C^*-ALGEBRAS OF RIGHT LCM MONOIDS AND THEIR EQUILIBRIUM STATES

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Abstract. We study the internal structure of C^*-algebras of right LCM monoids by means of isolating the core semigroup C^*-algebra as the coefficient algebra of a Fock-type module on which the full semigroup C^*-algebra admits a left action. If the semigroup has a generalised scale, we classify the KMS-states for the associated time evolution on the semigroup C^*-algebra, and provide sufficient conditions for uniqueness of the KMS_β-state at inverse temperature β in a critical interval.

1. Introduction

The role of the C^*-algebra $C^*(G)$ associated to a discrete group G is ubiquitous and fundamental in operator algebras. Likewise, there are indispensable examples of C^*-algebras naturally associated to semigroups through representations by isometries, such as the Toeplitz algebra $\mathcal{T}$ generated by a single isometry. While these prominent examples are well documented, a systematic theory built around C^*-algebras $C^*(S)$ associated to broad and abstract classes of discrete monoids S has only recently emerged, largely as a consequence of an accelerating supply of new classes of monoids with good properties, viewed from a C^*-algebraic perspective. The example-driven insights have often brought about understanding of general properties, and vice versa. The present paper aims to strengthen our understanding of the interplay between right LCM monoids and their C^*-algebras by drawing on a varied array of examples, including some that are unexplored in the literature.

Before we explain our aim in more detail, we review recent developments around semigroup C^*-algebras. We refer to [CELY17] for an excellent introduction, and we restrict ourselves to highlighting a non-exhaustive list of recent work in this area. As a word of warning on terminology, operator algebraists tend to not make a distinction between semigroups and monoids and casually refer to semigroup C^*-algebra in all situations. We shall adopt this habit, but since we are aware of big structural differences between the unital and the non-unital case in semigroup theory, we will specify monoid in all cases where the semigroup has an identity.

It was Nica's work in [Nic92] that decisively put forward C^*-algebras associated to nonabelian semigroups as a useful and versatile construction in operator algebras. Later work by Laca–Raeburn [LR96, LR10] and Cuntz [Cun08] provided new thrust in the form of new classes of C^*-algebras with interesting generating families of isometries

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and tractable properties. A general construction of full and reduced C^*-algebras for left cancellative monoids was introduced by Li, [Li12], and soon fundamental questions about the interplay between nuclearity of the C^*-algebra and amenability of the monoid or its left inverse hull were raised, see [Li13, Nor14]. Subsequent work on semigroup C^*-algebras that centered on computing K-theory revealed impressively rich connections to number theory, see [Li14, KT], and geometric group theory, see [ELR16].

While monoids in so-called quasi-lattice ordered pairs introduced by Nica dominated much of the early endeavours, the larger class of right LCM (for Least Common Multiple) monoids has in the past years received growing attention: The perspective from full C^*-algebras of Zappa-Szép products of right LCM monoids enabled the authors of [BRRW14] to unify different constructions of C^*-algebras from [LR10, Nek09, LRRW14]. The structure of full and reduced semigroup C^*-algebras and natural quotients for right LCM monoids were analysed further in [BLS17, Star15, BS16, BLS18, BOS18]. More recent work indicates that there are more directions to be explored in this context, see for example [Li18, LOS18, ACRS].

Supplementing the study of semigroup C^*-algebras from the point of view of amenability, nuclearity, and K-theory, there has been intense activity on describing KMS-states (in honour of Kubo and Martin–Schwinger) for certain natural flows on the C^*-algebras, see [LR10, BaHLR12, CaHR16] as well as the very recent [ABLS17, BLRS19, ALN].

The philosophy of unifying different case studies of C^*-algebras in a broad framework is the leitmotiv in [ABLS17], where a first step was taken towards uncovering internal structure of semigroup C^*-algebras via classification of KMS-states for what could be deemed as intrinsic one-parameter groups on $C^*(S)$. The main result of [ABLS17] provides general methods for analysing KMS-states for C^*-algebras associated to *admissible* right LCM monoids. This covers new cases such as algebraic dynamical systems [BLS18], and also offers a unified perspective onto this problem for C^*-algebras associated to the affine semigroup over the natural numbers [LR10], algebraic number fields with trivial class number [CDL13], integer dilation matrices [LRR11], self-similar group actions [LRRW14], and quasi-lattice ordered Baumslag–Solitar monoids [CaHR16].

The present work was triggered as we became aware of Spielberg's work [Spi12] on Baumslag–Solitar monoids that are not quasi-lattice ordered, but right LCM. This class went unnoticed until after a discussion with Jack Spielberg at a conference in Newcastle, Australia, where the authors of the present paper realised that it fell outside the scope of [ABLS17]. The current paper grew out of our attempts to gain insight into what happens for this special class of Baumslag–Solitar monoids. For reasons that will become clearer later, we refer to these as Baumslag–Solitar monoids with *positivity breaking.*

In this work we develop the full potential of classifying KMS-states for C^*-algebras of right LCM monoids as a means to uncover structural properties of $C^*(S)$. We employ new techniques, which will allow us to distill the admissibility assumption of [ABLS17] to its indispensable part, namely the existence of a generalised scale which gives rise to the intrinsic one-parameter group on $C^*(S)$. Roughly speaking, a generalised scale is a special monoidal homomorphism from S to the monoid $\mathbb{N}^\times$ of positive integers (with multiplication). The right LCM monoids covered in [ABLS17, Definition 3.1] are characterised by four admissibility conditions: the first two guarantee nice factorisation properties for the right LCM monoid S in question. The latter two conditions characterise the existence of a generalised scale. All the examples for right LCM monoids in [LR10, LRR11, BRRW14, CaHR16] suggested that these admissibility assumptions were quite modest. However, as discovered in Newcastle, the Baumslag–Solitar monoids with positivity breaking fail one of the two basic factorisation properties in an extreme way, see Section 7 for details.

The factorisation properties of an admissible right LCM monoid S are formulated with respect to its core submonoid $S_c := \{a \in S \mid aS \cap sS \neq \emptyset \text{ for all } s \in S\}$, first considered for right LCM monoids in [Star15] with roots in [CL07], and a counterpart of core irreducible elements S_{ci}, see [ABLS17, Subsection 2.1]. Faced with the absence of these factorisation properties for Baumslag–Solitar monoids with positivity breaking, we revert to techniques which stand in stark contrast to the ones in [ABLS17]. Our current analysis relies on the study of the quotient space $S/_\sim$, where $s \sim t$ if $sa = tb$ for some $a, b \in S_c$, and the action $\alpha\colon S_c \curvearrowright S/_\sim$ induced by left multiplication. With these ingredients at hand, we develop a framework which will encompass admissible right LCM monoids as well as the Baumslag–Solitar monoids with positivity breaking, but also new examples, especially from self-similar group actions from virtual endomorphisms that may not exhibit the finite-state property.

Our approach to classifying KMS-states depends crucially on a special representation of $C^*(S)$, arising from the construction of a Fock-type right-Hilbert $C^*(S_c)$-module $\mathfrak{M}$ that we call the *core Fock module*. We believe that this module is of independent interest as an abstract construction for arbitrary right LCM monoids, or equivalently, 0-bisimple inverse monoids. Given the success of the induction process for classifying KMS-states on Pimsner algebras for ordinary Hilbert bimodules from [LN04], it is not surprising that a Fock module would be useful. The striking feature of the core Fock module is that it exists for every right LCM monoid. Its predecessor was constructed in [ABLS17, Theorem 8.4] using the factorisation properties encoded in the admissibility of S, which we now prove to be superfluous as we provide a substantially refined and streamlined general construction.

To give a clue about the challenge for the new construction of $\mathfrak{M}$, let us point out that admissibility of S provides minimal elements for the equivalence classes of $S/_\sim$, similar to the natural numbers. Now, the Baumslag–Solitar monoids with positivity breaking are not admissible and so for these we must consider totally ordered equivalence classes that do not have minimal elements, similar to the integers. Thus, all the techniques in [ABLS17] that exploited the existence of minimal representatives must be substituted by solutions that would work with very little extra structure, namely, that equivalence classes are directed with respect to the partial order given by $s \geq_r t$ if $t \in sS_c$.

A further key asset of this work in comparison with [ABLS17] is the precision with which we identify when there is a unique KMS-state at inverse temperatures from the critical interval. Our main result in this respect, Theorem 3.3, is a generalisation of [ABLS17, Theorem 4.3]. The theorem is an improvement in two ways: admissibility is replaced by the existence of a generalised scale, and the sufficiency criteria for uniqueness of the KMS$_\beta$-state for β in the critical interval $[1, \beta_c]$ are weakened and unified at the same time. They appear to be very common and may in fact also be necessary conditions for uniqueness in many cases.

The essential idea behind these uniqueness conditions is a detailed analysis of the internal structure of $C^*(S)$ by means of the core submonoid S_c. Roughly speaking, these conditions can be explained as follows: the generalised scale N gives rise to a grading on $S/_\sim$ over the directed right LCM monoid $N(S)$. This provides a direction so that we can consider limits, and allows us to measure proportions of special equivalence classes among those that relate to a given $n \in N(S)$. For instance, we can look at fixed points for $a, b \in S_c$ under $\alpha\colon S_c \curvearrowright S/_\sim$ at level $n \in N(S)$, that is, $[s] \in S/_\sim$ with $s \in N^{-1}(n)$ and $asc = bsd$ for some $c, d \in S_c$. But it also makes sense to consider stronger fixed points, points that we call absorbing as they satisfy $asc = bsc$ for some $c \in S_c$. In other words, there is $s' \sim s$ with $as' = bs'$. The uniqueness criteria state, with varying particularities, that the proportion of non-absorbing fixed points at level $n \in N(S)$ tends to zero as $n \to \infty$.

In the case of faithful self-similar group actions, our condition can be given a measure-theoretic reformulation: The group acts on the space of infinite paths (starting from the root of the regular tree), which is equipped with the Borel probability measure determined by uniform distributions among the finite paths of equal length. Then the uniqueness condition for $\beta = 1$ holds if and only if the boundary of the set of fixed points has measure zero, see Proposition 8.3. The topological analogue, namely that the complement of this set is a dense G_δ-set, is easily seen to be true for every faithful self-similar group action. But the measure theoretic question is much more subtle, and there may be room for rather special specimens, which would, if existent, surely be of interest on their own.

At $\beta = 1$ we discover a characterisation of the KMS$_1$-states which could be summarised by saying that they correspond to the *thermally inert traces*. Explicitly, we construct the KMS$_\beta$-states on $C^*(S)$ for $\beta > 1$ via induction $\tau \mapsto \psi_{\beta,\tau}$ from normalised traces τ on $C^*(S_c)$ using the core Fock module. The KMS$_\beta$-state $\psi_{\beta,\tau}$ in turn restricts to a normalised trace on $C^*(S_c)$, once we precompose it with the $*$-homomorphism $\varphi\colon C^*(S_c) \to C^*(S)$ induced by the inclusion $S_c \subset S$. In this way, each inverse temperature $\beta > 1$ comes with a self-map χ_β of the trace simplex on $C^*(S_c)$. The analogue of such a self-map in the setting of finite-state self-similar actions of groupoids on graphs was recently shown to exhibit intriguing dynamical features, see [CS18]: the unique KMS-state is a universal attractor with regards to the dynamics defined by χ_β. Here we take a different route, namely we show in Theorem 3.6 that the KMS$_1$-states correspond precisely to the common fixed points of $\{\chi_\beta \mid \beta > 1\}$. We further show that the simplex of common fixed points of $\{\chi_\beta \mid \beta > \beta_0\}$ for every $\beta_0 \in [1, \infty)$ embeds into the simplex of KMS$_{\beta_0}$-states, which reveals a new continuity feature of the system $(C^*(S), \sigma)$.

We begin this article with a small background section, followed by a presentation of the main results on classification of KMS-states: Theorem 3.3 and Theorem 3.6. In Section 4 we start on the proof of the parametrisation part from Theorem 3.3. In particular, we prove our promised existence of a core Fock module $\mathfrak{M}$ and an associated $*$-homomorphism from $C^*(S)$ into $\mathcal{L}(\mathfrak{M})$, see Theorem 4.6. Section 5 provides a parametrisation of KMS-states using $\mathfrak{M}$-induced representations on $C^*(S)$ from GNS representations of $C^*(S_c)$ coming from normalised traces. In this section we also prove Theorem 3.6. In Section 6 we motivate our notions of core regular and summably core regular monoids, and prove that in different contexts they are sufficient conditions for achieving a unique KMS$_\beta$-state for β in the critical interval. Here we also complete the proof of Theorem 3.3. Sections 7–10 are devoted to applications. In Section 7, we illustrate the features around the phenomenon of positivity breaking in Baumslag–Solitar monoids $BS(c,d)^+$ with opposite signs for c and d.

Conditions for uniqueness of KMS$_\beta$-states in a critical interval bearing the flavour of ergodic theoretic considerations will be illustrated with examples arising from virtual endomorphism of the discrete Heisenberg group as in [BK13], see Proposition 8.3 and Corollary 8.8.

With the monoid of so-called shadowed natural numbers we provide the first examples of right LCM monoids whose C^*-algebras possess a unique KMS$_1$-state, despite the fact that the action $\alpha : S_c \curvearrowright S/_\sim$ is not faithful, see Proposition 9.1. In view of Proposition 6.8, uniqueness of the KMS$_1$-state is only possible due to failure of right cancellation. This degree of freedom was unavailable in other contexts, such as the finite, strongly-connected higher-rank graph case in [aHLRS15], whose *periodicity group* may be thought of as a measure for faithfulness of the action corresponding to α.

Finally, we also provide a new context for characterising uniqueness of the KMS-states in the critical interval with two examples that fail almost freeness of the action α, a property crucial in [ABLS17] as well as in several of the supporting case studies. In both examples the monoids are core regular and summably core regular, see Proposition 10.1 and Proposition 10.2.

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2. Preliminaries

2.1. **Right LCM monoids.** Let S be a right LCM monoid, by which we mean a left cancellative semigroup S with identity 1 so that for every $s,t \in S$, we have $sS \cap tS = rS$ for some $r \in S$ whenever $sS \cap tS \neq \emptyset$. We shall adopt the notation $s \Cap t$ from [Spi12] to indicate that $sS \cap tS \neq \emptyset$, and we write $s \perp t$ when $sS \cap tS = \emptyset$, where $s,t \in S$.

The *core* of S is the right LCM submonoid $S_c := \{a \in S \mid \forall s \in S : a \Cap s\}$, see [Star15]. Two elements $s,t \in S$ are said to be *core equivalent*, denoted $s \sim t$, if there are $a,b \in S_c$ satisfying $sa = tb$. The class of $t \in S$ will be denoted $[t]$. We recall from [ABLS17, Lemma 3.9] that left multiplication induces an action α of the semigroup S_c by bijections of $S/_\sim$, so $\alpha_a([t]) = [at]$ for $a \in S_c$ and $t \in S$.

According to [BRRW14], a finite subset F of S, denoted $F \subset\subset S$, is a foundation set if for every $s \in S$ there is $f \in F$ with $s \Cap f$. A foundation set F is accurate, cf. [BS16], if $f \perp f'$ holds for all $f,f' \in F$ with $f \neq f'$.

We will be interested in right LCM monoids S that admit a *generalised scale*.

Definition 2.1. ([ABLS17, Definition 3.1]) A generalised scale for a right LCM monoid S is a nontrivial homomorphism of monoids $N\colon S \to \mathbb{N}^\times$ such that $|N^{-1}(n)/_\sim| = n$ for all $n \in N(S)$, and for each $n \in N(S)$, every transversal of $N^{-1}(n)/_\sim$ is an accurate foundation set for S. For each $s \in S$ we write N_s for $N(s)$.

Although there is a wide range of examples of right LCM monoids which admit a generalised scale, its existence puts the focus on monoids with a particularly structured growth behaviour. This abstract assumption was given an explicit, combinatorial characterisation in [Sta19, Theorem 3.11].

In the sequel it will be useful to have a reference for the following simple observation.

Lemma 2.2. *Suppose that S is a right LCM semigroup with a generalised scale N. If s,t in S satisfy $s \Cap t$ and $N_s \in N_t N(S)$, then $sS \cap tS = saS$ for some $a \in S_c$.*

Proof. Assume $s,t \in S$ satisfy $sS \cap tS = ss'S$ with $ss' = tt'$ for some $s',t' \in S$. By [ABLS17, Proposition 3.6(iv)], $N_{ss'}$ is the right LCM of N_s and N_t in $N(S)$. So if $N_s \in N_t N(S)$, then we must have $N_{ss'} = N_s$, that is, $N_{s'} = 1$. Thus $s' \in S_c$ by [ABLS17, Proposition 3.6(i)]. □

Given S and N we define a partition function to be $\zeta(\beta) := \sum_{[s]\in S/_\sim} N_s^{-\beta}$ with $\beta \in \mathbb{R}$. The *critical inverse temperature* β_c is the smallest $\beta_0 \in \mathbb{R}\cup\{\infty\}$ such that $\zeta(\beta)$ converges for all $\beta > \beta_0$; thus $\beta_c = \infty$ refers to divergence of ζ on the real line.

Recall from [Sta19, Proposition 2.1] that every generalised scale N satisfies condition (A4) from [ABLS17, Definition 3.1], that is, $N(S)$ is freely generated by its irreducible elements. For $I \subset \operatorname{Irr}(N(S))$, we let $S_I := N^{-1}(\langle I\rangle) \subset S$ denote the submonoid of S corresponding to I, and remark that S_I is also a right LCM semigroup due to [ABLS17, Proposition 3.6(iv)]. The I-restricted ζ-function as introduced in [ABLS17, Definition 4.2] is given by

$$\zeta_I(\beta) = \sum_{[t]\in S_I/_\sim} N_t^{-\beta} = \sum_{n\in\langle I\rangle} \sum_{[t]:N_t=n} N_t^{-\beta} = \sum_{n\in\langle I\rangle} n^{1-\beta}, \tag{2.1}$$

and admits a product formula $\zeta_I(\beta) = \prod_{p\in I}(1-p^{1-\beta})^{-1}$ by [ABLS17, Remark 7.4]. For a finite subset F of I, we set $m_F := \prod_{n\in F} n$ and remark that, due to (A4), this is the right LCM of F. The finite subsets of I form a directed set with respect to inclusion, that is, $F \leq F' \Leftrightarrow F \subset F'$.

2.2. **C^*-algebras of right LCM monoids.** A general construction of [Li12] associates to any left cancellative monoid S a full semigroup C^*-algebra $C^*(S)$. In the present work we concentrate on right LCM monoids, in which case the presentation of $C^*(S)$ is somewhat simpler. The C^*-algebra $C^*(S)$ is the universal C^*-algebra generated by isometries $v_s, s \in S$ subject to the relations $v_s v_t = v_{st}$ and

$$v_s v_s^* v_t v_t^* = \begin{cases} v_r v_r^* & \text{if } sS \cap tS = rS; \\ 0 & \text{if } s \perp t, \end{cases}$$

for $s,t \in S$. It follows that $C^*(S)$ admits a dense spanning set $v_s v_t^*$ with $s,t \in S$, see [Nor14, BLS18]. We let e_{sS} denote the projection $v_s v_s^*$ in $C^*(S)$, for each $s \in S$.

The algebra $C^*(S)$ has a natural quotient, the *boundary quotient* $\mathcal{Q}(S)$ constructed in [BRRW14]: Let π be the quotient map $C^*(S) \to \mathcal{Q}(S)$ obtained by adding the relation $\prod_{s\in F}(1 - v_s v_s^*) = 0$ for every foundation set $F \Subset S$ to the defining relations of $C^*(S)$.

We denote by $w_a, a \in S_c$ the standard generating isometries of $C^*(S_c)$, and by $\varphi\colon C^*(S_c) \to C^*(S)$ the $*$-homomorphism induced by the inclusion $S_c \subset S$. We are grateful to Sergey Neshveyev for the observation that groupoid techniques show the following result, cf. [NS], which allows us to improve and simplify our construction of the core Fock module in subsection 4.3.

There exists a conditional expectation $E\colon C^*(S) \to C^*(S_c)$ given by

$$v_s v_t^* \mapsto \begin{cases} \varphi^{-1}(v_s v_t^*) & \text{if } s,t \in S_c; \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

In particular, the $*$-homomorphism $\varphi\colon C^*(S_c) \to C^*(S)$ is faithful since $E \circ \varphi = \mathrm{id}$.

Let us record the following observation on the existence of normalised traces:

Lemma 2.3. *For a right LCM monoid T, the C^*-algebra $C^*(T)$ admits a normalised trace if and only if T is left reversible, that is, $s \Cap t$ for all $s,t \in T$.*

Proof. Let T be a right LCM monoid. Suppose first that there exist $s,t \in T$ with $s \perp t$. If τ were a normalised trace on $C^*(T)$, we would get

$$1 = \tau(1) \geq \tau(v_s v_s^* + v_t v_t^*) = \tau(v_s v_s^*) + \tau(v_t v_t^*) = 2\tau(1) = 2,$$

a contradiction.

Now suppose that T is left reversible. Let $I_\ell(T)$ be the (0-bisimple) inverse monoid known as the left inverse hull of T, generated by the left translations $\lambda_t\colon T \to T, r \mapsto tr$ for $t \in T$. As observed in [Nor14, Lemma 3.29], the inverse monoid $I_\ell(T)$ is without 0 element. There is a maximal group homomorphic image of $I_\ell(T)$, denoted $G(T)$ in [Nor14, §3.4], accompanied by a quotient homomorphism $\alpha_T : I_\ell(T) \to G(T)$. At the level of C^*-algebras, there is a surjective $*$-homomorphism $\pi_T : C^*(I_\ell(T)) \to C^*(G(T))$. Since T is right LCM, then by following the proof of [Nor14, Proposition 3.27] we obtain a canonical isomorphism $\eta : C^*(T) \to C^*(I_\ell(T))$, $\eta(v_t) = \lambda_t$ for $t \in T$. We remark that [Nor14, Proposition 3.27] is stated for an arbitrary right LCM monoid, and then η maps onto $C_0^*(I_\ell(T))$, which is the quotient of $C^*(I_\ell(T))$ by the ideal corresponding to 0 in $I_\ell(T)$. However, as we already pointed out, when T is moreover left reversible then $0 \notin I_\ell(T)$, and the map η can be defined with image in $C^*(I_\ell(T))$.

By composing any normalised trace on $C^*(G(T))$ first with the $*$-homomorphism π_T, then the isomorphism η, we obtain a normalised trace on $C^*(T)$, as desired. ☐

Proposition 2.4. *The core S_c is the largest submonoid T of a right LCM monoid S such that $C^*(T)$ admits a normalised trace. Moreover, τ_0 given by $\tau_0(w_a w_b^*) = \delta_{a,b}$ for $a,b \in S_c$ defines a normalised trace on $C^*(S_c)$.*

Proof. This follows immediately from Lemma 2.3 and its proof as the group C^*-algebra $C^*(G(S_c))$ admits the trivial trace which induces τ_0 through π_{S_c}. ☐

2.3. **KMS-states on C^*-algebras of right LCM monoids.** We briefly review the notions of KMS- and ground states for a given C^*-dynamical system (A,σ) consisting of a C^*-algebra A and a time evolution $\sigma : \mathbb{R} \curvearrowright A$, see [BR97] for a standard introduction.

An element $x \in A$ is *analytic* for σ if the map $u \mapsto \sigma_u(x)$ with $u \in \mathbb{R}$ has a (unique) extension to an analytic function $z \mapsto \sigma_z(x)$ from $\mathbb{C}$ into A. For $\beta > 0$, a state ϕ of A is a σ-KMS$_\beta$-state, or simply a KMS$_\beta$-state, if it satisfies the KMS$_\beta$ condition

$$\phi(xy) = \phi(y\sigma_{i\beta}(x)) \tag{2.3}$$

for all analytic $x, y \in A$. To prove that a given σ-invariant state of A is a KMS$_\beta$-state, it suffices to check the KMS$_\beta$-condition for all x in a set of analytic elements that generate A as a C^*-algebra and all y in a dense spanning subspace of A, see [ALN, Lemma 1.9].

A *KMS$_\infty$-state* of A is a weak* limit of KMS$_{\beta_n}$-states as $\beta_n \to \infty$, see [CM06]. A state ϕ of A is a *ground state* of A if $z \mapsto \phi(x\sigma_z(y))$ is bounded on the upper-half plane for all analytic $x, y \in A$.

Given a right LCM monoid S with a generalised scale N, there is a time evolution $\sigma\colon \mathbb{R} \curvearrowright C^*(S)$ determined by $\sigma_u(v_s) := N_s^{iu} v_s$ for $s \in S, u \in \mathbb{R}$. Clearly, all isometries v_s, $s \in S$, are analytic.

Whenever ϕ is a KMS$_\beta$-state for $(C^*(S), \sigma)$, we let $(\pi_\phi, H_\phi, \xi_\phi)$ be the associated GNS representation and denote by $\tilde{\phi}$ its vector state extension to $\mathcal{L}(H_\phi)$. Note that $\tilde{\phi}$ is a KMS$_\beta$-state on $\pi_\phi(C^*(S))$. For $I \subset \mathrm{Irr}(N(S))$, we define a projection in $\pi_\phi(C^*(S))''$ by

$$Q_I := \prod_{n \in I} \big(1 - \sum_{[t] \in N^{-1}(n)/\sim} \pi_\phi(e_{tS})\big). \tag{2.4}$$

These are well-defined by the following lemma, and will play a key role for the reconstruction formula for KMS-states established in Subsection 4.2.

Lemma 2.5. *Let S be a right LCM semigroup and suppose σ is a time evolution on $C^*(S)$ such that v_a is an analytic isometry in $C^*(S)$ with $\phi(v_a v_a^*) = 1$ for all $a \in S_c$. If ϕ is a σ-KMS$_\beta$-state for some $\beta \in \mathbb{R}$, then for all analytic elements $x, y \in C^*(S)$ and $a \in S_c$ we have $\phi(xe_{aS}y) = \phi(xy)$. In particular, we have $\pi_\phi(e_{sS}) = \pi_\phi(e_{tS})$ whenever $s \sim t$.*

Proof. The assumption and the KMS$_\beta$-condition imply that

$$\phi(xe_{aS}y) = \phi(e_{aS}y\sigma_{i\beta}(x)e_{aS}) = \phi(y\sigma_{i\beta}(x)) = \phi(xy).$$

Now let $s \sim t$ in S. Thus $sa = tb$ for some $a, b \in S_c$, hence the previous step implies

$$\begin{aligned}\big(\pi_\phi(e_{sS}y)\xi_\phi \mid \pi_\phi(x)\xi_\phi\big) &= \phi(x^*e_{sS}y) = \phi(x^*e_{saS}y) \\ &= \phi(x^*e_{tbS}y) = \big(\pi_\phi(e_{tS}y)\xi_\phi \mid \pi_\phi(x)\xi_\phi\big)\end{aligned}$$

for all analytic elements $x, y \in C^*(S)$. Since the linear span of analytic elements is dense in $C^*(S)$, we conclude that $\pi_\phi(e_{sS}) = \pi_\phi(e_{tS})$. □

3. Main results on KMS states

In this section we summarise our results on KMS-states in the form of Theorems 3.3 and 3.6.

Theorem 3.3 gives a complete characterisation of the KMS$_\beta$-state structure of $C^*(S)$ for β outside the critical interval $[1, \beta_c]$, and provides sufficient conditions under which we get unique KMS$_\beta$-state at each β inside the critical interval. To prove Theorem 3.3 we follow the strategy of the proof of [ABLS17, Theorem 4.3], which deals with the more specialised case of admissible semigroups. The arguments of [ABLS17] that we cannot use — namely, those involving the factorisation properties of admissible monoids — have been amended, indeed sharpened, and are presented in Sections 4, 5 and 6.

To state the first result we need to introduce some notation. We keep the explanation for the following definitions at a minimum here, and refer to Section 6 for a thorough discussion. Throughout this section S will denote a right LCM monoid with generalised scale N. For $a, b \in S_c$ with $a \neq b$ and $n \in N(S)$ we define

$$F_n^{a,b} := \{[s] \in N^{-1}(n)/_\sim \mid asc = bsd \text{ for some } c, d \in S_c\}; \text{ and} \tag{3.1}$$

$$A_n^{a,b} := \{[s] \in N^{-1}(n)/_\sim \mid asc = bsc \text{ for some } c \in S_c\}. \tag{3.2}$$

We refer to the equivalence classes in $A_n^{a,b}$ as *absorbing* elements.

Note that if S is right cancellative, then these sets are empty. However, if S is not right cancellative then $A_n^{a,b}$ can be nonempty even though S_c is right cancellative, for example a group (as in the case of self-similar actions).

We now introduce our criterion for uniqueness of KMS_1-states.

Definition 3.1. Let S be a right LCM monoid with generalised scale N. For $a, b \in S_c$ we say that S is (a, b)*-regular* if

$$\frac{|F_n^{a,b} \setminus A_n^{a,b}|}{n} \stackrel{n\to\infty}{\longrightarrow} 0. \tag{3.3}$$

S is called *core regular* if it is (a, b)-regular for all $a, b \in S_c$.

Our second criterion is modelled on systems with a critical interval of the form $(1, \beta_c]$.

Definition 3.2. Let S be a right LCM monoid S with generalised scale N. For $\beta \in (1, \beta_c]$ and $a, b \in S_c$ we say that S is β*-summably* (a, b)*-regular* if

$$\lim_{I \Subset \mathrm{Irr}(N(S))} \zeta_I(\beta)^{-1} \sum_{n \in \langle I\rangle^+} n^{-\beta} |F_n^{a,b} \setminus A_n^{a,b}| = 0. \tag{3.4}$$

We say S is β*-summably core regular* if it is β-summably (a, b)-regular for all $a, b \in S_c$, and *summably core regular* if it is β-summably core regular for all $\beta \in (1, \beta_c]$.

Theorem 3.3. *Suppose S is a right LCM monoid with generalised scale N. For the time evolution σ on $C^*(S)$ determined by N, we have:*

(1) There are no KMS_β-states on $C^*(S)$ for $\beta < 1$.

(2) For $\beta > \beta_c$, there is an affine homeomorphism between KMS_β-states on $C^*(S)$ and normalised traces on $C^*(S_c)$.

(3) There is an affine homeomorphism between ground states on $C^*(S)$ and states on $C^*(S_c)$. In case that $\beta_c < \infty$, a ground state is a KMS_∞-state if and only if it corresponds to a normalised trace on $C^*(S_c)$.

(4) There is a KMS_1-state ψ_1 for $(C^*(S), \sigma)$ determined by

$$\psi_1(v_a v_b^*) = \lim_{n \in N(S)} \frac{|A_n^{a,b}|}{n} \quad \text{for } a, b \in S_c.$$

If S is core regular, then ψ_1 is the unique KMS_1-state.

(5) There is a KMS_β-state ψ_β for $(C^*(S), \sigma)$ for each $\beta \in (1, \beta_c]$ determined by

$$\psi_\beta(v_a v_b^*) = \lim_{I \Subset Irr(N(S))} \zeta_I(\beta)^{-1} \sum_{n \in \langle I\rangle^+} n^{-\beta} |A_n^{a,b}| \quad \text{for } a, b \in S_c.$$

If S is β-summably core regular, then ψ_β is the unique KMS$_\beta$-state.

It follows in particular that when S is core regular and summably core regular, there is a unique KMS$_\beta$-state for each $\beta \in [1, \beta_c]$.

Proof of Theorem 3.3. Part (1) is verbatim as in [ABLS17, Theorem 4.3]. Statement (2) is the content of Proposition 5.5. Part (3) combines Proposition 5.1 with the argument of the similar statement in [ABLS17, Theorem 4.3]. Part (4) is proved in Proposition 6.4, and (5) follows from Proposition 6.6. □

Remark 3.4. Recall that $\pi\colon C^*(S) \to \mathcal{Q}(S)$ denotes the $*$-epimorphism to the boundary quotient of S. Since σ descends to a time evolution on $\mathcal{Q}(S)$, we have that a KMS$_\beta$-state for $(C^*(S), \sigma)$ factors through π precisely when $\beta = 1$. Moreover, no ground state on $C^*(S)$ factors through π. The proofs of these two claims are the same as in [ABLS17, Theorem 4.3(5) and (8)] upon replacing π_p in (5) with π.

For every normalised trace τ on $C^*(S_c)$ and $\beta > 1$, Proposition 5.4 will produce a KMS$_\beta$-state $\psi_{\beta,\tau}$ given by

$$\psi_{\beta,\tau} = \lim_{I \Subset \mathrm{Irr}(N(S))} \zeta_I(\beta)^{-1} \sum_{n \in \langle I \rangle^+} n^{1-\beta} \psi_{\tau,n}$$

with $\psi_{\tau,n}(x) = n^{-1} \sum_{[s] \in N^{-1}(n)/\sim} \tau \circ E(v_s^* x v_s)$ for $x \in C^*(S)$, where $E\colon C^*(S) \to C^*(S_c)$ is the conditional expectation from (2.2). By Proposition 4.1, $\psi_{\beta,\tau} \circ \varphi$ is again a normalised trace on $C^*(S_c)$. We will be interested in the fixed points under the self-maps χ_β of the simplex of normalised traces on $C^*(S_c)$ given by $\tau \mapsto \psi_{\beta,\tau} \circ \varphi$ for $\beta \in (1, \infty)$.

Definition 3.5. For $\beta_0 \in [1, \infty)$, let $T_{>\beta_0}(C^*(S_c)) = \bigcap_{\beta \in (\beta_0, \infty)} T(C^*(S_c))^{\chi_\beta}$ denote the set of normalised traces τ on $C^*(S_c)$ satisfying

$$\chi_\beta(\tau) = \psi_{\beta,\tau} \circ \varphi = \tau \quad \text{for all } \beta > \beta_0. \tag{3.5}$$

We note that $T_{>\beta_0}(C^*(S_c))$ is a simplex.

Theorem 3.6. *Suppose S is a right LCM monoid with generalised scale N, and consider the time evolution σ on $C^*(S)$ determined by N. For every $\beta \in [1, \infty)$, the map $\tau \mapsto \psi_{\beta,\tau}$ is an embedding of the simplex $T_{>\beta}(C^*(S_c))$ into the simplex of* KMS$_\beta$*-states. For $\beta = 1$, this map is an affine homeomorphism between $T_{>1}(C^*(S_c))$ and the* KMS$_1$*-states.*

Theorem 3.6 is an entirely new result that explores the theory in a complementary direction to the findings of [CS18]. Its proof is presented at the end of Section 5.

Question 3.7. Under which conditions can one extend the characterisation of the KMS$_{\beta_0}$-states for $\beta_0 = 1$ from Theorem 3.6 to other values of β_0, say $\beta_0 \in (1, \beta_c]$?

4. The toolkit trilogy: algebraic constraints, a reconstruction formula, and the core Fock module

We assume throughout this section that S is a right LCM monoid with a generalised scale N and we let σ be the associated time evolution on $C^*(S)$.

4.1. **Algebraic constraints.** The results on algebraic constraints for KMS-states from [ABLS17, Section 6] apply to $(C^*(S),\sigma)$ with the exception [ABLS17, Proposition 6.4], which we shall now improve, and some minor parts of [ABLS17, Propositions 6.6 and 6.7], which we will not need. For completeness, we note that KMS_0-states are the traces on $C^*(S)$, but under our assumption $C^*(S)$ is traceless. Indeed, N is assumed nontrivial, so we can consider a transversal $\mathcal{T}_n$ for $N^{-1}(n)/_\sim$, where $n \in N(S)\setminus\{1\}$, and by the definition of N, $\mathcal{T}_n$ contains n mutually orthogonal elements. Thus a trace τ on $C^*(S)$ would satisfy $1=\tau(1)\geq \tau(\sum_{t\in\mathcal{T}_n} e_{tS}) = n$, which is impossible.

Proposition 4.1. *Let $\beta\in\mathbb{R}\setminus\{0\}$. A state ϕ on $C^*(S)$ is a KMS_β-state if and only if $\phi\circ\varphi$ is a normalised trace on $C^*(S_c)$ and*

$$\phi(v_sv_t^*) = \begin{cases} N_s^{-\beta}\phi(v_bv_a^*) & \text{if } s\sim t,\\ 0 & \text{otherwise}\end{cases} \tag{4.1}$$

for all $s,t\in S$.

Proof. Suppose that ϕ is a KMS_β-state. Since $N(a)=1$ for all $a\in S_c$, see [ABLS17, Proposition 3.6(i)], the KMS-condition implies that $\phi\circ\varphi$ is tracial. For $s,t\in S$, applying the KMS_β-condition twice gives

$$\phi(v_sv_t^*) = N_s^{-\beta}\phi(v_t^*v_s) = (N_t/N_s)^\beta\phi(v_sv_t^*).$$

Thus, if $N_s\neq N_t$, necessarily $\phi(v_sv_t^*)=0$. If $N_s=N_t$, then [ABLS17, Proposition 3.6(ii)] states that either $s\perp t$, in which case $\phi(v_sv_t^*)=N_s^{-\beta}\phi(v_t^*v_s)=0$, or $s\sim t$. In the later case, assume $sS\cap tS=saS, sa=tb$ for some $a,b\in S_c$. Then $\phi(v_sv_t^*)=N_s^{-\beta}\phi(v_t^*v_s)=N_s^{-\beta}\phi(v_bv_a^*)$. Suppose $c,d\in S_c$ also satisfy $sc=td$. Since sa is a right LCM of s and t, there is $f\in S_c$ such that $sc=saf, td=tbf$. Left cancellation implies $c=af$ and $d=bf$, and by Lemma 2.5 we obtain $\phi(v_dv_c^*)=\phi(v_be_{fS}v_a^*)=\phi(v_bv_a^*)$, showing (4.1).

Conversely, suppose ϕ is a state such that $\phi\circ\varphi$ is a trace and (4.1) holds. Note that ϕ is σ-invariant because $N_a=1$ for all $a\in S_c$. In view of [ALN, Lemma 1.9], it suffices to establish that

$$\phi(v_sv_tv_r^*) = N_s^{-\beta}\phi(v_tv_r^*v_s) \quad \text{for all } s,t,r\in S, \tag{4.2}$$

By (4.1), $\phi(v_sv_tv_r^*)=0$ unless $st\sim r$, and $v_r^*v_s=0$ unless $r \Cap s$. Assuming $rS\cap sS=ss'S, ss'=rr'$ for some $s',r'\in S$, we get $v_tv_r^*v_s=v_{tr'}v_{s'}^*$, so again by (4.1) $\phi(v_tv_r^*v_s)=0$ unless $tr'\sim s'$. Assume therefore that $tr'a=s'b$ for some $a,b\in S_c$. This leads to $str'a=ss'b=rr'b$, hence to $N_{st}=N_r$ because N is multiplicative on S and takes value 1 on S_c. It follows from [ABLS17, Proposition 3.6(ii)] that $st\sim r$ as well. So both sides of (4.2) are nonzero simultaneously, and it remains to show that they are equal in this case.

By Lemma 2.2, $r'\in S_c$. Using that $\phi\circ\varphi$ is a trace on $C^*(S_c)$, we get

$$\phi(v_tv_r^*v_s) = \phi(v_{tr'}v_{s'}^*) \overset{(4.1)}{=} N_t^{-\beta}\phi(v_bv_a^*) = N_t^{-\beta}\phi(v_{r'b}v_{r'a}^*) \overset{(4.1)}{=} N_s^\beta\phi(v_{st}v_r^*),$$

giving the desired claim. □

4.2. **The reconstruction formula.** The aim of this section is to obtain the reconstruction formula for KMS$_\beta$-states of [ABLS17, Lemma 7.5] for all right LCM semigroups S with generalised scale N. The result was first proved in [LR10, Lemma 10.1] for $S = \mathbb{N} \rtimes \mathbb{N}^\times$, and generalised to an admissible right LCM monoid S in [ABLS17, Lemma 7.5]. Here we show that the existence of a generalised scale alone suffices to obtain the formula. While the strategy of proof is the same as for [ABLS17, Lemma 7.5], the major difference is that the former result assumed the existence of minimal representatives for the equivalence classes in $S/_\sim$, see [ABLS17, Lemma 3.2], which allows one to perform most steps in $C^*(S)$, whereas here we need to work almost entirely in $\pi_\phi(C^*(S))$, for instance because the summations appearing in (2.4) would not be well-defined when replacing $\pi_\phi(e_{tS})$ by e_{tS}.

Lemma 4.2. *Let ϕ be a KMS$_\beta$-state on $C^*(S)$ for some $\beta \in \mathbb{R}$, let $(\pi_\phi, H_\phi, \xi_\phi)$ its GNS-representation and $\tilde{\phi} = (\cdot\ \xi_\phi \mid \xi_\phi)$ the vector state extension of ϕ to $\mathcal{L}(H_\phi)$. For every nonempty subset I of $\operatorname{Irr}(N(S))$, $Q_I := \lim_{F \Subset I} Q_F$ defines a projection in $\pi_\phi(C^*(S))''$. If $\zeta_I(\beta) < \infty$, then the following statements hold:*

(i) $\tilde{\phi}(Q_I) = \zeta_I(\beta)^{-1}$.

(ii) The map $y \mapsto \zeta_I(\beta)\ \tilde{\phi}(Q_I \pi_\phi(y) Q_I)$ defines a state ϕ_I on $C^*(S)$, for which $\phi_I \circ \varphi$ is a trace on $C^*(S_c)$.

(iii) The family $(\pi_\phi(v_s) Q_I \pi_\phi(v_s^*))_{[s] \in S_I/_\sim}$ consists of mutually orthogonal projections, and $Q^I := \sum_{[s]\in S_I/_\sim} \pi_\phi(v_s) Q_I \pi_\phi(v_s^*)$ defines a projection such that $\tilde{\phi}(Q^I) = 1$.

(iv) There is a reconstruction formula for ϕ given by

$$\phi(y) = \zeta_I(\beta)^{-1} \sum_{[s]\in S_I/_\sim} N_s^{-\beta}\ \phi_I(v_s^* y v_s) \quad \text{for all } y \in C^*(S). \tag{4.3}$$

Proof. The strategy of proof is similar to [ABLS17, Lemma 7.5], so we only indicate what changes can be made to avoid using core irreducible elements (whose collection in S may reduce to 1, see Proposition 7.2).

Note that $(Q_F)_{F \Subset I}$ with Q_F as in (2.4) is a family of commuting projections such that $F \subset F'$ implies $Q_F \geq Q_{F'}$. Thus they converge weakly to a projection $\lim_{F \Subset I} Q_F =: Q_I$. To obtain (i), we note that

$$Q_F = \sum_{A \subset F} (-1)^{|A|} \sum_{([t_n])_{n\in A} \in \prod\limits_{n \in A} N^{-1}(n)/_\sim} \pi_\phi(e_{\bigcap\limits_{n\in A} t_n S}) = \sum_{A \subset F} (-1)^{|A|} \sum_{[t] \in N^{-1}(m_A)/_\sim} \pi_\phi(e_{tS}) \tag{4.4}$$

for $F \Subset I$. Then the argument employed in [ABLS17, Lemma 7.5] gives that $\tilde{\phi}(Q_F) = \zeta_F(\beta)^{-1}$, so the claim (i) follows by continuity. This implies that ϕ_I is a state on $C^*(S)$. To prove (ii) it suffices, since ϕ_I is the w*-limit of $(\phi_F)_{F \Subset I}$, to show that $\phi_F \circ \varphi$ is tracial for an arbitrary, but fixed $F \Subset I$. We claim that

$$\phi_F(x) \;=\; \zeta_I(\beta) \sum_{A,B \subset F} (-1)^{|A|+|B|} \sum_{[r] \in N^{-1}(m_{A\cup B})/_\sim} N(r)^{-\beta} \phi(v_r^* x v_r),$$

for all $x \in \varphi(C^*(S_c))$. To prove this claim, note that by (4.4), $\phi_F(x)$ will contain summands of the form $\phi(e_{sS} x e_{tS})$ with $[s] \in N^{-1}(m_A)$, $[t] \in N^{-1}(m_B)$, which by the KMS-condition vanish unless $s \Cap t$. For any pair $[s],[t]$ with $sS \cap tS = rS$, we get summands $\phi(x e_{rS})$, which by Lemma 2.5 only depend on the equivalence class $[r]$ in $S/_\sim$. Further, since N preserves right LCMs by [ABLS17, Proposition 3.6], the right

LCM of m_A and m_B in $N(S)$ is given by $m_{A\cup B}$. Finally, the claim follows by applying the KMS$_\beta$-condition to $\phi(xe_{rS})$.

Next we recall that the bijection $\alpha_a\colon S/_\sim \to S/_\sim$ restricts to a bijection on $N^{-1}(n)/_\sim$ for $a \in S_c$, see [ABLS17, Lemma 3.9] (the proof given there only used the existence of a generalised scale on S, although the result was stated with further assumptions on S). Using this, for arbitrary $a,b,c,d \in S_c$ and $n \in N(S)$ we have

$$\sum_{[q]\in N^{-1}(n)/_\sim} \phi(v_q^*v_av_b^*v_cv_d^*v_q) \overset{\alpha_a}{=} \sum_{[q]\in N^{-1}(n)/_\sim} \phi(v_{bq}^*v_cv_d^*v_av_b^*v_{bq}) \overset{\alpha_b}{=} \sum_{[q]\in N^{-1}(n)/_\sim} \phi(v_q^*v_cv_d^*v_av_b^*v_q).$$

From this observation and the above claim it follows that $\phi_F \circ \varphi$ is a normalised trace on $C^*(S_c)$ for every $F \Subset I$, and hence $\phi_I \circ \varphi$ is also tracial.

To prove (iii), fix projections $\pi_\phi(v_s)Q_I\pi_\phi(v_s^*)$ and $\pi_\phi(v_t)Q_I\pi_\phi(v_t^*)$ for $[s] \neq [t]$ in $S_I/_\sim$. Then $[s] \in N^{-1}(n)/_\sim$ and $[t] \in N^{-1}(m)/_\sim$ for some $m,n \in N(S)$. Moreover, $s \not\sim t$. If $m = n$ we get $v_s^*v_t = 0$ exactly as in the proof of [ABLS17, Lemma 7.5]. Suppose that $m \neq n$. If $s \perp t$ then again $v_s^*v_t = 0$, so we may assume $sS\cap tS = ss'S, ss' = tt'$ for some $s',t' \in S$, and thus $v_s^*v_t = v_{s'}v_{t'}^*$. Moreover, since $s \not\sim t$, we have $N_{s'} > 1$ or $N_{t'} > 1$. Assume $N_{s'} > 1$ and pick a divisor $n' \in \mathrm{Irr}(N(S))$ of $N_{s'}$ in $N(S)$. Then $Q_I\pi_\phi(v_s^*v_t)Q_I$ contains a factor

$$\big(1 - \sum_{[r]\in N^{-1}(n')/_\sim} \pi_\phi(e_{rS})\big)\pi_\phi(e_{s'S}),$$

and there is at least one $r \in N^{-1}(n')$ satisfying $s'S \subset rS$. The latter yields $\pi_\phi(e_{rS}e_{s'S}) = \pi_\phi(e_{s'S})$, so the whole expression $Q_I\pi_\phi(v_s^*v_t)Q_I$ vanishes. The case of $N_{t'} > 1$ is analogous and orthogonality of the family follows. For I finite, we can now conclude that

$$\tilde\phi(Q^I) = \sum_{[s]\in S_I/_\sim} \tilde\phi(\pi_\phi(v_s)Q_I\pi_\phi(v_s)^*) = \sum_{[s]\in S_I/_\sim} N_s^{-\beta}\tilde\phi(Q_I) \overset{(i)}{=} 1$$

using $Q_I \in \pi_\phi(C^*(S))$ and the KMS-condition for $\tilde\phi$. Part (iii) is now completed for arbitrary I by invoking continuity.

Claim (iv) follows from (iii) together with the KMS-condition for $\tilde\phi$: For I finite and $y \in C^*(S)$, we first get

$$\begin{aligned}\phi(y) &= \sum_{[s],[t]\in S_I/_\sim} \tilde\phi(\pi_\phi(v_s)Q_I\pi_\phi(v_s^*yv_t)Q_I\pi_\phi(v_t)^*)\\ &= \sum_{[s],[t]\in S_I/_\sim} N_s^{-\beta}\tilde\phi(Q_I\pi_\phi(v_s^*yv_t)Q_I\pi_\phi(v_t)^*\pi_\phi(v_s)Q_I).\end{aligned}$$

According to (iii), the summands vanish unless $[s] = [t]$, which gives $\pi_\phi(v_s)Q_I\pi_\phi(v_s)^* = \pi_\phi(v_t)Q_I\pi_\phi(v_t)^*$. Thus we can let $t = s$ so that

$$\begin{aligned}\phi(y) &= \sum_{[s]\in S_I/_\sim} N_s^{-\beta}\tilde\phi(Q_I\pi_\phi(v_s^*yv_s)Q_I)\\ &\overset{(ii)}{=} \zeta_I(\beta)^{-1}\sum_{[s]\in S_I/_\sim} N_s^{-\beta}\phi_I(v_s^*yv_s).\end{aligned}$$

The case of arbitrary I is obtained by a continuity argument. □

Remark 4.3. Lemma 4.2 bears no implications for $\beta = 1$ as $\zeta_I(1) = \infty$ for all non-empty infinite $I \subset \mathrm{Irr}(N(S))$. However, we can obtain a simpler reconstruction formula, as follows. If ϕ is a KMS_1-state on $C^*(S)$ and $n \in N(S)$, then

$$\tilde{\phi}(1 - Q_{\{n\}}) = \sum_{[s] \in N^{-1}(n)/\sim} \phi(e_{sS}) = 1.$$

As in the proof of Lemma 4.2 (iv) we arrive at

$$\phi(y) = \sum_{[s] \in N^{-1}(n)/\sim} n^{-1} \phi(v_s^* y v_s) \quad \text{for all } y \in C^*(S), n \in N(S). \tag{4.5}$$

Remark 4.4. There is an apparent similarity between this type of reconstruction formula and [ALN, Theorem 6.8]. We need Lemma 4.2 in the style of [ABLS17] as we will make use of the intermediate results we get for finite subsets in the case where $1 < \beta \leq \beta_c$. This will be the key to proving Proposition 6.6, which is the uniqueness result for KMS-states of infinite type in the case of very high-dimensional dynamics, that is, where the critical inverse temperature is strictly larger than 1.

4.3. **The core Fock module.** The aim of this subsection is to introduce a $C^*(S)$-$C^*(S_c)$-module for every right LCM monoid S that can be employed to induce KMS-states on $C^*(S)$ from normalised traces on $C^*(S_c)$, and ground states from states.

For each $s \in S$, we let $\mathfrak{M}_{0,s}$ denote a copy of $C^*(S_c)$ equipped with the standard right $C^*(S_c)$-module structure. We write $\varphi_s \colon \mathfrak{M}_{0,s} \to C^*(S_c)$ for the natural identification map, which is an isometric isomorphism of Banach spaces, and set $\varphi_{s,t} := \varphi_s^{-1} \circ \varphi_t : \mathfrak{M}_{0,t} \to \mathfrak{M}_{0,s}$. For convenience, let $\tilde{\varphi}_t := \varphi \circ \varphi_t : \mathfrak{M}_{0,t} \to C^*(S)$.

Lemma 4.5. *On the right $C^*(S_c)$-module $\mathfrak{M}_0 := \bigoplus_{s \in S} \mathfrak{M}_{0,s}$, the map*

$$\begin{array}{rcl} \langle \cdot, \cdot \rangle \colon \mathfrak{M}_{0,s} \times \mathfrak{M}_{0,t} & \to & C^*(S_c) \\ (\xi, \eta) & \mapsto & E((v_s \tilde{\varphi}_s(\xi))^* v_t \tilde{\varphi}_t(\eta)) \end{array}$$

determines a positive semidefinite sesquilinear form on $\mathfrak{M}_0$.

Proof. It is straightforward to check that $\langle \cdot, \cdot \rangle$ is sesquilinear, so let $\xi = \sum_{s \in F} \xi_s$ with $\xi_s \in \mathfrak{M}_{0,s}$ for every $s \in F \Subset S$. Then $\varphi(\langle \xi, \xi \rangle) = \tilde{\xi}^* \tilde{\xi} \geq 0$ in $C^*(S)$ for $\tilde{\xi} := \sum_{s \in F} v_s \varphi_s(\xi_s)$. Since φ is faithful, we deduce $\langle \xi, \xi \rangle \geq 0$. □

We can thus form the right Hilbert $C^*(S_c)$-module $\mathfrak{M} := \overline{\mathfrak{M}_0/\mathfrak{N}}^{\|\cdot\|}$ for $\mathfrak{N} := \{\xi \in \mathfrak{M}_0 \mid \langle \xi, \xi \rangle = 0\}$ and the norm given by $\|\xi\|^2 := \|\langle \xi, \xi \rangle\|$.

It follows from the definition of $\langle \cdot, \cdot \rangle$ that for $\xi = \sum_{t \in S} \xi_t, \eta = \sum_{r \in S} \eta_r$ in $\mathfrak{M}_0$, if we let $\xi_{[s]} := \sum_{t \in [s]} \xi_t$ and similarly for $\eta_{[s]}$, then

$$\langle \xi, \eta \rangle = \sum_{[s] \in S/\sim} \sum_{t,r \in [s]} \langle \xi_t, \eta_r \rangle = \sum_{[s] \in S/\sim} \langle \xi_{[s]}, \eta_{[s]} \rangle, \tag{4.6}$$

because all terms $\langle \xi_t, \eta_r \rangle$ vanish for $[t] \neq [r]$. From this perspective, it is natural to view $\mathfrak{M}_0$ as $\mathfrak{M}_0 = \bigoplus_{[s] \in S/\sim} \mathfrak{M}_{0,[s]}$ with $\mathfrak{M}_{0,[s]} := \bigoplus_{t \in [s]} \mathfrak{M}_{0,t}$.

Theorem 4.6. *For every right LCM monoid S, $\mathfrak{M}$ is a $C^*(S) - C^*(S_c)$ right Hilbert bimodule with left action determined by $V_s([\xi]) := [\varphi_{st,t}(\xi)]$ for $\xi \in \mathfrak{M}_{0,t}$.*

Proof. We need to show that the linear maps $(V_s)_{s \in S}$ are well-defined and adjointable on $\mathfrak{M}_0/\mathfrak{N}$ so that they extend to adjointable, linear maps on $\mathfrak{M}$ that we again denote by V_s, and that these define a representation of $C^*(S)$. First let $V_{0,s}$ be the linear operator on $\mathfrak{M}$ given by $V_{0,s}(\xi_t) := \varphi_{st,t}(\xi_t)$ for $\xi_t \in \mathfrak{M}_{0,t}$ and $s,t \in S$. For $r \in S$ and $\xi \in \mathfrak{M}_{0,t}, \eta \in \mathfrak{M}_{0,r}$, we have

$$\begin{aligned}\langle V_{0,s}(\xi), V_{0,s}(\eta)\rangle &= E(\tilde{\varphi}_{st}(\varphi_{st,t}(\xi))^* v_{st}^* v_{sr} \tilde{\varphi}_{sr}(\varphi_{sr,r}(\eta)))\\ &= E(\tilde{\varphi}_t(\xi)^* v_t^* v_r \tilde{\varphi}_r(\eta))\\ &= \langle \xi, \eta\rangle,\end{aligned}$$

because $[st] = [sr]$ precisely when $[t] = [r]$ due to left cancellation. Thus $V_{0,s}$ induces a well-defined inner-product preserving linear operator V_s on $\mathfrak{M}_0/\mathfrak{N}$. In particular, if V_s is shown to be adjointable, it will be an isometry.

Fix $s \in S$. For each $t \in S$ such that $[t] \in sS/_\sim$, we fix $a_t \in S_c, \bar{t} \in S$ with the property that $sS \cap tS = ta_tS$ and $ta_t = s\bar{t}$. We then define a linear operator $V'_{0,s}$ on $\mathfrak{M}_0$ by sending $\xi_t \in \mathfrak{M}_{0,t}$ to

$$V'_{0,s}(\xi_t) = \begin{cases} \tilde{\varphi}_{\bar{t}}^{-1}(v_{a_t}^* \tilde{\varphi}_t(\xi_t)) & \text{if } sS \cap tS = ta_tS, ta_t = s\bar{t} \text{ with } a_t \in S_c, \bar{t} \in S;\\ 0 & \text{otherwise.}\end{cases} \tag{4.7}$$

We claim that $\langle V'_{0,s}(\xi), \eta\rangle = \langle \xi, V_{0,s}(\eta)\rangle$ for all $\xi,\eta \in \mathfrak{M}_0$. Due to the definition of $\langle \cdot,\cdot\rangle$ and linearity, it suffices to consider the case where $\xi \in \mathfrak{M}_{0,t}$ and $\eta \in \mathfrak{M}_{0,r}$ for arbitrary $t,r \in S$. We have $\langle V'_{0,s}(\xi), \eta\rangle = 0$ unless $[t] \in sS/_\sim$ and $\eta \in \mathfrak{M}_{0,r}$ with $r \sim \bar{t}$, that is, by left cancellation in S, unless $t \sim s\bar{t} \sim sr$. By (4.6), we see that $\langle \xi, V_{0,s}(\eta)\rangle = 0$ unless $sr \sim t$. Assuming $t \sim sr$, we compute that

$$\begin{aligned}\langle V'_{0,s}(\xi), \eta\rangle &= E(\tilde{\varphi}_t(\xi)^* v_{a_t} v_{\bar{t}}^* v_r \tilde{\varphi}_r(\eta))\\ &= E(\tilde{\varphi}_t(\xi)^* v_t^* v_{sr} \tilde{\varphi}_r(\eta)) \quad = \quad \langle \xi, V_{0,s}(\eta)\rangle.\end{aligned}$$

It follows that $\langle V'_{0,s}(\xi), \eta\rangle = \langle \xi, V_{0,s}(\eta)\rangle$ for all $\xi,\eta \in \mathfrak{M}_0$. In particular, $\mathfrak{N} \subset \ker\, V'_{0,s}$ for all $s \in S$ and $\|V'_{0,s}\| \leq 1$, so that we obtain a well-defined bounded linear operator V'_s on $\mathfrak{M}$. We conclude that V_s is an adjointable isometry on $\mathfrak{M}$ with $V'_s = V_s^*$ for every $s \in S$, and $V_sV_t = V_{st}$ for $s,t \in S$ is clear.

We note for later use that the expression for $V'_{0,s}(\xi_t)$ in (4.7) depends on the chosen pair a_t and $\bar{t}$. We can replace this by any other pair $f \in S_c$ and $w \in S$ such that $rS \cap sS = sfS$ and $tf = sw$ upon multiplying a_t and $\bar{t}$ on the right with an element x in S^*. The resulting $V'_{0,s}(\xi)$ will land in a summand $\mathfrak{M}_{0,\bar{t}x}$, but the final operator V_s^* will not, as it is defined on the module $\mathfrak{M}$.

In order to obtain a representation of $C^*(S)$, we need to show that

$$V_s^* V_t = \begin{cases} V_{s'} V_{t'}^* & \text{if } sS \cap tS = ss'S, ss' = tt';\\ 0 & \text{if } s \perp t.\end{cases} \tag{4.8}$$

for all $s,t \in S$. So let $s,t \in S$. If $s \perp t$, then $s \perp tr$ for all $r \in S$, and therefore $V_{0,t}(\mathfrak{M}_{0,r}) = \mathfrak{M}_{0,tr}$, implying $V_s^*V_t = 0$. So let us assume that there are $s',t' \in S$ with $sS \cap tS = ss'S, ss' = tt'$. Fix $r \in S$ and $\xi \in \mathfrak{M}_{0,r}$, and note that

$$sS \cap trS = sS \cap tS \cap trS = t(t'S \cap rS). \tag{4.9}$$

By our definition of adjoint in (4.7), we have $V_s^*V_t\,([\xi]) = 0$ unless $[tr] \in sS/_\sim$, and similarly $V_{s'}V_{t'}^*([\xi])$ vanishes unless $[r] \in t'S/_\sim$. In case the former inclusion holds, we have $trb = ss''$ for some $b \in S_c$, $s'' \in S$, thus $trb \in sS \cap tS = tt'S$, and hence by left cancellation $rb = t'r'$ for some $r' \in S$. This is exactly the condition $[r] \in t'S/_\sim$.

Conversely, from $[r] \in t'S/_\sim$ we get $[tr] \in sS/_\sim$ by (4.9). Thus it suffices to prove equality of the terms in (4.8) evaluated at $[\xi]$ under the assumption that they both are non-zero. We choose $a_r \in S_c$ and $\overline{r} \in S$ with $t'S \cap rS = ra_rS, ra_r = t'\overline{r}$, so that

$$V_{s'}V_{t'}^*([\xi]) = [\varphi_{s'\overline{r},\overline{r}}(\tilde{\varphi}_{\overline{r}}^{-1}(v_{a_r}^*\tilde{\varphi}_r(\xi)))].$$

Note that by left cancellation for t and (4.9), we equivalently have $sS \cap trS = tra_rS$ with $tra_r = tt'\overline{r} = ss'\overline{r}$. Similarly, choose $a_{tr} \in S_c$ and $\overline{tr} \in S$ such that $trS \cap sS = tra_{tr}S$ and $tra_{tr} = s\overline{tr}$ to obtain

$$V_s^*V_t\ ([\xi]) = [\tilde{\varphi}_{\overline{tr}}^{-1}(v_{a_{tr}}^*\tilde{\varphi}_{tr}(\varphi_{tr,r}(\xi)))].$$

Since we may compute V_s^* using the pair a_r, $s'\overline{r}$ or a_{tr}, $\overline{tr}$, we therefore get

$$\begin{aligned} V_{s'}V_{t'}^*([\xi]) &= [\tilde{\varphi}_{s'\overline{r}}^{-1}(v_{a_r}^*\tilde{\varphi}_r(\xi))] \\ &= [\tilde{\varphi}_{\overline{tr}}^{-1}(v_{a_{tr}}^*\tilde{\varphi}_{tr}(\varphi_{tr,r}(\xi)))] \;=\; V_s^*V_t\ ([\xi]) \end{aligned}$$

By continuity, we get (4.8), so that V defines a left action of $C^*(S)$ on $\mathfrak{M}$. □

Definition 4.7. For a right LCM monoid S, we let the *core Fock module* of S be the right Hilbert $C^*(S)$-$C^*(S_c)$-module $\mathfrak{M}$ of Theorem 4.6. We let $\pi_{\mathfrak{M}}$ denote the $*$-homomorphism $C^*(S) \to \mathcal{L}(\mathfrak{M})$ from Theorem 4.6.

By virtue of Theorem 4.6, every representation $\pi\colon C^*(S_c) \to B(H)$ of $C^*(S_c)$ on a Hilbert space H gives rise to an induced representation $\mathrm{Ind}^{\mathfrak{M}}\pi\colon C^*(S) \to \mathcal{L}(\mathfrak{M} \otimes_\pi H)$ of $C^*(S)$. In particular, this applies to the GNS-representation $(\pi_\rho, H_\rho, \xi_\rho)$ associated to a state ρ on $C^*(S_c)$.

Notation 4.8. For $r \in S$, let $\omega_r := [\varphi_r^{-1}(1)]$. For every state ρ on $C^*(S_c)$, define a state on $C^*(S)$ by $\chi_{\rho,r}(x) := \langle \mathrm{Ind}^{\mathfrak{M}}\pi_\rho(x)(\omega_r \otimes \xi_\rho), \omega_r \otimes \xi_\rho\rangle$ for $x \in C^*(S)$.

Lemma 4.9. *Let $r \in S$ and ρ be a state on $C^*(S_c)$. Then the following hold:*

- *(i) $\chi_{\rho,r}(x) = \rho(E(v_r^*xv_r))$ for all $r \in S, x \in C^*(S)$. In particular, $\chi_{\rho,1} \circ \varphi = \rho$.*
- *(ii) If ρ is a trace, then $\chi_{\rho,r} = \chi_{\rho,r'}$ whenever $r, r' \in S$ satisfy $r \sim r'$.*
- *(iii) If ρ is a trace and $s, t \in S$, then $\chi_{\rho,r}(v_sv_t^*) = 0$, unless $s \sim t$, say $sS \cap tS = saS, sa = tb$ for some $a, b \in S_c$, and there exists $r'' \in S$ with $sar'' = tbr'' \sim r$. The map $[r] \mapsto [r'']$ is a one-to-one correspondence between $saS/_\sim$ and $S/_\sim$ with $\chi_{\rho,r}(v_sv_t^*) = \chi_{\rho,r''}(v_a^*v_b)$.*

Proof. We first show that $\langle V_s\, V_t^*(\omega_r), \omega_r\rangle = E(v_r^*v_t\,v_s^*v_r)$ for all $s, t, r \in S$, which will then imply (i) as $C^*(S) = \overline{\mathrm{span}}\{v_sv_t^* \mid s, t \in S\}$. Due to (4.7), $V_t^*(\omega_r) = 0$ unless $r \in tS/_\sim$, in which case writing $rS \cap tS = ra_rS, ra_r = t\bar{r}$ for some $a_r \in S_c$ and $\bar{r} \in S$ implies that $V_t^*(\omega_r) \in \mathfrak{M}_{0,\bar{r}}$. Thus $\langle V_s\, V_t^*(\omega_r), \omega_r\rangle = 0$ unless we have $r \in tS/_\sim$ and $r \sim s\bar{r}$. Assume therefore $r \in tS/_\sim$ and $r \sim s\bar{r}$, with $a_r, \bar{r}$ defined as above. It follows as in the proof of (4.8) that

$$\langle V_s\, V_t^*(\omega_r), \omega_r\rangle = E(v_a v_{s\bar{r}}^* v_r) = E(v_r^*v_t\,v_s^*v_r).$$

Likewise, $v_r^*v_t\,v_s^*v_r$ is zero, unless $rS \cap tS = tt'S, rr' = tt'$ for some $r', t' \in S$. In this case, we have $v_r^*v_t\,v_s^*v_r = v_{r'}v_{st'}^*v_r$, which belongs to $\varphi(C^*(S_c))$ if and only if $st' \sim r$.

To prove part (ii) note that $\rho(w_aw_a^*) = 1$ for all $a \in S_c$ because ρ is a trace and w_a is an isometry, thus (i) implies $\chi_{\rho,ra} = \chi_{\rho,r}$ for all $r \in S$. As $s \sim t$ is equivalent to $sa = tb$ for some $a, b \in S_c$, we get (ii).

For (iii), let $s,t \in S$. Part (i) implies that $\chi_{\rho,r}(v_s v_t^*)$ vanishes unless $sr' \sim r \sim tr'$ for some $r' \in S$. If these equivalences hold, then $s \sim t$ by [ABLS17, Proposition 3.6(ii)] and $N_s N_{r'} = N_r = N_t N_{r'}$. So fix $a,b \in S_c$ satisfying $sS \cap tS = saS$ and $sa = tb$. Moreover, $sr' \sim tr'$ gives $sr'c = tr'd \in sS \cap tS = saS$ for suitable $c,d \in S_c$, so there is $[r''] \in S/_\sim$ with $[r] = [sr'] = [sar'']$ and $sar'' = tbr''$. Conversely, every $[r''] \in S$ yields a distinct class $[sar'']$. We thus obtain a one-to-one correspondence $[r] \mapsto [r'']$ between $saS/_\sim = tbS/_\sim$ and $S/_\sim$. Using this correspondence, we get

$$\chi_{\rho,r}(v_s v_t^*) \overset{(ii)}{=} \chi_{\rho,sar''}(v_s v_t^*) \overset{(i)}{=} \rho(E(v_{sar''}^* v_s v_t^* v_{tbr''})) \overset{(i)}{=} \chi_{\rho,r''}(v_a^* v_b).$$

□

5. Parametrisation of KMS-states

5.1. **Ground states and KMS-states of finite type.** With the core Fock module in place, we proceed with the discussion of KMS-states for $(C^*(S),\sigma)$. We first address ground states in Proposition 5.1, then we produce KMS$_\beta$-states from normalised traces on $C^*(S_c)$ in Proposition 5.4 for $\beta \in (1,\infty)$ and obtain a parametrisation for the KMS$_\beta$-states with $\beta \in (\beta_c,\infty)$ in Proposition 5.5. In the spirit of [ALN, Definition 6.4], we refer to these last ones as states of finite type.

Proposition 5.1. *There exists an affine homeomorphism between the states on $C^*(S_c)$ and the ground states on $C^*(S)$ given by $\rho \mapsto \psi_\rho := \chi_{\rho,1}$.*

Proof. Given a state ρ on $C^*(S_c)$, the map ψ_ρ is a state on $C^*(S)$ such that $0 \neq \psi_\rho(v_s v_t^*)$ forces $[1] \in (sS \cap tS)/_\sim$, see Lemma 4.9 (i). As this condition is equivalent to $s,t \in S_c$, the proof then follows as in [ABLS17, Proposition 6.2]. □

Before we construct KMS$_\beta$-states on $C^*(S)$ from traces τ on $C^*(S_c)$ by use of the GNS representation π_τ, we make note of an intermediate step which produces states on $C^*(S)$ from traces on $C^*(S_c)$. Recall that we view the finite subsets of $\mathrm{Irr}(N(S))$ as a (countable) directed set when ordered by inclusion.

Lemma 5.2. *Let τ be a normalised trace on $C^*(S_c)$. For every $n \in N(S)$, the state $\psi_{\tau,n} := n^{-1}\sum_{[s]\in N^{-1}(n)/_\sim} \chi_{\tau,s}$ on $C^*(S)$ restricts to a normalised trace $\psi_{\tau,n}\circ\varphi$ on $C^*(S_c)$. In particular, for all $\beta > 1$ and $I \subset Irr(N(S))$ with $\zeta_I(\beta) < \infty$,*

$$\psi_{\beta,\tau,I} := \zeta_I(\beta)^{-1} \sum_{[s]\in S_I/_\sim} N_s^{-\beta}\chi_{\tau,s} = \zeta_I(\beta)^{-1} \sum_{n\in\langle I\rangle^+} n^{1-\beta}\psi_{\tau,n}$$

defines a state on $C^*(S)$ such that $\psi_{\beta,\tau,I}\circ\varphi$ is tracial on $C^*(S_c)$.

Proof. By Lemma 4.9 (ii) and the trace property for τ, the map $\psi_{\tau,n}$ is a well-defined state on $C^*(S)$. We claim that

$$\sum_{[s]\in N^{-1}(n)/_\sim} \chi_{\tau,s}(v_a v_b^* v_c v_d^*) = \sum_{[s]\in N^{-1}(n)/_\sim} \chi_{\tau,s}(v_c v_d^* v_a v_b^*) \tag{5.1}$$

holds for all $a,b,c,d \in S_c$ and $n \in N(S)$. The term $\chi_{\tau,s}(v_a v_b^* v_c v_d^*)$ vanishes unless $[s] \in \mathrm{Fix}\ \alpha_a\alpha_b^{-1}\alpha_c\alpha_d^{-1}$, see Lemma 4.9 (i). Likewise, $\chi_{\tau,s'}(v_c v_d^* v_a v_b^*)$ is zero unless $[s'] \in \mathrm{Fix}\ \alpha_c\alpha_d^{-1}\alpha_a\alpha_b^{-1}$. These two sets are related by the bijections

$$\mathrm{Fix}\ \alpha_a\alpha_b^{-1}\alpha_c\alpha_d^{-1} \overset{\alpha_d}{\longleftarrow} \mathrm{Fix}\ \alpha_d^{-1}\alpha_a\alpha_b^{-1}\alpha_c \overset{\alpha_c}{\longrightarrow} \mathrm{Fix}\ \alpha_c\alpha_d^{-1}\alpha_a\alpha_b^{-1}. \tag{5.2}$$

We note that every $s \in S$ satisfies

$$\begin{aligned}(5.3)\qquad \chi_{\tau,ds}(v_a v_b^* v_c v_d^*) &= \tau(\varphi^{-1}(v_{ds}^* v_a v_b^* v_c v_d^* v_{ds})) = \tau(\varphi^{-1}(v_{ds}^* v_a v_b^* v_{cs})) \\ &= \tau(\varphi^{-1}(v_{cs}^* v_c v_d^* v_a v_b^* v_{cs})) = \chi_{\tau,cs}(v_c v_d^* v_a v_b^*).\end{aligned}$$

As α restricts to an action of S_c by bijections on $N^{-1}(n)/_\sim$, Lemma 4.9 (ii) gives

$$\{\chi_{\tau,es} \mid [s] \in N^{-1}(n)/_\sim\} = \{\chi_{\tau,s} \mid [s] \in N^{-1}(n)/_\sim\} \quad \text{for all } n \in N(S), e \in S_c.$$

This leads to

$$\begin{aligned}\sum_{[s]\in N^{-1}(n)/_\sim} \chi_{\tau,s}(v_a v_b^* v_c v_d^*) &= \sum_{\substack{[s]\in N^{-1}(n)/_\sim,\\ [s]\in \text{Fix } \alpha_a\alpha_b^{-1}\alpha_c\alpha_d^{-1}}} \chi_{\tau,s}(v_a v_b^* v_c v_d^*) \\ &= \sum_{\substack{[s]\in N^{-1}(n)/_\sim,\\ [s]\in \text{Fix } \alpha_d^{-1}\alpha_a\alpha_b^{-1}\alpha_c}} \chi_{\tau,ds}(v_a v_b^* v_c v_d^*) \\ &\overset{(5.2),(5.3)}{=} \sum_{\substack{[s]\in N^{-1}(n)/_\sim,\\ [s]\in \text{Fix } \alpha_d^{-1}\alpha_a\alpha_b^{-1}\alpha_c}} \chi_{\tau,cs}(v_c v_d^* v_a v_b^*) \\ &= \sum_{[s]\in N^{-1}(n)/_\sim} \chi_{\tau,s}(v_c v_d^* v_a v_b^*),\end{aligned}$$

which establishes (5.1). Hence $\psi_{\tau,n} \circ \varphi$ is a trace on $C^*(S_c)$. The claim concerning $\psi_{\beta,\tau,I}$ is an immediate consequence hereof. □

Proposition 5.3. *For every normalised trace τ on $C^*(S_c)$, $\psi_{1,\tau} := \lim_{n\in N(S)} \psi_{\tau,n}$ defines a* KMS_1*-state on $C^*(S)$.*

Proof. If the limit exists, it is necessarily a state whose restriction to $\varphi(C^*(S_c))$ is tracial. It thus suffices to show that (4.1) holds for any given $s,t \in S$ for $\psi_{\tau,n}$ for all large enough $n \in N(S)$. According to Lemma 4.9 (iii), $\chi_{\tau,r}(v_s v_t^*)$ vanishes for all r unless $s \sim t$, so let us assume that $sS \cap tS = saS, sa = tb$ for some $a,b \in S_c$. Let n be large enough so that $N_s = N_t$ divides n in $N(S)$, say $n = N_s m$ for some $m \in N(S)$. Next, we note that the one-to-one correspondence $[r] \mapsto [r'']$ from $saS/_\sim$ to $S/_\sim$ from Lemma 4.9 (iii) restricts to a bijection $\{[r] \in N^{-1}(n)/_\sim \mid s \Cap r\} \to N^{-1}(m)/_\sim$. We infer that

$$\psi_{\tau,n}(v_s v_t^*) = n^{-1} \sum_{[r]\in (N^{-1}(n)\cap saS)/_\sim} \chi_{\tau,r}(v_s v_t^*) \overset{(4.9)(iii)}{=} N_s^{-1}\psi_{\tau,m}(v_a^* v_b).$$

Since $\psi_{\tau,m} \circ \varphi$ is tracial and $n \nearrow \infty$ is the same as $m \nearrow \infty$, this establishes (4.1) for $\psi_{1,\tau}$. The limit exists by weak*-compactness: we first get a limit for some sequence $(n_k)_{k\in\mathbb{N}} \subset N(S)$ with $n_k \to \infty$, and then by using (4.5) we get that this limit does not depend on the choice of sequence. □

Proposition 5.4. *For $\beta > 1$ and every normalised trace τ on $C^*(S_c)$, there is a sequence $(I_k)_{k\geq 1}$ of finite subsets of $Irr(N(S))$ such that ψ_{β,τ,I_k} weak*-converges to a* KMS_β*-state $\psi_{\beta,\tau}$ on $C^*(S)$ as $I_k \nearrow Irr(N(S))$. For $\beta \in (\beta_c,\infty)$, the state $\psi_{\beta,\tau}$ is given by $\psi_{\beta,\tau,Irr(N(S))}$.*

Proof. Let $\beta > 1$ and τ be a normalised trace on $C^*(S_c)$. Due to weak*-compactness of the state space on $C^*(S)$, there is a sequence $(I_k)_{k\geq 1}$ of finite subsets of $\mathrm{Irr}(N(S))$

with $I_k \nearrow \mathrm{Irr}(N(S))$ such that $(\psi_{\beta,\tau,I_k})_{k\geq 1}$ obtained from Lemma 5.2 converges to some state $\psi_{\beta,\tau}$ in the weak* topology. By Proposition 4.1, $\psi_{\beta,\tau}$ is a KMS_β-state if and only if $\psi_{\beta,\tau} \circ \varphi$ defines a trace on $C^*(S_c)$ and (4.1) holds. Note that $\psi_{\beta,\tau} \circ \varphi$ is tracial on $C^*(S_c)$ because each of the ψ_{β,τ,I_k} has this property by Lemma 5.2.

It remains to prove (4.1). Let $s,t \in S$. For $k \geq 1$, Lemma 4.9 (iii) shows that $\chi_{\tau,r}(v_s v_t^*)$ vanishes for $[r] \in S_{I_k}/_\sim$ unless $s \sim t$ and $sr' \sim r \sim tr'$ for some $r' \in S$. In particular, k must be so large that $N_s = N_t \in \langle I_k\rangle^+$, which we assume from now on. Hence $\psi_{\beta,\tau}(v_s v_t^*) = 0$ unless $[s] = [t]$, so fix $a,b \in S_c$ satisfying $sS \cap tS = saS$ and $sa = tb$. Noting that the one-to-one correspondence $[r] \mapsto [r'']$ from Lemma 4.9 (iii) maps $saS_{I_k}/_\sim$ to $S_{I_k}/_\sim$ as $r \sim sar''$ and $N_r, N_s \in \langle I_k\rangle^+$ force $N_{r''} \in \langle I_k\rangle^+$, we conclude that the equality $\psi_{\beta,\tau,I_k}(v_s v_t^*) = \delta_{[s],[t]}\ N_s^{-\beta}\psi_{\beta,\tau,I_k}(v_a^* v_b)$ holds for k large enough. Thus the limit $\psi_{\beta,\tau}$ satisfies (4.1) for all $s,t \in S$, and hence is a KMS_β-state. The claim for $\beta \in (\beta_c,\infty)$ follows immediately from $I_k \nearrow \mathrm{Irr}(N(S))$ because the formula from Lemma 5.2 makes sense for $I = \mathrm{Irr}(N(S))$. □

Proposition 5.5. *Let $\beta \in (\beta_c,\infty)$. Then $\phi \mapsto \phi_{Irr(N(S))} \circ \varphi$ defines an affine homeomorphism between the* KMS_β*-states on $C^*(S)$ and the normalised traces on $C^*(S_c)$. Its inverse is given by $\tau \mapsto \psi_{\beta,\tau}$.*

Proof. We shall prove the following:

(i) If ϕ is a KMS_β-state on $C^*(S)$, then $\psi_{\beta,\phi_{\mathrm{Irr}(N(S))}\circ\varphi} = \phi$.

(ii) If $\psi_{\beta,\tau}$ denotes the KMS_β-state obtained in Proposition 5.4 for $\mathrm{Irr}(N(S))$ and a trace τ on $C^*(S_c)$, then $(\psi_{\beta,\tau})_{\mathrm{Irr}(N(S))} \circ \varphi = \tau$.

We start with (i): As $\phi_{\mathrm{Irr}(N(S))} \circ \varphi$ is a normalised trace on $C^*(S_c)$ by Lemma 4.2 (ii), $\psi_{\beta,\phi_{\mathrm{Irr}(N(S))}\circ\varphi}$ is a KMS_β-state, see Proposition 5.4. For $x \in C^*(S))$, we get

$$\psi_{\beta,\phi_{\mathrm{Irr}(N(S))}\circ\varphi}(x) \overset{4.9(i)}{=} \zeta(\beta)^{-1} \sum_{[s]\in S/_\sim} N_s^{-\beta}\phi_{\mathrm{Irr}(N(S))}(v_s^* x v_s) \overset{4.2\,(iv)}{=} \phi(x).$$

For (ii), we fix $x \in C^*(S_c)$ and get

$$(5.4)\qquad \begin{aligned} (\psi_{\beta,\tau})_{\mathrm{Irr}(N(S))}(\varphi(x)) &= \zeta(\beta)\tilde{\psi}_{\beta,\tau}(Q_{\mathrm{Irr}(N(S))}\pi_{\psi_{\beta,\tau}}(\varphi(x))Q_{\mathrm{Irr}(N(S))}) \\ &= \lim_{I\Subset\mathrm{Irr}(N(S))} \zeta(\beta)\tilde{\psi}_{\beta,\tau}(Q_I\pi_{\psi_{\beta,\tau}}(\varphi(x))Q_I). \end{aligned}$$

Let $(T_n)_{n\in I}$ be a family of transversals for $(N^{-1}(n)/_\sim)_{n\in I}$ (which we fix for the remainder of this proof). Note that by (2.4), the defect projection $q_I = \prod_{n\in I}(1-\sum_{t\in T_n} e_{tS})$ is a preimage of Q_I under $\pi_{\psi_{\beta,\tau}}$, so that $\tilde{\psi}_{\beta,\tau}(Q_I\pi_{\psi_{\beta,\tau}}(\varphi(x))Q_I) = \psi_{\beta,\tau}(q_I\varphi(x))q_I)$. By definition of $\psi_{\beta,\tau}$, we have

$$(5.5)\qquad \zeta(\beta)\psi_{\beta,\tau}(q_I\varphi(x)q_I) = \sum_{[s]\in S/_\sim} N_s^{-\beta}\chi_{\tau,s}(q_I\varphi(x)q_I).$$

We claim that $\chi_{\tau,s}(q_I\varphi(x)q_I) = \delta_{[s],S_c}\tau(x)$ for I large enough and all s. First, let $s \in S_c$. Since τ is a trace, Lemma 4.9 (ii) yields $\chi_{\tau,s}(q_I\varphi(x)q_I) = \chi_{\tau,1}(q_I\varphi(x)q_I) = \tau(E(q_I\varphi(x)q_I))$, where E is the conditional expectation from (2.2) given by $E(v_t v_r^*) = \chi_{S_c}(t)\chi_{S_c}(r)\ w_t w_r^*$. The term $q_I\varphi(x)q_I$ is of form $\varphi(x)+y$, where y corresponds to the collection of cross-terms. Thus y is a finite sum of elements of the form $\pm e_{tS}\varphi(x)e_{rS}$ with $t \notin S_c$ or $r \notin S_c$. But such terms belong to $\varphi(C^*(S_c))$ if and only if they vanish, so we get $\chi_{\tau,s}(q_I\varphi(x)q_I) = \tau(E(\varphi(x)) = \tau(x)$.

Now suppose $s \in S \setminus S_c$, that is, $N_s > 1$. For I large enough, there is $n \in I$ with $N_s \in nN(S)$. Since T_n is a foundation set, there is $t \in T_n$ such that $s \Cap t$, hence by Lemma 2.2 we can write $sS \cap tS = saS$ and $sa = tu$ for some $u \in S$ and $a \in S_c$. We claim that $s \perp t'$ for all $t' \in T_n$ such that $t \neq t'$. If this were not true, then from $s \Cap t'$ for $t' \in T_n$ with $t \neq t'$ we get as above $sS \cap t'S = sbS$ and $sb = t'r$ for some $r \in S$ and $b \in S_c$. Now pick $c, d \in S$ such that $ac = bd$ and note that $tuc = sac = sbd = t'rd$, which implies $t \Cap t'$ and contradicts the fact that T_n is accurate. It follows that $v_s^* q_I \varphi(x) q_I v_s$ begins with the factor $v_s^* q_{\{n\}}$, which by the claim is $v_s^*(1 - e_{tS})$. From the choice of a, it is easy to see that $v_s^*(1 - e_{tS}) = (1 - e_{aS})v_s^*$. By Lemma 4.9 (ii) we write $\chi_{\tau,s}(q_I \varphi(x) q_I) = \tau(E(v_s^* q_{\{n\}} q_{I\setminus\{n\}} \varphi(x) q_I v_s))$. Now denoting $z = q_{I\setminus\{n\}} \varphi(x) q_I v_s$, we obtain

$$\chi_{\tau,s}(q_I \varphi(x) q_I) = \tau(E((1 - e_{aS})v_s^* z)) = \tau((1 - w_a w_a^*)E(z)),$$

which vanishes because τ is a trace on $C^*(S_c)$. We have shown that for every $[s] \in S/_\sim, [s] \neq S_c$ the contribution $\chi_{\tau,s}(q_I \varphi(x) q_I)$ vanishes for all large enough I, and we conclude thus from (5.4) that $(\psi_{\beta,\tau})_{\mathrm{Irr}(N(S))}(\varphi(x)) = \tau(x)$ for all $x \in C^*(S_c)$. □

Proof of Theorem 3.6. Let $\beta_0 \geq 1$ and suppose τ is a normalised trace on $C^*(S_c)$ satisfying (3.5). By weak*- compactness of the state space, a subnet of $(\psi_{\beta,\tau})_{\beta>\beta_0}$, where the index set is ordered by the reverse order, converges to a state $\phi_{\beta_0,\tau}$. It follows from [BR97, Proposition 5.3.25], upon choosing $\beta_n \searrow \beta_0$, that $\phi_{\beta_0,\tau}$ is a KMS_{β_0} state. The hypothesis on τ ensures that $\phi_{\beta_0,\tau} \circ \varphi = \tau$. By Proposition 4.1, the latter identity determines the KMS_{β_0}-state completely, so every convergent subnet converges to $\phi_{\beta_0,\tau}$. Since $\psi_{\beta_0,\tau}$ determined by Proposition 5.4 is a weak*-limit of states, thus positive functionals, and since the limit over the finite subsets $\{I_k\}$ of $\mathrm{Irr}(N(S))$ does not depend on choice of subsets, we have that $\phi_{\beta_0,\tau} = \psi_{\beta_0,\tau}$. Moreover, $\phi_{\beta_0,\tau} \circ \varphi = \tau$ shows that we get an embedding of the simplex of normalised traces satisfying (3.5) into the simplex of KMS_{β_0}-states.

Now suppose $\beta_0 = 1$ and let ϕ be a KMS_1-state. In order to show that the embedding obtained before is a surjection, we will show that $\psi_{\beta,\phi\circ\varphi} \circ \varphi = \phi \circ \varphi$ for all $\beta > 1$. To do so, we claim first that for $x \in C^*(S_c)$ we have $v_s^* \varphi(x) v_s \in C^*(S_c)$ for all $s \in S$. To see this, it suffices to consider $x = w_a w_b^*$ with $a, b \in S_c$ such that $0 \neq v_s^* \varphi(x) v_s$. The latter amounts to $\alpha_a^{-1}([s]) = \alpha_b^{-1}([s])$, so that there exist $c, d \in S_c$ with $v_s^* \varphi(x) v_s = v_c v_d^*$. Hence

$$\chi_{\phi\circ\varphi,s}(\varphi(x)) = \phi \circ \varphi \circ E(v_s^* \varphi(x) v_s) = \phi(v_s^* \varphi(x) v_s).$$

Invoking Lemma 5.2, we get $\psi_{\phi\circ\varphi,n} \circ \varphi = n^{-1} \sum_{[s]\in N^{-1}(n)/_\sim} \chi_{\phi\circ\varphi,s} \circ \varphi \overset{(4.5)}{=} \phi \circ \varphi$, and then

$$\psi_{\beta,\phi\circ\varphi} \circ \varphi = \lim_{I \Subset \mathrm{Irr}(N(S))} \zeta_I(\beta)^{-1} \sum_{n \in \langle I \rangle^+} n^{1-\beta} \psi_{\phi\circ\varphi,n} \circ \varphi = \phi \circ \varphi$$

for all $\beta > 1$. Therefore, the normalised trace $\phi \circ \varphi$ satisfies (3.5) for $\beta_0 = 1$, and $\phi_{1,\phi\circ\varphi} \circ \varphi = \phi \circ \varphi$ entails $\phi_{1,\phi\circ\varphi} = \phi$ because of Proposition 4.1. □

Corollary 5.6. *The following hold:*

(i) *For $1 \leq \beta_1 \leq \beta_2$, the simplex $T_{>\beta_1}(C^*(S_c))$ is a face in $T_{>\beta_2}(C^*(S_c))$.*
(ii) *For every $\beta > 1$, the simplex of KMS_1-states embeds into the simplex of KMS_β-states.*

(iii) If $T_{>\beta_1}(C^*(S_c))$ is not a singleton, then $(C^*(S),\sigma)$ does not have a unique KMS$_{\beta_2}$-state for all $\beta_2 \geq \beta_1$.

Proof. The first claim is just an observation, which in combination with Theorem 3.6 proves both (ii) and (iii). ☐

6. Uniqueness for KMS-states in the critical interval

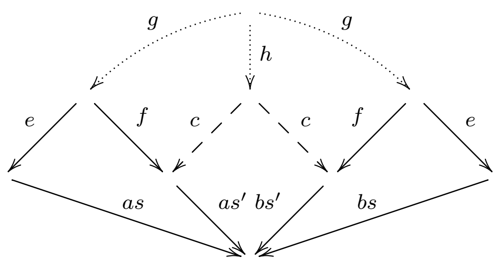
Let S be a right LCM monoid with generalised scale $N\colon S \to \mathbb{N}^\times$. It follows from Proposition 5.3 and Proposition 5.4 that every β in the critical interval $[1,\beta_c]$ has at least one KMS$_\beta$-state on $(C^*(S),\sigma)$. In this section, we will address sufficient conditions for uniqueness of these states.

Our first criterion is the notion of core regularity introduced in Definition 3.1. This is pertinent to uniqueness of KMS$_\beta$-state when the critical interval degenerates to the value $\beta = 1$. In order to justify our definition, first observe that if ϕ is a KMS$_1$-state, $a,b \in S_c$ and $n \in N(S)$, then by equation (4.5) and the KMS$_1$-condition we obtain

$$\phi(v_a v_b^*) = \phi(v_b^* v_a) = \sum_{[s]\in N^{-1}(n)/\sim} n^{-1}\phi(v_{bs}^* v_{as}) = \sum_{[s]\in N^{-1}(n)/\sim} n^{-1}\delta_{[as],[bs]}\ \phi(v_{bs}^* v_{as}). \tag{6.1}$$

The notion of core regularity says that for every choice of distinct elements $a,b \in S_c$, there may only be a small proportion of elements $[s] \in S/_\sim$ such that $[as] = [bs]$ but $asc \neq bsc$ for all $c \in S_c$.

Fix $a,b \in S_c$ and suppose $s \in S$ such that $[as] = [bs]$. We claim that if for some $s' \in [s]$ there is $c \in S_c$ such that $as'c = bs'c$, then there exists $d \in S_c$ such that $asd = bsd$. This will say that the property of absorbing the difference between a and b is invariant under $\sim$, and hence a property of $[s]$. To prove the claim, let $sS \cap s'S = seS, se = s'f$. Since S_c is itself right LCM, we can further take $fS_c \cap cS_c = fgS, fg = ch$ with $g,h \in S_c$. Then we get $aseg = as'fg = as'ch = bs'ch = bs'fg = bseg$, so that $d := eg$ yields the claim. Diagramatically, we can illustrate this as follows, with the bottom full arrows arising from the first equivalence $as \sim bs$:



Remark 6.1. Let $a,b,c \in S_c$ such that $a = bc$. Then left cancellation implies $F_n^{a,b} = F_n^{c,1}$ and $A_n^{a,b} = A_n^{c,1}$. If instead we have $a = cb$, then $[s] \in F_n^{a,b}$ is equivalent to $[bs] \in F_n^{c,1}$. Hence $F_n^{a,b} = \alpha_b^{-1}(F_n^{c,1})$. Likewise, $A_n^{a,b} = \alpha_b^{-1}(A_n^{c,1})$. It follows that if all pairs in S_c are comparable with respect to one of the partial orders given by $a \geq_r b \Leftrightarrow a \in bS_c$ and $a \geq_\ell b \Leftrightarrow a \in S_c b$, e.g. if S_c is a group or $\mathbb{N}$, then it suffices to consider $|F_n^{a,1} \setminus A_n^{a,1}|$ for arbitrary $a \in S_c$ in order to determine $|F_n^{a,b} \setminus A_n^{a,b}|$ for all $a,b \in S_c$. In this case, we shall simplify the notation to $F_n^a := F_n^{a,1}$ and $A_n^a := A_n^{a,1}$.

Remark 6.2. The sequence $(|A_n^{a,b}|/n)_{n\in N(S)} \subset [0,1]$ is monotone increasing for all $a,b \in S_c$ in the sense that if $[s] \in A_m^{a,b}$ and $t \in N^{-1}(n)$, then $[st] \in A_{mn}^{a,b}$. To see this, pick

$e \in S_c$ satisfying $ase = bse$, then $tS \cap eS = tfS, tf = et'$ for some $f \in S_c, t' \in S$. This implies $astf = aset' = bset' = bstf$, as needed.

Self-similar actions illustrate the meaning of the sets $A_n^{a,b} \subset F_n^{a,b}$ very nicely. (See Section 8 for more details on self-similar actions.)

Proposition 6.3. *Suppose $S = X^* \bowtie G$ for a self-similar action (G, X). For $g \in G$, the equivalence class $[(w, 1_G)] \in S/_\sim$ for a word $w \in X^*$ with $n := |X|^{\ell(w)}$ belongs to $F_n^{(\varnothing,g)}$ if and only if $g(w) = w$. It belongs to $A_n^{(\varnothing,g)}$ if and only if $g(w) = w$ and $g|_w = 1_G$.*

Proof. We have $S_c = \{\varnothing\} \times G$, and $[(w, h)] \in S/_\sim$ is determined by w. Therefore $\alpha_{(\varnothing,g)}([(w, 1_G)]) = [(g(w), g|_w)]$ shows the first claim. For the second part, one direction is obvious. So suppose there is $h \in G$ such that $(w, g|_w h) = (\varnothing, g)(w, h) = (w, h)$. By left cancellation we get $g|_w h = h$, which amounts to $g|_w = 1_G$ since h is invertible. □

Proposition 6.4. *There is a* KMS_1*-state ψ_1 for $(C^*(S), \sigma)$ determined by*

$$\psi_1(v_a v_b^*) = \lim_{n \in N(S)} \frac{|A_n^{a,b}|}{n} \quad \textit{for } a, b \in S_c.$$

If S is core regular, then ψ_1 is the unique KMS_1*-state.*

Proof. According to Proposition 2.4, τ_0 given by $\tau_0(w_a w_b^*) = \delta_{a,b}$ defines a normalised trace on $C^*(S_c)$. By Proposition 5.3, this yields a KMS_1-state $\psi_1 := \psi_{1,\tau_0}$. For $a, b \in S_c$ and $n \in N(S)$, we compute

$$\psi_{\tau_0,n}(v_a v_b^*) = \psi_{\tau_0,n}(v_b^* v_a) = n^{-1} \sum_{[r] \in N^{-1}(n)/_\sim} \tau_0 \circ E(v_{br}^* v_{ar}).$$

Since $\tau_0 \circ E(v_{br}^* v_{ar})$ vanishes unless $arc = brc$ for some $c \in S_c$, we arrive at

$$\psi_{\tau_0,n}(v_a v_b^*) = n^{-1} |A_n^{a,b}|, \tag{6.2}$$

which immediately implies the claim for ψ_1. Now suppose ϕ is any KMS_1-state on $C^*(S)$. Let $a, b \in S_c$ and $n \in N(S)$. According to (6.1),

$$\begin{aligned} \phi(v_a v_b^*) &= \sum_{[s] \in N^{-1}(n)/_\sim} n^{-1} \delta_{[as],[bs]} \phi(v_{bs}^* v_{as}) = \sum_{[s] \in F_n^{a,b}} n^{-1} \phi(v_{bs}^* v_{as}) \\ &= \sum_{[s] \in A_n^{a,b}} n^{-1} \phi(v_{bs}^* v_{as}) + \sum_{[s] \in F_n^{a,b} \setminus A_n^{a,b}} n^{-1} \phi(v_{bs}^* v_{as}). \end{aligned}$$

By Proposition 4.1, $\phi(v_{bs}^* v_{as})$ has form $\phi(v_g v_f^*)$ for some $f, g \in S_c$. Each summand corresponding to $[s] \in A_n^{a,b}$ will satisfy $g = f$, and since w_g is unitary in $C^*(S_c)$ and $\phi \circ \varphi$ is a trace on $C^*(S_c)$ we have $\phi(v_g v_f^*) = 1$. For the summands where $[s] \in F_n^{a,b} \setminus A_n^{a,b}$, we estimate that $|\phi(v_g v_f^*)| \leq 1$. Thus the assumption on S implies $\phi = \psi_1$, as needed. □

Proposition 6.4 improves [ABLS17, Proposition 9.2] significantly. If S satisfies the hypotheses of [ABLS17, Proposition 9.2], namely faithfulness of $\alpha \colon S_c \curvearrowright S/_\sim$ and finite propagation, then [ABLS17, Lemma 9.5] establishes that S is core regular. In Section 9 we present an example that has a unique KMS_1-state (with $\beta_c = 1$) and for which α is not faithful.

The value β_c, when finite, can be interpreted as a critical inverse temperature above which KMS_β-states are of Gibbs type in the sense of [ALN]. The value $\beta_1 = 1$, on the other hand, has the relevance of an inverse temperature below which no KMS_β-states can exist. The question of deciding in general when these values β_1 and β_c coincide is a very interesting one and far from fully answered, see though [ALN].

There are classes of examples where $\beta_1 < \beta_c$, and for this situation it is interesting to decide if there is uniqueness of KMS$_\beta$-states for $\beta \in (\beta_1, \beta_c]$. Our second criterion, that of summably core regular monoid from Definition 3.2, is tailored to the case of $\beta \in (1, \beta_c]$, which occurs for instance for the affine semigroup over the natural numbers from [LR10]. Up to now, it was only almost freeness of α that was used to detect uniqueness in this situation, compare [ABLS17, Proposition 9.1].

Lemma 6.5. *Let $\beta \in (1, \beta_c]$ and $a, b \in S_c$. The sequence $\{\zeta_I(\beta)^{-1} \sum_{n\in\langle I\rangle^+} n^{-\beta}|A_n^{a,b}|\}$ ranging over $I \Subset Irr(N(S))$ is monotone increasing, and hence converges to a limit*

$$\kappa_{\beta,a,b} := \lim_{I\Subset Irr(N(S))} \zeta_I(\beta)^{-1} \sum_{n\in\langle I\rangle^+} n^{-\beta}|A_n^{a,b}| \in [0, 1].$$

Proof. Suppose $I, J \Subset \mathrm{Irr}(N(S))$ are disjoint, then $\zeta_{I\cup J}(\beta) = \zeta_I(\beta)\zeta_J(\beta)$ by the definition of the restricted ζ-function in (2.1). Since $|A_{mn}^{a,b}| \geq |A_m^{a,b}|n$ for all $m, n \in N(S)$ due to Remark 6.2, we estimate that

$$\begin{aligned} \zeta_{I\cup J}(\beta)^{-1} \sum_{k\in\langle I\cup J\rangle^+} k^{-\beta}|A_k^{a,b}| &= \zeta_I(\beta)^{-1} \sum_{m\in\langle I\rangle^+} m^{-\beta}\zeta_J(\beta)^{-1} \sum_{n\in\langle J\rangle^+} n^{-\beta}|A_{mn}^{a,b}| \\ &\geq \zeta_I(\beta)^{-1} \sum_{m\in\langle I\rangle^+} m^{-\beta}|A_m^{a,b}|\zeta_J(\beta)^{-1} \sum_{n\in\langle J\rangle^+} n^{-\beta+1} \\ &= \zeta_I(\beta)^{-1} \sum_{m\in\langle I\rangle^+} m^{-\beta}|A_m^{a,b}|. \end{aligned}$$

Thus the sequence is monotone increasing as claimed, and since all its terms lie in $[0, 1]$ the lemma follows. □

Proposition 6.6. *There is a* KMS$_\beta$*-state ψ_β for $(C^*(S), \sigma)$ for each $\beta \in (1, \beta_c]$ determined by*

$$\psi_\beta(v_a v_b^*) = \lim_{I\Subset Irr(N(S))} \zeta_I(\beta)^{-1} \sum_{n\in\langle I\rangle^+} n^{-\beta}|A_n^{a,b}| \quad \text{for } a, b \in S_c.$$

If S is β-summably core regular, then ψ_β is the unique the KMS$_\beta$-state.

Proof. Let $\beta \in (1, \beta_c]$. As in the proof of Proposition 6.4, the normalised trace τ_0 given by $\tau_0(v_a v_b^*) = \delta_{a,b}$ gives rise to a KMS$_\beta$-state $\psi_\beta := \psi_{\beta,\tau_0}$ using Proposition 5.4. By construction, ψ_β is the weak* limit of the $\psi_{\beta,\tau,I} = \zeta_I(\beta)^{-1} \sum_{n\in\langle I\rangle^+} n^{1-\beta}\psi_{\tau,n}$ with $I \Subset \mathrm{Irr}(N(S))$. Therefore, (6.2) shows that ψ_β has the claimed form.

Now let ϕ be any KMS$_\beta$-state and $a, b \in S_c$ be fixed but arbitrary. The KMS$_\beta$-condition gives $\phi(v_a v_b^*) = \phi(v_b^* v_a)$, and we note that for all $s \in S$, the expression $v_{bs}^* v_{as}$ vanishes unless $[s] \in F_{N_s}^{a,b}$. Hence, by Lemma 4.2 (iv), we have

$$\begin{aligned} \phi(v_a v_b^*) &= \lim_{I\Subset \mathrm{Irr}(N(S))} \zeta_I(\beta)^{-1} \sum_{[s]\in S_I/\sim} N_s^{-\beta}\phi_I(v_{bs}^* v_{as}) \\ &= \lim_{I\Subset \mathrm{Irr}(N(S))} \zeta_I(\beta)^{-1} \sum_{n\in\langle I\rangle^+} \sum_{[s]\in F_n^{a,b}} n^{-\beta}\phi_I(v_{bs}^* v_{as}). \end{aligned}$$

Now we split the inner sum over $[s] \in F_n^{a,b}\setminus A_n^{a,b}$ and $[s] \in A_n^{a,b}$ and use that $\phi_I(v_{bs}^* v_{as}) = 1$ whenever $[s] \in A_n^{a,b}$ as $\phi_I \circ \varphi$ is tracial, which can be seen similar to the proof of

Proposition 6.4. Since S is β-summably core regular, we obtain

$$\begin{aligned}|\phi(v_a v_b^*) - \kappa_{\beta,a,b}| &= \lim_{I \Subset \mathrm{Irr}(N(S))} \zeta_I(\beta)^{-1} \sum_{n \in \langle I \rangle^+} \sum_{[s] \in F_n^{a,b} \setminus A_n^{a,b}} n^{-\beta} \underbrace{|\phi_I(v_{bs}^* v_{as})|}_{\leq 1} \\ &\leq \lim_{I \Subset \mathrm{Irr}(N(S))} \zeta_I(\beta)^{-1} \sum_{n \in \langle I \rangle^+} n^{-\beta} |F_n^{a,b} \setminus A_n^{a,b}| = 0.\end{aligned}$$

Since Proposition 4.1 says that ϕ is determined by its values on monomials $v_a v_b^*$, we conclude that there is a unique such state $\phi = \psi_\beta$. □

Proposition 6.7. *Suppose S is a right LCM monoid with generalised scale. If α is almost free, then S is core regular and summably core regular.*

Proof. The set $S/_\sim$ is infinite because the generalised scale is nontrivial by definition. If α is almost free, then, for given $a, b \in S_c, a \neq b$, the set $F_n^{a,b}$ is empty for almost all $n \in N(S)$. This implies that S is core regular. Next, suppose we have $\beta \in (1, \beta_c]$ and $a, b \in S_c$. Since $0 < n^{-\beta} \leq 1$, the series $\sum_{n \in \langle I \rangle^+} n^{-\beta} |F_n^{a,b} \setminus A_n^{a,b}|$ is bounded by the total number of fixed points for $\alpha_a^{-1}\alpha_b$, that is, $\sum_{n \in N(S)} |F_n^{a,b}| < \infty$. But $\zeta_I(\beta) \nearrow \infty$ as $I \nearrow \mathrm{Irr}(N(S))$ because $\beta \in (1, \beta_c]$, so S is β-summably (a,b)-regular. Since β, a and b were arbitrary, we get the claim. □

In Section 10 we will discuss a class of examples with $\beta_c = 2$ that satisfy both types of core regularity, but for which α is not almost free.

Proposition 6.8. *Suppose S is right cancellative. Then $A_n^{a,b} = \emptyset$ for all $a \neq b$ and $n \in N(S)$. If S is core regular, then α is faithful.*

Proof. The first claim follows from the definition. For the second part, suppose there are $a, b \in S_c, a \neq b$ with $\alpha_a = \alpha_b$. Then $F_n^{a,b} = N^{-1}(n)/_\sim$ for all $n \in N(S)$, thus $|F_n^{a,b}| = n$ and S fails to be (a,b)-regular. By Definition 3.1, S is not core regular. □

Finally, we establish a sufficient condition for S to be summably core regular. We start with a lemma, whose third part is a generalisation of [ABLS17, Equation (9.1)].

Lemma 6.9. *Let $a, b \in S_c$ and $m, n \in N(S)$. Then:*

(i) $F_{mn}^{a,b} = \{[st] \mid [s] \in F_m^{a,b}, [t] \in \alpha_c(F_n^{c,d})$ *for* $c, d \in S_c, asS \cap bsS = ascS, asc = bsd\}$.

(ii) $[s] \in A_m^{a,b}$ *implies* $[st] \in A_{mn}^{a,b}$ *for all* $t \in N^{-1}(n)$.

(iii) $F_{mn}^{a,b} \setminus A_{mn}^{a,b} = \{[st] \mid [s] \in F_m^{a,b} \setminus A_m^{a,b}$ *and* $[t] \in \alpha_c(F_n^{c,d} \setminus A_n^{c,d})\}$.

Proof. For (i), we first note that for all $m, n \in N(S)$, every element $[r] \in N^{-1}(mn)/_\sim$ is of the form $[st]$ for some $s \in N^{-1}(m), t \in N^{-1}(n)$. This follows from [ABLS17, Definition 3.1(A3)(b) and Proposition 3.6(iv)]. Let $s, t \in S$ be chosen accordingly for a fixed, but arbitrary $[r]$ as above with $[r] = [st] \in F_{mn}^{a,b}$, that is, there are $e, f \in S_c$ with $astS \cap bstS = asteS, aste = bstf$. Thus $as \not\perp bs$, and since $N_{as} = N_s = N_{bs}$ we obtain by [ABLS17, Proposition 3.6(ii)] that $as \sim bs$. That is, $[s] \in F_m^{a,b}$. Thus there are $c, d \in S_c$ with $asS \cap bsS = ascS, asc = bsd$, for which we can find $t', t'' \in S$ with

$$asct' = aste = bstf = bsdt''$$

because $aste = bstf \in asS \cap bsS$. By $asc = bsd$ and left cancellation, we get $t'' = t'$. Using left cancellation for as and bs, we further deduce that $\alpha_c([t']) = [t] = \alpha_d([t'])$, that is, $[t] \in \alpha_c(F_n^{c,d})$. This shows "$\subset$".

To prove the reverse inclusion, let $[s] \in F_m^{a,b}$ and $[t] \in \alpha_c(F_n^{c,d})$ for $c,d \in S_c$ as prescribed in (i). Thus for $t' \in S$ with $[t'] = \alpha_c^{-1}([t])$, we get $ast \sim asct' = bsdt' \sim bst$, that is, $[st] \in F_{mn}^{a,b}$.

For (ii), suppose there exists $e \in S_c$ such that $ase = bse$. Given $t \in N^{-1}(n)$, there are $t' \in S, f \in S_c$ with $eS \cap tS = et'S, et' = tf$. This yields $astf = aset' = bset' = bstf$, that is, $[st] \in A_{mn}^{a,b}$.

For part (iii), we claim that for $[st] \in F_{mn}^{a,b}$, $[st] \in A_{mn}^{a,b}$ is equivalent to $[t] \in \alpha_c(A_n^{c,d})$: If there is $e \in S_c$ such that $aste = bste$, then $aste = asct' = bsdt'$ for some $t' \in S$ with $ct' \sim t \sim dt'$ (by left cancellation), that is, $[t] \in \alpha_c(F_n^{c,d})$. Conversely, if there is $t' \in S$ with $ct' \sim t$ such that $ct' = dt'$, then $ct' = te$ for some $e \in S_c$ and $aste = asct' = bsdt' = bste$. Together with (i) and (ii), this yields (iii). □

Remark 6.10. The set $\alpha_c(F_n^{c,d})$ appearing in Lemma 6.9 (i) may depend on the choice of the representative $s \in [s]$, but the resulting sets are related via α: Let $a,b \in S_c$ and $m,n \in N(S)$, $[s] \in F_m^{a,b}$ and $s' \sim s$, say $s'S \cap sS = seS, se = s'f$. If we denote by $c,d,c',d' \in S_c$ elements that satisfy $asS \cap bsS = ascS, asc = bsd$ and $as'S \cap bs'S = as'c'S, asc' = bsd'$, then

$$\alpha_{c'}(F_n^{c',d'}) = \alpha_f \alpha_e^{-1} \alpha_c(F_n^{c,d}).$$

Indeed, for every $[t] \in F_n^{c,d}$ there is $t' \in S$ with $sctS \cap s'S = s(ctS \cap eS) = set'S$ and $et' \sim ct \sim dt$, which leads to

$$as'ft' = aset' \sim asct = bsdt \sim bsct \sim bset' = bs'ft'.$$

This yields $\alpha_f \alpha_e^{-1} \alpha_c(F_n^{c,d}) \subset \alpha_{c'}(F_n^{c',d'})$. Using the symmetry of the argument in s and s', and finiteness of the involved sets, we conclude that $\alpha_f \alpha_e^{-1} \alpha_c(F_n^{c,d}) = \alpha_{c'}(F_n^{c',d'})$.

Corollary 6.11. *Let S be a right LCM monoid S with generalised scale N and $a,b \in S_c$. If there exist a finite subset $E \Subset Irr(N(S))$ and a constant $C > 0$ such that $|F_n^{a,b} \setminus A_n^{a,b}| \leq C$ for all $n \in \langle Irr(N(S)) \setminus E\rangle^+$, then S is β-summably (a,b)-regular for every $\beta \in (\beta_c - 1, \beta_c]$.*

Proof. Fix $\beta \in (\beta_c - 1, \beta_c]$ and $I \Subset \mathrm{Irr}(N(S))$. Decomposing each $n \in \langle I\rangle^+$ as $n = n_1 n_2$ with $n_1 \in \langle I \setminus E\rangle^+$ and $n_2 \in \langle I \cap E\rangle^+$, we use Lemma 6.9(iii) to estimate that

$$\begin{aligned} \sum_{n \in \langle I\rangle^+} n^{-\beta} |F_n^{a,b} \setminus A_n^{a,b}| &= \sum_{n \in \langle I\rangle^+} n_1^{-\beta} \sum_{[s] \in F_{n_1}^{a,b} \setminus A_{n_1}^{a,b}} n_2^{-\beta} |F_{n_2}^{c(s),d(s)} \setminus \alpha_{c(s)}(A_{n_2}^{c(s),d(s)})| \\ &\leq C\zeta_{I\setminus E}(\beta+1)\zeta_{I\cap E}(\beta) \leq C\zeta_{\mathrm{Irr}(N(S))\setminus E}(\beta+1)\zeta_E(\beta) < \infty. \end{aligned}$$

On the other hand, we have $\zeta_I(\beta) \nearrow \infty$ as $I \nearrow \mathrm{Irr}(N(S))$, and so (3.4) is satisfied, as required for β-summably (a,b)-regular. □

7. Positivity breaking in Baumslag–Solitar monoids

For nonzero integers c and d, the Baumslag–Solitar group is the universal group $BS(c,d) := \langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{a}\mathfrak{b}^c = \mathfrak{b}^d\mathfrak{a}\rangle$ introduced in the 1960's in the influential note [BS62]. Here we are interested in the submonoid of $BS(c,d)$ generated by $\mathfrak{a}$ and $\mathfrak{b}$, the *Baumslag–Solitar monoid* $BS(c,d)^+$. Since these are finitely generated one-relator monoids, Adjan's criterion for embeddability of universal semigroups defined by generators and relations into the corresponding group applies, see [Adj66, Section II, Theorem 3]. Thus

$BS(c,d)^+$ is the universal monoid $\langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{a}\mathfrak{b}^c = \mathfrak{b}^d\mathfrak{a}\rangle^+$ for $cd > 0$, and $\langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{b}^{|d|}\mathfrak{a}\mathfrak{b}^{|c|} = \mathfrak{a}\rangle^+$ for $cd < 0$.

We record some facts about $BS(c,d)^+$ from [Spi12]. The *height* is the homomorphism $\theta\colon BS(c,d) \to \mathbb{Z}$ determined by $\mathfrak{a} \mapsto 1, \mathfrak{b} \mapsto 0$. By [Spi12, Proposition 2.3 (L)], each $s \in BS(c,d)^+$ admits a unique normal form

$$s = \mathfrak{b}^{i_1}\mathfrak{a}\mathfrak{b}^{i_2}\mathfrak{a}\ldots\mathfrak{a}\mathfrak{b}^{i_{\theta(s)}}\mathfrak{a}\mathfrak{b}^m, \quad \text{where } 0 \leq i_1, \ldots, i_{\theta(s)} < |d| \tag{7.1}$$

and $m \in \mathbb{N}$ in case $cd > 0$, while in the case $cd < 0$ and $\theta(s) > 0$ we have $m \in \mathbb{Z}$. Thus in particular, if $cd > 0$, then an element s of $BS(c,d)^+$ with trivial height must have normal form b^m with $m \in \mathbb{N}$. This normal form allows for efficient computation of right common multiples, see [Spi12, Proposition 2.10]: two elements $s, t \in BS(c,d)^+$ with normal forms $s = \mathfrak{b}^{i_1}\mathfrak{a}\mathfrak{b}^{i_2}\mathfrak{a}\ldots\mathfrak{a}\mathfrak{b}^{i_{\theta(s)}}\mathfrak{a}\mathfrak{b}^m$ and $t = \mathfrak{b}^{j_1}\mathfrak{a}\mathfrak{b}^{j_2}\mathfrak{a}\ldots\mathfrak{a}\mathfrak{b}^{j_{\theta(t)}}\mathfrak{a}\mathfrak{b}^n$ satisfy $s \Cap t$ if and only if $i_k = j_k$ for all $1 \leq k \leq \min(\theta(s), \theta(t))$. If this is the case, then there is a unique right LCM for s and t. In particular, it follows that $BS(c,d)^+$ is a right LCM semigroup for all $c, d \in \mathbb{Z}^\times$.

Spielberg proceeds by showing that the pair $(BS(c,d), BS(c,d)^+)$ is quasi lattice-ordered if and only if $cd > 0$, see [Spi12, Theorem 2.11]. The quasi lattice-order was the essential prerequisite for Clark–an Huef–Raeburn [CaHR16] to study the KMS-states for $C^*(BS(c,d)^+)$ with $cd > 0$ via the classical Nica-Toeplitz algebra construction.

Our aim here is to complete the classification of KMS-states for the semigroup C^*-algebras of Baumslag–Solitar monoids, by not only incorporating the case that $cd < 0$, but also fully describing the simplex of KMS$_\beta$-states at the critical value $\beta = 1$. We summarise first some features we shall need in the sequel.

Proposition 7.1. *Let $c, d \in \mathbb{Z}^\times$.*

(i) The core $(BS(c,d)^+)_c$ is canonically identified with the monoid $\langle \mathfrak{b}\rangle^+ \cong \mathbb{N}$ if $|d| > 1$, and coincides with $BS(c,d)^+$ if $d = \pm 1$.

(ii) There exists a generalised scale N on $BS(c,d)^+$ if and only if $|d| > 1$, in which case N is given by $s \mapsto |d|^{\theta(s)}$ for $s \in BS(c,d)^+$.

(iii) For $|d| > 1$, two elements $s, t \in BS(c,d)^+$ satisfy $s \sim t$ if and only if $\theta(s) = \theta(t) =: k \geq 1$ and, if $k > 1$, the respective normal forms $s = \mathfrak{b}^{i_1}\mathfrak{a}\mathfrak{b}^{i_2}\mathfrak{a}\ldots\mathfrak{a}\mathfrak{b}^{i_k}\mathfrak{a}\mathfrak{b}^m$ and $t = \mathfrak{b}^{j_1}\mathfrak{a}\mathfrak{b}^{j_2}\mathfrak{a}\ldots\mathfrak{a}\mathfrak{b}^{j_k}\mathfrak{a}\mathfrak{b}^n$ for $m, n \in \mathbb{Z}$ satisfy $i_\ell = j_\ell$ for all $1 \leq \ell \leq k$.

Proof. Part (i) follows from (7.1), as does (ii) when (7.1) is combined with [Sta19, Theorem 3.11]. Part (iii) follows from the proof of [Spi12, Proposition 2.10], which computes the right LCM of s and t when $s \Cap t$. □

We recall that a core irreducible element in a right LCM monoid S is an element s such that any decomposition $s = ta$ with $t \in S$ and $a \in S_c$ is only possible if a is an invertible element in S, see [ABLS17]. Further, S is core factorable if every element s admits an expression as the product of an element in S_c and one that is core irreducible.

Proposition 7.2. *For $c, d \in \mathbb{Z}^\times$ with $|d| > 1$, $BS(c,d)^+$ is core factorable if and only if $cd > 0$. For $cd < 0$, the identity is the only core irreducible element. The core irreducibles are always $\cap$-closed in $BS(c,d)^+$.*

Proof. The forward direction of the first claim is established in [ABLS17, Proposition 5.10], which ought to assume $c \geq 1, d > 1$ in place of $c, d \geq 1, cd > 1$. Now suppose $cd < 0$. Take any $s \in BS(c,d)^+$ with $\theta(s) > 0$ and let $n \in \mathbb{N}$ be arbitrary, then the element $t = s\mathfrak{b}^{-n}$ is in $BS(c,d)^+$ and satisfies $t \neq s = t\mathfrak{b}^n$. Hence the identity is the only core irreducible element in $BS(c,d)^+$. Note for completeness that when $cd > 0$, the core irreducible elements are precisely the *stems*, $t\mathfrak{b}^n$ with $n \in \mathbb{N}$, from [CaHR16]. In particular, $BS(c,d)^+$ is not core factorable when $cd < 0$ because the core is a proper submonoid. The last claim follows again from [ABLS17, Proposition 5.10] for $cd > 0$, and is trivial for $cd < 0$. □

Remark 7.3. In view of Proposition 7.2, the Baumslag–Solitar monoids $BS(c,d)^+$ with $cd < 0$ provide the first examples of right LCM monoids that are not core factorable. Hence the Zappa-Szép product $S_{ci} \bowtie S_c$ is a proper submonoid of S. The phenomenon behind this remarkable behaviour is a form of *positivity breaking*: In the simplest case of $(1,-2)$, the monoid $BS(c,d)^+$ can be thought of as a variant of $BS(1,2) \cong \mathbb{N} \rtimes_2 \mathbb{N}$ which does not act by an endomorphism of the positive cone $\mathbb{N} \subset \mathbb{Z}$, but a composition of such an endomorphism with the flip on $\mathbb{Z}$ restricted to the domain $\mathbb{N}$. In abstract terms, we combine an injective endomorphism of a positive cone P inside a group G with an automorphism of G that does not preserve P. This recipe opens the gates to a wide range of examples of (right LCM) semigroups with positivity breaking.

Remark 7.4. During the workshop in Newcastle mentioned earlier, it was observed by Xin Li that $BS(c,d)^+$ with $cd < 0$ are also the first examples of group embeddable monoids that do not admit a Toeplitz group embedding. The latter condition allows to regard semigroup crossed products by S as full corners in group crossed products for the respective group, see [CELY17, Proposition 5.8.5]. The failure of the Toeplitz condition for the embedding of $BS(c,d)^+$ in the respective group is due to the positivity breaking for $cd < 0$, which means that $(BS(c,d), BS(c,d)^+)$ is not quasi-lattice ordered.

Proposition 7.5. *For all nonzero integers c and d with $|d| > 1$, β_c equals 1, and the equilibrium states on $(C^*(BS(c,d)^+), \sigma)$ are as follows:*

(i) The ground states are parameterised by states on $C^*(\mathbb{N})$.
(ii) The KMS$_\beta$-states for $\beta \in (1,\infty]$ are parameterised by normalised traces on $C^*(\mathbb{Z})$.

Proof. As $|d| > 1$, $BS(c,d)^+$ admits a generalised scale given by $N_s = |d|^{\theta(s)}$ for $s \in BS(c,d)^+$. Note that in this case $\beta_c = 1$. Parts (i) and (ii) are due to Proposition 7.1(i) combined with Proposition 5.1 and Proposition 5.4, respectively. □

By [CaHR16, Proposition 7.1, Examples 7.3 and 7.4], if $c \geq 1$ and $d \geq 2$, the KMS$_1$-state is unique precisely when $c \notin d\mathbb{Z}$. In the case $c \in d\mathbb{Z}$, the authors of [CaHR16] constructed two distinct KMS$_1$-states. Furthermore, they observed that when $c = d$ there is an isomorphism $\mathcal{Q}(BS(c,d)^+) \cong \mathcal{O}_d \otimes C^*(\mathbb{Z})$ carrying the generating isometries $\pi(v_{\mathfrak{b}^j\mathfrak{a}})$ of the boundary quotient to $s_j \otimes 1$, where $s_1, \ldots, s_d$ are the universal generating isometries of $\mathcal{O}_d$, and taking $\pi(v_{\mathfrak{b}^d})$ to the unitary $1 \otimes u$, where u is the generating unitary of $C^*(Z)$.

We will now characterise uniqueness of the KMS$_1$-state in the case of arbitrary $c, d \in \mathbb{Z}^\times$ with $|d| > 1$ and also determine the simplex of KMS$_1$-states for $c \in d\mathbb{Z}$. For this purpose, let us denote by τ_0 the normalised trace on $C^*(\langle \mathfrak{b} \rangle^+) \cong C^*(\mathbb{N})$ determined by $\tau_0(v_{\mathfrak{b}^k}) = \delta_{0,k}$ for $k \in \mathbb{N}$. By right cancellation in $BS(c,d)^+$, we have $\psi_{\beta,\tau_0} \circ \varphi = \tau_0$ for all $\beta > 1$. Thus, in the notation of Theorem 3.6, we have $\tau_0 \in T_{>1}(C^*(\langle \mathfrak{b} \rangle^+))$.

Proposition 7.6. *Let $c, d \in \mathbb{Z}^\times, |d| > 1$.*

(i) There is a KMS$_1$-state ψ_1 determined by $\psi_1 \circ \varphi = \tau_0$.
(ii) For $c = md$ with $m \in \mathbb{Z}^\times$, the simplex of KMS$_1$-states is parameterised by normalised traces τ on $C^*(\mathbb{Z})$ such that $\tau(u^k) = \tau(u^{km^n})$ for all $k \in \mathbb{N}$ and $n \geq 1$.
(iii) The state ψ_1 is the unique KMS$_1$-state if and only if $c \notin d\mathbb{Z}$.

Proof. Part (i) is an application of Theorem 3.6 as $\tau_0 \in T_{>1}(C^*(\langle \mathfrak{b} \rangle^+))$. For (ii) we will also apply Theorem 3.6. We will therefore determine the constraints for normalised traces τ on $C^*(\langle \mathfrak{b} \rangle^+) \cong C^*(\mathbb{N})$ arising from $\tau \in T_{>1}(C^*(\mathbb{N}))$. As traces on $C^*(\mathbb{N})$ correspond to traces on $C^*(\mathbb{Z})$, the result is phrased in terms of the latter. So let $c = md$ and $\beta > 1$, and take $s \in BS(c,d)^+$ with $\theta(s) \geq 1$, that is, $s \notin S_c$. Given $k \in \mathbb{N}$, Proposition 7.1 (iii) implies that $\mathfrak{b}^k s \sim s$ if $k \in d\mathbb{N}$, and $\mathfrak{b}^k s \perp s$ otherwise. In the first case, the assumption $c = md$ implies that $\mathfrak{b}^k s = s\mathfrak{b}^{km^{\theta(s)}}$. Thus, if ϕ is a KMS$_1$-state, then by Remark 4.3 we obtain

$$\phi(v_{\mathfrak{b}^k}) = n^{-1} \sum_{[s] \in N^{-1}(n)/\sim} \phi(v_s^* v_{\mathfrak{b}^k s}) = \chi_{d\mathbb{N}}(k)\ \phi(v_{\mathfrak{b}^{km^\ell}}) \quad \text{for all } n = |d|^\ell \in N(S),$$

where $l = \theta(s)$ varies in $\mathbb{N}$ as s varies, and we have used that $v_s^* v_{\mathfrak{b}^k s}$ vanishes whenever $\mathfrak{b}^k s \perp s$. Upon identifying the simplex of normalised traces of $C^*(d\mathbb{N}) \cong C^*(\mathbb{N})$ with the one of $C^*(\mathbb{Z})$, this is precisely the claimed condition in (ii). To prove that the condition parametrises KMS$_1$-states we need to show that every normalised trace τ on $C^*(v_{\mathfrak{b}^d}) \subset C^*(BS(c,d)^+)$ with $\tau(v_{\mathfrak{b}^k}) = \tau(v_{\mathfrak{b}^{km^n}})$ for all $n \geq 1$ satisfies $\psi_{\beta,\tau} \circ \varphi = \tau$. To do so, we apply Lemma 4.9 (i) to get

$$\chi_{\tau,s}(v_{\mathfrak{b}^k}) = \tau \circ E(v_s^* v_{\mathfrak{b}^k s}) = \chi_{d\mathbb{N}}(k)\tau(v_{\mathfrak{b}^{km^{\theta(s)}}}).$$

Hence $\chi_{\tau,s}(v_{\mathfrak{b}^k}) = 0$ unless $k \in d\mathbb{N}$, and its value depends not on $[s]$, but only on $\theta(s)$. Therefore, with the notation of Lemma 5.2, we have $\psi_{\tau,n} = \chi_{\tau,s}$ for every $s \in N^{-1}(n)$. With this observation we compute

$$\psi_{\beta,\tau}(v_{\mathfrak{b}^k}) = \chi_{d\mathbb{N}}(k)\zeta(\beta)^{-1} \sum_{\ell \geq 0} (|d|^\ell)^{1-\beta} \tau(v_{\mathfrak{b}^{km^\ell}}) = \tau(v_{\mathfrak{b}^k})$$

for all $k \in \mathbb{N}$. In view of Theorem 3.6, this proves (ii).

For (iii), we note that α is almost free if $c \notin d\mathbb{Z}$, where the argument from [ABLS17, Proposition 5.10(i)] applies verbatim. Thus S is core regular by Proposition 6.7, and therefore uniqueness of the KMS$_1$-state follows from Proposition 6.4. If $c = md$ for some $m \in \mathbb{Z}^\times$, then

$$\tau_m(u^k) := |m|^{-1} \sum_{j \in \mathbb{Z}/m\mathbb{Z}} e^{2\pi i k j/m}$$

defines a normalised trace on $C^*(\mathbb{Z})$ with $\tau_m(u^{\ell m}) = 1 \neq 0 = \tau_0(u^{\ell m})$ for all $\ell \in \mathbb{Z}^\times$. Upon identifying τ_m and τ_0 with the corresponding normalised traces on $C^*(\langle \mathfrak{b} \rangle^+)$, Theorem 3.6 shows that τ_m and τ_0 give rise to distinct KMS$_1$-states on $C^*(BS(c,d)^+)$. □

Remark 7.7. As opposed to the case $c = d$, where all traces τ on $C^*(\mathbb{Z})$ yield distinct KMS$_1$-states for $C^*(BS(c,d)^+)$, the case $c = -d$ has the constraint that τ needs to be real valued on $u^k, k \in \mathbb{Z}$ because Proposition 7.6 (ii) states that $\tau(u^k) = \tau(u^{-k}) = \overline{\tau(u^k)}$. Therefore, the sign of the product of c and d has an impact on the KMS-state structure.

8. Virtual group endomorphisms and self-similar group actions

In this section we consider the problem of constructing and classifying KMS-states for natural dynamics on C^*-algebras associated to self-similar actions (G, X). This has been done earlier, cf. [LRRW14] using a Toeplitz-Pimsner C^*-algebra model based on the Hilbert C^*-correspondence from [Nek09], and [ABLS17] using a monoidal Zappa–Szép product $X^* \bowtie G$ and its C^*-algebra as in [BRRW14]. Uniqueness of the KMS_1-state at critical value $\beta_c = 1$ was linked in [LRRW14] to the property of (G, X) called finite-state, and in [ABLS17], it was phrased in terms of a condition called finite propagation.

Our approach here is to use the virtual endomorphism picture of self-similar actions as in [Nek02]. Using this setup, we illustrate with an example from [BK13] that the semigroup C^*-algebra of (G, X) cannot witness the finite-state property itself, especially not through uniqueness of the KMS_1-state, see Corollary 8.8 and Example 8.9.

In fact, noting that uniqueness of the KMS_1-state is actually a property of the virtual endomorphism rather than the particular associated self-similar action, we will show that the same is true for core regularity of the associated monoid. This is accomplished in Proposition 8.3 by providing a measure theoretic characterisation of core regularity.

Let G be a discrete group. Recall from [Nek02, Definition 2.7] that a *self-similar action* (G, X) is given by a finite alphabet X (in at least two letters) together with an action of G on the free monoid X^* in X with the following property: For all $x \in X, g \in G$, there are $y \in X, h \in G$ such that

$$g(xw) = yh(w) \quad \text{for all } w \in X^*.$$

When the action of G on X^* is faithful, the elements $y \in X^*$ and $h \in G$ are uniquely determined. Then h is referred to as the restriction of g with respect to x, denoted by $g|_x$. We will assume henceforth that (G, X) is faithful. Intuitively, we think of g as a level-preserving automorphism of the rooted $|X|$-regular tree X^*, in which case $g|_x$ is the action of g on the subtree attached at x. The recursive nature of self-similar actions of groups makes them extremely versatile, see [Nek05] and the references therein.

There is a close connection between self-similar actions and virtual endomorphisms of groups, see [Nek02]. A *virtual endomorphism* of a group G is a homomorphism $\phi\colon H \to G$ where H is a finite index subgroup of G. To every virtual endomorphism (G, ϕ, H) there is an associated self-similar action (G, X) with $|X| = [G : H]$ defined as follows: let X be an alphabet with $[G : H]$ distinct letters. Choose a transversal D for the left cosets in G/H (called the *digit set*), and label the elements as $D = \{q_x \mid x \in X\}$. The self-similar action associated to (G, ϕ, D) is then given by (G, X) with relation $g(xw) = yg|_x(w)$ for $g \in G, x \in X$ and all $w \in X^*$, where $y \in X$ is the unique letter satisfying

$$gq_x \in q_yH \text{ and } g|_x := \phi(q_y^{-1}gq_x). \tag{8.1}$$

Conversely, whenever (G, X) is a faithful self-similar action, one can construct a virtual endomorphism ϕ of G whose domain is the stabilizer group of a letter $x \in X$. If (G, X) is also recurrent (self-replicating), then (G, X) can be recovered as the self-similar action associated to (G, ϕ, D) for some digit set D, see [Nek05, §2.8] for more details.

When associating a self-similar action to a virtual endomorphism (G, ϕ, H), the choice of digit set may affect key properties of the resulting self-similar action. In [BK13, Section 4] Bondarenko and Kravchenko give an example with G equal the discrete Heisenberg group where changing only one of the digits leads to self-similar actions with and without the finite-state property. Nevertheless, all self-similar actions arising from a virtual endomorphism as in equation (8.1) are topologically conjugate, that is, their actions on the respective spaces $X^{\mathbb{N}}$ of one-sided infinite words are, see [Nek02, Proposition 4.19].

Recall from [BRRW14] that to a faithful self-similar action (G, X) there is an associated right LCM monoid $X^* \bowtie G$ called the Zappa-Szep product of X^* and G. We show next that when (G, X) arises from a virtual endomorphism (G, ϕ, H) the associated C^*-algebra $C^*(X^* \bowtie G)$ does not depend on the choice of digit set.

Definition 8.1. Given a virtual endomorphism ϕ of a discrete group G with domain $H \subset G$, we let $C^*(G, \phi)$ be the universal unital C^*-algebra generated by a unitary representation u of G and an isometry s_ϕ subject to the relations

$$u_h s_\phi = s_\phi u_{\phi(h)} \text{ for all } h \in H, \text{ and } \sum_{[g]\in G/H} u_g s_\phi s_\phi^* u_g^* \leq 1. \tag{8.2}$$

Theorem 8.2. *Let ϕ be a virtual endomorphism of G with domain H, choose a digit set D and let (G, X) be the associated self-similar action. Then*

$$u_g \mapsto v_{(\varnothing,g)},\ s_\phi \mapsto v_{(\varnothing,q_x^{-1})(x,1_G)},$$

with $g \in G$ and $x \in X$ arbitrary, defines an isomorphism $C^*(G, \phi) \cong C^*(X^* \bowtie G)$.

Proof. We use the identification $C^*(X^* \bowtie G) \cong \mathcal{T}(G, X)$ from [BRRW14] and employ the notation for $\mathcal{T}(G, X)$ from [LRRW14], that is, we also write u for the unitary representation $v_{(\varnothing,\cdot)}$ of G and s_x for the isometries $v_{(x,1_G)}, x \in X$. By [LRRW14, Proposition 3.2], the relations for $\mathcal{T}(G, X)$ are

$$u_g s_x = s_{g(x)} u_{g|_x} \text{ and } \sum_{x\in X} s_x s_x^* \leq 1. \tag{8.3}$$

First we check that u and $t_\phi := u_{q_x^{-1}} s_x$ for $x \in X$ define a representation of $C^*(G, \phi)$. To begin with, we show that t_ϕ does not depend on the choice of $x \in X$. Let $z \in X$. By (8.3), to have $u_{q_x^{-1}} s_x = u_{q_z^{-1}} s_z$ is equivalent to

$$s_{(q_z q_x^{-1})(x)} u_{(q_z q_x^{-1})|_x} = s_z.$$

This equality is satisfied if $(q_z q_x^{-1})(x) = z$ and $(q_z q_x^{-1})|_x = 1_G$, and these identities hold because the recipe in (8.1) implies that z is the unique element in X such that $(q_z q_x^{-1})q_x \in q_z H$ and

$$(q_z q_x^{-1})|_x = \phi(q_z^{-1}(q_z q_x^{-1})q_x) = \phi(1_G).$$

Let z be the unique element in X such that $q_z \in H$. Note that we then have $q_z^{-1}(z) = z$ and $q_z^{-1}|_z = \phi(q_z^{-1})$. For every $h \in H$, we have $hq_x^{-1}q_x = h \in q_z H$, and hence

$$\begin{aligned} u_h t_\phi &= u_{hq_x^{-1}} s_x = s_{(hq_x^{-1})(x)} u_{(hq_x^{-1})|_x} \\ &= s_z u_{\phi(q_z^{-1})} u_{\phi(h)} \\ &= t_\phi u_{\phi(h)}. \end{aligned}$$

This establishes the first relation from (8.2), and implies $u_h t_\phi t_\phi^* u_h^* = t_\phi t_\phi^*$ for all $h \in H$. We deduce that $u_g t_\phi t_\phi^* u_g^*$ does not depend on the choice of representative for $[g] \in G/H$, and since D is a transversal for G/H, there is a bijection between $\{u_g t_\phi t_\phi^* u_g^* \mid [g] \in G/H\}$ and $\{s_x s_x^* \mid x \in X\}$. Therefore

$$\sum_{[g]\in G/H} u_g t_\phi t_\phi^* u_g^* = \sum_{x\in X} s_x s_x^* \leq 1.$$

This gives rise to a surjective $*$-homomorphism $C^*(G,\phi) \to \mathcal{T}(G,X) \cong C^*(X^* \bowtie G)$. It is not hard to see that the assignment $u_g \mapsto u_g$ and $s_x \mapsto u_{q_x} s_\phi =: t_x$ defines a $*$-homomorphism on $\mathcal{T}(G,X)$ that is inverse to the first map: the Toeplitz-Cuntz relation is clear. For $g \in G$ and $x \in X$ there is a unique $y \in X$ such that $q_y^{-1} g q_x \in H$, hence by (8.2) implies that $u_g t_x = u_{q_y}(u_{q_y^{-1} g q_x} s_\phi) = t_y u_{g|_x}$, as required in (8.3). □

After this prelude, let us proceed with the analysis of KMS-states. The time evolution on $C^*(X^* \bowtie G)$ is the one from [ABLS17, Proposition 5.8], thus for $t \in \mathbb{R}$, $g \in G$ and $x \in X$, so that in particular $\ell(x) = 1$, we have

$$\sigma_t(v_{(\varnothing,g)}) = v_{(\varnothing,g)}\, \sigma_t(v_{(\varnothing,q_x^{-1})(x,1_G)}) = |X|^{it} v_{(\varnothing,q_x^{-1})(x,1_G)}.$$

The natural time evolution on $C^*(G,\phi)$ for a given virtual endomorphism (G,ϕ,H) is given by $\sigma_t(u_g) = u_g$ and $\sigma_t(s_\phi) = [G:H]^{it} s_\phi$ for all $t \in \mathbb{R}$ and $g \in G$. The isomorphisms between $C^*(X^* \bowtie G)$, $\mathcal{T}(G,X)$ and $C^*(G,\phi)$ from Theorem 8.2 are equivariant with respect to the respective time evolutions. Via the identification $C^*(X^* \bowtie G) \cong \mathcal{T}(G,X)$, recall from [LRRW14, Theorem 7.3(3)] that for every finite-state self-similar group action (G,X), the C^*-algebra $C^*(X^* \bowtie G)$ has a unique KMS-state at the critical inverse temperature, see also [ABLS17, Proposition 5.8(iii)]. As we shall now argue, core regularity improves the results in this direction. It not only covers new, non-finite state examples, but adds a new perspective in that it connects with so-called G-regular points of (G,X).

For a self-similar group action (G,X) with G countable, consider the induced action of G on the Cantor set $X^{\mathbb{N}}$ of right-infinite words in X (equipped with the product topology). Let $g \in G$. Following the more recent terminology of [Nek18, Definition 2.1] rather than [Nek09, Definition 3.3], a point $w \in X^{\mathbb{N}}$ is *g-regular* if either w is not fixed by g or it lies in the interior of the set of fixed points for g. The second condition means that there is a finite word $v \in X^*$ with $w \in vX^{\mathbb{N}}$ such that $v \in A^g_{|X|^{\ell(v)}}$. A point that is not g-regular is *g-singular*. Finally, a point is called *G-regular* if it is g-regular for all $g \in G$.

Let μ denote the Borel probability measure on $X^{\mathbb{N}}$ given by the product measure of uniform distribution on X at each stage, that is, $\mu[vX^{\mathbb{N}}] = |X|^{-\ell(v)}$ for every $v \in X^*$.

Proposition 8.3. *For a faithful self-similar action (G,X) with countable group G, the following are equivalent:*

(i) $X^* \bowtie G$ is core regular.
(ii) μ-almost every point in $X^{\mathbb{N}}$ is G-regular.

Proof. Let us fix $g \in G$ and note that $(S_n^g)_{n\geq 1}$ with $S_n^g := \bigsqcup_{v\in F^g_{|X|^n}\setminus A^g_{|X|^n}} vX^{\mathbb{N}}$ forms a decreasing sequence of Borel subsets, see Lemma 6.9. We claim that the set of g-singular points coincides with $\bigcap_{n\geq 1} S_n^g = \lim_{n\to\infty} S_n^g$.

Suppose that ω is g-singular and $v \in X^n$ is the beginning of ω of length n. Then $v \in F^g_{|X|^n}$ as $g(\omega) = \omega$, but $v \notin A^g_{|X|^n}$ as there exists $\omega' \in vX^{\mathbb{N}}$ with $g(\omega') \neq \omega'$. Thus every g-singular point lies in $\lim_{n\to\infty} S^g_n$. Conversely, let $\omega \in \lim_{n\to\infty} S^g_n$. Then every beginning of ω is fixed, and thus $g(\omega) \in vX^{\mathbb{N}}$ for every beginning v of ω. As the cylinder sets in $vX^{\mathbb{N}}$ separate ω from every other point ω' ($X^{\mathbb{N}}$ is Hausdorff with the specified topology), we conclude that $g(\omega) = \omega$. Moreover, if v is a beginning of ω of length n, then $v \in F^g_{|X|^n} \setminus A^g_{|X|^n}$, that is, $g(v) = v$ but $g_{|v} \neq 1_G$. By faithfulness of (G, X), there is $v' \in X^*$ such that $g(vv') \neq vv'$. Thus every $\omega' \in vv'X^{\mathbb{N}}$ satisfies $g(\omega') \neq \omega'$, and hence ω is g-singular.

With the above claim, we easily calculate

$$\mu[\{g\text{-singular}\}] = \lim_{n\to\infty} \mu[S^g_n] = \lim_{n\to\infty} \frac{|F^g_{|X|^n} \setminus A^g_{|X|^n}|}{|X|^n}$$

for every $g \in G$. Taking into account that $\{G\text{-regular}\} = \bigcap_{g\in G}\{g\text{-regular}\}$ and that G is countable, we therefore see that core regularity for $X^* \bowtie G$ is equivalent to G-regularity of μ-almost all points in $X^{\mathbb{N}}$. ☐

Remark 8.4. Every right LCM monoid S with generalised scale N admits an analogue of the measure μ on the spectrum $\widehat{\mathcal{D}}$ of the diagonal subalgebra $\mathcal{D} \subset C^*(S)$ generated by the commuting projections $e_{sS}, s \in S$. Its relevance in the context of KMS-states will be discussed in [NS].

Remark 8.5. It follows from [Nek05, Proposition 4.19] that for any two self-similar actions arising from the choice of two digit sets for a virtual endomorphism (G, ϕ), the respective actions $G \curvearrowright X^{\mathbb{N}}$ are topologically conjugate. In fact, the result includes an explicit description of the equivariant homeomorphism, say h, of $X^{\mathbb{N}}$, and we highlight that $h(vX^{\mathbb{N}}) = wX^{\mathbb{N}}$ with $\ell(w) = \ell(v)$ for every finite word v. This allows us to conclude that the probability measure μ appearing in Proposition 8.3 is h-invariant. Hence core regularity is a property of (G, ϕ), that is, it does not depend on the digit set.

In view of Theorem 8.2 and Remark 8.5, it seems that the natural approach to study the KMS-state structure of $(C^*(G \bowtie X^*), \sigma)$ is by transferring the question to $(C^*(G, \phi), \sigma)$ for a virtual endomorphism ϕ that allows for a realization of (G, X) via a suitably chosen digit set. Let us first note that injectivity of the virtual endomorphism simplifies the task:

Lemma 8.6. *Let ϕ be a virtual endomorphism of a group G with domain H, D a digit set for (G, ϕ, H) and (G, X) the self-similar action associated to (G, ϕ, D). If ϕ is injective, then $X^* \bowtie G$ is right cancellative, and in particular there are no nontrivial absorbing words for all $g \neq 1_G$.*

Proof. We have that $X^* \bowtie G$ is right cancellative if and only if there do not exist $g' \in G \setminus \{1_G\}$ and $w \in X^*$ with $g'(w) = w, g'_{|w} = 1_G$, that is, $w \in A^{g'}_{|X|^{\ell(w)}}$. If such g' and w existed, then we may assume the same phenomenon occurs already for some $g \in G \setminus \{1_G\}$ and $x \in X$. But $1_G = g|_x = \phi(q_x^{-1} g q_x)$ for $g \neq 1$ would contradict injectivity of ϕ. ☐

Corollary 8.7. *Suppose ϕ is a virtual endomorphism of a group G. If there exists a digit set D for which the associated $X^* \bowtie G$ is core regular, for instance if (G, X)*

is finite-state, then $(C^*(G,\phi),\sigma)$ has a unique KMS$_1$-state ψ_1. If ϕ is injective, then $\psi_1(u_g) = \delta_{g,1_G}$ for all $g \in G$.

Proof. Core regularity is implied by the finite-state property, see [ABLS17, Proposition 5.8(iii) and Lemma 9.5], and Proposition 6.4 implies the first claim. If ϕ is injective, Lemma 8.6 shows that there are no non-trivial absorbing elements. □

This applies to a class exhibited in [BK13, Theorem 2], to which we refer for a thorough explanation of the terminology.

Corollary 8.8. *Suppose ϕ is a surjective virtual endomorphism of a finitely generated torsion-free nilpotent group G with trivial ϕ-core. Let $\widehat{\phi}$ denote the differential of ϕ at the identity inside the Lie algebra of the Mal'cev completion of G. If the spectral radius of $\widehat{\phi}$ is at most 1, and the cells for eigenvalues of modulus 1 in the Jordan normal form of $\widehat{\phi}$ all have size 1, then $(C^*(G,\phi),\sigma)$ has a unique KMS$_1$-state ψ_1.*

Proof. By [BK13, Theorem 1], there exists a digit set for (G,ϕ) whose associated self-similar action is finite-state, so that Corollary 8.7 applies. □

Example 8.9. Consider the *discrete Heisenberg group*

$$G = \left\{ (x,y,z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{Z} \right\}.$$

For $m,n \in \mathbb{Z}^\times$, we consider the subgroup $H_{m,n} := \{(x,y,z) \in G \mid x \in m\mathbb{Z}, y \in n\mathbb{Z}, z \in mn\mathbb{Z}\}$ of index $(mn)^2$ and the virtual endomorphism $\phi_{m,n} \colon H_{m,n} \to G$ given by $\operatorname{diag}(1/m, 1/n, 1/mn)$, that is, $(mx, ny, mnz) \mapsto (x,y,z)$. In this case, $\widehat{\phi}_{m,n} = \operatorname{diag}(1/m, 1/n, 1/mn)$. According to [BK13, Theorems 1 and 2], the case $(m,n) = (1,2)$ allows for both finite-state and non-finite-state self-similar actions, which is demonstrated in [BK13, 4 Example] for $X := \{1,2,3,4\}$ by choosing $D_1 := \{0, e_2, e_3, e_2+e_3\}$ and $D_2 := \{e_1, e_2, e_3, e_2+e_3\}$, respectively.

Corollary 8.8 applies, so that $(C^*(G,\phi_{m,n}),\sigma)$ has a unique KMS$_1$-state, whenever $|mn| > 1$.

It is known that the G-regular points for a faithful self-similar action of a countable group always form a dense G_δ-set. It is natural to ask the following question: is there a countable discrete group G that admits a faithful self-similar action (G,X) with the property that the G-singular points have positive measure? Any such example is likely to be of great interest on its own as it shows quite distinct behaviour with respect to the two analogous statements from topology and ergodic theory.

9. Shadowed natural numbers

We introduce now a class of right LCM monoids that are ultimatively a mock version of $\mathbb{N} \rtimes \mathbb{N}^\times$, but we restrict our scope to a single generator $\mathfrak{c} \in \mathbb{N}^\times$ to keep technicalities and notation to a minimum. We replace $\mathbb{N}$ by the *left-absorbing monoid* $L_1 := \langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{a}\mathfrak{b} = \mathfrak{b}^2 \rangle^+$. If we think of $\mathfrak{b}$ as the generator of $\mathbb{N}$, then $\mathfrak{a}$ is merely a shadow that turns into $\mathfrak{b}$ as soon as it meets one on its right hand side.

For $n \geq 1$, let $L_n := \langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{a}\mathfrak{b}^n = \mathfrak{b}^{n+1} \rangle^+$ denote the *left absorbing* monoid, see [DDG+15, Reference Structure 8] for details. This is a right LCM monoid that is not right cancellative. In fact, every two elements in L_n have a right common multiple, so L_n is equal to its core, and hence does not admit a generalised scale. Nevertheless, we can use L_1, whose elements have a unique normal form $\mathfrak{b}^i\mathfrak{a}^j$ with $i,j \in \mathbb{N}$, to build an example with an interesting feature: For $m > 1$, let

$$S_m := L_1 \rtimes_m \mathbb{N} = \langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \mid \mathfrak{a}\mathfrak{b} = \mathfrak{b}^2, \mathfrak{c}\mathfrak{a} = \mathfrak{a}^m\mathfrak{c}, \mathfrak{c}\mathfrak{b} = \mathfrak{b}^m\mathfrak{c}\rangle^+.$$

Since the appearing endomorphism of L_1 is injective[1], S_m is left cancellative, and we claim that it is right LCM. So let $s = \mathfrak{b}^{i_1}\mathfrak{a}^{j_1}\mathfrak{c}^{k_1}, t = \mathfrak{b}^{i_2}\mathfrak{a}^{j_2}\mathfrak{c}^{k_2} \in S_m$ with $k_1 \leq k_2$. Then $s \Cap t$ if and only if $i_1 + j_1 - (i_2 + j_2) \in m^{k_1}\mathbb{Z}$. Suppose first that there are $r = \mathfrak{b}^{i_3}\mathfrak{a}^{j_3}\mathfrak{c}^{k_3}$ and $r' = \mathfrak{b}^{i_4}\mathfrak{a}^{j_4}\mathfrak{c}^{k_4}$ with $sr = tr'$. This yields

$$\mathfrak{b}^{i_1+j_1+m^{k_1}(i_3+j_3+m^{k_3})}\mathfrak{c}^{k_1+k_3} = sr\mathfrak{b} = tr'\mathfrak{b} = \mathfrak{b}^{i_2+j_2+m^{k_2}(i_4+j_4+m^{k_4})}\mathfrak{c}^{k_2+k_4}, \tag{9.1}$$

and as $k_2 \geq k_1$, we deduce that $i_1 + j_1 - (i_2 + j_2) \in m^{k_1}\mathbb{Z}$. Conversely, if this condition holds, then there are $j_3, j_4 \geq 0$ such that $i_1 + j_1 + m^{k_1}j_3 = i_2 + j_2 + m^{k_2}j_4$, which yields $s\mathfrak{b}^{j_3}\mathfrak{c}^{k_2-k_1} = t\mathfrak{b}^{j_4}$. In the case of $s \Cap t$, it is clear from (9.1) that, given $r \in sS_m \cap tS_m$, we can always find $r' \in sS_m \cap tS_m$ with $r \in r'S_m$ such that the power of $\mathfrak{c}$ in r' is k_2, i.e. the maximum of k_1 and k_2. To conclude that S_m is right LCM, we now only need to figure out that this case allows for exactly one minimal choice of a word in $\mathfrak{a}$ and $\mathfrak{b}$, which we leave to the reader as an exercise.

Borrowing terminology from [Sta19, Definitions 3.2,3.4], it follows that

(i) $(S_m)_c$ coincides with L_1;
(ii) for $s = \mathfrak{b}^{i_1}\mathfrak{a}^{j_1}\mathfrak{c}^{k_1}$ and $t = \mathfrak{b}^{i_2}\mathfrak{a}^{j_2}\mathfrak{c}^{k_2} \in S_m$, $s \sim t$ if and only if $k_1 = k_2$ and $i_1 + j_1 = i_2 + j_2 (\text{mod } m^{k_1})$;
(iii) an element $s = \mathfrak{b}^i\mathfrak{a}^j\mathfrak{c}^k$ is noncore irreducible if and only if $k = 1$;
(iv) the core graph $\Gamma(S_m)$ is the empty graph on m vertices.

As S_m is noncore factorable (and has balanced factorisation for trivial reasons), [Sta19, Theorem 3.11] implies that S_m has a generalised scale N determined by $\mathfrak{a}, \mathfrak{b} \mapsto 1, \mathfrak{c} \mapsto m$. In particular, $\beta_c = 1$ for all $m > 1$.

Proposition 9.1. *For every $m > 1$, the action $\alpha\colon L_1 \curvearrowright S_m$ is not faithful, but S_m is core regular, and therefore $(C^*(S_m), \sigma)$ has a unique KMS$_1$-state.*

Proof. Due to $\mathfrak{a}\mathfrak{b} = \mathfrak{b}^2$, every $[s] \in S/_\sim$ is represented by an element of the form $\mathfrak{b}^j\mathfrak{c}^k$ with $j > 0$. Thus α is not faithful at $\mathfrak{a} \neq \mathfrak{b}$. For $\mathfrak{b}^{i_1}\mathfrak{a}^{j_1}, \mathfrak{b}^{i_2}\mathfrak{a}^{j_2} \in L_1$, observation (ii) from the list above entails that, for $n = m^k$ large enough so that $i_1 + j_1, i_2 + j_2 < m^k$, the set $F_n^{\mathfrak{b}^{i_1}\mathfrak{a}^{j_1},\mathfrak{b}^{i_2}\mathfrak{a}^{j_2}}$ can only be nonempty if $i_1 + j_1 = i_2 + j_2$. But in this case we get

$$\mathfrak{b}^{i_1}\mathfrak{a}^{j_1}s \sim \mathfrak{b}^{i_1}\mathfrak{a}^{j_1}s\mathfrak{b} = \mathfrak{b}^{i_1+j_1+m^k}s = \mathfrak{b}^{i_2}\mathfrak{a}^{j_2}s\mathfrak{b} \sim \mathfrak{b}^{i_2}\mathfrak{a}^{j_2}s$$

for every $s \in N^{-1}(n)$, so that $A_n^{\mathfrak{b}^{i_1}\mathfrak{a}^{j_1},\mathfrak{b}^{i_2}\mathfrak{a}^{j_2}} = F_n^{\mathfrak{b}^{i_1}\mathfrak{a}^{j_1},\mathfrak{b}^{i_2}\mathfrak{a}^{j_2}} = N^{-1}(n)/_\sim$. Thus S_m is core regular and the KMS$_1$-state is unique by Proposition 6.4. $\square$

This example shows the strength of the approach taken in Proposition 6.4, as both results for uniqueness of the KMS$_1$-state in [ABLS17] require α to be faithful.

[1]Injectivity of such endomorphisms unfortunately only holds for $n = 1$. For instance, L_2 has $\mathfrak{a}\mathfrak{b} \neq \mathfrak{b}^2$ but $\mathfrak{c}\mathfrak{a}\mathfrak{b} = \mathfrak{b}^{2m}\mathfrak{c} = \mathfrak{c}\mathfrak{b}^2$.

10. Summable quasi-absorption in the absence of almost freeness

There are only few explicit examples of monoids S for which the critical inverse temperature β_c is strictly larger than the minimal value 1 and uniqueness of the KMS$_\beta$-state is known to hold for all β in the critical interval $[1, \beta_c]$, namely $\mathbb{N} \rtimes \mathbb{N}^\times$ through [LR10], $R \rtimes R^\times$ for rings of integers R in a number field, see [CDL13] or [Nes13]. In the case of $\mathbb{N} \rtimes \mathbb{N}^\times$, it was demonstrated in [ABLS17] that uniqueness is due to almost freeness of the core action α. This is no longer true even for $\mathbb{Z} \rtimes \mathbb{Z}^\times$, but this example is covered by the involved arguments of [CDL13]. Here, we show that one can simply apply our conditions of core regularity to get uniqueness for $\mathbb{Z} \rtimes \mathbb{Z}^\times$, see Proposition 10.1.

In a similar way, we treat an example built on a shift space over the finite cyclic groups involving a Frobenius automorphism in one direction, see Proposition 10.2. Thereby, we provide two examples of right LCM monoids that are core regular and summably core regular, while α fails to be almost free. In view of Proposition 6.8, uniqueness of the KMS$_1$-state is only possible due to failure of right cancellation. The reason why the monoids in these examples are core regular is that whenever $\alpha_a = \alpha_b$ for $a, b \in S_c, a \neq b$, then this is witnessed already by absorption in S, that is, for every $s \in S$ there are $c, d \in S_c$ with $asc = bsd$.

The first example relates to $\mathbb{N} \rtimes \mathbb{N}^\times$ of [LR10]: $S := \mathbb{Z} \rtimes \mathbb{Z}^\times$ with $S_c = S^* = \mathbb{Z} \rtimes \mathbb{Z}^*$.

Proposition 10.1. *The right LCM monoid S has a generalised scale with $\beta_c = 2$. The action $\alpha\colon S_c \curvearrowright S/_\sim$ is faithful, but not almost free. The monoid S is core regular and summably core regular. In particular, $(C^*(S), \sigma)$ has a unique KMS$_\beta$-state ψ_β for each $\beta \in [1, 2]$ determined by $\psi_\beta(v_g) = \delta_{g,1}$ for $g \in S^*$.*

Proof. Noting that S arises as the semidirect product of the algebraic dynamical system $(\mathbb{Z}, \theta, \mathbb{Z}^\times)$ (with θ being multiplication), we deduce from [ABLS17, Proposition 5.11] that S has a generalised scale, but $\alpha\colon \mathbb{Z} \rtimes \mathbb{Z}^* \curvearrowright S/_\sim$ is not almost free. Similar to the case of the monoid $\mathbb{N} \rtimes \mathbb{N}^\times$, we have $\beta_c = 2$ for S. Since S is cancellative, there are no nontrivial absorbing elements, see Proposition 6.8. Moreover, $S_c = S^*$ is a group, so Remark 6.1 reduces the question of core regularity to studying the fixed point sets $F_p^g = \{[(h,p)] \mid h + p\mathbb{Z} \in \mathbb{Z}/p\mathbb{Z}, \alpha_g([(h,p)]) = [(h,p)]\}$ for $g \in \mathbb{Z} \rtimes \mathbb{Z}^* \setminus \{0\}$ and $p \in N(S) = \mathbb{N}^\times$. We have that $\alpha_g([(h,p)]) = [(h,p)]$ for $g = (g', \pm 1)$ if and only if $g' \pm h - h \in p\mathbb{Z}$. The case of $g \in \mathbb{Z} \times \{1\}, g \neq (0,1) = 1_S$ resembles the case for $\mathbb{N} \rtimes \mathbb{N}^\times$: The map α_g only has fixed points of the form $[(h,p)]$ with $p \leq |g'|$. So the action α restricted to $\mathbb{Z} \times \{1\}$ is almost free. Hence both types of core regularity hold by Proposition 6.7. Thus let $g = (g', -1)$. We need to determine the number of cosets $h + pZ \in \mathbb{Z}/p\mathbb{Z}$ with $g' - 2h \in p\mathbb{Z}$.

Case $p \notin 2\mathbb{Z}$**:** then 2 defines a permutation of $\mathbb{Z}/p\mathbb{Z}$, so $h + p\mathbb{Z}$ with $g' - 2h \in p\mathbb{Z}$ is uniquely determined and $|F_p^g| = 1$.

Case $p \in 2\mathbb{Z}, g' \notin 2\mathbb{Z}$**:** there is no such h, so $F_p^g = \emptyset$.

Case $p \in 2\mathbb{Z}, g' \in 2\mathbb{Z}$**:** then $g' - 2h \in p\mathbb{Z}$ is equivalent to having $h \in (g'/2 + (p/2)\mathbb{Z}) \setminus p\mathbb{Z}$ or $h \in (g'/2 + p\mathbb{Z})$, so that $|F_p^g| = 2$.

Thus we see that $|F_p^g| \leq 2$ for all $g \in \mathbb{Z} \rtimes \{-1\}, p \in \mathbb{N}^\times$, and conclude that S is core regular. In addition, Corollary 6.11 with $E = \emptyset$ yields summable core regularity for these remaining cases. The claimed uniqueness of the KMS$_\beta$-state and its form for β inside the critical interval now follow from Proposition 6.4 and Proposition 6.6. □

The second example is inspired by [ABLS17, Example 5.13]. We decided to keep it explicit instead of aiming for the greatest generality, with the hope that the interested reader will be able to extract the essential ingredients for the recipe. Let us denote the set of all primes in $\mathbb{N}$ by $\mathfrak{P}$, and fix $q \in \mathfrak{P}$. For $p \in \mathfrak{P}$, we let $G_p := \bigoplus_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$ as a group. For each $p \in \mathfrak{P} \setminus \{q\}$, the Frobenius map $g \mapsto g^q$ defines an automorphism of $\mathbb{Z}/p\mathbb{Z}$, and hence an automomorphism of the group G_p in the natural way. Acting as the identity on G_q, we obtain an automorphism f_q of the group

$$G := \bigoplus_{p \in \mathfrak{P}} G_p = \{(g_{p,n})_{p \in \mathfrak{P}, n \in \mathbb{N}} \mid g_{p,n} \in \mathbb{Z}/p\mathbb{Z}\}.$$

In addition to f_q, we will let $\mathbb{N}^\times$ act on G via the shift maps Σ with $\Sigma_p|_{G \setminus G_p} = \mathrm{id}$ and $(g_1, g_2, g_3, \dots) \mapsto (0, g_1, g_2, g_3, \dots)$ on G_p. Each Σ_p is an injective endomorphism of G whose image has index p. As Σ_p commutes with f_q, it is easy to check that $(G, \Sigma \times f_q, \mathbb{N}^\times \times \mathbb{Z})$ forms an algebraic dynamical system in the sense of [BLS18]. Hence the induced semidirect product $S := G \rtimes_\theta (\mathbb{N}^\times \times \mathbb{Z})$ is a cancellative right LCM monoid with $S_c = S^* = G \rtimes_{f_q} \mathbb{Z}$.

Proposition 10.2. *For every $q \in \mathfrak{P}$, the right LCM semigroup S has a generalised scale with $\beta_c = 2$. The action $\alpha\colon S^* \curvearrowright S/_\sim$ is faithful, but not almost free. The monoid S is core regular and summably core regular. In particular, $(C^*(S), \sigma)$ has a unique KMS$_\beta$-state ψ_β for each $\beta \in [1, 2]$ determined by $\psi_\beta(v_g) = \delta_{g,1}$ for $g \in S^*$.*

Proof. The proof of Proposition 10.1 can be transferred to this setting. Therefore, we restrict ourselves to the analysis of F_p^g for $g \in G \rtimes_{f_q} \mathbb{Z}$ and $p \in \mathbb{N}^\times$. For $p \in \mathfrak{P}$ and $g = (g', 1)$, we find that $\alpha_g[(h, p)] = [(h, p)]$ if and only if $g'_{p,1} h_{p,1}^q = h_{p,1}$. As $\mathbb{Z}/p\mathbb{Z}$ is abelian, this is equivalent to $h_{p,1}^{q-1} = g'_{p,1}$. If $q - 1$ and p are relatively prime, the map $\tilde{h} \mapsto \tilde{h}^{q-1}$ defines an automorphism of $\mathbb{Z}/p\mathbb{Z}$, so that $h_{p,1}$ and hence $[(h, p)]$ is uniquely determined in this case. As $p \in \mathfrak{P}$, this will be true for almost all $p \in \mathfrak{P}$. More precisely, it fails for the finite set E_q given by q and the primes dividing $q - 1$. Then we conclude that $|F_p^g| = 1$ for all $p \in \mathfrak{P} \setminus E_q$ and all $g \in S^*$. Since $\mathfrak{P} \setminus E_q \neq \emptyset$, this shows core regularity. Faithfulness of α now follows from right cancellation and Proposition 6.8. Summable core regularity follows from Corollary 6.11. The claimed uniqueness of the KMS$_\beta$-state and its form for β inside the critical interval now follow from Proposition 6.4 and Proposition 6.6. □

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