Dynamic probabilistic forecasting with uncertainty

Fred Espen Benth \(^*\), Gleda Kutrolli\(^\dagger\), and Silvana Stefani\(^\ddagger\)

\(^1\)Department of Mathematics, University Of Oslo, P. O. Box 1053 Blindern, N-0316 Oslo, Norway
\(^2\)Department of Statistics and Quantitative Methods, University Of Milano - Bicocca, Piazza dell’Ateneo Nuovo, 1 - 20126, Milano, Italy

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Abstract

We introduce a dynamical model for the time evolution of probability density functions incorporating uncertainty in the parameters. The uncertainty follows stochastic processes, thereby defining a new class of stochastic processes with values in the space of probability densities. The purpose is to quantify uncertainty that can be used for probabilistic forecasting. Starting from a set of traded prices of equity indices we do some empirical studies. We apply our dynamic probabilistic forecasting to option pricing, where our proposed notion of model uncertainty reduces to uncertainty on future volatility. A distribution of option prices follows, reflecting the uncertainty on the distribution of the underlying prices. We associate measures of model uncertainty of prices in the sense of Cont.

Keywords: Probability density, model uncertainty, risk measure, volatility, option prices, stochastic processes in Banach space

\(^*\)fredb@math.uio.no
\(^\dagger\)g.kutrolli1@campus.unimib.it
\(^\ddagger\)silvana.stefani@unimib.it
1 Introduction and motivation

Forecasting the value of a financial asset or position is typically based on a stochastic model, where forecasts are derived as the mean or the quantiles. In this paper we propose forecasting based on a stochastic model of the probability distribution. Indeed, we suggest to incorporate model uncertainty by considering forecasting using dynamical stochastic evolutions of the probability distribution of the model in question.

To explain our idea, we recall that although uncertain future events are modelled by a random variable $X$, in practice one specifies the probability density $p$. Inference and forecasting is done with the use of $p$. Sometimes the density is complex, like in hierarchical Bayesian modelling where the Bayes formula is applied in modelling with conditional densities (Jokhadze & Schmidt 2020). An alternative is simulation, however, simulation methods like Monte Carlo often also rely on knowledge of the density.

In time series models, one starts out with a random variable as noise, and puts this into motion through some iterative scheme. A simple AR($p$)-time series is defined as first introducing a sequence of independent, identically distributed (i.i.d.) random variables $(\epsilon_n)$, and then defining

$$X(n + p) + \alpha_1X(n + p - 1) + \ldots + \alpha_pX(n) = \epsilon_n.$$  

However, the fundamental assumption in the modelling is usually a specification of the distribution of $\epsilon_n$, and not a representation of $\epsilon_n$ as a random variable. A rather frequent choice is to let $\epsilon_n$ be normally distributed. As such, we start out with a given distribution (e.g. the normal distribution), and through an inductive iteration (as in Equation (1)) we move the given distribution forward as a function of $n$. This gives a deterministic evolution of the prescribed distribution of $\epsilon_n$. In principle, we can completely determine the distribution of the system at any time step $n$, once specifying the distribution of $\epsilon_n$. As is well-known, for an AR($p$) model with normally distributed $\epsilon_n$, $X$ will be again normally distributed, completely characterised by $\epsilon$ and the regressors $\alpha_i$.

Stochastic differential equations (SDE) provide a mathematically very detailed description of the dynamics of the phenomena being modelled. Like time series models, SDEs also start out with a fundamental process, typically being a Brownian motion $B$ or a Lévy process with known distribution for its increments. The state of the SDE will at all future times be a deterministic functional of this process.

In real-world situations, we do not know, of course, the exact distribution of $\epsilon$, nor the driver in the SDE. Our specific choice is only our (best) model of the situation, and sometimes not even that, but a pragmatic choice to allow for analytic tractability of the model. But far more important, we do not know the exact
dynamics forward in time, that is, we do not know how the system is evolving dynamically. This is indeed also prone to uncertainty, and is specified pragmatically and by insight into the system based on the information we have available. There is no physical law, nor economical law that can prescribe to us how the prices of financial assets dynamically move in time.

SDEs can be very complex, and therefore it may be very hard to reveal the actual distribution and thereby make forecasts. Quantile forecasts may require simulation from the SDE, which is very time consuming if we look for tails of the distribution. We need many samples to accurately describe the tails of the distribution, along with iterations in time to reach the future forecasting point. Hence, although the SDE may feel attractive as a modeling device, it can lead to intractability when it comes to practical forecasting.

Investment decisions, risk evaluations of current positions or pricing of financial derivatives rely on knowledge of the future probability distribution of prices. If we decide for an SDE as the model for future asset prices, there is no room for uncertainty about the distribution that we move forward, nor any uncertainty on the moving forward dynamics. In this paper we propose an alternative approach: A dynamical model of the probability distribution where we include uncertainty, which follows a stochastic processes (indeed, an SDE).

Our suggested modelling paradigm generalises naturally classical probability theory in that we describe mathematically a random event through its probability distribution. We consider probability densities that evolve in time, \( t \mapsto p_t \). The added ingredient is that we allow for uncertainty in the distribution, and therefore consider \( p_t \) as a random variable in some appropriate state space of probability density functions, that is \( t \mapsto p_t \) is a stochastic process with values in a state space of probability density functions.

Classical SDEs are also covered by our approach. The solution, \( X_t \), has a probability law \( p_{X_t} \) that evolves deterministically via the Chapman-Kolmogorov (forward) equation. Indeed, we cover all Markovian processes, as these have associated Chapman-Kolmogorov equations for the evolution of their distribution densities. It also covers the model uncertainty framework by Cont (2006), where model uncertainty is prescribed through uncertainty about the probability measure. We apply our dynamic probabilistic forecasting to option pricing, where the notion of model uncertainty is reduced to uncertainty on future volatility and a distribution of option prices follows. This distribution reflects the uncertainty on the distribution of the underlying prices.

The idea put forward in this paper faces several challenges. First of all, we need to specify stochastic dynamical models for the probability density. Secondly, these models must be benchmarked against reality, that is, we need to be able to make inference on the model using data. We suggest several possible models, including an interpretation of a compound Poisson process. However, the focus
will be on Gaussian densities with mean and variance driven by continuous-time stochastic processes. In an empirical study using financial index data, we suggest a procedure to estimate the density dynamics for such models using Ornstein-Uhlenbeck processes. The impact on pricing of options in a Black-Scholes context is then analysed.

The paper is structured as follows. We start in Section 2 by discussing general density processes with values in $L^1(\mathbb{R}^d)$ and proposing various dynamical models introducing uncertainty in the probability density of a random event. A detailed empirical analysis of financial data from Europe and the US are performed in Section 3, including estimation of Ornstein-Uhlenbeck dynamical models assessing the uncertainty in the expected returns and their volatility. In the next section we focus on an application of our ideas to call option pricing where we quantify the uncertainty. Section 5 concludes with a discussion of our main findings.

2 Stochastic probability density dynamics in $L^1(\mathbb{R}^d)$

Assume we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. Furthermore, denote by $L^1(\mathbb{R}^d)$ the space of (equivalence classes of) real-valued measurable functions on $\mathbb{R}^d$, $d \in \mathbb{N}$, which are integrable. It is well-known that $L^1(\mathbb{R}^d)$ is a separable Banach space with respect to the norm $\|f\|_1 := \int_{\mathbb{R}^d} |f(x)| dx$. Moreover, it is closed under convolution and forms a commutative Banach algebra under the convolution product denoted by $\ast$.

We know that any probability density function $p$ is a non-negative integrable function on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} p(x) dx = 1$. Moreover, the convolution of two densities is again a density, being the density of the sum of two independent random variables. A family $(g_t)_{t \geq 0}$ is called an $L^1(\mathbb{R}^d)$-valued stochastic process if it is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, that is, if $g_t$ is an $\mathcal{F}_t$-measurable map from $\Omega$ into $L^1(\mathbb{R}^d)$ with the latter space equipped with the Borel $\sigma$-algebra. For a general account on the calculus for stochastic processes taking values in Banach spaces, we refer to Dinculeanu (2000).

**Definition 1.** An $L^1(\mathbb{R}^d)$-valued stochastic process $(p_t)_{t \geq 0}$ is said to be a density process if for any $t \geq 0$, $p_t(x) \geq 0$ for all $x \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} p_t(x) dx = 1$.

Consider now a parametric density function $f(\cdot, \theta)$, where $\theta \in C \subset \mathbb{R}^n$, $n \in \mathbb{N}$ with $C$ being an open set.

**Proposition 2.** Assume that $\Theta$ is a $C$-valued random variable. If for a.e. $x \in \mathbb{R}^d$, $\theta \mapsto f(x, \theta)$ is continuous and for any neighborhood $U \subset C$ around $\theta$ there exists a function $h_U \in L^1(\mathbb{R}^d)$, such that $|f(x, \theta)| \leq h_U(x)$ for $\theta \in U$, then $f(\cdot, \Theta)$ is an $L^1(\mathbb{R}^d)$-valued random variable.
Proof. For each fixed $\omega$, we have that $f(\cdot, \Theta(\omega)) \in L^1(\mathbb{R}^d)$. Moreover, as $\Theta$ is a random variable, it is a measurable map from $\Omega$ into the open set $C$. We must show that $f$ is a measurable map from $\Omega$ into $L^1(\mathbb{R}^d)$. Let $\theta_n$ be a sequence in $C$ that converges to $\theta$. Thus, for a given neighborhood $U$ around $\theta$, we find for sufficiently big $n$ that $\theta_n \in U$. But then we find from dominated convergence theorem that
\[
\lim_{n \to \infty} |f(\cdot, \theta_n) - f(\cdot, \theta)|_1 = \int_{\mathbb{R}^d} \lim_{n \to \infty} |f(x, \theta_n) - f(x, \theta)|dx = 0,
\]
and continuity in $L^1(\mathbb{R}^d)$ follows. The combination of a measurable map with a continuous function implies measurability.

We introduce next a sequence of $L^1(\mathbb{R}^d)$-valued random variables $(F_i)_{i \in \mathbb{N}}$, given as follows: Let $(\Theta_i)_{i \in \mathbb{N}}$ be an i.i.d. sequence of random variables. Define
\[
F_i(x) := f(x, \Theta_i),
\]
we have the following Proposition:

**Proposition 3.** Suppose that $\Theta$ has a density $p_\Theta$. If $(\Theta_i)_{i \in \mathbb{N}}$ is an i.i.d. sequence of random variables distributed according to $\Theta$, then $(F_i)_{i \in \mathbb{N}}$ in Equation (3) is i.i.d. $L^1(\mathbb{R}^d)$-valued random variables with the same distribution as $f(\cdot, \Theta)$.

Proof. Since $\Theta_i$ is distributed as $\Theta$, $F_i$ is distributed as $f(\cdot, \Theta)$ in $L^1(\mathbb{R}^d)$ for all $i \in \mathbb{N}$. By independence, we have that the density of $(\Theta_1, \Theta_2)$ is $p_{\Theta_1}(\cdot)p_{\Theta_2}(\cdot)$. For any $A, B \in B(L^1(\mathbb{R}^d))$, Borel sets of $L^1(\mathbb{R}^d)$, we have from conditioning
\[
P(F_1 \in A, F_2 \in B)
= \int_{C^2} P(F_1 \in A, F_2 \in B|\Theta_1 = \theta_1, \Theta_2 = \theta_2) p_{\Theta_1}(\theta_1)p_{\Theta_2}(\theta_2)d\theta_1d\theta_2.
\]

But for given $\theta_1, \theta_2$, we find that
\[
P(f_1(\cdot, \theta_1) \in A, f_2(\cdot, \theta_2) \in B) = P(f_1(\cdot, \theta_1) \in A)P(f_2(\cdot, \theta_2) \in B),
\]
as these probabilities are zero-one probabilities (either $f_i$ is in the set, or not). Thus, we find that
\[
P(F_1 \in A, F_2 \in B) = P(F_1 \in A)P(F_2 \in B),
\]
and the Proposition follows.
Let $N(t)$ be a Poisson process with values on $\mathbb{N} \cup \{0\}$, having an intensity $\lambda > 0$. Define the process

$$C(t) := g * \otimes_{i=1}^{N(t)} F_i,$$

(7)

where $g$ is a probability density function and $\otimes_{i=1}^{N(t)}$ signifies $N(t)$ times iterated use of the convolution product $*$ in $L^1(\mathbb{R}^d)$. We say that $C(t)$ is a convolved Poisson process, in some sense the natural analogue of a compound Poisson process for densities.

**Proposition 4.** The process $(C(t))_{t \geq 0}$ defined in (7) is a density process in $L^1(\mathbb{R}^d)$.

**Proof.** First, for each $\omega \in \Omega$, we have that $F_1 * F_2 * \cdots * F_n \in L^1(\mathbb{R}^d)$, being positive by the property of the convolution product, and with integral equal to 1. Hence, it will be a density function. The convolution product is also a continuous function on $L^1(\mathbb{R}^d)$, so $C_n := g * \otimes_{i=1}^{n} F_i$ will be a random walk time series with values in $L^1(\mathbb{R}^d)$ for $n \geq 1$. Define $C(t) := C_{N(t)}$, which is then a subordination of $C_n$. As $N$ is $\mathcal{F}_t$-adapted, we find that $C(t)$ is also $\mathcal{F}_t$-adapted. Thus the result follows.

The i.i.d. sequence $F_i$ and the convolved Poisson process defined above constitute examples of dynamical models for the stochastic evolution of a probability density. Important for our empirical studies is a class of Gaussian models, that we consider next.

### 2.1 A Gaussian model

Let $M$ and $\Sigma^2$ be two random variables with values on $\mathbb{R}$ and $\mathbb{R}_+$, respectively. We use the notation $\mathbb{R}_+$ for the positive real line not including origo. Define $f$ as

$$f(x, m, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - m)^2}{2\sigma^2}\right),$$

(8)

and consider the $L^1(\mathbb{R})$-valued random variable $f(\cdot, M, \Sigma^2)$. Thus, we use the Gaussian density function with mean $m$ and variance $\sigma^2$, combined with a parameter-valued bivariate random variable $(M, \Sigma^2)$. We note that we can bound this function around a neighborhood of any $(m, \sigma^2)$ by again a Gaussian function, and moreover, the map $(m, \sigma^2) \to f(x, m, \sigma^2)$ is continuous. Thus, $f(\cdot, M, \Sigma^2)$ is an $L^1(\mathbb{R})$-valued random variable by Proposition 2.

Consider now an example of pricing of call options. In the Black-Scholes Gaussian paradigm, the price of a call option is given by the Black-Scholes formula (Black and Scholes 1973):

$$BS(S, K, r, T, \sigma) = S \Phi(d_1) - Ke^{-rT} \Phi(d_2),$$

(9)
where $\Phi$ is the cumulative standard normal distribution function, $K$ is the strike price, $T$ the exercise time, $S$ the current stock price, $r$ the risk-free interest rate and
\[
d_{1,2} = \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.
\] (10)

This formula is based on a stock price with distribution at time $T$ being log-normal, $\ln S(T) - \ln S$ is normally distributed with mean $(r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2T$. However, if we are uncertain about the actual distribution at time $T$ of the log-price, we can consider the log-price being distributed according to the random density $f$. If we assume that $r$ is known with certainty, we have
\[
M = r - \frac{1}{2}\sum_{i=1}^2,\text{ and } \sum_{i=1}^2 \text{ is some random variable distributed in } \mathbb{R}_+.
\]
Then the option price will be $BS(S,K,r,T,\sum_{i=1}^2)$, with $d_{1,2}$ being random variables given in terms of $\sum_{i=1}^2$ (substituting $\sigma$ with $\sum_{i=1}^2$). This means that $BS(S,K,r,T,\sum_{i=1}^2)$ also becomes a random variable. We analyse the distribution of option prices empirically in Section 4.

Next, consider the convolved Poisson density process $(C(t))_{t \geq 0}$ defined in Equation (7), with $F_i := f(\cdot, M_i, \sum_{i=1}^2)$ for $(M_i, \sum_{i=1}^2)$ i.i.d. bivariate random variables. Let us look at the mean forecast at time $t$: By definition, the mean forecast is
\[
\hat{M}(t) = \int_{\mathbb{R}} xC(t)(x)dx.
\] (11)

The following holds:
\[
\hat{M}(t) = \int_{\mathbb{R}} x(g * \otimes_{i=1}^{N(t)} F_i(x))dx
\]
\[
= \int_{\mathbb{R}} xg(x)dx + \sum_{i=1}^{N(t)} \int_{\mathbb{R}} xF_i(x)dx
\]
\[
= \int_{\mathbb{R}} xg(x)dx + \sum_{i=1}^{N(t)} M_i.
\] (12)

In conclusion, the mean forecast becomes a compound Poisson process on $\mathbb{R}$, with intensity $\lambda$ and jump sizes given by $M$.

We can also define the probabilistic forecast of the variance at time $t$ as
\[
\hat{V}(t) := \int_{\mathbb{R}} (x - \hat{M}(t))^2C(t)(x)dx.
\] (13)

Indeed, as the normal density is closed under convolution by summing mean and variances, we find that
\[
C(t) = f(\cdot, m + \sum_{i=1}^{N(t)} M_i, \sigma^2 + \sum_{i=1}^{N(t)} \sum_{i=1}^2),
\] (14)
where $m = \int_{\mathbb{R}} x g(x) dx$ and $\sigma^2 = \int_{\mathbb{R}} (x - m)^2 g(x) dx$. Thus, we see that $C(t)$ is the density of a normal mean-variance mixture model, where the forecasted mean is given as the (real-valued) compound Poisson process $\hat{M}(t) = m + \sum_{i=1}^{N(t)} M_i$ and the forecasted variance given by

$$\hat{V}(t) = \sigma^2 + \sum_{i=1}^{N(t)} \Sigma_i^2,$$

i.e., a compound Poisson process with values on the positive half-line.

This representation motivates a new type of density processes in $L^1(\mathbb{R}^d)$, introduced in the following definition:

**Definition 5.** A conditional density process is given by $C(t) := f(\cdot, \Theta(t))$ where $f(\cdot, \theta) \in L^1(\mathbb{R}^d)$ is a probability density and $t \mapsto \Theta(t)$ is a stochastic process with values in the parameter space of $f$.

If $f(\cdot, m, v^2)$ is the normal density function, we see from Equation (14) that the convolved Poisson process $C(t) = g \ast \bigotimes_{i=1}^{N(t)} f_i(\cdot, M_i, \Sigma_i^2)$ can be represented as a conditional density process.

A natural extension of the convolved Poisson dynamics analysed above is to choose two stochastic processes $(X(t))_{t \geq 0}$ and $(Y(t))_{t \geq 0}$, with state spaces in $\mathbb{R}$ and $\mathbb{R}^+$, respectively. Define a conditional density process by

$$C(t) = f(\cdot, X(t), Y(t)),$$

with $f$ being the normal density function. We derive the following straightforward result:

**Corollary 6.** For the density process $(C(t))_{t \geq 0}$ defined in (16), the mean forecast is

$$\hat{M}(t) = X(t),$$

and the variance forecast is

$$\hat{V}(t) = Y(t).$$

**Proof.** The result follows by appealing to the elementary properties of the Gaussian distribution function: the mean forecast becomes

$$\hat{M}(t) = \int_{\mathbb{R}} x f(x, X(t), Y(t)) dx = X(t).$$

For the variance, we find

$$\hat{V}(t) = \int_{\mathbb{R}} (x - X(t))^2 f(x, X(t), Y(t)) dx = Y(t).$$

The result follows.
In Section 3 we choose to model both the mean and variance by Ornstein-Uhlenbeck processes, providing us stationary models of the uncertainty of these parameters. We shall use an Ornstein-Uhlenbeck process driven by a Brownian motion for $X$, whereas a subordinator drives the noise in $Y$ to ensure positivity of the variance model.

Let us end this section studying a quantile forecast at time $t$ for the density process in Equation (16). We define the $\alpha$-quantile, denoted $Q_\alpha(t)$ for $\alpha \in (0, 1)$, by the solution of the integral equation

$$\int_{-\infty}^{Q_\alpha(t)} f(x, X(t), Y(t)) dx = \alpha. \quad (21)$$

By changing variables $y = (x - X(t))/\sqrt{Y(t)}$, we find that

$$\int_{-\infty}^{Q_\alpha(t)} f(x, X(t), Y(t)) dx = \int_{-\infty}^{(Q_\alpha(t) - X(t))/\sqrt{Y(t)}} f(y, 0, 1) dy,$$

and therefore

$$Q_\alpha(t) = X(t) + \sqrt{Y(t)} q_\alpha, \quad (22)$$

with $q_\alpha$ being the $\alpha$-quantile of a standard normal distribution. This expression provides us with a dynamics for the Value-at-Risk (VaR), incorporating uncertainty on the mean and variance of the underlying.

3 Modeling stock indices by conditional density process

In this section we shall examine the uncertainty of the mean and variance in the conditional density of the logreturns of three equity indices, FTSEMIB, S&P500 and FTSE100. We recall the logreturns to be defined by,

$$R(t) := \log\left(\frac{S(t)}{S(t-1)}\right), \quad (23)$$

where $S(t)$ is the stock price at time $t$ and the time step in our studies is 1 day (daily logreturns). Table 1 summarizes some statistics for the daily logreturn series computed from the closing prices obtained from Investing.com and covering the period 1 January 2008 to 21 January 2019 (respectively 2820, 2781 and 2792 daily observations for FTSEMIB, S&P500 and FTSE100).

Figure 1 shows a time series plot of all the three indices, where we see some similarity between S&P500 and FTSE100 whereas FTSEMIB has a slightly different evolution. We observe from Figure 2 that the logreturns fluctuate randomly
<table>
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<tr>
<th></th>
<th>Mean</th>
<th>Var</th>
<th>Std Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>J-B</th>
</tr>
</thead>
<tbody>
<tr>
<td>FTSEMIB</td>
<td>0.0002</td>
<td>0.0003</td>
<td>0.0169</td>
<td>0.1934</td>
<td>4.6182</td>
<td>2522.67</td>
</tr>
<tr>
<td>S&amp;P500</td>
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<td>0.0002</td>
<td>0.0127</td>
<td>0.3472</td>
<td>10.8043</td>
<td>13577.43</td>
</tr>
<tr>
<td>FTSE100</td>
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<td>0.0002</td>
<td>0.0120</td>
<td>0.1459</td>
<td>8.3649</td>
<td>8147.03</td>
</tr>
</tbody>
</table>

Table 1: Statistics of the daily log-returns of FTSEMIB, S&P500 and FTSE100, 2008 - 2019.

Our aim in this section is to model the logreturns by a conditional density process of Definition 5 and for this purpose we focus on the Gaussian model. For the mean and variance, Equations (17) and (18), we shall use Ornstein-Uhlenbeck processes which we next introduce and discuss.

3.1 Dynamic modelling with Ornstein-Uhlenbeck processes

Representing the mean and variance in the normal distribution of the logreturn indices by Ornstein-Uhlenbeck processes allow us to model the uncertainty by
different speeds of mean reversion and incorporating a mixture of jump and diffusional behaviour. As we shall see, based on empirical findings, we suggest a Gaussian Ornstein-Uhlenbeck process for the mean and a non-Gaussian Ornstein-Uhlenbeck process with a gamma limiting distribution for the volatility.

The Gaussian Ornstein-Uhlenbeck process \( (X(t))_{t \geq 0} \) satisfies the following stochastic differential equation:

\[
dX(t) = \alpha(\mu - X(t))dt + \beta dB(t),
\]

where \( \alpha \) and \( \beta \) are positive constant parameters, \( \mu \) is the long-term mean of the process, and \( (B(t))_{t \geq 0} \) is a Brownian motion. A straightforward application of Ito’s Formula shows that the explicit solution is given by

\[
X(t) = X(0)e^{-\alpha t} + \mu(1 - e^{-\alpha t}) + \beta \int_0^t e^{-\alpha(t-s)}dB(s).
\]

The limiting distribution is normal, with mean \( \mu \) and variance \( \beta^2/2\alpha \).

A discretization of the Ornstein-Uhlenbeck process connects it to an AR(1) process, a first-order autoregressive model. Letting \( \Delta > 0 \) be the time-step and denoting \( x_k := X(k\Delta) \) for \( k \in \mathbb{N} \), we have

\[
x_k = \mu(1 - e^{-\alpha \Delta}) + e^{-\alpha \Delta} x_{k-1} + \epsilon_k.
\]

Here, \( x_0 = X(0) \) and \( (\epsilon_k)_{k \in \mathbb{N}} \) is an i.i.d. Gaussian random variables with zero mean and variance \( \beta^2(1 - e^{-2\alpha \Delta})/2\alpha \). Moreover, it is stationary as the autoregressive coefficient \( \exp(-\alpha \Delta) < 1 \).

For the model of the randomness of the variance, we shall assume the non-Gaussian Ornstein-Uhlenbeck dynamics

\[
dY(t) = -\xi Y(t)dt + dL(t),
\]

where \( \xi \) is a positive constant and \( (L(t))_{t \geq 0} \) is a subordinator (a non-decreasing Lévy process). This model is motivated from the Barndorff-Nielsen and Shephard volatility model (Barndorff-Nielsen & Shephard 2001). Its explicit representation is

\[
Y(t) = Y(0)e^{-\xi t} + \int_0^t e^{-\xi(t-s)}dL(s).
\]

As we shall see, a convenient assumption is to let \( (L(t))_{t \geq 0} \) be a compound Poisson process with exponential jumps, i.e.,

\[
L(t) = \sum_{i=1}^{N(t)} J_i,
\]
for a Poisson process \((N(t))_{t \geq 0}\) with jump intensity \(\lambda > 0\) and jumps \((J_i)_{i=1}^{\infty}\) being i.i.d. exponentially distributed random variables with parameter \(\zeta\). The limiting distribution of \(Y(t)\) as \(t \to \infty\) is in this case gamma distributed, that is, \(\lim_{t \to \infty} Y(t) \sim \Gamma(1/\zeta, \lambda/\xi)\) (Benth, Šaltyte Benth & Koekebakker 2004).

A similar discrete-time dynamics as for \(X\) is achieved by

\[
y_k = \exp(-\xi \Delta) y_{k-1} + \eta_k,
\]

with \(y_k := Y(k\Delta)\) and \(\eta_k\) being i.i.d. positively distributed random variables.

The Ornstein-Uhlenbeck models (or AR(1)-models) above raise the immediate question of generalizations, and the link to more sophisticated models like ARCH and GARCH, say, comes to mind (Bollerslev 1986 and Primiceri 2005). However, we emphasise here that we do not aim at modelling the volatility processes, but the randomness in the probability distribution dynamics. On the other hand, in future studies one may try out more sophisticated models for the mean and variance uncertainty in the probability distribution, following studies as in (Gneiting & Ranjan 2013; Billio, Casarin, Ravazzolo, & van Dijk 2013, and Geweke & Amisano 2010). Here one can find that a probabilistic forecast can be represented in the form of a predictive cumulative distribution function which can be discrete, discrete-continuous or continuous. However, still the focus in these papers is not on random probability distributions.

### 3.2 Uncertainty about the mean of logreturns

We study the uncertainty in the historical mean of logreturns based on the running mean. Recalling the definition of logreturns in Equation (23), the running mean defined over a time window of length \(N\) is

\[
\hat{m}_N(t_i) := \frac{1}{N} \sum_{k=0}^{N-1} R(t_{i-k}).
\]

We shall fit the Gaussian Ornstein-Uhlenbeck process \(X\) in Equation (24) to the running mean and study for this purpose the data in light of an AR(1) time series (Avellaneda and Lee 2010).

Before estimating the parameters in the Ornstein-Uhlenbeck processes to assess the uncertainty in the mean, the very first problem we have to resolve is to find the most appropriate length \(N\) of the rolling window we are going to use for calculating the running mean of logreturns. We appealed to the Durbin-Watson statistics (Durbin and Watson 1950 and 1951), which is a statistic testing for autocorrelation in the residuals from a regression analysis. It provides a number between 0 and 4, with 2 meaning that there is no autocorrelation in the sample.
Values from 0 to 2 indicate positive autocorrelation and values from 2 to 4 indicate negative autocorrelation. Figure 3 illustrates how the Durbin-Watson statistic changes for different $N$ for the three data series we study. By a close inspection of the estimated numbers behind Figure 3, we concluded that for FTSEMIB a good choice is $N = 11$, for S&P500 for $N = 7$, and for FTSE100 for $N = 9$. The Durbin-Watson statistics are, respectively, 1.99, 2.04 and 2.02.

Using these window sizes, we estimated the parameters $\alpha, \mu$ and $\beta$ for the dynamics of $X$ in Equation (24) by appealing to standard estimation procedures for AR(1) time series. In Table 2 we report our results. The parameter $\beta_{\infty}$ in Table 2 is the standard deviation of the limiting Gaussian distribution.

Figure 4 illustrates how the estimated AR(1) processes (in red) line fit the running mean of the logreturns (in blue). We observe a close resemblance with the original time series and it appears that a Gaussian Ornstein-Uhlenbeck model $X$ as in Equation (24) is a good model for the uncertain evolution of the mean in the conditional density process.
Table 2: Parameters of the Ornstein-Uhlenbeck process for the mean of the log-returns of FTSEMIB, S&P500 and FTSE100, 2008-2019.

<table>
<thead>
<tr>
<th>Index</th>
<th>$\alpha$</th>
<th>$\mu$</th>
<th>$\beta$</th>
<th>$\beta_\infty$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FTSEMIB</td>
<td>0.3283</td>
<td>-0.0394</td>
<td>0.4888</td>
<td>0.6033</td>
<td>77.6%</td>
</tr>
<tr>
<td>S&amp;P500</td>
<td>0.2271</td>
<td>-0.0322</td>
<td>0.5219</td>
<td>0.7744</td>
<td>61.6%</td>
</tr>
<tr>
<td>FTSE100</td>
<td>0.3747</td>
<td>-0.0130</td>
<td>0.4219</td>
<td>0.4874</td>
<td>70.6%</td>
</tr>
</tbody>
</table>

Figure 4: Fitted AR(1) process (red color) versus the running mean of the log-returns (blue color) for all indices.

3.3 Uncertainty about the variance of logreturns

We estimate the uncertainty in the variance of the conditional density of the log-returns by the running 1-day historical variance:

$$\hat{\sigma}_N^2(t_i) := \frac{1}{N-1} \sum_{k=1}^N (R(t_i-k) - \hat{m}_N(t_i))^2.$$  \hspace{0.5cm} (32)

We note that $(\hat{\sigma}_N(t_i))_i$ becomes a time series of the 1-day historical volatility, which is annualized after multiplying by the square root of the number of (trading) days in a year (fixed to be 252 in our analysis). The resulting time series are shown in Figure 5.

Based on the 1-day historical variance time series computed from Equation (32), we estimate a non-Gaussian Ornstein-Uhlenbeck process $Y(t)$ presented in Equation (27). For this, we first need to estimate the speed of mean reversion $\xi$. This is done by resorting to the $k$-lagged (stationary) autocorrelation function of $Y(t)$, which is on logarithmic scale equal to

$$\ln(\text{Corr}(Y(t), Y(t+k))) = -\xi k, \quad k = 1, 2, 3, \ldots$$  \hspace{0.5cm} (33)

Comparing with the empirical autocorrelations, we estimated the $\xi$ for the three indices by minimizing the negative log-likelihood function. The resulting estimates were $\hat{\xi} = 0.3683$ for FTSEMIB, $\hat{\xi} = 0.3661$ for S&P500 and $\hat{\xi} = 0.3679$ for FTSE100.
Next we analyse the residuals. Due to mean reversion we obtain (close to) zero correlated data by considering every \( \ell \)-th data point in the original variance time series. Based on the estimated speed of mean reversion, we chose \( \ell \) to be 25 (FTSEMIB), 16 (S&P500) and 30 (FTSE100), and treated these sampled data as independent. We fitted a gamma distribution to these data, where the estimated parameters are reported in Table 3. The histograms from the data along with the fitted gamma distributions are shown in Figure 6. The fit is reasonable, and we conclude that \( Y(t) \) in Equation (27) with a compound Poisson process having exponentially distributed jumps is a good model for the uncertainty dynamics of volatility.

Figure 5: Annualized volatility of logreturns for all indices.

Figure 6: Fitted gamma distribution for new sample (based on every 25th, 16th, 30th data of the original data for each index, respectively).

4 Application to option pricing

We shall next apply the Gaussian conditional density process, introduced in Section 2 and empirically fitted to data in Section 3, to price options. Our analysis is focused on at-the-money (ATM) call options where the volatility is uncertain. The option price is defined though the Black-Scholes formula in Equation (9).
Table 3: Estimated parameters of gamma distribution for FTSEMIB, S&P500 and FTSE100, 2008-2019 using the parametrisation in Subsection 3.1.

We fix the time to maturity of the ATM option to be one month for illustration (the exercise date under study will be 19th February, 2019), and consider the non-Gaussian Ornstein-Uhlenbeck model for the volatility empirically analysed in Subsection 3.3. The risk-free interest rates used in our calculations are 0.25% for the FTSEMIB, 2.69% for the S&P500 and 1.31% for the FTSE100. These rates are reported values of December 2018.

Figure 7: Distribution of option prices obtained from simulating the uncertain volatility based on the scheme in Equation (30) for the estimated Ornstein-Uhlenbeck process $Y(t)$ with exponential jumps in the driving compound Poisson process. We have included in the figure the Black-Scholes price based on the historical volatility from the complete set of logreturn data. (Note that $M$ refers to the median while $A$ means the average of prices shown by the red curves.)

In Figure 7 we plot the distribution of option prices obtained from simulating the uncertain volatility based on the scheme in Equation (30) for the estimated Ornstein-Uhlenbeck process $Y(t)$ with exponential jumps in the driving compound Poisson process. We have included in the figure the Black-Scholes price based on the historical volatility from the complete set of logreturn data as a reference point, depicted as a vertical line in each plot. Interestingly, the uncertainty in the volatility yields for all three indices a large variation in option prices.

around this reference point, with median and average relatively far away. Remark
that the negative prices shown in Figure 7 are an effect of the kernel smoother used
in depicting the distributions and not appearing in the actual simulated prices. The
number of simulations was 1000.

An illustration on how the option price changes over time for one-month ahead
contracts is shown in Figure 8. The prices are changing rather a lot in different tra-
jectories compared with the Black-Scholes prices based on the historical volatility
(red line). The historical volatility is estimated based on the data up to the present
time in the simulation. On the other hand, the deviation from the average price
obtained from the simulated (blue line) is not too big.

![Figure 8: Trajectories of option prices referring to the simulated volatility using an
Ornstein-Uhlenbeck process driven by compound Poisson innovations with expo-
nential jumps. The red lines indicate Black-Scholes prices based on the historical
volatility. The blue lines indicate the average price obtained from the simulated
trajectories.](image)

4.1 Model uncertainty

The distribution of option prices resulting from a conditional density model of
the logreturns is a probabilistic description of model uncertainty. Cont (2006)
and Avellaneda, Levy and Paras (1995) propose to quantify model uncertainty
by a worst case approach: Given that a payoff $X$ of a contingent claim has a
well-defined value in all pricing models $Q \in \mathcal{Q}$, where $\mathcal{Q}$ is the set of equivalent
martingale measures, one can define an upper and lower bound for the price of
$X$ by 
\[ \pi(X) = \sup_{Q \in \mathcal{Q}} E^Q[X] \quad \text{and} \quad \underline{\pi}(X) = \inf_{Q \in \mathcal{Q}} E^Q[X], \]
respectively. This definition clearly quantifies the extremes of model uncertainty, which is the main
purpose of Cont (2006). The model uncertainty measure is represented by the
difference $\pi(X) - \underline{\pi}(X)$.

In Table 4 we report the lowest and highest option price for the respective
indices based on our simulations of the Black-Scholes formula with uncertain
volatility performed above (recall Figure 7).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
 & $\pi$ & $\bar{\pi}$ \\
\hline
FTSE100 & 4.86 & 1587.74 \\
S&P500 & 5.93 & 278.88 \\
FTSEMIB & 7.47 & 516.92 \\
\hline
\end{tabular}
\caption{Model uncertainty for the ATM call option. $\pi$ is the price corresponding to the highest simulated volatility and $\bar{\pi}$ is the price corresponding to the lowest simulated volatility.}
\end{table}

The bounds are based on the worst case simulations of volatility, and therefore very sensitive to additional Monte Carlo simulations. Rather than using the worst case upper and lower prices to assess model uncertainty, it may be more sensible to calculate upper and lower bounds using a VaR measure instead. From our simulations, we report in Table 5 the 1%-VaR of the computed prices (respectively the 99%), which of course are the 1% and 99% quantiles of the simulated option prices. The uncertainty range is reduced, in particular as a result of the maximum

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
 & $\pi_{1\%}$ & $\pi_{99\%}$ \\
\hline
FTSE100 & 35.21 & 1238.67 \\
S&P500 & 5.94 & 183.41 \\
FTSEMIB & 9.56 & 369.25 \\
\hline
\end{tabular}
\caption{Model uncertainty for the call options by using 1% and 99% VaR.}
\end{table}

values. We obtain a much more reliable assessment of model uncertainty being more robust towards Monte Carlo error. Alternative risk measures over VaR could be expected shortfall, say. VaR is not sub-additive or convex, and it can lead to anomalous values (Artzner, Delbaen, Eber & Heath 1999 and Hull 2006). Bannör and Scherer (2013, 2014) assess model risk using convex risk measures. They quantify parameter risk, which is related to our uncertainty modelling of the volatility.

\section{Conclusion}

We have proposed a dynamical model of the probability distribution of random events adding uncertainty in the distribution. Further we have defined and discussed model uncertainty and risk measures in a stochastic manner illustrated by various examples applied to stock market indices such as FTSEMIB, S&P500 and FTSE100. It is demonstrated that model and parameter risk and uncertainty play
a prominent role. In our study the notion of model uncertainty is reduced to un-
certainty on future mean return and volatility. The dynamics of these parameters
in the evolution of a Gaussian distribution are modelled by Ornstein-Uhlenbeck
processes, the latter by a positive process. The uncertainty on volatility is applied
to option pricing, where a study of ATM call options on the three indices imply
a distribution of prices rather than a unique price. This is a quantification of the
volatility uncertainty in terms of its resulting option prices. Next we have im-
plemented and critically discussed the quantitative framework for measuring the
impact of model uncertainty on pricing options proposed by Cont (2006). We
have compared the uncertainty measure based on worst cases with our proposal
using a VaR-based measure giving more robust and slightly tighter estimates.

In future studies it may be interesting from a theoretical point of view to further
develop models for the stochastic evolution of probability densities. More applied
studies may reveal the need for other basic distributions beyond the Gaussian we
have focused on in this paper, for which different parameters can be randomized
using stochastic processes. Finally, it is a fundamental question to understand how
to estimate density models on data in a statistically sound way.

Data availability statement

The data that support the findings of this study are available from the correspond-
ing author upon reasonable request.

Acknowledgments

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which has led to significant improvements in the presentation of the paper.

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