Aristotelian aspirations, Fregean fears: Hossack on numbers as magnitudes

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1   Introduction

Hossack's new book, *Knowledge and the Philosophy of Number*, revives the ancient view that numbers are magnitudes and thus a special kind of property, which are instantiated in the physical world. In this way, Hossack makes a promising attempt to solve some stubborn metaphysical and epistemological problems about numbers.[1]

The book considers some of the most important number systems, such as the natural numbers, the reals, and the ordinals. To say that the numbers from each of these systems are magnitudes is to claim that the system is a linearly ordered family of properties that are instantiated by certain types of quantities. What are these quantities? For present purposes, the most important thing to note is the Aristotelian claim that "the most distinctive mark of quantity is that equality and inequality are predicated of it." (1941: 6a27) For example, some objects—or a "plurality", as I will often put it, purely for ease of communication—are a quantity. Two such quantities are equal just in case they are equinumerous, in the usual sense that they can be put in one-to-one correspondence. We can now formulate a simplified form of one of the central theses of the book.

> *The Magnitude Thesis (simplified form)*
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> A magnitude is a property that is shared by equivalent quantities.

For example, the natural number 2 is a property shared by all pairs.

The resulting conception of numbers has some attractive features, philosophically as well as mathematically. Since numbers are properties instantiated in the physical world, they are philosophically no more mysterious than any other properties. In particular, there is no

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[1] See also Peacocke (2019), especially chs. 2 and 5, for a closely related project, which, however, is less self-consciously Aristotelian than Hossack's.

mystery about their applicability or relevance to the physical world.[2] As for mathematical virtues, the conception offers a pleasingly unified framework for various kinds of numbers. As Aristotle observed, different types of quantity share some important algebraic features. These shared features entail, via the Magnitude Thesis, that all magnitudes have a shared algebraic structure—that of so-called *positive semigroups*, to be explained shortly. Thus, we obtain a unified treatment of different kinds of numbers, as was anticipated by Aristotle.[3] Different kinds of number are associated with different kinds of quantity, and additional structure on each of these kinds of quantity gives rise to additional structure on the associated numbers.

I applaud Hossack for attempting to revive some very promising ancient ideas that have received too little attention in recent decades. His discussion also brings to light some broadly Aristotelian insights which are important and valuable. I will argue, however, that Hossack is too hostile to Frege and too narrow in his development of the Aristotelian insights. There remains a strong pressure to generalize beyond the scope of Hossack's account. While the potential for generalizations is often a benefit, it can also expose problems. And indeed, when the needed generalizations are made, we reintroduce some of the problems that have haunted Frege and neo-Fregeans. I conclude that these problems cannot be skirted but need to be addressed head-on.

The structure of this article is as follows. First, I present Hossack's analysis of quantities. As we will see, these have both a mereological structure and support a notion of equality, which set the agenda for the next two sections. Then, I turn to Hossack's analysis of magnitudes. I query how different his magnitudes really are from mathematical objects, especially as these are understood in the Fregean tradition. Finally, I show that the pressure to generalize introduces a threat of paradox. Overall, I seek a rapprochement between Hossack's approach and that of my recent book, Linnebo (2018), in part by identifying some points of agreement, and in part by suggesting some ways in which his view might benefit from moving in the direction of my own.

2   Mereology based on an addition operation

Hossack's notion of quantity is explicitly Aristotelian. He quotes with approval the following

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[2] By contrast, (Field, 1989, pp. 18ff) and Kitcher (1978) argue that numbers, understood as abstract objects, would be mysterious in these ways.

[3] See his *Posterior Analytics* 74a18-25, quoted and discussed in Sect. 5.1.

definition:

> 'Quantity' means that which is divisible into two or more constituent parts (Metaphysics:1020a7) (Sect. 3.2).

In virtue of having "two or more constituent parts", quantities have a mereological, or partwhole, structure. This means that quantities, as Hossack understands them, have both a mereological structure and support a notion of equality. I will discuss these two types of structure in this section and the next, respectively.

Let us start by considering some examples of quantities. In addition to pluralities, mentioned above, Hossack provides two further canonical examples of quantities, which he calls *continua* and *series* (Sect. 3.1). Continua are the spatial regions occupied by stuffs, which in turn are the kinds of things to which mass nouns refer; for example, this milk, that beer, or all your gold. A series are some objects in a certain order. For example, a queue is a series. A series can thus be specified by means of an ordered list. Alice, Beth, and Cassie are a series, and so are Tom, Dick, and Harry. Here the order matters, unlike in the case of pluralities.[4]

Each of these types of quantity has a mereological structure. This is particularly clear in the case of pluralities and continua, which in fact share most of their mereological structure. We can talk about one plurality being part of another and about two pluralities overlapping or being disjoint.[5] Likewise, this gold can be part of all your gold, which in turn may overlap all the gold in the UK. Even series have a mereological structure, where parthood is understood as the notion $\unlhd$ *of* being an initial series; for example, we have:

$$\text{Alice, Beth} \unlhd \text{Alice, Beth, Cassie}$$

The mereological structure shared by pluralities and continua is a familiar one, namely that of Classical Extensional Mereology. This is the most widely studied mereological theory, often regarded as the default or classical account. But Hossack provides an alternative, less familiar axiomatization of this theory. Where the more usual approach uses parthood as its core notion, Hossack uses addition.He chooses this alternative axiomatization because he

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[4] See also the *serial logic* developed by Hewitt (2012).

[5] For a seminal article on plural logic, see Boolos (1984); for a survey, see Linnebo (2017). See also Hossack (2000), where the mereological structure of pluralities plays a central role.

finds it easy to understand and "clearly a priori" when interpreted as concerned with a system of quantities (ch. 4, first page).

Before we can state his axiomatization, we need some explanations. Let us write '&' for the operation of addition. For example, we have:

my part-time students & my full-time students = my students

We next observe that various other mereological notions can be defined in terms of this addition operation. First, *x is part of y*, written $x \leq y$, iff $x\&y = y$. Second, x is said to be a *proper part* of y iff $x \leq y$ but $y \nleq x$. Third, x and y *overlap*, written $x \circ y$, iff they have a common part; if not, x and y are said to be *disjoint*, written $x \perp y$. Finally, x is an *upper bound* for some quantities iff each of these quantities is part of x; and x is their *least upper bound* iff, whenever x' is an upper bound, then $x \leq x'$.

Hossack's axiomatization of Classical Extensional Mereology consists of eight principles which, echoing Euclid, he calls "common axioms".

(A1) *Closure*. Given x and y, there exists a quantity w such that $w = x\&y$.

(A2) *Idempotent Law*. $x\&x = x$.

(A3) *Commutative law*. $x\&y = y\&x$.

(A4) *Associative law*. $x\&(y\&z) = (x\&y)\&z$.

(A5) *Remainder*. If y is a proper part of x there always exists a quantity w disjoint from y such that $x = y\&w$.

(A6) *Overlap*. If z overlaps $x\&y$, then z overlaps x or z overlaps y.

(A7) *Universe*. There is a quantity of which every quantity is a part.

(A8) *Least upper bound*. If $\exists x\phi(x)$, then there is a least upper bound all x such that $\phi(x)$.[6]

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[6] I have slightly modified Hossack's formulation so as to avoid any reliance on Quinean "virtual classes".

As is well known, there are heated debates in metaphysics about the correct mereological structure of reality. Is this clay part of the statue, or vice versa, or perhaps both? Is there a sum of the clay and my bicycle? All these debates concern the mereological structure of *individuals*, Hossack contends, not *quantities*. He wisely sets these debates aside in order to focus on the mereological structure of different types of quantity. His concern is thus solely with what he regards as a purely logical notion of parthood, where things are more clear cut. Indeed, Hossack claims that, when the mereological notions are interpreted as concerned with pluralities or continua, not individuals, the axioms stated above are uncontroversial and *a priori*.

I agree that the mereological structure of quantities is less problematic than that of individuals and that many of Hossack's axioms are plausible on the mentioned interpretations. I am unable to follow him all the way, however, as I have argued in print against (A7) on the plural interpretation.[7] There is good reason, I argue, not to accept a universal plurality: a plurality has to be properly circumscribed in a way that the universe as a whole is not. Since this is a minority view, however, I set it aside for now and proceed on the assumption that Hossack's axioms are appropriate.

Pluralities and continua have further structure beyond that described by Classical Extensional Mereology. Say that x is *an atom* just in case x has no proper parts. Then, for pluralities, we want an axiom of atomicity, to the effect that every object has an atomic part:

(A9) Atomicity $\forall x \exists y (y \leq x \land \forall z (z \leq y \rightarrow z = y))$.

By contrast, continua are *gunky*, in the usual sense that every part contains a proper part. Thus, for continua, we add the following extra axiom:

(C9) Divisibility $\forall x \exists y\, y < x$.

Careful readers may have noticed that a single type of variables are used to range over individuals, pluralities, continua, and perhaps more. This is not a slip but fully intentional. Following Aristotle, Hossack holds that "'everything there is' divide into the predicables and the impredicables", that is, into properties and objects (Sect. 2.2). Indeed, he goes even further, adopting a type-free language the variables of which range freely over properties,

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[7] See Linnebo (2010) and Florio and Linnebo (forthcoming).

individuals, pluralities, continua and series: if we wish to speak about one category of things in particular, we can do so by simply restricting the variables to that category. (p.1 of ch.4)

It is natural to worry that this untyped language will lead to paradoxes. For example, the property of not self-instantiating appears to lead straight to a property-theoretic version of Russell's paradox. In response, Hossack points out that his properties are scarce, not abundant, which means that there is no obvious pressure on him to accept the mentioned property or others that would lead to trouble (ch.2).

What types of quantities are there? As we have seen, Hossack provides three examples: pluralities, continua, and series. He offers no systematic discussion, though, of whether there might be further examples. Why shouldn't there be? Various philosophers and logicians, including myself, have defended the idea of superplurals, that is, a doubly articulated form of reference that stands to plural reference the way this stands to ordinary singular reference. For example,

(1) My children, your children, and her children played against each other.

is naturally interpreted as involving superplural reference.[8] There are other possibilities too. Why shouldn't there be a "multiplural" form of reference that differs from ordinary plural reference, not by taking order into account, as in the case of serial reference, but by admitting multiple occurrences of the same object?[9] I may, for example, state that the winners of the race in the past three years are Alice, Beth, and Beth, thus conveying the fact that Alice won once and Beth twice. In short, there seem to be a variety of purely logical parthood relations and an associated formal mereology. A systematic theory of generalized parthood relations has been developed by Fine (2010).

I don't know to what extent Hossack would be willing to countenance other types of quantities beyond those that he discusses. But it is hard to see how he could block the proposed examples and generalizations just mentioned. Admittedly, he makes a point of the fact that natural language has "devices for plural reference, mass reference and serial reference" (ch. 3). So he might attempt to block further generalizations by arguing that there is no basis for this natural language. But this type of argument would be doubly problematic.

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[8] See (Linnebo, 2017, Sect. 2.4) for an overview and further references.

[9] For the *cognoscenti*: Multiplural reference would thus stand to the well understood notion of multiset as ordinary plural reference stands to sets.

First, it is doubtful that natural language is so limited, as the above examples illustrate. Second, and more importantly, appealing to what is available in a natural language such as English would anyway be too parochial for present purposes. One cannot assess whether a kind of quantity exists as a legitimate object of mathematical investigation by appealing to the structure of English or other natural languages.

3   Equality

As Hossack is fond of pointing out, Aristotle holds that "the most distinctive mark of quantity is that equality and inequality are predicated of it" (Categories: 6a27). For example, two pluralities can be equally numerous, and two masses, equally massive. To develop this idea, we need a notion of equality of quantities, which is distinct from identity. I will write this equivalence as $\sim$.[10]

What assumptions do we need concerning this equivalence? The basic idea is clear. We need it to be an equivalence relation, which "agrees with" our notions of addition and subtraction. When spelling this out, Hossack's source of inspiration are Euclid's famous Common Notions:

(1) Things which equal the same thing also equal one another.

(2) If equals are added to equals, then the wholes are equal.

(3) If equals are subtracted from equals, then the remainders are equal.

(4) Things which coincide with one another equal one another.

(5) The whole is greater than the part.

Seeking technical improvement, though, Hossack adopts some different axioms which entail, but go beyond, all the Common Notions except (4), which is not needed at this stage. These are his *Equality Axioms*:

(E1) Quantities equal to the same quantity are equal to one another.

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[10] Hossack uses $\approx$, but I would like to reserve this symbol for equinumerosity (or some restriction thereof), as is standard in much of the literature with which we will engage, and to use $\sim$ for the completely general notion of equivalence.

(E2) If disjoint equals are added to equals, the wholes are equal. That is:

$$x \perp y \wedge x' \perp y' \wedge x \sim x' \wedge y \sim y' \rightarrow x \& y \sim x' \& y'$$

(E3) (Trichotomy) Two quantities are unequal if and only if either the first is equal to a proper part of the second or the second is equal to a proper part of the first.

(E4) No quantity is equal to any of its proper parts.[11]

We can define a notion of a quantity being less than another by letting '$x \prec y$' abbreviate "x is equal to some proper part of y". The trichotomy axiom can now be formalized thus: for all quantities x and y, exactly one of the following is true: (i) $x \prec y$; (ii) $x = y$; (iii) $y \prec x$.

The basic idea, we recall, is that $\sim$ should be an equivalence relation that "agrees with" the other relations among quantities. The Equality Axioms ensure that. Hossack's Lemma 5.5 establishes that $\sim$ is an equivalence relation. Moreover, (E2) states that $\sim$ is a so-called congruence with respect to addition, provided that proper care is taken to ensure that appropriate quantities are disjoint; and the analogous claim can be proved concerning subtraction. We can also prove that $\sim$ is a congruence with respect to $\prec$.

I wish to end the section by resuming my case for further generalizations. First, consider pluralities with the equivalence of equinumerosity. Then (E4) states that no plurality can be equinumerous with a proper subplurality. This means that no plurality can be (Dedekind) infinite! Thus, Hossack's analysis rules out infinite pluralities and their cardinalities, which seem like perfectly good magnitudes.[12] Second, linear orders are no doubt important. But why

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[11] I have rephrased this axiom slightly for greater clarity.

[12] In fact, (E4) is not only unduly restrictive but also potentially in conflict with the rest of Hossack's theory. As we saw in the previous section, he accepts a universal plurality, that is, a plurality of everything whatsoever. Moreover, as his discussion of the natural numbers in Ch. 6 makes clear, each natural number is part of this universal plurality, which accordingly is an infinite plurality. One would therefore expect the universal plurality to have proper parts, or subpluralities, that are equivalent to itself, in violation of (E4). Hossack is saved from outright contradiction only by his unusually restrictive definition, in Sect. 6.2, of the relation $\approx$ of cardinality equivalence, or "tallying", between two pluralities.

Even setting aside worries about this relation being too restrictive, a problem remains: his proof in Sect. 6.3 that $\approx$ is an equivalence, in the sense of the equivalence axioms (E1)–(E4), is flawed. If the proof were correct, $\approx$ would be reflexive, by Lemma 5.5. But since only finite pluralities can tally, the universal plurality shows that $\approx$ isn't reflexive. To remedy this flaw, one option might be to abandon axiom (E4) and adopt a less restrictive notion of cardinality equivalence. Another option might be to redo the entire investigation in a way that replaces the work currently done by equivalence relations by so-called *partial* equivalence relations, that is, relations that are symmetric and transitive but not necessarily reflexive.

insist that *all* quantities be thus ordered? There are plausible examples of quantities that are not linearly ordered. Consider angles, which are naturally understood as quantities and support a familiar notion of addition. But the equivalence of angles is naturally understood *modulo* 2π, with the result that the ordering of angles fails to be linear. Or consider oriented line segments. These too can (in the Euclidean case) be shown to support natural notions of addition, subtraction, and equality.[13] What is shared by two line segments under this equivalence corresponds to a vector. But the order defined on these vectors in terms of our notion of addition is not linear.

I conclude that there is substantial pressure to generalize further than Hossack does. And again, it is unclear how he could resist this pressure, should he wish to do so.

4   Magnitudes

A magnitude, we recall, was explained as a property that is shared by equivalent quantities (Introduction, Sect. 5). Let us now be more precise. Let a *system of magnitudes* be a family of properties associated with one and the same relation of equality (Intro, ch. 1). We can now formulate what is perhaps the most important thesis of the entire book.

> *Magnitudes Thesis*
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> Whenever there is a standard of equality that satisfies the Common Notions, there
> is a corresponding system of magnitudes such that quantities are equal if and only
> if they instantiate the same magnitude of the system. (Introduction, Sect. 5)

Let the variable m range over the system of magnitudes associated with some fixed equivalence $\sim$, and write '$x\,\eta\,m$' for the claim that x instantiates m. Then the Magnitude Thesis tell us that:

(MT) $$\exists m(x\,\eta\,m \wedge y\,\eta\,m) \leftrightarrow x \sim y$$

There is another way to express this thesis as well. Let ϕ map a quantity to its magnitude.

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[13] For example, each such line segment can be represented as (P_1,P_2), where P_1 is its starting point and P_2 its end point. We define $(P_1,P_2)\&(Q_1,Q_2)$ as the segment (P_1,R), obtained by first shifting the segment (Q_1,Q_2) such that its starting point coincides with P_2 and then letting R by the point to which its end point has been moved. (As is well know, this operation of "parallel shift" is not available when the geometry is non-Euclidean.)

(MT) can then be written as Frege-style abstraction principle:

(AP) $$\phi(x) = \phi(y) \leftrightarrow x \sim y$$

This reveals an important structural similarity between the Magnitude Thesis and Fregean abstraction. The significance of this structural similarity will be a central concern in what follows.

As Hossack explains, we can now give an elegant account of the abstract structure of magnitudes. We have studied the two-fold structure of quantities, provided by mereology and the equivalence relation. The Magnitude Thesis enables us to transfer much of this structure from the quantities to the magnitudes that these instantiate.

We can, for example, define an operation of addition on the magnitudes. To see this, consider two magnitudes m_1 and m_2. Suppose it is possible to find two disjoint quantities x_1 and x_2 that instantiate these magnitudes. We wish to define the sum $m_1 + m_2$ as the magnitude of the sum $x_1 \& x_2$. For this definition to be permissible, however, the magnitude assigned to $m_1 + m_2$, namely $\phi(x_1 \& x_2)$, must be independent of our choice of disjoint quantities x_1 and x_2 to instantiate m_1 and m_2, respectively. So consider some alternative such choice, x'_1 and x'_2. Since (E2) states that $\sim$ is a congruence with respect to the operation of fusing disjoint quantities, we have $x_1 \& x_2 \sim x'_1 \& x'_2$. By the Magnitude Thesis, this ensures the desired independence:

$$\varphi(x_1 \& x_2) = \varphi(x'_1 \& x'_2)$$

Thus, our definition is permissible.

With the operation of addition of magnitudes defined, we can now proceed to investigate its properties. First, we prove that, for any disjoint quantities x_1 and x_2, the magnitude of their sum is the sum of their magnitudes:

(1) $$\phi(x_1 \& x_2) = \phi(x_1) + \phi(x_2)$$

This is only the beginning. Using the correspondence between quantities and magnitudes afforded by the Magnitude Thesis, Hossack derives some important algebraic properties of the magnitudes. In particular, we can show that addition of magnitudes obeys the associative law and a law known as "restricted subtraction":

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$a + (b + c) = (a + b) + c$

$a \neq b$ if and only if there is an element d such that either $b = a + d$ or $a = b + d$.

In technical parlance, we have thus shown that any system of magnitudes forms a so-called "positive semigroup". Hossack calls this important result *the homomorphism theorem*.[14]This theorem highlights two attractive features of Hossack's account, which were advertised already in Section 1. First, the theorem offers a pleasing unification of different types of numbers or magnitudes. Provided that a type of quantity satisfies the axioms of mereology and equality set out above, the system of magnitudes which the quantities instantiate is guaranteed to have a certain algebraic structure. Of course, if we consider quantities satisfying other axioms, then the corresponding magnitudes may satisfy different algebraic principles. Second, some attractive Aristotelian explanations become available. Magnitudes are no more mysterious than any other properties that are instantiated in the physical world. In particular, the algebraic structure of these magnitudes can be explained in terms of an analogous structure on the quantities whose magnitudes they are. The algebraic structure of the magnitudes is, as it were, "inherited" from an analogous structure on the quantities.

We observed above that the Magnitude Thesis is structurally similar to a Fregean abstraction principle: two quantities are associated with the same magnitude just in case they are equivalent. While this structural similarity is undeniable, its philosophical significance is obviously up for debate. Here Hossack and I disagree. He regards his broadly Aristotelian approach as fundamentally different from Fregean abstraction, for at least two reasons. First, there is an important structural difference, namely that magnitudes come with a linear ordering, whereas Fregean abstraction works on a greater variety of equivalence relations. Second, the things on the left-hand side of (AP) belong to the category of properties, not objects, as Fregeans would have it. Magnitudes are therefore acceptable to nominalists, who deny that there are abstract objects: for magnitudes aren't objects at all but properties instantiated by ordinary physical objects.

I deny that these alleged differences are very significant. First, how important is it that each system of magnitudes is linearly ordered? In the previous two sections, I have argued

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[14] It is unfortunate that Hossack does not compare his homomorphism theorem with analogous theorems from the established field of measurement theory; see e.g. Suppes (1951), especially Metatheorem A, or Krantz et al. (1971), for a more authoritative treatment.

that there is substantial pressure to generalize Hossack's account so as to take on board magnitudes that aren't linearly ordered, such as angles and vectors. I now wish to add a more principled, philosophical point. In my opinion, the single most valuable aspect of Hossack's approach is that it makes available some very attractive, broadly Aristotelian explanations. It explains how we can latch on to certain abstract features of reality, understood as properties that can be instantiated in the physical world. Moreover, the algebraic structure of these properties can be understood as "inherited" from an analogous structure on the entities whose properties they are. We are now in a position to make an important observation. *These attractive explanations have nothing to do with the equivalence on the relevant entities giving rise to a linear order!* On the contrary, the Aristotelian explanations apply to shapes, directions, and so on, just as much as to magnitudes. Consider the case of directions. A direction can be understood as a property of lines, such that two lines have the same direction just in case they are parallel:[15]

$$d(l_1) = d(l_2) \leftrightarrow l_1 \parallel l_2$$

Moreover, relations between directions, such as orthogonality (symbolized as '$\perp$'), can be seen as "inherited" from corresponding relations between the lines whose directions they are:

$$d(l_1) \perp d(l_2) \leftrightarrow l_1 \perp l_2$$

This shows that the attractive, broadly Aristotelian explanations have nothing to do with the properties in question being linearly ordered.

Second, how significant is it that magnitudes are properties rather than objects? Hossack attaches great significance to this distinction (see Introduction and Ch. 2). I disagree. To explain why, recall that Hossack uses an untyped logic where a single sort of variables is used to range over things of all types or categories, including objects, quantities, and properties. First-order variables are thus allowed to have magnitudes as their values, and magnitudes are permitted to figure as members of pluralities and sets. I believe this gives us all that Frege and his follows ever wanted when they defended the idea of numbers as objects. Frege's notion of

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[15] As usual, I ignore the slight awkwardness that Frege's "directions" lack orientation. To capture the ordinary notion of direction, we need to consider oriented lines or line segments under the equivalence relation of "co-orientation", defined as parallelism plus sameness of orientation.

an object is a broadly logical one, and the primary point of reifying numbers is to be able to treat them as objects for various logico-mathematical purposes. But the licence so to treat the numbers is granted to us anyway by Hossack's choice of an untyped logic.

To elaborate, consider what is often held up as a key advantage of the Fregean thesis that numbers are objects, namely that this thesis licences his famous bootstrapping argument for the existence of infinity many natural numbers. Here is the idea. Suppose we have established the existence of the natural numbers from 0 up through n. Since numbers are objects, this means we can consider the plurality of 0, 1, and so on up through n.[16] Since the number of this plurality is $n + 1$, we have now established the existence of one more number. For this argument to go through, however, there is no need for numbers to be objects in some robust metaphysical sense that some philosophers might find problematic. All it takes is that numbers can figure as members of the pluralities whose numbers we consider. And this is something that Hossack grants us anyway via his untyped logic. Thus, if Hossack's numbers fail to be objects, this would be in some purely metaphysical sense, which Fregeans probably never sought and certainly do not need.

All in all, I am inclined to regard Hossack's properties and objects as just two different kinds of Frege-objects. I also believe that the essence of his broadly Aristotelian metaphysics and epistemology of numbers can be co-opted by the Fregeans—and to a large extent is already part of their view, through their emphasis on numbers as reified cardinality properties and Frege's applicability constraint, which requires the applicability to counting to be part of the very nature of numbers, not only "grafted on" from the outside.[17]

5   The bad company problem

Suppose we generalize, as I have been urging, and allow a wider class of equivalence relations to define magnitudes or, more generally, to figure in (MT) and its variant (AP). Suppose further, following Hossack, that magnitudes and properties in general belong to the same logical type as objects. Then we encounter a well-known threat of paradox. For there are

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[16] How do we obtain the number 0 when numbers are understood as properties of pluralities, which presumably must have one or more members? Here we may follow Hossack's own lead and pick any non-number to serve as a proxy for 0 (see ch. 6).

[17] See Wright (2000) for a discussion and partial defense of this constraint.

many instances of the resulting abstraction scheme (AP) that are inconsistent or otherwise unacceptable. A famous example is (a plural variant of) Frege's Basic Law V

(V) $$\{xx\} = \{yy\} \leftrightarrow \forall z(z \prec xx \leftrightarrow z \prec yy)$$

which famously falls prey to Russell's paradox. This is the so-called *bad company problem*, which has received a great deal of attention in the Fregean tradition.

How should we respond to the problem? This is a huge question, which has generated a great deal of debate. I will end with some all too brief remarks about the options, including Hossack's and the one that I favor.

The neo-Fregeans take the lesson of the bad company problem to be that we need to restrict the kind of type-lowering involved in (AP)—where φ maps a property or a plurality to an object—to a limited range of instances, including the case of cardinality abstraction. How should the range of legitimate forms of abstraction be demarcated? Despite repeated attempts, there has been no satisfactory answer to this crucial question.[18]

In addition to this worrying lack of progress, Hossack has an independent reason to reject to the neo-Fregean response. As we have seen, he is committed to treating magnitudes and any generalizations thereof as things alongside ordinary objects, all in the range of his single untyped kind of variables. This means that there are no types to be lowered in the first place. Although this is a radical view, I find myself broadly in agreement. In particular, Hossack is right, it seems to me, that considerations about expressibility give us reason to lift the type restrictions in favor of a single untyped kind of variable.[19]

How should Hossack respond to the bad company problem? It is tempting to think that the problem arises only as a result of the generalizations I have been urging and that it can accordingly be avoided simply by refusing to generalize. This temptation is illusory, however, since the threat of paradox arises already for the kinds of abstraction that Hossack considers. To see this, recall that Hossack accepts unrestricted plural comprehension: any formula defines a plurality, provided there is at least one object that satisfies the formula. What about the well-ordered analogue of pluralities, namely series? One would expect a defender of

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[18] For this criticism, see Studd (2016) and (Linnebo, 2018, Sect. 3.2). For a recent survey of the options, see Cook and Linnebo (2018).

[19] See the passage quoted on p. 5, as well as ch. 2 in general.

unrestricted plural comprehension to be sympathetic also to unrestricted comprehension for series; that is, if a two-place formula defines a well-ordering without repetitions, then the formula defines as a series. But as Hossack is well aware, this form of comprehension would blow up his theory by allowing a version of the Burali-Forti paradox. By unrestricted series comprehensions, there would be a series of all the ordinals. By Hossack's theory, this series would define an ordinal, which would have to be greater than all the ordinals—including itself. This paradox is a special instance of the bad company problem that arises independently of the generalizations I have been urging.

Hossack's response is to impose restrictions on what series there can be; in particular, he denies that there can be a series of all ordinals. The only series there are, he argues, are those that are defined by *recursive* well-orderings. This response seems to me both unacceptably restrictive and ultimately *ad hoc*. To substantiate the former charge, consider all the real numbers. By the axiom of choice, this uncountable plurality can be well ordered. Why can't there be serial reference to these numbers in that order? This form of reference makes just as much sense, it seems to me, as plural reference to that uncountable lot of numbers, which Hossack accepts.

The *ad hoc* character of Hossack's response to the Burali-Forti paradox emerges when we compare it with his response to other paradoxes of naive set theory. Hossack responds to Russell's paradox by appealing to NFU—a version of Quine's New Foundations adapted so as to accommodate urelements. This means that there is absolutely no connection between his responses to the two paradoxes. The restriction to recursive well-orderings, which forms the heart of his response to the Burali-Forti paradox, plays no role whatsoever in his response to Russell's paradox.

As we have seen, Hossack's approach to numbers and magnitudes is heavily inspired by Aristotle. This makes it interesting, I think, to observe that my own response to the bad company can be seen as developing yet another Aristotelian idea, namely that magnitudes and generalizations thereof are *dependent entities*, which need to be accounted for in a bottomup manner. This is a central theme of my recent book Linnebo (2018). The existence of magnitudes and other abstracta is explained in terms of their instances; and their properties and relations are explained in terms of corresponding properties and relations among the instances, namely by developing the "inheritance" ideas adumbrated in the previous section.

This results in a hierarchical conception of numbers and other mathematical entities, where entities higher up in the hierarchy depend in various ways on ones lower down.

This hierarchical conception informs my view of what should be accepted as permissible plural and serial reference. If we are serious about the hierarchy, I argue, every form of reference must take place at some stage or other of the hierarchy. Thus, to refer plurally or serially to some things, all of these things need to be present at some stage or other. This in turn means that each plurality or series needs to be bounded by some stage or other. We thus arrive a uniform and moderately liberal view of both forms of reference, which contrasts with Hossack's unhappy combination of an extremely liberal form of plural reference and a severely restricted form of serial reference. This moderately liberal view can also be seen to guard against Russell's paradox and Burali-Forti's, again in a uniform way.

In sum, I have found that there is much to like about Hossack's neo-Aristotelian approach to numbers and magnitudes. But I have argued there is a push to generalize further, which introduces many of the problems discussed in the neo-Fregean tradition. Finally, I have suggested that these problems can be addressed by seeking inspiration from yet another Aristotelian idea, namely that numbers and other mathematical objects are dependent entities.

References

Boolos, G. (1984). To be is to be a value of a variable (or to be some values of some variables). *Journal of Philosophy*, 81(8):430–449.

Cook, R. T. and Linnebo, Ø. (2018). Cardinality and acceptable abstraction. *Notre Dame Journal of Formal Logic*, 59(1):61–74.

Field, H. (1989). *Realism, Mathematics, and Modality*. Blackwell, Oxford.

Fine, K. (2010). Towards a theory of part. *Journal of Philosophy*, 107(11):559–589.

Florio, S. and Linnebo, Ø. (forthcoming). *The Many and the One: A Philosophical Study*. Oxford University Press, Oxford.

Hewitt, S. (2012). The logic of finite order. *Notre Dame Journal of Formal Logic*, 53(3):297–318.

Hossack, K. (2000). Plurals and complexes. *British Journal for the Philosophy of Science*, 51(3):411–443.

Kitcher, P. (1978). The plight of the platonist. *Nouˆs*, 12:119–136.

Krantz, D., Luce, D., Suppes, P., and Tversky, A. (1971). *Foundations of Measurement, Vol. I: Additive and Polynomial Representations*. New York Academic Press.

Linnebo, Ø. (2010). Pluralities and sets. *Journal of Philosophy*, 107(3):144–164.

Linnebo, Ø. (2017). Plural quantification. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2017 edition.

Linnebo, Ø. (2018). *Thin Objects: An Abstractionist Account*. Oxford University Press, Oxford.

Peacocke, C. (2019). *The Primacy of Metaphysics*. Oxford University Press, Oxford.

Studd, J. P. (2016). Abstraction reconceived. *British Journal for the Philosophy of Science*, 67(2):579–615.

Suppes, P. (1951). A set of independent axioms for extensive quantities. *Portugaliae Mathematica*, 10(4):163–172.

Wright, C. (2000). Neo-Fregean foundations for analysis: Some reflections. *Notre Dame Journal of Formal Logic*, 41(4):317–34.