



A white noise approach to optimal insider control of systems with delay



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ABSTRACT

We use a white noise approach to study the problem of optimal insider control of a stochastic delay equation driven by a Brownian motion B and a Poisson random measure N . In particular, we use Hida-Malliavin calculus and the Donsker delta functional to study the problem. We establish a sufficient and a necessary maximum principle for the optimal control when the trader from the beginning has inside information about the future value of some random variable related to the system. These results are applied to the problem of optimal inside harvesting control in a population modelled by a stochastic delay equation. Next, we apply a direct white noise method to find the logarithmic utility optimal insider portfolio in a generalized Black-Scholes type financial market. A classical result of Pikovski and Karatzas shows that when the inside information is $B(T)$, where T is the terminal time of the trading period, then the market is not viable, i.e. the maximal utility is infinite. We consider two extensions to delay of this result and prove the following:

- If the risky asset price is given by a stochastic delay equation, the resulting insider market is still not viable.
- If, however, there is delay in the information flow to the insider, the market becomes viable.

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1. Introduction

Delay stochastic differential equations are a type of stochastic differential equation in which the derivative of the solution process at any given time depends not only on its value at the present time, but also on its values at previous times. They are also called systems with aftereffect.

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Many real life systems in for example engineering, biology, finance and communication networks include delay phenomena in their dynamics. Therefore delay stochastic differential equations have many applications, and the interest of delay stochastic differential equations continues to grow in all areas of science and in particular in control engineering.

In our paper, in addition to the delay, we suppose that we have an additional information related to the future value of the system. In other words, we consider insider control of delay stochastic differential equations. Insider control problems are treated in many papers. We mention [2–4,9,10]. Our paper differs from the above papers in two ways:

- We use a different insider control approach based on white noise theory. Specifically, we apply Hida-Malliavin calculus and the definition of the Donsker delta functional. This allows us to transform the insider delay control problem into a non-insider parameterized control problem. We prove two maximum principle theorems. Then we apply the results to an inside optimal harvesting problem in a population modelled by a delay equation.
- We apply a direct white noise method to study optimal inside control in the context of delay. In particular, an interesting question which, to the best of our knowledge, has not been studied before, is how inside information combines with the delay in the dynamics when it comes to performance, for example in finance. Specifically, we find an explicit expression for the optimal insider portfolio in a delay market and we prove that, perhaps surprisingly, even in the presence of the delay, the market is not viable when the inside information is given by $B(T)$, where T is the terminal value of the trading interval. See Section 5.
- Finally, in Section 6 we study the situation when there is delay in the information flow to the insider, and we prove that in this case the insider market is viable.

We now explain this in more detail.

In this paper we consider an insider's optimal control problem for a stochastic process $X(t) = X(t, Z) = X^u(t, Z)$ defined as the solution of a stochastic differential delay equation of the form

$$\begin{cases} dX(t) = dX(t, Z) = b(t, X(t, Z), Y(t, Z), u(t, Z), Z)dt + \sigma(t, X(t, Z), Y(t, Z), u(t, Z), Z)dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t, Z), Y(t, Z), u(t, Z), Z, \zeta) \tilde{N}(dt, d\zeta), & 0 \leq t \leq T \\ X(t) = \xi(t), & -\delta \leq t \leq 0, \end{cases} \quad (1.1)$$

where

$$Y(t, Z) = X(t - \delta, Z), \quad (1.2)$$

$\delta > 0$ being a fixed constant (the delay) and ξ is a deterministic function.

Here $B(t)$ and $\tilde{N}(dt, d\zeta)$ is a Brownian motion and an independent compensated Poisson random measure, respectively, jointly defined on a filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbf{P})$ satisfying the usual conditions where $\Omega = S'(\mathbb{R})$ the dual of Schwartz space and \mathbf{P} is the Gaussian measure on $S'(\mathbb{R})$. $T > 0$ is a given constant. We refer to [5] for more information about white noise theory and [12] for stochastic calculus for Itô-Lévy processes.

The process $u(t, Z) = u(t, x, z)_{z=Z}$ is our insider control process, where Z is a given \mathcal{F}_{T_0} -measurable random variable for some $T_0 > T$, representing the inside information available to the controller.

We assume that the inside information is of *initial enlargement* type. Specifically, we assume that the inside filtration \mathbb{H} has the form

$$\mathbb{H} = \{\mathcal{H}_t\}_{0 \leq t \leq T}, \text{ where } \mathcal{H}_t = \mathcal{F}_t \vee \sigma(Z) \quad (1.3)$$

for all $t \in [0, T]$, where Z is a given \mathcal{F}_{T_0} -measurable random variable, for some $T_0 > T$ (constant). Here and in the following we use the right-continuous version of \mathbb{H} , i.e. we put $\mathcal{H}_t = \mathcal{H}_{t+} = \bigcap_{s>t} \mathcal{H}_s$.

We assume that the value at time t of our insider control process $u(t)$ is allowed to depend on both Z and \mathcal{F}_t . In other words, $u(\cdot)$ is assumed to be \mathbb{H} -adapted, such that $u(\cdot, z)$ is \mathbb{F} -adapted for each $z \in \mathbb{R}$.

We also assume that the *Donsker delta functional* of Z exists. This assumption implies that the Jacod condition holds, and hence that $B(\cdot)$ and $N(\cdot, \cdot)$ are semimartingales with respect to \mathbb{H} therefore equation (1.1) is well defined. We will explain this with more details in the next Section.

Let \mathbb{U} denote the set of admissible control values. We assume that the functions

$$\begin{aligned} b(t, x, y, u, z) &= b(t, x, y, u, z, \omega) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \Omega \mapsto \mathbb{R}, \\ \sigma(t, x, y, u, z) &= \sigma(t, x, y, u, z, \omega) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \Omega \mapsto \mathbb{R}, \\ \gamma(t, x, y, u, z, \zeta) &= \gamma(t, x, y, u, z, \zeta, \omega) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \Omega \mapsto \mathbb{R}, \end{aligned}$$

are given C^1 functions with respect to x, y and u and adapted processes in (t, ω) for each given x, y, u, z, ζ . Let \mathcal{A} be a given family of admissible \mathbb{H} -adapted controls u . The *performance functional* $J(u)$ of a control process $u \in \mathcal{A}$ is defined by

$$J(u) = \mathbb{E}\left[\int_0^T f(t, X(t, Z), u(t, Z), Z)dt + g(X(T, Z), Z)\right], \tag{1.4}$$

where

$$\begin{aligned} f(t, x, u, z) &: [0, T] \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \mapsto \mathbb{R} \\ g(x, z) &: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \end{aligned} \tag{1.5}$$

are given functions, C^1 with respect to x and u . The functions f and g are called the *profit rate* and *terminal payoff*, respectively. For completeness of the presentation we allow these functions to depend explicitly on the future value Z also, although this would not be the typical case in applications. But it could be that f and g are influenced by the future value Z directly through the action of an insider, in addition to being influenced indirectly through the control process u and the corresponding state process X .

The problem we consider is the following:

Problem 1.1. Find $u^* \in \mathcal{A}$ such that

$$\sup_{u \in \mathcal{A}} J(u) = J(u^*). \tag{1.6}$$

2. Maximum principle theorems using the Donsker delta functional

To study the Problem 1.1 we use a white noise approach. In particular, we will use the definition of the Donsker delta functional which takes value in the stochastic Hida distribution space $(\mathcal{S})^*$ and the Hida-Malliavin derivatives D_t and $D_{t,z}$ with respect to the Brownian motion $B(\cdot)$ and the Poisson random measure $\tilde{N}(\cdot, \cdot)$, respectively. We refer to [5] for more information about $(\mathcal{S})^*$, white noise theory and the Hida-Malliavin derivative. First let us recall the definition and basic properties of the Donsker delta functional:

Definition 2.1. Let $Z : \Omega \rightarrow \mathbb{R}$ be a random variable which also belongs to $(\mathcal{S})^*$. Then a continuous functional

$$\delta_Z(\cdot) : \mathbb{R} \rightarrow (\mathcal{S})^* \tag{2.1}$$

is called a Donsker delta functional of Z if it has the property that

$$\int_{\mathbb{R}} g(z)\delta_Z(z)dz = g(Z) \quad \text{a.s.} \tag{2.2}$$

for all (measurable) $g : \mathbb{R} \rightarrow \mathbb{R}$ such that the integral converges.

Remark 2.2. If $Z \in L^2(S'(\mathbb{R}), \mathbf{P})$ then it belongs to $(\mathcal{S})^*$.

2.1. Donsker delta functional properties

Define the *regular conditional distribution* with respect to \mathcal{F}_t of a given real random variable Z , denoted by $Q_t(dz) = Q_t(\omega, dz)$, by the following properties:

- For any Borel set $\Lambda \subseteq \mathbb{R}$, $Q_t(\cdot, \Lambda)$ is a version of $\mathbb{E}[\mathbf{1}_{Z \in \Lambda} | \mathcal{F}_t]$.
- For each fixed ω , $Q_t(\omega, dz)$ is a probability measure on the Borel subsets of \mathbb{R} .

Our aim is to show that if Z has a Donsker delta functional then Jacod condition holds hence $B(\cdot)$ and $N(\cdot, \cdot)$ are semimartingales with respect to \mathbb{H} . Therefore the SDDE (1.1) is well defined.

First let us recall the Jacod condition [1,8]:

If there exists a σ -finite positive measure η such that the regular conditional distributions of the random variable Z given $\mathcal{F}_t, t \in [0, T]$ are absolutely continuous with respect to η , Jacod proves that every (P, \mathbb{F}) -martingale remains a (P, \mathbb{H}) -semimartingale on the interval $[0, T]$.

Now assume that the Donsker delta functional of Z exists. Equation (2.2) holds for all bounded g . Taking the conditional expectation of (2.2) with respect to \mathcal{F}_t we get

$$\int_{\mathbb{R}} g(z)E[\delta_Z(z)|\mathcal{F}_t]dz = E[g(Z)|\mathcal{F}_t] = \int_{\mathbb{R}} g(z)Q_t(\omega, dz) \tag{2.3}$$

Since this holds for all g , we conclude that we have the following identity of measures:

$$Q_t(\omega, dz) = E[\delta_Z(z)|\mathcal{F}_t]dz. \tag{2.4}$$

Hence $Q_t(\omega, dz)$ is absolutely continuous with respect to the Lebesgue measure. In particular, the Jacod condition holds then $B(\cdot)$ and $N(\cdot, \cdot)$ are semimartingales with respect to \mathbb{H} . Therefore the SDDE (1.1) is well defined.

From now on we assume that Z is a given random variable which also belongs to $(\mathcal{S})^*$, with a Donsker delta functional $\delta_Z(z) \in (\mathcal{S})^*$ satisfying

$$\mathbb{E}[\delta_Z(z)|\mathcal{F}_T] \in \mathbf{L}^2(\mathcal{F}_T, P). \tag{2.5}$$

2.2. Transforming the insider control problem to a related parameterized non-insider problem

Since $X(t)$ is \mathbb{H} -adapted, we get by using the definition of the Donsker delta functional $\delta_Z(z)$ of Z that

$$X(t) = X(t, Z) = X(t, z)_{z=Z} = \int_{\mathbb{R}} X(t, z)\delta_Z(z)dz, \tag{2.6}$$

for some z -parameterized process $X(t, z)$ which is \mathbb{F} -adapted for each z .

Then, again by the definition of the Donsker delta functional we can write, for $0 \leq t \leq T$

$$\begin{aligned}
 X(t) &= \xi(0) + \int_0^t b(s, X(s), Y(s), u(s, Z), Z)ds + \int_0^t \sigma(s, X(s), Y(s), u(s, Z), Z)dB(s) \\
 &+ \int_0^t \int_{\mathbb{R}} \gamma(s, X(s), Y(s), u(s, Z), Z, \zeta) \tilde{N}(ds, d\zeta) \\
 &= \int_{\mathbb{R}} \left\{ \xi(0) + \int_0^t b(s, X(s, z), Y(s, z), u(s, z), z)ds + \int_0^t \sigma(s, X(s, z), Y(s, z), u(s, z), z)dB(s) \right. \\
 &\left. + \int_0^t \int_{\mathbb{R}} \gamma(s, X(s, z), Y(s, z), u(s, z), z, \zeta) \tilde{N}(ds, d\zeta) \right\} \delta_Z(z) dz. \tag{2.7}
 \end{aligned}$$

Comparing (2.6) and (2.7) we see that (2.6) holds if we for each z choose $X(t, z)$ as the solution of the classical (but parameterized) SDDE

$$\begin{cases} dX(t, z) = b(t, X(t, z), Y(t, z), u(t, z), z)dt + \sigma(t, X(t, z), Y(t, z), u(t, z), z)dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t, z), Y(t, z), u(t, z), z, \zeta) \tilde{N}(dt, d\zeta); & t \in [0, T] \\ X(t, z) = \xi(t); & t \in [-\delta, 0]. \end{cases} \tag{2.8}$$

For results about existence and uniqueness of solutions of SDDE we refer to [11] and the references therein.

As before let \mathcal{A} be the given family of admissible \mathbb{H} -adapted controls u . Then in terms of $X(t, z)$ the performance functional $J(u)$ of a control process $u \in \mathcal{A}$ gets the form

$$\begin{aligned}
 J(u) &= \mathbb{E} \left[\int_0^T f(t, X(t, Z), u(t, Z), Z)dt + g(X(T, Z), Z) \right] \\
 &= \mathbb{E} \left[\int_0^T \left(\int_{\mathbb{R}} f(t, X(t, z), u(t, z), z) \mathbb{E}[\delta_Z(z) | \mathcal{F}_t] dz \right) dt + \int_{\mathbb{R}} g(X(T, z), z) \mathbb{E}[\delta_Z(z) | \mathcal{F}_T] dz \right] \\
 &= \int_{\mathbb{R}} j(u)(z) dz, \tag{2.9}
 \end{aligned}$$

where

$$\begin{aligned}
 j(u)(z) &:= \mathbb{E} \left[\int_0^T f(t, X(t, z), u(t, z), z) \mathbb{E}[\delta_Z(z) | \mathcal{F}_t] dt \right. \\
 &\left. + g(X(T, z), z) \mathbb{E}[\delta_Z(z) | \mathcal{F}_T] \right]. \tag{2.10}
 \end{aligned}$$

Thus we see that to maximize $J(u)$ it suffices to maximize $j(u)(z)$ for each value of the parameter $z \in \mathbb{R}$. Therefore Problem 1.1 is transformed into the problem

Problem 2.3. For each given $z \in \mathbb{R}$ find $u^* = u^*(t, z) \in \mathcal{A}$ such that

$$\sup_{u \in \mathcal{A}} j(u)(z) = j(u^*)(z). \tag{2.11}$$

2.3. A sufficient-type maximum principle

In this section we will establish a sufficient maximum principle for Problem 2.3.

Problem 2.3 is a stochastic control problem with a standard (parameterized) stochastic delay differential equation (2.8) for the state process $X(t, z)$, but with a non-standard performance functional given by (2.10). We can solve this problem by a modified maximum principle approach, as follows:

Define the *Hamiltonian* $H : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \rightarrow \mathbb{R}$ by

$$\begin{aligned} H(t, x, y, u, z, p, q, r) &= H(t, x, y, u, z, p, q, r, \omega) \\ &= \mathbb{E}[\delta_Z(z) | \mathcal{F}_t] f(t, x, u, z) + b(t, x, y, u, z) p \\ &\quad + \sigma(t, x, y, u, z) q + \int_{\mathbb{R}} \gamma(t, x, y, u, z, \zeta) r(\zeta) \nu(d\zeta). \end{aligned} \tag{2.12}$$

\mathcal{R} denotes the set of all functions $r(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ such that the last integral above converges. The quantities $p, q, r(\cdot)$ are called the *adjoint variables*. The *adjoint processes* $p(t, z), q(t, z), r(t, z, \zeta)$ are defined as the solution of the z -parameterized advanced backward stochastic differential equation (ABSDE)

$$\begin{cases} dp(t, z) = \mathbb{E}[\mu(t, z) | \mathcal{F}_t] dt + q(t, z) dB(t) + \int_{\mathbb{R}} r(t, z, \zeta) \tilde{N}(dt, d\zeta), & t \in [0, T] \\ p(T, z) = \frac{\partial g}{\partial x}(X(T, z)) \mathbb{E}[\delta_Z(z) | \mathcal{F}_T], \end{cases} \tag{2.13}$$

where

$$\begin{aligned} \mu(t, z) &= -\frac{\partial H}{\partial x}(t, X(t, z), Y(t, z), u(t, z), p(t, z), q(t, z), r(t, z, \cdot)) \\ &\quad - \frac{\partial H}{\partial y}(t + \delta, X(t + \delta, z), Y(t + \delta, z), u(t + \delta, z), p(t + \delta, z), q(t + \delta, z), r(t + \delta, z, \cdot)) \mathbf{1}_{[0, T-\delta]}(t). \end{aligned} \tag{2.14}$$

Let us now introduce a brief recall for the existence and uniqueness of the solution of a time-advanced BSDE. For more details see [13].

Given a positive constant δ , denote by $D([0, \delta], \mathbb{R})$ the space of all càdlàg paths from $[0, \delta]$ into \mathbb{R} . For a path $X(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$, X_t will denote the function defined by $X_t(s) = X(t + s)$ for $s \in [0, \delta]$. Set $\mathcal{H} = L^2(\nu)$.

Let $F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{H} \times \mathcal{H} \times \Omega \rightarrow \mathbb{R}$ be a predictable function. Introduce the following Lipschitz condition: There exists a constant C such that

$$\begin{aligned} &|F(t, p_1, p_2, q_1, q_2, r_1, r_2, \omega) - F(t, \bar{p}_1, \bar{p}_2, \bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2, \omega)| \\ &\leq C(|p_1 - \bar{p}_1| + |p_2 - \bar{p}_2| + |q_1 - \bar{q}_1| + |q_2 - \bar{q}_2| + |r_1 - \bar{r}_1|_{L^2(\nu)} + |r_2 - \bar{r}_2|_{L^2(\nu)}), \end{aligned} \tag{2.15}$$

where

$$|r|_{L^2(\nu)} = \left(\int_{\mathbb{R}} r^2(\zeta) \nu(d\zeta) \right)^{\frac{1}{2}}. \tag{2.16}$$

Consider the following time-advanced BSDE in the unknown \mathcal{F}_t adapted process $(p(t), q(t), r(t, \zeta))$

$$\begin{cases} dp(t) = E[F(t, p(t), p(t + \delta)1_{[0, T-\delta]}, q(t), q(t + \delta)1_{[0, T-\delta]}, r(t), r(t + \delta)1_{[0, T-\delta]}) | \mathcal{F}_t] dt \\ + q(t)dB(t) + \int_{\mathbb{R}} r(t, \zeta)\tilde{N}(dt, d\zeta), t \in [0, T] \\ p(T) = G, \end{cases} \tag{2.17}$$

where G is a given \mathcal{F}_T -measurable random variable such that $E[G^2] < \infty$.

Note that the time-advanced BSDE for the adjoint processes of the Hamiltonian is of this form. For this type of time advanced BSDEs, we have the following result

Theorem 2.4. [13] *Assume that condition (2.15) is satisfied. Then the BSDE (2.17) has a unique solution $(p(t), q(t), r(t, \zeta))$ such that*

$$E\left[\int_0^T \{p^2(t) + q^2(t) + \int_{\mathbb{R}} r^2(t, \zeta)\nu(d\zeta)\}dt\right] < \infty. \tag{2.18}$$

Moreover, the solution can be found by inductively solving a sequence of BSDEs backwards as follows.

Step 0. In the interval $[T - \delta, T]$ we let $p(t), q(t)$, and $r(t, \zeta)$ be defined as the solution of the classical BSDE

$$\begin{cases} dp(t) = F(t, p(t), 0, q(t), 0, r(t, \zeta), 0)dt + q(t)dB(t) + \int_{\mathbb{R}} r(t, \zeta)\tilde{N}(dt, d\zeta), t \in [T - \delta, T], \\ p(T) = G. \end{cases} \tag{2.19}$$

Step k, k ≥ 1. If the values of $(p(t), q(t), r(t, \zeta))$ have been found for $t \in [T - k\delta, T - (k - 1)\delta]$ then, if $t \in [T - (k + 1)\delta, T - k\delta]$, the values of $p(t + \delta), q(t + \delta)$ and $r(t + \delta, \zeta)$ are known and, hence, the BSDE

$$\begin{cases} dp(t) = E[F(t, p(t), p(t + \delta), q(t), q(t + \delta), r(t), r(t + \delta)) | \mathcal{F}_t] dt \\ + q(t)dB(t) + \int_{\mathbb{R}} r(t, \zeta)\tilde{N}(dt, d\zeta); t \in [T - (k + 1)\delta, T - k\delta], \\ p(T - k\delta) = \text{the value found in step } k - 1, \end{cases} \tag{2.20}$$

has a unique solution in $[T - (k + 1)\delta, T - k\delta]$. We proceed like this until k is such that $T - (k + 1)\delta \leq 0 < T - k\delta$ and then we solve the corresponding BSDE on the interval $[0, T - k\delta]$.

Let us now give some conditions on b, σ and γ in order to ensure that a unique solution of the advanced BSDE of the adjoint process (2.17) exists. In this case we have:

$$\begin{aligned} F(t, p_1, p_2, q_1, q_2, r_1, r_2, \omega) &= -\frac{\partial H}{\partial x}(t, X(t, z), Y(t, z), u(t, z), p_1, q_1, r_1) \\ &- \frac{\partial H}{\partial y}(t + \delta, X(t + \delta, z), Y(t + \delta, z), u(t + \delta, z), p_2, q_2, r_2)\mathbf{1}_{[0, T-\delta]}(t) \\ &= -\mathbb{E}[\delta_Z(z) | \mathcal{F}_t] \frac{\partial f}{\partial x}(t, X(t, z), u(t, z), z) - \frac{\partial b}{\partial x}(t, X(t, z), Y(t, z), u(t, z), z)p_1 \\ &- \frac{\partial \sigma}{\partial x}(t, X(t, z), Y(t, z), u(t, z), z)q_1 - \int_{\mathbb{R}} \frac{\partial \gamma}{\partial x}(t, X(t, z), Y(t, z), u(t, z), z, \zeta)r_1(\zeta)\nu(d\zeta) \\ &- \left\{ \frac{\partial b}{\partial y}(t, X(t + \delta, z), Y(t + \delta, z), u(t + \delta, z), z)p_2 + \frac{\partial \sigma}{\partial x}(t + \delta, X(t + \delta, z), Y(t + \delta, z), u(t + \delta, z), z)q_2 \right. \end{aligned}$$

$$+ \int_{\mathbb{R}} \frac{\partial \gamma}{\partial x}(t + \delta, X(t + \delta, z), Y(t + \delta, z), u(t + \delta, z), z, \zeta) r_2(\zeta) \nu(d\zeta) \mathbf{1}_{[0, T-\delta]}(t). \tag{2.21}$$

Now let us verify condition (2.15):

$$\begin{aligned} & |F(t, p_1, p_2, q_1, q_2, r_1, r_2, \omega) - F(t, \bar{p}_1, \bar{p}_2, \bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2, \omega)| \\ & \leq \left| \frac{\partial b}{\partial x}(t, X(t, z), Y(t, z), u(t, z), z) \right| |p_1 - \bar{p}_1| \\ & + \left| \frac{\partial \sigma}{\partial x}(t, X(t, z), Y(t, z), u(t, z), z) \right| |q_1 - \bar{q}_1| \\ & + \int_{\mathbb{R}} \left| \frac{\partial \gamma}{\partial x}(t, X(t, z), Y(t, z), u(t, z), z) \right| |r_1(\zeta) - \bar{r}_1(\zeta)| \nu(d\zeta) \\ & + \left| \frac{\partial b}{\partial y}(t, X(t, z), Y(t, z), u(t, z), z) \right| |p_2 - \bar{p}_2| \\ & + \left| \frac{\partial \sigma}{\partial y}(t, X(t, z), Y(t, z), u(t, z), z) \right| |q_2 - \bar{q}_2| \\ & + \int_{\mathbb{R}} \left| \frac{\partial \gamma}{\partial y}(t, X(t, z), Y(t, z), u(t, z), z) \right| |r_2(\zeta) - \bar{r}_2(\zeta)| \nu(d\zeta). \end{aligned} \tag{2.22}$$

To guarantee that the Lipschitz condition is verified we assume that the derivatives of b , σ and γ with respect to x and y are bounded.

We can now state the first maximum principle for our problem (2.11):

Theorem 2.5 (*Sufficient-type maximum principle*). *Let $\hat{u} \in \mathcal{A}$, and denote the associated solution of (2.8) and (2.13) by $\hat{X}(t, z)$ and $(\hat{p}(t, z), \hat{q}(t, z), \hat{r}(t, z, \zeta))$, respectively. Assume that the following hold:*

1. $x \rightarrow g(x, z)$ is concave for all z ,
2. $(x, y, u) \rightarrow H(t, x, y, u, z, \hat{p}(t, z), \hat{q}(t, z), \hat{r}(t, z, \zeta))$ is concave for all t, z, ζ ,
3. $\sup_{w \in \mathbb{U}} H(t, \hat{X}(t, z), \hat{Y}(t, z), w, \hat{p}(t, z), \hat{q}(t, z), \hat{r}(t, z, \zeta)) = H(t, \hat{X}(t, z), \hat{Y}(t, z), \hat{u}(t, z), \hat{p}(t, z), \hat{q}(t, z), \hat{r}(t, z, \zeta))$ for all t, z, ζ .

Then $\hat{u}(\cdot, z)$ is an optimal control for Problem 2.3.

Proof. We use the same techniques as the proof in [6] taking into account the new terms coming from the delay aspect. \square

2.4. A necessary-type maximum principle

In some cases the concavity conditions of Theorem 2.5 do not hold. In such situations a corresponding necessary-type maximum principle can be useful. For this, instead of the concavity conditions we need the following assumptions about the set of admissible control values:

- A_1 . For all $t_0 \in [0, T]$ and all bounded \mathcal{F}_{t_0} -measurable random variables $\alpha(z, \omega)$, the control $\theta(t, z, \omega) := \mathbf{1}_{[t_0, T]}(t) \alpha(z, \omega)$ belongs to \mathcal{A} .
- For all $u; \beta_0 \in \mathcal{A}$ with $\beta_0(t, z) \leq K < \infty$ for all t, z define

$$\delta(t, z) = \frac{1}{2K} \text{dist}((u(t, z), \partial \mathbb{U}) \wedge 1 > 0, \tag{2.23}$$

and put

$$\beta(t, z) = \delta(t, z)\beta_0(t, z). \tag{2.24}$$

Then the control

$$\tilde{u}(t, z) = u(t, z) + a\beta(t, z); \quad t \in [0, T]$$

belongs to \mathcal{A} for all $a \in (-1, 1)$.

- A3. For all β as in (2.24) the derivative process

$$\chi(t, z) := \frac{d}{da} X^{u+a\beta}(t, z)|_{a=0}$$

exists, and belongs to $\mathbf{L}^2(\lambda \times \mathbf{P})$ and

$$\begin{cases} d\chi(t, z) = [\frac{\partial b}{\partial x}(t, z)\chi(t, z) + \frac{\partial b}{\partial y}(t, z)\chi(t - \delta, z) + \frac{\partial b}{\partial u}(t, z)\beta(t, z)]dt \\ + [\frac{\partial \sigma}{\partial x}(t, z)\chi(t, z) + \frac{\partial \sigma}{\partial y}(t, z)\chi(t - \delta, z) + \frac{\partial \sigma}{\partial u}(t, z)\beta(t, z)]dB(t) \\ + \int_{\mathbb{R}} [\frac{\partial \gamma}{\partial x}(t, z, \zeta)\chi(t, z) + \frac{\partial \gamma}{\partial y}(t, z, \zeta)\chi(t - \delta, z) + \frac{\partial \gamma}{\partial u}(t, z, \zeta)\beta(t, z)]\tilde{N}(dt, d\zeta), \\ \chi(t, z) = 0 \quad \forall t \in [-\delta, 0]. \end{cases} \tag{2.25}$$

Theorem 2.6 (Necessary maximum principle). *Let $\hat{u} \in \mathcal{A}$. Then the following are equivalent:*

1. $\frac{d}{da} J((\hat{u} + a\beta)(\cdot, z))|_{a=0} = 0$ for all bounded $\beta \in \mathcal{A}$ of the form (2.24).
2. $\frac{\partial H}{\partial u}(t, z)_{u=\hat{u}} = 0$ for all $t \in [0, T]$.

Proof. We refer to [6] for the proof’s technique with considering the delay framework. \square

3. Optimal inside harvesting in a population modelled by a delay equation

Let us consider a population growth example. We denote by $X(t, Z)$ a single population at time t developing by a constant birth rate $\beta > 0$ and a constant death rate $\alpha > 0$ per inhabitant. Z here is an inside information about the future environment for example coming from global warming. In this model we take off immediately the dead from the population. We denote by the constant $r > 0$ the development period of each person ($r = 9$ months for example). A migration movement happens in this population and we assume that the global rate of the migration is distributed as a white noise $\sigma \dot{B}$. We denote by $u(t, Z)$ the harvesting rate of the population. The population change is given by the following SDDE:

$$\begin{cases} dX(t, Z) = (-\alpha X(t, Z) + \beta X(t - r, Z) - u(t, Z))dt + \sigma dB(t), \quad t \in [0, T] \\ X(s) = \eta(s), \quad -r \leq s \leq 0. \end{cases} \tag{3.1}$$

Here η is a deterministic function.

We denote by J the performance functional given by

$$J(u) = E[\int_0^T e^{-\rho t} \frac{1}{\gamma} u^\gamma(t, Z) dt + \theta X(T, Z)], \tag{3.2}$$

where θ is an \mathcal{F}_T -measurable strictly positive bounded random variable, $\rho > 0$ and $\gamma \in (0, 1)$.

Transforming the delayed stochastic control problem (3.1)-(3.2) into a z -parameterized \mathbb{F} -adapted delayed stochastic control problem we get the following z -parameterized SDDE:

$$\begin{cases} dX(t, z) = (-\alpha X(t, z) + \beta X(t - r, z) - u(t, z))dt + \sigma dB(t), & t \in [0, T] \\ X(s) = \eta(s), & -r \leq s \leq 0. \end{cases} \tag{3.3}$$

Let \mathcal{A} be the set of admissible controls, we require that $u(t, z) > 0$, that $E[\int_0^T u^2(t)dt] < \infty$.

Hypotheses (E_1) in [11] is easily verified in this case: since (3.3) is a linear SDDE with constant coefficient α and β then the Lipschitz condition is verified. Also we have $Y(t, z) = X(t - r, z)$ is \mathcal{F}_t -measurable then the drift function is \mathcal{F}_t -measurable. Therefore we get from Theorem I.1 in [11] the existence and uniqueness of the solution of the parameterized SDDE (3.3) such that $X(t, z)$ in $L^2(\Omega \times [0, T])$ for each z .

The transformed performance functional J is given by

$$J(u) = E\left[\int_0^T e^{-\rho t} \frac{1}{\gamma} u^\gamma(t, z) E[\delta_Z(z)|\mathcal{F}_t] dt + \theta X(T, z) E[\delta_Z(z)|\mathcal{F}_T]\right]. \tag{3.4}$$

In this case the Hamiltonian is given by

$$H(t, x, y, u, z, p, q) = E[\delta_Z(z)|\mathcal{F}_t] e^{-\rho t} \frac{1}{\gamma} u^\gamma(t, z) + (-\alpha x + \beta y - u)p + \sigma q. \tag{3.5}$$

We have

$$\frac{\partial H}{\partial x}(t, x, y, u, z, p, q) = -\alpha p, \quad \frac{\partial H}{\partial y}(t, x, y, u, z, p, q) = \beta p, \tag{3.6}$$

it follows that

$$\mu(t, z) = \alpha p(t, z) - \beta p(t + r, z) 1_{[0, T-r]}(t). \tag{3.7}$$

Therefore the advanced BSDE verified by the adjoint processes $(p(t), q(t))$ is given by

$$\begin{cases} dp(t, z) = (\alpha p(t, z) - \beta E[p(t + r, z) 1_{[0, T-r]}|\mathcal{F}_t])dt + q(t, z)dB(t), & t \in [0, T] \\ p(T, z) = \theta E[\delta_Z(z)|\mathcal{F}_T]. \end{cases} \tag{3.8}$$

Assume that $E[E[\delta_Z(z)|\mathcal{F}_T]^2] < \infty$. Equation (3.8) is a linear advanced BSDE then it is easy to verify that the solution exists and it is unique. We solve this BSDE recursively:

Step 1: If $t \in [T - r, T]$, the BSDE gets the form:

$$\begin{cases} dp(t, z) = \alpha p(t, z)dt + q(t, z)dB(t), & T - r \leq t \leq T \\ p(T, z) = \theta E[\delta_Z(z)|\mathcal{F}_T]. \end{cases} \tag{3.9}$$

This is a linear BSDE where the solution is given by

$$p(t, z) = e^{\alpha(t-T)} E[\theta E[\delta_Z(z)|\mathcal{F}_T]|\mathcal{F}_t], \quad t \in [T - r, T] \tag{3.10}$$

with corresponding $q(t, z)$ given by the martingale representation theorem.

Note that this solution is strictly positive since θ is a strictly positive bounded random variable and $E[\delta_Z(z)|\mathcal{F}_T]$ is strictly positive. Then $p(t, z)$ is strictly positive for all $t \in [T - r, T]$.

Step 2: If $t \in [T - 2r, T - r]$ and $T - 2r > 0$, then we get the BSDE:

$$\begin{cases} dp(t, z) = (\alpha p(t, z) - \beta E[p(t + r, z)|\mathcal{F}_t])dt + q(t, z)dB(t) \\ p(T - r, z) \text{ is known from step 1.} \end{cases} \tag{3.11}$$

We have also $p(t + r, z)$ is known from step 1. So this is a simple BSDE which can be solved for $p(t, z)$ and $q(t, z)$ for $t \in [T - 2r, T - r]$.

The solution of this BSDE is given by

$$p(t, z) = e^{\alpha t} E[p(T - r)e^{-\alpha(T-r)} + \beta \int_t^{T-r} e^{-\alpha s} E[p(s + r, z)|\mathcal{F}_s]ds|\mathcal{F}_t], \quad t \in [T - 2r, T - r] \tag{3.12}$$

Note that this solution is strictly positive for all $t \in [T - 2r, T - r]$.

We continue like this by induction until a step j where j satisfies $T - jr \leq 0 < T - (j - 1)r$. With this method we end up with a solution $p(t, z)$ of the BSDE (3.8).

The Hamiltonian H can have a finite maximum over all u only if

$$\frac{\partial H}{\partial u}(t) = E[\delta_Z(z)|\mathcal{F}_t]e^{-\rho t}u^{\gamma-1}(t, z) - p(t, z) = 0. \tag{3.13}$$

Then

$$\hat{u}(t, z) = \frac{e^{\frac{\rho t}{\gamma-1}}}{(E[\delta_Z(z)|\mathcal{F}_t])^{\frac{1}{\gamma-1}}} (\hat{p}(t, z))^{\frac{1}{\gamma-1}}. \tag{3.14}$$

Let us now verify the admissibility condition of $\hat{u}(t, z)$.

- The solution $p(t, z)$ of equation (3.8) is strictly positive since θ is a strictly positive bounded random variable and $E[\delta_Z(z)|\mathcal{F}_t] > 0$. Then $\hat{u}(t, z)$ in (3.14) is strictly positive.
- We will prove that $E[\int_0^T \hat{u}^2(t)dt] < \infty$ by steps.

Since θ is bounded positive random variable being away from 0 i.e. there exist $\theta_1, \theta_2 \in \mathbb{R}_+$ such that

$$\theta_1 \leq \theta(\omega) \leq \theta_2, \tag{3.15}$$

we have

- i) For $t \in [T - r, T]$ we have by (3.10)

$$p(t, z) \geq e^{\alpha(t-T)}\theta_1 E[\delta_Z(z)|\mathcal{F}_t]. \tag{3.16}$$

- ii) Then for $t \in [T - 2r, T - r]$ we have by (3.12) and (i)

$$p(t, z) \geq e^{-\alpha(T-r-t)} E[p(T - r)|\mathcal{F}_t] \geq e^{-\alpha(T-t)}\theta_1 E[E[\delta_Z(z)|\mathcal{F}_{T-r}|\mathcal{F}_t]] \geq e^{-\alpha(T-t)}\theta_1 E[\delta_Z(z)|\mathcal{F}_t]. \tag{3.17}$$

- iii) Proceeding like this by induction we end up with

$$p(t, z) \geq e^{-\alpha(T-t)}\theta_1 E[\delta_Z(z)|\mathcal{F}_t], \forall t \in [0, T]. \tag{3.18}$$

We conclude that

$$E\left[\int_0^T u^2(t)dt\right] \leq e^{-\frac{2\alpha(T-t)}{\gamma-1}} \theta_1^{\frac{2}{\gamma-1}} E\left[\int_0^T E[\delta_Z(z)|\mathcal{F}_t]^{2/(1-\gamma)} E[\delta_Z(z)|\mathcal{F}_t]^{2/(\gamma-1)} dt\right] = e^{-\frac{2\alpha(T-t)}{\gamma-1}} \theta_1^{\frac{2}{\gamma-1}} T. \tag{3.19}$$

Theorem 3.1. *Assume that $\gamma \in (0, 1)$ and $E[E[\delta_Z(z)|\mathcal{F}_T]^2] < \infty$. The harvesting control in the population growth example given by (3.1)-(3.2) is given by*

$$\hat{u}(t, Z) = \frac{e^{\frac{pt}{\gamma-1}}}{(E[\delta_Z(z)|\mathcal{F}_t]_{z=Z})^{\frac{1}{\gamma-1}}} (\hat{p}(t, Z))^{\frac{1}{\gamma-1}}, \tag{3.20}$$

where $p(t, z)$ is the solution of the advanced BSDE (3.8).

4. A direct white noise method for optimal insider portfolio in a financial market with delay

Consider the following SDDE:

$$\begin{cases} dX(t, Z) = X(t, Z)\pi(t, Z)[b(t)\theta(t, Z)dt + \sigma(t)\theta(t, Z)dB(t)], t \in [0, T] \\ X(t) = \xi(t), t \in [-r, 0]. \end{cases} \tag{4.1}$$

Here b, σ and ξ are deterministic bounded functions with $\xi(0) = 1$ and $\theta(t, Z) = \frac{X(t-r, Z)}{X(t, Z)} 1_{t < \tau_0}$ where

$$\tau_0 = \inf\{t > 0, X(t, Z) = 0\}. \tag{4.2}$$

The process π is our control process and assumed to be \mathbb{H} adapted.

Note that equation (4.1) is well defined since B is a semimartingale under the filtration \mathbb{H} as discussed in Section 2. Since we are working via Hida-Malliavin calculus we will interpret it using forward integral definition. Here a brief recall of the definition of forward integral:

Definition 4.1. We say that a stochastic process $\phi = \phi(t), t \in [0, T]$, is forward integrable (in the weak sense) over the interval $[0, T]$ with respect to B if there exists a process $I = I(t), t \in [0, T]$, such that

$$\sup_{t \in [0, T]} \left| \int_0^t \phi(s) \frac{B(s+\epsilon) - B(s)}{\epsilon} ds - I(t) \right| \rightarrow 0, \quad \epsilon \rightarrow 0^+ \tag{4.3}$$

in probability.

In this case we write $I(t) := \int_0^t \phi(s) d^-B(s), t \in [0, T]$, and call $I(t)$ the forward integral of ϕ with respect to B on $[0, t]$.

For π to be admissible in the set \mathcal{A} of admissible \mathbb{H} adapted controls, we require that

$$E\left[\int_0^{T \wedge \tau_0} \frac{X^2(s-r, Z)}{X^2(s, Z)} \pi^2(s, Z) ds\right] < \infty. \tag{4.4}$$

Condition (4.4) guarantees the existence and uniqueness of the solution of (4.1).

The solution of this SDDE is given by

$$\begin{aligned}
 X(t, Z) = & \exp\left(\int_0^{t \wedge \tau_0} \left(b(s) \frac{X(s-r, Z)}{X(s, Z)} \pi(s, Z) - \frac{1}{2} \sigma^2(s) \frac{X^2(s-r, Z)}{X^2(s, Z)} \pi^2(s, Z)\right) ds \right. \\
 & \left. + \int_0^{t \wedge \tau_0} \sigma(s) \frac{X(s-r, Z)}{X(s, Z)} \pi(s, Z) dB(s)\right), t \in [0, T].
 \end{aligned}
 \tag{4.5}$$

Problem. Our goal is to find $\pi^* \in \mathcal{A}$ which maximizes the following expected utility logarithmic function

$$\sup_{\pi \in \mathcal{A}} E\left[\ln(X^\pi(T \wedge \tau_0, Z))\right] = E\left[\ln(X^{\hat{\pi}}(T \wedge \tau_0, Z))\right].
 \tag{4.6}$$

We have

$$\begin{aligned}
 E\left[\ln(X^\pi(T \wedge \tau_0, Z))\right] &= E\left[\int_0^{T \wedge \tau_0} \left(b(s) \frac{X(s-r, Z)}{X(s, Z)} \pi(s, Z) - \frac{1}{2} \sigma^2(s) \frac{X^2(s-r, Z)}{X^2(s, Z)} \pi^2(s, Z)\right) \right. \\
 &+ \left. \sigma(s) E\left[D_s\left(\frac{X(s-r, Z)}{X(s, Z)} \pi(s, Z)\right) | \mathcal{F}_s\right] ds\right] \\
 &= E\left[\int_0^{T \wedge \tau_0} E\left[\left(b(s) \frac{X(s-r, Z)}{X(s, Z)} \pi(s, Z) - \frac{1}{2} \sigma^2(s) \frac{X^2(s-r, Z)}{X^2(s, Z)} \pi^2(s, Z)\right) \right. \right. \\
 &+ \left. \left. \sigma(s) D_s\left(\frac{X(s-r, Z)}{X(s, Z)} \pi(s, Z)\right) | \mathcal{F}_s\right] ds\right].
 \end{aligned}
 \tag{4.7}$$

Here and in the following we use the notation

$$D_s \pi(s) := D_{s+} \pi(s) := \lim_{t \rightarrow s^+} D_t \pi(s)
 \tag{4.8}$$

and the following result (see [5]):

Let ϕ be a càglàd and forward integrable process in $L^2([0, T] \times \Omega)$ then

$$E\left[\int_0^T \phi(s) d^- B(s)\right] = E\left[\int_0^T E[D_{s+} \phi(s) | \mathcal{F}_s] ds\right].
 \tag{4.9}$$

Let

$$J(\pi) = E\left[\left(b(s) \frac{X(s-r, Z)}{X(s, Z)} \pi(s, Z) - \frac{1}{2} \sigma^2(s) \frac{X^2(s-r, Z)}{X^2(s, Z)} \pi^2(s, Z) + \sigma(s) D_s\left(\frac{X(s-r, Z)}{X(s, Z)} \pi(s, Z)\right)\right) | \mathcal{F}_s\right].
 \tag{4.10}$$

Applying the Donsker delta functional to the previous equation we get

$$\begin{aligned}
 J(\pi) = & \int_{\mathbb{R}} \left(b(s) \frac{X(s-r, z)}{X(s, z)} \pi(s, z) E[\delta_Z(z) | \mathcal{F}_s] - \frac{1}{2} \sigma^2(s) \frac{X^2(s-r, z)}{X^2(s, z)} \pi^2(s, z) E[\delta_Z(z) | \mathcal{F}_s] \right. \\
 & \left. + \sigma(s) \frac{X(s-r, z)}{X(s, z)} \pi(s, z) E[D_s \delta_Z(z) | \mathcal{F}_s]\right) dz.
 \end{aligned}
 \tag{4.11}$$

We can maximize this over $\pi(s, z)$ for each s, z . Then we get that

$$\hat{\pi}(s, z) = \frac{X(s, z)}{\sigma(s)X(s-r, z)} \frac{E[D_s \delta_Z(z) | \mathcal{F}_s]}{E[\delta_Z(z) | \mathcal{F}_s]} + \frac{b(s)}{\sigma^2(s)} \frac{X(s, z)}{X(s-r, z)}, s \in [0, T \wedge \tau_0]. \quad (4.12)$$

Therefore we get

$$\begin{aligned} \hat{\pi}(s, Z) &= \frac{X(s, Z)}{\sigma(s)X(s-r, Z)} \frac{E[D_s \delta_Z(z) | \mathcal{F}_s]_{z=Z}}{E[\delta_Z(z) | \mathcal{F}_s]_{z=Z}} + \frac{b(s)}{\sigma^2(s)} \frac{X(s, Z)}{X(s-r, Z)}, s \in [0, T \wedge \tau_0] \\ &= \frac{X(s, Z)}{\sigma(s)X(s-r, Z)} \Phi(s, Z) + \frac{b(s)}{\sigma^2(s)} \frac{X(s, Z)}{X(s-r, Z)}, s \in [0, T \wedge \tau_0], \end{aligned} \quad (4.13)$$

where

$$\Phi(s, Z) = \frac{E[D_s \delta_Z(z) | \mathcal{F}_s]_{z=Z}}{E[\delta_Z(z) | \mathcal{F}_s]_{z=Z}}. \quad (4.14)$$

Replacing the expression of $\hat{\pi}$ in $\ln(X^\pi(T \wedge \tau_0, Z))$ we get

$$\begin{aligned} \ln(X^{\hat{\pi}}(T \wedge \tau_0, Z)) &= \int_0^{T \wedge \tau_0} \left(b(s) \left(\frac{\Phi(s, Z)}{\sigma(s)} + \frac{b(s)}{\sigma^2(s)} \right) - \frac{1}{2} \sigma^2(s) \left(\frac{\Phi(s, Z)}{\sigma(s)} + \frac{b(s)}{\sigma^2(s)} \right)^2 \right) ds \\ &\quad + \int_0^{T \wedge \tau_0} \sigma(s) \left(\frac{\Phi(s, Z)}{\sigma(s)} + \frac{b(s)}{\sigma^2(s)} \right) dB(s) > -\infty, \end{aligned} \quad (4.15)$$

and hence $X^{\hat{\pi}}(T \wedge \tau_0, Z) > 0$ a.s.

This is only possible if $\tau_0 > T$ a.s., which means that $X^{\hat{\pi}}(t) > 0$ for all $t \in [0, T]$ a.s. and our optimal indeed in the whole interval $[0, T]$. Let us now verify the admissibility condition:

$$E \left[\int_0^T \frac{X^2(s-r, Z)}{X^2(s, Z)} \hat{\pi}^2(s, Z) ds \right] = E \left[\int_0^T \left(\frac{E[D_s \delta_Z(z) | \mathcal{F}_s]_{z=Z}}{E[\delta_Z(z) | \mathcal{F}_s]_{z=Z}} + \frac{b(s)}{\sigma^2(s)} \right)^2 ds \right]. \quad (4.16)$$

This previous quantity is finite if we suppose that

$$E \left[\int_0^T \left(\frac{E[D_s \delta_Z(z) | \mathcal{F}_s]_{z=Z}}{E[\delta_Z(z) | \mathcal{F}_s]_{z=Z}} \right)^2 ds \right] < \infty. \quad (4.17)$$

We summarise what we have proved in the following Theorem:

Theorem 4.2. *Assume that*

$$E \left[\int_0^T \left(\frac{E[D_s \delta_Z(z) | \mathcal{F}_s]_{z=Z}}{E[\delta_Z(z) | \mathcal{F}_s]_{z=Z}} \right)^2 ds \right] < \infty. \quad (4.18)$$

Then the optimal portfolio $\hat{\pi}$ with respect to the logarithmic utility for an insider in the delay market (4.1) and with inside information (1.3) is given by

$$\hat{\pi}(s, Z) = \frac{X(s, Z)}{\sigma(s)X(s-r, Z)} \frac{E[D_s \delta_Z(z) | \mathcal{F}_s]_{z=Z}}{E[\delta_Z(z) | \mathcal{F}_s]_{z=Z}} + \frac{b(s)}{\sigma^2(s)} \frac{X(s, Z)}{X(s-r, Z)}, s \in [0, T] \quad (4.19)$$

In the following Corollary we treat a particular case:

Corollary 4.3. *Suppose that $Z = B(T_0)$, where $T_0 > T$. In this case we have*

$$\frac{E[D_s \delta_Z(z) | \mathcal{F}_s]_{z=Z}}{E[\delta_Z(z) | \mathcal{F}_s]_{z=Z}} = \frac{B(T_0) - B(s)}{T_0 - s} \tag{4.20}$$

(see [6] and [7]), and

$$E\left[\int_0^T \left(\frac{B(T_0) - B(s)}{T_0 - s}\right)^2 ds\right] = \int_0^T \frac{ds}{T_0 - s} < \infty \text{ since } T_0 > T. \tag{4.21}$$

Then

$$\hat{\pi}(s, B(T_0)) = \frac{X(s, B(T_0))}{\sigma(s)X(s-r, B(T_0))} \frac{B(T_0) - B(s)}{T_0 - s} + \frac{b(s)}{\sigma^2(s)} \frac{X(s, B(T_0))}{X(s-r, B(T_0))}, s \in [0, T]. \tag{4.22}$$

5. Viability of a market with delay

In this section we study the viability of the financial market (4.1)-(4.6). For this we define what is a viable market:

Definition 5.1. The market (4.1)-(4.6) is called viable if

$$\sup_{\pi \in \mathcal{A}} E[\ln X^\pi(T)] < \infty. \tag{5.1}$$

In the no delay case ($r = 0$) we have that

$$\hat{\pi}(s, B(T_0)) = \frac{B(T_0) - B(s)}{\sigma(s)(T_0 - s)} + \frac{b(s)}{\sigma^2(s)}, s \in [0, T]. \tag{5.2}$$

In this case we know from [14] that for $T_0 = T$, $E[\ln(X^{\hat{\pi}}(T, B(T)))]$ is infinite. In the case of delay for the market (4.1), we have the following result:

Theorem 5.2. *Suppose the market is given by (4.1)-(4.6) and, as before, the insider has at any time $t \in [0, T]$ access to the information \mathcal{H}_t given by (1.3). Then the market is not viable, i.e. we have*

$$\sup_{\pi \in \mathcal{A}} E[\ln(X^\pi(T, B(T)))] = \infty. \tag{5.3}$$

Proof. In fact, we can show that for $T_0 = T$ and $\hat{\pi}$ as in (4.22) then $E[\ln(X^{\hat{\pi}}(T, B(T)))] = \infty$. To see this, consider

$$\begin{aligned} E[\ln(X^{\hat{\pi}}(T, B(T)))] &= E\left[\int_0^T \left\{b(s)\left(\frac{\Phi(s, Z)}{\sigma(s)} + \frac{b(s)}{\sigma^2(s)}\right) - \frac{1}{2}\sigma^2(s)\left(\frac{\Phi(s, Z)}{\sigma(s)} + \frac{b(s)}{\sigma^2(s)}\right)^2\right\} ds\right] \\ &\quad + E\left[\int_0^T \sigma(s)\left(\frac{\Phi(s, Z)}{\sigma(s)} + \frac{b(s)}{\sigma^2(s)}\right) dB(s)\right] \end{aligned}$$

$$\begin{aligned}
&= E\left[\int_0^T \left\{b(s)\left(\frac{\Phi(s, Z)}{\sigma(s)} + \frac{b(s)}{\sigma^2(s)}\right) - \frac{1}{2}\sigma^2(s)\left(\frac{\Phi(s, Z)}{\sigma(s)} + \frac{b(s)}{\sigma^2(s)}\right)^2\right\} ds\right] \\
&+ E\left[\int_0^T D_s\Phi(s, Z) ds\right], \tag{5.4}
\end{aligned}$$

where b and σ are deterministic bounded. We have

$$D_s\Phi(s, Z) = D_s\left(\frac{B(T) - B(s)}{T - s}\right) = \frac{1}{T - s}1_{[0, T]}(s). \tag{5.5}$$

Then

$$\begin{aligned}
E[\ln(X^\pi)(T, B(T))] &= E\left[\int_0^T \left\{b(s)\left(\frac{B(T) - B(s)}{\sigma(s)(T - s)} + \frac{b(s)}{\sigma^2(s)}\right) - \frac{1}{2}\sigma^2(s)\left(\frac{B(T) - B(s)}{\sigma(s)(T - s)} + \frac{b(s)}{\sigma^2(s)}\right)^2\right\} ds\right] \\
&+ \int_0^T \frac{1}{T - s}1_{[0, T]}(s) ds \\
&= \int_0^T \left\{-\frac{1}{2}\frac{E[(B(T) - B(s))^2]}{(T - s)^2} - b(s)\frac{E[B(T) - B(s)]}{\sigma(s)(T - s)} + \frac{1}{2}\frac{b^2(s)}{\sigma^2(s)} + \frac{1}{T - s}\right\} ds \\
&= \frac{1}{2}\int_0^T \frac{ds}{(T - s)} + \frac{1}{2}\int_0^T \frac{b^2(s)}{\sigma^2(s)} ds = \infty. \tag{5.6}
\end{aligned}$$

So we conclude that even in a market with delay, the market is not viable when the inside information is $Z = B(T)$. \square

6. Optimal portfolio with delay in the information flow

Suppose that the insider knows at time t the value of $B(T)$ in addition to $\mathcal{F}_{(t-r)^+}$ for all $t \in [0, T]$ and $r > 0$ where $(t - r)^+ = (t - r) \vee 0$, i.e. the maximum of $t - r$ and 0. The inside information flow in this case is

$$\mathbb{G} = \{\mathcal{G}_t\}_{t \in [0, T]}, \quad \mathcal{G}_t = \mathcal{F}_{(t-r)^+} \vee \sigma(B(T)). \tag{6.1}$$

Consider the wealth process X^π satisfying the following SDE

$$\begin{cases} dX^\pi(t) = X^\pi(t)\pi(t, B(T))(\alpha(t)dt + \sigma(t)dB(t)), t \in [0, T] \\ X^\pi(0) = 1, \end{cases} \tag{6.2}$$

where π is an \mathbb{G} -adapted process such that for all $t \in [0, T]$ and $y \in \mathbb{R}$, $\pi(t, y)$ is $\mathcal{F}_{(t-r)^+}$ -measurable and α and σ are deterministic bounded functions such that there exists $\delta > 0$ such that $\sigma \geq \delta$. Let \mathcal{A} be the set of \mathbb{G} -adapted, self-financing portfolios π such that the forward equation (6.2) has a unique solution. We study the following insider optimal portfolio problem:

Problem 6.1. Find $\pi^* \in \mathcal{A}$ such that

$$J(\pi^*) = \sup_{\pi \in \mathcal{A}} J(\pi), \tag{6.3}$$

where

$$J(\pi) := E[\log(X^\pi(T))]. \tag{6.4}$$

In this situation we get:

Theorem 6.2. Suppose the market is given by (4.1)-(4.6) and that the insider has at any time $t \in [0, T]$ access to the information $\mathcal{G}_t = \mathcal{F}_{(t-r)^+} \vee \sigma(B(T))$. Then the market is viable, i.e. we have

$$\sup_{\pi \in \mathcal{A}} E[\ln(X^\pi(T, B(T)))] < \infty. \tag{6.5}$$

Proof. The solution of (6.2) is given by

$$X^\pi(t) = x \exp\left(\int_0^t (\alpha(s)\pi(s, B(T)) - \frac{1}{2}\sigma^2(s)\pi^2(s, B(T)))ds + \int_0^t \sigma(s)\pi(s, B(T))dB(s)\right). \tag{6.6}$$

Then we have

$$\begin{aligned} E[\log(X^\pi(T))] &= E\left[\int_0^T (\alpha(t)\pi(t, B(T)) - \frac{1}{2}\sigma^2(t)\pi^2(t, B(T)))ds + \int_0^T \sigma(t)\pi(t, B(T))dB(t)\right] \\ &= E\left[\int_0^T (\alpha(t)\pi(t, B(T)) - \frac{1}{2}\sigma^2(t)\pi^2(t, B(T)))ds + \int_0^T \sigma(t)D_{t^+}\pi(t, B(T))dt\right] \\ &= E\left[\int_0^T E[\alpha(t)\pi(t, B(T)) - \frac{1}{2}\sigma^2(t)\pi^2(t, B(T)) + \sigma(t)D_{t^+}\pi(t, B(T)) | \mathcal{F}_{(t-r)^+}]dt\right] \\ &= E\left[\int_0^T (\alpha(t)E[\pi(t, B(T)) | \mathcal{F}_{(t-r)^+}] - \frac{1}{2}\sigma^2(t)E[\pi^2(t, B(T)) | \mathcal{F}_{(t-r)^+}] \right. \\ &\quad \left. + \sigma(t)E[D_{t^+}\pi(t, B(T)) | \mathcal{F}_{(t-r)^+}])dt\right]. \end{aligned} \tag{6.7}$$

Let us denote by $J(\pi)$ the following equation

$$J(\pi) = \alpha(t)E[\pi(t, B(T)) | \mathcal{F}_{(t-r)^+}] - \frac{1}{2}\sigma^2(t)E[\pi^2(t, B(T)) | \mathcal{F}_{(t-r)^+}] + \sigma(t)E[D_{t^+}\pi(t, B(T)) | \mathcal{F}_{(t-r)^+}]. \tag{6.8}$$

Using the definition of the Donsker delta function we get

$$\begin{aligned} J(\pi) &= \int_{\mathbb{R}} (\alpha(t)\pi(t, y)E[\delta_{B(T)}(y) | \mathcal{F}_{(t-r)^+}] - \frac{1}{2}\sigma^2(t)\pi^2(t, y)E[\delta_{B(T)}(y) | \mathcal{F}_{(t-r)^+}] \\ &\quad + \sigma(t)\pi(t, y)E[D_{t^+}\delta_{B(T)}(y) | \mathcal{F}_{(t-r)^+}])dy. \end{aligned} \tag{6.9}$$

We maximize this over all $\pi(t, y)$ for all t, y , the maximizer $\hat{\pi}(t, y)$ will then be the optimal portfolio. It is given by:

$$\hat{\pi}(t, y) = \frac{\alpha(t)}{\sigma^2(t)} + \frac{E[D_{t+}\delta_{B(T)}(y)|\mathcal{F}_{(t-r)^+}]}{\sigma(t)E[\delta_{B(T)}(y)|\mathcal{F}_{(t-r)^+}]} \quad (6.10)$$

From [6] applied to the filtration $\mathcal{F}_{(t-r)^+}$ we have

$$\frac{E[D_{t+}\delta_{B(T)}(y)|\mathcal{F}_{(t-r)^+}]}{E[\delta_{B(T)}(y)|\mathcal{F}_{(t-r)^+}]} = \frac{y - B((t-r)^+)}{T - (t-r)^+} \quad (6.11)$$

Hence we get that

$$\hat{\pi}(t, y) = \frac{\alpha(t)}{\sigma^2(t)} + \frac{y - B(t-r)}{\sigma(t)(T-t+r)} \quad (6.12)$$

Substituting this quantity in the first line of expression (6.7) then we get

$$\begin{aligned} E[\log(X^{\hat{\pi}}(T))] &= E\left[\int_0^T \left(\frac{\alpha^2(t)}{2\sigma^2(t)} - \frac{1}{2}\sigma^2(t)\left(\frac{B(T) - B(t-r)}{\sigma(t)(T-t+r)}\right)^2\right)dt\right. \\ &\quad \left. + \int_0^T \frac{B(T) - B(t-r)}{(T-t+r)}dB(t)\right] \\ &= E\left[\int_0^T \left(\frac{\alpha^2(t)}{2\sigma^2(t)} - \frac{1}{2}\sigma^2(t)\left(\frac{B(T) - B(t-r)}{\sigma(t)(T-t+r)}\right)^2\right)dt\right. \\ &\quad \left. + \int_0^T D_{t+}\left(\frac{B(T) - B(t-r)}{(T-t+r)}\right)dt\right]. \end{aligned} \quad (6.13)$$

Now

$$D_{t+}\left(\frac{B(T) - B(t-r)}{(T-t+r)}\right) = \frac{1}{(T-t+r)}1_{[0,T]}(t), \quad (6.14)$$

and therefore

$$\begin{aligned} E[\log(X^{\hat{\pi}}(T))] &= E\left[\int_0^T \left(\frac{\alpha^2(t)}{2\sigma^2(t)} - \frac{1}{2}\sigma^2(t)\left(\frac{B(T) - B(t-r)}{\sigma(t)(T-t+r)}\right)^2\right)dt + \int_0^T \frac{1}{(T-t+r)}dt\right] \\ &= \frac{1}{2}\int_0^T \frac{1}{(T-t+r)}dt + \int_0^T \frac{\alpha^2(t)}{2\sigma^2(t)}dt < \infty, \end{aligned} \quad (6.15)$$

since α and β are deterministic bounded functions. \square

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