third arrow is the projection.

We will also need the definition of a parametrization:

Definition 3.2.8 ([BPØ20, Definition 7.2.6 and Definition 7.2.7]). Let (X, Z) be a smooth pair of schemes with an closed immersion $i : Z \to X$. An morphism $(f, f') : (X, Z) \to (X', Z')$ between smooth pairs is called *cartesian* if the commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \longrightarrow & X' \end{array}$$

is cartesian. A *parametrization* of the smooth pair (X, Z) is a smooth pair $(\mathbb{A}^{r+s}, \mathbb{A}^s)$ such that the cartesian morphism $(f, f') : (X, Z) \to (\mathbb{A}^{r+s}, \mathbb{A}^s)$ is étale.

Using deformation to the normal cone, we prove an important logarithmic analogue of the Gysin sequence in our setting (Theorem 3.2.9) to be used in **Step 2** and **Step 4** of Theorem 3.2.12. This result is particularly useful as it relates the compact log motive of a smooth log smooth log scheme with the underlying scheme and the divisor inducing the log structure. A priori, since the underlying scheme and the divisor are actual schemes, it reduces questions about log smooth log schemes to more classical (and hopefully easier) questions about schemes. Its proof uses uses induction on the number of component of the divisor, and the base case of the induction applies the technique of deformation to the normal cone to reduce to the nice case of the affine line with log structure only at the origin. We conclude by applying strict Nisnevich descent and dividing descent. The induction theorem uses a general theorem about triangulated categories by [May01], and concludes by the octahedral axiom.

Theorem 3.2.9. *Let X be a smooth scheme and Z a strict normal crossing divisor on X. Then there exists a distinguished triangle*

$$M^c(X,Z) \to M^c(X) \to M^c(Z)(1)[2] \to M^c(X,Z)[1]$$

in $\boldsymbol{logDM}^{\text{eff}}(k,\Lambda)$.

We note that by Proposition A.3.3 all smooth log smooth log schemes are isomorphic to a log scheme of the form (X,Z) where X is a smooth scheme and Z a strict normal crossing divisor on X. Hence the theorem gives a sequence for every smooth log smooth log scheme.

Proof. We do induction the number of components of $Z = Z_1 + \cdots + Z_r$, starting with $r = 1$. We then have to show that there is a cofiber sequence on the form

$$M^c(X,Z) \to M^c(X) \to M^c(Z)(1)[2], \tag{3.4}$$

where Z is a smooth irreducible divisor on X.

Following Theorem 7.5.4 in [BPØ20], Zariski locally on X we have a cartesian diagram

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle u} \\ \mathbb{A}^s & \hookrightarrow & \mathbb{A}^{s+1} \end{array}$$

where s denotes the dimension of Z (hence $s+1$ the dimension of X) over k and u is an étale morphism. Constructing $X_1 := \mathbb{A}^s_Z$, and defining Δ as the diagonal of Z over $\mathbb{A}^s$, we let

$$X_2 = X \times_{\mathbb{A}^s} X_1 - (X \times_{\mathbb{A}^s} Z - \Delta) \cup (X_1 \times_{\mathbb{A}^s} Z_\Delta).$$

This gives us a diagram

$$(X,Z) \longleftarrow (X_2,Z) \longrightarrow (X_1,Z),$$

and moreover a cartesian square

$$\begin{array}{ccc} D_Z X_2 & \longrightarrow & X_2 \times \overline{\square} \\ \downarrow & & \downarrow \\ D_Z X & \longrightarrow & X \times \overline{\square}, \end{array}$$

since blow-ups commute with flat base change. This shows that $D_Z X_2 \to D_Z X$ is étale, and since $X_2 \to X$ and $N_Z X_2 \to N_Z X$ are also étale, strict Nisnevich descent implies that there are isomorphisms

$$M^c((Bl_Z X) \to X) \xrightarrow{\simeq} M^c((Bl_Z X_2, E_2) \to X_2)$$
$$M^c((Bl_{Z\times\overline{\square}}(D_Z X)) \to D_Z X) \xrightarrow{\simeq} M^c((Bl_{Z\times\overline{\square}}(D_Z X_2), E_2^D) \to D_Z X_2)$$
$$M^c((Bl_Z(N_Z X)) \to N_Z X) \xrightarrow{\simeq} M^c((Bl_Z(N_Z X_2), E_2^N) \to N_Z X_2).$$

Applying deformation to the normal cone (Definition 3.2.7), we can thus replace X by X_2, and similarly we can repeat the argument to replace X_2 by X_1. This reduces the proof to showing the case when $X = \mathbb{A}^1$ and $Z = 0$.

Using the square Equation (2.4) and strict Nisnevich descent (Proposition 3.2.2) we get an isomorphism

$$M^c((Bl_0\mathbb{P}^1, E) \to \mathbb{P}^1) \xrightarrow{\simeq} M^c((Bl_0\,\mathbb{A}^1, E) \to \mathbb{A}^1). \tag{3.5}$$

Invoking Theorem 7.5.4 in [BPØ20] we have the cofiber sequence

$$M^c(Bl_0\mathbb{P}^1, E) \to M^c(\mathbb{P}^1) \to \Lambda(1)[2],$$

hence the cofiber sequence

$$M^c(Bl_0\,\mathbb{A}^1, E) \to M^c(\mathbb{A}^1) \to \Lambda(1)[2].$$

Applying dividing descent (Proposition 3.2.4) to the log modification

$$(Bl_0\,\mathbb{A}^1, E) \to (\mathbb{A}^1, 0)$$

we thus have our desired cofibre sequence

$$M^c(\mathbb{A}^1, 0) \to M^c(\mathbb{A}^1) \to \Lambda(1)[2].$$

Now for the induction step we have a commutative diagram

$$\begin{array}{ccccc}
M^c(X, Z_1 + \cdots + Z_r) & \longrightarrow & M^c(X, Z_r) & & \\
\downarrow & & \downarrow & & \\
M^c(X, Z_1 + \cdots + Z_{r-1}) & \longrightarrow & M^c(X) & \longrightarrow & M^c(Z_1 + \cdots + Z_{r-1})(1)[2] \\
 & & \downarrow & & \downarrow \\
 & & M^c(Z_r)(1)[2] & \rightarrow & M^c(Z_1 \cap \ldots \cap Z_r)(2)[4].
\end{array}$$

Then Lemma 5.7 in [May01] gives a pushpull square

$$\begin{array}{ccccc}
M^c(X) & & \longrightarrow & & M^c(Z_1 + \cdots + Z_{r-1})(1)[2] \\
\downarrow & \overset{c}{\searrow} & & \swarrow\nearrow & \downarrow \\
 & & M^c(Z_1 + \cdots + Z_r)(1)[2] & & \\
\downarrow & \swarrow\nearrow & & \searrow & \downarrow \\
M^c(Z_r)(1)[2] & & \longrightarrow & & M^c(Z_1 \cap \ldots \cap Z_r)(2)[4],
\end{array}$$

in which the octahedral axiom of triangulated categories (Definition 13.3.2 TR4 in [Sta21]) gives us the desired cofiber sequence

$$M^c(X, Z_1 + \cdots + Z_r) \to M^c(X) \xrightarrow{c} M^c(Z_1 + \cdots + Z_r)(1)[2].$$

□

Example 3.2.10. Using Theorem 3.2.9 we see that $M(\mathbb{P}^1, p_1 + \cdots + p_n)$, where the $p_i \simeq \operatorname{Spec} k$ denote different points, can be described by the distinguished triangle

$$M(\mathbb{P}^1, p_1 + \cdots + p_n) \to M(\mathbb{P}^1) \to M(p_1 + \cdots + p_n)(1)[2] \to M(\mathbb{P}^1, p_1 + \cdots + p_n)[1]$$

Using Theorem 3.2.9 we get our first calculations:

Corollary 3.2.11. *We have isomorphisms*

$$M^c(\mathbb{A}^1) \simeq \mathbb{Z}(1)[2], \tag{3.6}$$

and

$$M^c(\mathbb{A}_{\mathbb{N}}) \simeq \mathbb{Z}(1)[1]. \tag{3.7}$$

Proof. Applying Theorem 3.2.9 we find the distinguished triangles

$$M^c(\overline{\square}) \to M^c(\mathbb{P}^1) \to M^c(\operatorname{Spec} k)(1)[2] \to M^c(\overline{\square})[1]$$

and

$$M^c(\mathbb{A}_{\mathbb{N}}) \to M^c(\mathbb{A}^1) \to M^c(\operatorname{Spec} k)(1)[2] \to M^c(\overline{\square})[1].$$

Using the strict Nisnevich distinguished square from Example 2.2.4 and Proposition 3.2.4 on the morphisms $M^c(Bl_0\,\mathbb{A}^1, E) \to M^c(\mathbb{A}_{\mathbb{N}})$ and $M^c(Bl_0\mathbb{P}^1, E) \simeq M^c(\mathbb{P}^1, 0) \simeq \overline{\square}$, there is a naturally induced homotopy cartesian square

$$\begin{array}{ccc} M^c(\overline{\square}) & \longrightarrow & M^c(\mathbb{A}_{\mathbb{N}}) \\ \downarrow & & \downarrow \\ M^c(\mathbb{P}^1) & \longrightarrow & M^c(\mathbb{A}^1). \end{array}$$

where the vertical morphisms agree with those appearing in the above triangles. Hence we get a commutative square

$$\begin{array}{ccc} M^c(\operatorname{Spec} k)(1)[1] & \longrightarrow & M^c(\operatorname{Spec} k)(1)[1] \\ \downarrow & & \downarrow \\ M^c(\overline{\square}) & \longrightarrow & M^c(\mathbb{A}_{\mathbb{N}}) \\ \downarrow & & \downarrow \\ M^c(\mathbb{P}^1) & \longrightarrow & M^c(\mathbb{A}^1) \\ \downarrow & & \downarrow \\ M^c(\operatorname{Spec} k)(1)[2] & \longrightarrow & M^c(\operatorname{Spec} k)(1)[2], \end{array}$$

or equivalently

$$\begin{array}{ccc} \Lambda(1)[1] & \longrightarrow & \Lambda(1)[1] \\ \downarrow & & \downarrow \\ \Lambda & \longrightarrow & M^c(\mathbb{A}_{\mathbb{N}}) \\ \downarrow & & \downarrow \\ \Lambda \oplus \Lambda(1)[2] & \longrightarrow & M^c(\mathbb{A}^1) \\ \downarrow & & \downarrow \\ \Lambda(1)[2] & \longrightarrow & \Lambda(1)[2]. \end{array}$$

Using Proposition 3.2.2, the top and bottom horizontal maps are isomorphisms, which gives the identifications

$$M^c(\mathbb{A}_{\mathbb{N}}) \to \mathbb{Z}(1)[1],$$

and

$$M^c(\mathbb{A}^1) \to \mathbb{Z}(1)[2].$$

□

We are now in position to establish the one of the main results of this section.

Theorem 3.2.12 (Künneth formula)**.** *Assume that k admits resolution of singularities. Let X and Y be log smooth fs log schemes over k. Then there is an isomorphism*

$$M^c(X \times Y) \simeq M^c(X) \otimes M^c(Y). \tag{3.8}$$

Proof. **Step 1**, ($X, Y \in Sm/k$ and $X, Y \in lSm/k$ proper):
The case when X and Y have trivial log structure follows from the ordinary case ([MVW11, Corollary 16.16]), and if they are proper this follows from the monoidal structure in $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$ since we then have $M^c(X) \simeq M(X)$ and $M^c(Y) \simeq M(Y)$ by (3.3).

Step 2 ($X = (\underline{X}, Z) \in SmlSm/k, Y \in Sm/k$):
Using Theorem 3.2.9 we have cofiber sequences

$$M^c(\underline{X}, Z) \longrightarrow M^c(\underline{X}) \longrightarrow M^c(Z)(1)[2], \tag{3.9}$$

and

$$M^c(\underline{X} \times Y, Z \times Y) \longrightarrow M^c(\underline{X} \times Y) \longrightarrow M^c(Z \times Y)(1)[2]. \tag{3.10}$$

Tensoring (3.9) with $M^c(Y)$ we get a commutative diagram

$$\begin{array}{ccc} M^c(\underline{X}, Z) \otimes M^c(Y) & \longrightarrow & M^c(\underline{X} \times Y, Z \times Y) \\ \downarrow & & \downarrow \\ M^c(\underline{X}) \otimes M^c(Y) & \xrightarrow{\simeq} & M^c(\underline{X} \times Y) \\ \downarrow & & \downarrow \\ M^c(Z)(1)[2] \otimes M^c(Y) & \xrightarrow{\simeq} & M^c(Z \times Y)(1)[2]. \end{array}$$

The two last morphisms are isomorphisms by **Step 1**, hence from the fact that (3.9) and (3.10) are cofiber sequences, we conclude that the top morphism

$$M^c(\underline{X}, Z) \otimes M^c(Y) \longrightarrow M^c(\underline{X} \times Y, Z \times Y)$$

is an isomorphism as well.

Step 3, ($X \in lSm/k, Y \in Sm/k$):
When X is a log smooth fs log schemes we apply toric deformation (Proposition A.3.4) to find log modifications $X' \to X$ where $X' \in SmlSm/k$, and use

dividing descent (Proposition 3.2.4) to reduce to the case of $X \in SmlSm/k$. We then conclude by applying **Step 2**.

By symmetry, i.e., $M^c(X) \otimes M^c(Y) \simeq M^c(Y) \otimes M^c(X)$, we may assume that the above steps apply to Y as well when $X \in Sm/k$. Thus it only remains to show the general case of both X and Y are log smooth fs log schemes:

Step 4, $(X, Y \in lSm/k)$:
For the general case we again apply toric deformation (Proposition A.3.4) to find log modifications $X' \to X$ and $Y' \to Y$ where $X', Y' \in SmlSm/k$ to reduce to the case of X and Y in $SmlSm/k$. By Proposition A.3.3 we may assume that $X = (\underline{X}, Z)$ and $Y = (\underline{Y}, W)$ where Z and W are strict normal crossing divisors on $\underline{X}$ and $\underline{Y}$ respectfully. Then applying Theorem 3.2.9 we find two vertical cofiber sequences sitting in a commutative diagram

$$\begin{array}{ccc}
M^c(\underline{X}, Z) \otimes M^c(\underline{Y}, W) & \longrightarrow & M^c(\underline{X} \times \underline{Y}, \underline{X} \times W + \underline{Y} \times Z) \\
\downarrow & & \downarrow \\
M^c(\underline{X}, Z) \otimes M^c(\underline{Y}) & \longrightarrow & M^c(\underline{X} \times \underline{Y}, Z \times \underline{Y}) \\
\downarrow & & \downarrow \\
M^c(\underline{X}, Z) \otimes M^c(W)[1](2) & \longrightarrow & M^c(\underline{X} \times W, Z \ \times W)[1](2).
\end{array}$$

The two last morphisms are isomorphisms by **Step 3**, hence we conclude that the top arrow is an isomorphism as well. ☐

Note that **step 2** of in the proof of the Künneth formula shows $\overline{\square}$-invariance of the compact log motive. This is unlike the case in $\mathbf{DM}^{\text{eff}}(k, \Lambda)$ where the compact motive is not $\mathbb{A}^1$-invariant, since

$$M^c(\mathbb{A}^1) \ \simeq \Lambda(1)[2] \not\simeq M^c(\operatorname{Spec} k) = \Lambda$$

in $\mathbf{DM}^{\text{eff}}(k, \Lambda)$. We state this result as a corollary, along with some simple consequences.

Corollary 3.2.13 (Homotopy invariance)**.** *For every $X \in lSm/k$ there are isomorphisms*

$$M^c(X \times \overline{\square}) \simeq M^c(X),$$

$$M^c(X \times (\mathbb{P}^n, \mathbb{P}^{n-1})) \simeq M^c(X),$$

$$M^c(X \times \mathbb{P}^n) \simeq M^c(X) \otimes \bigoplus_{i=0}^{n} \Lambda(i)[2i],$$

$$M^c(\mathbb{A}^n) \simeq \Lambda(n)[2n],$$

and

$$M^c((\mathbb{A}_{\mathbb{N}})^n) \simeq \Lambda(n)[n].$$

Proof. By Theorem 3.2.12 we have the equivalences

$$M^c(X \times \overline{\square}) \simeq M^c(X) \otimes M^c(\overline{\square}),$$

$$M^c(X \times (\mathbb{P}^n, \mathbb{P}^{n-1})) \simeq M^c(X) \otimes M^c((\mathbb{P}^n, \mathbb{P}^{n-1})),$$

and

$$M^c(X \times \mathbb{P}^n) \simeq M^c(X) \otimes M^c(\mathbb{P}^n).$$

Since $\overline{\square}$, $(\mathbb{P}^n, \mathbb{P}^{n-1})$, and $\mathbb{P}^n$ are proper, we have

$$M^c(\overline{\square}) \simeq M(\overline{\square}),$$

$$M^c((\mathbb{P}^n, \mathbb{P}^{n-1})) \simeq M((\mathbb{P}^n, \mathbb{P}^{n-1})),$$

and

$$M^c(\mathbb{P}^n) \simeq M(\mathbb{P}^n)$$

by Proposition 3.1.6. We conclude by applying homotopy invariance of log motives, and the fact that $M(\mathbb{P}^n) \simeq \bigoplus_{i=0}^n \Lambda(i)[2i]$ from Proposition 8.3.4 [BPØ20].

The last two isomorphisms come from applying using Corollary 3.2.11 and Theorem 3.2.12 repeatably n-times. $\square$

Relations with $\mathbf{DM}^{\text{eff}}(k, \Lambda)$

Now that we have defined the logarithmic motives with compact support it is interesting to see how the theory relates to the classical theory. An answer is provided by the next theorem, which is a generalization of Proposition 8.2.6 in [BPØ20]. It enables us to relate a larger class of morphisms in $\mathbf{logDM}^{\text{eff}}(k, \Lambda)$ with those in $\mathbf{DM}^{\text{eff}}(k, \Lambda)$. As a consequence, we give an affirmative answer to a question raised in Remark 8.27 in [BPØ20]; which is that

$$\mathrm{Hom}_{\mathbf{logDM}^{\text{eff}}(k,\Lambda)}(M(Y)[i], M(X)) \simeq \mathrm{Hom}_{\mathbf{DM}^{\text{eff}}(k,\Lambda)}(M(Y - \partial Y)[i], M(X))$$

does not in general hold for smooth non-proper schemes X (Remark 3.2.15). We then prove an analogue of the classical duality theorem which we use to prove a cancellation theorem for log schemes.

We begin with a generalization of Theorem 8.2.6 in [BPØ20]. We note that this result, and the following corollary, is independent of the results in the previous section.

Theorem 3.2.14. *Assume that k admits resolution of singularities. Let X be a smooth scheme over k and Y an fs log scheme log smooth over k. Then for every integer $i \in \mathbb{Z}$ there is an isomorphism*

$$\mathrm{Hom}_{\boldsymbol{logDM}^{\text{eff}}(k,\Lambda)}(M(Y)[i], M^c(X)) \simeq \mathrm{Hom}_{\boldsymbol{DM}^{\text{eff}}(k,\Lambda)}(M(Y - \partial Y)[i], M^c(X)).$$

Proof. Lemma 3.1.18 gives an equivalence between

$$\mathrm{Hom}_{\mathbf{logDM}^{\text{eff}}(k,\Lambda)}(M(Y)[i], \omega^* \Lambda^c_{\text{tr}}(X))$$

and

$$\mathrm{Hom}_{\mathbf{logDM}^{\text{eff}}(k,\Lambda)}(M(Y)[i], \omega^* C_*^{\mathbb{A}^1} \Lambda^c_{\text{tr}}(X)).$$

The complex $C_*^{\mathbb{A}^1} \Lambda^c_{\text{tr}}(X)$ is strictly $\mathbb{A}^1$-local by Corollary 14.9 in [MVW11], hence it is strictly $\overline{\square}$-invariant in the dividing Nisnevich topology. Applying Proposition 5.2.3 in [BPØ20], there is an isomorphism

$$\mathrm{Hom}_{\mathbf{logDM}^{\text{eff}}(k,\Lambda)}(M(Y), \omega^* C_*^{\mathbb{A}^1} \Lambda^c_{\text{tr}}(X)[i]) \simeq \mathbf{H}^i_{\text{dNis}}(Y, \omega^* C_*^{\mathbb{A}^1} \Lambda^c_{\text{tr}}(X)[i]).$$

Applying Proposition 8.1.12 [BPØ20] we find

$$\mathbf{H}^i_{\mathrm{dNis}}(Y, \omega^* C_*^{\mathbb{A}^1} \Lambda^c_{\mathrm{tr}}(X)) \simeq \mathbf{H}^i_{\mathrm{Nis}}(Y - \partial Y, C_*^{\mathbb{A}^1} \Lambda^c_{\mathrm{tr}}(X)),$$

which by Proposition 14.16 [MVW11] gives an isomorphism

$$\mathbf{H}^i_{\mathrm{Nis}}(Y - \partial Y, C_*^{\mathbb{A}^1} \Lambda^c_{\mathrm{tr}}(X)) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M(Y - \partial Y), M^c(X)[i]).$$

Combining the equivalence above, and using that $\omega^* \Lambda^c_{\mathrm{tr}}(X) \simeq a^*_{\mathrm{dNis}} \Lambda^c_{\mathrm{ltr}}(X)$ from Proposition 3.1.17 completes the proof. □

Remark 3.2.15. Since we for general (non-proper) log schemes do not have $M^c(X) \simeq M(X)$, this provides an affirmative answer to the claim in [BPØ20, Remark 8.2.7] which states that the equivalence

$$\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M(Y)[i], M(X)) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M(Y - \partial Y)[i], M(X)).$$

does not hold in general for non-proper smooth schemes X.

An example is provided by taking $X = \mathbb{A}^1$. Although $M(\mathbb{A}^1)$ is unknown, we know from the fact that the category of reciprocity sheaves ([Kah+16]) embeds fully faithfully in $\mathbf{logDM}^{\mathrm{eff}}(k, \Lambda)$ ([Sai21]), that the relation between the Witt-vectors and $M(\mathbb{A}^1)$ forces $M(\mathbb{A}^1)$ to be very big. On the other hand, our calculation in Corollary 3.2.11 shows that $M^c(\mathbb{A}^1)$ is merely $\Lambda(1)[2]$.

Theorem 3.2.14 gives us a generalization of Corollary 8.2.8 in [BPØ20].

Corollary 3.2.16. *Assume that k admits resolution of singularities, and let X be a smooth scheme over k. Then the unit of the adjunction $id \to R\omega^* \omega_\sharp$ induces an isomorphism*

$$M^c(X) \simeq R\omega^* \omega_\sharp M^c(X).$$

Proof. It suffices to show that for every generator $M(Y)[i]$ for $Y \in lSm/k$ and $i \in \mathbb{Z}$ that there is an isomorphism between

$$\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M(Y)[i], M^c(X))$$

and

$$\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M(Y)[i], R\omega^* M^c(X)).$$

Using the isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M(Y)[i], R\omega^* M^c(X)) \\ \simeq\; &\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(\omega_\sharp M(Y)[i], M^c(X)) \\ \simeq\; &\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M(Y - \partial Y)[i], M^c(X)), \end{aligned}$$

the result now follows from Theorem 3.2.14. □

We then prove an analogue of the duality theorem Theorem 1.2.9.

Theorem 3.2.17. *Assume that k admits resolution of singularities. If $T \in lSm/k$ is of pure dimension d over k, $X \in Sm/k$, and $Y \in lSm/k$, then there are isomorphisms*

$$\mathrm{Hom}(M(Y \times T)[n], M^c(X)) \simeq \mathrm{Hom}(M(Y)(d)[2d + n], M^c(X \times (T - \partial T)))$$

in $\boldsymbol{logDM}^{\mathit{eff}}(k, \Lambda)$ for every $n \in \mathbb{Z}$.

Proof. By applying Theorem 3.2.14 there are isomorphisms between

$$\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M(Y \times T)[n], M^c(X))$$

and

$$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M((Y-\partial Y) \times (T \times \partial T))[n], M^c(X))$$

for every n. Using the classical duality statement (Theorem 1.2.9) we further have isomorphisms between

$$\mathrm{Hom}(M((Y-\partial Y) \times (T \times \partial T))[n], M^c(X))$$

and

$$\mathrm{Hom}(M(Y-\partial Y)(d)[2d+n], M^c(X \times (T-\partial T))).$$

Reapplying Theorem 3.2.14 we find isomorphisms between

$$\mathrm{Hom}(M(Y-\partial Y)(d)[2d+n], M^c(X \times (T-\partial T)))$$

and

$$\mathrm{Hom}(M(Y)(d)[2d+n], M^c(X \times (T-\partial T))).$$

Combining the isomorphisms finishes the proof. □

Having established Theorem 3.2.17 we have all we need to prove an analogue of the cancellation theorem (Theorem 1.2.10) for log schemes. It is interesting to see how many of our established results come together in the short proof of this result.

Theorem 3.2.18. *Assume that k admits resolution of singularities. Let M and N be two objects of $\boldsymbol{logDM}^{\mathit{eff}}(k,\Lambda)$. Then tensoring with $\Lambda(1)$ induces an isomorphism*

$$\mathrm{Hom}_{\boldsymbol{logDM}^{\mathit{eff}}(k,\Lambda)}(M,N) \simeq \mathrm{Hom}_{\boldsymbol{logDM}^{\mathit{eff}}(k,\Lambda)}(M(1),N(1)).$$

Proof. Let $X \in Sm/k$ be a proper scheme and $Y \in lSm/k$. We then have isomorphisms

$$\begin{aligned}
\mathrm{Hom}(M(Y)[n], M(X)) &\overset{(3.3)}{\simeq} \mathrm{Hom}(M(Y)[n], M^c(X)) \\
&\overset{3.2.13}{\simeq} \mathrm{Hom}(M(Y \times \overline{\square})[n], M^c(X)) \\
&\overset{3.2.17}{\simeq} \mathrm{Hom}(M(Y)(1)[2+n], M^c(X \times \mathbb{A}^1)) \\
&\overset{3.2.11}{\simeq} \mathrm{Hom}(M(Y)(1)[2], M^c(X)(1)[2]) \\
&\overset{(3.3)}{\simeq} \mathrm{Hom}(M(Y)(1)[2], M(X)(1)[2]).
\end{aligned}$$

where all Hom-groups are taken in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$.

Consider then the smooth log smooth fs log scheme $(\underline{X}, Z)$ whose underlying scheme $\underline{X}$ is proper. Then Theorem 3.2.9 gives a cofiber sequence

$$M(\underline{X}, Z) \to M(\underline{X}) \to M(Z)(1)[2] \tag{3.11}$$

that gives rise to the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(M(Y), M(\underline{X}, Z)) & \longrightarrow & \mathrm{Hom}(M(Y)(1), M(\underline{X}, Z)(1)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(M(Y), M(\underline{X})) & \longrightarrow & \mathrm{Hom}(M(Y)(1), M(\underline{X})(1)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(M(Y), M(Z)(1)[2]) & \longrightarrow & \mathrm{Hom}(M(Y)(1), M(Z)(2)[2]). \end{array}$$

Since (3.11) is a cofiber sequence, and the two bottom horizontal morphisms are isomorphisms, the top horizontal morphism is an isomorphism as well. Since these objects generate $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$, Proposition 5.2.5 in [BPØ20] allows us to conclude the statement for all objects of $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$.

Investing the explicit isomorphisms we used, we see similarly to the classical case, that this isomorphism is induced by tensoring with $\Lambda(1)$. □

Let $\mathbf{logDM}(k,\Lambda)$ denote the category obtained from $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ by inverting the Tate twist operation

$$M \mapsto M(1) \simeq M \otimes \Lambda(1).$$

Then Theorem 3.2.12 allow us to embed the category of *effective* logarithmic motives into the category of logarithmic motives.

Corollary 3.2.19. *Assume that k admits resolution of singularities. Then the localization functor*

$$\boldsymbol{logDM}^{\mathit{eff}}(k,\Lambda) \to \boldsymbol{logDM}(k,\Lambda)$$

is fully faithful.

3.3 Logarithmic motivic homotopy theory with compact support

Having the compact log motive at our disposal, and having established some of its properties, we are in position to introduce a new homology and cohomology theory for logarithmic schemes.

Definition 3.3.1. Let X be a log smooth fs log scheme and $i \geq 0$. Similarly to the classical case, we define *logarithmic motivic cohomology with compact support* with coefficients in Λ as

$$H^{n,i}_{lc}(X,\Lambda) = \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(M^c(X), \Lambda(i)[n]), \tag{3.12}$$

and (Borel–Moore) *logarithmic motivic homology with compact support*

$$H^{\mathrm{lBM}}_{n,i}(X,\Lambda) = \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(\Lambda(i)[n], M^c(X)). \tag{3.13}$$

Remark 3.3.2. This is a direct generalization of the *motivic cohomology with compact support* and *(Borel–Moore) motivic homology with compact support* introduced in [FV00].

It follows from Theorem 3.2.9 that motivic cohomology with compact support and logarithmic motivic homology with compact support satisfy a Gysin sequence, a Künneth formula by Theorem 3.2.12, a Mayer-Vietoris sequence by Proposition 3.2.2, are invariant under log modifications by Proposition A.3.5, and are homotopy invariant by Corollary 3.2.13.

We once again emphasize that if X is proper there are canonical isomorphisms

$$H^{n,i}_{lc}(X,\Lambda) \simeq H^{n,i}(X,\Lambda)$$

and

$$H^{\mathrm{lBM}}_{n,i}(X,\Lambda) = H_{n,i}(X,\Lambda)$$

in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$, where $H^{n,i}(X,\Lambda)$ and $H_{n,i}(X,\Lambda)$ are logarithmic motivic cohomology and logarithmic motivic homology of X with coefficients in $\mathbb{Z}$ (as defined in Definition 2.4.6).

Next, we define the logarithmic analogue of Borel–Moore fundamental classes.

Construction 3.3.3. (Assuming resolution of singularities) Let X be an integral smooth log smooth fs log scheme over k and $\nu : Z \to X$ a strict closed immersion of fs log schemes such that $\underline{Z}$ has pure codimension c in $\underline{X}$. Then by assuming resolution of singularities, Proposition 1.2.13 gives us a composition of maps

$$cl(\underline{Z}) \ : \ \Lambda(c)[2c] \xrightarrow{p^*} M^c(\underline{Z}) \xrightarrow{\nu_*} M^c(\underline{X}),$$

where the first map is the map induced by the structure morphism $p : \underline{Z} \to \operatorname{Spec} k$ and the second map induced by the closed embedding ν. This composition of maps represents an element in

$$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(\Lambda(c)[2c], M^c(\underline{X})),$$

i.e., a Borel–Moore motivic class in $H^{\mathrm{BM}}_{2c,c}(\underline{X},\Lambda)$. Using the isomorphism in Theorem 3.2.14 we can associate this with an element of

$$\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(\Lambda(c)[2c], M^c(\underline{X})).$$

Since $X \in SmlSm/k$, we have a canonical flat morphism $p_X : X \to \underline{X}$, where $\underline{X}$ denotes the underlying scheme of X with trivial log structure, and hence a map

$$\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(\Lambda(c)[2c], M^c(\underline{X})) \xrightarrow{p_X^*} \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(\Lambda(c)[2c], M^c(X)).$$

The *fundamental class* of Z in X is

$$p_X^*(cl(\underline{Z})).$$

As a special case, the fundamental class of $\operatorname{Spec} k$ in X we simply call the *fundamental class* of X.

In view of Definition 3.3.1 we see that a fundamental class of Y in X represents a class in $H^{\mathrm{lBM}}_{2c,c}(X,\Lambda)$ which we call *the Borel–Moore homology class* of Y in $H^{\mathrm{lBM}}_{2c,c}(X,\Lambda)$.

Example 3.3.4. An example of a fundamental class is that of $\mathbb{A}^n$ which gives the isomorphism

$$\Lambda(n)[2n] \xrightarrow{\simeq} M^c(\mathbb{A}^n)$$

in $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ induced from the isomorphism in $\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)$ [Nis89, p. 720].

Higher logarithmic Chow groups

We now define a generalization of the classical higher Chow groups to the logarithmic setting. We begin by recalling the definition of Bloch's higher Chow groups first defined in [Blo86] by S. Bloch.

Definition 3.3.5. Let X be an equidimensional scheme, and let $z^i(X,m)$ denote the free abelian group of cycles $Z \subset X \times \Delta^m$ of codimension i that intersects all faces of $X \times \Delta^j$ properly for all $j < m$. There are morphisms $z^i(X,m) \to z^i(X,j)$ induced by intersection of cycles which gives a simplicial group $m \mapsto z^i(X,m)$ denoted $z^i(X,\bullet)$. The resulting chain complex associated to $z^i(X,\bullet)$ we denote by $z^i(X,*)$, and we define the *higher Chow groups* as the homology of this chain complex, i.e.,

$$CH_i(X,m) = H_m(z^i(X,*)).$$

If $m = 0$, this is the ordinary Chow groups of codimension i-cycles of X.

Part of the usefulness of Borel–Moore motivic homology is the identification with higher Chow groups.

Proposition 3.3.6 ([MVW11, Proposition 19.18])**.** *Assume that k admits resolution of singularities and let X be an equidimensional scheme over k of dimensional d. Then for every $i \leq d$, and for all n, there is a canonical isomorphism*

$$CH_{d-i}(X,n) \simeq H^{BM}_{2i+n,i}(X,\mathbb{Z}).$$

We may now define Bloch's higher Chow groups in the logarithmic setting, for which Theorem 3.2.14 provide the natural definition:

Definition 3.3.7. For $X \in lSch/k$, and every m, we define the *higher logarithmic Chow groups* of X as

$$lCH_i(X,m) := CH_i(X - \partial X, m),$$

where $CH_i(-,m)$ is Bloch's higher Chow groups as defined in Definition 3.3.5.

When X is log smooth, the scheme $X - \partial X$ is smooth over k, and with $j = d - i$, we have

$$\begin{aligned} lCH_j(X,n) &:= CH_j(X - \partial X, n) \\ &\simeq H^{2j-n,j}(X - \partial X, \mathbb{Z}) \\ &\simeq H^{\mathrm{BM}}_{2i+n,i}(X - \partial X, \mathbb{Z}) \\ &= \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff},-}(k,\Lambda)}(\mathbb{Z}(i)[2i+n], M^c(X - \partial X)) \\ &\simeq \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff},-}(k,\Lambda)}(\mathbb{Z}(i)[2i+n], M^c(X - \partial X)) \\ &= H^{\mathrm{lBM}}_{2i+n,i}(X - \partial X, \mathbb{Z}), \end{aligned}$$

hence using Theorem 3.2.14 we recover a version of Proposition 3.3.6 in the logarithmic setting.

Of particular importance is the $\overline{\square}$-invariance of the logarithmic Chow groups. Indeed, we have

$$lCH_i(X \times \overline{\square}, m) = CH_i(X - \partial X) \times \mathbb{A}^1, m),$$

from which the $\overline{\square}$-invariance of the higher logarithmic Chow groups follows from the $\mathbb{A}^1$-invariance of the ordinary higher Chow groups [Blo86, Theorem 2.1]. Furthermore, applying our knowledge of $\Lambda^c_{\mathrm{ltr}}(-)$ we see that the logarithmic Chow groups are covariant with respect to closed embeddings, contravariant with respect to open embeddings.

Remark 3.3.8. There has been other attempts at generalizing Chow groups to the logarithmic setting. In [Bar20] (analogous to [HPS17]), Barrot presents a generalization of Chow groups to the logarithmic setting by extending the theory of b-Chow groups, and defines *logarithmic Chow groups* of an fs log scheme X as

$$CH_i(X) := \operatorname*{colim}_{Y \to X} CH_i(Y),$$

where the colimit is taken over locally free log blow-ups $Y \to X$. This theory is covariant with respect to proper maps, contravariant with respect to flat maps, has a Gysin morphism and an excision triangle. However it is unsatisfactory for our purposes, since it is not $\overline{\square}$-invariant.

3.4 Open questions and further developments

The theory developed here has many interesting, yet unexplored, territories and possible consequences: Indeed, much of our intuition for logarithmic motives comes from how it should behave with respect to the classical situation, and much of our work has been to recover generalization of such results. That being said, there are several classical results for which we have yet to find the right analogue:

One of the important applications of the motives with compact support in $\mathbf{DM}^{\mathrm{eff}}$ is that it provides a localization sequence (Theorem 1.2.7), for which we have yet to find a logarithmic analogue. Moreover, considering a localization sequence with $X = \overline{\square}$ and $U = \mathbb{A}^1$ we get a sequence that should involve the compact motive of a log point $\mathrm{pt}_{\mathbb{N}}$. Since the log point is not log smooth, finding a localization sequence of log schemes is limited by the yet to be developed theory of logarithmic motives over the log point. Establishing such a theory is likely to be very interesting in its own right, and relating it to our setting, it is likely to reveal new results and interesting connections to the general theory.

Another classical application of motive with compact support is the duality theorem (Theorem 16.24 in [MVW11]) and the cancellation theorem (Theorem 16.25 in [MVW11]). We proved analogues of these results in Theorem 3.2.17 and Theorem 1.2.10, but we believe there should be a more general analogue of the duality theorem. Such a result would have important consequences in $\mathbf{logDM}^{\mathrm{eff}}$, hence it would be very interesting to look further for log-analogues of that result. There should also be a Gysin sequence for more general subschemes, possibly of the form

$$M^c(X, Z) \to M^c(X) \to M^c(Z)(c)[2c] \to M^c(X, Z)[1],$$

where $Z \to X$ is a log smooth closed equidimensional subscheme of a smooth scheme X of codimension c.

For a flat morphism of schemes $f : X \to Y$ of relative dimension r there exists a morphism

$$f^! : M^c(Y)(r)[2r] \to M^c(X)$$

in $\mathbf{DM}^{\text{eff}}(k,\Lambda)$. It would be interesting to prove this statement for logarithmic motives with compact support, as it is likely that it would give a general duality theorem by following the abstract arguments of [PY08]. As a corollary we would get information about the dualizing object of $M(X)$ in $\mathbf{logDM}^{\text{eff}}(k,\Lambda)$ similarly to Proposition 20.3 in [MVW11], i.e., an isomorphism

$$M^c(X)(r)[-2d] \simeq \underline{RHom}(M(X), \Lambda(d+r))$$

induced by the diagonal $X \to X \times X$ for all $r \geq 0$. As a consequence we would get a natural isomorphism $\iota_M : M \xrightarrow{\simeq} M^{**}$.

Several of our results rely on the assumption that the base field k admits resolution of singularities. However, we believe that it might be possible to relax this assumption using De Jong's weaker notion which works in any characteristic ([De 96]).

In this thesis we only considered the logarithmic motive for the dividing Nisnevich topology. However, it would be interesting to also study the case for the *log étale*, *dividing étale*, and *Kummer étale* topology as well (see Definition 3.1.5 in [BPØ20]).

In Section 3.2 we investigated some relations to the classical theory of $\mathbf{DM}^{\text{eff}}(k,\Lambda)$ and to the general theory of $\mathbf{logDM}^{\text{eff}}(k,\Lambda)$. However, we believe that there is still much work to be conducted in this regard. One way of approaching this problem would be to further investigate the canonical morphism

$$M(X) \to M^c(X)$$

in $\mathbf{logDM}^{\text{eff}}(k,\Lambda)$. Wildeshaus ([Wil06]) initiated such an attempt in the classical setting by defining the *boundary motive* $\partial M(X)$ as the image of

$$\text{cone}(\Lambda_{\text{tr}}(X) \to \Lambda^c_{\text{tr}}(X))[-1]$$

in such a way that it sits in a distinguished triangle

$$\partial M(X) \to M(X) \to M^c(X) \to \partial M(X)[1]$$

in $\mathbf{DM}^{\text{eff}}(k,\Lambda)$. This approach saw some success in (among others) the papers [Wil13] and [Wil07]. We hope that a similar approach may shed some light on the general theory in the future.

We are optimistic that a better description of the interplay of $\mathbf{DM}^{eff}(k,\Lambda)$ with the theory presented here will illuminate interesting results in the classical setting. Theorem 3.2.14 suggests a first step in this regard, but we believe that there should be stronger analogues of this result.

APPENDIX A

Logarithmic Algebraic Geometry

Logarithmic algebraic geometry, often abbreviated *logarithmic geometry*, or simply *log geometry*, originates from work by Fontaine–Illusie, Deligne–Faltings and K. Kato in the late 1980s ([Ogu18]). The theory was developed to deal with essentially two concepts coming from algebraic geometry: Compactifications and degenerations. Later it was realized that logarithmic geometry has a beautiful theory on its own, as a generalization of algebraic geometry from rings to monoids.

In short, log geometry enables schemes with mild ("logarithmic") singularities to behave as if they were smooth by what Kato termed "magic"-stuff. Moreover, log geometry provides a way of describing schemes with boundary in a similar way as we describe manifolds with boundary.

For most of our purposes, a log scheme will be a scheme $(X, \mathcal{O}_X)$ together with some extra structure. This extra structure comes in the form of a morphism of sheaves of commutative monoids $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ that induces an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^\times) \xrightarrow{\simeq} \mathcal{O}_X^\times$. The corresponding log scheme is the triple $(X, \mathcal{O}_X, \alpha_X)$. The extra log structure typically "rememberers" the boundary ∂X of X, for example in the case of a compactification $U \hookrightarrow X$ where $X - \partial X = U$, or contains information about a family to which X is a fiber (in the case of a degeneration). The two cases are often linked, for example in compactifying a moduli space by adding degenerate objects. A specific example is provided by considering the moduli space of elliptic curves: We compactify such a space by adding the node, which has a logarithmic singularity at the origin.

The purpose of this appendix is to provide a self contained reference for logarithmic geometry, so that a reader familiar to algebraic geometry can follow the arguments given above by looking up definitions when necessary.

Most of the following material is taken from the standard textbook reference [Ogu18], otherwise I have tried to refer to the original source.

A.1 Basics on monoids

In order to describe sheaves of monoids we first begin with an introduction to the general theory of monoids.

Definition A.1.1. A (commutative) *monoid* $(M, +_M, e_M)$ is a set M with a commutative associative binary operation $+_M$, equipped with a unit e_M.

A *homomorphism of monoids* $\theta : M \to N$ is a morphism such that $\theta(e_M) = e_N$ and $\theta(m + m') = \theta(m) +_N \theta(m')$ for all elements m and m' in M.

We will often abuse notation and write M for $(M, +_M, e_M)$, $+$ for $+_M$ and 0 for e_M when no confusion is likely to arise.

Example A.1.2. Every group is a monoid.

Example A.1.3. Our prime example of a monoid that is not a group is the natural numbers $(\mathbb{N}, +)$ equipped with addition. If M is a monoid and $m \in M$ there is a unique morphism $\phi_m : \mathbb{N} \to M$ sending 1 to m. Thus $\mathbb{N}$ is the free monoid generated by 1. Note also that $(\mathbb{N}^*, *)$ equipped with multiplication is a monoid.

Example A.1.4. The multiplicative structure $(R^\times, *)$ of a (commutative) ring R is a monoid.

If M is a monoid, we let M^* be the set of inverses of M, i.e., all $m \in M$ such that there exists an element $n \in M$ for which $m + n = 0$, and call it *the group of units* in M. Elements $m \in M^*$ are called *units*. Similarly, we let M^+ denote the set of non-units of M.

Definition A.1.5. A monoid M is called

(i) *sharp* if $M^* = \{0\}$,

(ii) *u-integral* if $m \in M$, $u \in M^*$ and $m + u' \in M$ implies that $u' = 0$,

(iii) *quasi-integral* if $m, m' \in M$ and $m + m' \in M$ implies that $m' = 0$,

(iv) *integral* if $m, m', m'' \in M$ and $m + m' = m + m''$ implies that $m' = m''$.

It follows from the definition that every integral monoid is quasi-integral, and every quasi-integral monoid is u-integral.

If M is a monoid there is a universal morphism λ_M from M to a group M^{gp}. We call M^{gp} the *group completion* of M, and every morphism from M to a group factors uniquely through M^{gp}.

Example A.1.6. The smallest group containing $\mathbb{N}$ is $\mathbb{Z}$. Hence $\mathbb{N}^{\mathrm{gp}} = \mathbb{Z}$.

Definition A.1.7. A monoid M is called *fine* if it is integral and if M^{gp} is finitely generated as an abelian group, and *saturated* if it is integral and if whenever $m \in M^{\mathrm{gp}}$ is such that $mn \in M$ for some $n \in \mathbb{N}$, then we have $m \in M$.

The notion of fine and saturated monoids will be important for us, and we often abbreviates it as an fs monoid if it were to have both properties.

Definition A.1.8. An *ideal* of M is a subset I that is closed under addition from M. That is, if $k \in I$ and $m \in M$ it implies that $k + m \in I$. We call an ideal *prime* if $I \neq M$ and if $p + q \in I$ implies that either $p \in I$ or $q \in I$.

A *face* of a monoid M is a submonoid F such that $p + q \in F$ implies that both $p \in F$ and $q \in F$.

For a face F of a monoid M we let

$$M_F := \{(a, b) \in M \times F \mid (a, b) \sim (c, d) \text{ if } \exists p \in F \text{s.t.} p + a + b = p + b + c\}.$$

For an element $f \in M$ we let $\langle f \rangle$ denote the smallest face containing f, and we define the localization

$$M_f := M_{\langle f \rangle}.$$

A.2 Logarithmic schemes

In many cases the definitions appearing here are direct generalization of the definitions coming from algebraic geometry. However care needs to be taken in order to ensure that the logarithmic structure behaves nicely.

Definition A.2.1. A *sheaf of monoids* on a topological space X is a contravariant functor from the small site on X ($X_{\text{ét}}, X_{\text{Zar}}$, etc.) to the category of monoids.

A *monoidal space* is a pair $(X, \mathcal{M}_X)$ where X is a topological space and $\mathcal{M}_X$ is a sheaf of monoids on X.

A *morphism of monoidal spaces*

$$(f, f^\flat) : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$$

is a pair $(f, f^\flat)$, where $f : X \to Y$ is a continuous map,

$$f^\flat : f^*(\mathcal{M}_Y) \to \mathcal{M}_X$$

is a homomorphism of sheaves of monoids such that for every point $x \in X$ the induced map on the stalks $f^\flat_x : \mathcal{M}_{Y,f(x)} \to \mathcal{M}_{X,x}$ is a homomorphism of monoids satisfying $(f^\flat_x)^{-1}(\mathcal{M}^+_{X,x}) = \mathcal{M}^+_{Y,f(x)}$.

Definition A.2.2. Let Q be a monoid. We define the log scheme $\operatorname{Spec} Q$ as follows: The underlying space consists of faces of Q, and for an element $f \in Q$ we define

$$D(f) := \{F \in \operatorname{Spec} Q : f \notin F\}$$

as an open set. We then define the *Zariski topology* on $\operatorname{Spec} Q$ as the topology generated by all open sets, i.e.,

$$\mathcal{B} := \{D(f) : f \in Q\}.$$

The assignment

$$D(f) \mapsto Q_f$$

defines a presheaf of monoids, and its sheafication $\mathcal{M}_Q$ defines a sheaf on $\operatorname{Spec} Q$. This realizes Q as a monoidal space.

A locally monoidal space is *affine* if it is isomorphic to $\operatorname{Spec} Q$ for some monoid Q. A *monoscheme* is a locally monoidal space that admits an open cover of affine monoidal spaces.

Let $(X, \mathcal{O}_X)$ be a scheme and let $\mathbf{Mon}_X$ be the category of monoidal sheaves on $X_{\text{ét}}$ or X_{Zar}.

Definition A.2.3. A *prelogarithmic structure* on X is a homomorphism of sheaves of monoids $\alpha_X : \mathcal{P} \to \mathcal{O}_X$ on $X_{\text{ét}}$ (or X_{Zar}). A *logarithmic structure* on X is a prelogarithmic structure on X such that the induced homomorphism of monoids $\alpha_X^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism.

We often abbreviate logarithmic as log in order to shorten the terminology.

Definition A.2.4. A *log scheme* X is a scheme $(\underline{X}, \mathcal{O}_X)$ equipped with a logarithmic structure $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$. The scheme $\underline{X}$ is called the *underlying scheme* of X. The map α_X is sometimes denoted exp for *exponential* and the inverse $\mathcal{O}_X \to \mathcal{M}_X$ as log for *logarithmic*.

A *morphism of log schemes* $f = (\underline{f}, f^\flat)$ from X to Y is a morphism $\underline{f} : \underline{X} \to \underline{Y}$ of the underlying schemes together with a morphism of sheaves of monoids $f^\flat : \mathcal{M}_Y \to f_* \mathcal{M}_X$ such that the diagram

$$\begin{array}{ccc} \mathcal{M}_Y & \xrightarrow{f^\flat} & f_* \mathcal{M}_X \\ \downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle f_*(\alpha_X)} \\ \mathcal{O}_Y & \xrightarrow{f^\#} & f_* \mathcal{O}_X \end{array}$$

commutes.

Definition A.2.5. A morphism of log schemes $f : X \to Y$ is called *strict* if the induced homomorphism $f^*_{\log} \mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

Definition A.2.6. Let X be a log scheme and Q a monoid. A strict morphism $\alpha : Q \to \Gamma(X, \mathcal{M}_X)$ is called a *chart* of $\mathcal{M}_X$.

To compose finite log correspondences $V \in \mathrm{lCor}(X, Y)$ and $W \in \mathrm{lCor}(Y, Z)$ we need to construct log cycles on the normalization of $\underline{X} \times \underline{Z}$. This can be by considering cycles on $\underline{X} \times \underline{Y} \times \underline{Z}$ if X, Y and Z are solid (See Section 2.2 and 2.3 in [BPØ20]). We therefore provide the definition below.

Definition A.2.7. A coherent log scheme X is called *solid* if for any point $x \in X$ the induced map

$$\mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow \mathrm{Spec}(\mathcal{M}_{X,x})$$

is surjective.

Let $\mathcal{M}_{U/X}$ be the sheaf of monoids on X consisting of sections of $\mathcal{O}_X$ whose restriction to U is invertible.

Definition A.2.8. Let U be a nonempty Zariski open subset of a scheme X and $j : U \to X$ be the inclusion. The direct image log structure

$$\alpha_{U/X} : \mathcal{M}_{U/X} \to \mathcal{O}_X$$

is called the *compactifying log structure* associated with the open immersion j. A log structure α_X is called *compactifying* if its subset of triviality $X^* := \{x \in X : \mathcal{M}_{X,x} = 0\}$ is open and the natural map $\alpha_X \to \alpha_{X^*/X}$ is an isomorphism.

Definition A.2.9. A *Deligne–Faltings* structure on a scheme X is a finite sequence of homomorphisms $\gamma_i : \mathcal{L}_i \to \mathcal{O}_X$ where each $\mathcal{L}_i$ is an invertible sheaf on X.

The Deligne–Faltings structure gives a log structure on X and was first considered by Deligne and Faltings as a way of describing log schemes [Ogu18, §III.1.7]. Their theory was found less flexible than the one presented by Fontaine–Illusie and further developed by K. Kato. However, it gives an useful way of describing the compactifying log structure associated to the open immersion $\partial X \to X$.

Definition A.2.10. Letting $Z = n_1 Z_1 + \cdots + n_r Z_r$ be an effective Cartier divisor on a scheme X over k, each $n_i Z_i$ corresponding to the invertible sheaf I_i, an important example of a Deligne–Faltings log structure is that on (X, Z) given by the inclusions $I_i \to \mathcal{O}_X$ for $i = 1, \ldots, r$. Assuming all $n_i = 1$, this becomes the *compactifying* Deligne–Faltings log structure associated to the open immersion $X - Z \to X$.

Log geometry provides a natural language to describe compactifications of smooth schemes due to the following proposition.

Theorem A.2.11 ([BPØ20, Remark 9.5.2])**.** *Let $f : X \to Y$ be a morphism of log smooth fs log schemes over k. Then there exists an integral normal fs log scheme $\overline{X}$, an open immersion $j : X \hookrightarrow \overline{X}$ and a proper strict map $\overline{f} : \overline{X} \to Y$ factoring f, i.e.,*

$$f = \overline{f} \circ j : X \hookrightarrow \overline{X} \to Y,$$

such that $\overline{X}$ has the compactifying Deligne–Faltings log structure (Definition A.2.10) associated to the inclusion $\partial X \to X$, where ∂X is an effective Cartier divisor.

Proof. By Nagata's compactification theorem [Nag62] there exists a proper scheme X' such that $\underline{f}$ factorizes as an open immersion $\underline{X} \hookrightarrow X'$ followed by a proper map $\overline{f} : X' \to \underline{Y}$. Letting $\overline{X}$ be the fs log scheme whose log structure is $(\overline{f}^* \mathcal{M}_Y)_{\log}$ on X' we have a morphism $\overline{f} : \overline{X} \to Y$. Taking the normalization of $\underline{X}$ which does not change the interior, we may assume that X' is normal, and that ∂X is an effective Cartier divisor. ☐

Remark A.2.12. Assuming resolution of singularities we may assume that $\overline{X}$ constructed above is smooth log smooth over k.

As mentioned in the introduction, one of the important applications of log geometry to algebraic geometry is that of degeneration. Let us illustrate this with an example.

Example A.2.13 ([Tal15])**.** Assume that we have a family $X \to \mathbb{A}^1$ where the fibers X_t are all smooth except for the case $t = 0$, and the degenerate fiber X_0 is a simple normal crossing variety. Examples of this include the families $X = \{xy - t = 0\}$ and $X = \{y^2 - x^3 + tx = 0\}$. We can then equip X and $\mathbb{A}^1$ with log structures such that the map $X \to \mathbb{A}^1$ becomes a "well-behaved" (log-smooth) morphism. We achieve this by adding to X the log structure given by X_0 and to $\mathbb{A}^1$ that of $\{0\}$, and then constructing the morphism

$$(X, X_0) \longrightarrow (\mathbb{A}^1, 0)$$

which now is log smooth.

Definition A.2.14. A sheaf of monoids $\mathcal{M}$ on a topological space X is called *quasi-coherent* (resp. *coherent*) if X admits an open covering $\mathcal{U}$ such that the restriction of $\mathcal{M}$ to each U in $\mathcal{U}$ admits a chart $P \to \mathcal{M}$ (resp. a chart subordinate to a finitely generated monoid). The sheaf of monoids $\mathcal{M}$ is called *fine* (resp. saturated) if we can take P to be fine (resp. saturated).

If a log scheme X has a sheaf of monoid $\mathcal{M}$ that is both fine and saturated we call X a *fs* log scheme.

Since we will be working in the setting of homotopy theory with compact support, one of our primary notions is that of a proper fs log schemes.

Definition A.2.15. A quasi-compact quasi-separated fs monoscheme X is *separated* (resp. *proper*) if the naturally induced morphism

$$\mathrm{Hom}(\mathrm{Spec}\,\mathbb{N}, X) \to \mathrm{Hom}(\mathrm{Spec}\,\mathbb{Z}, X)$$

is injective (resp. bijective).

Remark A.2.16. We caution the reader that contrary to the case in algebraic geometry a proper morphism need not be closed. Indeed the summation morphism $\mathbb{N} \oplus \mathbb{N} \to \mathbb{N}$ gives an induced proper morphism of monoschemes, but the image is not closed.

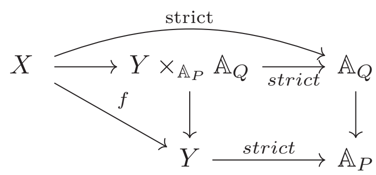
Definition A.2.17. A morphism $f: X \to Y$ of fs log schemes is called a *closed immersion* if the following two properties are satisfied:

(i) The underlying morphism $\underline{f}: \underline{X} \to \underline{Y}$ is a closed embedding,

(ii) The sheaf homomorphism $f^*_{\log}(\mathcal{M}_Y) \to M_X$ is surjective.

We now want to generalize smoothness to log schemes.

Definition A.2.18. A morphism of log schemes $f: X \to Y$ is called *log smooth* (resp. *log étale*) at $x \in X$ if locally at x there exists a chart of f of the form



such that

(i) the underlying scheme morphism $X \to Y \times_{\mathbb{A}_P} \mathbb{A}_Q$ is smooth (resp. étale) at x,

(ii) P and Q are fine,

(iii) $\ker(P^{\mathrm{gp}} \to Q^{\mathrm{gp}})$ is killed by an integer invertible at x,

(iv) The torsion part of $\mathrm{coker}(P^{\mathrm{gp}} \to Q^{\mathrm{gp}})$ is killed by an integer invertible at x (resp. $\mathrm{coker}(P^{\mathrm{gp}} \to Q^{\mathrm{gp}})$ is killed by an integer invertible at x).

We let lSm/k denote the category of fs log schemes that are log smooth over the trivial log point $\mathrm{Spec}\,k$.

The non-trivial part of the log structure lies in the quotient $\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{O}^*_X$, which we call the *characteristic monoid.*

Example A.2.19 ([BPØ20, Example A.6.4])**.** For a monoid P we can consider the monoidal ring $\mathbb{Z}[P]$, and the corresponding affine toric variety $\underline{\mathbb{A}_P} := \mathrm{Spec}\,\mathbb{Z}[P]$. Letting $\mathbb{A}_P := (\underline{\mathbb{A}_P}, \mathcal{M}_{\mathbb{A}_P})$ we for instance have the space $\mathbb{A}_\mathbb{N} \simeq (\mathbb{A}^1, 0)$ with the log structure associated to the origin.

If P^{gp} is torsion free, and $X = \mathbb{A}_P$ and $Y = \mathrm{Spec}\,\mathbb{Z}$ with trivial log structure, then a theorem of Kato ([Kat89, Theorem 3.5]) says that X is logarithmically smooth relative to Y even though the underlying toric variety may be singular. This is what Kato coined "the magic of log".

Example A.2.20 ([Tal15])**.** The theory of toric geometry plays an important part in logarithmic geometry. Indeed, every normal toric variety $X(\Delta)$ has a canonical log structure induced by the embedding $T \hookrightarrow X(\Delta)$ of the open torus. The monoids that appear as stalks are closely related to the lattice points in the dual cones of the cones appearing in the fan Δ. An important example is the toric variety $\mathbb{A}_P$ when P is a fine and saturated monoid.

Remark A.2.21 ([Tal15]). From this point on we will mostly be concerned with fs log schemes. This is the case where the log structure locally comes from the canonical log structure of $\mathbb{A}_P$ as above, i.e., étale locally there is a fine and saturated monoid P with a morphism of log schemes $f : X \to \mathbb{A}_P$ such that the log structure on X is isomorphic to $f^* \mathcal{M}_{\mathbb{A}_P}$. Because of details we will not cover here, it forces the set of points on X with trivial log structure $\underline{X} - \partial X$ to form an open subset of $\underline{X}$.

Equipping a scheme with the trivial log structure $\mathcal{M} = \mathcal{O}_X^*$, and letting α be the inclusion, induces a fully faithful embedding $\mathbf{Sch} \xrightarrow{\lambda} \mathbf{LogSch}$ of the category of schemes into the category of fs log schemes. Thus log geometry may be regarded as an enlargement of algebraic geometry. Moreover, mapping an fs log scheme $X \mapsto X - \partial X$, where $X - \partial X$ denotes the points of X with trivial log structure, we obtain the functor $\mathbf{LogSch} \xrightarrow{\omega} \mathbf{Sch}$ that is right adjoint to λ.

A.3 Properties of logarithmic schemes

The general theory of logarithmic geometry is rich and fruitful, but we will restrict ourself to definitions and results that will be of importance to us in the chapters above.

Recall that a strict normal crossing divisor $D \subset X$ is a divisor that étale locally looks like the union of k coordinate hyperplanes $\{x_1 \cdots x_k = 0\} \subset \mathbb{A}^n$.

Definition A.3.1. Given a smooth scheme X and a strict normal crossing divisor D on X we can associate a log scheme (X, D) with log structure induced by D, where

$$\mathcal{M}_X := \{f \in \mathcal{O}_X : f \text{ is invertible away from } D\}.$$

This log structure captures D acting as a sort of “boundary” of X, and by picking local coordinates for the branches of D through a point $x \in X$ we get local charts $\mathbb{N}^N$, where N denotes the number of branches of D at x.

Example A.3.2 ([Tal15])**.** Consider $\mathbb{A}^2$ with the log structure given by the two coordinate axes $\{x = 0\}, \{y = 0\}$. Then the stalk of the characteristic monoid $\overline{\mathcal{M}}_{\mathbb{A}^2}$ at a point $x \in \mathbb{A}^2$ is

- trivial if x is away from $xy = 0$,
- isomorphic to $\mathbb{N}$ if x is on one of the axes $x = 0$ or $y = 0$ (and not the origin),
- isomorphic to $\mathbb{N}^2$ if x is the origin.

The following proposition gives a nice characterization of log smooth fs log schemes X, such that the underlying scheme $\underline{X}$ is smooth over k. We denote the category of such log schemes $SmlSm/k$.

Proposition A.3.3 ([BPØ20, Lemma A.5.10]). *Suppose that X is a smooth log smooth fs log scheme. Then there exists a strict normal crossing divisor Z on $\underline{X}$ that induces the log structure on X, i.e., an isomorphism*

$$X \simeq (\underline{X}, Z)$$

of fs log schemes.

For this reason one can think of a smooth log scheme as a pair of a scheme and a (strict normal crossing) divisor, where the divisor acts as a sort of boundary of the scheme, and where the log structure determines the structure of the boundary.

Another useful property of log smooth fs log schemes is the following.

Proposition A.3.4 ([BPØ20, A.10.2]). *For $X \in lSm/k$, there is a log blow-up $Y \to X$ such that $Y \in SmlSm/k$.*

We will also often use the following identification.

Proposition A.3.5 ([BPØ20, Proposition A.11.9]). *Let $f : Y \to X$ be a morphism of fs log schemes. Then f is a log modification if and only if it is a surjective proper étale monomorphism.*

Similarly to the case for schemes we also have differentials on log schemes called *log differentials*.

Definition A.3.6 ([BPØ20, Definition A.7.1]). Given a morphism $f : X \to Y$ of fine log schemes, the *sheaf of relative logarithmic differentials* $\Omega^1_{X/Y}$ is

$$\Omega^1_{X/Y} := \left(\Omega_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{gp}})\right)/\mathcal{K},$$

where $\mathcal{K}$ is the $\mathcal{O}_X$-module generated by

- $(d\alpha(a)) - (0, \alpha(a) \otimes a)$ with $a \in \mathcal{M}_X$,
- $(0, 1 \ \otimes a)$ with $a \in \mathrm{im}(f^{-1}\mathcal{M}_Y \to \mathcal{M}_X)$.

The morphism $(\partial d, D)$ is the universal derivation given by

$$\partial : \mathcal{O}_X \xrightarrow{d} \Omega_{X/Y} \to \Omega^1_{X/Y}$$

and

$$D : \mathcal{M}_X \to \mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{gp}} \to \Omega^1_{X/Y}.$$

These differentials have properties similar to those of ordinary differentials, namely for a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

there is a natural exact sequence

$$f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \ \to \Omega^1_{Z/Y} \to 0,$$

and if f is log smooth this sequence is also left exact, and Ω^1_f is locally free as a $\mathcal{O}_X$-module of finite type.

We also have that the logarithmic differentials give rise to a sheaf, defined by the following definition.

Definition A.3.7. For every fs log scheme X over k, and for every $i \geq 0$, the *sheaf of logarithmic differentials* of order i is defined as

$$\Omega^i_{X/k} := \begin{cases} \mathcal{O}_X & i = 0, \\ \bigwedge^i \Omega^1_{X/k} & i > 0, \end{cases}$$

which we often just abbreviate as Ω^i. With the differentials above they also form a complex of sheaves denoted by $\Omega^\bullet$.

We can thus define cohomology groups $H^i_\tau(X, \Omega^j)$ for a topology τ on X. We note that there are isomorphisms

$$H^i_{Zar}(X, \Omega^j) \simeq H^i_{sNis}(X, \Omega^j) \simeq H^i_{s\,\text{ét}}(X, \Omega^j)$$

due to [BPØ20, s.174].

Example A.3.8. Let $X \in Sm/\mathbb{C}$ with a compactification $X \to \overline{X}$ such that $\overline{X}$ is smooth and proper, and the complement $\overline{X} - X = D$ is a strict normal crossing divisor on X. Then if $t_1, \ldots, t_n$ are local coordinates in a neighborhood of any point of D such that D is given by $t_1 \cdot \ldots \cdot t_r = 0$ for some $r \leq n$. Then by the open immersion

$$j : \underline{X} - \partial X \to \underline{X}$$

we define

$$\Omega^1_{\underline{X}}(\log \partial X) \subset j_* \Omega^1_X$$

as the locally free subsheaf generated by

$$\frac{dt_1}{t_1}, \ldots, \frac{dt_r}{t_r}, dt_{r+1} \ldots dt_n.$$

Since $\frac{dt_i}{t_i} = d \log t_i$, the *logarithmic forms* are precisely the meromorphic forms on $\overline{X}$ which are regular on X and with poles of order at most 1 along D. This is primary reason for the choosing the term "logarithmic" in logarithmic geometry.

Theorem A.3.9. *(Logarithmic de Rham theorem) In the situation above we have an isomorphism*

$$\mathbb{H}^*(\overline{X}, \Omega^\bullet_{\overline{X}}(\log D)) \simeq H^*(X(\mathbb{C}), \mathbb{C}).$$

This theorem gives the mixed Hodge structure on $H^*(X(\mathbb{C}), \mathbb{C})$ by the associated Hodge-to-de Rham spectral sequence.

One of the most important theorems in [BPØ20] is that Hodge cohomology is representable in $\mathbf{logDM}^{\text{eff}}(k, \Lambda)$, i.e.,

$$H^i(X, \Omega^j) \simeq \operatorname{Hom}_{\mathbf{logDM}^{\text{eff}}(k,\Lambda)}(M(X), \Omega^j[i])$$

as shown in Theorem 9.7.1 in [BPØ20]. Since the Hochchild–Kostant–Rosenberg Theorem (Theorem 3.4.12 in [Lod92]) gives a description of cyclic cohomology in terms of Hodge cohomology, namely

$$HC_n(X, \Omega^j) \simeq \bigoplus_{p \in \mathbb{Z}} H^{2p-n}_{Zar}(X, \Omega^{\leq 2p}),$$

we also get that cyclic homology is representable in $\mathbf{logDM}^{\text{eff}}(k, \Lambda)$ as

$$HC_n(X, \Omega^j) \simeq \operatorname{Hom}_{\mathbf{logDM}^{\text{eff}}(k,\Lambda)}(M(X), \bigoplus_{p \in \mathbb{Z}} \Omega^{\leq 2p}[2p - n]),$$

which is shown in Theorem 9.7.3 in [BPØ20].

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