NICA-TOEPLITZ ALGEBRAS ASSOCIATED WITH RIGHT TENSOR C^*-PRECATEGORIES OVER RIGHT LCM SEMIGROUPS

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ABSTRACT. We introduce and analyze the full $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ and the reduced $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ Nica-Toeplitz algebra associated to an ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ over a right LCM semigroup P. These C^*-algebras unify cross-sectional C^*-algebras associated to Fell bundles over discrete groups and Nica-Toeplitz C^*-algebras associated to product systems. They also allow a study of Doplicher-Roberts versions of the latter.

A new phenomenon is that when P is not right cancellative then the canonical conditional expectation takes values outside the ambient algebra. Our main result is a uniqueness theorem that gives sufficient conditions for a representation of $\mathcal{K}$ to generate a C^*-algebra naturally lying between $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ and $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$. We also characterize the situation when $\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \cong \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$. Unlike previous results for quasi-lattice monoids, P is allowed to contain nontrivial invertible elements, and we accommodate this by identifying an assumption of aperiodicity of an action of the group of invertible elements in P. One prominent condition for uniqueness is a geometric condition of Coburn's type, exploited in the work of Fowler, Laca and Raeburn. Here we shed new light on the role of this condition by relating it to a C^*-algebra associated to $\mathcal{L}$ itself.

1. Introduction

Tensor C^*-categories (or monoidal C^*-categories) and all the more right-tensor C^*-categories (also called semitensor C^*-categories) arise naturally in quantum field theory and duality theory of compact (quantum) groups [10], [11]. In particular, these structures play a fundamental role in numerous recent results with a flavor of geometric group theory, see e.g. [24], [30], [34] and references therein. Right-tensor C^*-(pre)categories proved also to be a very natural framework allowing efficient description of the structure of Cuntz-Pimsner and Nica-Toeplitz algebras associated to product systems, see [19], [22], [20]. In the present paper we initiate a systematic study of C^*-algebras associated to right tensor C^*-precategories inspired by this last class of examples. In fact, already in the context of Nica-Toeplitz algebras associated to product systems, our results extend substantially the existing theory, see [20]. We believe that the "categorial language" is well suited to the complicated analysis of C^*-algebras over semigroup structures, and has a potential to be used, for instance, in the study of Cuntz-Nica-Pimsner algebras [36] or Doplicher-Roberts algebras [10], [11] and their generalizations.

Our initial data consists of an ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ over a discrete, left cancellative and unital semigroup P. A C^*-precategory, as introduced in [19], is a non-unital version of a C^*-category. The important example that the reader may keep in mind is that of Banach spaces $\mathcal{L}(X_p, X_q)$ of adjointable operators between Hilbert A-modules X_p, X_q for $p, q \in P$, that form a product system $X = \bigsqcup_{p\in P} X_p$ over P. Then the right tensoring structure $\{\otimes 1_r\}_{r\in P}$ on $\mathcal{L} = \{\mathcal{L}(X_p, X_q)\}_{p,q\in P}$ is given by tensoring on the right with the unit 1_r in $\mathcal{L}(X_r)$. The Banach spaces $\mathcal{K}(X_p, X_q)$, $p, q \in P$, of generalized compacts form a

Date: 25 November 2016. Revised 5 June 2018 and 17 January 2019.

2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 46L05.

Key words and phrases. Nica-Toeplitz algebra, C^*-precategory, LCM semigroup, uniqueness theorem.

C^*-precategory $\mathcal{K}$, in fact an ideal in $\mathcal{L}$, which need not be preserved by the 'functors' $\otimes 1_r$, $r \in P$. In special cases such structures were considered in [19, Example 3.2], [22, Subsection 3.1]. We give a detailed analysis of this example in [20] where we also explain how the results of the present paper give a new insight to C^*-algebras associated with X. Another somewhat trivial but important and instructive example is when $P = G$ is a group. Then as we explain (see Section 12) our framework is equivalent to the theory of Fell bundles over discrete groups.

In general, there is a natural notion of a right-tensor representation of $\mathcal{K}$ which allows us to define the *Toeplitz algebra* $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$ of $\mathcal{K}$ as the universal C^*-algebra for these representations. We construct a canonical injective right-tensor representation T of $\mathcal{K}$ on an appropriate Fock module $\mathcal{F}_{\mathcal{K}}$. We call T *the Fock representation* of $\mathcal{K}$. We wish to study C^*-algebras which in general are quotients of $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$ obtained by considering additional relations coming from the Fock representation. It was Nica [31], who first identified and used explicitly such relations in the context of C^*-algebras associated to positive cones in quasi-lattice ordered groups. Fowler [14] generalized these conditions to product systems over semigroups studied by Nica. In particular, he introduced notions of *compact-alignment* and *Nica-covariance* for such objects. Recently, definitions of Nica covariant representations and the corresponding Nica-Toeplitz algebras were generalized to product systems over right LCM semigroups in [6].

In order to define Nica covariance we work under the assumption that P is a *right LCM semigroup*, a terminology introduced in [7]. Such semigroups appear also under the name of semigroups satisfying Clifford's condition, see [27] and [32]. We emphasize that passing from positive cones in quasi-lattice ordered groups to right LCM semigroups is not straightforward, and has a number of important consequences. First, it allows to develop a theory independent of the ambient group. In fact, the semigroups we consider need not be (right) cancellative, and hence they might not be embeddable into any group. Second, LCM semigroups allow invertible elements. This makes a number of problems much more delicate, but also allows us to cover a larger class of interesting examples. In particular LCM semigroups could be viewed as a unification of quasi-lattice ordered semigroups and groups.

Given a right-tensor C^*-precategory $\mathcal{L}$ over an LCM semigroup P we say that an ideal $\mathcal{K}$ in $\mathcal{L}$ is *well-aligned* if for every two 'morphisms' in $\mathcal{K}$ that can be tensored so that they can be composed, the composition is again in $\mathcal{K}$. This generalises the notion of compactly aligned product systems. For a well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ we introduce representations which we call *Nica covariant*. We show that the Fock representation of $\mathcal{K}$ is Nica covariant. Two C^*-algebras are then naturally associated to $\mathcal{K}$: a *Nica-Toeplitz algebra* $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ universal for Nica covariant representations, and a *reduced Nica-Toeplitz* algebra $\mathcal{NT}_{\mathcal{L}}^r(\mathcal{K})$ which is generated by the Fock representation of $\mathcal{K}$.

One important tool to study Nica-Toeplitz type C^*-algebras is a conditional expectation onto a natural *core* subalgebra. While core subalgebras of both $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ and $\mathcal{NT}_{\mathcal{L}}^r(\mathcal{K})$ are easy to write down, conditional expectations onto the respective cores do not obviously exist. In the generality of a right LCM semigroup, which need not be right cancellative, we find new ingredients, namely a self-adjoint operator space $B_{\mathcal{K}}$, which in general is not a subspace of $\mathcal{NT}_{\mathcal{L}}^r(\mathcal{K})$, and a faithful completely positive map E^T from $\mathcal{NT}_{\mathcal{L}}^r(\mathcal{K})$ onto $B_{\mathcal{K}}$. We refer to $B_{\mathcal{K}}$ as a *transcendental core* and to E^T as a *transcendental conditional expectation*. The map E^T becomes a genuine conditional expectation onto the core subalgebra of $\mathcal{NT}_{\mathcal{L}}^r(\mathcal{K})$ if and only if the semigroup P is cancellative. We note that a similar phenomenon was recently discovered in the context of C^*-algebras associated to actions of inverse semigroups of Hilbert bimodules in [8], where a notion of a weak conditional expectation is introduced (it is a genuine conditional expectation if and only if the space of units is closed in the dual transformation groupoid). We define an *exotic Nica-Toeplitz algebra* to be the C^*-algebra $C^*(\Phi(\mathcal{K}))$ generated by a

Nica covariant representation Φ such that there is a compatible transcendental conditional expectation from $C^*(\Phi(\mathcal{K}))$ onto $B_{\mathcal{K}}$. This is equivalent to existence of a $*$-homomorphism Φ_* making the following diagram commute:

$$\begin{tikzcd} \mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \arrow[r, "\Phi\rtimes P"] \arrow[rr, bend right, "T\rtimes P"'] & C^*(\Phi(\mathcal{K})) \arrow[r, "\Phi_*"] & \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K}) \end{tikzcd}$$

The uniqueness theorems we aim at require studying two somewhat independent problems, which are interesting in their own right. The first problem is to identify ideals $\mathcal{K}$ for which the regular representation $T \rtimes P$ is injective. The second one is to find conditions on a Nica covariant representation Φ so that $C^*(\Phi(\mathcal{K}))$ is an exotic Nica-Toeplitz algebra. Having these two ingredients we infer that both $\Phi \rtimes P$ and Φ_* are isomorphisms.

Amenability as a characterization of injectivity of the regular representation of the universal C^*-algebra for Nica covariant representations is prominent in work of Nica [31], Laca-Raeburn [25] and Fowler [14]. Motivated by this, we say that a well aligned ideal $\mathcal{K}$ of $\mathcal{L}$ is *amenable* if the regular representation $T \rtimes P$ is an isomorphism from $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ onto $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$. In Theorem 8.4 we prove a far-reaching generalization of [25, Proposition 4.2] and [14, Theorem 8.1]. Our result establishes necessary and sufficient conditions for amenability of $\mathcal{K}$ in terms of amenability of a Fell bundle that arises in a canonical way whenever there is a *controlled semigroup homomorphism* from P to another right LCM semigroup that sits inside a group G. In particular, if G can be chosen to be amenable, then any well-aligned ideal in a right-tensor C^*-precategory over P is amenable. As we show in Corollary 8.6 this applies to a large class of semigroups obtained from free products of right LCM semigroups.

A novel ingredient in our study is that we identify an algebraic condition which characterizes when a representation $\Phi \rtimes P$ of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ is injective on the core: we call this *Toeplitz covariance*. Such a condition was previously considered only in the case $P = \mathbb{N}$, in the context of relative Cuntz-Pimsner and relative Doplicher-Roberts algebras, cf. [19]. In Corollary 6.4 we prove that T is Toeplitz covariant, which equivalently means that $T \rtimes P$ is injective on the core of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$. The main technical result needed to achieve these characterizations is Theorem 6.1. It says that any controlled semigroup homomorphism from P to an arbitrary right LCM semigroup induces a C^*-subalgebra of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ containing the core, and gives conditions characterising when $\Phi \rtimes P$ is injective on the induced C^*-algebra. This result is inspired by [25, Lemma 4.1] and the proof of [14, Theorem 8.1]. Nevertheless, since we deal here with much more general situation our proof requires some new non-trivial steps.

Another new ingredient in our approach is related to a potential existence of invertible elements in a right LCM semigroup P. We show that for any well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ the restriction of right tensoring to the group of invertible elements P^* preserves $\mathcal{K}$. In Definition 10.8 we introduce the *aperiodicity condition* for this action of P^* on $\mathcal{K}$. If $P^* = \{e\}$ is trivial, this condition becomes vacuous. If $P^* = P$, that is, when P is a group, then $\mathcal{K}$ can be viewed as a Fell bundle $\mathcal{B}$ over P, and aperiodicity of $\mathcal{K}$ is equivalent to aperiodicity of $\mathcal{B}$ (see Section 12). By [21], if B_e is separable or of Type I, aperiodicity of $\mathcal{B}$ is equivalent to *topological freeness* of the dual partial action. Non-trivial examples where the aperiodicity condition holds come for instance from wreath products, see [20].

Our main results are the uniqueness theorems in Section 10 and Section 11. For a Nica covariant representation Φ of a well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ we identify two sources from which to extract characterizations of injectivity of $\Phi \rtimes P$. In Definition 10.1 we introduce a geometric condition on Φ, which we call *condition (C)*. It is closely related to the condition describing injectivity of representations of the Toeplitz algebra of a single C^*-correspondence, see [17, Theorem 2.1] or the condition for semigroup crossed products twisted by a product system, see [14, Equation (7.2)]. Theorem 10.12 is the main technical result on Φ that links condition (C), injectivity of $\Phi \rtimes P$ on the core, Toeplitz covariance, and generation of an exotic Nica-Toeplitz algebra. Our first uniqueness result, Corollary 10.14, says that when $\mathcal{K}$ is amenable and the action of P^* on $\mathcal{K}$ is aperiodic, then condition (C) on a representation Φ of $\mathcal{K}$ implies injectivity of $\Phi \rtimes P$. The converse holds if $\mathcal{K}$ is right-tensor invariant.

The most satisfactory uniqueness result is contained in Section 11. Here we reveal the true nature of condition (C). Under natural assumptions we show that the C^*-algebra $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ associated to $\mathcal{K}$ is a subalgebra of the C^*-algebra $\mathcal{NT}(\mathcal{L}) := \mathcal{NT}_{\mathcal{L}}(\mathcal{L})$ associated to $\mathcal{L}$ (the latter can be thought of as Doplicher-Roberts version of the former). Moreover, every Nica covariant representation Φ of $\mathcal{K}$ extends uniquely to a Nica covariant representation $\overline{\Phi}$ of $\mathcal{L}$. We prove, see Corollary 11.5, that condition (C) for Φ is equivalent to injectivity of the representation $\overline{\Phi} \rtimes P$ of the (Doplicher-Roberts) C^*-algebra $\mathcal{NT}(\mathcal{L})$. This also implies injectivity of $\Phi \rtimes P$ on $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$, as $\Phi \rtimes P$ is a restriction of $\overline{\Phi} \rtimes P$. In addition condition (C) is equivalent to $\overline{\Phi}$ being Toeplitz covariant and injective, cf. Theorem 10.15. It seems that in general, the geometric condition (C) is responsible for uniqueness of the C^*-algebra associated to $\mathcal{L}$ while uniqueness of C^*-algebra associated to $\mathcal{K}$ should be related with the algebraic condition of Toeplitz covariance, cf. also [20].

1.1. **Acknowledgements.** The research leading to these results has received funding from the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement number 621724. B.K. was partially supported by the NCN (National Centre of Science) grant number 2014/14/E/ST1/00525. This work was finished while he participated in the Simons Semester at IMPAN - Fundation grant 346300 and the Polish Government MNiSW 2015-2019 matching fund. Part of the work was done during the participation of both authors in the program "Classification of operator algebras: complexity, rigidity, and dynamics" at the Mittag-Leffler Institute (Sweden), and during the visit of N.L. at the University of Victoria (Canada). She thanks Marcelo Laca and the department at UVic for hospitality.

2. Preliminaries

2.1. **LCM semigroups.** We refer to [7] and [5] and the references therein for general facts about LCM semigroups. Throughout this paper P is a *left cancellative* semigroup with the *identity* element e. We let P^* be the group of *units*, or invertible elements, in P, where $x \in P$ is invertible if there exists (a necessarily unique) $x^{-1} \in P$ such that $xx^{-1} = x^{-1}x = e$. A *principal right ideal* in P is a right ideal in P of the form $pP = \{ps : s \in P\}$ for some $p \in P$. Occasionally, we will write $\langle p \rangle := pP$. The relation of inclusion on the principal right ideals induces a left invariant *preorder* on P given by

$$p \leq q \quad \stackrel{def}{\Longleftrightarrow} \quad qP \subseteq pP \quad \Longleftrightarrow \quad \exists r \in P \; q = pr.$$

This preorder is a partial order if and only if $P^* = \{e\}$. Left cancellation in P implies that for fixed $p, q \in P$, $pr = q$ determines $r \in P$ uniquely, motivating the notation:

$$p^{-1}q := r \quad \text{ if } \quad q = pr.$$

The following property of semigroups is sometimes called *Clifford's condition* [27], [32].

Definition 2.1. A semigroup P is a *right LCM semigroup* if it is left cancellative and the family $\{pP\}_{p\in P}$ of principal right ideals extended by the empty set is closed under intersections, that is if for every pair of elements $p,q\in P$ we have $pP\cap qP=\emptyset$ or $pP\cap qP=rP$ for some $r\in P$.

In the case that $pP\cap qP=rP$, the element r is a *right least common multiple (LCM)* of p and q. Note that a right LCM is determined by p and q up to multiplication from the right by an invertible element. Namely, if $pP\cap qP=rP$, then $pP\cap qP=tP$ if and only if there is $x\in P^*$ such that $t=rx$. If P is a right LCM semigroup we will refer to $J(P):=\{pP\}_{p\in P}\cup\{\emptyset\}$ as the *semilattice of principal right ideals* of P, see [7] and [28].

Example 2.2. One of the most known and studied examples of right LCM semigroups are positive cones in quasi-lattice ordered groups, introduced by Nica [31]. More precisely, suppose that P is a subsemigroup of a group G such that $P\cap P^*=\{e\}$. Then the partial order we defined on P extends to a left-invariant partial order on G where $g\le h$ if and only if $g^{-1}h\in P$ for all $g,h\in G$. The pair (G,P) is a *quasi-lattice ordered group* if every pair of elements $g,h\in G$ that has an upper bound in P admits a least upper bound in P. Note that if the upper bound exists, it is unique since $P^*=\{e\}$. In [13, Definition 32.1], Exel calls the pair (G,P) *weakly quasi-lattice ordered* if for each pair of elements $g,h\in P$ with an upper bound in P there exists a (necessarily unique) least upper bound in P.

Every positive cone P in a weakly quasi-lattice ordered group (G,P) is an LCM semigroup with $P^*=\{e\}$. Conversely, if P is an LCM subsemigroup of a group G such that $P^*=\{e\}$ then (G,P) is a weakly quasi-lattice ordered group.

Let P_i, $i\in I$, be a family of right LCM semigroups. The direct sum $\bigoplus_{i\in I}P_i$ is a right LCM semigroup with semilattice of principal right ideals isomorphic to the direct sum of semilattices $J(P_i)$, $i\in I$. It is slightly less obvious that also the free product $\prod^*_{i\in I}P_i$ is a right LCM semigroup. This follows from the proof of the following proposition, which is a generalization of [25, Proposition 4.3]. It links the two aforementioned constructions by a useful homomorphism.

Proposition 2.3. *Let P_i, $i\in I$, be a family of right LCM semigroups. Put $P:=\prod^*_{i\in I}P_i$ and $\mathcal{P}:=\bigoplus_{i\in I}P_i$, and let $\theta:P\to\mathcal{P}$ be the homomorphism which is the identity on each P_i, $i\in I$. Then $\theta(P^*)=\mathcal{P}^*$ and for any $s,t,r\in P$ with $sP\cap tP=rP$ we have*

$$\theta(s)\mathcal{P}\cap\theta(t)\mathcal{P}=\theta(r)\mathcal{P}\qquad\text{and}\qquad \theta(s)=\theta(t)\ \text{implies}\ s=t. \tag{2.1}$$

Proof. Note that $P^*=\prod^*_{i\in I}P_i^*$ and $\mathcal{P}^*=\bigoplus_{i\in I}P_i^*$. Hence $\theta(P^*)=\mathcal{P}^*$. Now, let $s=p_{i_1}\cdots p_{i_n}\in P$ be in reduced form, that is $p_{i_k}\in P_{i_k}$ and $i_k\neq i_{k+1}$ for all $k=1,\dots,n-1$. Similarly, write $t=q_{j_1}\cdots q_{j_m}\in P$. Without loss of generality we may assume that $m\ge n$. Suppose that $sP\cap tP\neq\emptyset$. This implies that $p_{i_1}\cdots p_{i_{n-1}}=q_{i_1}\cdots q_{i_{n-1}}$, $i_n=j_n$ and either

(1) $p_{i_n}\le q_{j_n}$, if $m>n$, in which case $sP\cap tP=tP$, or
(2) $m=n$ and $p_{i_n}P_{i_n}\cap q_{j_n}P_{i_n}=r_{i_n}P_{i_n}$ for some $r_{i_n}\in P_{i_n}$, in which case $sP\cap tP=sr_{i_n}P$.

Clearly, in both cases, we have $\theta(s)\mathcal{P}\cap\theta(t)\mathcal{P}=\theta(r)\mathcal{P}$. Moreover, $\theta(s)=\theta(t)$ implies that $m=n$ and $p_{i_n}=q_{j_n}$, that is, $s=t$. □

The property identified in (2.1) is important. We will formalise it in a definition. To indicate the analogy with similar concepts introduced in [25] and [9], we borrow Crisp and Laca's terminology of *controlled map*, see [9, Definition 4.1] and in particular conditions (C2), (C3) and (C5), but add the tag of right LCM semigroup.

Definition 2.4. A *controlled map of right LCM semigroups* is an identity preserving homomorphism $\theta : P \to \mathcal{P}$ between right LCM semigroups $P, \mathcal{P}$ such that $\theta(P^*) = \mathcal{P}^*$ and for all $s, t \in P$ with $sP \cap tP \neq \emptyset$ we have

$$\theta(s)\mathcal{P} \cap \theta(t)\mathcal{P} = \theta(r)\mathcal{P} \text{ whenever } r \text{ is a right LCM for } s, t \tag{2.2}$$

and

$$\theta(s) = \theta(t) \implies s = t. \tag{2.3}$$

Remark 2.5. (a) If $\theta : P \to \mathcal{P}$ is a controlled map of right LCM semigroups and $\mathcal{P}$ is right cancellative, then so is P. Indeed, if $pr = qr$ in P then $\theta(p)\theta(r) = \theta(q)\theta(r)$, so right cancellation in $\mathcal{P}$ implies $\theta(p) = \theta(q)$ giving $p = q$ by condition (2.3). In our applications, $\mathcal{P}$ will be contained in a group so it will necessarily be cancellative.

(b) A controlled map of right LCM semigroups seems to be different from a homomorphism of right LCM semigroups as considered in [6, Equation (3.1)]. This is because a homomorphism of right LCM semigroups need not preserve the group of units and, besides, it keeps track of pairs of elements in P and their images in $\mathcal{P}$ that do not have a right common upper bound as well as those who do.

2.2. **C^*-precategories.** We recall some background on C^*-precategories from [19]. C^*-precategories should be viewed as non-unital versions of C^*-categories, cf. [18], [10].

A *precategory* $\mathcal{L}$ consists of a set of objects $\mathrm{Ob}(\mathcal{L})$ and a collection $\{\mathcal{L}(\sigma, \rho)\}_{\sigma,\rho\in\mathrm{Ob}(\mathcal{L})}$ of sets of morphisms endowed with an associative composition. Explicitly, $\mathcal{L}(\sigma, \rho)$ stands for the space of morphisms from ρ to σ, and the composition $\mathcal{L}(\tau, \sigma) \times \mathcal{L}(\sigma, \rho) \to \mathcal{L}(\tau, \rho)$, $(a, b) \to ab$ must satisfy $(ab)c = a(bc)$ whenever the compositions of morphisms a, b, c are allowable. A morphism in $\mathcal{L}(\sigma, \rho)$ may be regarded as an arrow from ρ to σ. One can equip (if necessary) $\mathcal{L}(\sigma, \sigma)$ with identity morphisms in such a way that a given precategory $\mathcal{L}$ becomes a category. In the sequel, we will identify $\mathcal{L}$ with the collection of morphisms $\{\mathcal{L}(\sigma, \rho)\}_{\sigma,\rho\in\mathrm{Ob}(\mathcal{L})}$.

Definition 2.6. ([19, Definition 2.2]) A *C^*-precategory* is a precategory $\mathcal{L} = \{\mathcal{L}(\sigma, \rho)\}_{\sigma,\rho\in\mathrm{Ob}(\mathcal{L})}$ together with an operation $^* : \mathcal{L} \to \mathcal{L}$ such that the following hold:

(p1) each set of morphisms $\mathcal{L}(\sigma, \rho)$, $\sigma, \rho \in \mathcal{L}$ is a complex Banach space;
(p2) composition gives a bilinear map

$$\mathcal{L}(\tau, \sigma) \times \mathcal{L}(\sigma, \rho) \ni (a, b) \to ab \in \mathcal{L}(\tau, \rho),$$

which satisfies $\|ab\| \leq \|a\| \cdot \|b\|$;
(p3) for each σ, $\rho \in \mathcal{L}$, $a \to a^*$ is an antilinear map from $\mathcal{L}(\sigma, \rho)$ to $\mathcal{L}(\rho, \sigma)$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $b \in \mathcal{L}(\rho, \tau)$ and all $\tau \in \mathcal{L}$;
(p4) $\|a^*a\| = \|a\|^2$ for every $a \in \mathcal{L}(\sigma, \rho)$; and
(p5) for each $a \in \mathcal{L}(\sigma, \rho)$, we have $a^*a = b^*b$ for some $b \in \mathcal{L}(\rho, \rho)$.

We say that a C^*-precategory $\mathcal{L}$ is *C^*-category* if $\mathcal{L}$ is a category.

Note that (p3) says that the operation * is antilinear, involutive and contravariant. Moreover, (p1)–(p4) imply that $\mathcal{L}(\rho, \rho)$ is a C^*-algebra, and (p5) says that a^*a is positive as an element of $\mathcal{L}(\rho, \rho)$. Condition (p4) implies that the operation * is isometric on every space $\mathcal{L}(\sigma, \rho)$. A C^*-precategory $\mathcal{L}$ is a C^*-category if and only if every C^*-algebra $\mathcal{L}(\rho, \rho)$, $\rho \in \mathrm{Ob}(\mathcal{L})$, is unital.

Lemma 2.7. *Given a C^*-precategory $\mathcal{L}$ and objects σ, ρ we have*

$$\mathcal{L}(\sigma, \rho) = \mathcal{L}(\sigma, \sigma)\mathcal{L}(\sigma, \rho) = \mathcal{L}(\sigma, \rho)\mathcal{L}(\rho, \rho).$$

Proof. In a natural way, $\mathcal{L}(\sigma,\rho)$ is a left Banach $\mathcal{L}(\sigma,\sigma)$-module and a right Banach $\mathcal{L}(\rho,\rho)$-module. By [19, Lemma 2.5], these module actions are non-degenerate. Hence the assertion follows by the Cohen-Hewitt factorization theorem. □

Definition 2.8 (Definition 2.4 in [19])**.** An *ideal in a C^*-precategory* $\mathcal{L}$ is a collection $\mathcal{K} = \{\mathcal{K}(\sigma,\rho)\}_{\sigma,\rho\in \mathrm{Ob}(\mathcal{L})}$ of closed linear subspaces $\mathcal{K}(\sigma,\rho)$ of $\mathcal{L}(\sigma,\rho)$, $\rho,\sigma\in\mathrm{Ob}(\mathcal{L})$, such that

$$\mathcal{L}(\tau,\sigma)\mathcal{K}(\sigma,\rho)\subseteq\mathcal{K}(\tau,\rho) \quad \text{and} \quad \mathcal{K}(\tau,\sigma)\mathcal{L}(\sigma,\rho)\subseteq\mathcal{K}(\tau,\rho),$$

for all $\sigma,\rho,\tau\in\mathrm{Ob}(\mathcal{L})$.

An ideal $\mathcal{K}$ in a C^*-precategory $\mathcal{L}$ is automatically selfadjoint in the sense that $\mathcal{K}(\sigma,\rho)^* = \mathcal{K}(\rho,\sigma)$, for all $\sigma,\rho\in\mathrm{Ob}(\mathcal{L})$, cf. [18, Proposition 1.7]. Hence $\mathcal{K}$ is a C^*-precategory. Each space $\mathcal{K}(\rho,\rho)$ is a closed two-sided ideal in the C^*-algebra $\mathcal{L}(\rho,\rho)$. A useful fact is that $\mathcal{K}$ is uniquely determined by these diagonal ideals.

Proposition 2.9 (Theorem 2.6 in [19])**.** *If $\mathcal{K}$ is an ideal in a C^*-precategory $\mathcal{L}$, then for all σ and ρ the space $\mathcal{K}(\sigma,\rho)$ coincides with*

$$\{a\in\mathcal{L}(\sigma,\rho): a^*a\in\mathcal{K}(\rho,\rho)\} = \{a\in\mathcal{L}(\sigma,\rho): aa^*\in\mathcal{K}(\sigma,\sigma)\}. \tag{2.4}$$

Conversely, a collection of ideals $\mathcal{K}(\rho,\rho)$ in $\mathcal{L}(\rho,\rho)$, for $\rho\in\mathrm{Ob}(\mathcal{L})$, satisfying (2.4) *gives rise to an ideal $\mathcal{K}$ in $\mathcal{L}$.*

We generalize the notion of an essential ideal in a C^*-algebra as follows.

Definition 2.10. An ideal $\mathcal{K}$ in a C^*-precategory $\mathcal{L}$ is an *essential ideal* in $\mathcal{L}$ if

$$\mathcal{K}(\rho,\rho) \text{ is an essential ideal in } \mathcal{L}(\rho,\rho) \text{ for every } \rho\in\mathrm{Ob}(\mathcal{L}). \tag{2.5}$$

Homomorphisms between C^*-precategories are defined in a natural way. We recall from [19, Definition 2.8] that a *homomorphism* Φ from a C^*-precategory $\mathcal{L}$ to a C^*-precategory $\mathcal{S}$ consists of a map $\mathrm{Ob}(\mathcal{L})\ni\sigma\mapsto\Phi(\sigma)\in\mathrm{Ob}(\mathcal{S})$ and linear operators $\mathcal{L}(\sigma,\rho)\ni a\mapsto \Phi(a)\in\mathcal{S}(\Phi(\sigma),\Phi(\rho))$, $\sigma,\rho\in\mathrm{Ob}(\mathcal{L})$, such that $\Phi(a)\Phi(b)=\Phi(ab)$ and $\Phi(a^*)=\Phi(a)^*$ for all $a\in\mathcal{L}(\tau,\sigma)$, $b\in\mathcal{L}(\sigma,\rho)$ and all $\sigma,\rho,\tau\in\mathrm{Ob}(\mathcal{L})$. An *endomorphism* of $\mathcal{L}$ is a homomorphism from $\mathcal{L}$ to $\mathcal{L}$.

Note that composition of homomorphisms is again a homomorphism. If $\Phi:\mathcal{L}\to\mathcal{S}$ is a homomorphism between C^*-precategories, then $\Phi:\mathcal{L}(\rho,\rho)\to\mathcal{S}(\Phi(\rho),\Phi(\rho))$ are *-homomorphisms of C^*-algebras. Using this observation one gets the following fact, cf. [19, Proposition 2.9].

Lemma 2.11. *For any homomorphism $\Phi:\mathcal{L}\to\mathcal{S}$ of C^*-precategories the operators*

$$\Phi:\mathcal{L}(\sigma,\rho)\to\mathcal{S}(\Phi(\sigma),\Phi(\rho)), \qquad \sigma,\rho\in\mathrm{Ob}(\mathcal{L}), \tag{2.6}$$

are contractions. Moreover, all the maps in (2.6) *are isometric if and only if all the maps in* (2.6) *with $\sigma=\rho$ are injective.*

Definition 2.12. A *representation of a C^*-precategory* $\mathcal{L}$ in a C^*-algebra B is a homomorphism $\Phi:\mathcal{L}\to B$ where B is considered as a C^*-precategory with a single object. Equivalently, Φ may be viewed as a collection $\{\Phi_{\sigma,\rho}\}_{\sigma,\rho\in\mathrm{Ob}(\mathcal{L})}$ of linear operators $\Phi_{\sigma,\rho}:\mathcal{L}(\sigma,\rho)\to B$ such that

$$\Phi_{\sigma,\rho}(a)^*=\Phi_{\rho,\sigma}(a^*), \quad \text{and} \quad \Phi_{\tau,\rho}(ab)=\Phi_{\tau,\sigma}(a)\Phi_{\sigma,\rho}(b),$$

for all $a\in\mathcal{L}(\tau,\sigma)$, $b\in\mathcal{L}(\sigma,\rho)$. A *representation* of $\mathcal{L}$ *on a Hilbert space* H is a representation of $\mathcal{L}$ in the C^*-algebra $\mathcal{B}(H)$ of all bounded operators on H. We say that a representation $\{\Phi_{\sigma,\rho}\}_{\sigma,\rho\in\mathrm{Ob}(\mathcal{L})}$ is *injective* if all $\Phi_{\rho,\rho}$ for $\rho\in\mathrm{Ob}(\mathcal{L})$ are injective (then all the maps $\Phi_{\sigma,\rho}$, $\sigma\in\mathrm{Ob}(\mathcal{L})$, are isometric by Lemma 2.11).

Let $\{\Phi_{\sigma,\rho}\}_{\sigma,\rho\in\mathcal{L}}$ be a representation of a C^*-precategory $\mathcal{L}$. If $\mathcal{K}$ is an ideal in $\mathcal{L}$ then $\{\Phi_{\sigma,\rho}|_{\mathcal{K}(\sigma,\rho)}\}_{\sigma,\rho\in\mathcal{L}}$ is a representation of $\mathcal{K}$. In the converse direction we have the following result, cf. [19, Proposition 2.13].

Proposition 2.13. *Suppose that $\mathcal{K}$ is an ideal in a C^*-precategory $\mathcal{L}$ and let $\Phi = \{\Phi_{\sigma,\rho}\}_{\sigma,\rho\in\mathrm{Ob}(\mathcal{L})}$ be a representation of $\mathcal{K}$ on a Hilbert space H. There is a unique extension $\overline{\Phi} = \{\overline{\Phi}_{\sigma,\rho}\}_{\sigma,\rho\in\mathrm{Ob}(\mathcal{L})}$ of Φ to a representation of $\mathcal{L}$ such that the essential subspace of $\overline{\Phi}_{\sigma,\rho}$ is contained in the essential subspace of $\Phi_{\sigma,\rho}$, for every $\sigma,\rho\in\mathrm{Ob}(\mathcal{L})$. Namely,*

$$\overline{\Phi}_{\sigma,\rho}(a)(\Phi_{\rho,\rho}(\mathcal{K}(\rho,\rho))H)^{\perp} = 0, \quad \textit{and} \quad \overline{\Phi}_{\sigma,\rho}(a)\Phi_{\rho,\rho}(b)h = \Phi_{\sigma,\rho}(ab)h \tag{2.7}$$

for all $a\in\mathcal{L}(\sigma,\rho)$, $b\in\mathcal{K}(\rho,\rho)$, $h\in H$. Moreover,

$$\ker\overline{\Phi}_{\sigma,\rho} = \{a\in\mathcal{L}(\sigma,\rho) : a\mathcal{K}(\rho,\rho)\subseteq\ker\Phi_{\sigma,\rho}\}. \tag{2.8}$$

In particular, $\overline{\Phi}$ is injective if and only if Φ is injective and $\mathcal{K}$ is an essential ideal in $\mathcal{L}$.

Proof. The existence and uniqueness of $\overline{\Phi}$ satisfying (2.7) are guaranteed by [19, Proposition 2.13]. Let $a\in\mathcal{L}(\sigma,\rho)$, $\sigma,\rho\in\mathrm{Ob}(\mathcal{L})$. By (2.7), we have

$$a\in\ker\overline{\Phi}_{\sigma,\rho} \iff \Phi_{\sigma,\rho}(a\mathcal{K}(\rho,\rho)) = \{0\} \iff a\mathcal{K}(\rho,\rho)\subseteq\ker\Phi_{\sigma,\rho},$$

which proves the second part of the assertion. □

Example 2.14. The prototypical examples of C^*-precategories arise from adjointable maps between Hilbert modules. Specifically, let $X = \{X_p\}_{p\in S}$ be a family of right Hilbert modules over a C^*-algebra A, indexed by a set S. Then the families

$$\mathcal{K}(p,q) := \mathcal{K}(X_q, X_p), \qquad \mathcal{L}(p,q) := \mathcal{L}(X_q, X_p)$$

for $p,q\in S$ with operations inherited from the corresponding spaces form C^*-precategories. In fact, $\mathcal{L}$ is a C^*-category and $\mathcal{K}$ is an essential ideal in $\mathcal{L}$.

3. Nica covariant representations and Nica-Toeplitz algebra

Right-tensor C^*-precategories over the semigroup $\mathbb{N}$ were introduced in [19], where they were shown to provide a good framework for studying Pimsner and Doplicher-Roberts type C^*-algebras. Here we notice that [19, Definition 3.1] makes sense for an arbitrary semigroup P, and we set out to study associated C^*-algebras.

Definition 3.1. A *right-tensor C^*-precategory* is a C^*-precategory $\mathcal{L} = \{\mathcal{L}(p,q)\}_{p,q\in P}$ whose objects form a semigroup P with identity e and which is equipped with a semigroup $\{\otimes 1_r\}_{r\in P}$ of endomorphisms of $\mathcal{L}$ such that $\otimes 1_r$ acts on P by sending p to pr, for all $p,r\in P$, and $\otimes 1_e = \mathrm{id}$. For a morphism $a\in\mathcal{L}(p,q)$ we denote the value of $\otimes 1_r$ on a by $a\otimes 1_r$, and note that it belongs to $\mathcal{L}(pr,qr)$. We refer to $\{\otimes 1_r\}_{r\in P}$ as to a *right tensoring* on $\mathcal{L} = \{\mathcal{L}(p,q)\}_{p,q\in P}$.

Note in particular that for all $a\in\mathcal{L}(p,q)$, $b\in\mathcal{L}(q,s)$, and $p,q,r,s\in P$ we have

$$((a\otimes 1_r)\otimes 1_s) = a\otimes 1_{rs}, \qquad (a\otimes 1_r)^* = a^*\otimes 1_r, \qquad (a\otimes 1_r)(b\otimes 1_r) = (ab)\otimes 1_r.$$

The following definition is a semigroup generalization of [19, Definition 3.6].

Definition 3.2. Let $\mathcal{K}$ be an ideal in a right-tensor C^*-precategory $\mathcal{T}$. We say that a representation $\Phi:\mathcal{K}\to B$ of $\mathcal{K}$ in a C^*-algebra B is a *right-tensor representation* if for all $a\in\mathcal{K}(p,q)$ and $b\in\mathcal{K}(s,t)$ such that $sP\subseteq qP$ we have

$$\Phi(a)\Phi(b) = \Phi\left((a\otimes 1_{q^{-1}s})b\right). \tag{3.1}$$

We let $C^*(\Phi(\mathcal{K}))$ be the C^*-algebra generated by the spaces $\Phi(\mathcal{K}(p,q))$, $p,q\in P$. We call it the C^*-algebra generated by Φ.

Remark 3.3. Since $\mathcal{K}$ is an ideal the right hand side of (3.1) makes sense. Furthermore, by taking adjoints one gets the symmetrized version of this equation:

$$\Phi(a)\Phi(b) = \Phi(a(b \otimes 1_{s^{-1}q})),$$

where $a \in \mathcal{K}(p,q)$, $b \in \mathcal{K}(s,t)$, $qP \subseteq sP$, $p,q,s,t \in P$.

Standard arguments coupled with our proof of existence of an injective Nica covariant representation, see Proposition 5.2 below, show that the following proposition holds.

Proposition 3.4. *Let $\mathcal{K}$ be an ideal in a right-tensor C^*-precategory $\mathcal{L}$. There are a C^*-algebra $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$ and an injective right-tensor representation $t_{\mathcal{K}} : \mathcal{K} \to \mathcal{T}_{\mathcal{L}}(\mathcal{K})$, such that*

(a) *for every right-tensor representation Φ of $\mathcal{K}$ there is a homomorphism $\Phi \times P$ of $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$ such that $(\Phi \times P) \circ t_{\mathcal{K}} = \Phi$; and*

(b) *$\mathcal{T}_{\mathcal{L}}(\mathcal{K}) = C^*(t_{\mathcal{K}}(\mathcal{K}))$.*

The C^*-algebra $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$ is unique up to canonical isomorphism.

Proof. Since every representation $\Phi : \mathcal{K} \to B$ is automatically contractive, cf. Lemma 2.11, a direct sum of right-tensor representations of $\mathcal{K}$ is a right-tensor representation. Thus existence and uniqueness of $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$ follow from [3, Section 1]. Injectivity of $t_{\mathcal{K}}$ follows from Proposition 5.2 that we prove below. □

Definition 3.5. Given an ideal $\mathcal{K}$ in a right-tensor C^*-precategory $(\mathcal{L}, \{\otimes 1_r\}_{r \in P})$, the C^*-algebra $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$ described in Proposition 3.4 is the *Toeplitz algebra* of $\mathcal{K}$.

The Toeplitz algebra $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$ in general is very large. It lacks a version of 'Wick ordering' and therefore its structure is hardly accessible. This is the main reason why in the present paper we will study C^*-algebras generated by representations satisfying a condition of Nica type, which is stronger than (3.1). Since such conditions are (so far) established only for right LCM semigroups, from now on (with the exception of Section 4) we will *always assume that P is a right LCM semigroup*.

Definition 3.6. Let $(\mathcal{L}, \{\otimes 1_r\}_{r \in P})$ be a right-tensor C^*-precategory over a right LCM semigroup P. An ideal $\mathcal{K}$ in $\mathcal{L}$ is *well-aligned* in $(\mathcal{L}, \{\otimes 1_r\}_{r \in P})$ if for all $a \in \mathcal{K}(p,p)$, $b \in \mathcal{K}(q,q)$ we have

$$(a \otimes 1_{p^{-1}r})(b \otimes 1_{q^{-1}r}) \in \mathcal{K}(r,r) \qquad \text{whenever} \quad pP \cap qP = rP. \tag{3.2}$$

An ideal $\mathcal{K}$ in $\mathcal{L}$ is $\otimes 1$-*invariant* if $\mathcal{K}(p,p) \otimes 1_r \subseteq \mathcal{K}(pr,pr)$ for all $p,r \in P$. We denote this property of $\mathcal{K}$ as $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$.

Note that by Proposition 2.9, if $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$, then $\mathcal{K}(p,q) \otimes 1_r \subseteq \mathcal{K}(pr,qr)$ for all $p,q,r \in P$. Plainly, if $\mathcal{K}$ is a $\otimes 1$-invariant ideal in a right-tensor C^*-precategory $\mathcal{L}$, then $\mathcal{K}$ is itself a right-tensor C^*-precategory, and $\mathcal{K}$ is well-aligned both in $\mathcal{L}$ and in $\mathcal{K}$. The condition in (3.2) is a generalization of the notion of compact alignment for product systems of C^*-correspondences from [14, Definition 5.7], cf. [6], and see [20] for details. The next lemma shows that (3.2) captures more than just *diagonal* fibres $\mathcal{K}(p,p)$ for $p \in P$.

Lemma 3.7. *Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $\mathcal{L}$. For all $a \in \mathcal{K}(p,q)$, $b \in \mathcal{K}(s,t)$ we have*

$$(a \otimes 1_{q^{-1}r})(b \otimes 1_{s^{-1}r}) \in \mathcal{K}(pq^{-1}r, ts^{-1}r) \qquad \textit{whenever} \quad qP \cap sP = rP. \tag{3.3}$$

Proof. By Lemma 2.7 we have $a = a'a''$ and $b = b'b''$ where $a' \in \mathcal{K}(p,q)$, $b' \in \mathcal{K}(s,t)$ and $a'' \in \mathcal{K}(q,q)$, $b'' \in \mathcal{K}(s,s)$. If $qP \cap sP = rP$ then by (3.2) we have $(a'' \otimes 1_{q^{-1}r})(b'' \otimes 1_{s^{-1}r}) \in \mathcal{K}(r,r)$. Composing this from the left by $(a' \otimes 1_{q^{-1}r}) \in \mathcal{L}(pq^{-1}r, r)$ and from the right by $(b' \otimes 1_{s^{-1}r}) \in \mathcal{L}(r, ts^{-1}r)$ and using that $\mathcal{K}$ is an ideal in $\mathcal{L}$ gives (3.3). □

The notion of Nica covariance for a representation of a compactly aligned product system of C^*-correspondences was introduced in [14] in the context of quasi-lattice ordered groups, and was extended to right LCM semigroups in [6]. Lemma 3.7 allows us to extend this concept to C^*-precategories over right LCM semigroups. In our generalization below, Nica covariance will be imposed in subspaces $\mathcal{K}(p,q)$ for $p,q \in P$ that are not necessarily diagonal, in the sense that p need not equal q.

Definition 3.8. Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $\mathcal{L}$. A representation $\Phi : \mathcal{K} \to B$ of $\mathcal{K}$ in a C^*-algebra B is *Nica covariant* if for all $a \in \mathcal{K}(p,q)$, $b \in \mathcal{K}(s,t)$ we have

$$\Phi(a)\Phi(b) = \begin{cases} \Phi\left((a \otimes 1_{q^{-1}r})(b \otimes 1_{s^{-1}r})\right) & \text{if } qP \cap sP = rP \text{ for some } r \in P, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Remark 3.9. If Φ is a Nica covariant representation of a well-aligned ideal $\mathcal{K}$, then Φ is a right-tensor representation and moreover the space

$$C^*(\Phi(\mathcal{K}))^0 := \operatorname{span}\{\bigcup_{p,q\in P} \Phi(\mathcal{K}(p,q))\} \tag{3.5}$$

is a dense $*$-subalgebra of $C^*(\Phi(\mathcal{K}))$; it is clearly closed under taking adjoints and it is closed under multiplication by (3.4). Hence $C^*(\Phi(\mathcal{K})) = \overline{\operatorname{span}}\{\bigcup_{p,q\in P} \Phi(\mathcal{K}(p,q))\}$.

Since a right-tensor C^*-precategory $\mathcal{L}$ is well-aligned in itself we may always talk about Nica covariant representations of $\mathcal{L}$. For every Nica covariant representation $\Phi : \mathcal{L} \to B$ of $\mathcal{L}$ and every well-aligned ideal $\mathcal{K}$ in $\mathcal{L}$ the restriction $\Phi : \mathcal{K} \to B$ is a Nica covariant representation of $\mathcal{K}$. Moreover, (3.4) readily implies that $C^*(\Phi(\mathcal{K}))$ is a C^*-subalgebra of $C^*(\Phi(\mathcal{L}))$, and if $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$, then $C^*(\Phi(\mathcal{K}))$ is in fact an ideal in $C^*(\Phi(\mathcal{L}))$.

Since the element r in the right hand side of (3.4) is determined only up to invertible elements in P^*, Nica covariant representations behave in a special way with respect to $\otimes 1_x$, $x \in P^*$.

Lemma 3.10. *Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $\mathcal{L}$. For every $x \in P^*$, $\otimes 1_x$ is an automorphism of $\mathcal{L}$ which maps $\mathcal{K}(p,q)$ onto $\mathcal{K}(px,qx)$ for every $p,q \in P$, thus it restricts to an automorphism of $\mathcal{K}$.*

Moreover, if $x \in P^*$ and $\Phi : \mathcal{K} \to B$ is a Nica covariant representation of $\mathcal{K}$, then $\Phi(a) = \Phi(a \otimes 1_x)$ for all $a \in \mathcal{K}(p,q)$, $p,q \in P$.

Proof. Let $x \in P^*$. Clearly, $\otimes 1_x$ is an automorphism of $\mathcal{L}$ since $\otimes 1_{x^{-1}}$ acts as an inverse. Take $a \in \mathcal{K}(p,q)$. As in the proof of Lemma 3.7, write $a = a'a''$ where $a' \in \mathcal{K}(p,q)$ and $a'' \in \mathcal{K}(q,q)$. Since $qP = qxP$, by (3.3) we get $a \otimes 1_x = (a'a'') \otimes 1_x = (a' \otimes 1_x)(a'' \otimes 1_x) \in \mathcal{K}(px,qx)$. If $\Phi : \mathcal{K} \to B$ is a Nica covariant representation of $\mathcal{K}$, using (3.4) we get $\Phi(a) = \Phi(a')\Phi(a'') = \Phi((a' \otimes 1_x)(a'' \otimes 1_x)) = \Phi(a \otimes 1_x)$. □

Existence (and uniqueness) of the universal C^*-algebra described in the following proposition can be shown as in the proof of Proposition 3.4, or by considering a quotient of the Toeplitz algebra $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$. Injectivity of the universal Nica covariant representation follows from Proposition 5.2 below.

Proposition 3.11. *Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $\mathcal{L}$. There are a C^*-algebra $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ and an injective Nica covariant representation $i_{\mathcal{K}} : \mathcal{K} \to \mathcal{NT}_{\mathcal{L}}(\mathcal{K})$, such that*

(a) *for every Nica covariant representation Φ of $\mathcal{K}$ there is a homomorphism $\Phi \rtimes P$ of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ such that $(\Phi \rtimes P) \circ i_{\mathcal{K}} = \Phi$; and*

(b) $\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) = C^*(i_{\mathcal{K}}(\mathcal{K}))$.

The C^*-algebra $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ is unique up to canonical isomorphism.

Definition 3.12. Given a well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$, the C^*-algebra $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ described in Proposition 3.11 is called the *Nica-Toeplitz algebra* of $\mathcal{K}$. We write $\mathcal{NT}(\mathcal{L})$ for the Nica-Toeplitz algebra $\mathcal{NT}_{\mathcal{L}}(\mathcal{L})$ associated to $\mathcal{L}$, viewed as a well-aligned ideal in itself.

Remark 3.13. The universal property of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ ensures existence of a homomorphism

$$\iota : \mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \longmapsto \mathcal{NT}(\mathcal{L}), \qquad i_{\mathcal{K}}(a) \longmapsto i_{\mathcal{L}}|_{\mathcal{K}}(a), \tag{3.6}$$

for $a \in \mathcal{K}(p,q)$ and $p,q \in P$. We note that ι is injective whenever the universal representation $i_{\mathcal{K}} : \mathcal{K} \to \mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ can be extended to a Nica covariant representation $\overline{i_{\mathcal{K}}} : \mathcal{L} \to B$ where B is a C^*-algebra containing $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$. Indeed, in this case we have $(\overline{i_{\mathcal{K}}} \rtimes P) \circ i_{\mathcal{L}} = \overline{i_{\mathcal{K}}}$, from which it follows that $(\overline{i_{\mathcal{K}}} \rtimes P) \circ \iota = \mathrm{id}_{\mathcal{NT}_{\mathcal{L}}(\mathcal{K})}$, showing the claimed injectivity. We will explore these issues in more detail in Section 11.

Remark 3.14. If $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$, then $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ does not depend on $\mathcal{L}$. Indeed, in this case, the definition of Nica covariant representations involves only elements of $\mathcal{K}$. Therefore, with $\mathcal{NT}(\mathcal{K})$ denoting the Nica-Toeplitz algebra of the right-tensor C^*-category $\mathcal{K}$, we have $\mathcal{NT}(\mathcal{K}) = \mathcal{NT}_{\mathcal{L}}(\mathcal{K})$. In general, $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ depends only on the C^*-precategory structure of $\mathcal{K}$ equipped with a family of mappings $\{N_r\}_{r\in P}$ where N_r, $r \in P$, is defined for quadruples $p,q,s,t \in P$ such that $qP \cap sP = rP$ by the formula

$$\mathcal{K}(p,q) \times \mathcal{K}(s,t) \ni (a,b) \longmapsto N_r(a,b) := (a \otimes 1_{q^{-1}r})(b \otimes 1_{s^{-1}r}) \in \mathcal{K}(pq^{-1}r, ts^{-1}r).$$

Note that $N_r(a,b)^* = N_r(a^*,b^*)$ and $N_e(a,b) = ab$ if $q = s$. Moreover, for any $c \in \mathcal{K}(u,w)$ where $uP \cap ts^{-1}rP = zP$, for some $z \in P$, we have

$$N_z(N_r(a,b),c) = N_{y^{-1}z}(a, N_y(b,c))$$

for any $y \in P$ such that $tP \cap uP = yP$ (we then necessarily have $qP \cap st^{-1}yP = y^{-1}zP$). Nevertheless, in this paper we will not pursue this more general intrinsic description of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ for three reasons. Firstly, such a theory would be technically more involved. Secondly, we do not have good examples that require such an approach. Thirdly, the relationship between $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ and $\mathcal{NT}(\mathcal{L})$ is interesting in its own right (in the context of Doplicher-Roberts algebras such a relationship was studied, for instance, in [11], [15], [19]). The latter problem, in our setting, will be addressed in Section 11.

Obviously, the Nica-Toeplitz algebra $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ may be viewed as a quotient of the Toeplitz algebra $\mathcal{T}_{\mathcal{L}}(\mathcal{K})$. If every two elements in P are comparable, then every ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ over P is automatically well-aligned and right-tensor representations of $\mathcal{K}$ coincide with Nica covariant representations. Hence $\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \cong \mathcal{T}_{\mathcal{L}}(\mathcal{K})$ in this case.

Example 3.15 (The case when $P = \mathbb{N}$)**.** Let $\mathcal{L}$ be a right-tensor C^*-precategory over $\mathbb{N}$ and $\mathcal{K}$ an ideal in $\mathcal{L}$. Due to above discussion $\mathcal{K}$ is automatically well-aligned, and a representation Φ of $\mathcal{K}$ is Nica covariant if and only if Φ is a right-tensor representation. For any ideal $\mathcal{J}$ in $J(\mathcal{K}) := \otimes 1^{-1}(\mathcal{K}) \cap \mathcal{K}$, C^*-algebras $\mathcal{O}_{\mathcal{L}}(\mathcal{K},\mathcal{J})$ were introduced in [19] as universal C^*-algebras with respect to right-tensor representations Φ of $\mathcal{K}$ satisfying $\Phi_{n,m}(a) = \Phi_{n+1,m+1}(a \otimes 1)$ for all $a \in \mathcal{J}(n,m)$ and $n,m,\in \mathbb{N}$. Therefore,

$$\mathcal{T}_{\mathcal{L}}(\mathcal{K}) \cong \mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \cong \mathcal{O}_{\mathcal{L}}(\mathcal{K},\{0\})$$

and every C^*-algebra $\mathcal{O}_{\mathcal{L}}(\mathcal{K},\mathcal{J})$ is a quotient of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$.

Example 3.16 (Product systems)**.** Let $X = \bigsqcup_{p\in P} X_p$ be a product system as defined in [14]. In [20, Section 2.1], cf. the introduction, we associate to X the right-tensor C^*-precategory $\mathcal{L}_X$. Then $\mathcal{K}_X = \{\mathcal{K}(X_p, X_q)\}_{p,q\in P}$ is an essential ideal in $\mathcal{L}_X$. By [20, Proposition 2.8] we have a natural isomorphism $\mathcal{T}_{\mathcal{L}_X}(\mathcal{K}_X) \cong \mathcal{T}(X)$ where $\mathcal{T}(X)$ is the Toeplitz algebra of X defined in [14]. Assume that P is a right LCM semigroup. Then $\mathcal{K}_X$ is well-aligned if and only if X is compactly aligned. In this case, [20, Proposition 2.10] gives

$$\mathcal{NT}_{\mathcal{L}_X}(\mathcal{K}_X) \cong \mathcal{NT}(X),$$

where $\mathcal{NT}(X)$ is the *Nica-Toeplitz algebra* associated to X, see [6], [14]. In [20] we also analyze a Doplicher-Roberts version $\mathcal{DR}(\mathcal{NT}(X))$ of $\mathcal{NT}(X)$, which by definition is $\mathcal{NT}(\mathcal{L}_X)$.

In view of the following lemma we may always assume that a well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ generates $\mathcal{L}$ as a right-tensor C^*-precategory.

Lemma 3.17. *For every well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ the spaces*

$$\mathcal{L}_{\mathcal{K}}(p,q) = \overline{\operatorname{span}}\{\mathcal{K}(s,t)\otimes 1_r : sr = p, tr = q, \text{ for } s,t,r \in P\}, \qquad p,q\in P,$$

define the minimal right-tensor sub-C^*-precategory of $\mathcal{L}$ containing $\mathcal{K}$. In particular, we have $\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \cong \mathcal{NT}_{\mathcal{L}_{\mathcal{K}}}(\mathcal{K})$.

Proof. Plainly, the family $\{\mathcal{L}_{\mathcal{K}}(p,q)\}_{p,q\in P}$ is closed under right-tensoring $\otimes 1$ and under taking adjoints. Suppose that $a\otimes 1_r \in \mathcal{L}_{\mathcal{K}}(p,q)$ and $b\otimes 1_w \in \mathcal{K}(q,z)$ where $a\in\mathcal{K}(s,t)$, $b\in \mathcal{K}(u,v)$, $sr = p$, $tr = q$, $uw = q$, $vw = z$. Then $tP\cap uP = yP$ for some $y\in P$, which implies that $(t^{-1}y)^{-1}r = y^{-1}q$ and $(u^{-1}y)^{-1}w = y^{-1}q$. Since $\mathcal{K}$ is well-aligned we have $(a\otimes 1_{t^{-1}y})(b\otimes 1_{u^{-1}y}) \in \mathcal{K}(st^{-1}y, vu^{-1}y)$. Using this we obtain

$$\begin{aligned}(a\otimes 1_r)(b\otimes 1_w) &= \Big((a\otimes 1_{t^{-1}y})\otimes 1_{y^{-1}q}\Big)\Big((b\otimes 1_{u^{-1}y})\otimes 1_{y^{-1}q}\Big)\\ &= \Big((a\otimes 1_{t^{-1}y})(b\otimes 1_{u^{-1}y})\Big)\otimes 1_{y^{-1}q} \in \mathcal{L}_{\mathcal{K}}(p,z).\end{aligned}$$

Hence $\{\mathcal{L}_{\mathcal{K}}(p,q)\}_{p,q\in P}$ is a right-tensor sub-C^*-precategory of $\mathcal{L}$. Clearly, it is the smallest sub-C^*-precategory of $\mathcal{L}$ containing $\mathcal{K}$ and invariant under $\otimes 1$. ☐

An important role in the theory is played by the following core C^*-algebra.

Definition 3.18. Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $\mathcal{L}$. For an arbitrary Nica covariant representation Φ of $\mathcal{K}$ the space

$$B_e^{\Phi} := \overline{\operatorname{span}}\Big\{\bigcup_{p\in P}\Phi(\mathcal{K}(p,p))\Big\}$$

is a C^*-algebra. We call B_e^{Φ} the *core C^*-subalgebra* of $C^*(\Phi(\mathcal{K}))$.

Remark 3.19. For any Nica covariant representation $\Phi:\mathcal{K}\to B$ the core C^*-algebra B_e^{Φ} is a non-degenerate subalgebra of $C^*(\Phi(\mathcal{K}))$. Indeed, by Lemma 2.7, every $a\in\mathcal{K}(p,q)$ can be written as $a = a_p a' a_q$ where $a_p\in\mathcal{K}(p,p)$, $a'\in\mathcal{K}(p,q)$, $a_q\in\mathcal{K}(q,q)$. Hence $\Phi(a)\in B_e^{\Phi}C^*(\Phi(\mathcal{K}))B_e^{\Phi}$. In particular, every multiplier $m\in M(B_e^{\Phi})$ of B_e^{Φ} extends via the formula $m\Phi(a) := (m\Phi(a_p))\Phi(a'a_q)$ to a multiplier of $C^*(\Phi(\mathcal{K}))$. We will use this embedding in the sequel to identify $M(B_e^{\Phi})$ as a subalgebra of $M(C^*(\Phi(\mathcal{K})))$.

4. Fock representation

Only for the purposes of this section we fix an *arbitrary* left cancellative semigroup P, a right-tensor C^*-precategory $(\mathcal{L}, \{\otimes 1_r\}_{r\in P})$ and an *arbitrary* ideal $\mathcal{K}$ in $\mathcal{L}$. We will construct a canonical right-tensor representation of $\mathcal{L}$ associated to $\mathcal{K}$, whose restriction to $\mathcal{K}$ is injective. This construction will proceed in two steps. First we associate a representation to each $t \in P$, regarded as a fixed source, and then we consider the direct sum of these representations as we vary t.

We fix $t \in P$. For each $s \in P$ the space $X_{s,t} := \mathcal{K}(s,t)$ is naturally equipped with a structure of a right Hilbert module over $A_t := \mathcal{K}(t,t)$ given by

$$x \cdot a := xa, \quad \langle x, y\rangle := x^* y, \qquad x, y \in X_{s,t},\ a \in A_t.$$

Thus we may consider the following direct sum right Hilbert A_t-module:

$$\mathcal{F}^t_{\mathcal{K}} := \bigoplus_{s\in P} X_{s,t}.$$

We will construct representations of $\mathcal{L}$ using the maps defined in the following lemma.

Lemma 4.1. *Let $a \in \mathcal{L}(p,q)$ for $p, q \in P$. For each $s \in qP$ there is a well defined operator $T^{s,t}_{p,q}(a) \in \mathcal{L}(X_{s,t}, X_{pq^{-1}s,t})$ given by*

$$T^{s,t}_{p,q}(a)x := (a \otimes 1_{q^{-1}s})x, \qquad x \in X_{s,t}.$$

The adjoint is $T^{pq^{-1}s,t}_{q,p}(a^*) \in \mathcal{L}(X_{pq^{-1}s,t}, X_{s,t})$.

Proof. Let $x \in X_{s,t}$ and $y \in X_{pq^{-1}s,t}$. Clearly, $(a \otimes 1_{q^{-1}s})x \in X_{pq^{-1}s,t} = \mathcal{K}(pq^{-1}s, t)$, and

$$\begin{aligned}\langle T^{s,t}_{p,q}(a)x, y\rangle &= \langle (a \otimes 1_{q^{-1}s})x, y\rangle = x^*(a^* \otimes 1_{q^{-1}s})y = \langle x, (a^* \otimes 1_{q^{-1}s})y\rangle \\ &= \langle x, T^{pq^{-1}s,t}_{q,p}(a^*)y\rangle.\end{aligned}$$

□

Let $a \in \mathcal{L}(p,q)$ for $p, q \in P$. By Lemma 4.1, under the obvious identifications of the Hilbert modules $X_{s,t}$, $s \in P$, with the corresponding submodules of $\mathcal{F}^t_{\mathcal{K}}$, we have $T^{s,t}_{p,q}(a) \in \mathcal{L}(X_{s,t}, X_{pq^{-1}s,t}) \subseteq \mathcal{L}(\mathcal{F}^t_{\mathcal{K}})$. Since $\|T^{s,t}_{p,q}(a)\| \le \|a\|$ and the map $qP \ni s \to pq^{-1}s \in pP$ is a bijection, the direct sum

$$\overline{T}^t_{p,q}(a) := \bigoplus_{s\in qP} T^{s,t}_{p,q}(a) \tag{4.1}$$

is an operator in $\mathcal{L}(\mathcal{F}^t_{\mathcal{K}})$ with norm bounded by $\|a\|$. In other words, we get a contractive mapping $\overline{T}^t_{p,q} : \mathcal{L}(p,q) \to \mathcal{L}(\mathcal{F}^t_{\mathcal{K}})$, which satisfies

$$\overline{T}^t_{p,q}(a)x = \begin{cases} (a \otimes 1_{q^{-1}s})x & \text{if } s \in qP, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

for every $a \in \mathcal{L}(p,q)$, $s \in P$ and $x \in X_{s,t}$.

Lemma 4.2. *For each $t \in P$, the family of maps $\overline{T}^t = \{\overline{T}^t_{p,q}\}_{p,q\in P}$ given by (4.2) is a right-tensor representation $\overline{T}^t : \mathcal{L} \to \mathcal{L}(\mathcal{F}^t_{\mathcal{K}})$.*

Proof. By Lemma 4.1 we have $(T^{s,t}_{p,q}(a))^* = T^{pq^{-1}s,t}_{q,p}(a^*)$ for $a \in \mathcal{L}(p,q)$, and therefore it follows from (4.1) that $\overline{T}^t_{p,q}(a)^* = \overline{T}^t_{q,p}(a^*)$. Clearly, the maps $\overline{T}^t_{p,q}$, $p,q \in P$ are linear. Moreover, for any $a \in \mathcal{L}(r,p)$, $b \in \mathcal{L}(p,q)$, $x \in X_{s,t}$, where $r,p,q \in P$ with $s \in qP$, we have

$$T^{pq^{-1}s,t}_{r,p}(a)T^{s,t}_{p,q}(b)x = (a \otimes 1_{q^{-1}s})(b \otimes 1_{q^{-1}s})x = (ab \otimes 1_{q^{-1}s})x = T^{s,t}_{r,q}(ab)x.$$

Hence $\overline{T}^t_{r,p}(a)\overline{T}^t_{p,q}(b) = \overline{T}^t_{r,q}(ab)$ and thus $\overline{T}^t : \mathcal{L} \to \mathcal{L}(\mathcal{F}^t_{\mathcal{K}})$ is a representation of C^*-precategories.

To see that $\overline{T}^t$ is a right-tensor representation let $a \in \mathcal{L}(p,q)$ and $b \in \mathcal{L}(s,l)$ for $p,q,s,l \in P$ such that $sP \subseteq qP$. Note that if $w \notin lP$ then both $\overline{T}^t_{p,q}(a)\overline{T}^t_{s,l}(b)$ and $\overline{T}^t_{pq^{-1}s,l}((a \otimes 1_{q^{-1}s})b)$ act as zero on $X_{w,t}$. Assume then that $w \in lP$ and let $x \in X_{w,t}$. Then $(b \otimes 1_{l^{-1}w})x \in X_{sl^{-1}w,t}$ and $sl^{-1}w \in qP$. Thus we have

$$\begin{aligned}\overline{T}^t_{p,q}(a)\overline{T}^t_{s,l}(b)x &= \overline{T}^t_{p,q}(a)(b \otimes 1_{l^{-1}w})x = (a \otimes 1_{q^{-1}sl^{-1}w})(b \otimes 1_{l^{-1}w})x \\ &= \Big(\big((a \otimes 1_{q^{-1}s})b\big) \otimes 1_{l^{-1}w}\Big)x = \overline{T}^t_{pq^{-1}s,l}((a \otimes 1_{q^{-1}s})b)x.\end{aligned}$$

Accordingly, $\overline{T}^t_{p,q}(a)\overline{T}^t_{s,l}(b)$ and $\overline{T}^t_{pq^{-1}s,l}((a \otimes 1_{q^{-1}s})b)$ coincide, and $\overline{T}^t : \mathcal{L} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ is a right-tensor representation. □

Note that for each $t \in P$, we may view $\mathcal{F}^t_{\mathcal{K}}$ as a right Hilbert module over the C^*-algebra $A := \bigoplus_{p\in P} A_p$, where multiplication on the right by an element of the summand A_p for $p \neq t$ is defined to be zero. We define the *Fock module* of $\mathcal{K}$ to be the direct sum Hilbert A-module of $\mathcal{F}^t_{\mathcal{K}}$ as $t \in P$:

$$\mathcal{F}_{\mathcal{K}} := \bigoplus_{t\in P} \mathcal{F}^t_{\mathcal{K}} = \bigoplus_{s,t\in P} X_{s,t}.$$

Accordingly, $\mathcal{F}_{\mathcal{K}}$ consists of elements $\bigoplus_{s,t\in P} x_{s,t}$ where $x_{s,t} \in \mathcal{K}(s,t)$, $s,t \in P$, and the element $\bigoplus_{t\in P}\Big(\sum_{s\in P} x^*_{s,t}x_{s,t}\Big)$ belongs to the C^*-algebraic direct sum $\bigoplus_{t\in P} \mathcal{K}(t,t)$. We will treat the C^*-algebraic direct product $\prod_{t\in P} \mathcal{L}(\mathcal{F}^t_{\mathcal{K}})$ as a C^*-subalgebra of $\mathcal{L}(\mathcal{F}_{\mathcal{K}})$.

Proposition 4.3. *The direct sum of representations from Lemma 4.2 as t varies in P yields a right-tensor representation $\overline{T} : \mathcal{L} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ determined by the formula*

$$\overline{T}_{p,q}(a)x = \begin{cases}(a \otimes 1_{q^{-1}s})x & \text{if } s \in qP,\\ 0 & \text{otherwise,}\end{cases} \tag{4.3}$$

for $a \in \mathcal{L}(p,q)$, $x \in X_{s,t}$ and $p,q,s,t \in P$. Furthermore, the restriction of $\overline{T}$ to $\mathcal{K}$ yields an injective right-tensor representation $T : \mathcal{K} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}})$.

Proof. It is immediate that the direct sum of right-tensor representations: $\overline{T}_{p,q} := \bigoplus_{t\in P} \overline{T}^t_{p,q} = \bigoplus_{s\in qP, t\in P} T^{s,t}_{p,q}$, $p,q \in P$, yields a right-tensor representation $\overline{T} : \mathcal{L} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ which satisfies (4.3). Let $T : \mathcal{K} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ be its restriction to $\mathcal{K}$. For every $p \in P$ the map $T^{p,p}_{p,p} : \mathcal{K}(p,p) \to \mathcal{L}(X_{p,p})$ is injective and hence $T_{p,p} : \mathcal{K}(p,p) \to \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ is injective. □

Definition 4.4. We call the right-tensor representation $T : \mathcal{K} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ from Proposition 4.3 the *Fock representation* of the ideal $\mathcal{K}$ in the right-tensor C^*-precategory $\mathcal{L}$.

Remark 4.5. For each $t \in P$, we may view the restriction $T^t : \mathcal{K} \to \mathcal{L}(\mathcal{F}^t_{\mathcal{K}})$ of $\overline{T}^t$ to $\mathcal{K}$ as a Fock representation of $\mathcal{K}$ with fixed source t. However, T^t is injective if and only if for every $a \in \mathcal{K}(p,p)$ and $p \in P$ there is $r \in P$ such that $(a \otimes 1_r)\mathcal{K}(pr,t) \neq 0$; and this may fail.

Remark 4.6. The Fock representation is the direct sum $T = \bigoplus_{t\in P} T^t$ of *t-th Fock representations* $T^t : \mathcal{K} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)$, $t \in P$. So by projecting, for each $t \in P$, we get a surjective homomorphism $h_t : C^*(T(\mathcal{K})) \to C^*(T^t(\mathcal{K}))$, where $h_t \circ T = T^t$. We will show that h_e is an isomorphism for Fell bundles (cf. Proposition 7.6 below) and for right-tensor C^*-precategories arising from compactly aligned product systems, see [20]. Thus our Fock representation generalizes those for product systems and Fell bundles.

The grading of the Fock Hilbert module $\mathcal{F}_{\mathcal{K}}$ yields natural conditional expectations. To make this explicit we introduce some notation. For $w, t \in P$ we let $Q_w^t \in \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)$ be the projection onto $X_{w,t}$, and $Q_w := \bigoplus_{s\in P} Q_w^s \in \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ be the projection onto $\bigoplus_{s\in P} X_{w,s}$. Note that Q_w projects onto the subspace of $\mathcal{F}_{\mathcal{K}}$ of fixed range w.

Lemma 4.7. *Let $T : \mathcal{L} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ be the Fock representation of $\mathcal{K}$.*

(a) *For each $t \in P$, the space $D^t := \{S \in \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t) : Q_w^t S = S Q_w^t \text{ for every } w \in P\}$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)$ and the map $E^t : \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t) \mapsto D^t$ given by*

$$E^t(S) = \sum_{w\in P} Q_w^t S Q_w^t, \qquad S \in \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t), \tag{4.4}$$

is a faithful conditional expectation.

(b) *The space $D := \{S \in \mathcal{L}(\mathcal{F}_{\mathcal{K}}) : Q_w S = S Q_w \text{ for every } w \in P\}$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{F}_{\mathcal{K}})$ and the map $E : \mathcal{L}(\mathcal{F}_{\mathcal{K}}) \mapsto D$ given by*

$$E(S) = \sum_{w\in P} Q_w S Q_w, \qquad S \in \mathcal{L}(\mathcal{F}_{\mathcal{K}}), \tag{4.5}$$

is a faithful conditional expectation. Furthermore, $E|_{\prod_{t\in P}\mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)} = \bigoplus_{t\in P} E^t$.

Proof. For part (a), clearly D^t is a C^*-subalgebra of $\mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)$. The map $a \mapsto Q_w^t a Q_w^t$ from $\mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)$ to $\mathcal{L}(X_{w,t}) \subseteq \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)$ is a contractive completely positive map with range in D^t. The direct sum of these maps is a well defined contractive completely positive map $E^t : \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t) \to D^t \subseteq \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)$ which is the identity on D^t. Hence (4.4) defines a conditional expectation as claimed. That E^t is faithful follows because if $a \in \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)$ is such that $E^t(a^*a) = 0$ then for every $x \in X_{w,t}$, $w \in P$,

$$\|ax\|^2 = \|\langle ax, ax\rangle\| = \|\langle x, a^*ax\rangle\| = \|\langle x, E^t(a^*a)x\rangle\| = 0.$$

Since the elements $x \in X_{w,t}$, $w \in P$, span $\mathcal{F}_{\mathcal{K}}^t$ it follows that $a = 0$. The proof of part (b) is analogous to (a) and is left to the reader. □

5. Reduced Nica-Toeplitz algebra and the transcendental core

In this section, we come back to our standing assumption that P is a right LCM semigroup. We fix a right-tensor C^*-precategory $(\mathcal{L}, \{\otimes 1_r\}_{r\in P})$ and a well-aligned ideal $\mathcal{K}$ in $\mathcal{L}$. Under these assumptions, the Fock representation is Nica covariant:

Lemma 5.1. *For each $t \in P$, the family of maps $\overline{T}^t = \{\overline{T}^t_{p,q}\}_{p,q\in P}$ given by* (4.2) *is a Nica covariant representation $\overline{T}^t : \mathcal{L} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}}^t)$.*

Proof. By Lemma 4.2 we only need to show that $\overline{T}^t$ is Nica covariant. Let $a \in \mathcal{L}(p, q)$ and $b \in \mathcal{L}(s, l)$ for $p, q, s, l \in P$. Note that if $w \in lP$ and $x \in X_{w,t}$, then $T^{w,t}_{s,l}(b)$ is in $X_{sl^{-1}w,t}$. Since $T^{u,t}_{p,q}(a)$ acts in $X_{u,t}$, we have

$$T^{u,t}_{p,q}(a) T^{w,t}_{s,l}(b) \neq 0 \implies u = sl^{-1}w \text{ and } u \in qP. \tag{5.1}$$

In particular, $T_{p,q}^{u,t}(a)T_{s,l}^{w,t}(b)\neq 0$ implies that $qP\cap sP=rP$ for some $r\in P$ such that $u\in rP$. Hence, if $qP\cap sP=\emptyset$, then $\overline{T}_{p,q}^{t}(a)\overline{T}_{s,l}^{t}(b)=0$. Assume now $qP\cap sP=rP$. Using (5.1) and the fact that $ls^{-1}rP\ni w\to sl^{-1}w\in rP$ is a bijection, we get

$$\begin{aligned}\overline{T}_{p,q}^{t}(a)\overline{T}_{s,l}^{t}(b)&=\bigoplus_{u\in qP}T_{p,q}^{u,t}(a)\bigoplus_{w\in lP}T_{s,l}^{w,t}(b)=\bigoplus_{u\in rP}T_{p,q}^{u,t}(a)\bigoplus_{w\in ls^{-1}rP}T_{s,l}^{w,t}(b)\\&=\bigoplus_{w\in ls^{-1}rP}T_{p,q}^{sl^{-1}w,t}(a)T_{s,l}^{w,t}(b).\end{aligned}$$

Moreover, for every $w\in ls^{-1}rP$ and every $x\in X_{w,t}$ we have

$$\begin{aligned}T_{p,q}^{sl^{-1}w,t}(a)T_{s,l}^{w,t}(b)x&=(a\otimes 1_{q^{-1}sl^{-1}w})(b\otimes 1_{l^{-1}w})x\\&=(a\otimes 1_{q^{-1}r(ls^{-1}r)^{-1}w})(b\otimes 1_{s^{-1}r(ls^{-1}r)^{-1}w})x\\&=T_{pq^{-1}r,ls^{-1}r}^{w,t}\left((a\otimes 1_{q^{-1}r})(b\otimes 1_{s^{-1}r})\right)x.\end{aligned}$$

Accordingly, $\overline{T}_{p,q}^{t}(a)\overline{T}_{s,l}^{t}(b)=T_{pq^{-1}r,ls^{-1}r}^{t}\left((a\otimes 1_{q^{-1}r})(b\otimes 1_{s^{-1}r})\right)$. Thus $\overline{T}^{t}:\mathcal{L}\to\mathcal{L}(\mathcal{F}_{\mathcal{K}})$ is Nica covariant. □

Proposition 5.2. *The right-tensor representation $\overline{T}:\mathcal{L}\to\mathcal{L}(\mathcal{F}_{\mathcal{K}})$ given by* (4.3) *is Nica covariant. In particular, the Fock representation $T:\mathcal{K}\to\mathcal{L}(\mathcal{F}_{\mathcal{K}})$ is an injective Nica covariant representation.*

Proof. Since a direct sum of Nica covariant representations is a Nica covariant representation, the first part follows from Lemma 5.1. The second part follows from Proposition 4.3 and the fact that restriction of a Nica covariant representation to a well-aligned ideal is Nica covariant. □

Definition 5.3. We define the *reduced Nica-Toeplitz algebra* of the well-aligned ideal $\mathcal{K}$ in the right-tensor C^*-precategory $\mathcal{L}$ to be the C^*-algebra

$$\mathcal{NT}_{\mathcal{L}}^{r}(\mathcal{K}):=C^*(T(\mathcal{K})),$$

and we call $T\rtimes P:\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\to\mathcal{NT}_{\mathcal{L}}^{r}(\mathcal{K})$ *the regular representation* of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$, cf. Proposition 3.11. When $\mathcal{K}=\mathcal{L}$, we also write $\mathcal{NT}^{r}(\mathcal{L}):=\mathcal{NT}_{\mathcal{L}}^{r}(\mathcal{L})$.

It turns out that if P is right cancellative, in particular if it is a group, then the conditional expectation (4.5) restricts to a conditional expectation of the C^*-algebra $\mathcal{NT}_{\mathcal{L}}^{r}(\mathcal{K})$ onto the core C^*-subalgebra B_e^T of $\mathcal{NT}_{\mathcal{L}}^{r}(\mathcal{K})$. If P is not right cancellative, this is no longer true and in particular E may not preserve the C^*-algebra $\mathcal{NT}_{\mathcal{L}}^{r}(\mathcal{K})$.

Proposition 5.4. *The conditional expectation defined by* (4.5) *restricts to a faithful contractive completely positive map $E^T:\mathcal{NT}_{\mathcal{L}}^{r}(\mathcal{K})\to\mathcal{L}(\mathcal{F}_{\mathcal{K}})$ given by*

$$E^T\Big(\sum_{p,q\in F}T(a_{p,q})\Big)=\sum_{p,q\in F}\bigoplus_{\substack{w\in pP\cap qP,t\in P\\ p^{-1}w=q^{-1}w}}T_{p,q}^{w,t}(a_{p,q})\tag{5.2}$$

for $F\subseteq P$ finite, $a_{p,q}\in\mathcal{K}(p,q)$, $p,q\in F$. The range of E^T is the following self-adjoint operator space:

$$B_{\mathcal{K}}:=\overline{\operatorname{span}}\Big\{\bigoplus_{\substack{w\in pP\cap qP,t\in P\\ p^{-1}w=q^{-1}w}}T_{p,q}^{w,t}(a):a\in\mathcal{K}(p,q),p,q\in P\Big\}\subseteq\mathcal{L}(\mathcal{F}_{\mathcal{K}}).\tag{5.3}$$

If P is cancellative, then $B_{\mathcal{K}} = \overline{\operatorname{span}}\left\{\bigcup_{p\in P} T_{p,p}(\mathcal{K}(p,p))\right\}$ equals the core C^*-subalgebra B_e^T of $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ and E^T is a faithful conditional expectation onto $B_{\mathcal{K}}$ given by the formula

$$E^T\Big(\sum_{p,q\in F} T(a_{p,q})\Big) = \sum_{p\in F} T(a_{p,p}), \tag{5.4}$$

for all $a_{p,q} \in \mathcal{K}(p,q)$, $p,q \in F$ and $F \subseteq P$ finite.

Proof. The crucial observation is the following claim: for $a \in \mathcal{L}(p,q)$, $w \in qP$ and $p,q,t \in P$ we have

$$Q_w T^{w,t}_{p,q}(a) Q_w = \begin{cases} T^{w,t}_{p,q}(a) & \text{if } w \in pP\cap qP \text{ and } p^{-1}w = q^{-1}w \\ 0 & \text{otherwise.}\end{cases} \tag{5.5}$$

To see this, recall that $T^{w,t}_{p,q}(a)$ acts as an adjointable operator from $X_{w,t}$ to $X_{pq^{-1}w,t}$. Since Q_w is the projection onto fixed range w, the only possibility to have $Q_w T^{w,t}_{p,q}(a) Q_w$ nonzero is that $w = pq^{-1}w$, in which case it equals $T^{w,t}_{p,q}(a)$. Thus to prove (5.5) it remains to see that if $p,q \in P$ with $w \in qP$, then $w = pq^{-1}w$ is equivalent to $w \in pP \cap qP$ and $p^{-1}w = q^{-1}w$. However, the non-trivial left to right implication follows since $w \in pP$ implies $w = ps$ for some $s \in P$ and so left cancellation gives $s = q^{-1}w = p^{-1}w$.

Let now $t \in P$, $F \subseteq P$ finite, $a_{p,q} \in \mathcal{K}(p,q)$ for $p,q \in F$. By (5.5), $E(T_{p,q}(a_{p,q})) = \{0\}$ when $pP \cap qP = \emptyset$, and if $pP \cap qP \neq \emptyset$, then

$$E(T_{p,q}(a_{p,q})) = E\left(\bigoplus_{s\in qP, t\in P} T^{s,t}_{p,q}(a_{p,q})\right) = \bigoplus_{\substack{w\in pP\cap qP, t\in P\\ p^{-1}w=q^{-1}w}} T^{w,t}_{p,q}(a_{p,q}).$$

Thus $E^T = E|_{\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})}$ maps $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ onto $B_{\mathcal{K}}$ according to the formula (5.2).

Now suppose that P is right cancellative. Note that $w \in pP\cap qP$ and $p^{-1}w = q^{-1}w$ if and only if $p = q$ and $w \in pP = qP$, where the non-trivial left to right implication follows upon invoking right cancellation in $w = p(p^{-1}w) = q(q^{-1}w)$. By using this observation one sees that (5.2) reduces to (5.4) and $B_{\mathcal{K}} = B_e^T$. Clearly, E^T is an idempotent map and therefore a conditional expectation onto $B_{\mathcal{K}}$. □

Definition 5.5. Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $\mathcal{L}$. We call the space $B_{\mathcal{K}}$ given by (5.3) the *transcendental core* for $\mathcal{K}$, and the map E^T given by (5.2) the *transcendental conditional expectation* from $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ onto $B_{\mathcal{K}}$.

Remark 5.6. The transcendental core $B_{\mathcal{K}}$ always contains the core C^*-subalgebra B_e^T, see Definition 3.18. For semigroups P that are not right cancellative, we may have $B_e^T \subsetneq B_{\mathcal{K}}$, and then it is not clear whether there is a conditional expectation from $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ onto B_e^T.

Remark 5.7. In view of Lemma 4.7 and Proposition 5.4 one sees that for each $t \in P$, the conditional expectation defined by (4.4) restricts to a faithful contractive completely positive map $E^{T,t}$ on $C^*(T^t(\mathcal{K}))$ with range the subspace of $B_{\mathcal{K}}$ where t is fixed. In fact, we have $E^T = \bigoplus_{t\in P} E^{T,t}$.

The semilattice of projections introduced in the following lemma is one of the key tools to analyze the structure of the reduced Nica-Toeplitz algebra.

Lemma 5.8. *For each $p \in P$, let $Q^T_{\langle p\rangle} \in \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ be the projection*

$$Q^T_{\langle p\rangle}\Big(\bigoplus_{s,t\in P} x_{s,t}\Big) := \bigoplus_{s\in pP, t\in P} x_{s,t}, \qquad \bigoplus_{s,t\in P} x_{s,t} \in \mathcal{F}_{\mathcal{K}}.$$

The assignment $pP \mapsto Q^T_{\langle p\rangle}$ *and* $\emptyset \mapsto 0$ *forms a semilattice homomorphism* $J(P) \longmapsto \operatorname{Proj}(\mathcal{L}(\mathcal{F}_{\mathcal{K}}))$*, meaning that*

$$Q^T_{\langle p\rangle}Q^T_{\langle q\rangle} = \begin{cases} Q^T_{\langle r\rangle}, & \text{if } pP \cap qP = rP \text{ for some } r \in P \\ 0, & \text{if } pP \cap qP = \emptyset \end{cases} \tag{5.6}$$

for all $p, q \in P$*. In particular, we have* $\mathcal{J}(P) \cong \{Q^T_{\langle p\rangle} : p \in P\} \cup \{0\}$*.*

Proof. The proof is immediate from the definition of $Q^T_{\langle p\rangle}$. □

Lemma 5.9. *Let* $\overline{T} : \mathcal{L} \to \mathcal{L}(\mathcal{F}_{\mathcal{K}})$ *be the representation given by* (4.3)*. Then the projections introduced in Lemma 5.8 satisfy the relation:*

$$\overline{T}(a)Q^T_{\langle p\rangle} = \begin{cases} \overline{T}(a \otimes 1_{q^{-1}w}) & \text{if } qP \cap pP = wP \text{ for some } w \in P, \\ 0 & \text{otherwise}, \end{cases} \tag{5.7}$$

for all $a \in \mathcal{L}(r, q)$*,* $r, p, q \in P$*.*

Proof. Let $a \in \mathcal{L}(r, q)$ and $x_{u,v} \in X_{u,v}$ where $u, v, r, q \in P$. Let $p \in P$. If $\overline{T}(a)Q^T_{\langle p\rangle}x_{u,v} \neq 0$ then by the definition of $Q^T_{\langle p\rangle}$ and (4.3) we necessarily have that $u \in pP$ and $u \in qP$, which implies that $qP \cap pP = wP$ for some $w \in P$ where $u \in wP$. Assume then that $qP \cap sP = wP$ for some $w \in P$. Note that $\overline{T}(a \otimes 1_{q^{-1}w})x_{u,v} \neq 0$ implies that $u \in wP$. Thus both sides of (5.7) are zero when $u \notin wP$. It remains to verify that equality holds when $u \in wP$. This follows from two applications of (4.3):

$$\begin{aligned} \overline{T}(a)x_{u,v} &\overset{(4.3)}{=} (a \otimes 1_{q^{-1}u})x_{u,v} = (a \otimes 1_{q^{-1}(w(w^{-1}u))})x_{u,v} \\ &= \Big((a \otimes 1_{q^{-1}w}) \otimes 1_{w^{-1}u}\Big)x_{u,v} \overset{(4.3)}{=} \overline{T}(a \otimes 1_{q^{-1}w})x_{u,v}. \end{aligned}$$

□

Lemma 5.10. *The projections* $\{Q^T_{\langle p\rangle}\}_{p\in P}$ *may be treated as multipliers of both* $C^*(\overline{T}(\mathcal{L}))$ *and* $B^{\overline{T}}_e$*. If* $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$ *in the sense of Definition 3.6, then* $\{Q^T_{\langle p\rangle}\}_{p\in P}$ *may be treated as multipliers of both* $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ *and* B^T_e*.*

Proof. By Lemma 2.7 we have $T(\mathcal{K}(p, p))X_{p,t} = X_{p,t}$ for all $p, t \in P$, which implies that B^T_e and therefore also $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$, $C^*(\overline{T}(\mathcal{L}))$ and $B^{\overline{T}}_e$ act on $\mathcal{F}_{\mathcal{K}}$ in a non-degenerate way. Using (5.7) we see that $C^*(\overline{T}(\mathcal{L}))Q^T_{\langle s\rangle} \subseteq C^*(\overline{T}(\mathcal{L}))$ and $B^{\overline{T}}_e Q^T_{\langle p\rangle} \subseteq B^{\overline{T}}_e$. Thus, since $Q^T_{\langle p\rangle}$ is self-adjoint, we may treat $Q^T_{\langle p\rangle}$ as a multiplier of both $C^*(\overline{T}(\mathcal{L}))$ and $B^{\overline{T}}_e$, cf. [26, Proposition 2.3]. If $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$, then (5.7) implies that $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})Q^T_{\langle p\rangle} \subseteq \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ and $B^T_e Q^T_{\langle p\rangle} \subseteq B^T_e$, which finishes the proof. □

We end this section by establishing certain norm formulas for elements in the transcendental core $B_{\mathcal{K}}$ which can be considered far reaching generalizations of similar formulas obtained in [14, Lemma 7.4].

We begin by introducing some notation, which is inspired by [25, Remark 1.5] and [16, Remark 5.2], where quasi-ordered groups were considered, cf. also [14] and [5]. Suppose that C is a finite subset of P. We put $\sigma(C) := e$ if $C = \emptyset$. If C is non-empty, then either $\bigcap_{c\in C} cP = \emptyset$ or $\bigcap_{c\in C} cP = c'P$ for an element $c' \in P$ (determined by C up to multiplication from the right by elements of P^*). In the latter case we write $\sigma(C) = c'$. Let F be a finite subset of P. A subset C of F is an *initial segment* of F if $\sigma(C)$ exists in P and

$C = \{t \in F : t \leq \sigma(C)\}$. We denote by $\mathrm{In}(F)$ the collection of all initial segments of F. For each $C \in \mathrm{In}(F)$, the set

$$P_{F,C} := \{t \in P : \sigma(C) \leq t \text{ and } f \not\leq t \text{ for all } f \in F \setminus C\}$$

is non-empty. Note that neither definition of initial segment nor of the set $P_{F,C}$ depends on the choice of $\sigma(C)$. Moreover, $\{P_{F,C} : C \in \mathrm{In}(F)\}$ form a decomposition of the set P; in particular, $s \in P_{F,C}$ if and only if $C = \{t \in F : t \leq s\}$ (the latter set is in $\mathrm{In}(F)$ for every $s \in P$). Now, for every finite set $F \subseteq P$ and any $C \in \mathrm{In}(F)$,

$$Q^T_{F,C} := Q^T_{\langle \sigma(C) \rangle} \prod_{s \in F \setminus C} (1 - Q^T_{\langle s \rangle})$$

are mutually orthogonal projections that sum up to the identity in $\mathcal{L}(\mathcal{F}_{\mathcal{K}})$ as C varies. We use this partition of the identity to prove the following lemma.

Lemma 5.11. *Suppose that $\mathcal{K}$ is a well-aligned ideal in a right-tensor C^*-precategory $\mathcal{L}$ and consider an element*

$$Z = \sum_{p,q \in F} \bigoplus_{\substack{w \in pP \cap qP, t \in P \\ p^{-1}w = q^{-1}w}} T^{w,t}_{p,q}(a_{p,q}) \in B_{\mathcal{K}}, \tag{5.8}$$

where $F \subseteq P$ is a finite set and $a_{p,q} \in \mathcal{K}(p,q)$ for $p, q \in F$. Then

$$\|Z\| = \max_{C \in \mathrm{In}(F)} \sup_{w \in P_{F,C}} \Big\| T^{w,w}_{w,w}\big(\sum_{\substack{p,q \in C \\ p^{-1}w = q^{-1}w}} a_{p,q} \otimes 1_{q^{-1}w} \big) \Big\|. \tag{5.9}$$

Proof. For any projection $Q^T_{\langle r \rangle}$, $r \in P$, and every $a \in \mathcal{K}(p,q)$, $p, q \in P$, we have

$$\bigoplus_{\substack{w \in pP \cap qP, t \in P \\ p^{-1}w = q^{-1}w}} T^{w,t}_{p,q}(a) Q^T_{\langle r \rangle} = \bigoplus_{\substack{w \in pP \cap qP \cap rP, t \in P \\ p^{-1}w = q^{-1}w}} T^{w,t}_{p,q}(a) = Q^T_{\langle r \rangle} \bigoplus_{\substack{w \in pP \cap qP, t \in P \\ p^{-1}w = q^{-1}w}} T^{w,t}_{p,q}(a).$$

Thus the projections $Q^T_{F,C}$, where $C \in \mathrm{In}(F)$, commute with Z. Since they form a partition of identity and

$$ZQ^T_{F,C} = \sum_{p,q \in C} \bigoplus_{\substack{w \in P_{F,C}, t \in P \\ p^{-1}w = q^{-1}w}} T^{w,t}_{p,q}(a_{p,q}) = \bigoplus_{w \in P_{F,C}, t \in P} \sum_{\substack{p,q \in C \\ p^{-1}w = q^{-1}w}} T^{w,t}_{p,q}(a_{p,q}) \tag{5.10}$$

we get

$$\begin{aligned} \|Z\| &= \max \Big\{ \Big\| \bigoplus_{w \in P_{F,C}, t \in P} \sum_{\substack{p,q \in C \\ p^{-1}w = q^{-1}w}} T^{w,t}_{p,q}(a_{p,q}) \Big\| : C \in \mathrm{In}(F) \Big\} \\ &= \max_{C \in \mathrm{In}(F)} \sup_{w \in P_{F,C}, t \in P} \Big\| \sum_{\substack{p,q \in C \\ p^{-1}w = q^{-1}w}} T^{w,t}_{p,q}(a_{p,q}) \Big\|. \end{aligned}$$

Now, in the summation above over $p, q \in C$ we have $a_{p,q} \otimes 1_{q^{-1}w} \in \mathcal{L}(w,w)$, and therefore

$$\|Z\| = \max_{C \in \mathrm{In}(F)} \sup_{w \in P_{F,C}, t \in P} \Big\| T^{w,t}_{w,w}\Big(\sum_{\substack{p,q \in C \\ p^{-1}w = q^{-1}w}} a_{p,q} \otimes 1_{q^{-1}w} \Big) \Big\|. \tag{5.11}$$

Next note that for any $w, t \in P$ and $a \in \mathcal{L}(w,w)$ we have $\ker T^{w,w}_{w,w} \subseteq \ker T^{w,t}_{w,w}$ due to

$$\begin{aligned} a \in \ker T^{w,w}_{w,w} &\iff a\mathcal{K}(w,w) = \{0\} \implies a\mathcal{K}(w,t)\mathcal{K}(t,w) = \{0\} \\ &\iff a\mathcal{K}(w,t) = \{0\} \iff a \in \ker T^{w,t}_{w,w}. \end{aligned} \tag{5.12}$$

Therefore, the supremum in (5.11) is attained for $t = w$. This proves (5.9). ☐

Corollary 5.12. *With the assumptions from Lemma 5.11, if $\mathcal{K}$ is essential in the right-tensor sub-C^*-precategory $\mathcal{L}_{\mathcal{K}}$ of $\mathcal{L}$ generated by $\mathcal{K}$, cf. Lemma 3.17, then* (5.9) *reduces to*

$$\|Z\| = \max_{C\in \mathrm{In}(F)} \sup_{w\in P_{F,C}} \Big\| \sum_{\substack{p,q\in C\\ p^{-1}w=q^{-1}w}} a_{p,q}\otimes 1_{q^{-1}w}\Big\|. \tag{5.13}$$

If, additionally, $Z = \sum_{p\in F} T_{p,p}(a_{p,p}) \in B_e^T$, then

$$\|Z\| = \max\Big\{\Big\| \sum_{p\in C} a_{p,p}\otimes 1_{p^{-1}\sigma(C)}\Big\| : C \in \mathrm{In}(F)\Big\}. \tag{5.14}$$

Proof. Note that elements of the form $\sum_{\substack{p,q\in C\\ p^{-1}w=q^{-1}w}} a_{p,q}\otimes 1_{q^{-1}w}$ belong to $\mathcal{L}_{\mathcal{K}}$. If $\mathcal{K}$ is essential in $\mathcal{L}_{\mathcal{K}}$ then the maps $T_{w,w}^{w,w}$ are injective on $\mathcal{L}_{\mathcal{K}}(w,w)$, and therefore (5.9) reduces to (5.13). If, in addition, $Z = \sum_{p\in F} T_{p,p}(a_{p,p}) \in B_e^T$, then (5.13) reduces to (5.14) because for every $C \in \mathrm{In}(F)$ the supremum $\sup_{w\in\sigma(C)P} \|\sum_{p\in C} a_{p,p}\otimes 1_{p^{-1}w}\|$ is attained at $w = \sigma(C)$. ☐

6. Faithfulness of representations on core C^*-subalgebras

The goal of this section is to prove Theorem 6.1, which is inspired by certain results used to obtain amenability criteria, cf. [25, Lemma 4.1], [16, Theorem 6.1], [14, Theorem 8.1]. In this section we use it to detect necessary and sufficient conditions for injectivity of representations $\Phi \rtimes P$ on the core C^*-subalgebra $B_e^{i_{\mathcal{K}}} \subseteq \mathcal{NT}_{\mathcal{L}}(\mathcal{K})$. We will apply Theorem 6.1, in its full force, in Section 8 where we discuss the problem of amenability.

Theorem 6.1. *Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $\mathcal{L}$ over a right LCM semigroup P. Suppose that $\theta : P \to \mathcal{P}$ is a controlled map of right LCM semigroups, cf. Definition 2.4.*

(a) *The subspace of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) = C^*(i_{\mathcal{K}}(\mathcal{K}))$ defined as*

$$\overline{\mathrm{span}}\Big\{ i_{\mathcal{K}}(\mathcal{K}(p,q)) : p,q\in P, \theta(p)=\theta(q)\Big\} \tag{6.1}$$

is a C^*-subalgebra of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ on which the regular representation $T \rtimes P$ is faithful.

(b) *If Φ is a Nica covariant representation of $\mathcal{K}$, the representation $\Phi \rtimes P$ is faithful on* (6.1) *if and only if Φ is injective and satisfies*

$$\begin{aligned}&\overline{\mathrm{span}}\{\Phi(\mathcal{K}(p,q)) : \theta(p)=\theta(q)=u\}\cap \overline{\mathrm{span}}\{\Phi(\mathcal{K}(s,t)) : \theta(s)=\theta(t)\in F\} = \{0\}\\ &\textit{for all } u\in\mathcal{P} \textit{ and all finite sets } F\subseteq \mathcal{P} \textit{ such that } u \not\geq v \textit{ for every } v\in F.\end{aligned} \tag{6.2}$$

Before we pass to the proof of Theorem 6.1 let us derive some consequences in the case when the homomorphism θ is the identity map. To facilitate the discussion we introduce a name for the condition in (6.2) in the case θ is injective.

Definition 6.2. A Nica covariant representation $\Phi : \mathcal{K} \to B$ is *Toeplitz covariant* if

$$\begin{aligned}&\text{for every } p\in P \text{ and } q_1,...,q_n\in P \text{ such that } p\not\geq q_i \text{ for all } i=1,...,n,\ n\in\mathbb{N},\\ &\text{we have } \Phi(\mathcal{K}(p,p))\cap\overline{\mathrm{span}}\{\Phi(\mathcal{K}(q_i,q_i)) : i=1,...,n\} = \{0\}.\end{aligned} \tag{6.3}$$

In short we will call such representations *Nica-Toeplitz covariant* representations of $\mathcal{K}$.

Corollary 6.3. *Let $\Phi : \mathcal{K} \to B$ be a Nica covariant representation. The representation $\Phi \rtimes P$ of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ restricts to an isomorphism*

$$(\Phi\rtimes P)|_{B_e^{i_{\mathcal{K}}}} : B_e^{i_{\mathcal{K}}} \xrightarrow{\cong} B_e^{\Phi}$$

if and only if Φ is injective and Toeplitz covariant.

Proof. Applying Theorem 6.1 with $\theta = \mathrm{id}$, we see that the C^*-algebra in (6.1) equals $B_e^{i_\mathcal{K}}$, and condition (6.2) collapses to (6.3). □

Corollary 6.4. *The Fock representation $T : \mathcal{K} \to \mathcal{L}(\mathcal{F}_\mathcal{K})$ is an injective Nica-Toeplitz covariant representation of $\mathcal{K}$. Equivalently, the regular representation $T \rtimes P$ restricts to an isomorphism of core subalgebras $(T \rtimes P)|_{B_e^{i_\mathcal{K}}} : B_e^{i_\mathcal{K}} \xrightarrow{\cong} B_e^T$.*

Proof. Apply Theorem 6.1 with $\theta = \mathrm{id}$. □

Corollary 6.5. *If P is cancellative, then the formula*

$$E\Big(\sum_{p,q\in F} i_\mathcal{K}(a_{p,q})\Big) = \sum_{p\in F} i_\mathcal{K}(a_{p,p}), \qquad a_{p,q} \in \mathcal{K}(p,q), p, q \in F \subseteq P, \tag{6.4}$$

defines a conditional expectation $E : \mathcal{NT}_\mathcal{L}(\mathcal{K}) \to B_e^{i_\mathcal{K}} \subseteq \mathcal{NT}_\mathcal{L}(\mathcal{K})$.

Proof. Since P is cancellative, $B_\mathcal{K} = B_e^T$. By Corollary 6.4, $T \rtimes P$ restricts to an isomorphism of $B_e^{i_\mathcal{K}}$ onto B_e^T. Denote by $(T \rtimes P)^{-1}$ the inverse to this isomorphism. Taking into account (5.4) one sees that $E = (T \rtimes P)^{-1} \circ E^T \circ (T \rtimes P)$ is a conditional expectation satisfying (6.4). □

The overall strategy of the proof of Theorem 6.1 is comparable to that behind the proofs of the quoted results in [25], [16], [14]. Nevertheless, we deal here with a much more general situation, which will require new insight. One of the new difficulties is the presence of invertible elements in the semigroup P. To deal with them we consider the following equivalence relation on P:

$$p \sim q \iff \quad p = qx \text{ for some } x \in P^*.$$

We denote by $[p]$ the equivalence class of $p \in P$ in $P/_\sim$. Note that $\sim$ might not be a congruence and therefore $P/_\sim$ might not inherit the semigroup structure from P, cf. [5, Proposition 2.7]. However, $P/_\sim$ inherits the preorder. In fact,

$$[p] \le [q] \iff q \in pP$$

yields a partial order on $P/_\sim$. Moreover, $[p]$ and $[q]$ have a common upper bound if and only if $pP \cap qP = rP$ for some $r \in P$, and if this holds then $[r]$ is the unique least common upper bound of $[p]$ and $[q]$. In the latter situation we write $[p] \vee [q] := [r]$.

The following two lemmas play a key role in the proof of Theorem 6.1.

Lemma 6.6. *Retain the assumptions of Theorem 6.1 (possibly without condition (2.3)). For every subset $F \subseteq \mathcal{P}/_\sim$ such that $[u] \vee [v] \in F$ whenever $[u], [v] \in F$, the space*

$$\mathcal{K}_F := \overline{\mathrm{span}}\,\big\{ i_\mathcal{K}(\mathcal{K}(p,q)) : p, q \in P, \theta(p) = \theta(q), [\theta(p)] \in F \big\} \tag{6.5}$$

is a C^*-subalgebra of $\mathcal{NT}_\mathcal{L}(\mathcal{K})$ such that for any $F_0 \subseteq \mathcal{P}$ with $F = \{[u] : u \in F_0\}$ we have

$$\mathcal{K}_F = \overline{\mathrm{span}}\,\big\{ i_\mathcal{K}(\mathcal{K}(p,q)) : p, q \in P, \theta(p) = \theta(q) \in F_0 \big\}. \tag{6.6}$$

Proof. To see that $\mathcal{K}_F$ is a C^*-algebra, it suffices to show that $\mathcal{K}_F$ is closed under multiplication. Let $u, v \in \mathcal{P}$ with $[u], [v] \in F$ and suppose $\theta(p) = \theta(q) = u$ and $\theta(s) = \theta(t) = v$. Let $a \in \mathcal{K}(p,q)$ and $b \in \mathcal{K}(s,t)$. Since $i_\mathcal{K}$ is Nica covariant the product $i_\mathcal{K}(a)i_\mathcal{K}(b)$ is either zero or $qP \cap sP = rP$, for some $r \in P$, and then

$$i_\mathcal{K}(a)i_\mathcal{K}(b) = i_\mathcal{K}\big((a \otimes 1_{q^{-1}r})(b \otimes 1_{s^{-1}r})\big).$$

In the latter case we have $(a \otimes 1_{q^{-1}r})(b \otimes 1_{s^{-1}r}) \in \mathcal{K}(pq^{-1}r, ts^{-1}r)$ and using (2.2) we get $[\theta(r)] = [u] \vee [v] \in F$. Since $\theta(q)\theta(q^{-1}r) = \theta(r)$ we have $\theta(q^{-1}r) = \theta(q)^{-1}\theta(r)$ and therefore

$$\theta(pq^{-1}r) = \theta(p)\theta(q^{-1}r) = \theta(p)\theta(q)^{-1}\theta(r) = \theta(r).$$

Similarly, $\theta(ts^{-1}r) = \theta(r)$. Thus $i_{\mathcal{K}}(a)i_{\mathcal{K}}(b) \in \mathcal{K}_F$. Hence $\mathcal{K}_F$ is a C^*-algebra.

To prove (6.6), let F_0 be a transversal for F and suppose that $\theta(p) = \theta(q)$ with $[\theta(p)] \in F$ for $p, q \in P$. Since $\theta(P^*) = \mathcal{P}^*$ there is $x \in P^*$ such that $\theta(px) \in F_0$. By Lemma 3.10 we have $i_{\mathcal{K}}(\mathcal{K}(px, qx)) = i_{\mathcal{K}}(\mathcal{K}(p, q))$. This finishes the proof. □

We will prove injectivity in Theorem 6.1 by induction over the size of finite subsets of $\mathcal{P}/_{\sim}$. In particular, it is crucial to establish the claim on sets with one element. As a matter of notation, if $F = \{[u]\} \subset \mathcal{P}/_{\sim}$, we write $\mathcal{K}_{[u]}$ instead of $\mathcal{K}_F$.

Lemma 6.7. *Retain the assumptions of Theorem 6.1. If Φ is an injective Nica covariant representation of $\mathcal{K}$, then the homomorphism $\Phi \rtimes P$ of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ is faithful on $\mathcal{K}_{[u]}$ for every $u \in \mathcal{P}$.*

Proof. Let $u \in \mathcal{P}$. Then $\mathcal{K}_{[u]} = \overline{\operatorname{span}}\Big\{\bigcup_{p,q \in \theta^{-1}(u)} i_{\mathcal{K}}(\mathcal{K}(p,q))\Big\}$ by (6.6).

Suppose that Φ is an injective Nica covariant representation of $\mathcal{K}$. Note that whenever $s, t \in \theta^{-1}(u)$ with $s \neq t$, (2.3) implies that $sP \cap tP = \emptyset$, and so by Nica covariance the C^*-subalgebras $i_{\mathcal{K}}(\mathcal{K}(s,s))$ and $i_{\mathcal{K}}(\mathcal{K}(t,t))$ of $\mathcal{K}_{[u]}$ are orthogonal. Hence we have a direct sum C^*-subalgebra $D_u := \bigoplus_{s \in \theta^{-1}(u)} i_{\mathcal{K}}(\mathcal{K}(s,s))$ in $\mathcal{K}_{[u]}$. Since $\Phi \rtimes P$ is faithful on each summand it is also faithful on D_u.

We claim that for every surjective $*$-homomorphism $\pi : \mathcal{K}_{[u]} \to B$ for B a C^*-algebra, the formula

$$E_\pi\Big(\pi\big(\sum_{s,t \in I} i_{\mathcal{K}}(a_{s,t})\big)\Big) := \sum_{s \in I} \pi\big(i_{\mathcal{K}}(a_{s,s})\big), \tag{6.7}$$

where $a_{s,t} \in \mathcal{K}(s,t)$ and $I \subseteq \theta^{-1}(u)$ is a finite set, defines a faithful conditional expectation from $\pi(\mathcal{K}_{[u]})$ onto $\pi(D_u)$. By choosing a faithful and non-degenerate representation of B on a Hilbert space H, we may assume that $\pi : \mathcal{K}_{[u]} \to \mathcal{B}(H)$ is a non-degenerate representation.

For each $r \in \theta^{-1}(u)$ let $Q_r \in \mathcal{B}(H)$ be the projection onto the essential space $\pi(i_{\mathcal{K}}(\mathcal{K}(r,r)))H$ for the C^*-algebra $\pi(i_{\mathcal{K}}(\mathcal{K}(r,r)))$. Note that the projections Q_r, $r \in \theta^{-1}(u)$, are pairwise orthogonal and their ranges span H. Thus, exactly as in the proof of Lemma 4.7, we conclude that the formula $E_\pi(a) = \sum_{r \in \theta^{-1}(u)} Q_r(a)Q_r$, for $a \in \mathcal{B}(H)$, defines a faithful conditional expectation from $\mathcal{B}(H)$ onto the commutant of $\{Q_r\}_{r \in \theta^{-1}(u)}$. Thus it suffices to check that this map satisfies (6.7). But this is easy. Indeed, clearly we have $Q_s\pi\big(i_{\mathcal{K}}(a_{s,s})\big)Q_s = \pi\big(i_{\mathcal{K}}(a_{s,s})\big)$ for every $s \in I$. On the other hand, if $r \in \theta^{-1}(u)$ is such that either $r \neq s$ or $r \neq t$, then $Q_r\pi\big(i_{\mathcal{K}}(a_{s,t})\big)Q_r = 0$, since, by Lemma 2.7, we have $\pi\big(i_{\mathcal{K}}(a_{s,t})\big) \in \pi\big(i_{\mathcal{K}}(\mathcal{K}(s,s))i_{\mathcal{K}}(\mathcal{K}(s,t))i_{\mathcal{K}}(\mathcal{K}(t,t))\big) \subseteq Q_s\mathcal{B}(H)Q_t$.

The above claim applied separately to id and the $*$-homomorphism $\Phi \rtimes P$ gives a commutative diagram

$$\begin{array}{ccc} \mathcal{K}_{[u]} & \xrightarrow{\Phi \rtimes P} & (\Phi \rtimes P)(\mathcal{K}_{[u]}) \\ {\scriptstyle E_{\mathrm{id}}}\big\downarrow & & \big\downarrow{\scriptstyle E_{\Phi \rtimes P}} \\ D_u & \xrightarrow{\Phi \rtimes P} & (\Phi \rtimes P)(D_u) \end{array}$$

in which the vertical arrows and the bottom horizontal arrow are faithful. Therefore, also $(\Phi \rtimes P) : \mathcal{K}_{[u]} \to (\Phi \rtimes P)(\mathcal{K}_{[u]})$ is faithful. □

Proof of Theorem 6.1. We begin by proving that the Fock representation T satisfies (6.2). To this end, let $u \in \mathcal{P}$ and $F \subseteq \mathcal{P}$ be a finite set such that $u \not\geq v$ for every $v \in F$. Suppose that $p, q \in P$ satisfy $\theta(p) = \theta(q) = u$. By (5.5), the projection Q_q in $\mathcal{L}(\mathcal{F}_\mathcal{K})$ onto $\bigoplus_{t\in P} X_{q,t}$ satisfies $Q_q\big(T(\mathcal{K}(s,t))\big)Q_q = 0$ whenever s, t are in P such that $q \notin sP \cap tP$. Thus, for all $s, t \in P$ with $\theta(s) = \theta(t) \in F$, we have $Q_q\big(T(\mathcal{K}(s,t))\big)Q_q = 0$ by the choice of u and F.

An arbitrary element a in $\overline{\operatorname{span}}\{T(\mathcal{K}(p,q)) : \theta(p) = \theta(q) = u\}$ has the form

$$a = \sum_{q\in I} a_q \quad \text{where} \quad a_q \in \overline{\operatorname{span}}\Big\{ \bigcup_{p\in\theta^{-1}(u)} T(\mathcal{K}(p,q))\Big\}$$

where $I \subseteq \theta^{-1}(u)$ is finite. Suppose that $a \in \overline{\operatorname{span}}\{T(\mathcal{K}(s,t)) : \theta(s) = \theta(t) \in F\}$. The considerations of the previous paragraph imply that $aQ_q = 0$. On the other hand, $aQ_q = (\sum_{r\in I} a_r)Q_q$. For each $r \in I$ with $r \neq q$, the assumption (2.3) implies that $rP \cap qP = \emptyset$, and so the considerations above show that $a_rQ_q = 0$. Therefore $aQ_q = a_qQ_q$ for every $q \in I$. This implies that $a_q^*a_qQ_q = 0$. However, since $a_q^*a_q \in T(\mathcal{K}(q,q))$ and $T(\mathcal{K}(q,q))$ acts faithfully on $\oplus_{t\in P}X_{q,t}$ (because $\mathcal{K}(q,q)$ acts faithfully on the subspace $X_{q,q}$) we get $a_q^*a_q = 0$. Thus $a_q = 0$ for every $q \in I$. Accordingly, $a = 0$ and (6.2) is proved. Thus the injectivity claim in part (a) of the theorem will follow from part (b).

Sufficiency in part (b). Let $\Phi : \mathcal{K} \to B$ be an injective Nica covariant representation satisfying (6.2).

Let $\mathcal{F}$ denote the collection of all finite subsets $F \subseteq \mathcal{P}/_\sim$ such that $[u] \vee [v] \in F$ whenever $[u], [v] \in F$. For any finite $F_0 \subseteq \mathcal{P}/_\sim$ the set F consisting of least upper bounds of all finite sub-collections of elements in F_0 is finite, contains F_0 and is closed under $\vee$. Thus $\mathcal{F}$ is a directed set. Accordingly, the corresponding C^*-algebras (6.5) form an inductive system $\{\mathcal{K}_F : F \in \mathcal{F}\}$ with limit $\overline{\bigcup_{F\in\mathcal{F}} \mathcal{K}_F} = \overline{\operatorname{span}}\Big\{\bigcup_{\substack{p,q\in P,\\ \theta(p)=\theta(q)}} i_\mathcal{K}(\mathcal{K}(p,q))\Big\}$. Hence (6.1) is a C^*-algebra, and to prove faithfulness of $\Phi \rtimes P$ on $\overline{\bigcup_{F\in\mathcal{F}} \mathcal{K}_F}$ it suffices to show that $\Phi \rtimes P$ is faithful on $\mathcal{K}_F$ for each $F \in \mathcal{F}$, see e.g. [1, Lemma 1.3].

By Lemma 6.7, $\Phi \rtimes P$ is faithful on $\mathcal{K}_F$ if $F = \{[u]\}$ for some $u \in \mathcal{P}$. For the inductive step, let $F \in \mathcal{F}$ and suppose that $\Phi \rtimes P$ is faithful on $\mathcal{K}_{F'}$ whenever $F' \in \mathcal{F}$ and $|F'| < |F|$; we aim to prove that $\Phi \rtimes P$ is faithful on $\mathcal{K}_F$. Let $Z \in \mathcal{K}_F$ be a finite sum of the form

$$Z = \sum_{[u]\in F} Z_{[u]} \qquad \text{where } Z_{[u]} \in \mathcal{K}_{[u]} \ \text{ for } \ [u] \in F.$$

Suppose that $\Phi \rtimes P(Z) = 0$. We will show that $Z = 0$, giving the desired injectivity.

Since F is finite it has a minimal element, that is, there is $[u_0] \in F$ such that $[u] \not\leq [u_0]$ for every $[u] \in F \setminus \{[u_0]\}$. It is immediate from considerations concerning products of elements in $\mathcal{K}_F$, cf. the proof of Lemma 6.6, that the C^*-algebra $\mathcal{K}_{F\setminus\{[u_0]\}}$ is an ideal in $\mathcal{K}_F$. Hence $(\Phi \rtimes P)(\mathcal{K}_{F\setminus\{[u_0]\}})$ is an ideal in $(\Phi \rtimes P)(\mathcal{K}_F)$. Let

$$\rho : (\Phi \rtimes P)(\mathcal{K}_F) \to (\Phi \rtimes P)(\mathcal{K}_F)/(\Phi \rtimes P)(\mathcal{K}_{F\setminus\{[u_0]\}})$$

be the quotient map. We claim that ρ is injective on $(\Phi \rtimes P)(\mathcal{K}_{[u_0]})$. Indeed, by (6.6) we have

$$(\Phi \rtimes P)(\mathcal{K}_{[u_0]}) = \overline{\operatorname{span}}\{\Phi(\mathcal{K}(p,q)) : \theta(p) = \theta(q) = u_0\}$$

and for any $F_0 \subseteq \mathcal{P}$ finite set such that $F \setminus \{[u_0]\} = \{[u] : u \in F_0\}$, we have

$$(\Phi \rtimes P)(\mathcal{K}_{F\setminus\{[u_0]\}}) = \overline{\operatorname{span}}\{\Phi(\mathcal{K}(s,t)) : \theta(s) = \theta(t) \in F_0\}.$$

Note that $v \not\leq u_0$ for every $v \in F_0$. Thus condition (6.2) applied to u_0 and F_0 proves our claim. In particular, ρ is isometric on the C^*-algebra $\Phi \rtimes P(\mathcal{K}_{[u_0]})$. Thus

$$\|(\Phi \rtimes P)(Z_{[u_0]})\| = \|\rho\Big((\Phi \rtimes P)(Z_{[u_0]})\Big)\| = \|\rho\Big((\Phi \rtimes P)(Z)\Big)\| = 0.$$

This implies that $Z_{[u_0]} = 0$, as $\Phi \rtimes P$ is isometric on $\mathcal{K}_{[u_0]}$ by the first inductive step. Hence $Z = \sum_{[u] \in F\setminus\{u_0\}} Z_{[u]} \in \mathcal{K}_{F\setminus\{[u_0]\}}$. Therefore, $Z = 0$ by the inductive hypothesis.

Necessity in part (b). Since T satisfies (6.2), we conclude by sufficiency in part (b) that the regular representation $T \rtimes P$ is faithful on the C^*-algebra in (6.1). It is not difficult to see that this forces the universal representation $i_{\mathcal{K}}$ to satisfy (6.2): take u and F as specified for (6.2) and suppose that an element C in $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ satisfies $C = i_{\mathcal{K}}(C_1) = i_{\mathcal{K}}(C_2)$ where $C_1 \in \mathcal{K}(p, q)$ is in the closed span determined by u and $C_2 \in \mathcal{K}(s, t)$ is in the closed span determined by F. Then $T(C_1) = (T \rtimes P) \circ i_{\mathcal{K}}(C_1) = (T \rtimes P) \circ i_{\mathcal{K}}(C_2) = T(C_2)$ and so, first, $T(C_1) = T(C_2) = 0$ because T satisfies (6.2), and second, $i_{\mathcal{K}}(C_1) = i_{\mathcal{K}}(C_2) = 0$ because both are in the C^*-subalgebra (6.1) on which $T \rtimes P$ is faithful. This gives $C = 0$ in this particular case, and the general case follows from here. Now, since $i_{\mathcal{K}}$ satisfies (6.2), then every Nica covariant representation whose integrated form is faithful on (6.1) has to satisfy (6.2). This concludes the proof of the theorem. □

7. Exotic Nica-Toeplitz C^*-algebras

We fix a well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-category $\mathcal{L}$. In this section, we introduce Nica-Toeplitz C^*-algebras of $\mathcal{K}$ that sit between $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ and $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$. Inspired by the terminology introduced in the context of group C^*-algebras in [4], we shall call these C^*-algebras *exotic*. In view of Example 12.12 below, our exotic Nica-Toeplitz C^*-algebras are in fact generalizations of exotic crossed products for actions of discrete groups.

Definition 7.1. We say that a Nica covariant representation $\Phi : \mathcal{K} \to B$ *generates an exotic Nica-Toeplitz C^*-algebra* if $\ker(\Phi \rtimes P) \subseteq \ker(T \rtimes P)$. If this is the case, we refer to $C^*(\Phi(\mathcal{K}))$ as an *exotic Nica-Toeplitz C^*-algebra* of $\mathcal{K}$.

Remark 7.2. Clearly, Φ generates an exotic Nica-Toeplitz C^*-algebra if and only if there is a homomorphism $\Phi_* : C^*(\Phi(\mathcal{K})) \to \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ making the following diagram commute:

$$\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \xrightarrow{\Phi \rtimes P} C^*(\Phi(\mathcal{K})) \xrightarrow{\Phi_*} \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K}), \qquad \mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \xrightarrow{T \rtimes P} \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K}) \tag{7.1}$$

Proposition 7.3. *If $\Phi : \mathcal{K} \to B$ is a Nica covariant representation that generates an exotic Nica-Toeplitz C^*-algebra, then Φ is injective and Toeplitz covariant.*

Proof. The diagram (7.1) restricts to a commuting diagram of core subalgebras

$$B_e^{i_{\mathcal{K}}} \xrightarrow{\Phi \rtimes P} B_e^{\Phi} \xrightarrow{\Phi_*} B_e^{T}, \qquad B_e^{i_{\mathcal{K}}} \xrightarrow{T \rtimes P} B_e^{T} \tag{7.2}$$

By Corollary 6.4, the map $T \rtimes P$ restricted to $B_e^{i_{\mathcal{K}}}$ is an isomorphism onto B_e^T. Thus $\Phi \rtimes P : B_e^{i_{\mathcal{K}}} \to B_e^{\Phi}$ is an isomorphism and Corollary 6.3 implies the assertion. □

Nica covariant representations generating exotic Nica-Toeplitz algebras can be characterized as representations which admit a *transcendental conditional expectation*:

Proposition 7.4. *Let $B_{\mathcal{K}}$ be the self-adjoint operator subspace of $\mathcal{L}(\mathcal{F}_{\mathcal{K}})$ introduced in* (5.2). *For a Nica covariant representation $\Phi : \mathcal{K} \to B$ the following conditions are equivalent:*

(i) *$\Phi : \mathcal{K} \to B$ generates an exotic Nica-Toeplitz C^*-algebra of $\mathcal{K}$,*
(ii) *There is a bounded map $E^\Phi : C^*(\Phi(\mathcal{K})) \to B_{\mathcal{K}}$ satisfying*

$$E^\Phi\Big(\sum_{p,q\in F}\Phi(a_{p,q})\Big) = \sum_{p,q\in F}\bigoplus_{\substack{w\in pP\cap qP, t\in P\\ p^{-1}w=q^{-1}w}} T^{w,t}_{p,q}(a_{p,q}) \tag{7.3}$$

for every $a_{p,q}\in\mathcal{K}(p,q)$ and finite $F\subseteq P$.

If (i) and (ii) hold then E^Φ is a contractive completely positive map making the diagram

$$\begin{array}{ccc} C^*(\Phi(\mathcal{K})) & \xrightarrow{\Phi_*} & \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K}) \\ & {}_{E^\Phi}\searrow \quad \swarrow_{E^T} & \\ & B_{\mathcal{K}} & \end{array} \tag{7.4}$$

commute. Moreover, Φ_* is an isomorphism if and only if E^Φ is faithful.

Proof. The implication (i)⇒(ii) follows by letting $E^\Phi := E^T\circ\Phi_*$. In particular, E^Φ is a contractive completely positive map and the diagram (7.4) commutes. Moreover, since E^T is faithful, Φ_* is faithful if and only if E^Φ is faithful.

We prove next the implication (ii)⇒(i). Let us put $E^{i_\mathcal{K}} := E^T\circ(T\rtimes P)$. Then

$$\ker(T\rtimes P) = \{a\in\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) : E^{i_\mathcal{K}}(a^*a)=0\}.$$

Indeed, inclusion $\ker(T\rtimes P)\subseteq\{a\in\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) : E^{i_\mathcal{K}}(a^*a)=0\}$ is trivial. The reverse inclusion follows, because if $a\in\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ is such that $E^{i_\mathcal{K}}(a^*a)=0$, then $E^T\big((T\rtimes P)(a)^*(T\rtimes P)(a)\big) = 0$ and therefore $(T\rtimes P)(a)=0$ by faithfulness of E^T. The assumption in (ii) implies that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{NT}_{\mathcal{L}}(\mathcal{K}) & \xrightarrow{\Phi\rtimes P} & C^*(\Phi(\mathcal{K})) \\ & {}_{E^{i_\mathcal{K}}}\searrow \quad \swarrow_{E^\Phi} & \\ & B_{\mathcal{K}} & \end{array}$$

Indeed, this diagram commutes when restricted to the dense C^*-subalgebra $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})^0 = \operatorname{span}\{\bigcup_{p,q\in P} i_\mathcal{K}(\mathcal{K}(p,q))\}$ of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$, cf. (3.5). Hence by continuity the diagram commutes on $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$. Now,

$$a\in\ker(\Phi\rtimes P) \implies E^\Phi\big((\Phi\rtimes P)(a^*a)\big)=0 \implies E^{i_\mathcal{K}}(a^*a)=0.$$

Thus $\ker(\Phi\rtimes P)\subseteq\{a\in\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) : E^{i_\mathcal{K}}(a^*a)=0\} = \ker(T\rtimes P)$. □

Remark 7.5. Suppose that P is cancellative. Then the right-hand side of (7.3) reduces to the right-hand side of (5.4) and $B_{\mathcal{K}} = B^T_e$, cf. Proposition 5.4. Accordingly, Proposition 7.4 reduces to the following statement: a Nica covariant representation Φ of $\mathcal{K}$ generates an exotic Nica-Toeplitz C^*-algebra if and only if the formula

$$E_\Phi\Big(\sum_{p,q\in F}\Phi(a_{p,q})\Big) = \sum_{p\in F}\Phi(a_{p,p}), \qquad a_{p,q}\in\mathcal{K}(p,q),\ p,q\in F\subseteq P,\ F \text{ finite},$$

defines a genuine conditional expectation from $C^*(\Phi(\mathcal{K}))$ onto its core subalgebra B_e^Φ. In fact, if Φ generates an exotic Nica-Toeplitz C^*-algebra, we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{NT}_{\mathcal{L}}(\mathcal{K}) & \xrightarrow{\Phi\rtimes P} & C^*(\Phi(\mathcal{K})) & \xrightarrow{\Phi_*} & \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K}) \\ \downarrow E & & \downarrow E_\Phi & & \downarrow E^T \\ B_e^{i_\mathcal{K}} & \xrightarrow{\Phi\rtimes P} & B_e^\Phi & \xrightarrow{\Phi_*} & B_e^T \end{array} \tag{7.5}$$

where the bottom horizontal arrows are isomorphisms and $E_\Phi = (\Phi_*|_{B_e^\Phi})^{-1} \circ E^\Phi$. Moreover, $\Phi_* : C^*(\Phi(\mathcal{K})) \to \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ is an isomorphism if and only if E^Φ is faithful.

As a first application of Proposition 7.4, we show that in order to study $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$, one may in some cases use the t-th Fock subrepresentation T^t of T (usually T^e will work, see [20]).

Proposition 7.6. *Let $t \in P$ and suppose that the well-aligned ideal $\mathcal{K}$ satisfies the condition:*

$$\forall_{p\in P}\ \exists_{x\in P^*}\ \overline{\mathcal{K}(px,t)\mathcal{K}(t,px)} \text{ is an essential ideal in the } C^*\text{-algebra } \mathcal{L}_\mathcal{K}(px,px). \tag{7.6}$$

Then the t-th Fock representation $T^t : \mathcal{K} \to \mathcal{L}(\mathcal{F}^t_\mathcal{K})$ generates a copy of $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$, in the sense that there is an isomorphism $h_t : \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K}) \to C^*(T^t(\mathcal{K}))$ such that $h_t \circ T = T^t$.

Proof. In view of Remark 5.7 and the last part of Proposition 7.4, it suffices to show that we have an isometry

$$B_\mathcal{K} \in Z = \sum_{p,q\in F} \bigoplus_{\substack{w\in pP\cap qP, s\in P\\ p^{-1}w=q^{-1}w}} T^{w,s}_{p,q}(a_{p,q}) \longmapsto Z^t := \sum_{p,q\in F} \bigoplus_{\substack{w\in pP\cap qP\\ p^{-1}w=q^{-1}w}} T^{w,t}_{p,q}(a_{p,q}) \in B^t_\mathcal{K}, \tag{7.7}$$

This map is a well defined contraction, as it is the restriction of the projection from $\bigoplus_{s\in P} \mathcal{L}(\mathcal{F}^s_\mathcal{K})$ to $\mathcal{L}(\mathcal{F}^t_\mathcal{K})$. Now, the argument leading to (5.11) gives us that

$$\|Z^t\| = \max_{C\in \mathrm{In}(F)} \sup_{w\in P_{F,C}} \Big\| T^{w,t}_{w,w}\big(\sum_{\substack{p,q\in C\\ p^{-1}w=q^{-1}w}} a_{p,q}\otimes 1_{q^{-1}w}\big)\Big\|.$$

Let us fix $w \in P_{F,C}$. Note that the sum under $T^{w,t}_{w,w}$ belongs to $\mathcal{L}_\mathcal{K}(px,px)$. By (7.6) there is $x \in P^*$ such that the homomorphism $T^{wx,t}_{wx,wx} : \mathcal{L}_\mathcal{K}(wx,wx) \to \mathcal{L}(X_{wx,t})$ is injective (and hence isometric), cf. (5.12). Since $\mathcal{L}_\mathcal{K}\otimes 1 \subseteq \mathcal{L}_\mathcal{K}$ and the homomorphism $\otimes 1_x$ is isometric we get

$$\Big\| \sum_{\substack{p,q\in C\\ p^{-1}w=q^{-1}w}} a_{p,q}\otimes 1_{q^{-1}w}\Big\| = \Big\| \sum_{\substack{p,q\in C\\ p^{-1}wx=q^{-1}wx}} a_{p,q}\otimes 1_{q^{-1}wx}\Big\|$$
$$= \Big\| T^{wx,t}_{wx,wx}\Big(\sum_{\substack{p,q\in C\\ p^{-1}wx=q^{-1}wx}} a_{p,q}\otimes 1_{q^{-1}wx}\Big)\Big\|.$$

Clearly, $wx \in P_{F,C}$ and therefore $\|Z^t\|$ is not greater than $\|Z\|$ by (5.9). Hence (7.7) is an isometry. ∎

8. Amenability and Fell Bundles

We fix a well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $(\mathcal{L}, \{\otimes 1_r\}_{r\in P})$ over a right LCM semigroup P. In this section, we study the properties under which all exotic Nica-Toeplitz algebras of $\mathcal{K}$ are naturally isomorphic.

Definition 8.1. A well-aligned ideal $\mathcal{K}$ in $(\mathcal{L},\{\otimes 1_r\}_{r\in P})$ is called *amenable* if the regular representation $T\rtimes P:\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\longrightarrow \mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ is an isomorphism. A right-tensor C^*-precategory $\mathcal{L}$ is *amenable* if it is amenable as an ideal in itself.

We have the following simple characterization of amenability in terms of the natural map on the transcendental core $B_{\mathcal{K}}$.

Lemma 8.2. *A well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ is amenable if and only if the map $E^{i_{\mathcal{K}}}=E^T\circ(T\rtimes P):\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\to B_{\mathcal{K}}$ is faithful.*

If P is cancellative, then $\mathcal{K}$ is amenable if and only if the conditional expectation E from $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ onto $B_e^{i_{\mathcal{K}}}$ given by (6.4) *is faithful.*

Proof. Apply the last part of Proposition 7.4. □

We obtain more efficient amenability criteria using the theory of Fell bundles. We first recall the definition of a full coaction of G on a C^*-algebra A. An unadorned tensor product of C^*-algebras will denote the minimal tensor product. We write $g\mapsto i_G(g)$ for the canonical inclusion of G as unitaries in the full group C^*-algebra $C^*(G)$. There is a homomorphism $\delta_G:C^*(G)\to C^*(G)\otimes C^*(G)$ given by $\delta_G(g)=i_G(g)\otimes i_G(g)$. A *full coaction* of G on A is an injective, non-degenerate homomorphism $\delta:A\to A\otimes C^*(G)$ that satisfies the coaction identity $(\delta\otimes id_{C^*(G)})\circ\delta=(id_A\otimes\delta_G)\circ\delta$.

Proposition 8.3. *Suppose that $\theta:P\to G$ is a unital semigroup homomorphism from P into a group G. There is a full coaction δ of G on $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ such that*

$$\delta(i_{\mathcal{K}}(a))=i_{\mathcal{K}}(a)\otimes i_G(\theta(p)\theta(q)^{-1}) \text{ for every } a\in\mathcal{K}(p,q),\ p,q\in P.$$

The Fell bundle $\mathcal{B}^\theta=\{B_g^\theta\}_{g\in G}$ associated to δ has fibers given by

$$B_g^\theta=\overline{\operatorname{span}}\left\{i_{\mathcal{K}}\left(\mathcal{K}(p,q)\right):p,q\in P, g=\theta(p)\theta(q)^{-1}\right\}\quad \text{if}\quad g\in\theta(P)\theta(P)^{-1},$$

and $B_g^\theta=\{0\}$ if $g\notin\theta(P)\theta(P)^{-1}$. In particular, $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\cong C^*(\mathcal{B}^\theta)$ with the isomorphism which is identity on the spaces B_g^θ, $g\in G$.

Proof. We claim that the maps $\mathcal{K}(p,q)\ni a\longmapsto i_{\mathcal{K}}(a)\otimes i_G(\theta(p)\theta(q)^{-1})$ yield a Nica covariant representation $\Phi:\mathcal{K}\to\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\otimes C^*(G)$. Indeed, let $a\in\mathcal{K}(p,q)$ and $b\in K(s,t)$, $p,q,s,t\in P$. If $qP\cap sP=\emptyset$, then $i_{\mathcal{K}}(a)i_{\mathcal{K}}(b)=0$ and therefore $\Phi(a)\Phi(b)=0$. Assume that $qP\cap sP=rP$ for some $r\in P$. Writing for example $qq'=ss'=r$ for some $s',q'\in P$ shows that $\theta(p)\theta(q)^{-1}\theta(s)\theta(t)^{-1}=\theta(pq')\theta(ts')^{-1}$. With $q'=q^{-1}r$ and $s'=s^{-1}r$, we therefore have

$$\begin{aligned}\Phi(a)\Phi(b)&=\Big(i_{\mathcal{K}}(a)i_{\mathcal{K}}(b)\Big)\otimes\Big(i_G(\theta(p)\theta(q)^{-1})i_G(\theta(s)\theta(t)^{-1})\Big)\\&=i_{\mathcal{K}}\Big((a\otimes 1_{q^{-1}r})(b\otimes 1_{s^{-1}r})\Big)\otimes\Big(\theta(pq^{-1}r)\theta(ts^{-1}r)^{-1})\Big)\\&=\Phi\Big((a\otimes 1_{q^{-1}r})(b\otimes 1_{s^{-1}r})\Big).\end{aligned}$$

Thus Φ integrates to a homomorphism $\delta=\Phi\rtimes P:\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\to\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\otimes C^*(G)$. By Lemma 2.7 and the Hewitt-Cohen factorization theorem every element $a\in\mathcal{K}(p,q)$, $p,q\in P$, can be written as $a=a'a''$ where $a'\in\mathcal{K}(p,p)$ and $a''\in\mathcal{K}(p,q)$. Thus

$$i_{\mathcal{K}}(a)\otimes i_G(\theta(p)\theta(q)^{-1})=\Big(i_{\mathcal{K}}(a')\otimes i_G(e)\Big)\Big(i_{\mathcal{K}}(a'')\otimes i_G(\theta(p)\theta(q)^{-1})\Big)\in\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\otimes C^*(G).$$

This implies that δ is non-degenerate. For every $a \in \mathcal{K}(p,q)$, $p,q \in P$ we have

$$\begin{aligned}\Big((\delta \otimes id_{C^*(G)}) \circ \delta\Big)(i_{\mathcal{K}}(a)) &= (\delta \otimes id_{C^*(G)})\big(i_{\mathcal{K}}(a) \otimes i_G(\theta(p)\theta(q)^{-1})\big) \\ &= i_{\mathcal{K}}(a) \otimes i_G\big(\theta(p)\theta(q)^{-1}\big) \otimes i_G\big(\theta(p)\theta(q)^{-1}\big) \\ &= \Big((id_A \otimes \delta_G) \circ \delta\Big)(i_{\mathcal{K}}(a)).\end{aligned}$$

Hence δ is a full coaction. It is readily seen that the spectral subspaces B_g^θ, $g \in G$, for the coaction δ are of the claimed form. To see that $\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \cong C^*(\mathcal{B}^\theta)$ it suffices to note that $C^*(\mathcal{B}^\theta)$ is generated by a universal Nica covariant representation. Now $C^*(\mathcal{B}^\theta)$ is generated by the spaces $i_{\mathcal{K}}(\mathcal{K}(p,q))$, $p,q \in G$, and for a Nica covariant representation Φ of $\mathcal{K}$ the map $\Phi \rtimes G : \bigoplus_{g\in G} B_g^\theta \to C^*(\Phi(\mathcal{K}))$ given by $(\Phi \rtimes G)(\oplus_{g\in G} b_g) := \sum_{g\in G} (\Phi \rtimes P)(b_g)$ is a $*$-homomorphism. Since $C^*(\mathcal{B}^\theta)$ is the completion of $\bigoplus_{g\in G} B_g$ in the maximal C^*-norm, the homomorphism $\Phi \rtimes G$ extends to the epimorphism $\Phi \rtimes G : C^*(\mathcal{B}^\theta) \to C^*(\Phi(\mathcal{K}))$. □

Theorem 8.4. *Suppose that $\theta : P \to \mathcal{P} \subseteq G$ is a controlled map of right LCM semigroups such that $\mathcal{P}$ is a subsemigroup of a group G. Let $\mathcal{B}^\theta$ be the Fell bundle described in Proposition 8.3. We have a commutative diagram*

$$\begin{array}{ccc} C^*(\mathcal{B}^\theta) & \xrightarrow{\quad\Lambda\quad} & C_r^*(\mathcal{B}^\theta) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathcal{NT}_{\mathcal{L}}(\mathcal{K}) & \xrightarrow{\quad T\rtimes P\quad} & \mathcal{NT}_{\mathcal{L}}^r(\mathcal{K}) \end{array}$$

where vertical arrows are isomorphisms and horizontal ones are regular representations. In particular, $\mathcal{B}^\theta$ is amenable if and only if $\mathcal{K}$ is amenable as an ideal of $\mathcal{L}$.

Proof. Let $\Pi : \mathcal{NT}_{\mathcal{L}}(\mathcal{K}) \to C^*(\mathcal{B}^\theta)$ be the isomorphism given by Proposition 8.3. It is easy to see that $\Phi := \Lambda \circ \Pi \circ i_{\mathcal{K}}$ is a Nica covariant representation of $\mathcal{K}$ such that $C^*(\Phi(\mathcal{K}))$ is equal to $C_r^*(\mathcal{B}^\theta)$. There are canonical conditional expectations $E^\delta : C^*(\mathcal{B}^\theta) \to B_e^\theta$ and $E^{\delta,r} : C_r^*(\mathcal{B}^\theta) \to B_e^\theta$. The existence of a controlled map into a semigroup $\mathcal{P}$ that is right cancellative (being a subsemigroup of G) guarantees that P is cancellative, see Remark 2.5 (a). By Proposition 5.4, the transcendental core $B_{\mathcal{K}}$ is equal to B_e^T, hence it is a subspace of $(T \rtimes P)(B_e^\theta) = \overline{\operatorname{span}}\big\{T(\mathcal{K}(p,q)) : p,q \in P, \theta(p) = \theta(q)\big\}$. By Theorem 6.1 the $*$-homomorphism $T \rtimes P : B_e^\theta \to (T \rtimes P)(B_e^\theta)$ is an isomorphism. Hence $E^\Phi := E^T \circ (T \rtimes P) \circ E^{\delta,r}$ is a faithful completely positive map from $C_r^*(\mathcal{B}^\theta)$ onto $B_{\mathcal{K}} = B_e^T$. We claim that E^Φ satisfies equation (7.3). Note first that

$$E^\delta \circ \Pi(i_{\mathcal{K}}(a)) := \begin{cases} \Pi(i_{\mathcal{K}}(a)) & \text{if } \theta(p) = \theta(q), \\ 0 & \text{if } \theta(p) \neq \theta(q), \end{cases} \qquad \text{for every } a \in \mathcal{K}(p,q),\ p,q \in P. \tag{8.1}$$

Then for every choice of finite family $a_{p,q}$ in $\mathcal{K}(p,q)$, where $p,q \in F$ finite, we have

$$\begin{aligned} E^{\Phi}\big(\sum_{p,q\in F}\Phi(a_{p,q})\big) &= E^{T}\circ (T\rtimes P)\circ E^{\delta,r}\big(\sum_{p,q\in F}\Lambda\circ\Pi\circ i_{\mathcal{K}}(a_{p,q})\big)\\ &= E^{T}\circ (T\rtimes P)\circ E^{\delta}\big(\sum_{p,q\in F}\Pi\circ i_{\mathcal{K}}(a_{p,q})\big)\\ &= E^{T}\big(\sum_{\{p,q\in F:\theta(p)=\theta(q)\}} T(a_{p,q})\big), \end{aligned}$$

which is the term in the right-hand side of (7.3) (which in this case reduces to the right-hand side of (5.4)). Thus Proposition 7.4 implies that there is a $*$-homomorphism Φ_* from $C_r^*(\mathcal{B}^\theta)$ onto $\mathcal{NT}_{\mathcal{L}}^r(\mathcal{K})$ such that $T\rtimes P = \Phi_*\circ(\Phi\rtimes P)$. Since E^{Φ} is faithful, the same proposition shows that in fact Φ_* is an isomorphism. This implies the assertion. □

Remark 8.5. Under the assumptions of Theorem 8.4, the Fell bundle $\mathcal{B}^\theta$ can be constructed using any injective Nica covariant representation Φ satisfying (6.2). More specifically, given such a representation Φ we define

$$B_g^{\Phi,\theta} := \overline{\operatorname{span}}\left\{\Phi\left(\mathcal{K}(p,q)\right) : p,q\in P, g=\theta(p)\theta(q)^{-1}\right\} \text{ if } g\in\theta(P)\theta(P)^{-1}$$

and $B_g^{\Phi,\theta} := \{0\}$ for $g\notin\theta(P)\theta(P)^{-1}$. Then $\mathcal{B}^{\Phi,\theta} := \{B_g^{\Phi,\theta}\}_{g\in G}$ is a Fell bundle isomorphic to $\mathcal{B}^\theta$. Indeed, $\Phi\rtimes P : B_e^\theta \to B_e^{\Phi,\theta}$ is an isomorphism by Theorem 6.1, and hence by the C^*-equality all the mappings $\Phi\rtimes P : B_g^\theta\to B_g^{\Phi,\theta}$, $g\in G$, are isomorphisms.

The condition of amenability of $\mathcal{B}^\theta$ in Theorem 8.4 is satisfied for instance when G is amenable or $\mathcal{B}^\theta$ has the approximation property, [12]. In particular, we get the following generalization of [14, Corollary 8.2].

Corollary 8.6. *Suppose that $P := \prod_{i\in I}^* P_i$ is the free product of a family of right LCM semigroups P_i, $i\in I$, where each P_i is a subsemigroup of an amenable group G_i. Any well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ over P is amenable.*

Proof. The direct sum $G := \bigoplus_{i\in I} G_i$ is an amenable group. By Proposition 2.3, we have a homomorphism $\theta : P\to G$ which satisfies the assumptions of Theorem 8.4. Hence the assertion follows from Theorem 8.4. □

9. Projections associated to Nica covariant representations

In this section we fix a Nica covariant representation $\Phi : \mathcal{K}\to\mathcal{B}(H)$ of a well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$. We let $\overline{\Phi} : \mathcal{L}\to\mathcal{B}(H)$ be the extension of Φ from Proposition 2.13. Our goal is to investigate two families of projections associated to Φ: $\{Q_p^\Phi\}_{p\in P}$ and $\{Q_{\langle p\rangle}^\Phi\}_{p\in P}$. The former are projections onto the essential spaces we used to define $\overline{\Phi}$. The latter can be considered analogues of projections we associated to the Fock representation in Lemma 5.8. In general, the family $\{Q_p^\Phi\}_{p\in P}$ has some good properties that $\{Q_{\langle p\rangle}^\Phi\}_{p\in P}$ lacks and vice versa. Thus the case when these families coincide is desirable and we give a natural condition implying that.

Definition 9.1. For each $p\in P$, we denote by Q_p^Φ and $Q_{\langle p\rangle}^\Phi$ projections in $\mathcal{B}(H)$ defined by

$$Q_p^\Phi H = \begin{cases}\Phi(\mathcal{K}(p,p))H & \text{if } p\in P\setminus P^*,\\ C^*(\Phi(\mathcal{K}))H & \text{if } p\in P^*.\end{cases} \tag{9.1}$$

and $Q^{\Phi}_{\langle p\rangle}H = \overline{\operatorname{span}}\{\Phi(\mathcal{K}(w,w))H : w \in pP\}$ (so we have $Q^{\Phi}_{\langle p\rangle} = \bigvee_{w\geq p} Q^{\Phi}_{w}$).

Lemma 9.2. *There is a well defined mapping $J(P) \longmapsto \operatorname{Proj}(\mathcal{B}(H))$ given by the assignment*

$$pP \longmapsto Q^{\Phi}_{p} \quad \text{and} \quad \emptyset \longmapsto 0 \tag{9.2}$$

which sends comparable ideals (in the sense of inclusion) to commuting projections and has the property that $Q^{\Phi}_{p}Q^{\Phi}_{q} = 0$ whenever $pP \cap qP = \emptyset$ for $p,q \in P$. Moreover, for every $p,q,s \in P$ and $a \in \mathcal{K}(p,q)$ we have

$$\Phi(a)Q^{\Phi}_{s} = \begin{cases} \overline{\Phi}(a \otimes 1_{q^{-1}r})Q^{\Phi}_{s} & \text{if } sP \cap qP = rP \text{ for some } r \in P, \\ 0 & \text{if } sP \cap qP = \emptyset. \end{cases} \tag{9.3}$$

If additionally $d \in \mathcal{L}(s,s)$ and $t \geq s$, then $\overline{\Phi}(d)Q^{\Phi}_{t} = Q^{\Phi}_{t}\overline{\Phi}(d)$ and

$$\Phi(a)\overline{\Phi}(d) = \begin{cases} \overline{\Phi}(a \otimes 1_{q^{-1}r})\overline{\Phi}(d) & \text{if } sP \cap qP = rP \text{ for some } r \in P, \\ 0 & \text{if } sP \cap qP = \emptyset. \end{cases} \tag{9.4}$$

Proof. Note first that for every $p \in P \setminus P^*$ the projection Q^{Φ}_{p} is the strong limit of the net $\{\Phi(\mu^{p}_{\lambda})\}_{\lambda\in\Lambda}$, where $\{\mu^{p}_{\lambda}\}_{\lambda\in\Lambda}$ is an approximate unit in $\mathcal{K}(p,p)$.

If $pP = qP$ for some $p,q \in P$ then $q = px$ for some $x \in P^*$ and therefore $\Phi(\mathcal{K}(q,q)) = \Phi(\mathcal{K}(p,p) \otimes 1_x) = \Phi(\mathcal{K}(p,p))$ by Lemma 3.10. Thus $Q^{\Phi}_{p} = Q^{\Phi}_{q}$, so the map in (9.2) is well defined. If $pP \cap qP = \emptyset$, then $\Phi(\mathcal{K}(p,p))\Phi(\mathcal{K}(q,q)) = 0$ by Nica covariance. Thus $Q^{\Phi}_{p}Q^{\Phi}_{q} = 0$, as claimed. Now suppose that $p \leq q$ for some $p,q \in P$, and let $a \in \mathcal{K}(p,p)$. Then the equality $\Phi_{p,p}(a)\Phi_{q,q}(b) = \Phi_{q,q}((a \otimes 1_{p^{-1}q})b)$ implies that $\Phi_{p,p}(a)Q^{\Phi}_{q} = Q^{\Phi}_{q}\Phi_{p,p}(a)Q^{\Phi}_{q}$. By passing to adjoints we get $Q^{\Phi}_{q}\Phi_{p,p}(a) = Q^{\Phi}_{q}\Phi_{p,p}(a)Q^{\Phi}_{q}$. Thus $Q^{\Phi}_{q} \in \Phi_{p,p}(\mathcal{K}(p,p))'$. It follows from the definition of $\overline{\Phi}_{p,p}$, cf. (2.7), that $\Phi_{p,p}(\mathcal{K}(p,p))' \subseteq \overline{\Phi}_{p,p}(\mathcal{L}(p,p))'$. Hence

$$Q^{\Phi}_{q} \in \overline{\Phi}_{p,p}(\mathcal{L}(p,p))', \tag{9.5}$$

and in particular, $Q^{\Phi}_{p}Q^{\Phi}_{q} = Q^{\Phi}_{q}Q^{\Phi}_{p}$.

Let now $p,q,s \in P$ and $a \in \mathcal{K}(p,q)$. If $sP \cap qP = \emptyset$, then $\Phi(a)Q^{\Phi}_{s} = \Phi(a)Q^{\Phi}_{q}Q^{\Phi}_{s} = 0$ by Lemma 2.7. Assume then that $sP \cap qP = rP$. For any $b \in \mathcal{K}(s,s)$ we have

$$\begin{aligned} \Phi(a)\Phi(b) &= \Phi\big((a \otimes 1_{q^{-1}r})(b \otimes 1_{s^{-1}r})\big) = \overline{\Phi}(a \otimes 1_{q^{-1}r})\overline{\Phi}(b \otimes 1_{s^{-1}r}) \\ &= \text{s-}\lim \overline{\Phi}(a \otimes 1_{q^{-1}r})\Phi(\mu^{r}_{\lambda})\overline{\Phi}(b \otimes 1_{s^{-1}r}) \\ &= \text{s-}\lim \overline{\Phi}(a \otimes 1_{q^{-1}r})\Phi(\mu^{r}_{\lambda}(b \otimes 1_{s^{-1}r})) \\ &= \text{s-}\lim \overline{\Phi}(a \otimes 1_{q^{-1}r})\Phi(\mu^{r}_{\lambda})\Phi(b) \\ &= \overline{\Phi}(a \otimes 1_{q^{-1}r})\Phi(b) \end{aligned}$$

by construction of $\overline{\Phi}$ in (2.7), and Nica covariance of Φ. Claim (9.3) follows. It implies (9.4) because $\overline{\Phi}(d) = Q^{\Phi}_{s}\overline{\Phi}(d)$. If $t \geq s$, then $\overline{\Phi}(d)Q^{\Phi}_{t} = Q^{\Phi}_{t}\overline{\Phi}(d)$ by (9.5). □

Lemma 9.3. *The assignment $pP \mapsto Q^{\Phi}_{\langle p\rangle}$ and $\emptyset \mapsto 0$ is a homomorphism $J(P) \longmapsto \operatorname{Proj}(\mathcal{B}(H))$ of semilattices, i.e. $Q^{\Phi}_{\langle p\rangle}Q^{\Phi}_{\langle q\rangle} = Q^{\Phi}_{\langle r\rangle}$ when $pP \cap qP = rP$ and $Q^{\Phi}_{\langle p\rangle}Q^{\Phi}_{\langle q\rangle} = 0$ for $pP \cap qP = \emptyset$. In particular, $Q^{\Phi}_{\langle p\rangle}$ for $p \in P$ are mutually commuting projections. Moreover,*

$$Q^{\Phi}_{\langle q\rangle} \in \Phi(\mathcal{K}(p,p))' \quad \text{for all } p,q \in P. \tag{9.6}$$

If $\overline{\Phi}:\mathcal{L}\to\mathcal{B}(H)$ is Nica covariant (which happens for instance when $\mathcal{K}=\mathcal{L}$) then

$$\overline{\Phi}(a)Q^{\Phi}_{\langle s\rangle}=\begin{cases}\overline{\Phi}(a\otimes 1_{q^{-1}r}) & \text{if } sP\cap qP=rP \text{ for some } r\in P,\\ 0 & \text{if } sP\cap qP=\emptyset\end{cases}\tag{9.7}$$

for every $p,q,s\in P$ and $a\in\mathcal{L}(p,q)$. In particular, $\{Q^{\Phi}_{\langle p\rangle}\}_{p\in P}\subseteq M(B_e^{\overline{\Phi}})\subseteq M(C^*(\overline{\Phi}(\mathcal{L})))$.

Proof. By Nica covariance of Φ, $C^*_p(\Phi):=\overline{\operatorname{span}}\{\Phi(\mathcal{K}(w,w)):w\in pP\}$ is a C^*-algebra, for each $p\in P$. We claim that, for every $p,q\in P$ we have

$$C^*_p(\Phi)C^*_q(\Phi)=C^*_p(\Phi)\cap C^*_q(\Phi)=\begin{cases}C^*_r(\Phi) & \text{if } pP\cap qP=rP \text{ for some } r\in P,\\ 0 & \text{otherwise.}\end{cases}\tag{9.8}$$

Indeed, if $pP\cap qP=\emptyset$, then for any $w\in pP$ and $v\in qP$ we have $wP\cap vP=\emptyset$ and hence $\Phi(\mathcal{K}(w,w))\Phi(\mathcal{K}(v,v))=\{0\}$ by Nica covariance. Accordingly, in this case $C^*_p(\Phi)C^*_q(\Phi)=\{0\}$. Assume next that $pP\cap qP=rP$ for some $r\in P$. Then $C^*_r(\Phi)\subseteq C^*_q(\Phi)\cap C^*_p(\Phi)\subseteq C^*_q(\Phi)C^*_p(\Phi)$. The reverse inclusion $C^*_q(\Phi)C^*_p(\Phi)\subseteq C^*_r(\Phi)$ readily follows from Nica covariance of Φ.

Plainly, for every $p\in P$, the essential space $C^*_p(\Phi)H$ for $C^*_p(\Phi)$ is equal to $Q^{\Phi}_{\langle p\rangle}H$. In particular, $Q^{\Phi}_{\langle p\rangle}$ is a strong limit of any approximate unit in $C^*_p(\Phi)$, and (9.8) justifies the claim about the semilattice homomorphism. Morever, (9.8) implies that for every $p,q\in P$ and $a\in\mathcal{K}(p,p)$ we have $\Phi(a)Q^{\Phi}_{\langle q\rangle}=Q^{\Phi}_{\langle q\rangle}\Phi(a)Q^{\Phi}_{\langle q\rangle}$. By passing to adjoints we get $Q^{\Phi}_{\langle q\rangle}\Phi(a)=Q^{\Phi}_{\langle q\rangle}\Phi(a)Q^{\Phi}_{\langle q\rangle}$ and therefore $\Phi(a)Q^{\Phi}_{\langle q\rangle}=Q^{\Phi}_{\langle q\rangle}\Phi(a)$ which proves (9.6).

Suppose now that $\overline{\Phi}:\mathcal{L}\to\mathcal{B}(H)$ is Nica covariant. Let $p,q,s\in P$ and $a\in\mathcal{L}(p,q)$. If $sP\cap qP=\emptyset$, then $\overline{\Phi}(a)C^*_s(\Phi)=\{0\}$ by Nica covariance, and therefore $\overline{\Phi}(a)Q^{\Phi}_{\langle s\rangle}=0$. Assume that $sP\cap qP=rP$. Let $w\in sP$ and $b\in\mathcal{K}(w,w)$. Since $rP\cap wP=(sP\cap qP)\cap wP=sP\cap(qP\cap wP)=sP\cap tP$ for some $t\in P$, using Nica covariance of $\overline{\Phi}$ twice we get

$$\overline{\Phi}(a)\Phi(b)=\overline{\Phi}\big((a\otimes 1_{q^{-1}t})(b\otimes 1_{w^{-1}t})\big)=\overline{\Phi}(a\otimes 1_{q^{-1}r})\Phi(b).$$

This implies that $\overline{\Phi}(a)Q^{\Phi}_{\langle s\rangle}=\overline{\Phi}(a\otimes 1_{q^{-1}r})Q^{\Phi}_{\langle s\rangle}$. Since $\overline{\Phi}(a\otimes 1_{q^{-1}r})=\overline{\Phi}(a\otimes 1_{q^{-1}r})Q^{\Phi}_{\langle r\rangle}$ and $Q^{\Phi}_{\langle r\rangle}\le Q^{\Phi}_{\langle s\rangle}$, we get $\overline{\Phi}(a)Q^{\Phi}_{\langle s\rangle}=\overline{\Phi}(a\otimes 1_{q^{-1}r})$. This proves (9.7).

Relation (9.7) readily implies $B_e^{\overline{\Phi}}Q^{\Phi}_{\langle p\rangle}\subseteq B_e^{\overline{\Phi}}$. By taking adjoints we obtain $Q^{\Phi}_{\langle p\rangle}B_e^{\overline{\Phi}}\subseteq B_e^{\overline{\Phi}}$. Thus assuming the standard identification $M(B_e^{\overline{\Phi}})=\{a\in Q^{\Phi}_{\langle e\rangle}\mathcal{B}(H)Q^{\Phi}_{\langle e\rangle}: aB_e^{\overline{\Phi}},\,B_e^{\overline{\Phi}}a\subseteq B_e^{\overline{\Phi}}\}$ we conclude that $Q^{\Phi}_{\langle p\rangle}\in M(B_e^{\overline{\Phi}})$. □

Proposition 9.4. *The following conditions are equivalent:*

(i) *$Q^{\Phi}_p=Q^{\Phi}_{\langle p\rangle}$ for every $p\in P$;*
(ii) *The map (9.2) is a semilattice homomorphism;*
(iii) *The map (9.2) is a pre-order homomorphism ($pP\subseteq qP$ implies $Q^{\Phi}_p\le Q^{\Phi}_q$);*
(iv) *The extension $\overline{\Phi}:\mathcal{L}\to\mathcal{B}(H)$ of Φ is Nica covariant, and*

$$\overline{\Phi}(a)Q^{\Phi}_{s}=\begin{cases}\overline{\Phi}(a\otimes 1_{q^{-1}r}) & \text{if } sP\cap qP=rP \text{ for some } r\in P,\\ 0 & \text{if } sP\cap qP=\emptyset,\end{cases}\tag{9.9}$$

for every $p,q,s\in P$ and $a\in\mathcal{L}(p,q)$.

In particular, if the above equivalent conditions hold, then $Q^{\Phi}_p=Q^{\Phi}_{\langle p\rangle}\in\overline{\Phi}(\mathcal{L}(q,q))'$ for all $p,q\in P$, and we have $\{Q^{\Phi}_p\}_{p\in P}=\{Q^{\Phi}_{\langle p\rangle}\}_{p\in P}\subseteq M(B_e^{\overline{\Phi}})\subseteq M\big(C^*(\overline{\Phi}(\mathcal{L}))\big)$.

Proof. Implication (i)⇒(ii) follows from Lemma 9.3. Implication (ii)⇒(iii) is trivial.

We prove that (iii)⇒(iv). Let $p, q, s \in P$ and $a \in \mathcal{L}(p, q)$. If $sP \cap qP = \emptyset$, then $Q_q^\Phi Q_s^\Phi = 0$ by Lemma 9.2, and hence $\overline{\Phi}(a)Q_s^\Phi = \overline{\Phi}(a)Q_q^\Phi Q_s^\Phi = 0$. Assume then that $sP \cap qP = rP$. Note that $\overline{\Phi}(a)$ is a strong limit of the net $\Phi(a\mu_\lambda^q) = \overline{\Phi}(a)\Phi(\mu_\lambda^q)$, where $\{\mu_\lambda^q\}_{\lambda\in\Lambda}$ is an approximate unit in $\mathcal{K}(q, q)$. Thus (9.9) follows from the calculations

$$\begin{aligned}\overline{\Phi}(a)Q_s^\Phi &= \text{s-}\lim \Phi(a\mu_\lambda^q)Q_s^\Phi \overset{(9.3)}{=} \text{s-}\lim \overline{\Phi}\big((a\mu_\lambda^q)\otimes 1_{q^{-1}r}\big)Q_s^\Phi \\ &= \text{s-}\lim \overline{\Phi}(a\otimes 1_{q^{-1}r})\overline{\Phi}(\mu_\lambda^q\otimes 1_{q^{-1}r})Q_s^\Phi \overset{(9.3)}{=} \text{s-}\lim \overline{\Phi}(a\otimes 1_{q^{-1}r})\Phi(\mu_\lambda^q)Q_s^\Phi \\ &= \overline{\Phi}(a\otimes 1_{q^{-1}r})Q_r^\Phi Q_q^\Phi Q_s^\Phi \overset{(iii)}{=} \overline{\Phi}(a\otimes 1_{q^{-1}r})Q_r^\Phi = \overline{\Phi}(a\otimes 1_{q^{-1}r}).\end{aligned}$$

To prove Nica covariance, let $a \in \mathcal{L}(p, q)$, $b \in \mathcal{L}(s, t)$. By (i), if $qP \cap sP = rP$, then

$$\overline{\Phi}(a)\overline{\Phi}(b) = \overline{\Phi}(a)Q_q^\Phi Q_s^\Phi \overline{\Phi}(b) = \overline{\Phi}(a)Q_r^\Phi \overline{\Phi}(b),$$

which is $\overline{\Phi}\left((a\otimes 1_{q^{-1}r})(b\otimes 1_{s^{-1}r})\right)$ by the previous paragraph. The calculations above also show that $\overline{\Phi}(a)\overline{\Phi}(b) = 0$ when $qP \cap sP = \emptyset$.

To see that (iv)⇒(i), note that (9.7) and (9.9) imply that for every $p, q \in P$ and $a \in \mathcal{K}(p, p)$ we have $Q_{\langle p\rangle}^\Phi \Phi(a) = Q_p^\Phi \Phi(a)$. Since $Q_{\langle p\rangle}^\Phi$ and Q_p^Φ are zero on the orthogonal complement of $C^*(\Phi(\mathcal{K}))H$, this implies that $Q_{\langle p\rangle}^\Phi = Q_p^\Phi$.

This proves the equivalence of (i)-(iv). Applying (9.9) and its adjoint to $a \in \mathcal{L}(q, q)$ we get $Q_p^\Phi \in \overline{\Phi}(\mathcal{L}(p, p))'$, for all $p, q \in P$. The remaining part follows from Lemma 9.3. □

It is possible to cook up an example where the above equivalent conditions fail:

Example 9.5. Let $\mathcal{L} = \mathcal{K} = \{\mathcal{K}(n, m)\}_{n,m\in\mathbb{N}}$ be a C^*-precategory where $\mathcal{K}(n, m) = \{0\}$ for all $n \neq m$ and $\mathcal{K}(n, n)$ are arbitrary (non-zero) C^*-algebras. The multiplication in $\mathcal{L}$ is zero. We equip $\mathcal{L}$ with a right tensoring which is also zero. Then $\mathcal{NT}_{\mathcal{L}}(\mathcal{K}) = \mathcal{NT}(\mathcal{L}) = \mathcal{T}(\mathcal{L})$ is naturally isomorphic with the direct sum $\bigoplus_{n\in\mathbb{N}} \mathcal{K}(n, n)$. In particular, taking any faithful representations $\Phi_{n,n} : \mathcal{K}(n, n) \to \mathcal{B}(H_n)$ and setting $H := \bigoplus_n H_n$ and $\Phi_{n,m} := 0$ for $n \neq m$, $n, m \in \mathbb{N}$, we get an injective Nica covariant representation Φ of $\mathcal{K}$. For each $n \in \mathbb{N}$, Q_n^Φ is the projection onto H_n and $Q_{\langle n\rangle}^\Phi$ is the projection onto $\bigoplus_{k\geq n} H_k$. Note that $\overline{\Phi} = \Phi$ is Nica covariant and (9.7) is satisfied. However, (9.9) fails.

In order to avoid situations as in Example 9.5, a nondegeneracy condition was introduced in [19, Definition 3.8][1], in the case $P = \mathbb{N}$. We generalize this notion to arbitrary LCM semigroups. Virtually all examples considered in [20] will satisfy this condition.

Definition 9.6. An ideal $\mathcal{K}$ in a right-tensor C^*-precategory $\mathcal{L}$ is $\otimes 1$-*nondegenerate* if

$$(\mathcal{K}(p, p) \otimes 1_r)\mathcal{K}(pr, pr) = \mathcal{K}(pr, pr) \text{ for every } p \in P \setminus P^* \text{ and } r \in P.$$

Proposition 9.7. *If $\mathcal{K}$ is an $\otimes 1$-nondegenerate ideal in $\mathcal{L}$ then for every Nica covariant representation Φ of $\mathcal{K}$ the equivalent conditions in Proposition 9.4 hold true.*

Proof. We show condition (i) in Proposition 9.4. By definitions, $Q_p^\Phi \leq Q_{\langle p\rangle}^\Phi$ for all $p \in P \setminus P^*$ and $Q_p^\Phi = Q_{\langle p\rangle}^\Phi$ for all $p \in P^*$. By nondegeneracy of $\mathcal{K}$, for any $p \in P \setminus P^*$ and any $w \in pP$ we have $\mathcal{K}(w, w) = (\mathcal{K}(p, p) \otimes 1_{p^{-1}w})\mathcal{K}(w, w)$. Thus by Nica covariance we get $\Phi(\mathcal{K}(w, w)) = \Phi(\mathcal{K}(p, p) \otimes 1_{p^{-1}w})\mathcal{K}(w, w) = \Phi(\mathcal{K}(p, p))\Phi(\mathcal{K}(w, w))$, which implies that $Q_p^\Phi \geq Q_{\langle p\rangle}^\Phi$. □

[1]We note that there are typos in [19, Definition 3.8], one should put there $m = n$.

10. Representations generating exotic C^*-algebras - uniqueness theorem

We fix a well-aligned ideal $\mathcal{K}$ in a right-tensor C^*-precategory $(\mathcal{L}, \{\otimes 1_r\}_{r\in P})$. In this section we study conditions implying that a Nica covariant representation of $\mathcal{K}$ generates an exotic Nica-Toeplitz C^*-algebra of $\mathcal{K}$. In the presence of amenability this will lead to isomorphism theorems for the universal Nica-Toeplitz algebra $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ as well.

We start by introducing the key condition that we call (C). Here the letter C stands for both compression and Coburn.

Definition 10.1. Let $\Phi : \mathcal{K} \to \mathcal{B}(H)$ be a Nica covariant representation of $\mathcal{K}$ on a Hilbert space, and let $\{Q^{\Phi}_{\langle p\rangle}\}_{p\in P} \subseteq \mathcal{B}(H)$ be the projections introduced in Definition (9.1). We say that Φ satisfies *condition* (C) if

$$\text{(10.1)}\qquad \begin{array}{l}\text{for every } p \in P \text{ and } q_1, ..., q_n \in P \text{ such that } p \not\geq q_i, \text{ for } i = 1, ..., n,\ n \in \mathbb{N},\\ \text{the representation } \mathcal{K}(p,p) \ni a \longmapsto \Phi(a)\prod_{i=1}^n (1 - Q^{\Phi}_{\langle q_i\rangle}) \text{ is faithful.}\end{array}$$

Remark 10.2. Since projections $\{Q^{\Phi}_{\langle p\rangle}\}_{p\in P}$ mutually commute, we have $(1 - \bigvee_{i=1}^n Q^{\Phi}_{\langle q_i\rangle}) = \prod_{i=1}^n (1 - Q^{\Phi}_{\langle q_i\rangle})$. By (9.6), $(1 - Q^{\Phi}_{\langle q_i\rangle}) \in \Phi(\mathcal{K}(p,p))'$ and hence $\mathcal{K}(p,p) \ni a \longmapsto \Phi(a)\prod_{i=1}^n (1 - Q^{\Phi}_{\langle q_i\rangle})$ is indeed a representation of the C^*-algebra $\mathcal{K}(p,p)$.

Every well-aligned ideal $\mathcal{K}$ admits a representation satisfying condition (C).

Proposition 10.3. *Let π be a representation of $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ induced by the Fock Hilbert module $\mathcal{F}_{\mathcal{K}}$ from a faithful representation $\pi_0 : \bigoplus_{t\in P} \mathcal{K}(t,t) \to \mathcal{B}(H)$. Then $\pi \circ T : \mathcal{K} \to \mathcal{B}(\mathcal{F}_{\mathcal{K}} \otimes_{\pi_0} H)$ is a Nica covariant representation satisfying condition (C).*

Proof. Recall that $\pi : \mathcal{L}(\mathcal{F}_{\mathcal{K}}) \to \mathcal{B}(\mathcal{F}_{\mathcal{K}} \otimes_{\pi_0} H)$ is a faithful representation given by the formula $a(x \otimes_{\pi_0} h) := (ax) \otimes_{\pi_0} h$, $a \in \mathcal{L}(\mathcal{F}_{\mathcal{K}})$, $x \in \mathcal{F}_{\mathcal{K}}$, $h \in H$.

Let $p \in P$ and $q_1, ..., q_n \in P$ be such that $p \not\geq q_i$, for $i = 1, ..., n$. Consider projections $\{Q^T_{\langle q_i\rangle}\}_{i=1}^n$ introduced in Lemma 5.8. Since the representation $\mathcal{K}(p,p) \ni a \longmapsto T^{p,p}_{p,p}(a) \in \mathcal{L}(X_{p,p})$ is faithful, so is $\mathcal{K}(p,p) \ni a \longmapsto T(a)\prod_{i=1}^n (1 - Q^T_{\langle q_i\rangle})$. With the definition of projections from Lemma 9.3, it readily follows that $Q^{\pi\circ T}_{\langle q_i\rangle} \leq \pi(Q^T_{\langle q_i\rangle})$, for $i = 1, ..., n$. Hence $\pi(\prod_{i=1}^n (1 - Q^T_{\langle q_i\rangle})) \leq \prod_{i=1}^n (1 - Q^{\pi\circ T}_{\langle q_i\rangle})$ and therefore

$$T(a)\prod_{i=1}^n (1 - Q^T_{\langle q_i\rangle}) \neq 0 \implies \pi\Big(T(a)\prod_{i=1}^n (1 - Q^T_{\langle q_i\rangle})\Big) \neq 0 \implies \pi(T(a))\prod_{i=1}^n (1 - Q^{\pi\circ T}_{\langle q_i\rangle}) \neq 0.$$

Thus $\mathcal{K}(p,p) \ni a \longmapsto \pi(T(a))\prod_{i=1}^n (1 - Q^{\pi\circ T}_{\langle q_i\rangle})$ is faithful. □

Plainly, every Nica covariant representation satisfying condition (C) is injective and Toeplitz covariant in the sense of (6.3). In the converse direction we have the following:

Proposition 10.4. *If $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$, then a Nica covariant representation $\Phi : \mathcal{K} \to B(H)$ satisfies condition (C) if and only if Φ is injective and Toeplitz covariant.*

Proof. The 'only if' part is clear. Suppose than that Φ is injective and Toeplitz covariant. By Corollaries 6.3 and 6.4 there is an isomorphism $\Phi_* : B^{\Phi}_e \to B^T_e$ making the diagram (7.2) commute. Denote by $\overline{\Phi}_* : M(B^{\Phi}_e) \to M(B^T_e)$ the strictly continuous extension of Φ_*. Let $p \in P$ and $q_1, ..., q_n \in P$ be such that $p \not\geq q_i$. We noticed in the proof of Proposition 10.3 that the representation $\mathcal{K}(p,p) \ni a \longmapsto T_{p,p}(a)\prod_{k=1}^n (1 - Q^T_{\langle q_i\rangle})$ is faithful. By Lemma 5.10 we may view $Q^T_{\langle q_i\rangle}$ as elements of $M(B_e^T)$. Similarly, by Lemma 9.3 applied to $\mathcal{L}=\mathcal{K}$, one concludes that we may treat projections $Q^\Phi_{\langle q_i\rangle}$ as elements of $M(B_e^\Phi)$, and then $\overline{\Phi}_*(Q^\Phi_{\langle q_i\rangle})=Q^T_{\langle q_i\rangle}$. Thus

$$T_{p,p}(a)\prod_{k=1}^{n}(1-Q^T_{\langle q_i\rangle})=\overline{\Phi}_*\Big(\Phi_{p,p}(a)\prod_{k=1}^{n}(1-Q^\Phi_{\langle q_i\rangle})\Big)\quad \text{for all } a\in\mathcal{K}(p,p).$$

Therefore $\mathcal{K}(p,p)\ni a\longmapsto \Phi_{p,p}(a)\prod_{k=1}^{n}(1-Q^\Phi_{\langle q_i\rangle})$ is faithful, and Φ satisfies (C). □

Corollary 10.5. *Let $\Phi:\mathcal{K}\to B(H)$ be a Nica covariant representation. Suppose that $\mathcal{K}$ is essential in $\mathcal{L}$ and that the extended representation $\overline{\Phi}:\mathcal{L}\to B(H)$ is Nica covariant (which is automatic when $\mathcal{K}$ is $\otimes 1$-non-degenerate). The following conditions are equivalent:*
(i) Φ *satisfies condition (C);*
(ii) $\overline{\Phi}$ *satisfies condition (C);*
(iii) $\overline{\Phi}$ *is injective and Toeplitz covariant.*

Proof. Equivalence (ii)⇔(iii) follows from Proposition 10.4 applied to $\mathcal{L}$ and $\overline{\Phi}$. To see that (i)⇔(ii) let $p\in P$ and $q_1,...,q_n\in P$ such that $p\not\geq q_i$, for $i=1,...,n$. Note that $Q^\Phi_{\langle q\rangle}=Q^{\overline{\Phi}}_{\langle q\rangle}$, for $q\in P$. Hence since $\mathcal{K}(p,p)$ is an essential ideal in the C^*-algebra $\mathcal{L}(p,p)$, the representation $\mathcal{K}(p,p)\ni a\longmapsto \Phi(a)\prod_{i=1}^{n}(1-Q^\Phi_{\langle q_i\rangle})$ is faithful if and only if the representation $\mathcal{L}(p,p)\ni a\longmapsto \overline{\Phi}(a)\prod_{i=1}^{n}(1-Q^{\overline{\Phi}}_{\langle q_i\rangle})$ is faithful. □

Next we introduce an auxiliary condition which describes properties of the (not necessarily Nica covariant) representation of $\mathcal{L}$ that extends a Nica covariant representation of $\mathcal{K}$.

Definition 10.6. Let $\Phi:\mathcal{K}\to\mathcal{B}(H)$ be a Nica covariant representation on a Hilbert space, and let $\overline{\Phi}:\mathcal{L}\to\mathcal{B}(H)$ be the extension from Proposition 2.13. Let $\{Q_p^\Phi\}_{p\in P}\subseteq\mathcal{B}(H)$ be the projections given by (9.1). We say that $\overline{\Phi}$ satisfies *condition* (C') if

(10.2)
$$\begin{gathered}\text{for every } p\in P \text{ and } q_1,...,q_n\in P \text{ such that } p\not\geq q_i, \text{ for } i=1,...,n,\\ \|(1-\textstyle\bigvee_{i=1}^{n}Q^\Phi_{q_i})\overline{\Phi}(a)(1-\bigvee_{i=1}^{n}Q^\Phi_{q_i})\|=\|\overline{\Phi}(a)\| \text{ for all } a\in\mathcal{L}(p,p).\end{gathered}$$

Proposition 10.7. *Let $\Phi:\mathcal{K}\to\mathcal{B}(H)$ be a Nica covariant representation of $\mathcal{K}$ and let $\overline{\Phi}:\mathcal{L}\to\mathcal{B}(H)$ be the extended representation of $\mathcal{L}$. Consider the following assertions:*
(i) Φ *satisfies condition (C);*
(ii) Φ *is injective and $\overline{\Phi}$ satisfies condition (C');*

Then (i)⇒(ii). If $\mathcal{K}$ is $\otimes 1$-non-degenerate or $\mathcal{K}\otimes 1\subseteq\mathcal{K}$, then (i)⇔(ii).

Proof. (i)⇒(ii). Plainly, condition (C) implies that Φ is injective. Let $p\in P$ and $q_1,...,q_n\in P$ be such that $p\not\geq q_i$. Let $T^{p,p}_{p,p}:\mathcal{L}(p,p)\to\mathcal{L}(X_{p,p})=\mathcal{L}(\mathcal{K}(p,p))$ be the homomorphism given by multiplication from left, cf. Lemma 4.1. By (2.8) and the construction of T, we have that $\ker T^{p,p}_{p,p}=\ker\overline{\Phi}_{p,p}$. Similarly, using (10.1), we see that the kernel of the representation $\mathcal{L}(p,p)\ni a\longmapsto\overline{\Phi}(a)\prod_{i=1}^{n}(1-Q^\Phi_{\langle q_i\rangle})$ coincides with the kernel of $T^{p,p}_{p,p}$. Thus $\|\overline{\Phi}(a)\prod_{i=1}^{n}(1-Q^\Phi_{\langle q_i\rangle})\|=\|\overline{\Phi}(a)\|$ for all $a\in\mathcal{L}(p,p)$. Since $\prod_{i=1}^{n}(1-Q^\Phi_{\langle q_i\rangle})\leq(1-\bigvee_{i=1}^{n}Q^\Phi_{q_i})$, this implies that $\|(1-\bigvee_{i=1}^{n}Q^\Phi_{q_i})\overline{\Phi}(a)(1-\bigvee_{i=1}^{n}Q^\Phi_{q_i})\|\geq\|\overline{\Phi}(a)\|$ for all $a\in\mathcal{L}(p,p)$. The reverse inequality is clear.

(ii)⇒(i). Let $p\in P$ and $q_1,...,q_n\in P$ be such that $p\not\geq q_i$. For every $a\in\mathcal{K}(q_i,q_i)$, $j=i,\dots,n$, we have $\|(1-\bigvee_{i=1}^{n}Q^\Phi_{q_i})\overline{\Phi}(a)(1-\bigvee_{i=1}^{n}Q^\Phi_{q_i})\|=0$. Hence condition (C') implies that Φ is Toeplitz covariant, and therefore if $\mathcal{K}\otimes 1\subseteq\mathcal{K}$ then Φ satisfies condition (C) by

Proposition 10.4. Suppose then that $\mathcal{K}$ is $\otimes 1$-non-degenerate. By Proposition 9.7 we have $(1-\bigvee_{i=1}^n Q_{q_i}^{\Phi})=\prod_{i=1}^n(1-Q_{\langle q_i\rangle}^{\Phi})$. Hence for any $a\in\mathcal{K}(p,p)$ we get

$$\|\Phi(a)\prod_{i=1}^n(1-Q_{\langle q_i\rangle}^{\Phi})\|=\|(1-\bigvee_{i=1}^n Q_{q_i}^{\Phi})\overline{\Phi}(a)(1-\bigvee_{i=1}^n Q_{q_i}^{\Phi})\|=\|\Phi(a)\|=\|a\|.$$

Thus Φ satisfies condition (C). □

In order to get a uniqueness result in the case when the group $P^*\subseteq P$ is non-trivial, we need to impose certain additional conditions on $\mathcal{K}$ and $\mathcal{L}$. It seems that there is no single candidate for such a condition available on the scene. However, there are several natural conditions that can be applied in different situations. For instance, if P can be embedded into a group G via a monomorphism $\theta: P\to G$, and $\mathcal{B}^\theta$ is the Fell bundle described in Proposition 8.3, then Theorem 8.4 and results of [21], [23] indicate that a natural condition is aperiodicity of the Fell bundle $\mathcal{B}^\theta$.

We recall from [23, Definition 4.1] that a Fell bundle $\mathcal{B}=\{B_g\}_{g\in G}$ is *aperiodic* if for each $t\in G\setminus\{e\}$, each $b_t\in B_t$ and every non-zero hereditary subalgebra D of B_e,

$$\inf\{\|db_td\| : d\in D^+,\ \|d\|=1\}=0.$$

For general semigroups P we may consider the following modification (and in fact generalization) of this notion expressed in terms of $\mathcal{L}$ and $\mathcal{K}$. We recall, see Lemma 3.10, that the group P^* of invertible elements acts on $\mathcal{K}$ by automorphisms.

Definition 10.8. We say that the group $\{\otimes 1_x\}_{x\in P^*}$ of automorphisms of $\mathcal{L}$ is *aperiodic on* $\mathcal{K}$ if for every $p\in P$, every non-zero hereditary C^*-subalgebra $D\subseteq\mathcal{K}(p,p)$ and every $b\in\mathcal{K}(px,p)$ where $x\in P^*\setminus\{e\}$ we have

$$\inf\{\|(a\otimes 1_x)ba\| : a\in D^+,\ \|a\|=1\}=0. \tag{10.3}$$

Remark 10.9. For a fixed $p\in P$ the spaces $B_x^{(p)}:=\mathcal{K}(px,p)$, $x\in P^*$, equipped with multiplication and involution defined by $b_x\cdot b_y:=b_x\otimes 1_y b_y$ and $(b_x)^*:=(b_x^*)\otimes 1_{x^{-1}}$, for $b_x\in B_x^{(p)}, b_y\in B_y^{(p)}$, form a Fell bundle $\mathcal{B}^{(p)}:=\{B_x^{(p)}\}_{x\in P^*}$ over P^*. In particular, the group $\{\otimes 1_x\}_{x\in P^*}$ is aperiodic on $\mathcal{K}$ if and only if for each $p\in P$ the Fell bundle $\mathcal{B}^{(p)}$ is aperiodic.

The following lemma can be proved using a standard argument, cf. the proof of [33, Lemma 5.2]. In view of Remark 10.9, it is a corollary of [23, Lemma 4.2].

Lemma 10.10. *Suppose that the group $\{\otimes 1_x\}_{x\in P^*}$ of automorphisms of $\mathcal{K}$ is aperiodic. Take any $p\in P$ and let $F\subseteq P^*$ be a finite set containing e. For every family of elements $a_x\in\mathcal{K}(px,p)$ for $x\in F$ with $a_e=a_e^*$, and every $\varepsilon>0$ there is an element $d\in\overline{a_e\mathcal{K}(p,p)a_e}$, $\|d\|=1$, such that $\|(d\otimes 1_x)a_xd\|<\varepsilon$ for every $x\in F\setminus\{e\}$ and $\|da_ed\|>\|a_e\|-\varepsilon$.*

Proof. By passing to $-a_e$ if necessary, we may assume that $\|a_e\|$ is a spectral value of a_e. Then applying to a_e a non-decreasing continuous function that vanishes on $(-\infty,0]$ and is identity on a neighborhood of $\|a_e\|$ we may assume a_e is positive. Then the assertion follows from [23, Lemma 4.2] applied to the Fell bundle $\mathcal{B}^{(p)}$ described in Remark 10.9. □

We will also need the following fact.

Lemma 10.11. *Suppose that $\mathcal{K}$ is an essential ideal in $\mathcal{L}$. Then the group $\{\otimes 1_x\}_{x\in P^*}$ of automorphisms is aperiodic on $\mathcal{K}$ if and only if it is aperiodic on $\mathcal{L}$.*

Proof. In view of Remark 10.9 the assertion follows from [21, Corollary 6.9]. □

Our main results relate the properties of a Nica covariant representation of $\mathcal{K}$ on a Hilbert space that have been introduced so far. Recall that Toeplitz covariance was defined in (6.3); we think of it as being an algebraic condition. By contrast, condition (C) can be viewed as a geometric condition.

Theorem 10.12. *Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $(\mathcal{L}, \{\otimes 1_r\}_{r\in P})$. Consider the following conditions on a Nica covariant representation $\Phi : \mathcal{K} \to B(H)$:*

(i) *Φ satisfies condition (C);*
(ii) *Φ is injective and $\overline{\Phi} : \mathcal{L} \to \mathcal{B}(H)$ satisfies condition (C');*
(iii) *Φ generates an exotic Nica-Toeplitz C^*-algebra $C^*(\Phi(\mathcal{K}))$ of $\mathcal{K}$;*
(iv) *$\Phi \rtimes P$ is injective on the core C^*-subalgebra $B_e^{i_{\mathcal{K}}}$ of $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$;*
(v) *Φ is injective and Toeplitz covariant.*

Then (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v). If one of the following conditions holds:

(1) *$P^* = \{e\}$,*
(2) *the group $\{\otimes 1_x\}_{x\in P^*}$ is aperiodic on $\mathcal{K}$ and $\mathcal{K}$ is essential in $\mathcal{L}_{\mathcal{K}}$, cf. Lemma 3.17,*
(3) *there is a unital monomorphism $\theta : P \to G$ into a group G and the Fell bundle $\mathcal{B}^\theta$ from Proposition 8.3 is aperiodic,*

then (ii) $\Rightarrow$ (iii). Further, if $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$, then all five conditions are equivalent.

Proof. Implications (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(v)$\Leftrightarrow$(iv) follow respectively from Propositions 10.7 and 7.3 and Corollary 6.3. If $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$ we also have (v)$\Rightarrow$(i) by Proposition 10.4. Thus it remains to show that (ii)$\Rightarrow$(iii), provided one of the conditions (1)-(3) holds.

Suppose therefore that Φ is injective and $\overline{\Phi} : \mathcal{L} \to \mathcal{B}(H)$ satisfies condition (C'). In view of Proposition 7.4, it suffices to show that (7.3) defines a bounded map E^Φ on the dense $*$-subalgebra $C^*(\Phi(\mathcal{K}))^0$ of $C^*(\Phi(\mathcal{K}))$, cf. (3.5). Moreover, it suffices to define a bounded map E^Φ on the $\mathbb{R}$-linear subspace $C^*(\Phi(\mathcal{K}))^0_{sa}$ of $C^*(\Phi(\mathcal{K}))^0$ consisting of elements of the form

$$a = \sum_{p,q\in F} \Phi(a_{p,q}), \quad \text{where } a_{p,q} \in \mathcal{K}(p,q),\ a^*_{p,q} = a_{q,p}, \tag{10.4}$$

for $F \subseteq P$ finite and $p, q \in F$. Indeed, such map on $C^*(\Phi(\mathcal{K}))^0_{sa}$ extends via the formula $E^\Phi(a) = E^\Phi(\frac{a+a^*}{2}) + iE^\Phi(\frac{a-a^*}{2i})$ to a bounded map E^Φ on $C^*(\Phi(\mathcal{K}))^0$ satisfying (7.3).

Let us then fix a self-adjoint element a given by (10.4). Denote by Z the right hand side of (7.3), cf. also (5.8). Note that $Z^* = Z$. Our strategy is to show (by consecutive compressions of a and its compressions) that for every $\varepsilon > 0$ we have

$$\|Z\| - \varepsilon \le \|a\|. \tag{10.5}$$

To this end, we fix $\varepsilon > 0$, and note that by Lemma 5.11 we may find an initial segment C of F and $w \in \sigma(C)P$ with $w \in P_{F,C}$ such that

$$\|Z\| - \varepsilon/2 \le \Big\| T^{w,w}_{w,w}\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q} \otimes 1_{q^{-1}w}\Big)\Big\|.$$

As noticed in the proof of Proposition 10.7, faithfulness of Φ implies that $\ker T^{w,w}_{w,w} = \ker \overline{\Phi}_{w,w}$. Thus we obtain

$$\|Z\| - \varepsilon/2 \le \Big\| \overline{\Phi}\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q} \otimes 1_{q^{-1}w}\Big)\Big\|. \tag{10.6}$$

Clearly, (10.5) will follow from (10.6) if we prove that

$$\Big\|\overline{\Phi}\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)\Big\| - \frac{\varepsilon}{4} \le \|a\|. \tag{10.7}$$

We fix the above C and w. Put

$$\begin{aligned} F_{rest} &:= \{r\in P : tP\cap wP = rP, t\in F\setminus C\},\\ F_{>w} &:= \{pq^{-1}w : pq^{-1}w = wx,\ x\in P\setminus P^*, p,q\in C\},\\ F_w &:= \{pq^{-1}w : pq^{-1}w = wx,\ x\in P^*, p,q\in C\}. \end{aligned}$$

Note that for $s\in F_w$ we have $s\ge w\ge s$ and for $s\in F_{>w}$ we have $s\ge w$ and $w\not\ge s$. Since $\omega\notin\bigcup_{t\in F\setminus C} tP$, for $r\in F_{rest}$ we have $r\ge w$ and $w\not\ge r$. Now suppose that $d\in\mathcal{L}(w,w)$. Using (9.4) (and its adjoint) and that $\overline{\Phi}$ is a representation we get

$$\begin{aligned} \overline{\Phi}(d)a\overline{\Phi}(d) = &\sum_{\substack{p,q\in C\\ pq^{-1}w\in F_w}} \overline{\Phi}(d)\overline{\Phi}(a_{p,q}\otimes 1_{q^{-1}w})\overline{\Phi}(d)\\ &+\sum_{\substack{p,q\in C\\ pq^{-1}w\in F_{>w}}} \overline{\Phi}(d)\overline{\Phi}(a_{p,q}\otimes 1_{q^{-1}w})\overline{\Phi}(d)\\ &+\sum_{\substack{p\in F, q\in F\setminus C\\ qP\cap wP=rP, r\in F_{rest}}} \overline{\Phi}(d)\overline{\Phi}(a_{p,q}\otimes 1_{q^{-1}r})\overline{\Phi}(d)\\ &+\sum_{\substack{p\in F\setminus C, q\in C\\ pP\cap wP=rP, r\in F_{rest}}} \overline{\Phi}(d)\overline{\Phi}(a_{p,q}\otimes 1_{p^{-1}r})\overline{\Phi}(d). \end{aligned}$$

We put $F_0 := F_{rest}\cup F_{>w}$ and consider the projection

$$Q^\Phi_{F_0} := \bigvee_{s\in F_0} Q^\Phi_s.$$

Note that $(Q^\Phi_{F_0})^\perp := I - Q^\Phi_{F_0}$ is a nonzero projection. For any $s\in F_0$ we have $s\ge w$ and therefore $\overline{\Phi}(d)$ and Q^Φ_s commute, by the last part of Lemma 9.2. This implies that

$$(Q^\Phi_{F_0})^\perp\overline{\Phi}(d)Q^\Phi_s = 0 \quad\text{and}\quad Q^\Phi_s\overline{\Phi}(d)(Q^\Phi_{F_0})^\perp = 0 \quad\text{for every } s\in F_0.$$

Applying this observation to $Q^\Phi_w a Q^\Phi_w$ we get

$$(Q^\Phi_{F_0})^\perp\overline{\Phi}(d)a\overline{\Phi}(d)(Q^\Phi_{F_0})^\perp = (Q^\Phi_{F_0})^\perp \sum_{\substack{p,q\in C\\ pq^{-1}w\in F_w}} \overline{\Phi}(d)\overline{\Phi}(a_{p,q}\otimes 1_{q^{-1}w})\overline{\Phi}(d)(Q^\Phi_{F_0})^\perp. \tag{10.8}$$

To deduce (10.7) from here we consider the cases (1)-(3).
Case (1). Assume that $P^*=\{e\}$. Then $F_w=\{w\}$. In particular, allowing d in (10.8) to run through an approximate unit in $\mathcal{K}(w,w)$ and taking strong limit we get

$$(Q^\Phi_{F_0})^\perp Q^\Phi_w a Q^\Phi_w (Q^\Phi_{F_0})^\perp = (Q^\Phi_{F_0})^\perp\overline{\Phi}\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)(Q^\Phi_{F_0})^\perp.$$

Employing this equality and condition (C') we have

$$\|a\| \ \geq \ \|(Q^{\Phi}_{F_0})^{\perp} Q^{\Phi}_w a Q^{\Phi}_w (Q^{\Phi}_{F_0})^{\perp}\| = \|(Q^{\Phi}_{F_0})^{\perp}\overline{\Phi}\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)(Q^{\Phi}_{F_0})^{\perp}\|$$

$$\overset{(10.2)}{=} \|\overline{\Phi}\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)\|.$$

This proves (10.7) and finishes the proof under hypothesis (1).

Case (2). Suppose that the group $\{\otimes 1_x\}_{x\in P^*}$ is aperiodic on $\mathcal{K}$ and that $\mathcal{K}$ is essential in $\mathcal{L}_{\mathcal{K}}$. Then $\{\otimes 1_x\}_{x\in P^*}$ is also aperiodic on $\mathcal{L}_{\mathcal{K}}$, by Lemma 10.11. Since Z is self-adjoint, it follows from (5.10) that $\sum_{\{p,q\in C: pq^{-1}w=w\}} a_{p,q}\otimes 1_{q^{-1}w}\in \mathcal{L}_{\mathcal{K}}(w,w)$ is also self-adjoint. Hence by Lemma 10.10 there is $d\in \mathcal{L}_{\mathcal{K}}(w,w)$, $\|d\|=1$, such that

$$\|(d\otimes 1_x)(a_{p,q}\otimes 1_{q^{-1}w})d\| < \frac{\varepsilon}{8|C|^2} \tag{10.9}$$

for every $p,q\in C$ and $x\in P^*\setminus\{e\}$ where $pq^{-1}w=wx$, and

$$\|d\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)d\| > \|\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\| - \frac{\varepsilon}{8}. \tag{10.10}$$

We now return to the computation (10.8) with d chosen above, and note that

$$(Q^{\Phi}_{F_0})^{\perp}\overline{\Phi}(d)a\overline{\Phi}(d)(Q^{\Phi}_{F_0})^{\perp} = (Q^{\Phi}_{F_0})^{\perp}\overline{\Phi}\Big(d\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)d\Big)(Q^{\Phi}_{F_0})^{\perp}$$

$$+ (Q^{\Phi}_{F_0})^{\perp}\sum_{\substack{p,q\in C, x\in P^*\setminus\{e\}\\ pq^{-1}w=wx}} \overline{\Phi}\Big((d\otimes 1_x)(a_{p,q}\otimes 1_{q^{-1}w})d\Big)(Q^{\Phi}_{F_0})^{\perp}.$$

Since $\mathcal{K}$ is essential in $\mathcal{L}_{\mathcal{K}}$ and Φ is injective, $\overline{\Phi}$ is isometric on $\mathcal{L}_{\mathcal{K}}$, by the last part of Proposition 2.13. Note that $d\big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\big)d\in \mathcal{L}_{\mathcal{K}}(w,w)$. Thus we obtain the estimates

$$\begin{aligned}
\|a\| \ &\geq \ \|(Q^{\Phi}_{F_0})^{\perp}\overline{\Phi}(d)a\overline{\Phi}(d)Q^{\Phi}_{F_0})^{\perp}\| \\
&\overset{(10.9)}{>} \|(Q^{\Phi}_{F_0})^{\perp}\overline{\Phi}\Big(d\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)d\Big)(Q^{\Phi}_{F_0})^{\perp}\| - \frac{\varepsilon}{8} \\
&\overset{(10.2)}{=} \|\overline{\Phi}\Big(d\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)d\Big)\| - \frac{\varepsilon}{8} \\
&= \|d\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)d\| - \frac{\varepsilon}{8} \\
&\overset{(10.10)}{\geq} \|\overline{\Phi}\Big(\sum_{\substack{p,q\in C\\ pq^{-1}w=w}} a_{p,q}\otimes 1_{q^{-1}w}\Big)\| - \frac{\varepsilon}{4}.
\end{aligned}$$

This proves (10.7) and finishes the proof under hypothesis (2).

Case (3). Suppose that $\theta : P\to G$ is a unital monomorphism and the Fell bundle $\mathcal{B}^{\theta}$ is aperiodic. Then $\theta : P\to \theta(P)\subseteq G$ is a controlled map of right LCM semigroups, P is cancellative, and $B_e^\theta = B_e^{i_\mathcal{K}}$. Injectivity of Φ and condition (C') imply that Φ is a Toeplitz covariant representation, see the proof of Proposition 10.7. Hence by Corollary 6.3 (and the C^*-equality) the maps $\Phi \rtimes P : B_g^\theta \to C^*(\Phi(\mathcal{K}))$, $g \in G$, are injective. Putting $F_G := \{\theta(p)\theta(q)^{-1} : p, q \in F\}$ we have

$$a = (\Phi \rtimes P)(\oplus_{g\in F_G} b_g) \quad \text{where} \quad b_g = \sum_{\substack{p,q\in F,\\ \theta(p)\theta(q)^{-1}=g}} i_\mathcal{K}(a_{p,q}),\ g \in F_G.$$

By (10.4), b_e is self-adjoint. Hence by [23, Lemma 4.2] there is $d \in B_e$ such that $\|d\| = 1$, $\|db_ed - dad\| < \varepsilon/2$ and $\|db_ed\| > \|b_e\| - \varepsilon/2$. Thus we get

$$\|b_e\| - \varepsilon/2 < \|db_ed\| = \|\Phi \rtimes P(db_ed)\| \le \|\Phi \rtimes P(dad)\| + \varepsilon/2 \le \|a\| + \varepsilon/2. \tag{10.11}$$

But since P is cancellative and θ injective we also have

$$Z = \sum_{p\in F} T_{p,p}(a_{p,p}) = \sum_{\substack{p,q\in F\\ \theta(p)\theta(q)^{-1}=e}} T_{p,q}(a_{p,q}) = T \rtimes P(b_e).$$

Hence $\|Z\| = \|b_e\|$ and (10.11) implies (10.5). □

Remark 10.13. Under condition (3) in Theorem 10.12, also the implication $(v) \Rightarrow (iii)$ holds. Indeed, given Nica covariant Φ that is injective and Toeplitz covariant, assume $\theta : P \to G$ is an injective controlled homomorphism such that the associated Fell bundle B^θ is aperiodic. Corollary 6.3 implies that $\Phi \rtimes P$ is injective on $B_e^{i_\mathcal{K}} \cong B_e^\theta$ and, therefore $\ker(\Phi \rtimes P) \subseteq \ker \Lambda$ by Proposition 12.10 below. Thus Φ generates an exotic Nica-Toeplitz C^*-algebra by Theorem 8.4.

Corollary 10.14 (Uniqueness Theorem I)**.** *Suppose that $\mathcal{K}$ is an amenable well-aligned ideal in a C^*-precategory $\mathcal{L}$ and that one of conditions $(1) - (3)$ in Theorem 10.12 holds. Consider the following properties of a Nica covariant representation $\Phi : \mathcal{K} \to B(H)$:*

(i) *Φ satisfies condition (C);*
(ii) *The map $\Phi \rtimes P$ is an isomorphism $\mathcal{NT}_\mathcal{L}(\mathcal{K}) \cong C^*(\Phi(\mathcal{K}))$.*

Then $(i) \Rightarrow (ii)$. If $\mathcal{K} \otimes 1 \subseteq \mathcal{K}$, then $(i) \Leftrightarrow (ii)$.

Proof. Since $\mathcal{K}$ is amenable, the regular representation $T \rtimes P$ is injective. If any of the conditions $(1) - (3)$ in Theorem 10.12 is satisfied, we infer from that result that Φ generates an exotic Nica-Toeplitz C^*-algebra $C^*(\Phi(\mathcal{K}))$. Now the very definition of an exotic Nica-Toeplitz algebra means that $\Phi \rtimes P$ is injective. □

By Proposition 9.7, if the ideal $\mathcal{K}$ is $\otimes 1$-nondegenerate, then for every Nica covariant representation Φ of $\mathcal{K}$ the extended representation $\overline{\Phi}$ of $\mathcal{L}$ is Nica covariant. Moreover, if $\mathcal{K}$ is essential in $\mathcal{L}$ and Φ is injective, then $\overline{\Phi}$ is injective, see Proposition 2.13. This observation leads us to the following reformulation of Theorem 10.12:

Theorem 10.15. *Suppose that the well-aligned ideal $\mathcal{K}$ in $\mathcal{L}$ is essential and $\otimes 1$-nondegenerate. Assume also that either $P^* = \{e\}$ or the group $\{\otimes 1_x\}_{x\in P^*}$ is aperiodic on $\mathcal{K}$. For every Nica covariant representation $\Phi : \mathcal{K} \to B(H)$ the extended representation $\overline{\Phi}$ of $\mathcal{L}$ is Nica covariant and the following statements are equivalent:*

(i) *Φ satisfies condition (C);*
(ii) *Φ generates an exotic Nica-Toeplitz C^*-algebra $C^*(\Phi(\mathcal{K}))$ of $\mathcal{K}$, and $\overline{\Phi}$ generates an exotic Nica-Toeplitz C^*-algebra $C^*(\overline{\Phi}(\mathcal{L}))$ of $\mathcal{L}$;*
(iii) *$\overline{\Phi}$ is injective and Toeplitz covariant;*
(iv) *The map $\overline{\Phi} \rtimes P$ is injective on the core C^*-subalgebra $B_e^{i_\mathcal{L}}$ of $\mathcal{NT}(\mathcal{L})$.*

Proof. By Theorem 10.12 condition (i) implies that Φ generates an exotic Nica-Toeplitz C^*-algebra $C^*(\Phi(\mathcal{K}))$ of $\mathcal{K}$. By Lemma 10.11, if the group $\{\otimes 1_x\}_{x\in P^*}$ is aperiodic on $\mathcal{K}$, then it is also aperiodic on $\mathcal{L}$. By Corollary 10.5, Φ satisfies condition (C) if and only if $\overline{\Phi}$ satisfies this condition. Hence we may apply Theorem 10.12 to the extended representation $\overline{\Phi}:\mathcal{L}\to B(H)$. Then we get that each of conditions (i), (iii) and (iv) is equivalent to that $\overline{\Phi}$ generates an exotic Nica-Toeplitz C^*-algebra $C^*(\overline{\Phi}(\mathcal{L}))$ of $\mathcal{L}$. □

The above theorem explains why in general condition (C) is stronger then the "uniqueness" for Nica-Toeplitz algebras: (C) implies uniqueness not only for a representation of $\mathcal{K}$ but also for the extended representation of $\mathcal{L}$. We make this comment more formal in next section, see Corollary 11.5.

11. On the relationship between Nica-Toeplitz algebras of $\mathcal{K}$ and $\mathcal{L}$

Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $\mathcal{L}$. In this section we collect results which reflect relationship between the full and reduced Nica-Toeplitz algebras associated to $\mathcal{K}$ and $\mathcal{L}$. Recall from (3.6) the existence of a homomorphism ι from $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ to $\mathcal{NT}(\mathcal{L})$. We write $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\hookrightarrow\mathcal{NT}(\mathcal{L})$ whenever ι is injective.

Lemma 11.1. *If $\mathcal{K}$ is $\otimes 1$-nondegenerate, then $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\hookrightarrow\mathcal{NT}(\mathcal{L})$.*

Proof. Let $i_{\mathcal{K}}:\mathcal{K}\to\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ be the universal covariant representation of $\mathcal{K}$. Suppose that $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\subseteq B(H)$ (for example in the usual universal representation as a C^*-algebra). Then Proposition 9.7 and Proposition 9.4(ii) imply that the extension $\overline{i_{\mathcal{K}}}:\mathcal{L}\to B(H)$ of $i_{\mathcal{K}}$ is Nica covariant. Then injectivity of ι follows as explained in Remark 3.13. □

Proposition 11.2. *If $\mathcal{K}$ is amenable, then $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})\hookrightarrow\mathcal{NT}(\mathcal{L})$.*

Proof. The Fock representation T of $\mathcal{K}$ is the restriction of a Nica covariant representation $\overline{T}:\mathcal{L}\to\mathcal{L}(\mathcal{F}_{\mathcal{K}})$, cf. Proposition 4.3. The hypothesis on $\mathcal{K}$ means that $T\rtimes P$ is an isomorphism from $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ onto $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$. With η denoting the composition of the inclusion map from $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ into $\mathcal{L}(\mathcal{F}_{\mathcal{K}})$ with $T\rtimes P$, it follows that $i_{\mathcal{K}}:\mathcal{K}\to\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ admits the extension $\overline{T}:\mathcal{L}\to\eta(\mathcal{NT}_{\mathcal{L}}(\mathcal{K}))$, which is a Nica covariant representation of $\mathcal{L}$. Then injectivity follows by Remark 3.13. □

Theorem 11.3. *Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $(\mathcal{L},\{\otimes 1_r\}_{r\in P})$. Suppose that either P is cancellative or $\mathcal{K}$ is essential in the right-tensor C^*-precategory $\mathcal{L}_{\mathcal{K}}$ generated by $\mathcal{K}$. There is an embedding of C^*-algebras:*

$$\iota^r:\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})\hookrightarrow\mathcal{NT}^r(\mathcal{L})$$

determined by $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})\ni T(a)\to S(a)\in\mathcal{NT}^r(\mathcal{L})$, $a\in\mathcal{K}(p,q)$, $p,q\in P$, where T and S are Fock representations of $\mathcal{K}$ and $\mathcal{L}$, respectively. Moreover, ι^r intertwines the conditional expectations $E^T:\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})\to B_{\mathcal{K}}$ and $E^S:\mathcal{NT}^r(\mathcal{L})\to B_{\mathcal{L}}$, in the sense that $E^S\circ\iota^r=\iota^r|_{B_{\mathcal{K}}}\circ E^T$.

Proof. Suppose first that P is cancellative. Then E^T and E^S are faithful conditional expectations onto $B_{\mathcal{K}}=B^T_e$ and $B_{\mathcal{L}}=B^S_e$ respectively, and are given by (5.4). Since S is an injective Nica-Toeplitz representation of $\mathcal{L}$, it restricts to an injective Nica-Toeplitz representation $\Phi:\mathcal{K}\to\mathcal{L}(\mathcal{F}_{\mathcal{L}})$, and E^S restricts to a faithful conditional expectation E_Φ : $C^*(\Phi(\mathcal{K}))\to B^\Phi_e\subseteq C^*(\Phi(\mathcal{K}))$. Applications of Corollaries 6.3 and 6.4 yield an isomorphism $h:B_{\mathcal{K}}\to B^\Phi_e\subseteq B_{\mathcal{L}}$ such that $\Phi=h\circ T$ on each space $\mathcal{K}(p,p)$, $p\in P$. The composition $E^\Phi:=h^{-1}\circ E_\Phi$ is a faithful completely positive map from $C^*(\Phi(\mathcal{K}))$ onto $B_{\mathcal{K}}$ satisfying equation (7.3). Hence Proposition 7.4 gives an isomorphism Φ_* from $C^*(\Phi(\mathcal{K}))$ onto $\mathcal{NT}^r_{\mathcal{K}}(\mathcal{L})$.

Composing $(\Phi_*)^{-1}$ with the embedding of $C^*(\Phi(\mathcal{K})) = C^*(S(\mathcal{K}))$ into $\mathcal{NT}^r(\mathcal{L})$ yields an embedding ι^r. Since $E^S \circ \iota^r = E_\Phi \circ (\Phi_*)^{-1}$ and $\iota^r|_{B_\mathcal{K}} \circ E^T = h \circ E^T$, the claim about intertwining conditional expectations follows.

Suppose now that $\mathcal{K}$ is essential in $\mathcal{L}_\mathcal{K}$. Let $Z \in B_\mathcal{K}$ be as in (5.8) and put

$$Z_S := \sum_{p,q\in F} \bigoplus_{\substack{w\in pP\cap qP, t\in P\\ p^{-1}w=q^{-1}w}} S_{p,q}^{w,t}(a_{p,q}) \in B_\mathcal{L},$$

where the adjointable maps $S_{p,q}^{w,t}$ are as defined in Lemma 4.1 (for $\mathcal{K} = \mathcal{L}$). By Corollary 5.12, $\|Z\| = \|Z_S\|$, so the map $Z \to Z_S$ extends to an isometry $h : B_\mathcal{K} \to h(B_\mathcal{K}) \subseteq \mathcal{B}_\mathcal{L}$. Let $\Phi : \mathcal{K} \to \mathcal{L}(\mathcal{F}_\mathcal{L})$ be the restriction of S. Then E^S restricts to a faithful map $E_\Phi : C^*(\Phi(\mathcal{K})) \to h(B_\mathcal{K}) \subseteq C^*(\Phi(\mathcal{K}))$. The composition $E^\Phi := h^{-1} \circ E_\Phi$ is a faithful completely positive map satisfying (7.3) and the proof if completed as in the case of P cancellative. □

Corollary 11.4. *Suppose that $\mathcal{L}$ is amenable and that either P is cancellative or $\mathcal{K}$ is essential in the right-tensor C^*-precategory $\mathcal{L}_\mathcal{K}$ generated by $\mathcal{K}$. If $\mathcal{NT}_\mathcal{L}(\mathcal{K}) \hookrightarrow \mathcal{NT}(\mathcal{L})$, then $\mathcal{K}$ is amenable.*

Proof. By our assumptions and Theorem 11.3 we have a commutative diagram

$$\begin{array}{ccc} \mathcal{NT}_\mathcal{L}(\mathcal{K}) & \xrightarrow{\ \iota\ } & \mathcal{NT}(\mathcal{L}) \\ {\scriptstyle T\rtimes P}\downarrow & & \downarrow{\scriptstyle S\rtimes P} \\ \mathcal{NT}^r_\mathcal{L}(\mathcal{K}) & \xrightarrow{\ \iota^r\ } & \mathcal{NT}^r_\mathcal{L}(\mathcal{K}) \end{array}$$

where ι, ι^r and $S \rtimes P$ are injective. Hence $T \rtimes P$ is injective. □

Corollary 11.5 (Uniqueness Theorem II). *Suppose that the well-aligned ideal $\mathcal{K}$ in $\mathcal{L}$ is essential and $\otimes 1$-nondegenerate. Assume also that either $P^* = \{e\}$ or that the group $\{\otimes 1_x\}_{x\in P^*}$ is aperiodic on $\mathcal{K}$. If $\mathcal{L}$ is amenable, then $\mathcal{K}$ is also amenable and for any Nica covariant representation $\Phi : \mathcal{K} \to B(H)$ the following statements are equivalent:*

(i) *Φ satisfies condition (C);*
(ii) *$\overline{\Phi}$ generates the universal Nica-Toeplitz algebra $\mathcal{NT}(\mathcal{L})$, i.e. $C^*(\overline{\Phi}(\mathcal{L})) \cong \mathcal{NT}(\mathcal{L})$.*

Under the embedding $\mathcal{NT}_\mathcal{L}(\mathcal{K}) \hookrightarrow \mathcal{NT}(\mathcal{L})$, the isomorphism in (ii) restricts to an isomorphism $C^*(\Phi(\mathcal{K})) \cong \mathcal{NT}_\mathcal{L}(\mathcal{K})$.

Proof. Lemma 11.1 implies that $\mathcal{NT}_\mathcal{L}(\mathcal{K}) \hookrightarrow \mathcal{NT}(\mathcal{L})$, hence $\mathcal{K}$ is amenable by Corollary 11.4. The claim follows from Theorem 10.15. □

12. Fell bundles and right-tensor C^*-precategories over groups

In this section we show that the theory of right-tensor C^*-precategories over groups is equivalent to the theory of Fell bundles over (discrete) groups. On one hand, we explain how results about C^*-algebras associated to Fell bundles follow from our results. On the other hand, we will see the origin of some of our assumptions. Throughout this section we assume that P equals a (discrete) group G.

Proposition 12.1. *If $(\mathcal{L}, \{\otimes 1_r\}_{r\in G})$ is a right-tensor C^*-precategory over G, then the Banach spaces $B_g := \mathcal{L}(g, e)$ for $g \in G$ equipped with the operations*

$$\circ : B_g \times B_h \ni (b_g, b_h) \longrightarrow b_g \circ b_h := (b_g \otimes 1_h)b_h \in B_{gh} \tag{12.1}$$

$$^\star : B_g \ni b_g \longrightarrow b_g^\star := b_g^* \otimes 1_{g^{-1}} \in B_{g^{-1}}, \tag{12.2}$$

for $g,h\in G$, give rise to a Fell bundle $\mathcal{B}^{\mathcal{L}}:=\{B_g\}_{g\in G}$ isomorphic to the Fell bundle $\mathcal{B}^{\mathrm{id}}$ associated to $\mathcal{L}$ and id *as in Proposition 8.3, and so*

$$C_r^*(\mathcal{B}^{\mathcal{L}})\cong\mathcal{NT}^r(\mathcal{L}),\qquad C^*(\mathcal{B}^{\mathcal{L}})\cong\mathcal{NT}(\mathcal{L}). \tag{12.3}$$

Conversely, for any Fell bundle $\mathcal{B}=\{B_g\}_{g\in G}$ over a discrete group G the spaces

$$\mathcal{L}(g,h):=B_{gh^{-1}},\qquad g,h\in G, \tag{12.4}$$

with composition and involution inherited from $\mathcal{B}$ and right-tensoring given by

$$\otimes 1_x:\mathcal{L}(g,h)=B_{gh^{-1}}\ni b\longrightarrow b\otimes 1_x:=b\in B_{gx(hx)^{-1}}=\mathcal{L}(gx,hx) \tag{12.5}$$

for $g,h,x\in G$, yield a right-tensor C^*-precategory $\mathcal{L}:=\{\mathcal{L}(g,h)\}_{g,h\in G}$ such that $\mathcal{B}=\mathcal{B}^{\mathcal{L}}$.

Proof. Let $(\mathcal{L},\{\otimes 1_r\}_{r\in G})$ be a right-tensor C^*-precategory over G. By Lemma 3.10, for every $g,p,q\in G$ the map $\otimes 1_g:\mathcal{L}(p,q)\to\mathcal{L}(pg,qg)$ is an isometric isomorphism and for every $a\in\mathcal{L}(p,q)$ we have $i_{\mathcal{L}}(a\otimes 1_g)=i_{\mathcal{L}}(a)$. This readily implies that the maps $B_g=\mathcal{L}(g,e)\ni a\mapsto i_{\mathcal{L}}(a)\in B_g^{\mathrm{id}}$, $g\in G$, are isometric isomorphisms. Under these maps, the operations in the Fell bundle $\mathcal{B}^{\mathrm{id}}=\{\mathcal{B}_g^{\mathrm{id}}\}_{g\in G}$ translate to (12.1), (12.2). Hence, $\mathcal{B}$ is a Fell bundle isomorphic to $\mathcal{B}^{\mathrm{id}}$. The isomorphisms (12.3) follow from Theorem 8.4. The remaining claims of the proposition are straightforward and left to the reader. □

Remark 12.2. To clarify relationships between the constructions in Proposition 12.1, let $\mathcal{B}^{\mathcal{L}}=\{\mathcal{L}(g,e)\}_{g\in G}$ denote the Fell bundle associated to a right-tensor C^*-precategory $\mathcal{L}$ and let $\mathcal{L}_{\mathcal{B}}=\{B_{gh^{-1}}\}_{g,h\in G}$ be the right-tensor C^*-precategory associated to a Fell bundle $\mathcal{B}$. Then $\mathcal{B}^{\mathcal{L}_{\mathcal{B}}}=\mathcal{B}$ and $\mathcal{L}_{\mathcal{B}^{\mathcal{L}}}\cong\mathcal{L}$ as right-tensor C^*-precategories: the maps $\otimes 1_h:\mathcal{L}(gh^{-1},e)\to\mathcal{L}(g,h)$ for $g,h\in G$ implement an isomorphism $\mathcal{L}_{\mathcal{B}^{\mathcal{L}}}\cong\mathcal{L}$ of C^*-precategories which intertwines right-tensorings.

Remark 12.3. When $P=G$ is a group and $\mathcal{L}$ is any right-tensor C^*-precategory over G, then for every $t\in G$ condition (7.6) is trivially satisfied (one may take $x=p^{-1}t$). Thus, by Proposition 7.6, $\mathcal{NT}_{\mathcal{L}}^r(\mathcal{K})\cong C^*(T^t(\mathcal{K}))$, for all $t\in G$, with $T^t:\mathcal{L}\to\mathcal{L}(\mathcal{F}_{\mathcal{L}}^t)$ denoting the t-th Fock representation on $\mathcal{L}$. Therefore, if $\mathcal{B}=\{B_g\}_{g\in G}$ is a Fell bundle and $\mathcal{L}=\{B_{gh^{-1}}\}_{g,h\in G}$ is the associated C^*-precategory from Proposition 12.1, then the Fock representation T^e of $\mathcal{L}$, viewed as a representation of $\mathcal{B}$, coincides with the usual Fock representation of $\mathcal{B}$ introduced in [12].

For right-tensor C^*-precategories over groups well-aligned ideals have nice structure.

Lemma 12.4. *Let $\mathcal{K}$ be a well-aligned ideal in a right-tensor C^*-precategory $(\mathcal{L},\{\otimes 1_r\}_{r\in G})$ over a group G. Then $\mathcal{K}$ is automatically $\otimes 1$-invariant and $\otimes 1$-nondegenerate. Moreover, $\mathcal{K}$ is essential in $\mathcal{L}$ if and only if $\mathcal{K}(e,e)$ is essential in $\mathcal{L}(e,e)$.*

Proof. Lemma 3.10 gives directly that $\mathcal{K}$ is $\otimes 1$-invariant and $\otimes 1$-nondegenerate. If $\mathcal{K}(e,e)$ is essential in $\mathcal{L}(e,e)$, then $\mathcal{K}(p,p)=\mathcal{K}(e,e)\otimes 1_p$ is essential in $\mathcal{L}(p,p)=\mathcal{L}(e,e)\otimes 1_p$, which proves the second part of the assertion. □

Lemma 12.5. *Let $(\mathcal{L},\{\otimes 1_r\}_{r\in G})$ be a right-tensor C^*-precategory and $\mathcal{B}^{\mathcal{L}}=\{\mathcal{L}(g,e)\}_{g\in G}$ the associated Fell bundle. Then there is a bijective correspondence between ideals $\mathcal{I}=\{I_g\}_{g\in G}$ in $\mathcal{B}$ and well-aligned ideals $\mathcal{K}$ in $\mathcal{L}$, given by $I_g=\mathcal{K}(g,e)$ for all $g\in G$.*

Proof. In view of Remark 12.2, we may assume that $\mathcal{L}=\{B_{gh^{-1}}\}_{g,h\in G}$ and the right tensoring is given by identity maps (12.5). Now it is clear that for any $\otimes 1$-invariant ideal $\mathcal{K}$ in $\mathcal{L}$, $I_g=\mathcal{K}(g,e)$ defines an ideal in $\mathcal{B}^{\mathcal{L}}$. Conversely, let $\mathcal{I}$ be an ideal in $\mathcal{B}^{\mathcal{L}}$ and put $\mathcal{K}(g,h):=I_{gh^{-1}}$, for all $g,h\in G$. Clearly, $\mathcal{K}$ is an ideal in $\mathcal{L}$. It is also the smallest $\otimes 1$-invariant ideal in $\mathcal{B}$

satisfying $I_g = \mathcal{K}(g,e)$ for all g. However, by Lemma 12.4 well-alignment and $\otimes 1$-invariance coincide. Hence this proves the assertion. □

As a first application of our results we obtain embedding of cross-sectional algebras of ideals in Fell bundles. The latter problem for sub-bundles is studied in [13, Section 21].

Proposition 12.6. *Given a right-tensor C^*-precategory $(\mathcal{L}, \{\otimes 1_r\}_{r\in G})$ over a group G and a well-aligned ideal $\mathcal{K}$ in $\mathcal{L}$, there are natural embeddings*

$$\mathcal{NT}^r(\mathcal{K}) \hookrightarrow \mathcal{NT}^r(\mathcal{L}) \qquad \textit{and} \qquad \mathcal{NT}(\mathcal{K}) \hookrightarrow \mathcal{NT}(\mathcal{L}).$$

Equivalently, for any ideal $\mathcal{I}$ in a Fell bundle $\mathcal{B}$ over a discrete group G we have

$$C^*_r(\mathcal{I}) \hookrightarrow C^*_r(\mathcal{B}) \qquad \textit{and} \qquad C^*(\mathcal{I}) \hookrightarrow C^*(\mathcal{B}).$$

Proof. Theorem 11.3 gives $\mathcal{NT}^r(\mathcal{K}) \hookrightarrow \mathcal{NT}^r(\mathcal{L})$. By Lemmas 12.4 and 11.1 we obtain $\mathcal{NT}(\mathcal{K}) \hookrightarrow \mathcal{NT}(\mathcal{L})$. In view of Proposition 12.1 and Lemma 12.5, the second part of the assertion is equivalent to the first one. □

Remark 12.7. It is shown in [13, Theorem 21.13] that the embedding $C^*(\mathcal{I}) \hookrightarrow C^*(\mathcal{B})$ is valid, not only for ideals but also for hereditary sub-bundles $\mathcal{I}$ of $C^*(\mathcal{B})$. In fact, Proposition 12.6 could be deduced from [13, Theorem 21.13 and Proposition 21.3]

As a next application we get a correct version of [23, Proposition 3.15], see Remark 12.9 below.

Proposition 12.8. *Let $\mathcal{B} = \{B_g\}_{g\in G}$ be a Fell bundle and let $\Psi : C^*(\mathcal{B}) \to \mathcal{B}(H)$ be a representation injective on B_e. The following conditions are equivalent*

(i) $\ker\Psi \subseteq \ker\Lambda$ *where $\Lambda : C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$ is the regular representation*;
(ii) $\overline{\Psi(\bigoplus_{g\in G} B_g)}$ *is topologically graded, that is we have $\|b_e\| \leq \|\Psi(\bigoplus_{g\in G} b_g)\|$ for any $\bigoplus_{g\in G} b_g \in \bigoplus_{g\in G} B_g$;*
(iii) *condition (ii) is satisfied for any positive $\bigoplus_{g\in G} b_g \in \bigoplus_{g\in G} B_g$.*

Proof. Let $\mathcal{L} = \{B_{gh^{-1}}\}_{g,h\in G}$ be the right-tensor C^*-precategory associated to $\mathcal{B}$, as in Proposition 12.1. It is straightforward that the maps $\Phi_{g,h} := \Psi|_{B_{gh^{-1}}}$ for $g,h \in G$ form a representation Φ of $\mathcal{L}$ such that $\Phi_{gr,hr}(a \otimes 1_r) = \Phi_{g,h}(a)$ for $a \in \mathcal{L}(g,h)$ and all $g,h,r \in G$. Accordingly, Φ is Nica covariant. Identifying $\mathcal{NT}(\mathcal{L})$ with $C^*(\mathcal{B})$, via (12.3), the core of $\mathcal{NT}(\mathcal{L})$ is $\mathcal{L}(e,e) = B_e$ and $\Psi = \Phi \rtimes P$. Thus Proposition 7.4 applied to Φ gives the equivalence of (i) and (ii). Equivalence between (ii) and (iii) is explained in the beginning of the proof of Theorem 10.12. □

Remark 12.9. Retain the notation of Proposition 12.8. Unless $\mathcal{B}$ is amenable, condition (i) is stronger than condition (i'): $\Lambda(\ker\Psi) \cap B_e = \{0\}$. Unfortunately, [23, Proposition 3.15] says that counterparts of conditions (i'), (ii), (iii) are equivalent. In particular, the proof of implication (i)⇒(ii) in [23, Proposition 3.15] is incorrect. This mistake does not affect the remaining results of [23].

Let $\mathcal{B}$ be a Fell bundle over G and $\mathcal{L}$ the associated right-tensor C^*-precategory, given by (12.4) and (12.5). It is immediate that the action of $\{\otimes 1_g\}_{g\in G}$ on $\mathcal{L}$ is aperiodic if and only if $\mathcal{B}$ is aperiodic. By (12.3), amenability of $\mathcal{B}$ is equivalent to amenability of $\mathcal{L}$. Moreover, since $g \leq h$ holds for any g, h in G, conditions (C) and (C') are void. Thus Theorem 10.12 reduces to the following uniqueness-type result, which in view of Remark 12.9 is stronger than [23, Corollary 4.3].

Proposition 12.10. *Let $\mathcal{B} = \{B_g\}_{g\in G}$ be an aperiodic Fell bundle. For any representation $\Psi : C^*(\mathcal{B}) \to \mathcal{B}(H)$ injective on B_e we have $\ker \Psi \subseteq \ker \Lambda$.*

Remark 12.11. By [21, Theorem 9.11], when $G = \mathbb{Z}$ or $G = \mathbb{Z}_n$ for a square free number $n > 0$, aperiodicity of $\mathcal{B}$ is not only sufficient but also necessary for Proposition 12.10 to hold, at least when B_e contains an essential ideal which is separable or of Type I.

Example 12.12 (Crossed products by group actions)**.** Let $\alpha : G \to \operatorname{Aut}(A)$ be an action of a discrete group G on a C^*-algebra A. Consider the C^*-precategory $\mathcal{L} = \{\mathcal{L}(g, h)\}_{g,h\in G}$ where for each $g, h \in G$, $\mathcal{L}(g, h)$ equals A as a Banach space, and composition and involution in $\mathcal{L}$ are given by multiplication and involution in A. Then $a \longrightarrow a \otimes 1_r := \alpha_r(a)$ for $a \in A = \mathcal{L}(g, h)$ and $g, h, r \in G$ defines a right-tensoring on $\mathcal{L}$. The corresponding Fell bundle $\mathcal{B}^{\mathcal{L}}$ coincides with the semi-direct product bundle associated to α, cf. [13]. Hence

$$\mathcal{NT}^r(\mathcal{L}) \cong A \rtimes^r_\alpha G \quad \text{and} \quad \mathcal{NT}(\mathcal{L}) \cong A \rtimes_\alpha G.$$

We have a natural bijective correspondence between α-invariant ideals I in A and well-aligned ($\otimes 1$-invariant) ideals $\mathcal{K}$ in $\mathcal{L}$. Moreover, assuming that A contains an essential ideal which is separable or of Type I, by [21, Theorems 2.13, 8.1 and Corollary 9.10], the following conditions are equivalent:

(i) the action of $\{\otimes 1_g\}_{g\in G}$ on $\mathcal{L}$ is aperiodic;
(ii) α is pointwise properly outer, i.e. each α_g for $g \in G \setminus \{e\}$ is properly outer;
(iii) the dual action $\widehat{\alpha} : G \to \operatorname{Homeo}(\widehat{A})$ is topologically free, i.e. for any $g_1, \dots, g_n \in G\setminus\{e\}$ the set $\bigcap_{i=1}^n \{\pi \in \widehat{A} : \widehat{\alpha}_{g_i}(\pi) = \pi\}$ has empty interior.

Using this, one recovers [2, Theorem 1] from Proposition 12.10.

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