Existence of weak solutions to the Keller–Segel chemotaxis system with additional cross-diffusion

Gurusamy Arumugam\textsuperscript{a}, André H. Erhardt\textsuperscript{b,}\textsuperscript{*}, Indurekha Eswaramoorthy\textsuperscript{c}, Balachandran Krishnan\textsuperscript{c}

\textsuperscript{a} Discipline of Mathematics, Indian Institute of Technology Gandhinagar, Gandhinagar - 382355, Gujarat, India
\textsuperscript{b} Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, 0316 Oslo, Norway
\textsuperscript{c} Department of Mathematics, Bharathiar University, Coimbatore - 641046, India

\textbf{ARTICLE INFO}

\textbf{Article history:}
Received 12 July 2019
Received in revised form 17 December 2019
Accepted 31 December 2019
Available online xxxx

\textbf{Keywords:}
Keller–Segel system
Weak solutions
Fixed point theorem

\textbf{A B S T R A C T}

In this paper, we consider the Keller–Segel chemotaxis system with additional cross-diffusion term in the second equation. This system is consisting of a fully nonlinear reaction-diffusion equations with additional cross-diffusion. We establish the existence of weak solutions to the considered system by using Schauder’s fixed point theorem, a priori energy estimates and the compactness results.

© 2020 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

\section{1. Introduction}

The term chemotaxis is used to denote cell movement towards or away from a chemical source, classified as positive and negative chemotaxis, respectively. According to the common, broad definition of chemotaxis, it is the study of any cell motion affected by a chemical gradient in a way that results in net propagation up a chemoattractant gradient or down a chemorepellent gradient. Typical examples include aggregation processes such as slime mold formation in Dictyostelium Discoideum \cite{1} or pattern formation like in colonies of Salmonella typhimurium \cite{2}, and also medically relevant processes such as tumour invasion \cite{3,4} and self-organisation during embryonic development \cite{5}. For some more detailed (biological and mathematical modelling) background we refer also to \cite{6,7}. In 1970s, Keller and Segel \cite{8} proposed a classical and fundamental mathematical systems to model chemotaxis. The general form of a chemotaxis system is given by

\begin{equation}
\begin{aligned}
&u_t = \nabla \cdot (\psi(u,v)\nabla u - \phi(u,v)\nabla v) + k_3(u,v), \\
v_t = d_v \Delta v + k_4(u,v)u - k_5(u,v)v,
\end{aligned}
\end{equation}

\textsuperscript{*} Corresponding author.
E-mail address: andreerh@math.uio.no (A.H. Erhardt).

https://doi.org/10.1016/j.nonrwa.2020.103090
1468-1218/© 2020 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).
where \( u \) denotes the density of cell population and \( v \) is the chemical attractant concentration. Moreover, \( \psi(u,v) \) represents the diffusivity of the cells, \( \phi(u,v) \) denotes the chemotactic sensitivity, while \( k_3(u,v) \) describes the cell growth and death. \( d_v \) represents the diffusion coefficient of chemical attractant, while \( k_4(u,v) \) and \( k_5(u,v) \) describe the production and degradation of the chemical signal.

After their pioneering works for chemotaxis the literature is extensive. In the last decades, there has been considerable amount of work done in the analysis of various particular cases of system (1.1). The main focus on this model is its solvability.

In some special cases even more subtle analytical results on global existence of solutions of system (1.1) are available, such as the existence of weak solutions and their regularity proved by Bendahmane et al. in [9] for the case

\[
\psi(u,v) = a(u), \quad \phi(u,v) = \chi uf(u), \quad k_3(u,v) = 0, \quad k_4(u,v) = \alpha \quad \text{and} \quad k_5(u,v) = \beta,
\]

where \( \alpha, \beta \geq 0 \) and \( d_v \) is a constant. Furthermore, \( f(u) \) is a density dependent probability that a cell in position \( x \) at time \( t \) finds space in its neighbouring location. The cells are attracted by the chemical and \( \chi \) denotes their chemotactic sensitivity. In addition, the functions \( a(u) \) and \( f(u) \) are assumed to be sufficiently smooth. In [10] Nakaguchi and Osaki studied global existence of solutions to a parabolic–parabolic system for chemotaxis with a logistic source in a two-dimensional domain, where the degradation order of the logistic source is weaker than quadratic.

Bendahmane et al. [11] proved the existence and regularity of weak solutions for a fully parabolic model of chemotaxis, with prevention of overcrowding, that degenerates in a two-sided fashion, including an extra nonlinearity represented by a p-Laplacian diffusion term. Wang et al. [12] extended a previous results from [13] on global existence of solutions to quasilinear chemotaxis models under the condition that \( m > 2 - \frac{2}{n} \) can be relaxed to \( m > 2 - \frac{6}{n+4} \) in the case \( \psi(u,v) = D(u) \geq c_D u^{m-1} \) for all \( u > 0 \) with some \( c_D > 0, m > 1, d_v = 1, \phi(u,v) = u, k_3(u,v) = k_4(u,v) = 0 \) and \( k_5(u,v) = u \) in dimension \( n \geq 3 \).

Moreover, Wang et al. [14] dealt with the global existence and boundedness of solutions for the quasilinear chemotaxis system in the case when \( \psi(u,v) \geq D_0(u+1)^{m-1} \) for all \( u \geq 0, \phi(u,v) = u\chi(v), k_4(u,v) = -f(v) \) and \( k_3(u,v) = k_5(u,v) = 0 \). In addition, the given functions \( D(s), \chi(s) \) and \( f(s) \) are supposed to be sufficiently smooth for all \( s \geq 0 \) and that \( f(0) = 0 \).

Shangerganesh et al. [15] analysed the existence and uniqueness of a weak solution of the strongly coupled chemotaxis model with Dirichlet boundary conditions. In [16] Khelghati and Baghaei studied the classical solutions in a quasilinear parabolic–elliptic chemotaxis system with logistic source that are uniformly in-time-bounded without any restrictions on \( m \) and \( b \). Moreover, they assumed that \( \psi(u,v) = D(u) \geq 0 \) for \( u \geq 0, D(u) \geq C_D u^{m-1} \) for \( u > 0 \) and \( k_3(u,v) = g(u), g(0) \geq 0, g(u) \leq a - bu^\gamma, u > 0, d_v = 1, k_4(u,v) = k_5(u,v) = 1, v_t = 0, \) with constants \( C_D > 0, m \geq 1, a \geq 0, b > 0 \) and \( \gamma > 2 \). For the case \( \psi(u,v) = 1, \phi(u,v) = \chi u, k_3(u,v) = g(u) = Au - bu^\alpha \) with \( \alpha > 1, A \geq 0 \) and \( b > 0, k_4(u,v) = k_5(u,v) = 1, d_v = 1 \) and \( v_t = 0 \).

Furthermore, in [17] the existence of weak solutions to the following parabolic–elliptic system

\[
\begin{cases}
    u_t = \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, \\
    0 = \Delta v + u - v,
\end{cases}
\]

under homogeneous Neumann boundary conditions was established for arbitrary \( \mu > 0 \), while classical smooth solutions exist only for \( \mu > \frac{n+2}{n} \). In addition to it Winkler introduced in [18] the concept of very weak solutions and established (as a special case) the global existence of very weak solutions for any nonnegative initial data \( u_0 \in L^1(\Omega) \). Notice that his result is more general since instead of the logistic source term \( \kappa u - \mu u^2 \) he considers a more general logistic source term \( g(u) \approx \kappa u - \mu u^\alpha, \alpha > 2 - \frac{2}{n}, \kappa \geq 0 \) and \( \mu > 0 \). Moreover, boundedness properties of the constructed solutions are studied. Then, in [19] Vigilaloro
proved the existence of very weak solutions to the corresponding parabolic–parabolic logistic Keller–Segel system with smooth initial data and $\alpha \geq 1$. Furthermore, very recently Winkler used the concept of very weak solutions to investigate a general class of systems involving superlinear degradation in [20], which finds application to logistic Keller–Segel systems.

In [21] Lankeit proved the existence of global weak solutions to the chemotaxis system

$$\begin{cases}
u_t = \Delta u - \nabla \cdot (u\nabla v) + \kappa u - \mu v^2, \\
u_t = \Delta v + u - v,
\end{cases} \tag{1.2}$$

under homogeneous Neumann boundary conditions in a smooth bounded convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, for arbitrarily small values of $\mu > 0$. He also showed that in the three-dimensional setting, after some time, these solutions become classical solutions, provided that $\kappa$ is not too large. Moreover, in [22] the authors establish the existence of a global bounded classical solution for suitably large $\mu$ and prove that for any $\mu > 0$ there exists a weak solution. These are showcases that the concept of (very) weak solutions can be used to prove more general problems in the sense that one does not have to impose stronger assumptions on the system.

Furthermore, in [23] Winkler proved that system (1.1) has unique globally bounded classical solution under suitable assumptions on the initial data, provided $\psi(u, v) = 1$, $k_4(u, v) = k_5(u, v) = 1$, $d_v = 1$, $\phi(u, v) = \chi(v)$, where $0 < \chi(v) \leq \chi_0/(1 + \alpha v)^k$ with some $\alpha > 0$ and $k > 1$ for any $\chi_0 > 0$. Fujie et al. [24] extended the results from [23] to the singular case $0 < \chi(v) \leq \chi_0/v^k$ for some $\chi_0 > 0$ and $k \geq 1$. In [25] Fujie solved the open problem of uniform-in-time boundedness of solutions for $\chi < \sqrt{2/n}$, which was conjectured by Winkler [26].

Moreover, Zheng et al. [27] showed the unique global classical solution is uniformly bounded in $\Omega \times (0, \infty)$ for quasilinear parabolic–elliptic chemotaxis system with signal dependent sensitivity under the assumptions $\psi(u, v) = \varphi(u) \geq c_0(u + 1)^{m-1}$ for all $u \geq 0$ with $0 < c_0 \leq 1$, $m \in \mathbb{R}$, $\psi(u, v) = u\chi(v)$, $0 < \chi(v) \leq \chi_0/v^k$ with $k \geq 1$ and $\chi_0 > 0$, $k_4(u, v) = 0$, $\nu_t = 0$, $k_4(u, v) = k_5(u, v) = 1$, $d_v = 1$. He also proved nonnegative global-in-time bounded weak solution when $\psi(u, v) = \varphi(u)$ is degenerate.

The authors of [28] studied the global existence of solution for the quasilinear chemotaxis system with Dirichlet boundary condition and also established that the blow-up properties of the solution depend only on the first eigenvalue. Moreover, Stinner and Winkler [29] showed the global existence of nonnegative radially symmetric solutions to

$$\begin{cases}
u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v}\nabla v\right), \\
u_t = \Delta v + u - v,
\end{cases} \tag{1.3}$$

where $\chi > 0$. Additionally, Lankeit and Winkler designed a novel concept of generalised solvability and proved global solvability for large nonradial data [30].

Finally, we want to highlight that very recently Lankeit and Lankeit extended the study of the two previous chemotaxis systems (1.2) and (1.3) and established the existence of classical solutions and generalised global solutions, respectively, to the following system

$$\begin{cases}
u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v}\nabla v\right) + \kappa u - \mu v^2, \\
u_t = \Delta v - uv,
\end{cases}$$

for suitable choices of $\chi$, $\kappa$ and $\mu$, see [31, 32].

The literature related to Keller–Segel chemotaxis system is very rich and the given overview does not cover everything. Nevertheless, we believe that this overview gives a brief insight in the development and importance of this topic.

During the past decades, the main issue of the investigation of system (1.1) is whether the chemotaxis model allows for a chemotactic collapse, that is, if it possesses solutions that blow up in infinite or in finite time. However, in this paper we will prove global existence of solutions to system (1.4) (from below)
under the assumption (1.8) which rules out the chemotactic collapse. This work adds the significance by the introduction of cross-diffusion terms into classical diffusion systems as it allows the mathematical model to capture much more important features of many phenomena in physics, biology, ecology and engineering sciences.

In general, the chemotaxis model has become one of the best-studied models in mathematical biology and the main issue of the investigation was whether the models allow for solutions that blow-up or exist globally. Moreover, other contributions in this direction is also tremendous and there is a fairly rich literature addressing boundedness and blow-up issues in such systems of both parabolic–parabolic and parabolic–elliptic type (see [11,33–39]) and the references therein. In addition to that for stability and pattern formation one can refer [40,41] and the references therein.

In this paper, we consider a special case of the classical Keller–Segel chemotaxis system for the cell density $u$ and the concentration of the chemical signal $v$ with an additional cross-diffusion term $\delta \Delta u$ with $\delta > 0$, which reads as follows:

$$\begin{align*}
\partial_t u &= \nabla \cdot (d_1 \nabla u - \phi(u,v) \nabla v) + h(u,v) \\
\partial_t v &= d_2 \Delta v + \delta \Delta u + g(u,v)
\end{align*}$$

on $Q_T = \Omega \times (0,T)$,

(1.4)

where $T > 0$ is a fixed time and $\Omega$ is a bounded domain in $\mathbb{R}^n, n \geq 2$ with smooth boundary $\partial \Omega$, and $\psi(u,v) = d_1 \phi(u,v) = \chi \frac{u}{\gamma + v}$. The positive constant $\chi$ is called the chemotactic coefficient, $d_1$, $d_2 > 0$ are the diffusion coefficients and $\gamma \geq 1$. Furthermore, we also assume the logistic growth term $h(u,v) = u(1-u)$ and $g(u,v) = \alpha u - \beta v$. The nonnegative constant $\alpha$ is called the creation rate and $\beta u$ the chemical degradation with $\beta > 0$. Additionally, system (1.4) is supplemented by homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{on} \quad \Gamma_T := \partial \Omega \times (0,T),$$

(1.5)

where $\eta$ is the exterior unit normal to $\partial \Omega$; a no-flux boundary condition is implied on $\partial \Omega$ such that the ecosystem is closed to the exterior environment and with the initial conditions

$u(x,0) = u_0(x) \in L^2(\Omega), \quad v(x,0) = v_0(x) \in L^2(\Omega)$ \quad \text{for all} \quad x \in \Omega.$

(1.6)

This model is a special case of the general chemotaxis system with a cross-diffusion term $\delta \Delta u$, where $\delta > 0$. This term describes the movement of chemical signal along cell concentration towards or away from the cell density. Biologically, it models some diffusive effects in the equation of the chemical signal due to variations of the cell density, a kind of feedback effects. Additionally, for the classical Keller–Segel system

$$\begin{align*}
\partial_t u &= \nabla \cdot (\nabla u - \chi u \nabla v) \\
\partial_t v &= \Delta v - v + u
\end{align*}$$

in $\Omega \times (0,T)$

under homogeneous Neumann conditions and $\chi > 0$ one may expect a finite-time or infinite-time blow-up of solutions, cf. e.g [42,43]. For more details on blow-up we refer to the survey paper of Lankeit and Winkler [44].

However, from a biological point of view a blow-up of solutions is not realistic. Therefore, one has to introduce ‘something’ realistic to prevent such blow-up. This might be achieved by introducing a certain logistic growth or an additional cross-diffusion. In our study we consider both, indeed the main focus is on the additional cross-diffusion term.

There are some known facts about system (1.4) in presence of the cross-diffusion term $\delta \Delta u$. Let us mention briefly some interesting literatures. Hittmeir and Jüngel [45] were the first who introduced a particular case of system (1.4) in $2-D$ with $\phi(u,v) = \chi u$, $d_1 = \chi = 1$, $g(u,v) = \mu u - v$ with $\mu > 0$, and the logistic growth term $h(u,v) = 0$ in order to avoid the blow-up phenomena.

Xiang [46] studied the existence of global weak solutions to system (1.4) when $\phi(u,v) = \chi u$ and his results provided global dynamics and insights on how the biological parameters, especially, the cross-diffusion,
affects the pattern formations. Meyries [47] investigated the existence of travelling waves for a model similar to system (1.4).

The analysis of system (1.4) with diffusion matrix

\[
A(u, v) = \begin{pmatrix}
d_1 & -\chi \frac{u}{\gamma + v} \\
\delta & d_2
\end{pmatrix},
\]

(1.7)

faces a number of mathematical challenges. First of all, we can rewrite system (1.4) as

\[
\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \nabla \cdot \left( A(u, v) \nabla \begin{pmatrix} u \\ v \end{pmatrix} \right) + F(u, v),
\]

where

\[
F(u, v) := \begin{pmatrix} F_1(u, v) \\ F_2(u, v) \end{pmatrix} = \begin{pmatrix} h(u, v) \\ g(u, v) \end{pmatrix} = \begin{pmatrix} u(1 - u) \\ \alpha u - \beta v \end{pmatrix}.
\]

Note that system (1.4) is strongly coupled so that standard tools, like maximum principles and regularity theory, generally do not apply, cf. [48]. Many existence results for similar parabolic rely on Amann’s theory, generally do not apply, cf. [49]. Many existence results for similar parabolic rely on Amann’s theorem [50, Section 1] for general reaction–diffusion system. However, in our situation this theorem is not directly applicable due to (1) the choice of our initial values \( u_0, v_0 \in L^2(\Omega) \), (2) the additional cross-diffusion term, and (3) since the diffusion matrix \( A(u, v) \) is not positive definite.

Hence, we apply fixed point method for system (1.4), cf. [50]. For using this concept, the diffusion matrix \( A(u, v) \) of system (1.4) should be symmetric, positive definite and uniformly bounded. However, in our case, the matrix is non-symmetric and in fact, it is uniformly bounded only when certain \( L^\infty \) bounds are available.

Nevertheless, we are able to give an existence result for weak solutions of (1.4) under the conditions that there exists a positive constant \( K \), such that

\[
d := \min\{d_1, d_2\} > \frac{\delta}{2} + \frac{\chi}{2\gamma} K.
\]

Additionally, we supposed that \( \gamma \geq 1 \) and \( \chi, d_1, d_2 \) and \( \delta \) are positive constants.

Under this hypotheses, it turns out that the diffusion matrix is positive definite. The lack of uniform upper bounds for the diffusion terms is compensated by a weaker definition of solution, cf. Definition 1.1. Before we state our main results, let us introduce the definition of weak solutions to system (1.4).

**Definition 1.1.** A pair of nonnegative functions \((u, v)\) is said to be a weak solution of (1.4)–(1.6) if and only if

\[
u, v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad u_t, v_t \in L^2(0, T; (W^{1, \infty}(\Omega))'),
\]

such that for all test functions \( \varphi, \zeta \in L^2(0, T; W^{1, \infty}(\Omega)) \) the following holds true:

\[
\int_0^T \langle u_t, \varphi \rangle dt + \int_{Q_T} \left( d_1 \nabla u - \chi \frac{u}{\gamma + v} \nabla v \right) \cdot \nabla \varphi dx dt = \int_{Q_T} F_1(u, v) \varphi dx dt
\]

and

\[
\int_0^T \langle v_t, \zeta \rangle dt + \int_{Q_T} \left( d_2 \nabla v + \delta \nabla u \right) \cdot \nabla \zeta dx dt = \int_{Q_T} F_2(u, v) \zeta dx dt,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( (W^{1, \infty}(\Omega))' \) and \( W^{1, \infty}(\Omega) \), and \( T \in (0, \infty) \).

Now, we state our main theorem as follows.

**Theorem 1.1.** Assume that (1.8) holds. If \( u_0, v_0 \in L^2(\Omega) \), then problem (1.4)–(1.6) possesses a weak solution.

The outline of this article is as follows: In Section 2, we establish the existence of weak solutions to the regularised system (2.2) by using Schauder’s fixed point theorem. Then, in Section 3, we prove the existence of weak solutions to (1.4)–(1.6) by letting regularisation parameter \( \varepsilon \to 0 \).
2. Existence of weak solutions

In this section, we will prove the existence of weak solutions to (1.4)–(1.6) satisfying (1.8). In order to prove the main result, we will first prove the existence of weak solutions to the regularised system (2.2) of (1.4)–(1.6) which can be obtained by using Schauder’s fixed-point theorem, a priori estimates and using the compactness results. We start by introducing for a small number $\varepsilon > 0$ the following:

$$
\begin{align*}
F_{i,\varepsilon} &:= \frac{F_i}{1 + \varepsilon |F_i|} \quad \text{for} \quad i = 1, 2, \\
f_{\varepsilon}(a) &:= \frac{a}{1 + \varepsilon |a|} \quad \text{and} \quad b^+ := \max(0, b) \quad \text{for any} \quad a, b \in \mathbb{R}.
\end{align*}
$$

(2.1)

Then, let us define the following regularised system:

$$
\begin{align*}
\begin{cases}
\xi_t - \nabla \cdot (a_{11} \nabla u + a_{12} \nabla v) = F_{1,\varepsilon}(u^+, v^+) \\
v_t - \nabla \cdot (a_{21} \nabla v + a_{22} \nabla u) = F_{2,\varepsilon}(u^+, v^+)
\end{cases}
\quad \text{on} \quad Q_T,
\end{align*}
$$

(2.2)

where the diffusion matrix $A$ is given by

$$
A(u^+, v^+) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} d_1 + \frac{f^+_\varepsilon(u)}{\gamma + f^+_\varepsilon(v)} \\ \delta \end{pmatrix}.
$$

(2.3)

We supplement system (2.2) with no-flux boundary condition (1.5) and the initial data (1.6). Moreover, we have

$$
0 \leq f^+_\varepsilon(s) = \frac{s^+}{1 + \varepsilon s^+} \leq \min\{s^+, \varepsilon^{-1}\} \quad \text{for any} \quad s \in \mathbb{R}
$$

and $f^+_\varepsilon(s) \to s^+$ pointwise in $\mathbb{R}$ as $\varepsilon \to 0$. Hence, we can conclude that there exists a positive constant $K$, such that

$$
\frac{f^+_\varepsilon(u)}{\gamma + f^+_\varepsilon(v)} \leq \frac{1}{\gamma} f^+_\varepsilon(u) = \frac{1}{\gamma} \frac{u^+}{1 + \varepsilon u^+} \leq \frac{1}{\gamma} \min\{u^+, \varepsilon^{-1}\} \leq \frac{K}{\gamma} \quad \implies \quad -\chi \frac{f^+_\varepsilon(u)}{\gamma + f^+_\varepsilon(v)} \geq -\frac{\chi K}{\gamma}
$$

and thus, it follows that

$$
\xi^T A \xi = d_1 \xi_1^2 + d_2 \xi_2^2 + \left( \delta - \chi \frac{f^+_\varepsilon(u)}{\gamma + f^+_\varepsilon(v)} \right) \xi_1 \xi_2 \geq \left( d - \frac{\delta}{2} - \frac{\chi K}{2\gamma} \right) \left( \xi_1^2 + \xi_2^2 \right) \geq 0
$$

(2.4)

for any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, where we used (1.8) in combination with $a \cdot b = \frac{1}{2} (a + b)^2 - \frac{1}{2} (a^2 + b^2) \geq -\frac{1}{2} (a^2 + b^2)$ or $-a \cdot b = \frac{1}{2} (a - b)^2 - \frac{1}{2} (a^2 + b^2) \geq -\frac{1}{2} (a^2 + b^2)$ for all $a, b \in \mathbb{R}$. Therefore, the matrix $A(u^+, v^+)$ is uniformly positive definite, provided (1.8) is fulfilled. Note that we will frequently use (2.4) to prove the existence and nonnegativity of weak solutions. Additionally, we want to mention that similar holds true if we consider bounded nonnegative functions $d_1(u, v)$ and $d_2(u, v)$ which are sufficiently smooth.

**Remark 2.1.** If condition (1.8) fails, we may loose the ellipticity condition (2.4) which leads to unbounded solutions in finite time that is called blow-up. Thus, the assumption (1.8) allows us to actually prove the existence of weak solutions. The case $\phi(u, v) = \delta$ would also imply that the diffusion matrix $A(u, v)$ is uniformly positive definite.

Our next aim is to establish an existence result to the fixed problem. Here, we omit the dependence of the solutions on the parameter $\varepsilon$. We prove, for each fixed $\varepsilon > 0$, the existence of solutions to the fixed problem (2.2) by applying the Schauder’s fixed-point theorem. Since we use Schauder’s fixed-point theorem, we need to introduce the following closed subset of the Banach space $L^2(Q_T)$:

$$
S = \left\{ U = (u, v) \in L^2(Q_T) : \|U\|_{L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))} \leq \delta \right\},
$$

(2.5)
where \( \tilde{\delta} > 0 \) is a constant. With \( (\bar{u}, \bar{v}) \in S \) fixed, let \((u, v)\) be the unique solution of the system

\[
\begin{align*}
\frac{du}{dt} &= \nabla \cdot \left( d_1 \nabla u - \frac{f^+_e (\bar{u})}{\gamma + f^+_e (\bar{v})} \nabla v \right) + F_{1,\varepsilon}(\bar{u}^+, \bar{v}^+) \quad \text{on } Q_T, \\
\frac{dv}{dt} &= \nabla \cdot \left( d_2 \nabla v + \delta \nabla u \right) + F_{2,\varepsilon}(\bar{u}^+, \bar{v}^+) \quad \text{on } Q_T.
\end{align*}
\]  

(2.6)

The existence of a unique solution is guaranteed by [51, Chapter VII] due to (2.4) and \( \mathbf{A}_{ij} \in L^\infty(Q_T) \) and \( F_{i,\varepsilon}(\bar{u}^+, \bar{v}^+) \in L^2(Q_T), \ 1 \leq i, j \leq 2 \), cf. also [52].

Next, we introduce a map \( T : S \to S \) such that \( T(u, \bar{v}) = (u, v) \), where \((u, v)\) solves (2.6). By using the Schauder’s fixed-point theorem, we prove that the map \( T \) has a fixed point for (2.6). First, we show that the map \( T \) is continuous. For this, let \((\bar{u}_n, \bar{v}_n)\) be a sequence in \( S \). Furthermore, let \((\bar{u}, \bar{v}) \in S \) be such that \((\bar{u}_n, \bar{v}_n) \to (\bar{u}, \bar{v}) \) in \( L^2(Q_T) \) as \( n \to \infty \). Moreover, we define \((u_n, v_n) = T(\bar{u}_n, \bar{v}_n)\). The goal is to show that \((u_n, v_n)\) converges to \( T(\bar{u}, \bar{v}) \) in \( L^2(Q_T) \).

To this end, we need the following lemma:

**Lemma 2.1.** The solution \((u_n, v_n)\) to system (2.6) satisfies

(i) the sequence \((u_n, v_n)\) is bounded in \( L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \),

(ii) the sequence \((u_n, v_n)\) is relatively compact in \( L^2(Q_T) \).

**Proof.** Multiplying the first equation in system (2.6) by \( u_n \) and integrate over \( \Omega \), we derive at

\[
\int_\Omega u_n u_n dx - \int_\Omega \nabla \cdot (d_1 \nabla u_n) u_n - \nabla \cdot \left( \frac{f^+_e (\bar{u})}{\gamma + f^+_e (\bar{v})} \nabla v_n \right) u_n dx = \int_\Omega F_{1,\varepsilon} u_n dx.
\]

Integration by parts yields

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_n|^2 dx + \int_\Omega d_1 |\nabla u_n|^2 - \left( \frac{f^+_e (\bar{u})}{\gamma + f^+_e (\bar{v})} \nabla v_n \right) \cdot \nabla u_n dx = \int_\Omega F_{1,\varepsilon} u_n dx.
\]  

(2.7)

Similarly, we derive from the second equation in (2.6) multiplying by \( v_n \) the following:

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |v_n|^2 dx + \int_\Omega d_2 |\nabla v_n|^2 + \int_\Omega \delta \nabla u_n \cdot \nabla v_n dx = \int_\Omega F_{2,\varepsilon} v_n dx.
\]  

(2.8)

Adding (2.7) and (2.8), by the definition and boundedness of \( F_{i,\varepsilon} \), cf. (2.1), and using (2.4), we get

\[
\frac{d}{dt} \int_\Omega \left( |u_n|^2 + |v_n|^2 \right) dx + \left( d - \frac{\delta}{2} - \frac{\chi}{2\gamma} \right) \int_\Omega \left( |\nabla u_n|^2 + |\nabla v_n|^2 \right) dx \leq C \int_\Omega \left( |u_n|^2 + |v_n|^2 \right) dx
\]  

(2.9)

for some constant \( C > 0 \). Using Gronwall’s inequality in (2.9) and taking sup over \((0, T)\), we obtain that

\[
\sup_{t \in (0, T)} \int_\Omega \left( |u_n|^2 + |v_n|^2 \right) dx \leq \exp \left( \int_0^T \left[ \int_\Omega \left( |u_0|^2 + |v_0|^2 \right) dx \right] ds \right) \leq \exp (CT) \left( \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right),
\]  

(2.10)

which proves the first part of (i). Furthermore, from (2.9)–(2.10), we can conclude that

\[
\left( d - \frac{\delta}{2} - \frac{\chi}{2\gamma} \right) \int_\Omega \left( |\nabla u_n|^2 + |\nabla v_n|^2 \right) dx \leq \exp (CT) \left( \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right).
\]

Now, integrating over \((0, T)\), we have

\[
\int_0^T \int_\Omega \left( |\nabla u_n|^2 + |\nabla v_n|^2 \right) dx dt \leq \frac{\exp (CT)}{d - \frac{\delta}{2} - \frac{\chi}{2\gamma} \kappa} \int_0^T \left( \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right) dt \leq \frac{T \exp (CT)}{d - \frac{\delta}{2} - \frac{\chi}{2\gamma} \kappa} \left( \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right).
\]  

(2.11)
which proves (i). Next, we multiply the first equation of (2.6) by \( \varphi \in L^2(0, T; H^1(\Omega)) \) and integrate over \( Q_T \) to get

\[
\left| \int_0^T \langle u_{n,t}, \varphi \rangle \, dt \right| \leq \left| \int \int_{Q_T} d_1 \nabla u_n \cdot \nabla \varphi \, dx \, dt \right| + \left| \int \int_{Q_T} \left( \frac{f^+(u)}{\gamma + f^+(v)} \right) \nabla \varphi \, dx \, dt \right|
\]

\[
+ \left| \int \int_{Q_T} F_{1, \varepsilon}(\bar{u}^+, \bar{v}^+) \varphi \, dx \, dt \right|.
\]

(2.12)

Then, using the boundedness of \( f^+ \) and \( F_{i, \varepsilon} \) for \( i = 1, 2 \), and (2.11), there exists a constant \( C = C(\varepsilon, d_1, \chi, \gamma) > 0 \) such that we can conclude from (2.12) that

\[
\left| \int_0^T \langle u_{n,t}, \varphi \rangle \, dt \right| \leq C\| \varphi \|_{L^2(0, T; H^1(\Omega))}.
\]

(2.13)

Similarly, multiplying the second equation in (2.6) by \( \zeta \in L^2(0, T; H^1(\Omega)) \) and integrating over \( Q_T \), we get

\[
\left| \int_0^T \langle v_{n,t}, \zeta \rangle \, dt \right| \leq \left| \int \int_{Q_T} d_2 \nabla v \cdot \nabla \zeta \, dx \, dt \right| + \left| \int \int_{Q_T} \delta \nabla u \cdot \nabla \zeta \, dx \, dt \right|
\]

\[
+ \left| \int \int_{Q_T} F_{2, \varepsilon}(\bar{u}^+, \bar{v}^+) \zeta \, dx \, dt \right|.
\]

(2.14)

Again, using the boundedness of \( f^+ \) and \( F_{i, \varepsilon} \) for \( i = 1, 2 \), and (2.11), there exists a constant \( C = C(\varepsilon, d_2, \delta) > 0 \) such that Eq. (2.14) becomes

\[
\left| \int_0^T \langle v_{n,t}, \zeta \rangle \, dt \right| \leq C\| \zeta \|_{L^2(0, T; H^1(\Omega))}.
\]

(2.15)

Finally, (ii) is a consequence of the boundedness of \( (u_n, v_n)_n \), cf. (i), and the uniform bounds (2.13) and (2.15) of \( (u_n, v_n)_n \) in \( L^2(0, T; (H^1(\Omega))')^2 \). \( \square \)

Thus, from Lemma 2.1, we can conclude that there exist functions \( (u_n, v_n) \in (L^2(0, T; H^1(\Omega)))^2 \) such that (up to a subsequence)

\[
(u_n, v_n) \to (u, v) \quad \text{in} \quad (L^2(Q_T))^2 \quad \text{strongly},
\]

which shows the continuity of \( T \) on \( S \). Further, from Lemma 2.1, we can conclude that \( T(S) \) is bounded in

\[\mathcal{X} = \{ u \in L^2(0, T; H^1(\Omega)) : u_t \in L^2(0, T; (H^1(\Omega))') \} .\]

Moreover, since \( T(S) \) is bounded and the embedding \( \mathcal{X} \hookrightarrow L^2(Q_T) \) is compact (by the results of [53]), we know that, \( T \) is compact. Thus, \( T \) satisfies all assumptions of Schauder’s fixed-point theorem. Therefore, the operator \( T \) has a fixed point \( (u_\varepsilon, v_\varepsilon) \) such that \( T(u_\varepsilon, v_\varepsilon) = (u_\varepsilon, v_\varepsilon) \). Then, there exists a solution \( (u_\varepsilon, v_\varepsilon) \) of

\[
\int_0^T \langle u_{\varepsilon,t}, \varphi \rangle \, dt + \int \int_{Q_T} \left( d_1 \nabla u_\varepsilon - \chi \frac{f^+(u)}{\gamma + f^+(v)} \nabla v \right) \cdot \nabla \varphi \, dx \, dt = \int \int_{Q_T} F_{1, \varepsilon}(u_\varepsilon^+, v_\varepsilon^+) \varphi \, dx \, dt
\]

(2.16)

and

\[
\int_0^T \langle v_{\varepsilon,t}, \zeta \rangle \, dt + \int \int_{Q_T} d_2 \nabla v_\varepsilon \cdot \nabla \zeta \, dx \, dt + \int \int_{Q_T} \delta \nabla u \cdot \nabla \zeta \, dx \, dt = \int \int_{Q_T} F_{2, \varepsilon}(u_\varepsilon^+, v_\varepsilon^+) \zeta \, dx \, dt
\]

(2.17)

for all \( \varphi, \zeta \in L^2(0, T; H^1(\Omega)) \). Furthermore, the solution \( (u_\varepsilon, v_\varepsilon) \) is nonnegative and satisfies certain regularity estimate as stated in the following lemma.
Lemma 2.2. Assume that condition (1.8) holds. If the initial conditions $u_0 \in L^2(\Omega)$ and $v_0 \in L^2(\Omega)$ are positive, then the solution $(u_\varepsilon, v_\varepsilon)$ is nonnegative. Moreover, there exist constants $C_1, C_2, C_3, C_4 > 0$ not depending on $\varepsilon$ such that
\begin{align}
\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} &\leq C_1, \\
\|F_{1,\varepsilon}(u_\varepsilon, v_\varepsilon)\|_{L^2(Q_T)} + \|F_{2,\varepsilon}(u_\varepsilon, v_\varepsilon)\|_{L^2(Q_T)} &\leq C_2, \\
\|\nabla u_\varepsilon\|_{L^2(Q_T)} + \|\nabla v_\varepsilon\|_{L^2(Q_T)} &\leq C_3, \\
\|u_\varepsilon\|_{L^2(0,T;W^{1,\infty}(\Omega))} + \|v_\varepsilon\|_{L^2(0,T;W^{1,\infty}(\Omega))} &\leq C_4. 
\end{align}

Proof. Choosing $\varphi = -u_\varepsilon = \frac{u_\varepsilon - |u_\varepsilon|}{2}$ in (2.16) and integrate over $\Omega$ instead $Q_T$. Using [54, Proposition 1.2] yields
\begin{align}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| u_\varepsilon \right|^2 dx + \int_{\Omega} \left( d_1 \nabla u_\varepsilon - \frac{\chi}{\gamma + \varepsilon} \nabla v_\varepsilon \right) \cdot \nabla u_\varepsilon dx = \int_{\Omega} F_{1,\varepsilon}(u_\varepsilon^+, v_\varepsilon^+) u_\varepsilon^+ dx. 
\end{align}
Similarly, in (2.17), we take $\zeta = -v_\varepsilon = \frac{v_\varepsilon - |v_\varepsilon|}{2}$ and integrate over $\Omega$ instead of $Q_T$ to obtain
\begin{align}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| v_\varepsilon \right|^2 dx + \int_{\Omega} \left( d_2 \nabla v_\varepsilon - \delta \nabla u_\varepsilon \right) \cdot \nabla v_\varepsilon dx = \int_{\Omega} F_{2,\varepsilon}(u_\varepsilon^+, v_\varepsilon^+) v_\varepsilon^+ dx.
\end{align}
Adding (2.22) and (2.23), using (2.4) and the definition of $F_{i,\varepsilon}$, we conclude that
\begin{align}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |u_\varepsilon|^2 + |v_\varepsilon|^2 \right) dx + \left( d - \delta - \frac{\chi}{2\gamma} \right) \int_{\Omega} \left( |\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 \right) dx &\leq C \int_{\Omega} \left( |u_\varepsilon|^2 + |v_\varepsilon|^2 \right) dx
\end{align}
for some constant $C > 0$. Utilising Gronwall’s lemma, we get
\begin{align}
\int_{\Omega} \left( |u_\varepsilon|^2 + |v_\varepsilon|^2 \right) dx &\leq 0
\end{align}
for a.e. $t \in (0,T)$. This yields the nonnegativity of $u_\varepsilon$ and $v_\varepsilon$. Additionally, for $i = 1, 2$, we have
\begin{align}
|F_{i,\varepsilon}(u_\varepsilon, v_\varepsilon)| &\leq C_2 \left( |u_\varepsilon|^2 + |v_\varepsilon|^2 \right)
\end{align}
for some constant $C_2 > 0$. Now, using (2.18) in (2.25), we get (2.19). Because of the weak lower semicontinuity properties of norms, estimates (2.9) and (2.10) hold for $u_\varepsilon$ and $v_\varepsilon$, respectively. Thus, we obtain (2.18) and (2.20). Finally, using Hölder’s inequality, (2.18) and (2.20), we can deduce from (2.16) for all $\varphi \in L^2(0,T;W^{1,\infty}(\Omega))$ that
\begin{align}
\left| \int_{0}^{T} \langle u_\varepsilon, \varphi \rangle dt \right| &\leq C \|\nabla u_\varepsilon\|_{L^2(Q_T)} \|\nabla \varphi\|_{L^2(Q_T)} \\
&\quad + C' \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \|\nabla v_\varepsilon\|_{L^2(Q_T)} \|\nabla \varphi\|_{L^2(0,T;L^\infty(\Omega))} \\
&\quad + C'' \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \|v_\varepsilon\|_{L^2(Q_T)} \|\varphi\|_{L^2(0,T;L^\infty(\Omega))} \\
&\quad + C''' \|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \|v_\varepsilon\|_{L^2(Q_T)} \|\varphi\|_{L^2(0,T;L^\infty(\Omega))} \\
&\leq C''' \|\varphi\|_{L^2(0,T;W^{1,\infty}(\Omega))}
\end{align}
for some constant $C, C', C'', C''' > 0$ independent of $\varepsilon$. From this, the following bound
\begin{align}
\|u_\varepsilon\|_{L^2(0,T;W^{1,\infty}(\Omega))'} \leq C'''
\end{align}
derives. Similarly, we can conclude from the weak formulation (2.17) that we have for all $\zeta \in L^2(0,T;W^{1,\infty}(\Omega))$
\begin{align}
\left| \int_{0}^{T} \langle v_\varepsilon, \zeta \rangle dt \right| &\leq C''' \|\zeta\|_{L^2(0,T;W^{1,\infty}(\Omega))}.
\end{align}
From this, we deduce the bound
\[ \|v_\varepsilon\|_{L^2(0,T;(W^{1,\infty}(\Omega))')} \leq C'''. \]  
(2.29)
Adding (2.27) and (2.29), we derive (2.21). This completes the proof of the lemma. □

3. Proof of Theorem 1.1

In this section, we prove the existence of weak solutions to (1.4)–(1.6), since until now we only showed that approximate system (2.2) admits a solution \((u_\varepsilon, v_\varepsilon)\), which is nonnegative. The next goal is to send the regularisation parameter \(\varepsilon\) to zero in sequences of such solution to obtain weak solutions of the original system (1.4)–(1.6).

From Lemma 2.2 and Aubin’s lemma, we can conclude that there exist limit functions \((u, v)\) such that as \(\varepsilon \to 0\) the following convergences (up to a subsequence) hold:
\[
\begin{cases}
(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{a.e. in } Q_T \text{ and strongly in } L^2(Q_T), \text{ weakly in } L^2(0,T; H^1(\Omega)), \\
F_{t,\varepsilon}(u_\varepsilon, v_\varepsilon) \to F_t(u, v) \quad \text{a.e. in } Q_T \text{ and strongly in } L^1(Q_T), \\
(u_\varepsilon, v_\varepsilon) \to (u_\varepsilon, v_\varepsilon) \quad \text{weakly in } L^2(0,T; (W^{1,\infty}(\Omega))').
\end{cases}
\]  
(3.1)
Furthermore, we have
\[
\|f_\varepsilon(u_\varepsilon) - u\|_{L^2(Q_T)} = \|f_\varepsilon(u_\varepsilon) - u_\varepsilon + u_\varepsilon - u\|_{L^2(Q_T)} \\
\leq \sqrt{2}\|u_\varepsilon - u\|_{L^2(Q_T)} + \sqrt{2}\|f_\varepsilon(u_\varepsilon) - u_\varepsilon\|_{L^2(Q_T)} \\
= \sqrt{2}\|u_\varepsilon - u\|_{L^2(Q_T)} + \sqrt{2}\varepsilon\|f_\varepsilon(u_\varepsilon)\|_{L^2(Q_T)} \to 0 \quad \text{as } \varepsilon \to 0,
\]  
(3.2)
which implies
\[ f_\varepsilon(u_\varepsilon) \to u \quad \text{a.e. in } Q_T \text{ and strongly in } L^2(Q_T). \]
In addition, we can conclude using Hölder’s inequality the following
\[ \|f_\varepsilon(u_\varepsilon) - u\|_{L^1(Q_T)} \leq \sqrt{|Q_T|}\|f_\varepsilon(u_\varepsilon) - u\|_{L^2(Q_T)} \to 0 \quad \text{as } \varepsilon \to 0, \]
which implies \(f_\varepsilon(u_\varepsilon) \to u\) a.e. in \(Q_T\) and strongly in \(L^p(Q_T)\) for all \(1 \leq p \leq 2\). Finally, by passing to the limits as \(\varepsilon \to 0\) in the weak formulation (2.16)–(2.17) with \(\varphi \in L^2(0,T; W^{1,\infty}(\Omega))\) and \(\zeta \in L^2(0,T; W^{1,\infty}(\Omega))\), we could obtain in this way that the limit \((u, v)\) is a solution of the system (1.4)–(1.6) satisfying (1.8) in the sense of Definition 1.1.

4. Conclusions

In this paper, we considered a Keller–Segel chemotaxis system with additional cross-diffusion and proved the existence of weak solutions to the regularised system of (2.2) by using Schauder’s fixed point theorem. Then, the existence of weak solutions to system (1.4)–(1.6) satisfying (1.8) has been achieved by a priori estimates and compactness arguments.

Acknowledgements

G.A. and B.K. are supported by Defence Research and Development Organisation, Government of India, Grant Number: DLS/86/50011/DRDO-BU Center Phase II. Furthermore, the authors wish to thank the anonymous referee for the careful reading of the original manuscript and the comments that eventually led to an improved presentation. Open access funding was provided by the University of Oslo.
References


