On the Necessity of Horizons

Disputing the Existence of Hawking Radiation from Horizon-Less Objects

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Thesis submitted for the degree of
Master in Theoretical Physics
60 credits

Department of Physics
Faculty of Mathematics and Natural Sciences

UNIVERSITY OF OSLO

Spring 2021

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http://www.duo.uio.no/

Printed: Reprosentralen, University of Oslo

Abstract

As an attempt to complete black hole thermodynamics, Stephen Hawking showed in 1974 that black holes emit thermal radiation [1]. This result has since been vastly disputed, both because it violates the classical notion that black holes are regions of spacetime from which nothing can escape, but also because it leads to a paradox: The thermal radiation which causes the black hole to evaporate contains no information about the black hole. Thus, information appears to be lost – despite the deterministic nature of the physical theories describing the phenomenon [2]. As an attempt to solve this paradox, some researchers have suggested that horizon-formation is avoided in stellar collapse models due to the presence of a so-called pre-Hawking radiation [3, 4]. This proposal is based on claims that Hawking-like radiation also occurs in collapse models where a horizon never forms [5–8]. It has further been proposed that this radiation may prevent black holes from forming at all [9]. On the other hand, others claim that such a radiation is too weak to play a crucial role in the course of stellar collapse and that its existence leads to serious physical inconsistencies [10–13]. The question of whether a horizon is needed in order for Hawking radiation to occur therefore seems to be at the very heart of this discussion. This further seems to be closely related to the questions of where and when the Hawking particles are created. Because of the global nature of event horizons these latter concerns are intrinsically difficult to address. We do not hope to solve this problem in this thesis. Rather, we aim to narrow down the discussion to its essentials. In the attempt to do just that, we study the possibility of Hawking radiation from a subclass of exotic compact objects (ECOs) which look exactly like black holes to distant observers [14]. From our highly idealised models we do not find evidence of Hawking radiation from horizonless objects. However, more advanced and realistic models are needed in order to properly conclude that event horizons are necessary for particle creation, and thus the dispute remains unresolved.

Acknowledgements

There are many people which deserve to be thanked properly for their contributions to the completion of this thesis. First and foremost, I owe to express my gratitude to my supervisor, Torsten, for invaluable aid, inspiring and instructive discussions and for believing in me throughout the work on this thesis. I should further thank all my friends and family for their high spirits and moral support. Specifically, I would like to thank Sunniva and Sara for keeping me social, making me laugh and for listening to all my ideas and dreams that never get realized. Also, I want to send my gratitude to my flatmates, Inga, Emilie, Erlend and Jake, for making my days a little brighter.

Further, I want to thank Argument for inspiring me to write more often, Lillefy for being a social and great place to study during my bachelor's degree, Blindern studenterhjem for introducing me to the student life of Oslo and all of my fellow students that have made these seven years at the University of Oslo a superb experience. It has been a ride.

Lastly, I want to thank Erlend for being the greatest support and company during the course of this year, for working side by side with me on your own thesis and for discussing many of the physical and mathematical properties of this thesis with me. Getting through this thesis would have been considerably more difficult without you by my side.

Contents

Chapter 1

Introduction

A black hole is a region of spacetime from which nothing can escape – not even light. It was therefore rather surprising when Stephen Hawking proposed in 1974 that black holes radiate particles [1, 15]. This thermal radiation of particles, which today is widely known as Hawking radiation, was found to arise in semi-classical models of black hole formation from stellar collapse. Being a semi-classical quantum effect, the controversy between the well-established theory of black holes in general relativity and the possibility of black hole radiation from quantum field theory has since inspired numerous researchers to investigate the theoretical framework around the phenomenon of black hole particle creation. This has led to many different calculations of Hawking radiation, and a deeper insight into its more fundamental nature; already the year after Hawking's proposal, William Unruh showed that the Hawking effect is not merely a gravitational artifact by demonstrating that also flat spacetimes constitute a similar radiation – a phenomenon now called the Unruh effect [16]. The fundamentality of the Hawking effect is further supported by results from analogue models of general relativity, such as acoustic models of trans-sonic fluid flow [17–19].

Assuming that Hawking radiation exists does, however, also lead to serious physical inconsistencies. Ranging highest in popularity is the so-called *information loss paradox*: If black holes radiate thermally, leading to black hole evaporation, information will be lost. Yet, because quantum mechanics postulate unitary time evolution and general relativity obeys causality and energy-momentum conservation, such a loss of information simply *cannot* take place. We thus encounter a paradox [2, 20].

As an attempt to solve this paradox it has been proposed that a Hawking-like radiation prevents the formation of an event horizon in models of stellar collapse [3, 4] – a result which may further suggest that black holes do not exist [9]. The main assumption upon which these results rest is part of a long-standing and still unresolved dispute on the necessity of horizons in the calculations of Hawking radiation. From the ongoing discussion it seems like the question of whether horizons are necessary for Hawking radiation is closely related to the question of *where* and *when* the Hawking particles are created. In the original derivation of Hawking radiation and much of the well-established literature on the topic, the entire history of the spacetime is needed in order to obtain particle production [15, 21, 22]. However, as pointed out by Visser in his attempt to boil the discussion down to its essentials, one should be "a little alarmed if the question of whether or not a black hole is radiating *now* depends on what it is doing in the infinite future" [18]. Hence, if the generation of particles starts before the star has collapsed to a black hole, it seems rather puzzling that its existence depends on whether an event horizon forms in the future. This argument has led some researchers to conjecture that only the existence of a locally defined *apparent* horizon is necessary to obtain particle creation [18, 23]. Others further suggest that no horizon of any kind is needed for particles to be created [6, 7, 24, 25], and that there exists a so-called *pre-Hawking radiation* which may alter the collapse of the star [3, 4, 9]. In spite of claiming interesting phenomena, it has been argued that the above-mentioned consequences of a pre-Hawking radiation lead to physical fallacies. Particularly, it has been claimed that the stress-energy tensor of such a radiation is far too small to alter stellar collapse and further that this radiation leads to tachyonic collapse velocities [10–13].

Despite numerous theoretical derivations of Hawking radiation, its existence has not yet been verified by observations. Moreover, distinguishing event horizons from very compact objects through measurements is intrinsically difficult [26]. Hence, if Hawking radiation is found to exist by experimental detection, it is still not clear whether the radiation actually comes from a black hole or from an object arbitrarily close to the Schwarzschild radius, which looks like a black hole to distant observers [3, 4, 14]. Nevertheless, with the discovery of gravitational waves and recent advancements in observations within the field, a fair possibility to measure both the existence of Hawking radiation and the imprint of a horizon or a near-horizon surface, which will serve useful to separate the former from the latter, now seems to be at hand [27, 28].

In the strive to understand the bewilderment of whether or not horizons are important for Hawking radiation, the aim of this thesis is to narrow down the discussion in order to get a better idea of its essentials. We will do this by having a closer look at the need for horizons in the original derivation of Hawking radiation, and investigate whether a similar argumentation can be used to obtain particle creation in different horizon-less scenarios. The outline of this thesis will therefore be as follows: Starting out with a condensed introduction of the main properties and consequences of quantum field theory in curved spacetime in chapter 2, we will in chapter 3 continue by presenting the derivations of Hawking radiation in the spacetime of an eternal Schwarzschild black hole and in the spacetime of a star collapsing to form a Schwarzschild black hole. Motivated by the discussion of Hawking radiation from horizonless objects and recent discoveries pointing in the direction of the possibility of Hawking radiation from exotic compact objects (ECOs), chapter 4 will be devoted to studying special classes of stellar collapse forming ECOs and static ECOs. As we will see, little evidence will be found for the existence of Hawking radiation in the studied horizon-less scenarios. In chapter 5 follows a discussion of what we have found and possible implications, and finally in chapter 6 we summarize and propose directions of further investigation.

We will mostly work in two dimensional spacetimes with metric signature $(+-)$, meaning that the time component is defined to have a positive sign in the metric, whereas the spatial component has a negative sign. Throughout this thesis we will also set $G = \hbar = c = 1$.

Chapter 2

Quantum Field Theory in Curved Spacetime

Quantum field theory in curved spacetime may be considered as a linear order approach to a unification of quantum field theory and general relativity. In general terms, this theoretical investigation assumes that a metric on a given manifold can be split into a background metric, $g^{b}_{\mu\nu}$, and a small perturbation to this metric, $\bar{g}_{\mu\nu}$, as

$$g_{\mu\nu} = g^{b}_{\mu\nu} + \bar{g}_{\mu\nu}. \tag{2.1}$$

From Einstein's equations, which reveal how the curvature of spacetime reacts to the presence of matter and energy, the perturbation to the background metric can equally be interpreted as part of the energy-momentum tensor, $T_{\mu\nu}$. In this sense, a first order approach to quantum gravity can be introduced as an approach in which the arbitrary, classical background metric, $g^{b}_{\mu\nu}$, stays fixed and the perturbation, $\bar{g}_{\mu\nu}$, represented by the fields propagating on top of this background, is quantized. This quantization method, also known as the "background field" method, was first used by DeWitt [21].

For an n-dimensional spacetime that is infinitely differentiable, globally hyperbolic and pseudo-Riemannian, the corresponding line element can be expressed as

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}, \quad \mu, \nu = 0, 1, 2, ..., (n-1), \tag{2.2}$$

where $g_{\mu\nu}$ is the pseudo-Riemannian metric with signature $(+ - - \, ... \, -)$, where the plus sign is assigned to the time dimension and each minus sign is assigned to one of the $(n-1)$ spatial dimensions [21].

With such a general spacetime in mind, a natural starting point for introducing quantum field theory in curved spacetime is to observe how a scalar field evolves in the spacetime.

2.1 Scalar Field on a Curved Background

Since the motion of a scalar field can be traced back to the action,

$$S = \int d^n x \mathcal{L}, \tag{2.3}$$

and the action depends on the Lagrangian density, $\mathcal{L}$, one is first prompted to obtain a description of the Lagrangian density in order to describe the evolution of the scalar field. The Lagrangian density of a scalar field ϕ, propagating in a spacetime with metric $g_{\mu\nu}$, is given by

$$\mathcal{L} = \frac{1}{2}\sqrt{|g|}\left\{g^{\mu\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi - \left(m^2 + \xi R\right)\phi^2\right\}, \tag{2.4}$$

where $g \equiv \det g_{\mu\nu}$, m is the mass of the scalar field and R is the curvature scalar. The term $\xi R \phi^2$ governs the coupling between the scalar field and the gravitational field, parameterized by the constant ξ. This constant may take multiple values, but two numerical values are of special interest in the literature: the minimal coupling for $\xi = 0$, and the conformal coupling for $\xi = (n-2)/(4(n-1))$. The names are well suited, as setting $\xi = 0$ in Eq. (2.4) gives the Lagrangian density one obtains when using the principal of minimal coupling to generalize the theory in flat spacetime to curved spacetime, and the latter choice of ξ yields a conformally invariant action when the mass of the scalar field is zero. A conformal transformation is a transformation which shrinks or stretches spacetime in the sense that

$$\bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \tag{2.5}$$

for a non-vanishing function $\Omega(x)$.

Setting the variation of the action in Eq. (2.3) with respect to the field ϕ to zero, we obtain the field equation

$$\left(\Box + m^2 + \xi R\right)\phi = 0, \tag{2.6}$$

which is also known as the Klein–Gordon equation. The operator $\Box$ acting on the scalar field ϕ, is defined as

$$\Box\phi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{1}{\sqrt{|g|}}\partial_\mu\left[\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right]. \tag{2.7}$$

To get to the last equality we have used that the covariant divergence of a vector V^μ can be written as [29]

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}V^\mu\right), \tag{2.8}$$

together with metric compatibility and that partial derivatives and covariant derivatives coincide when acting on a scalar field.

We further define the scalar product between two scalar fields as

$$(\phi_1, \phi_2) = -i\int_\Sigma \sqrt{|g_\Sigma|}d\Sigma^\mu \phi_1 \overset{\leftrightarrow}{\partial}_\mu \phi_2^*, \tag{2.9}$$

where $d\Sigma^\mu$ is a future directed volume element orthogonal to the spacelike hypersurface Σ, and g_Σ is the determinant of the induced metric on the hypersurface. This product is independent of the choice of spacelike hypersurface Σ, as can be shown by making use of Gauss' theorem. Also,

$$\phi_1 \overset{\leftrightarrow}{\partial}_\mu \phi_2^* \equiv \phi_1\partial_\mu\phi_2^* - \phi_2^*\partial_\mu\phi_1. \tag{2.10}$$

Solving Eq. (2.6) we find that there exists a complete set of mode solutions $f_{\mathbf{k}}(x)$ orthonormal in the scalar product given in Eq. (2.9), so that

$$(f_{\mathbf{k}}, f_{\mathbf{k}'}) = \delta^{n-1}(\mathbf{k}-\mathbf{k}'); \quad (f_{\mathbf{k}}^*, f_{\mathbf{k}'}^*) = -\delta^{n-1}(\mathbf{k}-\mathbf{k}'); \quad (f_{\mathbf{k}}, f_{\mathbf{k}'}^*) = 0. \tag{2.11}$$

Here we have used a continuous normalization of the modes $f_{\mathbf{k}}$. A discrete description of the modes may be obtained by solving Eq. (2.6) in a box of dimensions L, and impose periodic boundary conditions on the field ϕ. Then the Dirac delta functions in Eq. (2.11) reduce to Kronecker delta functions and the integral $\int d^{n-1}k$ in Eq. (2.12) is substituted by the sum $(2\pi/L)^{n-1}\sum_{\mathbf{k}}$ instead. The integral is reattained from the sum by letting $L \to \infty$ [1]. This set of modes is complete, so we can write the field ϕ as

$$\phi = \int d^{n-1}k\left(a_{\mathbf{k}}f_{\mathbf{k}} + a_{\mathbf{k}}^\dagger f_{\mathbf{k}}^*\right). \tag{2.12}$$

Since we want to study a *quantum* field propagating on top of a classical spacetime, we need to impose equal time commutation relations for the coefficients $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$. Thus these coefficients must obey the following, equal time commutation relations

$$\left[a_{\mathbf{k}}, a_{\mathbf{k}'}\right] = 0; \quad \left[a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}^\dagger\right] = 0; \quad \left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger\right] = \delta^{n-1}(\mathbf{k}-\mathbf{k}'). \tag{2.13}$$

[1]In the literature it is common to introduce quantum field theory in curved spacetime with a discrete normalization of the field modes. However, I have chosen to use a continuum normalization in this introductory chapter in order to stay consistent with later calculations.

Additionally, there exists a vacuum state $|0_f\rangle$ which is annihilated by all operators $a_{\mathbf{k}}$ so that,

$$a_{\mathbf{k}} |0_f\rangle = 0, \ \forall \mathbf{k}. \tag{2.14}$$

From this vacuum state a complete Fock basis of the corresponding Hilbert space of possible quantum states can be created by repeated action of the operators $a_{\mathbf{k}}^{\dagger}$ on the vacuum state. The operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$ are interpreted as annihilation and creation operators, respectively, for the modes $f_{\mathbf{k}}$.

Generally, there is nothing unique with the set of modes $f_{\mathbf{k}}$. As a matter of fact, we are free to choose a different complete and orthonormal set of mode solutions to Eq. (2.6). Naming these modes $g_{\mathbf{k}}$, we may expand the scalar field ϕ in terms of these modes and their complex conjugates, as

$$\phi = \int d^{n-1}k \left(b_{\mathbf{k}} g_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger} g_{\mathbf{k}}^{*} \right). \tag{2.15}$$

As in the previous case, the coefficients $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$ are interpreted as annihilation and creation operators for the $g_{\mathbf{k}}$-modes, and satisfy the commutation relations

$$\left[b_{\mathbf{k}}, b_{\mathbf{k}'} \right] = 0; \ \left[b_{\mathbf{k}}^{\dagger}, b_{\mathbf{k}'}^{\dagger} \right] = 0; \ \left[b_{\mathbf{k}}, b_{\mathbf{k}'}^{\dagger} \right] = \delta^{n-1}(\mathbf{k} - \mathbf{k}'). \tag{2.16}$$

Moreover, the vacuum state $|0_g\rangle$ is defined such that

$$b_{\mathbf{k}} |0_g\rangle = 0, \ \forall \mathbf{k}, \tag{2.17}$$

and repeated action by $b_{\mathbf{k}}^{\dagger}$ yields an entire Fock basis for the *same* Hilbert space as before.

Since the set $\{f_{\mathbf{k}}\}$ serves as a complete set of solutions to Eq. (2.6), we may express the modes $g_{\mathbf{k}}$ as linear combinations of these. We thus write

$$g_{\mathbf{k}} = \int d^{n-1}k' \left(\alpha_{\mathbf{k}\mathbf{k}'} f_{\mathbf{k}'} + \beta_{\mathbf{k}\mathbf{k}'} f_{\mathbf{k}'}^{*} \right), \tag{2.18}$$

with some complex coefficients $\alpha_{\mathbf{k}\mathbf{k}'}$ and $\beta_{\mathbf{k}\mathbf{k}'}$. Using Eq. (2.18) and the orthonormality of the modes $f_{\mathbf{k}}$ in Eq. (2.11), we may continue by observing that the coefficients $\alpha_{\mathbf{k}\mathbf{k}'}$ and $\beta_{\mathbf{k}\mathbf{k}'}$ can be obtained by computing the scalar products

$$\begin{aligned} (g_{\mathbf{k}}, f_{\mathbf{k}'}) &= \int d^{n-1}k'' \left\{ \alpha_{\mathbf{k}\mathbf{k}''} (f_{\mathbf{k}''}, f_{\mathbf{k}'}) + \beta_{\mathbf{k}\mathbf{k}''} (f_{\mathbf{k}''}^{*}, f_{\mathbf{k}'}) \right\} = \alpha_{\mathbf{k}\mathbf{k}'}, \\ (g_{\mathbf{k}}, f_{\mathbf{k}'}^{*}) &= \int d^{n-1}k'' \left\{ \alpha_{\mathbf{k}\mathbf{k}''} (f_{\mathbf{k}''}, f_{\mathbf{k}'}^{*}) + \beta_{\mathbf{k}\mathbf{k}''} (f_{\mathbf{k}''}^{*}, f_{\mathbf{k}'}^{*}) \right\} = -\beta_{\mathbf{k}\mathbf{k}'}. \end{aligned} \tag{2.19}$$

Because of the orthonormality in the sets of modes $g_{\mathbf{k}}$ and $f_{\mathbf{k}}$, as well as the anti-linearity of the second argument in the scalar product defined in Eq. (2.9), the coefficients $\alpha_{\mathbf{k}\mathbf{k}'}$ and $\beta_{\mathbf{k}\mathbf{k}'}$ must obey the following normalization conditions:

$$\begin{aligned} &\int d^{n-1}k' \left(\alpha_{\mathbf{k}\mathbf{k}'} \alpha_{\mathbf{k}''\mathbf{k}'}^{*} - \beta_{\mathbf{k}\mathbf{k}'} \beta_{\mathbf{k}''\mathbf{k}'}^{*} \right) = \delta^{n-1}(\mathbf{k}'' - \mathbf{k}), \\ &\int d^{n-1}k' \left(\alpha_{\mathbf{k}\mathbf{k}'} \beta_{\mathbf{k}''\mathbf{k}'} - \beta_{\mathbf{k}\mathbf{k}'} \alpha_{\mathbf{k}''\mathbf{k}'} \right) = 0. \end{aligned} \tag{2.20}$$

From this we see that we can express the modes $f_{\mathbf{k}}$ in terms of the modes $g_{\mathbf{k}}$ in a similar way, namely as

$$f_{\mathbf{k}} = \int d^{n-1}k' \left(\alpha_{\mathbf{k}'\mathbf{k}}^{*} g_{\mathbf{k}'} - \beta_{\mathbf{k}'\mathbf{k}} g_{\mathbf{k}'}^{*} \right). \tag{2.21}$$

The relations given in Eq. (2.18) and Eq. (2.21) are called Bogolubov transformations. Naturally, then, the coefficients $\alpha_{\mathbf{k}\mathbf{k}'}$ and $\beta_{\mathbf{k}\mathbf{k}'}$ are called Bogolubov coefficients [21, 29].

Equating Eq. (2.12) and Eq. (2.15), taking the inner product with $g_{\mathbf{k}}$ on both sides and making use of the orthonormality in the modes $g_{\mathbf{k}}$, we obtain the following expression for the operator $b_{\mathbf{k}}$,

$$\begin{aligned} b_{\mathbf{k}} &= \int d^{n-1}k' \left\{ a_{\mathbf{k}'}(f_{\mathbf{k}'}, g_{\mathbf{k}}) + a^{\dagger}_{\mathbf{k}'}(f^{*}_{\mathbf{k}'}, g_{\mathbf{k}}) \right\} \\ &= \int d^{n-1}k' \left(\alpha^{*}_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}'} - \beta^{*}_{\mathbf{k}\mathbf{k}'} a^{\dagger}_{\mathbf{k}'} \right), \end{aligned} \tag{2.22}$$

where we in the last step have made use of Eq. (2.21). Likewise, the operator $a_{\mathbf{k}}$ can be expressed in terms of $b_{\mathbf{k}}$ as

$$a_{\mathbf{k}} = \int d^{n-1}k' \left(\alpha_{\mathbf{k}'\mathbf{k}} b_{\mathbf{k}'} + \beta^{*}_{\mathbf{k}'\mathbf{k}} b^{\dagger}_{\mathbf{k}'} \right). \tag{2.23}$$

Let us now assume that two distinct static observers use the modes $g_{\mathbf{k}}$ and $f_{\mathbf{k}}$ respectively to describe the same Hilbert space. For the observer using the modes $f_{\mathbf{k}}$, the expectation value of $f_{\mathbf{k}}$-mode particles in the vacuum state $|0_f\rangle$ is

$$\langle 0_f | a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} | 0_f \rangle = 0, \quad \forall \mathbf{k}, \tag{2.24}$$

as readily follows from Eq. (2.14). The observer with the modes $g_{\mathbf{k}}$ and the operators $b_{\mathbf{k}}$ and $b^{\dagger}_{\mathbf{k}}$, however, measures that the number of $g_{\mathbf{k}}$-mode particles in the state $|0_f\rangle$ is

$$\begin{aligned} \langle 0_f | b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} | 0_f \rangle &= \int d^{n-1}k' d^{n-1}k'' \, \langle 0_f | \left(\alpha_{\mathbf{k}\mathbf{k}'} a^{\dagger}_{\mathbf{k}} - \beta_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}} \right) \left(\alpha^{*}_{\mathbf{k}\mathbf{k}''} a_{\mathbf{k}''} - \beta^{*}_{\mathbf{k}\mathbf{k}''} a^{\dagger}_{\mathbf{k}''} \right) | 0_f \rangle \\ &= \int d^{n-1}k' d^{n-1}k'' \beta_{\mathbf{k}\mathbf{k}'} \beta^{*}_{\mathbf{k}\mathbf{k}''} \, \langle 0_f | a_{\mathbf{k}'} a^{\dagger}_{\mathbf{k}''} | 0_f \rangle \\ &= \int d^{n-1}k' |\beta_{\mathbf{k}\mathbf{k}'}|^2, \end{aligned} \tag{2.25}$$

where we have used Eq. (2.22) and the relation

$$a_{\mathbf{k}'} a^{\dagger}_{\mathbf{k}''} - a^{\dagger}_{\mathbf{k}''} a_{\mathbf{k}'} = \delta^{n-1}(\mathbf{k}' - \mathbf{k}''). \tag{2.26}$$

Consequently, instead of a vacuum state devoid of particles, the stationary observer using the modes $g_{\mathbf{k}}$ measures that there are

$$\langle 0_f | b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} | 0_f \rangle = \int d^{n-1}k' |\beta_{\mathbf{k}\mathbf{k}'}|^2 \tag{2.27}$$

$g_{\mathbf{k}}$-mode particles in the vacuum state $|0_f\rangle$. In general, $\beta_{\mathbf{k}\mathbf{k}'} \neq 0$, so the state $|0_f\rangle$ does not appear vacuous to this observer.

At this point it may be suitable to stop and ponder: How can observers disagree on the number of particles in a given state? The answer seems to be hidden in our understanding of the *concept of particles*.

2.2 The Meaning of Vacuum and Particles

The fact that different observers can measure an unequal amount of particles in a given state is not some special feature of curved spacetimes: Up to this very point we have kept the discussion quite general. Nothing stops us from choosing the metric in Eq. (2.2) to describe flat spacetime, and the above derivation of vacuum expectation values for different observers holds in flat spacetimes as well.

Nevertheless, flat spacetimes conduct one special feature that cannot be generalized to curved spacetimes: The existence of a timelike Killing vector. This single property ensures that the concept of vacuum and particles is well-defined in flat spacetimes. We will illustrate this for Minkowski spacetime in the following.

A set of solutions to Eq. (2.6) in four-dimensional Minkowski spacetime (setting $\xi = 0$) takes the form

$$f_{\mathbf{k}} \propto e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}. \tag{2.28}$$

In this spacetime we can always find coordinates in which the metric is independent of the time coordinate, so there exists a timelike Killing vector, ∂_t, throughout this spacetime. Insisting that the angular frequency, ω, of the mode oscillations always stays positive, the mode solutions in Eq. (2.28) can be divided into so-called positive- and negative frequency modes [2] defined respectively as

$$\begin{aligned} \partial_t f_{\mathbf{k}} &= -i\omega f_{\mathbf{k}}, \qquad \omega > 0, \\ \partial_t f_{\mathbf{k}}^* &= i\omega f_{\mathbf{k}}^*, \qquad \omega > 0. \end{aligned} \tag{2.29}$$

Performing a Lorentz transformation or a translation on the frame in which Eq. (2.29) is defined, will not change the signs of the positive- and negative frequency modes. This can be illustrated by a boost of Eq. (2.29) into a frame of constant velocity $\mathbf{v}$. Then the spacetime coordinate x^μ is boosted into a new spacetime coordinate x'^μ, and the boosted time derivative takes the form

$$\partial_{t'} = \frac{\partial x^\mu}{\partial t'}\partial_\mu = \gamma(1+\mathbf{v})\partial_\mu. \tag{2.30}$$

Here we have used the inverse Lorentz transformations $t = \gamma(t' + \mathbf{v}\cdot\mathbf{x}')$ and $\mathbf{x} = \gamma(\mathbf{x}' + \mathbf{v}t')$ with $\gamma = 1/\sqrt{1-v^2}$. Thus the boosted version of Eq. (2.29) is

$$\begin{aligned} \partial_{t'} f_{\mathbf{k}} &= \frac{\partial x^\mu}{\partial t'}\partial_\mu f_{\mathbf{k}} = -i\omega' f_{\mathbf{k}}, \quad \omega' > 0, \\ \partial_t f_{\mathbf{k}}^* &= \frac{\partial x^\mu}{\partial t'}\partial_\mu f_{\mathbf{k}}^* = i\omega' f_{\mathbf{k}}^*, \qquad \omega' > 0, \end{aligned} \tag{2.31}$$

with boosted frequency $\omega' \equiv \gamma(\omega - \mathbf{v}\cdot\mathbf{k})$.

Hence, in this spacetime there exists a natural set of inertial observers, related to each other through actions of the Poincaré group (Lorentz transformations and translations), which all agree on the division into positive- and negative frequency modes. Going back to the calculations earlier, then, in order for the modes $f_{\mathbf{k}}$ and $g_{\mathbf{k}}$ to both satisfy Eq. (2.29), they cannot be described by admixtures of creation and annihilation operators. Hence the coefficient $\beta_{\mathbf{k}\mathbf{k}'}$ in Eq. (2.18) and Eq. (2.21) must vanish and the vacuum state $|0_f\rangle$ of the observer with the modes $f_{\mathbf{k}}$ coincides with the vacuum state of the observer with basis modes $g_{\mathbf{k}}$.

Because the metric of curved spacetimes cannot generally be expressed in a way such that it is independent of the time coordinate, it is normally not possible to find a timelike Killing vector – from which we can define positive and negative frequency modes – all over the spacetime. Thus, in curved spacetimes, a natural set of inertial observers agreeing on what is vacuum can not, in general, be found. When going from flat spacetimes to curved spacetimes we have lost the reason to prefer any set of observers to the others.

This does not, of course, mean that the measurements from a given particle detector cannot be trusted. A particle detector following a specific trajectory will indeed measure the number of particles that it encounters along its trajectory. In fact, such a detector will define positive- and negative frequency modes with respect to the directional covariant derivative along its trajectory, using its proper time τ. Thus, according to this detector, positive frequency modes are all modes $f_{\mathbf{k}}$ that satisfy

$$\frac{dx^\mu}{d\tau}\nabla_\mu f_{\mathbf{k}} = -i\omega f_{\mathbf{k}}. \tag{2.32}$$

Particles observed by this detector are perfectly well-defined. As a matter of fact, Eq. (2.32) is just a generalized version of the first equation in Eq. (2.29). In other words, the above statement is just a

[2] Despite their names, these modes do *not* have negative or positive frequencies (we do indeed demand $\omega > 0$). Rather, their names correspond to the sign of the eigenvalue of the time derivative-operator acting on the modes $f_{\mathbf{k}}$. The time derivative of the negative frequency modes pulls down a factor $+i\omega$, as opposed to the minus sign which is common for standard planar waves.

general way of stating that particles are well-defined for all freely falling observers. Nevertheless, a different observer will not generally be able to find such modes $f_{\mathbf{k}}$ that satisfy Eq. (2.32).

There do, however, exist spacetimes for which a separation into positive- and negative frequency modes is well-defined throughout the spacetime. This can, for example, be done in a static spacetime. In such spacetimes there exists a hypersurface-orthogonal timelike Killing vector, K^{μ} [29]. From this Killing vector we are always able to find a set of coordinates $f_{\mathbf{k}}$ that satisfy the coordinate-invariant version of Eq. (2.29),

$$\begin{aligned} \mathcal{L}_K f_{\mathbf{k}} &= K^{\mu}\partial_{\mu} f_{\mathbf{k}} = -i\omega f_{\mathbf{k}}, \qquad \omega > 0, \\ \mathcal{L}_K f^*_{\mathbf{k}} &= K^{\mu}\partial_{\mu} f^*_{\mathbf{k}} = i\omega f^*_{\mathbf{k}}, \qquad \omega > 0. \end{aligned} \tag{2.33}$$

2.3 Gravitational Particle Production

Having discussed the meaning of particles and vacuum in general spacetimes, we are now ready to examine how to understand the disagreement on the number of particles in the state $|0_f\rangle$. A physical interpretation of the result in Eq. (2.27) is that the observer using the modes $g_{\mathbf{k}}$ actually measures a radiation of particles in the vacuum state of the observer using the modes $f_{\mathbf{k}}$. Exactly what gives rise to these particles physically, depends on the properties of the spacetimes in question. In particular, as we discussed in the previous section, the possibility to choose different sets of observers from which the concept of particles can be defined in a natural way, is related to the existence of timelike Killing vectors in the given spacetime. We call such sets of observers *natural*. In the following we will look at an example of how to interpret the radiation of particles for a specific class of spacetimes.

Assume that in a given spacetime there exist two asymptotic regions in the future and in the past that both contain a natural set of observers, from which a privileged vacuum state can be defined. We call these two regions the *in*- and *out*-region of the spacetime, and their corresponding vacuum states the *in*- and *out*-vacuum. In the *in*-region, the *in*-vacuum will remain devoid of particles to all natural observers in this region. Outside this region, however, freely-falling particle detectors will not in general measure that this state is devoid of particles. Specifically, this *in*-vacuum will not coincide with the *out*-vacuum, which is devoid of particles as observed by the natural set of observers in the *out*-region. Despite having made no assumptions on the region of spacetime in between the *in*- and *out*-region, we may say that the particles observed in the *in*-vacuum by natural observers in the *out*-region, comes from the changing gravitational field of the spacetime [21].

By the equivalence principle, we should therefore expect that an accelerated observer in flat spacetime measures a radiation of particles in the vacuum state of an inertial observer. As we will demonstrate in the following, this is indeed the case. For simplicity, this demonstration will be restricted to a class of uniformly accelerated observers in two-dimensional Minkowski spacetime.

An inertial observer in a two-dimensional Minkowski spacetime will have a line element of the form

$$ds^2 = dt^2 - dx^2. \tag{2.34}$$

In two dimensions, minimal- and conformal coupling coincide. Hence, we need not worry about the coupling between the curvature scalar and the scalar field ϕ in Eq. (2.6), as this term vanishes anyway. Thus, since we are considering a *massless* scalar field, we are looking for solutions of the differential equation

$$\Box\phi = \left(\partial_t^2 - \partial_x^2\right)\phi = 0. \tag{2.35}$$

A set of solutions to this differential equation consists of orthonormal mode solutions

$$f_k = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega t + ikx}, \tag{2.36}$$

which we can combine to form the following expression for the field ϕ:

$$\phi = \int dk \, \left(a_k f_k + a_k^{\dagger} f_k^*\right). \tag{2.37}$$

Consequently, the vacuum state for inertial observers in Minkowski space is defined as the state satisfying

$$a_k |0_M\rangle = 0 \quad \forall k. \tag{2.38}$$

An observer moving with a constant acceleration, α, in the x-direction of Minkowski spacetime, will follow a hyperbolic trajectory parameterized by

$$x^2 - t^2 = \frac{1}{\alpha^2}. \tag{2.39}$$

This equation is solved by hyperbolic functions $t = \alpha^{-1}\sinh \alpha\tau$ and $x = \alpha^{-1}\cosh \alpha\tau$. By inserting these hyperbolic expressions for x and t into the metric given in Eq. (2.34), we see that the parameter τ is actually the proper time of the accelerating observer. Defining coordinates (η, ξ) such that

$$\alpha = ae^{-a\xi}; \qquad \tau = e^{a\xi}\eta, \tag{2.40}$$

with $-\infty < \eta, \xi < \infty$, the coordinates t and x satisfying Eq. (2.39) can be recast as

$$\begin{aligned} t &= \frac{1}{a} e^{a\xi} \sinh a\eta \\ x &= \frac{1}{a} e^{a\xi} \cosh a\eta. \end{aligned} \tag{2.41}$$

Here we have chosen the solution of Eq. (2.39) with $x > |t|$. The other solution, with $x < |t|$, is obtained by flipping the signs of x and t in Eq. (2.41).

In terms of these coordinates, the line element in Eq. (2.34) takes the form

$$ds^2 = e^{2a\xi}\left(d\eta^2 - d\xi^2\right). \tag{2.42}$$

Minkowski spacetime expressed in terms of the coordinates (η, ξ) defined in Eq. (2.40) is known as Rindler space. An observer moving along a path of constant acceleration in this spacetime is called a Rindler observer. Such an observer follows a path satisfying $\xi =$ constant, which corresponds to a hyperbola. Paths of constant η are straight lines in this spacetime. The causal structure of Minkowski spacetime expressed in Rindler coordinates is depicted in the Minkowski diagram in figure 2.1. Following the path of a Rindler observer throughout this spacetime, we may notice that the observer's acceleration is confined by a Cauchy horizon[3] both in the future and in the past. These horizons are denoted H^+ and H^- in figure 2.1, and correspond to the lines $x = t$ in inertial coordinates. According to an inertial Minkowski observer, Rindler observers thus approach the speed of light as $\eta \to \pm\infty$.

Making use of the relation for the action of the d'Alembertian on a scalar field given in Eq. (2.7), we obtain the following wave equation expressed in Rindler coordinates:

$$\Box\phi = e^{-2a\xi}\left(\partial_\eta^2 - \partial_\xi^2\right)\phi = 0. \tag{2.43}$$

Since the metric in Eq. (2.42) is just a conformal transformation of the metric of Minkowski spacetime given in Eq. (2.34), a normalized set of solutions to the Klein-Gordon equation, Eq. (2.43), consists of plane waves of the form

$$g_k = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi}, \tag{2.44}$$

with $\omega = |k| > 0$ and $-\infty < k < \infty$.

From Eq. (2.42) we may observe that the metric of Rindler space is independent of the time coordinate η. Hence, the operator ∂_η is a timelike Killing vector in this spacetime, which we can use to define positive- and negative frequency modes. However, this Killing vector only points in the

[3]A Cauchy horizon marks the boundary of all future or all past causally connected events inside a given domain of the spacetime in question. Unlike an event horizon, a Cauchy horizon does not mark a surface beyond which timelike curves cannot escape to infinity [29].

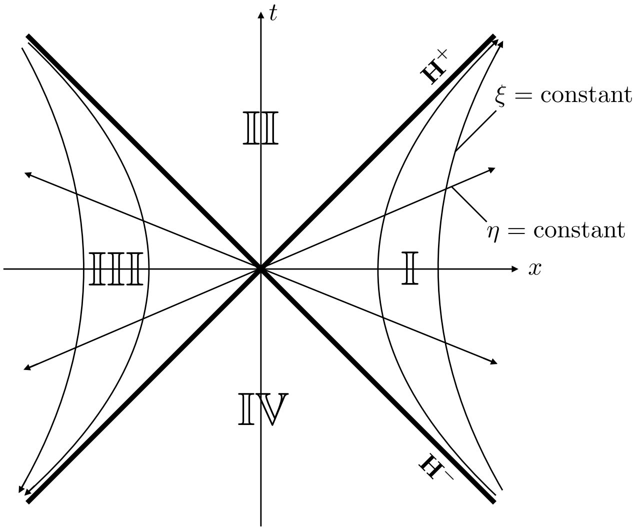


Figure 2.1: Rindler space. Curves of constant η are straight lines and curves of constant ξ are hyperbolas. Region I corresponds to the coordinate transformation given in Eq. (2.41), whereas flipping the sign of the coordinates x and t in the same transformation gives region II. All Rindler observers are asymptotic to the future- and past Cauchy horizons H^+ and H^-.

future timelike direction in region I in Eq. (2.1). In region IV, this vector points in the past timelike direction. A future timelike Killing vector in region IV is thus $\partial_{-\eta} = -\partial_\eta$. In other words, we need to divide the solutions in Eq. (2.44) into two sets of solutions, one that corresponds to positive frequency modes in region I, and another that corresponds to positive frequency modes in region IV. Thus we write the mode solutions as

$$\begin{aligned} g_k^{(1)} &= \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi} & \mathrm{I} \\ 0 & \mathrm{IV} \end{cases}, \\ g_k^{(2)} &= \begin{cases} 0 & \mathrm{I} \\ \frac{1}{\sqrt{4\pi\omega}} e^{i\omega\eta + ik\xi} & \mathrm{IV} \end{cases}. \end{aligned} \tag{2.45}$$

It is important to note that the coordinates (η, ξ) used in Eq. (2.45) cannot, strictly speaking, be used simultaneously for both region I and region IV, since they range from $-\infty$ to ∞ in both cases. However, because we want Eq. (2.42) to apply in both regions, we keep this notation – but make an explicit distinction between the two regions, when needed.

The two sets of mode solutions in Eq. (2.45) form, together with their complex conjugates, a complete set of basis modes for the Hilbert space of solutions to Eq. (2.43). This is due to the fact that these modes can be analytically extended into the spacelike regions II and III in figure 2.1. Hence, we may expand the field ϕ as

$$\phi = \int dk \left(b_k^{(1)} g_k^{(1)} + b_k^{(1)\dagger} g_k^{(1)*} + b_k^{(2)} g_k^{(2)} + b_k^{(2)\dagger} g_k^{(2)*} \right). \tag{2.46}$$

The vacuum state corresponding to this expansion satisfies

$$b_k^{(1)} |0_R\rangle = b_k^{(2)} |0_R\rangle = 0 \quad \forall k. \tag{2.47}$$

This vacuum state does not coincide with the vacuum state $|0_M\rangle$ given in Eq. (2.38) for an inertial Minkowski observer. To see this, follow the surface $t = 0$ in figure 2.1 and observe from Eq. (2.45) that the modes $g_k^{(1)}$ are non-zero only for $x > 0$. Thus we cannot expand these modes in terms of only positive-frequency Minkowski modes all over Minkowski space. The annihilation operator

$b_k^{(1)}$ must therefore be an admixture of the inertial Minkowski observer's annihilation and creation operators, corresponding to a non-zero $\beta_{kk'}$-coefficient in Eq. (2.22). Hence, the vacuum expectation value of these Rindler modes in the Minkowski vacuum is non-zero, as we can see by comparison with Eq. (2.27). A similar argument can be used for the modes $g_k^{(2)}$ and their corresponding annihilation operator, $b_k^{(2)}$.

Now we may continue by the same reasoning as presented in the previous section and find the Bogolubov coefficients connecting the Minkowski modes to the Rindler modes, and then calculate the vacuum expectation value for the Rindler observer in the Minkowski vacuum. However, a different, less cumbersome approach can be found by realizing that we can create two linear combinations of the modes in Eq. (2.45) which share the same vacuum state as the inertial Minkowski observer. These modes take the form [29]

$$\begin{aligned} h_k^{(1)} &= \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}}\left(e^{\pi\omega/2a}g_k^{(1)} + e^{-\pi\omega/2a}g_{-k}^{(2)*}\right), \\ h_k^{(2)} &= \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}}\left(e^{\pi\omega/2a}g_k^{(2)} + e^{-\pi\omega/2a}g_{-k}^{(1)*}\right). \end{aligned} \tag{2.48}$$

Again, we may expand the scalar field ϕ in terms of these basis modes as

$$\phi = \int dk\ \left(c_k^{(1)}h_k^{(1)} + c_k^{(1)\dagger}h_k^{(1)*} + c_k^{(2)}h_k^{(2)} + c_k^{(2)\dagger}h_k^{(2)*}\right), \tag{2.49}$$

where now

$$c_k^{(1)}|0_M\rangle = c_k^{(2)}|0_M\rangle = 0 \quad \forall k. \tag{2.50}$$

This latter statement, that the observer using the modes in Eq. (2.48) share vacuum state with inertial Minkowski observers, comes from the fact that these modes can be fully expressed in terms of positive-frequency Minkowski modes. By writing the $g_k^{(1,2)}$-modes in Eq. (2.48) in terms of the coordinates x and t, this becomes evident.

In the same way as we obtained the relations given in Eq. (2.22) and Eq. (2.23) in the previous section, we may now write the operators $b_k^{(1)}$ and $b_k^{(2)}$ in terms of the operators $c_k^{(1)}$ and $c_k^{(2)}$ by equating Eq. (2.46) with Eq. (2.49) and taking the inner products with the modes $g_k^{(1)}$ and $g_k^{(2)}$, respectively. This yields the relations

$$\begin{aligned} b_k^{(1)} &= \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}}\left(e^{\pi\omega/2a}c_k^{(1)} + e^{-\pi\omega/2a}c_{-k}^{(2)*}\right), \\ b_k^{(2)} &= \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}}\left(e^{\pi\omega/2a}c_k^{(2)} + e^{-\pi\omega/2a}c_{-k}^{(1)*}\right). \end{aligned} \tag{2.51}$$

A Rindler observer in region I will thus measure that the number of particles in mode k is

$$\begin{aligned} \langle 0_M| b_k^{(1)\dagger}b_k^{(1)}|0_M\rangle &= \frac{1}{2\sinh\frac{\pi\omega}{a}}\langle 0_M| e^{-\pi\omega/a}c_{-k}^{(1)}c_{-k}^{(1)\dagger}|0_M\rangle \\ &= \frac{1}{e^{2\pi\omega/a}-1}\delta(0), \end{aligned} \tag{2.52}$$

where we have used Eq. (2.50) and that the state $c_k^{(1)\dagger}|0_M\rangle$ is a normalized one-particle state. We obtain the same result for region II if we instead use the number operator $b_k^{(2)\dagger}b_k^{(2)}$ in Eq. (2.52). The infinity encountered in the delta-function above is an artifact of the chosen normalization of the basis modes, given in Eq. (2.11), and hence the normalization of the operators defined in Eq. (2.13). If we instead chose to normalize these modes as wave packets centered around a peak frequency, we would have gotten a finite result in Eq. (2.52).

The vacuum expectation value in Eq. (2.52) is a Planck spectrum of radiation at temperature $T = a/2\pi k_B$, where k_B is the Boltzmann constant [21]. What this relation actually tells us, then, is that a uniformly accelerated observer in Minkowski spacetime will observe a thermal spectrum of particles in the vacuum state of an inertial Minkowski observer [16]. Because Unruh was the first to point out this effect, it has been baptized the *Unruh effect*. The temperature $T = a/2\pi k_B$ is in reality the temperature of the radiation as measured by an accelerated observer following the path $\xi = 0$, which from Eq. (2.40) feels an acceleration $\alpha = a$. Accelerated observers following different paths of constant ξ will thus in general observe a temperature proportional to their acceleration α, namely [21, 29]

$$T = \frac{\alpha}{2\pi k_B}. \tag{2.53}$$

In this chapter we have found that some observers measure particles where others observe none. This surely sounds like science fiction, and not something that should be taken seriously. Indeed, when computing the expectation value of the normal-ordered energy-momentum tensor of the scalar field ϕ in the Minkowski vacuum, the result is zero – for all observers. Hence, observers measuring a flux of particles in the Minkowski vacuum will also find that the vacuum expectation value of the energy-momentum tensor is zero. This seems to violate energy conservation, and we may be tempted to conclude that these observed particles are not real [21].

One way out of this apparent paradox is to consider what causes the acceleration in the first place. In order for an accelerated detector to maintain constant acceleration, some sort of constant work must be performed on it. This work gives rise to the creation of particles. From the point of view of an inertial Minkowski observer, the accelerating detector both emits and absorbs particles. Hence the detected particles are created as a consequence of the constant acceleration [21, 29].

Chapter 3

Derivation of Hawking Radiation

In 1975, Stephen Hawking found that black holes radiate; a thermal radiation which later has been known as Hawking radiation. That black holes radiate is a rather surprising result, given the definition of black holes in the classical theory of general relativity: Black holes are regions of spacetime where gravity is so strong that nothing can escape from them – not even light. As a consequence, Hawking's result has since its first presentation been vastly disputed.

Because the creation of particles was first derived in the spacetime of a spherically symmetric star collapsing to a Schwarzschild black hole [15], many readers have been led to the incorrect conclusion that Hawking radiation is a manifestly gravitational artifact. However, already the year after Hawking's result, Unruh showed that also uniformly accelerated observers in flat spacetimes observe a thermal spectrum of particles in the traditional Minkowski spacetime [16]. Moreover, research over the recent years has shown that Hawking radiation can be derived through many different approaches, some of which have nothing to do with gravity [18]. Despite this intriguing remark, we will, for the purpose of this thesis, restrict the derivations of Hawking radiation to the spacetimes of an eternal black hole and a collapsing star.

3.1 Hawking Radiation from Eternal Black Holes

In the previous chapter we saw that an observer which accelerates constantly registers particles in the vacuum state of an inertial Minkowski observer. From the equivalence principle, we know that there is no way in which we can separate constant acceleration from a curved background spacetime, locally. Therefore we expect that a freely-falling observer propagating through a static, curved spacetime which is asymptotically flat, will observe particles being created by the spacetime curvature. This is indeed the case for an eternal black hole.

An eternal black hole can be described by the unique vacuum solution to Einstein's equations with spherical symmetry: the Schwarzschild metric. For simplicity, we will keep the following discussion in two dimensions. The four-dimensional treatment and results are essentially the same, and the two-dimensional approach corresponds to treating each two-sphere in the four-dimensional spacetime as a single point in the two-dimensional spacetime. In two dimensions, the Schwarzschild line element for a black hole of mass M is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) du dv, \tag{3.1}$$

where the null coordinates u and v are defined as

$$\begin{aligned} u &= t - r^* \\ v &= t + r^*. \end{aligned} \tag{3.2}$$

Also, the radial coordinate satisfies

$$r^* = r + 2M \ln \left| \frac{r}{2M} - 1 \right|. \tag{3.3}$$

This spacetime is asymptotically flat, which can easily be shown by sending $r \to \infty$ in Eq. (3.1) and Eq. (3.3). Thus for a massless scalar field ϕ in the asymptotic region of this spacetime, one natural basis of solutions to the wave equation reduces to a set of plane waves proportional to $e^{-i\omega u}$ and $e^{-i\omega v}$. Following the reasoning from the previous chapter, these modes will have a corresponding vacuum state $|0_S\rangle$ that is devoid of particles for all inertial observers in these regions of spacetime.

From Eq. (3.1) we see that the metric becomes singular at the radial coordinate distance $r = 2M$. These modes will oscillate rapidly at the event horizon of the black hole, situated at $r = 2M$. Transforming to Kruskal coordinates [21],

$$\begin{aligned} \bar{u} &= -4Me^{-u/4M} \\ \bar{v} &= 4Me^{v/4M}, \end{aligned} \tag{3.4}$$

the two-dimensional line element becomes

$$ds^2 = \frac{2M}{r} e^{-r/2M} d\bar{u} d\bar{v}. \tag{3.5}$$

This line element is regular all over the Schwarzschild spacetime, except for at the physical singularity at $r = 0$.

The line element of Schwarzschild spacetime expressed in terms of Kruskal coordinates is conformal to Minkowski spacetime. Thus there exists a Killing vector in these coordinates from which positive- and negative-frequency modes can be defined. A different set of modes serving as a natural basis of solutions to the wave equation in the Schwarzschild spacetime, can therefore be found. These modes are plane waves of the form $e^{-i\omega\bar{u}}$ and $e^{-i\omega\bar{v}}$. Moreover, there exists a vacuum state $|0_K\rangle$ for these modes which does not coincide with the vacuum state $|0_S\rangle$.

Now we may proceed as before by expressing the scalar field ϕ in both sets of modes, and find a relation between the annihilation- and creation operators of the two observers. Having obtained such a relation, we may further find the expectation value of the number operator in the vacuum state of the observer using the Kruskal coordinates in Eq. (3.4). Rather than going through the details of this calculation, we will use the equivalence principle to argue that a Schwarzschild observer will find a thermal radiation of particles in the Kruskal vacuum.

The reasoning goes as follows: Firstly, we may observe that the Minkowski line element in Eq. (2.34) can be written in terms of null coordinates $(\tilde{u}, \tilde{v})$ as

$$ds^2 = dt^2 - dx^2 = d\tilde{u} d\tilde{v}. \tag{3.6}$$

Using Eq. (2.41) and defining Rindler null coordinates $u' = \eta - \xi$ and $v' = \eta + \xi$, the coordinates $(\tilde{u}, \tilde{v})$ can be written as

$$\begin{aligned} \tilde{u} &= t - x = -\frac{1}{a} e^{-au'} \\ \tilde{v} &= t + x = \frac{1}{a} e^{av'}. \end{aligned} \tag{3.7}$$

Comparing Eq. (3.7) with Eq. (3.4), we see that setting $a = 1/4M$ yields the same relation between the Minkowski null coordinates $(\tilde{u}, \tilde{v})$ and the Rindler null coordinates (u', v') as between the Kruskal coordinates $(\bar{u}, \bar{v})$ and the Schwarzschild coordinates (u, v). Furthermore, $\kappa = 1/4M$ is the surface gravity of a Schwarzschild black hole of mass M, so the acceleration, a, of a Rindler observer following the path $\xi = 0$ in Rindler space is equivalent to the surface gravity, κ, on the event horizon of the Schwarzschild black hole. This is thus in complete agreement with the equivalence principle, which states that constant acceleration cannot be distinguished from gravitation locally.

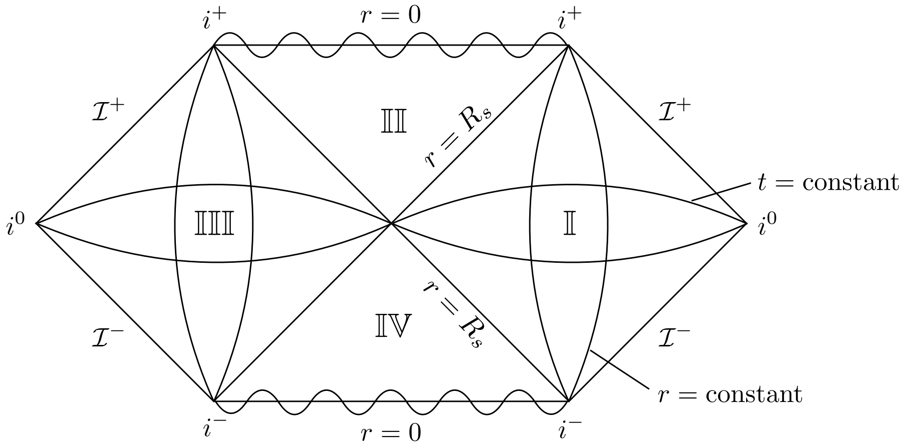


Figure 3.1: Penrose diagram of Schwarzschild spacetime. The future and past event horizons are marked by the lines of radial coordinate $r = R_s$. These horizons divide the spacetime into four separate regions: Regions II and IV correspond to a black and a white hole, respectively, and regions I and III represent two causally disconnected, asymptotically flat spacetime regions. The singularities at $r = 0$ are proper singularities of the spacetime. Null rays travel along lines parallel to the event horizons.

As a matter of fact, the Penrose diagram figure 2.1 depicts the exact same causal structure as that of a Schwarzschild black hole spacetime, shown in figure 3.1. More specifically, the horizon structure is identical in both spacetimes. As one can readily see from figure 3.1, the event horizons in the Schwarzschild spacetime divide the spacetime into four separate regions. Regions II and IV correspond to a black and a white hole, respectively, and regions I and III represent two causally disconnected, asymptotically flat spacetime regions. The Schwarzschild modes can be defined in these latter two regions in a similar manner as the Rindler modes in Eq. (2.45) [16]. Additionally, these modes can be extended analytically into the whole spacetime, yielding a similar field expansion for a massless scalar field ϕ, as in Eq. (2.46). Hence the vacuum state $|0_S\rangle$ for a Schwarzschild observer is closely correlated with the vacuum state $|0_R\rangle$ defined in Eq. (2.47) for a Rindler observer. Similarly, the Kruskal vacuum $|0_K\rangle$ corresponds to the Minkowski vacuum $|0_M\rangle$. Thus the vacuum states $|0_S\rangle$ and $|0_K\rangle$ do not coincide.

Adopting the same arguments as for the accelerated observer in Minkowski spacetime, we are able to combine the Schwarzschild modes into a set of modes which share vacuum state with the Kruskal modes. Therefore, we may relate the operator transformations of the Schwarzschild- and Kruskal coordinates in the exact same way as for the Rindler- and Minkowski observers in Eq. (2.51). In particular, a static Schwarzschild observer at a radius somewhat larger than the Schwarzschild radius will observe – over sufficiently short length- and timescales – a thermal flux of particles of temperature $T = a/2\pi k_B$ emerging from the vacuum state of a freely falling observer. In a static spacetime that is asymptotically flat, the surface gravity is just the red-shifted acceleration of a static observer close to the event horizon, as measured by an observer at infinity. Hence, a static observer far away from the Schwarzschild black hole will observe a spectrum of thermal radiation of temperature

$$T = \frac{\kappa}{2\pi k_B}, \tag{3.8}$$

emerging from the black hole [29].

At the end of the previous chapter we discussed the origin of the particles observed by a uniformly accelerated detector in the vacuum state of an inertial Minkowski observer. As mentioned, one possible interpretation of the phenomenon was that the detected particles were created as a consequence of the work needed to keep the detector's acceleration constant. This interpretation does not translate very well to curved spacetimes, as it is restricted to local measurements only. Instead, a more useful interpretation can be found by looking at the global derivation of Hawking radiation: In a black hole spacetime the Kruskal modes $e^{-i\omega\bar{u}}$ are positive frequency with respect to the operator $\partial_{\bar{u}}$ which is a Killing vector on the black hole past horizon H^-. Similarly, the Schwarzschild modes proportional to $e^{-i\omega v}$ are positive frequency with respect to the Killing vector ∂_v on the surface $\mathcal{I}^-$. Both these sets of modes can thus be described as modes propagating into the asymptotically flat regions of Schwarzschild spacetime, corresponding to a flux of particles emerging out from the black hole. Similarly, the time reversed modes $e^{-i\omega\bar{v}}$ on H^+ and $e^{-i\omega u}$ on $\mathcal{I}^+$ correspond to thermal radiation entering the black hole. Hence, an eternal black hole emits and absorbs thermal radiation, being in thermal equilibrium with its surroundings. This thermal radiation is most famously known as Hawking radiation [21].

Nothing in the argumentation above demands any time dependence of the spacetime in question. This discussion thus suggests that the Hawking effect is a consequence of the causal and topological structure of spacetime, rather than the geometry in question [21]. An eternal black hole is nevertheless not a physically realistic object. Rather, black holes are thought to be formed from collapsing stars. This is the main reason for disposing with the model of an eternal black hole in the following, and instead focus on Hawking radiation from stars collapsing to black holes. We will, however, use a simplified collapse model for the further discussion, letting the star undergo a spherically symmetric collapse to a Schwarzschild black hole. Such collapse models have been studied in great detail, which will make it a lot easier to pin-point the time and place in which the Hawking radiation may occur later on.

3.2 Hawking Radiation from a Collapsing Object

As already mentioned, multiple methods have been developed to show the existence of Hawking radiation. In the following we will mainly follow the derivation presented in Birrel and Davies' *Quantum fields in curved space* [21]. Many of the arguments presented in this section will be tweaked slightly in chapter 4 on Hawking radiation from ECOs. For this reason, the following discussion will be rather detailed.

Let us start out with a collapsing spherical star of mass M. Initially, this star is at rest with a surface fixed at $r = R_0$. At the time $t = \tau = 0$, the star starts to collapse. Outside the object the spacetime takes the form

$$ds^2 = C(r)du dv, \tag{3.9}$$

where $C(r) = (1 - 2M/r)$ and the null coordinates (u, v) are defined by

$$\begin{aligned} u &= t - \int_{R_0}^{r} \frac{dr'}{C(r')} = t - r^* + R_0^*, \\ v &= t - \int_{R_0}^{r} \frac{dr'}{C(r')} = t + r^* - R_0^*. \end{aligned} \tag{3.10}$$

The time coordinate t is the coordinate used by an inertial observer, far away from the body.

Similarly, the metric inside the star can be written in terms of null coordinates (U, V) as

$$ds^2 = A(U, V)dUdV \tag{3.11}$$

for some function A, where the null coordinates are defined by

$$\begin{aligned} U &= \tau - r + R_0, \\ V &= \tau + r - R_0. \end{aligned} \tag{3.12}$$

The coordinates u, v, U and V are chosen such that at some given time $t = \tau = 0$, we have $u = U = v = V = 0$ at the surface of the star.

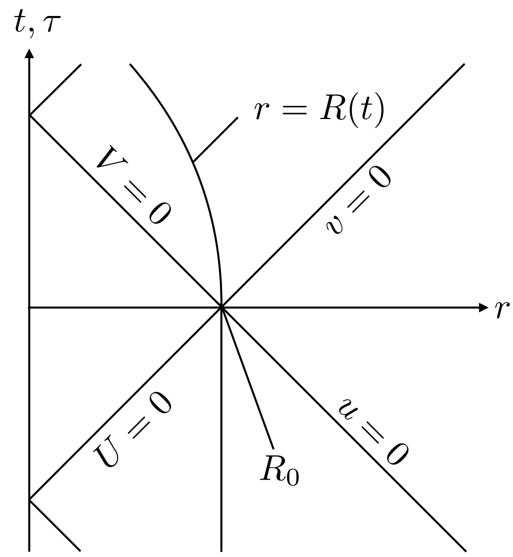


Figure 3.2: Minkowski diagram showing the wordline of the star's surface, $r = R(\tau)$. All null rays crossing the surface at the time $t = \tau = 0$ are defined to have the constant value zero. The coordinates V and U mark incoming and outgoing null rays, respectively, on the inside of the star, whereas the coordinates v and u correspond to incoming and outgoing null rays outside the star.

We will study null rays starting at $\mathcal{J}^-$, propagating through the star and ending up on $\mathcal{J}^+$. Null rays going out to $\mathcal{J}^+$ run along worldlines of constant null coordinates u and U, and incoming null rays from $\mathcal{J}^-$ run along worldlines described by constant v and V. There will exist some relation between the inside and outside null coordinates [21, 24]. We express these transformations as

$$\begin{aligned} U &= \alpha(u), \\ v &= \beta(V), \end{aligned} \tag{3.13}$$

where we have assumed no reflections on the surface of the star so that the outgoing internal null coordinate smoothly turns into the outgoing external coordinate, and equivalently for the incoming null coordinates. At the point $r = 0$, we observe from Eq. (3.12) that

$$V = U - 2R_0.$$

We are thus able to relate the null coordinate v to the null coordinate u at the point $r = 0$ by further noting that

$$v = \beta(V) = \beta(U - 2R_0) = \beta(\alpha(u) - 2R_0). \tag{3.14}$$

Our task now is to solve the wave equation in Eq. (2.6) for a massless scalar field ϕ in this spacetime. Because this spacetime is asymptotically flat, the mode solutions reduce to standard exponential functions on $\mathcal{J}^-$ and $\mathcal{J}^+$, so these solutions must be proportional to

$$e^{-i\omega v} + e^{-i\omega u}, \tag{3.15}$$

where $\omega \equiv \sqrt{k^2 + m^2}$ and $k \equiv |\mathbf{k}|$. Furthermore, the modes proportional to $e^{-i\omega v}$ correspond to incoming modes from $\mathcal{J}^-$ and the modes proportional to $e^{-i\omega u}$ correspond to outgoing modes on $\mathcal{J}^+$. Instead of reflecting the metrics in Eq. (3.11) and Eq. (3.9) at $r = 0$, we restrict our treatment to the region $r \geqslant 0$, imposing the boundary condition $\phi = 0$ at $r = 0$. Thus incoming null rays are reflected smoothly into outgoing null rays at this point. Using Eq. (3.14) and the boundary condition for ϕ, we see from Eq. (3.15) that the outgoing modes can be expressed as

$$e^{-i\omega u}|_{r=0} = -e^{-i\omega v}|_{r=0} = -e^{-i\omega\beta(\alpha(u)-2R_0)}. \tag{3.16}$$

Inserting this into Eq. (3.15) and normalizing the modes in the scalar product Eq. (2.9), we obtain the mode solutions

$$\frac{i}{\sqrt{4\pi\omega}}\left(e^{-i\omega v} - e^{-i\omega\beta(\alpha(u)-2R_0)}\right), \tag{3.17}$$

where the modes of the form $e^{-i\omega v}$ are incoming waves which then get transferred into outgoing waves of the form $e^{-i\omega\beta(\alpha(u)-2R_0)}$.

We seek continuity between the mathematical description of the interior of the star with the exterior. Specifically, we impose that the two line elements Eq. (3.9) and Eq. (3.11) are equal on the surface of the star, at $r = R(\tau)$. This yields

$$\frac{dU}{du}\Big|_{r=R(\tau)} = \frac{C(r)}{A(U,V)}\frac{dv}{dV}\Big|_{r=R(\tau)}. \tag{3.18}$$

At the surface $r = R(\tau)$ these derivatives can further be expressed as

$$\begin{aligned}
\frac{dU}{du}\Big|_{r=R(\tau)} &= \left(\frac{du}{d\tau}\right)^{-1}\frac{dU}{d\tau}\Big|_{r=R(\tau)} = \left(\frac{dt}{d\tau} - \frac{\dot{R}}{C}\right)^{-1}\left(1-\dot{R}\right)\Big|_{r=R(\tau)} \\
\frac{dv}{dV}\Big|_{r=R(\tau)} &= \left(\frac{dV}{d\tau}\right)^{-1}\frac{dv}{d\tau}\Big|_{r=R(\tau)} = \left(1+\dot{R}\right)^{-1}\left(\frac{dt}{d\tau} + \frac{\dot{R}}{C}\right)\Big|_{r=R(\tau)},
\end{aligned} \tag{3.19}$$

where we have set $\dot{R} \equiv dR/d\tau$ and made use of Eq. (3.10) and Eq. (3.12). Inserting these expressions into Eq. (3.18) we find an expression for $dt/d\tau$, which when inserted back into Eq. (3.19) yields the following relations

$$\begin{aligned}
\frac{d}{du}\alpha(u)\big|_{r=R(\tau)} &= \frac{dU}{du}\Big|_{r=R(\tau)} = \frac{C\left(1-\dot{R}\right)}{\sqrt{AC\left(1-\dot{R}^2\right)+\dot{R}^2} - \dot{R}}, \\
\frac{d}{dV}\beta(V)\big|_{r=R(\tau)} &= \frac{dv}{dV}\Big|_{r=R(\tau)} = \frac{\sqrt{AC\left(1-\dot{R}^2\right)+\dot{R}^2} + \dot{R}}{C\left(1+\dot{R}\right)}.
\end{aligned} \tag{3.20}$$

where the parameters C, U and V are evaluated on the surface $r = R(\tau)$.

As we approach the event horizon, $C \to 0$. Hence, we may expand Eq. (3.20) in the limit $AC(1-\dot{R}^2)/\dot{R}^2 \ll 1$. Additionally, because the star is collapsing, $\dot{R} < 0$ and thus $\sqrt{\dot{R}^2} = -\dot{R}$. To leading order in $x = AC(1-\dot{R}^2)/\dot{R}^2$, this yields

$$\begin{aligned}
\frac{d}{du}\alpha(u)\big|_{r=R(\tau)} &= \frac{dU}{du}\Big|_{r=R(\tau)} \sim \frac{C\left(\dot{R}-1\right)}{2\dot{R}}, \\
\frac{d}{dV}\beta(V)\big|_{r=R(\tau)} &= \frac{dv}{dV}\Big|_{r=R(\tau)} \sim \frac{A\left(1-\dot{R}\right)}{2\dot{R}},
\end{aligned} \tag{3.21}$$

where we have absorbed a minus sign into the order of magnitude estimate in the equation for the derivative of β.

Before integrating Eq. (3.21), we need to know more about the functions C and A on the surface $r = R(\tau)$. First, we expand the function $C(R)$ around its value on the event horizon $R = R_s$:

$$C(R) = C(R_s) + \frac{\partial C}{\partial r}\Big|_{r=R_s}(R-R_s) + \mathcal{O}((R-R_s)^2). \tag{3.22}$$

From the first equation in Eq. (3.21), we may further observe that to $\mathcal{O}(R-R_s)$ we can write

$$\frac{\dot{R}}{\dot{R}-1}dU = \frac{1}{2}C(R)du = \frac{1}{2}\frac{\partial C}{\partial r}\Big|_{r=R_s}(R-R_s)du, \tag{3.23}$$

where we have used that $C(R_s) = 0$.

Observing from Eq. (3.12) that $dU/d\tau = 1 - \dot{R}$ and that $\dot{R}d\tau = dR$, we may write Eq. (3.23) as

$$\frac{1}{R_s - R(\tau)} dR = \kappa du, \tag{3.24}$$

where we have defined

$$\kappa \equiv \frac{1}{2} \frac{\partial C}{\partial r}\bigg|_{r=R_s}. \tag{3.25}$$

Physically, the quantity κ is interpreted as the surface gravity of the black hole [21]. For a Schwarzschild black hole, we have $\kappa = 1/2R_s = 1/4M$. Integrating Eq. (3.24) we further obtain the relation

$$\kappa u = -\ln |R_s - R(\tau)| + \text{constant}. \tag{3.26}$$

We may also expand $R(\tau)$ close to the event horizon. This expansion is given by

$$R(\tau) = R_s + \gamma(\tau_s - \tau) + \mathcal{O}((\tau_s - \tau)^2), \tag{3.27}$$

where $R_s = R(\tau_s)$ and $\gamma \equiv -\dot{R}(\tau_s)$. Making use of this expansion together with the definition of U given in Eq. (3.12) we have, to $\mathcal{O}(\tau_s - \tau)$ – which from Eq. (3.27) is the same as expanding to $\mathcal{O}(R - R_s)$, that

$$U(\tau_s, R_s) - U(\tau, R) = (\tau_s - \tau) + (R - R_s) = \frac{1+\gamma}{\gamma}(R - R_s). \tag{3.28}$$

Thus we may rewrite Eq. (3.26) as

$$\kappa u = -\ln |U + R_s - R_0 - \tau_s| + \text{constant}. \tag{3.29}$$

Inverting Eq. (3.29) we get that

$$U = \alpha(u) \propto e^{-\kappa u} + \text{constant} \tag{3.30}$$

for late times (i.e. for $\tau \approx \tau_s$).

As $U \to \tau_s - R_s + R_0$, we see from Eq. (3.29) that $u \to \infty$. The null ray $u = \infty$ coincides with the event horizon of the black hole. Following this ray backwards in time, we see that it turns into the last null ray $v = v_0$ emerging from past null infinity that propagates through the center of the star and escapes the star before it turns into a black hole. This is depicted in the Penrose diagram 3.3. Here the pink line represents the null ray $v = v_0$ turning into the ray $u = \infty$, which coincides with the horizon of the black hole. The orange lines are null rays leaving $\mathcal{I}^-$ for values $v < v_0$ and $v > v_0$, respectively. The former propagates through the star to $\mathcal{I}^+$ and the latter gets trapped by the event horizon of the black hole.

Denote the pink null ray in figure 3.3 γ and the orange null ray that escapes to $\mathcal{I}^+$ γ'. Further assume that this latter ray leaves $\mathcal{I}^-$ just before the ray γ leaves this null surface. Following the ray γ from $\mathcal{I}^-$ all the way to future null infinity, we see that it emerges from a finite value $v = v_0$ on $\mathcal{I}^-$ and ends up at $\mathcal{I}^+$ at an infinite value of u. The ray γ', on the other hand, leaves $\mathcal{I}^-$ for a finite value v' which is a little bit smaller than v_0, and ends up at $\mathcal{I}^+$ at a finite value $u = u'$. Hence there must be an infinite number of equispaced null rays $u =$ constant between the null coordinates $u = u'$ and $u = \infty$. When tracing these null rays backwards in time from $\mathcal{I}^+$ to $\mathcal{I}^-$, they pile up between the rays γ' and γ. Consequently there is only a narrow range of values for v and V that make up the late time asymptotic region of $\mathcal{I}^+$.

Hence, to compute the experience of an observer at late times (i.e. for large values of u), we may set A to be approximately constant in the null coordinate V in Eq. (3.21). Integrating this equation we get that

$$v = \beta(V) \sim -AV\frac{1+\gamma}{2\gamma} + \text{constant}, \tag{3.31}$$

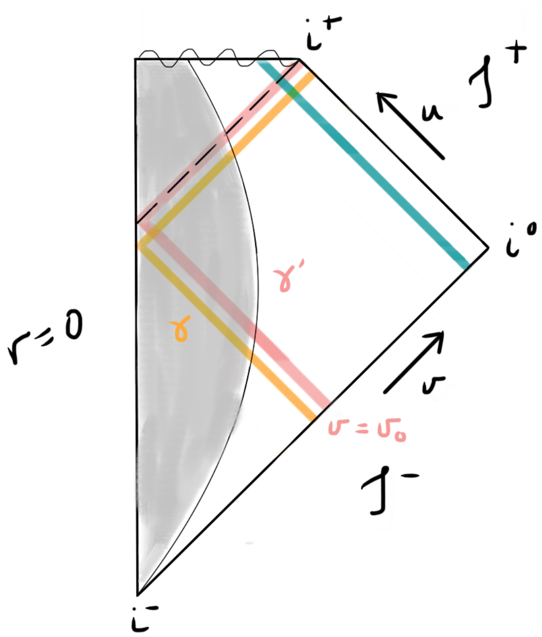


Figure 3.3: Penrose diagram of a collapsing star. The pink line depicts the last null ray originating on $\mathcal{J}^-$ that propagates through the collapsing star and escapes to $\mathcal{J}^+$ just before the star collapses to a black hole. Furthermore, the dotted line marks the horizon of the black hole, and the grey area depicts the interior of the star. The orange line corresponds to a null ray that leaves $\mathcal{J}^-$ for $v < v_0$, ending up on $\mathcal{J}^+$, and the blue line corresponds to a null ray that leaves $\mathcal{J}^-$ for $v > v_0$ and thus continues through the event horizon of the black hole, towards the singularity at $r = 0$. The wavy line depicts the singularity of the spacetime, and the line $r = 0$ is the origin of spherical coordinates.

where we have set $\dot{R} = -\gamma$ since we look at a very narrow range of the spacetime.

Using Eq. (3.14) together with Eq. (3.30) and Eq. (3.31) we see that at late times, the phase factor of the outgoing modes in Eq. (3.17) is given by

$$v = \beta(\alpha(u) - 2R_0) = -A(e^{-\kappa u} + \text{constant} - 2R_0)\frac{1+\gamma}{2\gamma} + \text{constant} = -ce^{-\kappa u} + d, \qquad (3.32)$$

where c and d are constants. The mode solutions Eq. (3.17) thus take the form

$$\frac{i}{\sqrt{4\pi\omega}}\left(e^{-i\omega v} - e^{i\omega(ce^{-\kappa u} - d)}\right) \qquad (3.33)$$

In other words, the outgoing modes suffer an increasing redshift of e-folding time κ^{-1}.

Since the mode solutions to the wave equation in this spacetime are simple plane waves on $\mathcal{J}^-$, an inertial observer measuring the number of particles in the vacuum state will observe that it is devoid of particles. An inertial observer in the region $\mathcal{J}^+$, on the other hand, will measure a flux of particles coming out from the vacuum of the observer on $\mathcal{J}^-$. This is due to the redshift in the outgoing modes. As before, the spectrum of the flux of particles going out from this vacuum state can be found by computing the Bogolubov coefficients relating the modes Eq. (3.33) and the modes used by an inertial observer in the region $\mathcal{J}^+$.

However, instead of using the late time modes defined on $\mathcal{J}^+$, we will rather invert the mode functions Eq. (3.33) and compute the Bogolubov transformations close to the null surface $\mathcal{J}^-$. The reason for making this transition is that on $\mathcal{J}^-$ there is a natural bound on the coordinate v giving rise to radiation on $\mathcal{J}^+$, since all rays leaving $\mathcal{J}^-$ after some coordinate $v = v_0$ will enter the black hole horizon and never get out. Thus we aim to write the outgoing mode solutions of the Klein-Gordon equation such that the modes are standard outgoing modes on $\mathcal{I}^+$, but take a more complicated form on $\mathcal{I}^-$. To make this transition we must therefore invert the modes Eq. (3.33).

Defining $F(u) \equiv \beta(\alpha(u) - 2R_0)$, we may write these inverted modes as

$$\frac{i}{\sqrt{4\pi\omega}} \left(e^{-i\omega F^{-1}(\nu)} - e^{-i\omega F^{-1}(F(u))} \right). \tag{3.34}$$

By definition, $F^{-1}(F(u)) = u$. Also, from Eq. (3.32) we see that $u = F^{-1}(\nu)$. Thus, we must have that

$$u = F^{-1}(\nu) = -\frac{1}{\kappa} \ln\left(\frac{d - \nu}{c} \right). \tag{3.35}$$

Additionally, since all modes leaving $\mathcal{I}^-$ for $\nu > \nu_0$ end up inside the black hole, these will not give any contribution to the modes that propagate to $\mathcal{I}^+$. Hence, up to a phase factor, we may write the modes Eq. (3.33) on $\mathcal{I}^-$ as

$$\frac{i}{\sqrt{4\pi\omega}} \left(e^{i\omega \ln((\nu_0 - \nu)/c)/\kappa} - e^{-i\omega u} \right), \quad \nu < \nu_0. \tag{3.36}$$

At this point it is important to note that we have made use of a very specific assumption in the calculations above. The fact that we can relate the modes on $\mathcal{I}^+$ to the modes on $\mathcal{I}^-$ by tracing the null rays backwards in time, is actually due to geometrical optics. Implicitly, we have assumed that the null rays in question are propagating along geodesics, so that the affine parameter distance between different null rays on $\mathcal{I}^+$ is proportional to the affine parameter distance between the same modes on $\mathcal{I}^-$. See Appendix B for details.

We can use the assumption that the null rays follow geodesics as long as the rays oscillate rapidly. The rays leaving $\mathcal{I}^-$ just before the horizon formation, will suffer a large redshift when propagating from the star to $\mathcal{I}^+$. Physical rays on $\mathcal{I}^+$ that escape the star just before the collapse to a black hole, must have had a lot larger frequencies on $\mathcal{I}^-$. We may therefore safely use the geometric optics approximation for these rays.

In the following we will denote the modes incoming from $\mathcal{I}^-$ that are standard plane waves on past null infinity,

$$f_\omega = \frac{i}{\sqrt{4\pi\omega}} e^{-i\omega\nu}, \tag{3.37}$$

and the modes that are standard plane waves on $\mathcal{I}^+$ but which become complicated on $\mathcal{I}^-$ as

$$p_\omega = \frac{i}{\sqrt{4\pi\omega}} e^{i\omega \ln((\nu_0 - \nu)/c)/\kappa}. \tag{3.38}$$

To find the form of the spectrum of radiation we need to calculate the Bogolubov coefficients relating these modes.

There are different ways in which we may proceed to find the Bogolubov coefficients. One possibility is to compute the scalar products

$$\begin{aligned} \alpha_{\omega\omega'} &= (f_{\omega'}, p_\omega) \\ \beta_{\omega\omega'} &= -(f_{\omega'}, p^*_\omega) \end{aligned} \tag{3.39}$$

on a suitable spacelike hypersurface. This method is straight-forward, but may in some cases be a bit cumbersome – especially if there exist no symmetries of the spacetime from which a specific surface of integration can be chosen that simplifies the calculations considerably. A different approach, which is the method presented by Hawking in his first paper on the matter [15], is to instead make use of Fourier transformations. This is the method we will use in the following.

In analogy with Eq. (2.18), we can write the modes p_ω in terms of the modes f_ω and their complex conjugates. Equating Eq. (3.38) with this expansion, we get that

$$\frac{i}{\sqrt{4\pi\omega}} e^{i\omega \ln((\nu_0 - \nu)/c)/\kappa} = \int_{-\infty}^{\infty} d\omega'' \left(\frac{i}{\sqrt{4\pi\omega''}} \alpha_{\omega\omega''} e^{-i\omega''\nu} - \frac{i}{\sqrt{4\pi\omega''}} \beta_{\omega\omega''} e^{i\omega''\nu} \right). \tag{3.40}$$

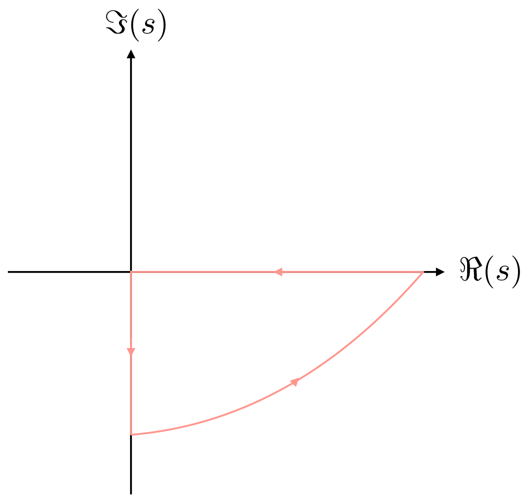


Figure 3.4: Contour of integration for the $\alpha_{\omega\omega'}$-integral in the complex s-plane. The contour is marked with a pink line. Since the integral vanishes on the boundary at infinity and there are no poles of the integrand inside the contour, the integral along the real axis from ∞ to 0 is the same as the integral from 0 to $-i\infty$ along the imaginary axis. Similarly, the contour for the $\beta_{\omega\omega'}$-integral is chosen to be in the lower left quadrant of the complex s-plane.

Fourier transforming both sides of this equation, as well as performing an integration over the arising delta functions further yields

$$\alpha_{\omega\omega'} = \frac{1}{2\pi}\int_{-\infty}^{\nu_0} d\nu \sqrt{\frac{\omega'}{\omega}} e^{i\omega'\nu} e^{i\omega \ln((\nu_0-\nu)/c)/\kappa}, \tag{3.41}$$

for the integral over all positive values of ω'', and

$$\beta_{\omega\omega'} = -\frac{1}{2\pi}\int_{-\infty}^{\nu_0} d\nu \sqrt{\frac{\omega'}{\omega}} e^{-i\omega'\nu} e^{i\omega \ln((\nu_0-\nu)/c)/\kappa}, \tag{3.42}$$

for the negative values of ω''. Also, we have cut the upper integration boundary on $\nu = \nu_0$, since all values of ν larger than this gives zero contribution to the integrals. Instead of computing these integrals directly, we will continue by arguing that we can relate the coefficients $\alpha_{\omega\omega'}$ to the coefficients $\beta_{\omega\omega'}$ by a constant factor.

First we observe that by defining a new parameter $s \equiv \nu_0 - \nu$ in Eq. (3.41) and $s \equiv \nu - \nu_0$ in Eq. (3.42), we may rewrite these integrals as

$$\begin{aligned} \alpha_{\omega\omega'} &= -\frac{1}{2\pi}\int_{\infty}^{0} ds \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' s} e^{i\omega'\nu_0} e^{i\omega \ln(s/c)/\kappa}, \\ \beta_{\omega\omega'} &= -\frac{1}{2\pi}\int_{-\infty}^{0} ds \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' s} e^{-i\omega'\nu_0} e^{i\omega \ln(-s/c)/\kappa}. \end{aligned} \tag{3.43}$$

The contour of the integral $\alpha_{\omega\omega'}$ in Eq. (3.43) over values of s ranging from 0 to ∞ can be joined at infinity by a quarter of a circle in the complex s-plane with the contour of the complex integral ranging from $-i\infty$ to 0. These three segments together make a closed contour in the lower, right quadrant of the complex s-plane. Figure 3.4 shows the contour of integration. Inside this contour, the integrand of the same integral has no poles. Thus the contour can be analytically compressed into one single point. Since the integrand does not have any poles in this quadrant, the integral along the whole contour in the complex s-plane must vanish. Additionally, since the integrand vanishes on the boundary at infinity, i.e. along the quarter of a circle connecting the imaginary and real axes, the integral along the real axis must equal the integral along the imaginary s-axis. A similar argument can be used for the integral $\beta_{\omega\omega'}$. We can thus define $s \equiv is'$. Moreover, since the integrals Eq. (3.43) are confined to the lower complex plane, we know $s' < 0$. Hence we may write Eq. (3.43) as

$$\begin{aligned} \alpha_{\omega\omega'} &= \frac{i}{2\pi} e^{i\omega' v_0} e^{\pi\omega/2\kappa} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{0} ds' e^{\omega' s'} e^{i\omega \ln(|s'|/c)/\kappa}, \\ \beta_{\omega\omega'} &= \frac{i}{2\pi} e^{-i\omega' v_0} e^{-\pi\omega/2\kappa} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{0} ds' e^{\omega' s'} e^{i\omega \ln(|s'|/c)/\kappa} \end{aligned} \tag{3.44}$$

where we have used that

$$\ln(\pm is'/K) = \mp(i\pi/2) + \ln(|s'|/K). \tag{3.45}$$

Thus we may write

$$|\alpha_{\omega\omega'}|^2 = e^{2\pi\omega/\kappa} |\beta_{\omega\omega'}|^2. \tag{3.46}$$

From Eq. (2.27) we see that an observer using the modes p_ω measures a flux of particles from the vacuum state of an observer using the modes f_ω, given by

$$\langle 0_f | p_\omega^\dagger p_\omega | 0_f \rangle = \int d\omega' |\beta_{\omega\omega'}|^2. \tag{3.47}$$

Evaluating this integral straight forwardly-yields an infinite amount of particles in the vacuum state, which can be seen by making use of the first relation in Eq. (2.20) for $\omega'' = \omega$. This infinity should not come as a surprise, as the expectation value Eq. (3.47) is defined for all times. Because there is a steady flux of particles out from this state, we expect an accumulation of these for late times, at $\mathcal{I}^+$.

To get around this infinity, we may insert the Fourier transform of the delta function,

$$\delta(\omega - \omega'') = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T/2}^{T/2} dt\ e^{i(\omega - \omega'')t}, \tag{3.48}$$

into Eq. (2.20), so that we can write

$$\begin{aligned} \lim_{T\to\infty} \frac{T}{2\pi} &= \int d\omega' \left(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2 \right) \\ &= \left(e^{2\pi\omega/\kappa} - 1 \right) \int d\omega' |\beta_{\omega\omega'}|^2, \end{aligned} \tag{3.49}$$

for $\omega'' = \omega$. In the last equality we have used Eq. (3.46).

Further inserting Eq. (3.49) into Eq. (3.47) yields the vacuum expectation value

$$\langle 0_f | p_\omega^\dagger p_\omega | 0_f \rangle = \lim_{T\to\infty} \frac{T}{2\pi} \frac{1}{e^{2\pi\omega/\kappa} - 1}. \tag{3.50}$$

Thus, the number of particles per unit time and frequency that pass through the surface of the star is

$$\frac{1}{e^{\omega/k_B T} - 1}, \tag{3.51}$$

where we have defined the temperature

$$T = \frac{\kappa}{2\pi k_B}. \tag{3.52}$$

Hence, the two-dimensional model of a star collapsing to a black hole radiates thermally with a temperature given by Eq. (3.52). This is the exact same thermal spectrum of radiation as the ones found in the case of an eternal Schwarzschild black hole and for an accelerated observer in Minkowski spacetime.

The same result can be found in four dimensions. Essentially, the arguments of that calculation follows the same steps as for the two-dimensional model given here. This is demonstrated in Appendix A.

3.3 Where and When are the Hawking Particles Created?

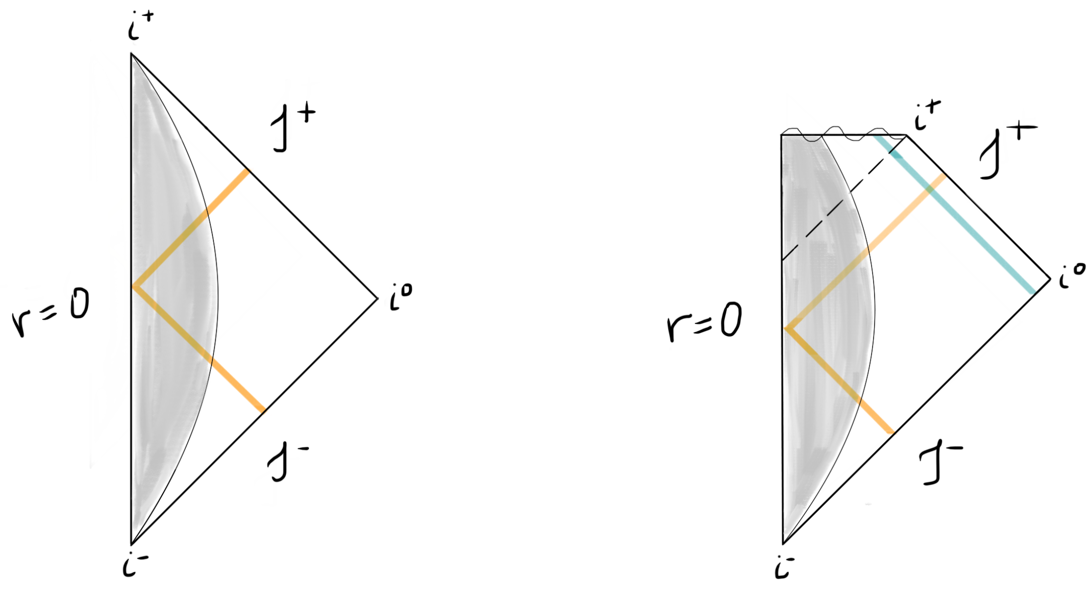
In the previous derivations of Hawking radiation, being either in a flat spacetime with an accelerated observer, in the spacetime of an eternal black hole or in the spacetime of a collapsing star, one specific aspect has been crucial in order to obtain the predicted thermal spectrum of particles – the existence of a horizon. When calculating the spectrum of particles from a star collapsing to a Schwarzschild black hole, the event horizon is essential in the argument for a pile-up of null rays on $\mathcal{I}^-$, leading to Eq. (3.31). Also, the redshift in the modes propagating from past to future null infinity in Eq. (3.33) scales exponentially with the inverse of the surface gravity, κ, of the black hole horizon. Thus, as one can readily see from Eq. (3.52), the Hawking temperature observed on $\mathcal{I}^+$ is proportional to this surface gravity. An equal relation for the temperature and the redshift can also be found in the spacetime of an eternal Schwarzschild black hole, which is reflected in the temperature of the radiation given in Eq. (3.8). Similarly, the uniformly accelerated Rindler observer accelerates to the speed of light at the Rindler horizons. Hence, the constant acceleration, α, of the Rindler observer serves the same role as the surface gravity in the previous examples, being proportional to the temperature of the Unruh radiation given in Eq. (2.53).

Moreover, both in the spacetime of an eternal Schwarzschild black hole and in Rindler spacetime the horizon – which coincides with the Killing horizon for a timelike Killing vector – gives rise to an additional natural set of positive frequency mode solutions to the wave equation. These sets differ from the positive frequency modes defined with respect to the timelike Killing vectors in the metrics of the Schwarzschild observer and the inertial Minkowski observer, given respectively in Eq. (3.1) and Eq. (2.34). In the black hole spacetime the past horizon serves as a Killing horizon for the Kruskal time coordinate, whereas the future horizon in the Rindler spacetime is a Killing horizon for the Rindler time coordinate. For both sets of observers the concept of particles is well-defined. Hence, the fact that we can find two natural sets of observers in these two spacetimes yields the possibility to state that particles are being produced.

We are also able to find two such natural sets of observers in the spacetime of a star collapsing to a black hole – one set of observers on $\mathcal{I}^-$, and one set on $\mathcal{I}^+$. By comparing the Penrose diagram of a stellar collapse which forms a black hole with that of a static star, we see that the modes that propagate from past null infinity in the former case, either end up on future null infinity or on the future event horizon. In the latter case, all modes starting on past null infinity eventually end up on future null infinity. Thus it seems plausible to draw the conclusion that in the former case, the two sets of inertial observers on $\mathcal{I}^-$ and $\mathcal{I}^+$ are different in the collapsing spacetime because of the existence of a horizon: Some of the modes which originate on past null infinity propagate through the event horizon of the black hole, instead of ending up on future null infinity. Thus, $\mathcal{I}^+$ is not a Cauchy horizon such as $\mathcal{I}^-$, but constitutes a complete set of mode solutions to the wave equation only in union with the future event horizon. From this it seems like the existence of a horizon may be a sufficient condition for particle creation to occur.

The existence of a thermal radiation from horizons is also compelling in connection with black hole thermodynamics. In fact, one of the main reasons why Hawking originally studied the matter was to find a more consistent thermodynamic interpretation of the Bekenstein entropy [15, 18]. Black holes are like the vacuum cleaners of spacetime: If an object crosses the black hole event horizon, it will never be able to get out. This feature of black holes, seemingly removing entropy from the spacetime by translating highly entropic objects into three degrees of freedom: mass, electric charge and angular momentum, violates the second law of thermodynamics stating that the entropy of the universe increases. Hawking radiation, being almost exactly thermal, solves this problem: The thermal radiation emitted from the black hole does not contain any information about the objects that have fallen into it, and does therefore cause a higher entropy in the universe [15, 29].

Following Hawking's original derivation of particle creation from collapsing stars, however, there is no explicit use of the existence of an event horizon. Rather, his arguments seem to lean on the fact that the gravitational field from a soon-to-be black hole is so strong that, by geometrical optics, the null rays close to the event horizon travel on null geodesics [15]. From this approximation one can



(a) Static star.

(b) Collapsing star.

Figure 3.5: Penrose diagrams of (a) a static star and (b) a star collapsing to a Schwarzschild black hole. The regions exterior to the stars are described by the Schwarzschild metric. The orange lines mark null geodesic propagating from $\mathcal{J}^-$ to $\mathcal{J}^+$, through the center of the respective stars. The blue line in (b) denotes a null ray propagating from $\mathcal{J}^-$ and t hrough the event horizon of the black hole.

thereby relate the affine parameter distance between two null rays on past null infinity to their affine parameter distance on future null infinity, leading to a redshift factor of the same form as the one we found in Eq. (3.32) (see Appendix B for further details). It is this exponential relation between the internal and external outgoing null coordinates of the star that typically is connected to the existence of Hawking radiation [18]. On these grounds, one may suspect that for a star collapsing to a compact object with radius very close to the black hole event horizon, this relation still holds.

A different answer to the above-mentioned question can therefore be that particles are created in the spacetime of a star collapsing to a black hole because of the greatly changing gravitational field – not because of the loss of information into a horizon. In a series of articles, the authors Barceló, Liberati, Sonego and Visser have tried to show that stars collapsing to objects with radii larger than the event horizon are indeed accompanied with the production of particles [5, 24, 30]. Specifically, they show that for a star collapsing adiabatically to form a compact, horizonless object, there will be an approximately exponential relation between the outgoing null coordinates on the inside and outside of the star, and thus a Planck-distributed, Hawking-like flux of particles will occur – even though the object never collapses to form a horizon. Also other authors claim that no horizon is needed in order for particle production to take place, and show that spherically symmetric shells of matter collapsing to extremely compact objects, with radii very close to – but slightly larger than – the event horizon of the would-be black hole, still possess particle creation (see e.g. [6], [7] or [8]). This Hawking-like radiation, which sometimes is referred to as pre-Hawking radiation, has further led some authors to suggest that the event horizon never forms [3, 4] – and hence question the very existence of black holes.

If Hawking radiation does not emerge from the event horizon itself, but rather somewhere in its vicinity, one should expect radiation to occur also from static objects with a radius slightly larger than the event horizon of the spacetime. Opening up for this possibility yields, however, a set of follow-up questions that need to be answered in order to get a complete picture of the situation. Firstly, if particles are created when stars collapse to objects without horizon – how large can these objects be before the geometrical optics approximation is no longer valid? Secondly, when looking at Penrose diagrams of collapse-models such as the one in figure 3.3, we see that also null rays leaving $\mathcal{I}^-$ at very early times, for $v \ll v_0$, propagate through the collapsing body and end up on $\mathcal{I}^+$. However, we are only considering null rays observed at late times on $\mathcal{I}^+$ in the proposed spectrum. Thus, how close to v_0 must the null rays on $\mathcal{I}^-$ be in order to be a part of this spectrum?

The literature on the topic is vast. In order to derive Hawking radiation from spacetimes of stars collapsing to form black holes, we need to know the entire history of the spacetime in question. This is connected to the global nature of event horizons, but also to the fact that we define Hawking radiation with respect to asymptotic observers. Nevertheless, it seems that the essential part of the discussion can be boiled down to the following question: *Where* and *when*, exactly, are the Hawking particles created? Because of the global properties of Hawking radiation, these questions are intrinsically difficult to answer. We thus do not hope to come to a final answer to these questions in this thesis. Instead, by addressing these questions in the following, we aim to narrow down the discussion of where and when Hawking radiation occurs to more fundamental properties.

Chapter 4

Hawking Radiation From Exotic Compact Objects (ECOs)

Event horizons are generally not observable, as such observations require a knowledge of the entire spacetime of the black hole – including the infinite future [26]. This is in contrast to apparent horizons and trapped surfaces, which can – in theory – be detected by quasi-local measurements. Since an asymptotic observer never actually sees the formation of an event horizon from a collapsing star, this observer cannot distinguish such an astrophysical black hole from a very compact object – at least not through measurements of light rays [3, 4, 14]. Such objects, which look like black holes to distant observers but lack event horizons, are called Exotic Compact Objects (ECOs). Depending on how close the object's surface is to the would-be horizon, and the interior of the object, different ECOs can be defined. Examples of ECOs are gravastars and wormholes [14]. The "exoticness" of this classification is related to the dubious nature of the objects in question. This far, ECOs are merely theoretical curiosities – they have not yet been observed, though some scientists claim to have found suggestions of their existence [28]. Furthermore, there are no known physical processes as of today which are able to inhibit a collapsing star from falling past the event horizon when it is this close to the event of horizon-formation: The gravitational pull in this region seems to overcome all known pressure effects inside the collapsing matter. Though there exist multiple suggestions to what these objects are and how they are formed, their physical significance thus remains unclear and their stability is disputed [31]. For the sake of the discussion of where and when the Hawking radiation takes place, however, they may provide useful insight. We will therefore look into the details regarding Hawking radiation from a subclass of ECOs in the following.

Before delving into these scenarios, we will try to give a more quantitative motivation for the calculations to come. As pointed out in the previous chapter most of the derivation of Hawking radiation in the scenario of a collapsing star does not explicitly make use of the existence of an event horizon for the creation of particles. Rather, the discussion is typically restricted to the vicinity of the horizons. It may therefore seem strange that there should be a qualitative difference between two objects with the same mass, but slightly different radii. Except for the obvious dissimilarity at the event horizon, we would not expect any physical difference between the region *just outside* the event horizon of a Schwarzschild black hole and the immediate vicinity of a static ECO. More specifically, we expect the geometric optics approximation – which enables us to treat null rays with high frequencies as null geodesics – to be valid also outside an ECO, given that the radius of the ECO is small enough. If this is indeed the case, then many of the arguments leading to Hawking radiation from black holes can be transferred directly to ECOs. It would therefore be helpful for the discussion to see whether the motivation can be envisioned by quantitative arguments as well. Put simply; we would like to know exactly *how* close to the event horizon this approximation holds.

4.1 Range of Validity for the Geometric Optics Approximation

Consider a given domain of space of length L and suppose that high-frequency null rays with wavelengths λ propagate through this domain. If the wavelengths λ are a lot smaller than the size L of the domain, we may treat the background on top of which the null rays propagate as approximately flat. This allows us to use the geometric optics approximation, which is valid as long as the distance over which the geometry varies is a lot larger than the wavelengths of the studied light rays [32]. Since we are interested in quantifying how far from the Schwarzschild radius this approximation can be used, we seek to quantify the distance $L - R_s$ where L satisfies

$$L \gg \lambda, \tag{4.1}$$

for small λ. We thus first need to find expressions for the characteristic length, L, and the characteristic wavelength, λ, which satisfy Eq. (4.1) in Schwarzschild spacetime.

Since the Riemann curvature tensor contains information about the curvature of the spacetime around a given object of mass M, we can relate the characteristic distance L to this tensor through $R_{\alpha\beta\gamma\delta} \sim L^{-2}$ [32]. In an orthonormal basis, we thus have [33]

$$L(r) \sim \sqrt{\frac{1}{R_{trtr}}} = \sqrt{\frac{r^3}{R_s}}. \tag{4.2}$$

To define the characteristic wavelength λ, we may use the wavelength that corresponds to the peak of the energy distribution observed in the spectrum on $\mathcal{I}^+$. We denote this wavelength by λ_{peak}. Close to the Schwarzschild radius, λ_{peak} will be highly blueshifted as a consequence of the strong gravitational field in this area. As a function of the radial Schwarzschild coordinate, we may therefore write the characteristic wavelength as

$$\lambda(r) = \sqrt{1 - \frac{R_s}{r}}\, \lambda_{\text{peak}}. \tag{4.3}$$

Assuming that the spectrum on $\mathcal{I}^+$ is completely thermal, we can describe it as a Planck spectrum with a spectral energy density

$$B(\omega, T) = \frac{\omega^3}{4\pi^3} \frac{1}{e^{\omega/k_B T} - 1}, \tag{4.4}$$

where the constant k_B is the Boltzmann constant, T is the temperature of the radiating body and ω is an angular frequency. For a given temperature of radiation, we can find the specific angular frequency that corresponds to the maximum of Eq. (4.4). The rays with this angular frequency contribute to the maximal energy in the thermal spectrum observed on $\mathcal{I}^+$. Hence, for a thermally radiating body we define λ_{peak} to be the wavelength on $\mathcal{I}^+$ that corresponds to this angular frequency. To find the wavelength λ_{peak} we must solve the equation $dB(\omega, T)/d\omega = 0$ for a given temperature T, and use that $\omega_{\text{peak}} = 2\pi/\lambda_{\text{peak}}$. For a body of temperature $T = \kappa/2\pi k_B$, this yields

$$\lambda_{\text{peak}} = \frac{8\pi^2 R_s}{x}, \tag{4.5}$$

where we have defined $x \equiv \omega_{\text{peak}}/k_B T = 2.82144$ and used that the surface gravity of a Schwarzschild black hole can be written as $\kappa = 1/2R_s$.

Having found suitable expressions for the characteristic length of the domain and the characteristic wavelength of the radiation, we are ready to quantify the range of validity for the relation in Eq. (4.1) in Schwarzschild spacetime. Instead of dealing explicitly with this inequality, we will rather write the relation as $L(r) = s\lambda(r)$, where s is a real, positive number. The inequality is reattained by demanding $s \gg 1$. Inserting Eq. (4.2) and Eq. (4.3) into this relation, we obtain the following polynomial equation,

$$\hat{r}^4 - s^2 \hat{\lambda}^2_{\text{peak}} \hat{r} + s^2 \hat{\lambda}^2_{\text{peak}} = 0, \tag{4.6}$$

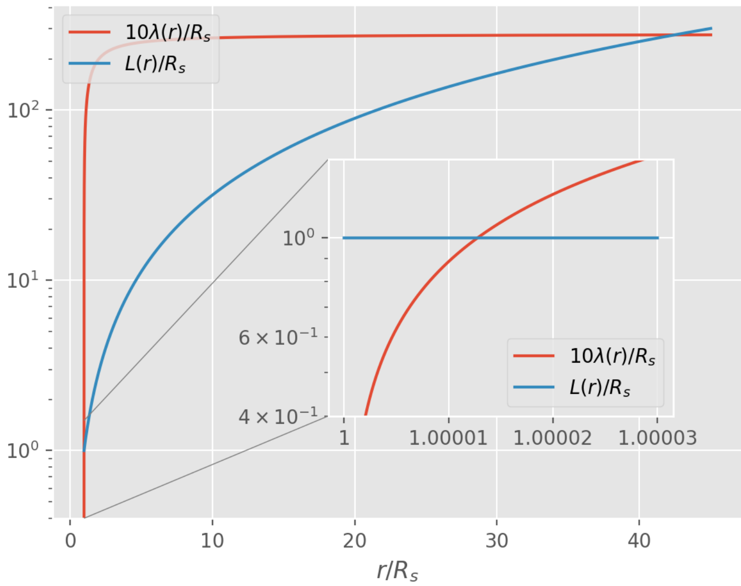


Figure 4.1: Intersections between the curves $L(r)$ and $s\lambda(r)$ for $s = 10$, given relatively to the Schwarzschild radius R_s. The intersection points are found to be at $r = 1.0000128R_s$ and $r = 42.445125R_s$. The subplot shows a close-up of the intersection near $r = Rs$.

where we have defined $\hat{r} = r/R_s$ and $\hat{\lambda}_{\text{peak}} = \lambda_{\text{peak}}/R_s$. Solving this equation without specifying the value of s gives a rather unpleasant expression. Therefore, we choose to specify what we mean by $s \gg 1$ instead. As we are working in natural units, setting $s = 10$ should suffice. In this case we find that for $r < 1.0000128R_s$ and $r > 42.445125R_s$ the distances over which the geometry changes is a lot larger than the wavelengths of the null rays propagating in the respective areas. Figure 4.1 shows the points of intersection between the curves $L(r)/R_s$ and $s\lambda(r)/R_s$, for $s = 10$. The solution at $r \approx 42R_s$ should not come as a surprise: Schwarzschild spacetime is asymptotically flat. Thus, far from the Schwarzschild black hole, we do indeed expect null rays of very high frequencies to propagate by geometrical optics.

From the zoom-in of the smallest root in Fig.4.1, we clearly see that the relation $L \gg \lambda$ also holds in a small region, extremely close to the Schwarzschild radius. Hence, within some finite distance from the event horizon the geometrical optics approximation is valid. The maximal relative distance from the Schwarzschild black hole where this approximation holds, is given by

$$\frac{L_{\text{max}} - R_s}{R_s} \approx 3.84 \cdot 10^{-5}, \tag{4.7}$$

where we have defined $L_{\text{max}} \equiv \sqrt{r_1^3/R_s}$ and $r_1 \equiv 1.0000128R_s$ is the intersection point closest to the event horizon. As a consequence, all ECOs of relative radii smaller than the quantity given in Eq. (4.7) can possibly serve the same role as a Schwarzschild black hole. Therefore, Eq. (4.7) yields a clear boundary on the size of the ECOs that can replace the black hole in the calculations of Hawking radiation presented in chapter 3.

Equivalently, close to the Schwarzschild radius the light rays that give rise to the thermal spectrum on $\mathcal{I}^+$ can maximally have a wavelength of size

$$\frac{\lambda_{\text{max}}}{\lambda_{\text{peak}}} \approx 0.00358, \tag{4.8}$$

relative to the wavelength λ_{peak} measured on $\mathcal{I}^+$, in order for the geometrical optics approximation to hold. Here we have defined $\lambda_{\text{max}} \equiv \sqrt{1 - R_s/r_1}\lambda_{\text{peak}}$.

Together with the unobservable nature of event horizons, these explicit calculations showing that the geometric optics approximation is valid also in the vicinity of the event horizon of a Schwarzschild black hole, definitely stimulates the search for Hawking radiation from ECOs. We will study the implications of this in the following.

4.2 Hawking Radiation from Static ECOs

From the discussion in the beginning of the previous chapter we saw that eternal black holes radiate Hawking particles. On conceptual grounds, one may therefore ponder over whether static ECOs radiate too, and if not; what separates the former from the latter? In the following we will thus try to calculate the tentative spectrum of particles on $\mathcal{I}^+$ created by a static, spherically symmetric star of radius $r = R_*$, where R_* is constant.

By Birkhoff's theorem we know that the exterior region will be described by Schwarzschild spacetime. We will not make any assumption for the interior region other than spherical symmetry. For simplicity and for comparison with the case of a collapsing star, we will work in two dimensions. Let therefore the exterior metric be that of two-dimensional Schwarzschild spacetime presented in Eq. (3.9), and the interior metric take the form presented in Eq. (3.11).

For a static star of radius $r = R_*$ the collapse velocity of the surface, $\dot{R}$, is zero. Going back to the transformation equations between the interior and exterior coordinates on the surface of the star described in Eq. (3.20), we thus see that they become

$$\begin{aligned} \frac{d}{du}\alpha(u)\Big|_{r=R_*} &= \frac{dU}{du}\Big|_{r=R_*} = \sqrt{\frac{C}{A}}\Big|_{r=R_*}, \\ \frac{d}{dV}\beta(V)\Big|_{r=R_*} &= \frac{dv}{dV}\Big|_{r=R_*} = \sqrt{\frac{A}{C}}\Big|_{r=R_*}. \end{aligned} \tag{4.9}$$

On the surface $r = R_*$, $C = C_*$ is constant. The function $A(U(\tau, R_*), V(\tau, R_*))$ on the other hand, is not necessarily constant since $U(\tau, R_*) = V(\tau, R_*) = \tau$. As opposed to the scenario of a star collapsing to a black hole, there is no boundary on the time null rays must leave $\mathcal{I}^-$ in order to reach $\mathcal{I}^+$ in the spacetime of a static star. This becomes evident when looking at figure 3.5, where the null rays in the spacetime of the collapsing body in figure 3.5a are separated into two regions: one region consisting of null rays that end up on $\mathcal{I}^+$, and one with null rays entering the black hole horizon, eventually facing the singularity of the spacetime.

Because there is no boundary on $\mathcal{I}^-$ for which null rays propagate to $\mathcal{I}^+$, there is no pile-up of null coordinates on past null infinity. Thus we cannot simply adopt the argument from the collapsing spacetime stating that $A(U, V)$ must be constant in the null coordinate V, and the integral in Eq. (3.31) is not trivial. However, since we are looking at a compact object with radius very close to the event horizon, and because the star is static, we may suspect that the function $A(U, V)$ can be approximated with a constant value at the surface of the star also in this scenario.

With the assumption that $A(U, V)$ is constant at the star's surface, both C and A in Eq. (4.9) are constant. Solving these equations thus yields the relation

$$u = F^{-1}(v) = v - \text{constant}, \tag{4.10}$$

between the null coordinates u and v (see Eq. (3.35) for comparison). Since outgoing null rays are defined by a constant null coordinate u, the mode solutions to the wave equation on $\mathcal{I}^+$ therefore become

$$\frac{i}{\sqrt{4\pi\omega}}\left(e^{-i\omega(v - \text{constant})} - e^{-i\omega u}\right), \tag{4.11}$$

in this scenario. From the general discussion of chapter 2 we know that adding a constant phase to the plane wave modes of an inertial Minkowski observer is equivalent to performing a translation on the modes. Hence, since the modes used by an inertial observer on $\mathcal{I}^+$ is just a translation of the modes used by an inertial observer on $\mathcal{I}^-$, they must share the same vacuum state. In other words, particles will not be created in this case. To see this explicitly, we can make use of the following arguments: First, we name the outgoing modes in Eq. (4.11) p_ω, such that on $\mathcal{I}^-$ they are expressed by

$$p_\omega = \frac{i}{\sqrt{4\pi\omega}} e^{-i\omega(\nu - c)}, \tag{4.12}$$

where c is a constant. Then we continue by naming the set of mode solutions to the wave equation that are planar waves on $\mathcal{I}^-$

$$f_\omega = \frac{i}{\sqrt{4\pi\omega}} e^{-i\omega\nu}. \tag{4.13}$$

Together with their complex conjugates, these modes constitute a complete set of solutions to the wave equation. Thus we may express all other solutions to the same equation as linear combinations of these modes. In other words, we may write

$$p_\omega = \int d\omega' \left(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}\right). \tag{4.14}$$

Equating Eq. (4.12) and Eq. (4.14) in addition to taking the Fourier transform of both sides of the equation, we find that

$$\begin{aligned} \alpha_{\omega\omega'} &= \frac{1}{2\pi} e^{i\omega c} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\infty} d\nu \; e^{i(\omega' - \omega)\nu}, \\ \beta_{\omega\omega'} &= -\frac{1}{2\pi} e^{i\omega c} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\infty} d\nu \; e^{-i(\omega' + \omega)\nu}. \end{aligned} \tag{4.15}$$

Since we demand all angular frequencies to be positive so that positive- and negative-frequency modes are defined by Eq. (2.29), we must have $\delta(\omega' + \omega) = 0 \;\; \forall(\omega, \omega')$. Hence, computing the square of the absolute value of these Bogolubov coefficients, we get that

$$\begin{aligned} |\alpha_{\omega\omega'}|^2 &= \left|\frac{\omega'}{\omega}\right| \delta(\omega' - \omega)\delta(\omega' - \omega), \\ |\beta_{\omega\omega'}|^2 &= 0. \end{aligned} \tag{4.16}$$

Going back to Eq. (2.27) we see that if $|\beta_{\omega\omega'}|^2 = 0$, the vacuum expectation value of the number operator for the modes p_ω is zero in the vacuum of the modes f_ω. In other words, a stationary, asymptotic observer using the modes p_ω observes no particle production in the vacuum of a stationary, asymptotic observer using the modes f_ω in the case of a static star with a constant interior metric.

Thus, static ECOs do not seem to emit Hawking radiation, in spite of this being the case for eternal black holes. One crucial difference between a static ECO and an eternal black hole is that in the spacetime of the latter, there exist two different sets of natural, inertial observers; one set is defined with respect to the timelike Killing vector of a Schwarzschild observer on future null infinity, and another set is defined with respect to the timelike Killing vector of a Kruskal observer on the past event horizon [16]. These observers do not have coinciding vacuum states. Looking at the Penrose diagram of the spacetime of an eternal Schwarzschild black hole in figure 3.1, we see that the null rays emerging from the past event horizon must propagate to future null infinity, whereas the null rays emerging from past null infinity are confined to propagation through the future event horizon. Hence, the modes originating on $\mathcal{I}^-$ are causally disconnected from the modes originating on the past event horizon. This is not the case for the spacetime of a static ECO, which can readily be seen by examining figure 3.5a. Here all null rays must emerge from past null infinity, and consequently end up on future null infinity. The modes on $\mathcal{I}^-$ thus necessarily need to be related to the modes on $\mathcal{I}^+$, somehow. Since the spacetime of the ECO is static, only the gravitational field of the ECO can change the form of the propagating modes. However, because the modes propagate through the body from $\mathcal{I}^-$ to $\mathcal{I}^+$, the blueshift experienced by the modes falling in towards the static ECO from

$\mathcal{I}^-$ must exactly compensate the redshift of the same modes when they propagate from the center of the ECO and out to $\mathcal{I}^+$ [21]. Thus the modes on $\mathcal{I}^-$ must be directly related to the modes on $\mathcal{I}^+$, motivating the non-existence of Hawking radiation in this scenario.

4.3 Hawking Radiation from Dynamical ECOs

As we saw in the previous section, a static object with radius larger than the event horizon does not seem to radiate Hawking particles. In a collapsing spacetime, on the other hand, the modes propagating from the collapsing star to $\mathcal{I}^+$ will be more redshifted than they are blueshifted on their way from $\mathcal{I}^-$ to the star, due to the increase in the surface gravity of the star as the star's size decreases. If objects without horizons are to emit radiation, we may therefore suspect that this radiation is related to the dynamics of the actual collapse.

Hence, in the following we will study a star collapsing to a finite object of radius somewhat larger than the event horizon of the would-be black hole, if the star had collapsed to a Schwarzschild black hole. We must therefore go back to the derivatives of the transformation equations $\alpha(u)$ and $\beta(V)$ between the interior and exterior coordinates in a collapsing star, presented in Eq. (3.20). As before, the exterior metric is described by Eq. (3.9) and the interior metric is given by Eq. (3.11). More specifically, we will look at two collapse scenarios in the following: a fast collapse and a slow collapse.

4.3.1 Fast Collapse

For an interior metric that is finite and regular everywhere, the condition $x = AC(1-\dot{R}^2)/\dot{R}^2 \ll 1$ leading to the simplified equations of motion given in Eq. (3.21) can be satisfied in two ways: Either the surface of the star approaches the singularity of the exterior metric, yielding $C \to 0$, or the collapse velocity of the surface of the star approaches the speed of light, so that $\dot{R} \to -1$[1]. As demonstrated in the previous chapter, the former scenario leads to the creation of Hawking particles. Can something similar be found for the latter case?

We will assume that the star collapses to a radial coordinate distance $R_* = R_s(1+\epsilon)$ from the singularity of the spacetime, where $\epsilon \ll 1$. Then $C = \mathcal{O}(\epsilon)$. Further, we will presume that the collapse velocity is given by $\dot{R} = -1 + \delta$ for $\delta \ll 1$ close to this surface. To make sure that we are indeed investigating the limit where $\dot{R} \to -1$ and not where $C \to 0$, we must also assume that $\delta/\epsilon \ll 1$. Then the relation $x = AC(1-\dot{R}^2)/\dot{R}^2 \ll 1$ is satisfied because the term $(1-\dot{R}^2)$ is very small. Thus, we may simplify Eq. (3.20) as before, to obtain the same relations as in Eq. (3.21).

Expanding the surface of the collapsing star around $R_* = R_s(1+\epsilon)$ for $\epsilon > 0$, we find that

$$R(\tau) = R_* + \gamma_*(\tau_* - \tau) + \mathcal{O}\left((\tau_* - \tau)^2\right), \tag{4.17}$$

where $\tau_* \equiv \tau_s - \tau_\epsilon$ and $R_* \equiv R(\tau_*)$. The parameter γ_* is defined as $\gamma_* \equiv -\dot{R}(\tau_*)$. Similarly, the parameter $C(R)$ can be expanded as

$$C(R) = C_* + 2\kappa_*(R - R_*) + \mathcal{O}((R-R_*)^2), \tag{4.18}$$

where $C_* \equiv C(R_*)$ and $2\kappa_* \equiv \partial C/\partial r|_{r=R_*}$. To $\mathcal{O}(\tau_* - \tau)$ we thus want to solve the following equations:

$$\begin{aligned} \left.\frac{dU}{du}\right|_{r=R(\tau)} &\sim [C_* + 2\kappa_*(R-R_*)]\,\frac{\dot{R}-1}{2\dot{R}}, \\ \left.\frac{dv}{dV}\right|_{r=R(\tau)} &\sim A_*\frac{1-\dot{R}}{2\dot{R}}. \end{aligned} \tag{4.19}$$

Using that $dU = d\tau(1-\dot{R}) = -dR(\dot{R}-1)/\dot{R}$ and observing from Eq. (3.12) and Eq. (4.17) that to $\mathcal{O}(\tau_* - \tau)$ we have

$$\begin{aligned} U - U_* &= -(1+\gamma_*)(\tau_* - \tau), \\ V - V_* &= -(1-\gamma_*)(\tau_* - \tau), \end{aligned} \tag{4.20}$$

[1]Remember that we are looking at a collapsing star, so the velocity of the surface must be negative.

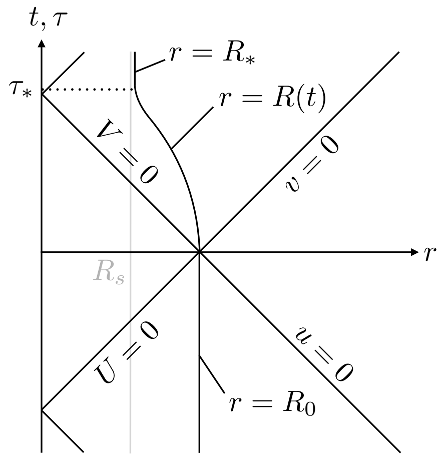


Figure 4.2: Fast collapse of a star to an ECO of radius R_*.

the first equation in Eq. (4.19) integrates to give

$$U \propto \frac{1+\gamma_*}{2\kappa_*\gamma_*}\left(e^{-\kappa_* u} - C_*\right) + \text{constant}, \tag{4.21}$$

where we have used Eq. (4.17) and Eq. (4.20) to write Eq. (4.21) in terms of the interior null coordinate U instead of the coordinate R. To solve the second equation we must first use that to $\mathcal{O}(\delta)$, we can write $(1-\dot{R})/\dot{R} = -2-\delta$ for $\dot{R} = -1+\delta$. Then we may integrate the second equation in Eq. (4.19) to get

$$v = -\left(1+\frac{\delta}{2}\right) A_* V + \text{constant}. \tag{4.22}$$

Combining Eq. (4.22) with Eq. (4.21) through the transformation equation, Eq. (3.32), further yields the relation

$$u = -\frac{1}{\kappa_*}\ln\left(\frac{a-v}{c}\right). \tag{4.23}$$

Here we have defined the constants $c = A_*(1+\delta)/\kappa_*$ and $a = \text{constant} - C_* c$, and used that $\gamma_* \equiv -\dot{R}(\tau_*) = 1-\delta$. This relation takes the same form as the inverse function in Eq. (3.35) which led to the mode solutions in Eq. (3.36). As we saw in the subsequent discussion, such a relation between the null coordinates u and v led to the creation of Hawking particles. We want to investigate whether Hawking particles may be created also in the present scenario. Adopting the reasoning in the previous chapter we therefore want to relate the following integrals,

$$\begin{aligned} \alpha_{\omega\omega'} &= D\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{\infty} ds\; e^{i\omega \ln(s/c)/\kappa_*} e^{-i\omega' s} e^{i\omega' a}, \\ \beta_{\omega\omega'} &= D\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{\infty} ds\; e^{i\omega \ln(-s/c)/\kappa_*} e^{-i\omega' s} e^{-i\omega' a}, \end{aligned} \tag{4.24}$$

where we have defined $s \equiv a - v$ in the $\alpha_{\omega\omega'}$-integral and $s \equiv v - a$ in the $\beta_{\omega\omega'}$-integral. These integrands have no poles in the lower complex s-plane, and we can therefore choose the contour of integration to lie in this domain, connecting the real axis with a boundary at infinity. Additionally, by computing the following limit for $s \equiv ze^{-i\phi}$,

$$\lim_{z\to\infty} e^{-i\omega' s} = \lim_{z\to\infty} e^{-i\omega' z\cos\phi} e^{\omega' z \sin\phi} = 0 \iff \phi \in (-\pi, 0), \tag{4.25}$$

we see that both integrals in Eq. (4.24) vanish on the boundary at infinity. Thus, the integrals are zero on the real axis, so both Bogolubov coefficients are zero. This can also be seen by examining

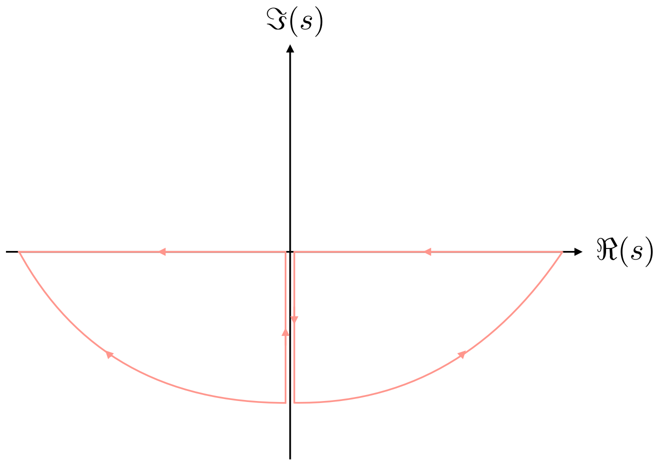


Figure 4.3: Contours of integration in the complex s-plane. These two segments are equivalent to equating the whole contour along the real axis with a contour at infinity in the lower complex plane. Because the integrals in Eq. (4.24) vanish on the whole boundary at infinity, the integrals over the whole real axis becomes zero.

the resulting complex contours of integration given in figure 4.3. Dividing the integrals in Eq. (4.24) into two parts, respectively, the integrals ranging from $-\infty$ to zero along the real axis is equal to the integral from zero to $-i\infty$ along the negative complex axis. The other part of the real integral, ranging from zero to ∞, can be exchanged with the integral from $-i\infty$ to zero: The integrals cancel each other out on the negative imaginary axis. From this we clearly see that $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ must be zero. This result is in itself problematic, as it means that the modes redshifted with Eq. (4.23) on $\mathcal{I}^-$ cannot be written as linear combinations of the modes that are plane waves on $\mathcal{I}^-$. What may be even more disturbing, is that this yields the relation $|\alpha_{\omega\omega'}|^2 = |\beta_{\omega\omega'}|^2$, which is simply not allowed by the normalization condition for the Bogolubov coefficients in Eq. (2.20).

The only difference between the integrals in Eq. (4.24) and the integrals found to produce Hawking radiation in Eq. (3.43) seems to be the boundary $v < v_0$ on the null rays on $\mathcal{I}^-$ that propagate through the body to reach $\mathcal{I}^+$. As described in section 3.2, this boundary is directly related to the existence of an event horizon, which causally separates the modes in the late time spectrum on $\mathcal{I}^-$ from the modes entering the horizon.

Acknowledging that the collapse scenario described above is rather unphysical as the star collapses at a speed close to the speed of light at the event that it settles to an ECO, we will in the following instead assume that the star has zero velocity at this event. Then we see from Eq. (4.17) that the term $R - R_* = 0$ in the first equation in Eq. (4.19). Performing the integration with $\dot{R} = -1 + \delta$ thus yields the relation

$$U = \left(1 + \frac{\delta}{2}\right) u + \text{constant}. \tag{4.26}$$

Still looking at the scenario with a constant interior and extracting a minus sign from the order of magnitude estimate in the second equation in Eq. (4.19), the null coordinates u and v can be related through

$$u = \frac{v - d}{c}, \tag{4.27}$$

for constants c and d. As we saw in the case of a static star, the null coordinate u is here just a scaled translation of the null coordinate v and inertial observers on $\mathcal{I}^+$ and $\mathcal{I}^-$ will therefore share the same vacuum state. Hence, in this scenario there will be no particle production.

4.3.2 Slow Collapse

If a star collapses to a finite radius $r = R_*$ without the advent of new physics, it may be more realistic to assume that it approaches this radius with a very slow velocity. We will therefore investigate the limit of slow collapse velocities in the following.

Again, going back to Eq. (3.20), a very slow collapse is envisaged by letting $\dot{R} \to 0$. Assuming that the star collapses to a finite radius $R_* = R_s(1+\epsilon)$, where $\epsilon \ll 1$ and positive, we may use that $C(R_*) = \mathcal{O}(\epsilon)$ is lager than the collapse velocity $\dot{R}$, which becomes zero on the surface $R = R_*$. Defining the parameter $x \equiv AC(1-\dot{R}^2)/\dot{R}^2$, as before, we will therefore perform an expansion around $1/x$ in this limit. Since $\dot{R}^2 \ll 1$, we may neglect this term inside the parenthesis, and expand around $\bar{x} = \dot{R}^2/AC \ll 1$ instead. To leading order, this yields the following equations,

$$\begin{aligned} \frac{dU}{du}\Big|_{r=R(\tau)} &= \sqrt{\frac{C}{A}}\left(1-\dot{R}\right), \\ \frac{dv}{dV}\Big|_{r=R(\tau)} &= \sqrt{\frac{A}{C}}\frac{1}{1+\dot{R}}. \end{aligned} \tag{4.28}$$

From the definitions of U and V given in Eq. (3.12), we see that we can write $dU = d\tau(1-\dot{R})$ and $dV = d\tau(1+\dot{R})$. Inserting these expressions into Eq. (4.28), we are therefore left to integrate

$$\begin{aligned} du &= \sqrt{\frac{A}{C}}d\tau, \\ dv &= \sqrt{\frac{A}{C}}d\tau. \end{aligned} \tag{4.29}$$

Close to $r = R_*$, we may use the expansions in Eq. (4.17) and Eq. (4.18) for the parameters $R(\tau)$ and $C(R)$, respectively. To $\mathcal{O}((\tau_* - \tau))$, we can thus write $R - R_* = \gamma_*(\tau_* - \tau)$. Furthermore, we assume that the interior metric is constant. Hence, in terms of the proper time, τ, of an observer on the surface of the collapsing star, we must solve the integrals

$$\int \sqrt{\frac{A_*}{C_* + 2\kappa_*\gamma_*(\tau_* - \tau)}}d\tau. \tag{4.30}$$

If the star's collapse velocity is exactly zero at the time τ_* that the surface reaches the radial coordinate $r = R_*$, we must have $\gamma_* \equiv -\dot{R}(\tau_*) = 0$. Then the integrals in Eq. (4.30) are the exact same integrals as for the static star of radius $r = R_*$ discussed at the beginning of this chapter, leading to the Bogolubov coefficients presented in Eq. (4.16). Thus, if the star has zero velocity at the time it reaches the surface R_*, there will be no Hawking radiation.

We therefore want to investigate the situation where γ_* is very small, but non-zero at the event of ECO-formation. For a non-zero γ_* the integrals in Eq. (4.30) are solved to obtain

$$-\frac{\sqrt{A_*}}{\gamma_*\kappa_*}\sqrt{C_* - 2\gamma_*\kappa_*(\tau_* - \tau)} + \text{constant}. \tag{4.31}$$

Substituting $\tau_* - \tau$ in Eq. (4.31) with the expressions for $U - U_*$ and $V - V_*$ given in Eq. (4.20), respectively, further yields

$$\begin{aligned} U &= U_* + \frac{1+\gamma_*}{2\kappa_*\gamma_*}\left[\left(-\frac{\kappa_*\gamma_*}{\sqrt{A_*}}(u - \text{constant})\right)^2 - C_*\right], \\ v &= \frac{-\sqrt{A_*}}{\kappa_*\gamma_*}\sqrt{C_* + 2\kappa_*\frac{\gamma_*}{1-\gamma_*}(V - V_*)} + \text{constant}. \end{aligned} \tag{4.32}$$

Using the transformation equation $v = \beta(V) = \beta(U - 2R_0)$ from Eq. (3.14), and inverting the relation so that we get an expression for u as a function of v instead, we finally obtain the following

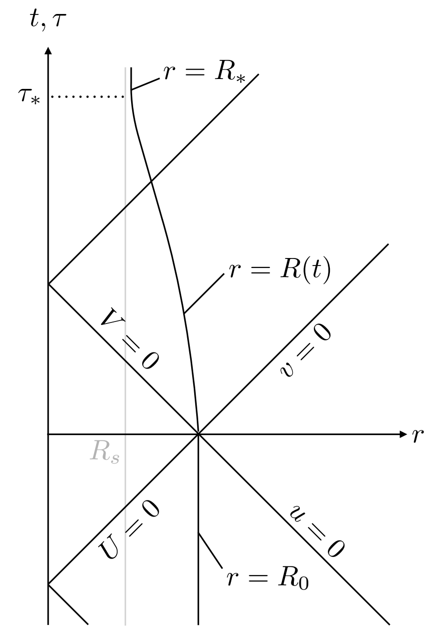


Figure 4.4: Minkowksi diagram of a star collapsing slowly to form an ECO of radius R_*.

outgoing mode solutions on $\mathcal{I}^-$ in the limit $\gamma_* \to 0$,

$$\frac{i}{\sqrt{4\pi\omega}} \exp\left\{-i\omega\left(\sqrt{(v-c)^2 + \frac{b}{\gamma_*}} + d\right)\right\}, \tag{4.33}$$

where $b = 2A_*(C_* - 2R_*\kappa_*)/\kappa_*^2$ and c, d are constants.

Defining a parameter s such that $s \equiv c - v$ in the $\alpha_{\omega\omega'}$-integral and $s \equiv v - c$ in the $\beta_{\omega\omega'}$-integral, we get that

$$\begin{aligned} \alpha_{\omega\omega'} &= \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\infty} ds\; e^{-i\omega\left(\sqrt{(-s)^2+\gamma_*^{-1}b}+d\right)} e^{-i\omega' s} e^{i\omega' c}, \\ \beta_{\omega\omega'} &= \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\infty} ds\; e^{-i\omega\left(\sqrt{s^2+\gamma_*^{-1}b}+d\right)} e^{-i\omega' s} e^{-i\omega' c}. \end{aligned} \tag{4.34}$$

These integrals are identical, yielding the relation $|\alpha_{\omega\omega'}|^2 = |\beta_{\omega\omega'}|^2$ which violates the normalization condition for the Bogolubov coefficients.

In all illustrated scenarios above we have assumed that the internal metric has been constant on the surface of the object in question. For a static ECO or a very slowly collapsing star we may be able to argue in favor of such an assumption. For the scenario of a fast collapse, on the other hand, the physicality of making such an assumption becomes less obvious. However, seeing that this latter scenario is rather unrealistic anyways, we have not made too big of a deal out of this assumption here. Nevertheless, a non-constant interior metric on the surface of the object in question can be implemented in the calculations above, for example by expanding the parameter $A(U, V)$ to first order around the null coordinates $U_* \equiv U(\tau_*, R_*)$, $V_* \equiv V(\tau_*, R_*)$ as

$$A = A_* + \frac{\partial A}{\partial U}\Big|_{U_*,V_*} (U - U_*) + \frac{\partial A}{\partial V}\Big|_{U_*,V_*} (V - V_*)\,, \tag{4.35}$$

where $A_* \equiv A(U_*, V_*)$. Using (4.20) we may further simplify the expansion in Eq. (4.35) by

$$A = A_* - \chi_*(\tau_* - \tau), \tag{4.36}$$

where we have defined

$$\chi_* = (1+\gamma_*)\frac{\partial A}{\partial U}\Big|_{U_*,V_*} + (1-\gamma_*)\frac{\partial A}{\partial V}\Big|_{U_*,V_*}. \tag{4.37}$$

Implementing these relations in the scenarios above seems to yield the same relation between the Bogolubov coefficients as in the studied scenarios of fast collapse, namely that $|\alpha_{\omega\omega'}|^2 = |\beta_{\omega\omega'}|^2$. As already pointed out, such a relation violates the normalization condition for the Bogolubov coefficients given in Eq. (2.20). At this stage it is not entirely clear what we are to make out of such a result. As we will discuss in chapter 5, it may imply that a clear boundary on the null coordinates on past null infinity – such as from an event horizon – is not only a sufficient condition, but may in fact be a necessary condition in order for Hawking radiation to occur.

Chapter 5

Discussion

Based on the original derivation of Hawking radiation from stellar collapse, as well as on more recent research on the topic, we have displayed both qualitative and quantitative arguments supporting the possibility of Hawking radiation from compact, horizonless objects. Most importantly, we showed in section 4.1 that the geometric optics approximation is valid also in the vicinity of the event horizon of a Schwarzschild black hole. This approximation allows us to treat the null rays travelling through the collapsing body at late times as null geodesics, thus enabling us to relate the affine parameter distance between two null coordinates on past null infinity to the affine parameter distance between the same two null coordinates on future null infinity. In Appendix B this procedure is shown in detail for a four dimensional Schwarzschild spacetime. Because the geometric optics approximation only requires the null rays in question to be extremely *close to*, but not exactly *on* the event horizon, it is also valid for highly compact objects such as a bounded class of ECOs – which may suggest that they radiate in a similar manner to black holes.

In spite of this, the results from chapter 4 seem to point in a less optimistic direction regarding ECOs' ability to radiate. We study three main scenarios in this thesis: a static ECO, a rapidly collapsing star settling to a static ECO and a slowly collapsing star settling to a static ECO. The examination of the first scenario is motivated by the similarities between a Schwarzschild black hole and a static ECO, whereas the latter two bear similarities with the collapse model presented in chapter 3. In the static as well as the slowly collapsing scenarios we find little evidence for Hawking radiation. Similarly, for a rapidly collapsing star with zero collapse velocity at the ECO formation-event, Hawking radiation does not seem to occur. In all these scenarios we obtain the same results as the ones given in Eq. (4.16), which show that the Bogolubov coefficients $\beta_{\omega\omega'}$ and $\alpha_{\omega\omega'}$ are found to be respectively zero and infinite. By Eq. (2.22) and Eq. (2.23) a zero β-coefficient leads to no admixture of annihilation and creation operators, and hence no disagreement between different inertial observers on the definition of vacuum and particles.

That this is indeed the case can be seen explicitly from Eq. (2.27), where a β-coefficient that is zero leads to no particle production in the vacuum state of an inertial observer on $\mathcal{I}^-$, as seen by an inertial observer on $\mathcal{I}^+$. The fact that the α-coefficient is infinite in these scenarios only means that during the whole history of the given spacetime an infinite amount of particles propagate from $\mathcal{I}^-$, through the static body and all the way to $\mathcal{I}^+$. This is reasonable: we do expect null rays to emerge from $\mathcal{I}^-$ and propagate to $\mathcal{I}^+$ during the whole history of the spacetime in question, and because the vacuum state of inertial observers in the asymptotic past and future coincide, no *new* particles should be created from the propagation through the collapsing body.

Based on what we find in section 4.3.2, we have therefore no reason to believe that slowly collapsing stars forming compact, horizonless objects emit Hawking radiation. This result may be surprising, at least in regard of the recent discussion on Hawking radiation from collapsing stars. For example, as mentioned earlier, Barceló et al. argue that a star collapsing to a static, horizonless object will emit

Hawking-like radiation, as long as the collapse satisfies a certain adiabatic condition[1] [5, 24].

For a star collapsing at nearly the speed of light at the event of ECO-formation, on the other hand, we find a relation between the null coordinates on past and future null infinity that resembles the relation in Eq. (3.35) which, as described in chapter 3, leads to the existence of Hawking radiation. Nevertheless, in the former scenario we find that the resulting Bogolubov coefficients are equal up to a phase factor, thus violating the normalization conditions for the Bogolubov coefficients given in Eq. (2.20). As mentioned at the end of chapter 4, we get similar relations between the Bogolubov coefficients in all the above-mentioned scenarios if we include a slightly changing interior metric at the surface of the respective bodies. The invalid nature of these results is problematic, and may suggest that some of our fundamental assumptions are wrong.

However, a comparison between the integrals in Eq. (4.24) and the Bogolubov coefficients in Eq. (3.43) from which we find Hawking radiation, reveals that up to a phase factor the only difference between the respective integrals are the integration boundaries. In Eq. (4.24) we have no physical reason to expect the null coordinates on $\mathcal{I}^-$ to be bounded, so the integrals are defined on the whole real axis. For the coefficients in Eq. (3.43), the event horizon creates a clear boundary on which null coordinates are included in the integral. In fact, allowing the integrals in Eq. (3.43) to be integrated over the whole real axis yields the same relation between the Bogolubov coefficients as in the case of a fast collapse to an ECO, and therefore violate the normalization conditions in Eq. (2.20). Likewise, restricting the integrals in Eq. (4.24) to be valid only for half of the real axis yields the same relation between the Bogolubov coefficients as in Eq. (3.46), which leads to Hawking radiation.

This discontinuity in the validity of the integrals can be traced back to the fact that the null coordinates we are considering are restricted to real valued functions. Forcing the null coordinate u to be real in Eq. (4.23) indeed makes sure that the integrals in Eq. (4.24) make sense only for a restricted part of the real axis. Also in the scenario of a star collapsing to a Schwarzschild black hole, such a restriction must be implemented for the exterior outgoing null coordinate given in Eq. (3.35). In this latter scenario, the problem sorts itself out rather easily as the event horizon defines a clear, physical bound on the null coordinates on $\mathcal{I}^-$. From this we see that the argument in the logarithm, which enters into the complicated phase factor in Eq. (3.36), always stays positive.

Since the spacetimes of ECOs do not comprise event horizons, the integrals in Eq. (4.24) do not generally make sense. Nevertheless, we may ask ourselves whether there exist different physical arguments enabling us to bound the null coordinates on past null infinity, also in these spacetimes. Assume, for example, that some null ray v_* is the last null ray to leave $\mathcal{I}^-$, pass through the collapsing body and end up on $\mathcal{I}^+$ before the collapsing star settles to a static ECO. All rays leaving $\mathcal{I}^-$ after this, i.e. for $v > v_*$, will effectively propagate thorough a static ECO which, as we saw in section 4.2, does not give rise to any Hawking radiation – at least not if the interior metric is constant. Then we would actually find that the Bogolubov coefficients related to the modes leaving past null infinity for null coordinates smaller than v_* are related thorough a factor of $e^{2\pi\omega/\kappa_*}$, which yields a Planck spectrum of the form given in Eq. (3.51), at a temperature $T = \kappa_*/2\pi k_B$. Thus, under the above-mentioned assumptions, a star collapsing rapidly to form an ECO will emit Hawking radiation. However, it is not clear whether this argumentation actually holds. In the spacetime of a star collapsing to a black hole the modes that leave $\mathcal{I}^-$ for null coordinates $v > v_0$ enter the event horizon. Thus, these modes propagate to a part of spacetime that is causally disconnected from the spacetime at $\mathcal{I}^+$. The Bogolubov coefficients related to the spectrum observed on $\mathcal{I}^+$ are therefore solely made up of the null rays leaving $\mathcal{I}^-$ for $v < v_0$. This is not the case in the spacetime of an ECO where all null rays emerging from $\mathcal{I}^-$ must, eventually, end up on $\mathcal{I}^+$, and therefore enter into the expressions for the Bogolubov coefficients.

What we have found suggests that in most horizonless scenarios, Hawking radiation does not occur. In the case of a static ECO, it nonetheless seems strange that there should be a qualitative difference between two objects with the same mass, but slightly different radii: A static ECO of radius

[1] They define this adiabatic condition as $|d\kappa(u_*)/du|_{u_*} \ll \kappa(u_*)^2$, where $\kappa(u) \equiv -(d^2U/du^2)/(dU/du)$, and $\kappa(u_*)$ can be interpreted by the regular notion of surface gravity only in an asymptotically settled black hole spacetime.

$R_s(1+\epsilon)$ with $\epsilon \ll 1$, and a Schwarzschild black hole with radius R_s have very similar gravitational potentials. However, one conceptually distinct feature of black holes that separate them from static, compact objects is the event horizon – a surface beyond which no information escapes to future null infinity. Thus, even though the gravitational potential in the region *just outside* the event horizon of a Schwarzschild black hole and the immediate vicinity of a static ECO are essentially the same, the former has an infinitely deep gravitational well whereas the latter extends only to a large, but finite negative value. Looking at the Penrose diagrams of a static star and a star collapsing to a Schwarzschild black hole presented in figure 3.5, this distinction becomes more transparent. Without a horizon, there is nothing that can possibly separate the modes on $\mathcal{I}^-$ from the modes on $\mathcal{I}^+$ in terms of linear combinations of operators, and their vacuum states coincide.

As already stressed, our result yielding no radiation from slowly collapsing stars is in disagreement with many of the newer papers on Hawking radiation [6, 7, 24, 30]. This controversy may be connected to flaws in our argumentation or defining assumptions. Firstly, the model used in section 4.3.2 for a slow collapse does not take into account that the second derivative may change. In fact, we have restricted our calculations to $\mathcal{O}(\tau_* - \tau)$ when defining the worldline, $R(\tau)$, of the surface of the collapsing star. For a realistic collapse scenario we would expect that the velocity of the surface slows down as it approaches the final radius of the evolving ECO. Adding a non-zero second derivative into the description of the collapse may therefore give some further insight into what happens close to the formation of the ECO. Secondly, we needed to make some assumptions about the interior metric of the star to get to the results showing no Hawking radiation. Investigating what happens when the interior metric changes more drastically – which may be a reasonable assumption in the case of a fast collapse – would therefore also be interesting, and may lead to different results from the ones found in section 4.3.2.

However, in much of the literature leading to Hawking radiation from collapsing objects without horizons, the collapse models used have been that of a spherical shell of dust (see e.g. [6, 7]). Hence, in these scenarios the interior metric has been assumed to be that of Minkowski spacetime. A Minkowski interior metric is not a good approximation of the interior of a realistic, astronomical star. One may therefore ponder on whether the use of a Minkowski interior can be problematic in its own right, and that the inclusion of a more advanced interior metric yields different results. This has been investigated by Barceló et al. in a rather recent paper, but the conclusion of the final state of a more general collapse of this kind seems to depend highly on the properties of the interior metric [8]. Despite having had many supporters over the past twenty years, there have also been strong evidence in the opposite direction, claiming that Hawking radiation cannot occur unless horizons are formed [10–13].

As far as we know, there exists no realistic collapse scenario that leads to a static, horizonless object if the collapse velocity is close to the speed of light when the final object forms. The model we use to probe the physics connected to a fast collapse in section 4.3.1 is therefore highly unrealistic. Including a very large second derivative at the event that the ECO forms may be a step in the more realistic direction, but will still not suffice as a realistic model of a collapse without having to include new physics. Nevertheless, despite not being too realistic in its own nature, we may be able to point out some important differences between a spacetime with and without a horizon from this result. Because the existence of a horizon in a collapse scenario does not yield any constraints on the collapse velocity of the object at the event of horizon formation (besides the natural constraint of the speed never passing the speed of light), the existence of Hawking radiation in such a scenario, as opposed to the scenarios of slowly collapsing objects without event horizons, may be traced back to the fact that the velocity of the collapse can be close to the speed of light in the former scenario but not in the latter – unless we allow for the advent of new physics. That Hawking radiation occurs for stars collapsing at the speed of light is also something that can be found both in the literature [25].

Finally, it is worth stressing that the discussion in this thesis has been restricted to $1+1$ dimensions. However, as detailed in Appendix A, none of the qualitative conclusions presented here are expected to change when considering black holes and ECOs in full $3+1$ dimensional spacetimes instead.

Chapter 6

Outlook

Motivated by the question of whether the existence of a horizon is necessary in order for Hawking radiation to occur, we started this thesis by showing how the ambiguity of the concept of particles in quantum field theory in curved spacetimes leads to particle production. More specifically, we saw that in spacetimes where there exists a timelike Killing vector, a natural set of observers can be found from which the notion of vacuum is well-defined. From this we showed that an accelerated observer in Minkowski spacetime measures a thermal flux of particles in the vacuum state of an inertial Minkowski observer. Through the equivalence principle we further argued that such a thermal flux also arise in the spacetime of an eternal Schwarzschild black hole, where a freely falling observer detects particles in the vacuum of a static observer. We demonstrated subsequently that eternal black holes radiate particles thermally, and showed that the temperature of radiation scales with the gravitational acceleration of the freely falling observer as seen by a distant Schwarzschild observer.

Because black holes are thought to be created from stellar collapse, we continued our discussion in the spacetime of a star collapsing to form a Schwarzschild black hole. With the ambiguity of particles in mind, we argued that each asymptotic region in this spacetime constitute a natural set of observers with corresponding vacuum states. These vacuum states will in general not coincide, and an observer in the future asymptotic region will therefore generally register particles in the vacuum state of an observer in the asymptotic region in the past. We used this to show that a black hole forming from stellar collapse emits Hawking radiation. The purpose of showing this calculation was to pinpoint exactly where in this argumentation the existence of an event horizon is needed. We found that the event horizon is essential in order to use the geometrical optics approximation and to define a clear boundary on which null coordinates on past null infinity will propagate to $\mathcal{I}^+$ through the collapsing star.

Arguing both qualitatively and quantitatively that the geometric optic approximation is valid also in the vicinity of the event horizon, we further investigated the possibility of Hawking radiation from a chosen class of exotic compact objects (ECOs). The motivation for doing this was to understand whether the arguments connected to the horizon could be transferred to horizonless spacetimes. We studied three main scenarios: a static ECO followed by a fast and slowly collapsing star forming a static ECO, respectively. In all scenarios we found little evidence of Hawking radiation. As discussed in chapter 5, the result of no Hawking-like radiation from slowly collapsing stars forming horizonless objects seems to contravene with recent literature on the topic [6, 7, 24, 30]. However, we also find support for our result [10–13].

For the scenario of a star collapsing close to the speed of light at the event of ECO-formation, we found that the relation between the resulting Bogolubov coefficients violated the normalization condition of these coefficients. This was also the result in all above-mentioned scenarios if we let the interior metric change slightly on the surface of the given object. We acknowledge that the ill-defined nature of these results may be a consequence of badly chosen assumptions. Nonetheless, our results seem to be more reasonable once we bound the values of the null coordinate on past null infinity. In fact, in the scenario of the fast collapse such a boundary leads to the same integrals for the Bogolubov coefficients from which we found Hawking radiation in the scenario of a star collapsing to a black hole. This may suggests that a clear boundary on the modes that propagate through the star and end up on future null infinity is needed in order to obtain Hawking radiation.

Our results and discussion seem to point in the direction favouring the necessity of horizons in the calculations of Hawking radiation. From classical general relativity we know that the event horizon is a rather unique astronomical concept. As opposed to other kinds of horizons, it marks a boundary of no return: Inside the event horizon everything travels towards the black hole singularity. Since the event horizon is manifestly different from the surface of a star or an ECO, we may find the qualitative difference between the spacetimes, in that Hawking radiation exists in the former but not in the latter, less puzzling. Nonetheless, we have not made any attempt to explicitly distinguish between event horizons and apparent horizon, trapped surfaces and the like in this thesis. A natural follow-up on the discussion in question would therefore be to investigate whether the same argumentation leading to Hawking radiation in spacetimes with event horizons can be employed also for other types of horizons. Specifically, since much of the literature on the topic investigates the possibility of Hawking radiation from collapsing shells a more thorough examination of a collapse with a non-trivial metric should be carried out properly. Also, investigating more realistic collapse scenarios numerically, preferably in spacetimes with less degrees of symmetries, may give further insight into the current discussion.

In conclusion, the discussion of whether or not horizons are needed for the existence of Hawking radiation is yet to be finished.

Appendices

Appendix A

Dimensional Differences and Limitations of the 2D-Model

Throughout this paper we have kept our discussion to two dimensions. From our two-dimensional models we have then drawn conclusions regarding the nature of objects that in reality are four-dimensional. Without further explanation, we have stated that the conclusions found in two dimensions easily translate to four dimensions. We therefore seek to concretize what we mean by this frequently used statement. To do this, we must go all the way back to the Schwarzschild metric and the massless Klein-Gordon equation. In $3+1$-dimensions the Schwarzschild metric takes the form

$$ds^2 = \left(1 - \frac{2M}{r}\right) du dv - r^2 d\Omega^2, \tag{A.1}$$

where the quantity $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ denotes a two dimensional sphere of radius r, and the null coordinates (u, v) are defined as

$$\begin{aligned} u &= t - \int_{R_0}^{r} \left(1 - \frac{2M}{r'}\right)^{-1} dr' = t - r^* + R_0^* \\ v &= t + \int_{R_0}^{r} \left(1 - \frac{2M}{r'}\right)^{-1} dr' = t + r^* - R_0^*, \end{aligned} \tag{A.2}$$

where

$$\frac{dr^*}{dr} = \frac{dR_0^*}{dR_0} = \left(1 - \frac{2M}{r}\right)^{-1}. \tag{A.3}$$

Comparing with Eq. (3.1), we see that the two-dimensional spacetime is a foliation of the four-dimensional spacetime where each point in the former corresponds to a two-sphere in the latter.

Due to spherical symmetry in four-dimensional Schwarzschild spacetime, the solutions to the Klein-Gordon equation defined in Eq. (2.6) with the metric given in Eq. (A.1) can for a massless scalar field be separated into a radial, angular and time component. The solutions are thus of the form [21]

$$\frac{1}{r} Y_{lm}(\theta, \phi) R_{\omega l}(r) e^{-i\omega t}, \tag{A.4}$$

where Y_{lm} is a spherical harmonic and the radial function $R_{\omega l}$ satisfies the equation

$$\frac{dR_{\omega l}^2}{dr^{*2}} + \left\{\omega^2 - \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right]\left(1 - \frac{2M}{r}\right)\right\} R_{\omega l} = 0. \tag{A.5}$$

In $3+1$ dimensions we cannot write the solutions to this radial function in terms of known functions [21]. Observing that Eq. (3.3) has the form of a one-dimensional wave equation with a potential term, we may get around this by artificially setting the term in the square brackets to zero. This corresponds to neglecting backscattering of the field modes on the spacetime curvature [21]. Then we can solve the radial equation straight-forwardly, to obtain the normalized mode solutions

$$\frac{Y_{lm}}{\sqrt{8\pi^2\omega r}}\left(e^{-i\omega v}+e^{-i\omega u}\right). \tag{A.6}$$

Far from the black hole event horizon these modes reduce to flat spacetime modes, which can be seen by sending $r \to \infty$ in Eq. (A.2). By the same reasoning that lead to the redshifted modes in Eq. (3.36), we may write the modes that become complicated on $\mathcal{I}^-$ but which are standard plane waves on $\mathcal{I}^+$ as

$$\frac{Y_{lm}}{\sqrt{8\pi^2\omega r}}e^{i\omega \ln((v_0-v)/c)/\kappa}, \quad v < v_0, \tag{A.7}$$

where c is a constant and $\kappa = 1/4M$ is the surface gravity of the Schwarzschild black hole event horizon. As before, all null rays leaving $\mathcal{I}^-$ for $v > v_0$ enter the event horizon of the black hole, and can thus not be observed in the spectrum on $\mathcal{I}^+$.

Writing these modes as linear combinations of the modes proportional to $\exp(-i\omega v)$ on $\mathcal{I}^-$ and taking the Fourier transform of both sides yields the following Bogolubov coefficients for the four-dimensional Schwarzschild spacetime,

$$\begin{aligned}\alpha_{\omega\omega'} &= C\int_{-\infty}^{v_0} dv\sqrt{\frac{\omega'}{\omega}}e^{i\omega' v}e^{i\omega \ln((v_0-v)/c)/\kappa}\\ \beta_{\omega\omega'} &= C\int_{-\infty}^{v_0} dv\sqrt{\frac{\omega'}{\omega}}e^{i\omega' v}e^{i\omega \ln((v_0-v)/c)/\kappa}\end{aligned} \tag{A.8}$$

for a constant C. These coefficients look very much like the Bogolubov coefficients in the two-dimensional case, given in Eq. (3.41) and Eq. (3.42). Indeed, also in this case we can use the method presented in chapter 3 by writing the integrals in the complex plane to find that $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are related by

$$|\alpha_{\omega\omega'}|^2 = e^{2\pi\omega/\kappa}|\beta_{\omega\omega'}|^2. \tag{A.9}$$

From the normalization condition of the Bogolubov coefficients given in Eq. (2.20), this relation yields a thermal spectrum of the same form as in Eq. (3.50). However, Eq. (3.49) is only valid if all of the modes observed as standard plane waves for late times on $\mathcal{I}^+$ have propagated through the collapsing star on their way from $\mathcal{I}^-$. This is typically not the case in more than two dimensions. Remembering that we neglected the backscattering of field modes from the gravitational potential in Eq. (A.5), we must subtract these scattered modes from the spectrum on $\mathcal{I}^+$ for a complete description of the radiation from the collapsing object. This can be done by assuming that only a fraction $\Gamma(\omega)$ of the particles that reach $\mathcal{I}^+$ has propagated through the collapsing body from $\mathcal{I}^-$. The rest of the particles observed in a given wavepacket on $\mathcal{I}^+$ come from modes that have scattered off the effective gravitational potential on their way from $\mathcal{I}^-$. Since these modes do not propagate through the collapsing body, their angular frequencies will stay approximately the same over their whole paths of propagation. More importantly, they will not give rise to the thermal radiation of Hawking particles and and should thus be omitted from the expression above. Naming these modes $\bar{p}_\omega$, an observed wavepacket of planar waves on $\mathcal{I}^+$ can thus be written as

$$P_\omega = p_\omega + \bar{p}_\omega. \tag{A.10}$$

Because the modes p_ω and $\bar{p}_\omega$ come from disjoint regions on $\mathcal{I}^-$, their inner product must vanish all over the spacetime. Hence,

$$(P_\omega, P_{\omega''}) = (p_\omega, p_{\omega''}) + (\bar{p}_\omega, \bar{p}_{\omega''}) = \delta(\omega-\omega''). \tag{A.11}$$

Defining $\Gamma(\omega)$ as the fraction of an outgoing wavepacket on $\mathcal{I}^+$ that when propagated backwards in time pass through the collapsing body and emerge to $\mathcal{I}^-$, we must have that

$$\begin{aligned}(p_\omega, p_{\omega''}) &= \Gamma(\omega)\delta(\omega-\omega''),\\ (\bar{p}_\omega, \bar{p}_{\omega''}) &= (1-\Gamma(\omega))\,\delta(\omega-\omega'').\end{aligned} \tag{A.12}$$

The normalization condition Eq. (2.20) for the Bogolubov coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ therefore becomes

$$\Gamma(\omega)\delta(\omega-\omega'') = \int d\omega' \left(\alpha_{\omega\omega'}\alpha^*_{\omega''\omega'} - \beta_{\omega\omega'}\beta^*_{\omega''\omega'}\right). \tag{A.13}$$

Inserting Eq. (A.13) for $\omega'' = \omega$ into Eq. (3.49) and using Eq. (A.9), we thus get that

$$\int d\omega' |\beta_{\omega\omega'}|^2 = \lim_{T\to\infty} \frac{T}{2\pi} \frac{\Gamma(\omega)}{e^{2\pi\omega/\kappa} - 1}, \tag{A.14}$$

which by Eq. (3.47) yields a spectrum of the form

$$\langle 0_f | p^\dagger_\omega p_\omega | 0_f \rangle = \lim_{T\to\infty} \frac{T}{2\pi} \frac{\Gamma(\omega)}{e^{2\pi\omega/\kappa} - 1}. \tag{A.15}$$

That this spectrum is thermal is revealed by regarding the eventual black hole as an object in thermal equilibrium with its surroundings. Then the same fraction $(\Gamma(\omega) - 1)$ of the incoming modes that back-scatters off the gravitational field to reach $\mathcal{I}^+$ is identical to the fraction of outgoing from the black hole that scatters back into the forming black hole. Hence, the resulting black hole emits and absorbs radiation like a grey-body of absorptivity $\Gamma(\omega)$, at a temperature

$$T = \frac{2\pi\omega}{\kappa}. \tag{A.16}$$

In two dimensions the null rays can only propagate radially, and will therefore not be able to scatter off the effective potential to reach $\mathcal{I}^+$. Thus, the radial equation in the two-dimensional scenario is exactly the one in Eq. (A.5) without the gravitational potential term in the square-brackets, and the introduction of a grey-body factor is therefore not relevant in this scenario.

Appendix B

Relating Null Coordinates Through the Geometric Optics Approximation

In the following we will see how the relation between the null coordinates u and v can be related by explicitly making use of the geometric optics approximation. This method is similar to the one used by Hawking in his original derivation of Hawking radiation [15], and the following presentation follows Parker and Toms rather closely [22].

We want to study the null rays which originate on $\mathcal{I}^-$, enter the collapsing body and escape from the body just before the collapse, terminating on $\mathcal{I}^+$. Because the concept of particles adapted from quantum field theory in general becomes ill-defined when the theory is transferred to curved spacetimes, we would like to go to a limit where the spacetime is approximately flat. Instead of propagating the null rays forward in time from $\mathcal{I}^-$, we thus only consider the asymptotic configurations of the null rays. Therefore we need a way to relate the null rays ending on future null infinity to the null rays originating on past null infinity. In the following we will show how this procedure is done for a Schwarzschild black hole. The approach is similar for other types of black holes.

Geodesics of Schwarzschild Spacetime

For simplicity we will in the following assume that the star collapses to a black hole in a nearly spherically symmetric spacetime, and that the spacetime outside the collapsing body is vacuum. By Birkhoff's theorem we thus know that the spacetime in question is the Schwarzschild spacetime. The Schwarzschild line element is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - d\Omega^2, \tag{B.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the volume element of a two dimensional sphere.

At $r = 2M$ the line element Eq. (B.1) becomes singular. However, observing that the contraction of the Riemann tensor with itself yields a scalar that is non-singular for $r = 2M$ reveals that this singularity is an artifact of the coordinate system, and not a true singularity. Instead one can show that for $r \leqslant 2M$ all future directed paths are forced to move in the direction of decreasing r. Therefore, $r = 2M$ is interpreted as the event horizon of the black hole.

Our mission is to relate the radial incoming null geodesics at $\mathcal{I}^+$ to the outgoing radial null geodesics at $\mathcal{I}^-$. Thus a reasonable place to start is to describe the radial null rays of Schwarzschild spacetime. This is exactly what we will do in the following.

A geodesic is defined as the extremal path between two points, and is the path followed by a freely falling observer; an observer that feels no acceleration. Finding the geodesic between two points a and b in a given spacetime is equivalent to requiring that the variation of the action between the points is zero. For null geodesics this can be stated mathematically as

$$\delta S = \delta\left(\int_a^b \mathcal{L} d\lambda\right) = 0,$$

where

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}. \tag{B.2}$$

The quantity $\mathcal{L}$ can be regarded as the Lagrangian of a classical particle with coordinates x^μ and velocity $dx^\mu/d\lambda$.

Looking at the Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial(dx^\mu/d\lambda)}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{B.3}$$

that one obtains by following this variation procedure, it becomes clear that if the metric $g_{\mu\nu}$ is independent of a given coordinate x^μ, then from Eq. (B.2) $\mathcal{L}$ is independent of this coordinate too, and the corresponding conjugate momentum

$$p_\mu \equiv \frac{\partial \mathcal{L}}{\partial(dx^\mu/d\lambda)} = g_{\mu\nu}\frac{dx^\nu}{d\lambda} \tag{B.4}$$

is thus constant along the geodesic. Given the Schwarzschild line element Eq. (B.1), we see that the metric of this spacetime is independent of the time coordinate, t, and the angular coordinate, ϕ. Thus the conjugate momenta to these coordinates are constant along the geodesic. That the metric is independent of the angular coordinate further implies that the geodesics all lie in the same plane. From spherical symmetry we may thus choose this plane to be $\theta = \pi/2$, as it is always possible to rotate the coordinate system so that the geodesics lie in this exact plane.

Having chosen the plane of geodesics to coincide with the plane $\theta = \pi/2$, we may continue to find the constants of motion from the conjugate momenta p_t and p_ϕ. Using Eq. (B.4) we find that the conserved quantities are

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} = E \tag{B.5}$$

and

$$r^2 \frac{d\phi}{d\lambda} = L. \tag{B.6}$$

It is natural to interpret the constant parameters E and L as the energy and angular momentum along the given geodesic.

Radial Null Geodesics

Null geodesics are defined as the geodesics that have a vanishing line element. The null geodesics of Schwarzschild spacetime are therefore the paths that satisfy

$$\left(\frac{ds}{d\lambda}\right)^2 = \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - \left(\frac{d\Omega}{d\lambda}\right)^2 = 0 \tag{B.7}$$

Now, since we are free to choose the geodesics to lie in the plane $\theta = \pi/2$, we know that $d\theta/d\lambda = 0$ and that $\sin\theta = 1$. Expressing Eq. (B.7) in terms of the conserved quantities E and L, we obtain the following equation

$$\left(\frac{dr}{d\lambda}\right)^2 + \left(\frac{L}{r}\right)^2\left(1 - \frac{2M}{r}\right) = E^2. \tag{B.8}$$

Since we are only interested in radial null rays, we must have $L = 0$, and Eq. (B.8) further simplifies to

$$\frac{dr}{d\lambda} = \pm E. \tag{B.9}$$

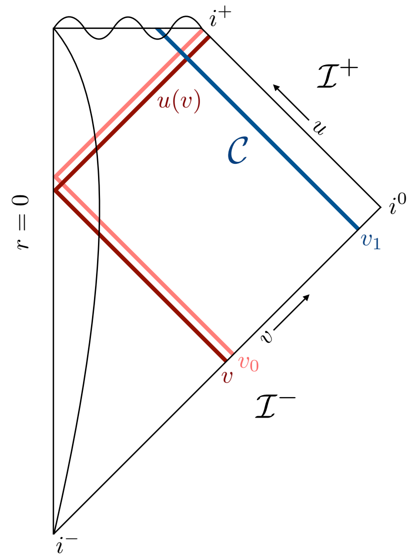


Figure B.1: Penrose diagram of a star collapsing to a black hole. The pink line at $v = v_0$ marks the last null ray that leaves $\mathcal{I}^-$, propagates through the collapsing body and reaches infinity. All rays leaving $\mathcal{I}^-$ for $v < v_0$ end up on $\mathcal{I}^+$, and the rays leaving $\mathcal{I}^-$ close to the null ray $v = v_0$ (marked with a red line) make up the late time spectrum observed on $\mathcal{I}^+$. The blue line marks a null curve $\mathcal{C}$ which leaves $\mathcal{I}^-$ for $v > v_0$, thus propagating past the event horizon of the black hole towards the singularity of the spacetime.

The upper sign corresponds to outgoing geodesics with respect to the black hole horizon (for $r > 2M$), and the lower sign corresponds to incoming geodesics. Inserting Eq. (B.9) into Eq. (B.5) we get that

$$\frac{d}{d\lambda}(t \mp r^*) = 0,$$

where the tortoise coordinate r^* is defined as

$$\frac{dr^*}{dr} = \left(1 - \frac{2M}{r}\right)^{-1}. \tag{B.10}$$

As r approaches $2M$ from above, r^* moves towards $-\infty$, and as r approaches ∞, r^* and r are approximately equal. Defining a set of null coordinates (u, v) as

$$u = t - r^* \tag{B.11}$$

and

$$v = t + r^* \tag{B.12}$$

we see that u remains constant along any outgoing, radial null geodesic, and that v remains constant along any incoming, radial null geodesic.

Relating Null Coordinates u and v

Having obtained a mathematical description of the radial null geodesics of Schwarzschild spacetime, we want to find a way to express the null coordinates u and v in terms of an affine parameter λ. Let the curve $\mathcal{C}$, shown as a blue line in figure B.1, be an incoming null geodesic defined by the null coordinate $v = v_1$ on $\mathcal{I}^-$. This null geodesic enters the horizon of the Schwarzschild black hole. Let λ be an affine parameter along this geodesic.

From figure B.1 we see that the null coordinate u can be expressed along $\mathcal{C}$ as a function of λ. Thus from Eq. (B.11) we have that

$$\frac{du}{d\lambda} = \frac{dt}{d\lambda} - \frac{dr^*}{d\lambda}$$

along this curve. Applying energy conservation along a geodesic, given in Eq. (B.5), together with the relation

$$\frac{dr^*}{d\lambda} = \frac{dr^*}{dr}\frac{dr}{d\lambda} = -\left(1 - \frac{2M}{r}\right)^{-1} E$$

from Eq. (B.10) and Eq. (B.9) for incoming null geodesics, we may continue to write

$$\frac{du}{d\lambda} = 2\left(1 - \frac{2M}{r}\right)^{-1} E. \tag{B.13}$$

Additionally, integrating Eq. (B.9) over all values of λ outside the black hole, we get that

$$\Delta r = r - R_s = r - 2M = -E\lambda, \tag{B.14}$$

where R_s is the Schwarzschild radius and λ is defined to be zero on the event horizon. This interval is always positive since we are restricted to the region $r > 2M$. Thus λ must be negative in the region outside the black hole event horizon.

Dividing by r on both sides of the last equality in Eq. (B.14), inverting the equation and using that $r = 2M - E\lambda$, one obtains the relation

$$\left(1 - \frac{2M}{r}\right)^{-1} = 1 - \frac{2M}{E\lambda}.$$

Inserting this into Eq. (B.13) and integrating over all λ along the incoming null geodesic $\mathcal{C}$ leads to the following expression for the null coordinate u,

$$u(\lambda) = 2E\lambda - 4M \ln\left(\frac{\lambda}{K_1}\right), \tag{B.15}$$

where K_1 is a negative constant (since $\lambda < 0$ for $r > 2M$).

Far from the event horizon, i.e. for $r \gg 2M$, we see from Eq. (B.14) that $\lambda \to -\infty$. Thus Eq. (B.15) shows that $u(\lambda) \approx 2E\lambda$ at such large distances from the black hole event horizon, since the logarithm of a quantity approaches infinity more slowly than the quantity itself.

Close to the event horizon, $\lambda \approx 0$, so

$$u(\lambda) = -4M \ln \frac{\lambda}{K_1}. \tag{B.16}$$

Our remaining task is to relate the affine parameter λ to the incoming null coordinate v in order to find an expression for u in terms of v.

First of all we observe from figure B.1 that null geodesics which originate on $\mathcal{I}^-$ for values $v > v_0$ will enter the black hole through its event horizon, and run into the singularity. These rays will never reach $\mathcal{I}^+$, and can not be a part of the spectrum observed there. Furthermore, the null geodesic parameterized by $v = v_0$ generates the event horizon, and reaches $\mathcal{I}^+$ in an infinite amount of time. Hence the null geodesics of interest for the late time spectrum on $\mathcal{I}^+$ are the rays with constant incoming null coordinate $v \lesssim v_0$.

Moreover, the affine parameter λ along all incoming radial null geodesics that pass through the event horizon, such as the geodesic $\mathcal{C}$, can be chosen so that it satisfies relation Eq. (B.16) near the event horizon. Since the relation holds for all such incoming radial null curves, this means that the affine parameter distance between any two outgoing radial null geodesics, e.g. $u(v_0)$ and $u(v)$, will remain constant along their entire lengths, as measured by the change in λ along any such incoming null ray intersecting the two outgoing rays.

Following these outgoing null geodesics backwards in time, through the collapsing body, one ends up on the incoming null geodesics that originate on past null infinity. For the rays $u(v_0)$ and $u(v)$, these incoming null geodesics are parameterized by v_0 and v. The affine separation along the null direction between the incoming null geodesics v_0 and v on $\mathcal{I}^-$ can be chosen so that it remains the same as for the rays' outgoing counterparts. In other words the affine separation between v and v_0 at $\mathcal{I}^-$ can be chosen so that it is the same as the affine separation between $u(v)$ and $u(v_0)$ at $\mathcal{I}^+$ [22].

Now, since past null infinity is far from the event horizon in both time and space, the coordinate v is itself an affine parameter along $\mathcal{I}^-$. There must therefore be a linear relation between the null coordinate distance $v - v_0$ in the null direction and the affine separation λ between $u(v)$ and $u(v_0)$. We write this relation as

$$v_0 - v = K_2 \lambda,$$

where K_2 is a negative constant (since $v_0 > v$ and $\lambda < 0$). Inserting this relation into Eq. (B.16), we obtain the expression

$$u(v) = -4M \ln \left(\frac{v_0 - v}{K} \right) \tag{B.17}$$

close to the event horizon, with $K = K_1 K_2$. This is the exact same relation as in Eq. (3.36), with $\kappa = 1/4M$.

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