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New and extended design moment formulations for slender columns in frames with sway

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A R T I C L E I N F O

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A B S T R A C T

Present first-order based design code formulations for slender columns in frames with sway, employ sway magnification factors (for global second-order effects) also as moment magnifiers for the individual columns of the frame (storey). This approach ignores differences in magnification of individual column moments caused by local second-order effects in the columns. This difference can be significant. Better understanding of this aspect will strengthen approximate first-order based elastic methods, for which the important superposition principle is valid. Towards this end, local second-order effects are considered for shears, end moments and maximum moments, applicable over the full range of axial loads. Specifically, end moment and maximum moment expressions of columns with sway are derived. These represent novel contributions that are suitable in typical design code formats, and in practical design work. They will allow more rational column assessments, and will allow more economical designs than present structural code procedures. Proposals are verified by comparison with results from second-order in-plane elastic analyses of single restrained columns and columns in frame panels. Also, extensions to the general case of load combinations that include both gravity and sideways loading are briefly presented.

1. Introduction

Elastic second-order frame analysis methods are important in practical design for bending and stability of both reinforced concrete and steel columns (beam-columns). For reinforced concrete structures, effects on section stiffnesses of nonlinear material behaviour, cracking, creep, etc., are generally accounted for through the use of reduced, “effective” linear (elastic) section stiffnesses (e.g., [1,2]). For steel structures, nominal section stiffnesses are often used (e.g., [3] App. 8, [4]), or reduced stiffnesses that reflect yielding and other effects (e.g., [3]). Possible local, cross-section dependent instability effects, ductility, inelasticity and out-of-plane buckling phenomena, such as covered for instance in [5–8], are not considered in such methods themselves.

Sidesway due to lateral and vertical loading of reinforced concrete and steel structures can be handled by existing second-order analysis computer programs, and also by approximate methods, that have been, and still are, important in parallel with, or as a complement to, more exact methods. In approximate, elastic based methods for frame structures, global (overturning) second-order effects of the vertical loads are often accounted for by a storey sway magnification factor that are applied to the first-order end moments of the columns of the frame (or storeys of the frame).

Such approximate procedures, which is in line with practice in major design codes (e.g., ACI [1], AISC [3], Eurocode 2 [2], Eurocode 3 [4], etc.), does not presently reflect the fact that the sidesway actually may affect the column moments differently in the different columns on the same level of the frame. Normally, the larger end moment increase will be smaller than that reflected by the sway magnification factor.

Some, but limited, attention to this matter can be found in the literature. The AISC Commentary dealt with a reduction in the maximum moment due to sidesway already in 1969, and later in the 1978 edition of the Commentary. The same reduction factor expression was suggested for the sway action of all columns of the storey, and it was allowed to approximate it by a constant factor (0.85). Both the expression and the constant are unconservative in the general case. In later commentary revisions, this reduction factor is omitted. Hellesland [9], as recapitulated in a slightly different form elsewhere [10], derived an expression for end moments in individual column axes for special cases. At about the same time, LeMessurier [11] derived a similar expression and proposed an extension to a general column of a sway frame. However, the extension is incorrect in that it yields the same moment magnifier for moments at both ends of all columns of the storey. Hellesland and MacGregor (Mechanics and design of columns in sway frames, 1981, unpublished draft) presented an extension of the previous work by Hellesland. Lui [12] also suggested an approach allowing for different moment magnification factors in different column axes. The

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Nomenclature

B_b	approximate rising moment branch factor
B_m	approximate maximum moment magnification factor
B_{max}	exact maximum moment magnification factor moments
B_s	system (storey) sway magnification factor
B_v	shear multiplication factor
B_1, B_2	end moment multiplication factors
EI, EI_b	cross-sectional stiffness of columns, and of beams
G_j	scaled rotational restraint flexibility at member end j
L, L_b	length of columns, and of restraining beam(s)
M_{0j}, M_j	moment from first-order and second-order analysis, respectively, at end j
N	axial (normal) force
N_{cb}, N_{cs}	critical load of columns considered fully braced, and free-to-sway, respectively
N_E	Euler buckling load of pin-ended column (= $\pi^2 EI/L^2$)
S_B	lateral stiffness of external bracing(s)
V_0, V	first-order and total (first + second-order) shear force in a column, respectively
k_j	rotational restraint stiffness (spring stiffness) at end j
α_b, α_s	load index of column considered fully braced, and free-to-sway, respectively
α_{ss}	system (storey) stability index
α_E	nominal axial load index of a column (= N/N_E)
β	effective (buckling) length factor
β_b, β_s	effective length factor corresponding to N_{cb} and N_{cs}, respectively
Δ_0, Δ	first-order and total lateral displacement, respectively
γ_n	axial load dependent flexibility factor
γ_s, γ_0	flexibility factor at free-sway and at zero axial load, respectively
κ_j	relative rotational restraint stiffness at end j (=$k_j/(EI/L)$).

approach does not seem to be well founded, and breaks down in the general case. Better understanding of this aspect would strengthen the base for approximate first-order based elastic methods, for which the important superposition principle is valid.

The main attention of the present paper is directed towards developing approximate shear and moment formulations, suitable in design code formats, for slender elastic columns in frames with sidesway. The full range of axial loads are considered, thus covering both supporting and supported columns. More specifically, the main objectives are (1) to derive appropriate multiplication factors that account for global and local second-order elastic effects on column end moments, and, not least, (2) to establish improved, more realistic, magnification factors for the prediction of maximum moments between column ends. Towards this end, some basic column mechanics are reviewed and exact second-order analysis results are presented for evaluation of the proposals. The scope is limited to elastic bending in the plane of loading.

2. Mechanics of moment formation and definitions

Fig. 1(a) shows an unbraced, laterally loaded three bay frame subjected a lateral load (H) and vertical loads (P_i). The load H, alone, gives rise to a first-order sway displacement (Δ_0), equal in all axes when axial beam deformations are negligible, and to first-order column shears (V_{0i}) that are proportional to the relative lateral stiffness of the columns. The effect of the axial loading is to increase the sway displacement beyond Δ_0 to $\Delta = B_s\Delta_0$, and to cause a redistribution of the shears from their first-order values to their final values (V_i) in the respective column axes i. B_s is the system (storey) sway displacement magnifier, that reflects global second-order (overturning "$N\Delta$") effects.

The formation of moments in the columns will depend on the axial loading (local second-order effects). Fig. 1(b) shows possible moment distributions along the individual column axes. In the general case, shears (V), end moments (M_1 and M_2) and maximum column moments M_{max} can be given in terms of first-order quantities (identified by subscripts zero) as follows:

$$V = B_v(B_sV_0) \tag{1}$$

$$M_1 = B_1(B_sM_{01}) \tag{2}$$

$$M_2 = B_2(B_sM_{0j}) \tag{3}$$

$$M_{max} = B_{max}(B_sM_{02}) \tag{4}$$

While the same global second-order B_s factor is applied to all columns, the factors B_v, B_1, B_2 and B_{max}, are individual column factors, different in the different axes. Above, it is chosen to express M_{max} in terms of M_{02} at end 2, which is defined as the end at which the first-order moment has its largest absolute value.

Analytical expressions will be established for all of these B factors. Prior to this, consider some basic cases with reference to Fig. 1(b).

Column 1, with no axial force, is affected by the global second-order effects, but has no local second-order effects. Thus, local B factors are all equal to unity. The column has a positive shear, implying that it contributes to the bracing of the frame. The moment diagram stays linear, and the final moments are directly proportional to the sway displacement $\Delta = B_s\Delta_0$.

Although its moment diagram varies linearly along the axis, it does not comply with the definition of a first-order moment ("obtained from equilibrium based on the undeformed geometry"), since it includes global second-order effects through B_s. For the sake of precision and distinction, B_sM_0 is labelled "*sway-magnified first-order moment*" [30].

Column 2 has moderate local second-order effects. All local B factors, but for B_{max}, are different from unity. It has a positive shear, and thus it contributes to lateral frame support (bracing).

Column 3 has significant local second-order effects, and the

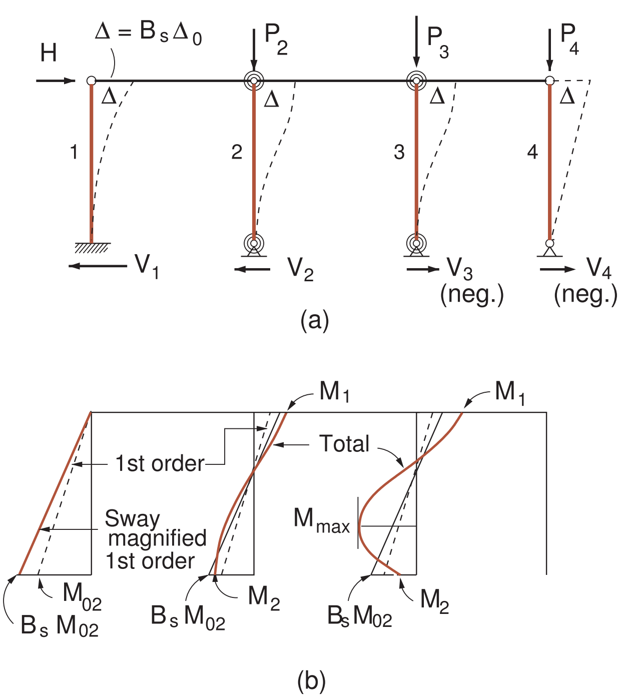

Fig. 1. (a) Laterally loaded, unbraced three-bay frame; (b) Possible moment diagrams in laterally supporting and supported columns.

maximum moment may form away from the column end. All local B factors are different from unity. Its shear force is negative, implying that it requires lateral support.

The straight, pin-ended Column 4 in the figure, with negative shears, requires lateral support. It contributes to the overturning global second-order effects, but has no local second-order effects itself. In practice, though, moments will be introduced through imperfect hinges and through imperfections along the axis.

3. Global second-order effects

It is appropriate for the sake of perspective and completeness, and useful for later discussions, with a brief review of a suitable approximate storey (system) sway magnifier. With reference to Fig. 2, where beam deformations are assumed to be negligible, B_s can be defined by

$$B_s = \frac{1}{1 - \alpha_{ss}} \tag{5}$$

where α_{ss} is the storey (system) stability index defined by

$$\alpha_{ss} = \frac{\sum (\gamma_n N/L)}{(H/\Delta_0)} \quad \text{or} \quad \alpha_{ss} = \frac{\sum (\gamma_n N/L)}{\sum (\gamma_s N_{cs}/L) + S_B} \tag{6a,b}$$

Here, the ratio H/Δ_0 is the first-order lateral storey stiffness (including possible external bracing S_B). In multistorey structures, Δ_0 is the interstorey first-order displacement and H the corresponding storey shear. The γ factors, reflecting local second-order effects, will be discussed later (General shear formulation). The summations are over all interacting columns. N_{cs} is the pseudo-critical load defined below. Provided N_{cs} is determined with the same first-order restraints implicit in the H/Δ_0 calculation, the two B_s expressions above are equivalent.

These forms were presented by Hellesland [10]. With the γ_n included, sway-braced column interaction can be accounted for. Several authors and codes give similar sway magnifiers. They can all be derived from simplifications of the α_{ss} in Eq. (6a,b), as discussed in [10]. Common, and often justified, simplifications are constant column lengths, neglect or introduction of constant values for γ_n and γ_s (typically about 1.15).

4. Local second-order effects and analyses

4.1. Second-order effects and load indices

The local member second-order ($N\delta$) effects can be quantified by the ratios of results obtained from second-order analyses of a column with a specified sidesway $\Delta = B_s\Delta_0$ and given axial loads N and results from the same analyses for $N = 0$ (first-order, determined from an analysis that is based on the original, geometry).

The local second-order factors (B_v, B_1, B_2, and B_{max}), defined previously through Eqs. (1)–(4), computed using the analysis method presented briefly below, will be given for verification purposes in a later section.

For frames with sway due to lateral and axial loading only, these coefficients depend on (1) the end restraints, which then uniquely define the first-order moment gradient, and on (2) the axial load level defined for instance by the nondimensional load parameters α_s, α_b or α_E. For an elastic member of length L, uniform axial load and sectional stiffness EI along the length, they can be defined by

$$\alpha_s = \frac{N}{N_{cs}};\ \alpha_b = \frac{N}{N_{cb}};\ \alpha_E = \frac{N}{N_E} \tag{7a-c}$$

where

$$N_{cs} = \frac{N_E}{\beta_s^2};\quad N_{cb} = \frac{N_E}{\beta_b^2};\quad N_E = \frac{\pi^2 EI}{L^2} \tag{8a-c}$$

Here, N_{cs} and N_{cb} are the free-to-sway and the fully braced critical load, respectively, and β_s and β_b are the corresponding effective length (buckling) factors. They are computed for the columns considered in isolation with representative restraints. Except when the frame consists of a single column, these are strictly pseudo-critical loads, that are useful in column characterisation and discussion. N_E is the socalled Euler load (critical load of a pin-ended column), which is a convenient reference load parameter in several contexts. As defined above, the load indices are interrelated. For instance, $\alpha_s = \alpha_E \beta_s^2$, $\alpha_b = \alpha_E \beta_b^2$, $\alpha_b = \alpha_s(\beta_b/\beta_s)^2$.

4.2. Local second-order analysis

For the verification of the proposals of the present paper, results were obtained using the second-order analyses presented in detail in Hellesland [13]. Columns considered are either single restrained columns, as shown in Fig. 3(b), or columns that are part of a panel frame, such as shown in Fig. 3(c). The columns are initially straight and have lengths L and uniform section stiffnesses EI. The deflection shapes indicated by the solid lines in the figure, are those due to an initial, imposed top (joint) displacement $\Delta = B_s\Delta_0$. For this Δ to remain constant for increasing axial loading, the column shears V (lateral loading) will have to decrease (see Fig. 3(b)) to compensate for the increased overturning effect of the vertical loading acting on the relative joint displacement. No gravity load induced moments (such as from loading on beams) are included. The dashed lines are deflection shapes developing as the critical axial loading is approached. For the panel, other deflection shapes may result depending on the relative stiffness and axial loading in the members.

The restrained single column may be the complete structure, or it can be considered isolated from the two-column panel, or from a greater frame. In the latter case, the rotational end restraints should reflect the rotational interaction at the joints with restraining beams ("horizontal interaction"), and possible other columns framing into the considered joint ("vertical interaction"). Procedures for this are given in structural codes, and discussed in the literature (e.g. [10,16]).

The rotational end restraints can conveniently be represented by rotational restraint stiffnesses (or spring stiffnesses) labelled k_1 and k_2 (equal to the moments required to give a unit rotation), or in nondimensional form by κ_j at end $j = 1$ and $j = 2$ defined by

$$\kappa_j = \frac{k_j}{(EI/L)} \qquad j = 1, 2 \tag{9}$$

In the present paper, end restraints are due to beams. Then, k_j is equal to the rotational stiffness k_{bj} of all beams connected to the joint considered. Thus,

$$k_j = k_{bj} = \left(\sum bEI_b/L_b\right)_j \tag{10}$$

where b is the bending stiffness coefficient, typically equal to 3, 4, 2 and 6 for beams (with negligible axial forces) pinned at the far end, fixed at the far end, bent into symmetrical single curvature bending and antisymmetrical double curvature bending, repectively.

Alternatively, instead of κ_j, inverse flexibility factors ($1/\kappa_j$) may be

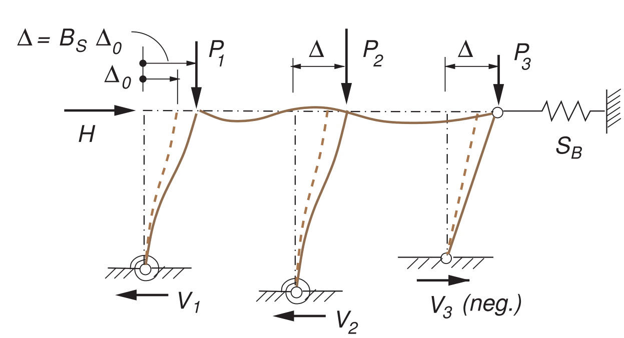

Fig. 2. Partially braced frame with sidesway.

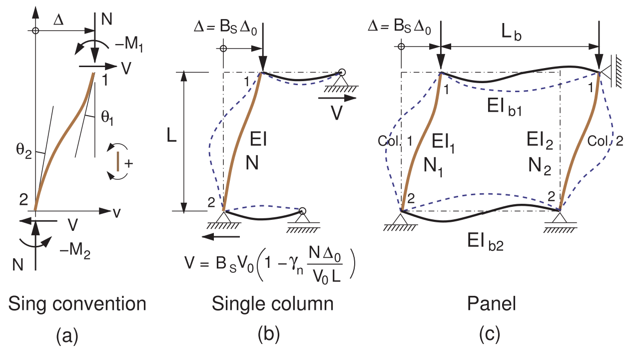

Fig. 3. (a) Sign convention; (b) Single column model; (c) Two-column panel model. Dashed curves show possible deflection modes at member instability (one of several for the panel).

used, or *scaled* flexibility factors, such as the well known G factors. In their generalized forms [10], they can be defined by

$$G_j = b_o \frac{(EI/L)}{k_j} \quad \left(= \frac{b_o}{\kappa_j}\right) \qquad j = 1, 2 \tag{11}$$

where b_o is simply a reference (datum) factor by which the relative restraint flexibilities are scaled. Datum values adopted for instance in ACI [1] and AISC [3], are $b_o = 6$ and $b_o = 2$ for unbraced and braced frames, respectively. It should be noted that $b_o = 6$ is used throughout this paper, unless otherwise noted.

4.3. Typical column response in frames with sway

Characteristics study. In a another study [13], response characteristics were presented and discussed for single columns and panels with various restraint combinations. Also presented were closed form expressions defining a number of key characteristics, or ”landmarks” in the moment versus axial load “map”, useful for for enabling a quick establishment of moment-axial load relationship of laterally displaced columns. Moment results from that study, to be used for the verification of approximated moment proposals, and points of interest for the present paper, will be briefly recaptured and discussed where appropriate.

Single columns. Typical moment and shear responses versus increasing axial load, computed with the second-order analysis, are shown in Fig. 4 and Fig. 5 for single columns with unequal, rotational end restraints with different degree of rotational stiffness, as given in the insert in the upper left hand corner of the figures. In a real case, the column top displacement $B_s\Delta_0$ will be maintained by the action of the overall frame of which the column may be considered isolated from.

Similar results for two single columns, isolated from the panel by assuming hinged supports at the first-order inflection points (near midlengths) of the panel beams, are shown in Figs. 6 and 7. This allows comparison with the response of the panel columns. The isolated column is seen to describe the panel end moment response almost exactly up to fairly high axial load levels. More detailed discussion of response characteristics of this and a panel with a stiffer Column 2, are given elsewhere [13].

The moments and shear are shown nondimensionally in terms of the respective B factors, Eqs. (1)–(4). Axial forces are given nondimensionally in terms of axial load indices α_s and α_E (Eq. (7)). All moment results in the figures are given in terms of B_sM_{02}. Therefore, B_1 and B_{1lin} are multiplied by the ratio M_{01}/M_{02} in the figures.

Both moments and shears approach infinity (in either the positive or negative direction) as $\alpha_b = N/N_{cb} = 1$ (or $\alpha_E = 1/\beta_b^2$) is approached. N_{cb} for an elastic column is independent of whether the column is fully braced at zero or at a non-zero end displacement. The results in the figures are independent of the sway magnification factor B_s, as stated previously. However, if the considered column was part of a larger frame, it is worth noting that B_s may reach unacceptable values well before the present “local instability” is reached (at “$\alpha_b = 1$” in the figures).

Panel columns. The panel columns are rigidly connected to beams at the top and bottom. EI/L values of the panel members are EI/L, $1.1EI/L$, $0.333EI/L$ and $1.667EI/L$ for Column 1 (left hand), Column 2 (right hand), top beam and bottom beam, respectively. The bottom beam is considerably stiffer than the upper beam, and will attract the larger first-order end moments. Axial forces are neglected in the beams. The columns have the same axial force N. Then, because of the stiffness difference, the load index in Column 1 (left hand) becomes 1.1 times that in Column 2 ($\alpha_{E1} = 1.1\alpha_{E2}$). Thus, Column 1 is the more flexible of the two panel columns, and the one at which system instability will be initiated.

The results in Fig. 6 are for the most flexible Column 1 in the panel (broken lines) and for the isolated Column 1 (solid lines). Results are plotted versus α_{E1} (α_E for Column 1). The α_{s1} ($=\alpha_{E1}\beta_{s1}^2$) abscissa, added for convenience and information, is computed with the free-sway critical load of the isolated Column 1 defined above (with effective length factor $\beta_{s1} = 1.483$). Similarly for the stiffer Column 2, the α_{s2} ($=\alpha_{E2}\beta_{s2}^2$) abscissa (with $\beta_{s2} = 1.522$), is added in Fig. 7.

Stationary versus non-stationary restraints. Unlike restraints of single columns, those of the panel columns will not remain stationary in the general case. As a consequence, the critical loads of the panel columns are lower than those of the isolated columns. This is due to the reduced restraints offered by the panel beams as the axial column loading increases towards the braced critical load. This is indicated in Fig. 3(c), and in the inserts in Figs. 6 and 7, by the unwinding of the beams from nearly antisymmetrical double curvature ($k \approx 6EI_b/L_b$) to nearly symmetrical single curvature bending ($k \approx 2EI_b/L_b$). Associated with the reduction in beam restraints, a rather sudden reversal of end moments is seen to take place in the stiffer Column 2 for loads close to the critical panel load.

End moments. The largest first-order end moment (absolute value) is obtained at end 2 with the largest rotational restraint stiffness. End moments at the two ends become equal at $\alpha_E = 1$ for single columns, and for common frame (panel) columns [13]. The linear approximations, B_{2lin} and B_{1lin}, in the figures are discussed later (Section 7.5).

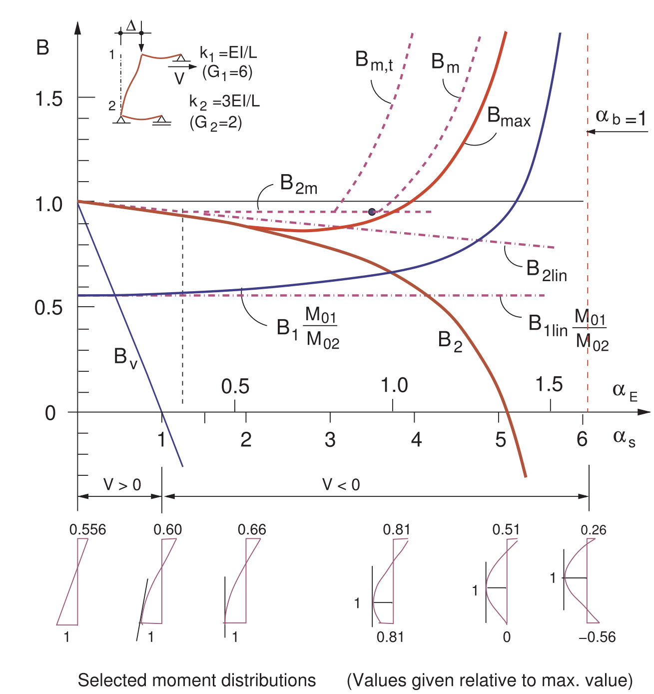

Fig. 4. Moments and shear versus axial load level in column with unequal, relative flexible end restraints ($\beta_s = 1.932$, $\beta_b = 0.785$, $\alpha_E = 0.268\alpha_s$).

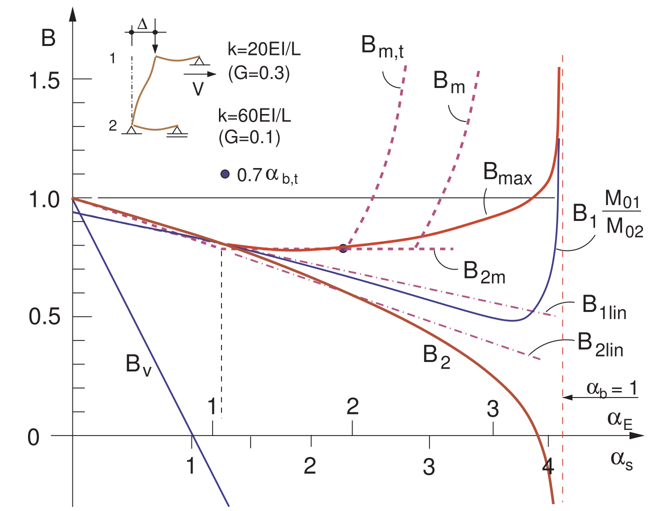

Fig. 5. Moments and shear versus axial load level in column with unequal, very stiff end restraints ($\beta_s = 1.065$, $\beta_b = 0.532$, $\alpha_E = 0.88\alpha_s$).

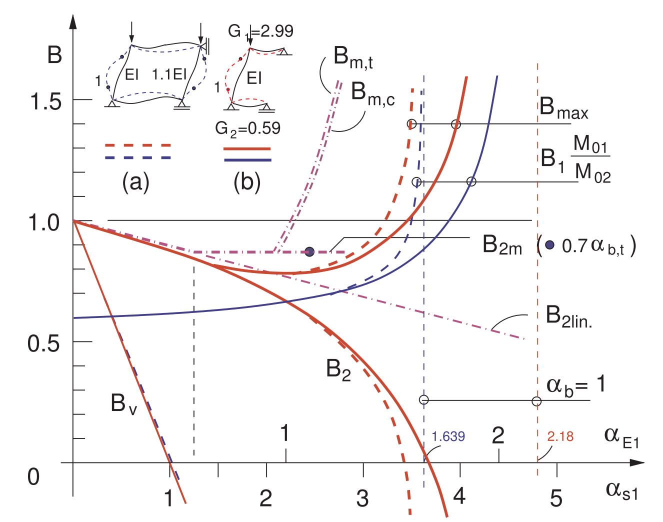

Fig. 6. Moments and shears for two cases: (a) Left hand Column 1 of the panel (dashed lines); (b) Column 1 considered in isolation (solid lines). B_{2lin} are secant predictions of B_2.

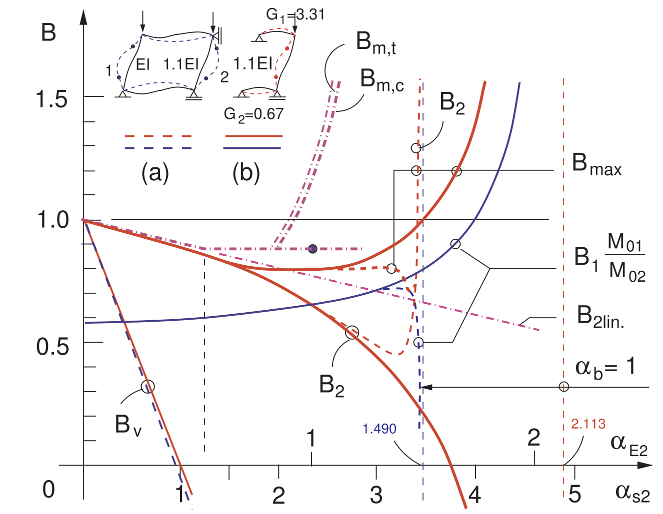

Fig. 7. Moments and shears for two cases: (a) Right hand Column 2 of the panel (dashed lines); (b) Column 2 considered in isolation (solid lines). B_{2lin} are secant predictions of B_2.

Maximum moment. The maximum moment (M_{max}, B_{max}) is initially equal to the larger end moment (M_2, B_2). Following an initial decrease along the B_2 path, B_{max} starts forming away from end 2. Thereafter, following a continued small decrease, maximum moments increase and approach infinity for loads approaching the braced critical load ($\alpha_b = 1$).

The B_{2m} curves, and bullets on the curves, will be discussed later (Section 8). So will the maximum moment factor approximations, B_m, $B_{m,t}$ and $B_{m,c}$ (Section 9).

Shears. For the range of shears shown in the figures, the B_v variation is close to linear ($B_v = 1 - \alpha_s$, Eq. (12b) for $\gamma_n = \gamma_s$).

5. General shear formulation

In order to establish end moments, it is necessary to know the shear force. A general shear expression $V = B_v B_s V_0$ (Eq. (1)) can be taken, with B_v according to Hellesland [10], as

$$B_v = 1 - \frac{\gamma_n N \Delta_0}{V_0 L} \tag{12a}$$

or, equivalently,

$$B_v = 1 - \alpha_s \frac{\gamma_n}{\gamma_s} \tag{12b}$$

The novelty of Eq. (12) is represented by the distinction between γ_n and γ_s (= γ_n at $\alpha_s = 1$), and with the γ_n factor as defined below. Both factors represent the increased column flexibility caused by the column axial load acting on the deflection of the column away from the chord through its ends ($N\delta$ effects).

An approximate γ_n factor, reflecting the full transition from free-sway to nearly fully braced, was proposed by Hellesland [10] in terms of γ_s and two additional flexibility terms:

$$\gamma_n = \gamma_s + \Delta\gamma_1 + \Delta\gamma_2 \quad (\geqslant \gamma_s) \tag{13a}$$

$$\Delta\gamma_1 = 0.12(\gamma_s - 1)(\alpha_s - 1); \quad \Delta\gamma_2 = 0.6\alpha_{s,b}\left(\frac{\alpha_s - 1}{\alpha_{s,b}}\right)^8 \tag{13b,c}$$

$$\alpha_{s,b} = \frac{N_{cb}}{N_{cs}} = \left(\frac{\beta_s}{\beta_b}\right)^2 \tag{13d}$$

For a cantilever column fixed at the base and with zero axial load, Eq. (13a)) gives $\gamma_n = 1.19$ (exact 1.20). The limitation on γ_n ($\geqslant \gamma_s$) may be adopted when this represents a simplification. For pin-ended columns ($N_{cs} = 0$), $\gamma_n = 1$. This value may also be taken in the rare case of a column with axial tensile loading.

The γ_n factor may take on large values as the braced critical load is approached. In other cases, the two first terms of Eq. (13a)) may be adequate. Often, the first term is sufficient (for frames with reasonably similar columns and loads in the various axes). A linear approximation of γ_n beyond γ_s was derived theoretically [14], but the expression was rather cumbersome, and not very convenient in practical contexts. The first to suggest γ values in excess of γ_s, was possibly Stevens [15].

The γ_s factor, defined below, varies between 1 and 1.216 (1.22) for columns with positive end restraints, but may become greater in single curvature regions of multistorey frames.

Several diagrams have been established for easy determination of γ_s [9–11,16]. Furthermore, general γ_s expressions are available, notably one first given by Rubin [17], and in a modified form by Lai and MacGregor [18]. A detailed discussion and summary of available γ_s diagrams and expressions are given in [16]. A rather simple approximate expression, proposed initially by the author in 1981 during a research stay at Univ. of Alberta, Edmonton, based on observations of the variation of γ_s with changing restraints, such as illustrated in Fig. 8, is adequate in the present context. It is given by

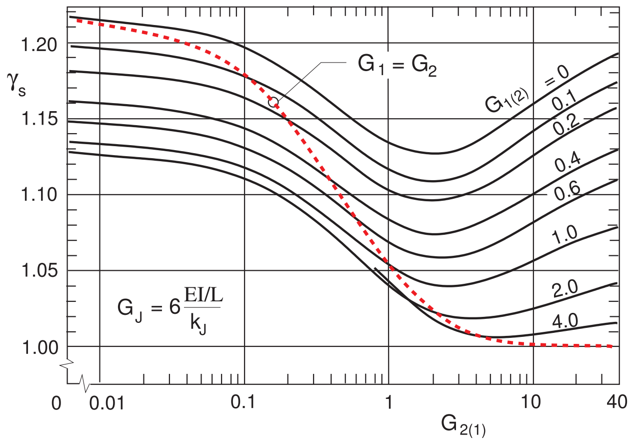

Fig. 8. The flexibility factor γ_s at the free-sway condition versus positive rotational end restraints in terms of G factors (reproduced from Hellesland [9]).

$$\gamma_s = 1 + 0.108\frac{1 + [1 - (0.5G_{max})^p]^3}{(1 + 0.5G_{min})^2} \tag{14}$$

where $p = 1$ for $G_{\max} \leq 2$ ($\kappa \geqslant 3$) and $p = -1$ for $G_{\max} > 2$ ($\kappa < 3$). $G_{\max}$ is the larger and $G_{\min}$ the smaller of the G factors at the two column ends. Eq. (14) breaks down into a pin-ended case considered later (in conjunction with Eq. (20)).

6. General end moment formulation

From moment equilibrium of a laterally loaded (displaced) column (Fig. 3(a)), $M_1 + M_2 + N\Delta + VL = 0$, where $\Delta = B_s\Delta_0$ and V is the shear given by Eq. (12b), the end moment sum may be written as

$$\frac{M_1 + M_2}{B_s(M_{01} + M_{02})} = 1 - \frac{\gamma_n - 1}{\gamma_s}\cdot\alpha_s \tag{15}$$

From this equation, end moments can be computed directly in cases in which there is only one unknown end moment. Two such cases are: (1) columns pinned at end 1 ($M_{01} = M_1 = 0$), and (2) columns with equal end restraints ($M_1 = M_2$). In these cases, Eq. (15) reduces to

$$\frac{M_2}{B_sM_{02}} = 1 - \frac{\gamma_n - 1}{\gamma_s}\cdot\alpha_s \tag{16}$$

End moment predictions with this equation are shown in Fig. 9 for two columns pinned at the top and base restraints as shown. The agreement with second-order theory, labelled "Exact" in the figure (reproduced from Hellesland [10]), is seen to be good, in particular in the positive B_2 range. Maximum moment predictions (B_m) will be discussed later.

In the general case, with unequal end restraints at the two ends, Eq. (15) is not directly useful. The distribution of the moment sum to the two ends must be established before individual end moments can be calculated. Efforts at accomplishing this have not been successful, and remains a task for future research.

The simpler task of establishing moment expressions that are valid for the important class of supporting columns ($\alpha_s < 1$), and over a load range somewhat beyond this, is pursued below.

7. Linearized end moment formulations

7.1. Formulations

End moments will generally be discussed in terms of the end moment factors B_1 and B_2. Linear moment relations for the individual end moments, taken as secant approximations (between $\alpha_s = 0$ to 1.0) can be expressed by

$$B_{1lin} = \frac{M_1}{B_sM_{01}} = 1 - (1 - B_{1s})\alpha_s \tag{17}$$

$$B_{2lin} = \frac{M_2}{B_sM_{02}} = 1 - (1 - B_{2s})\alpha_s \tag{18}$$

Here, B_{1s} and B_{2s} are end moment factor values at $\alpha_s = 1$ as shown in Fig. 10. B_{1lin} and B_{2lin} predictions with these factors have been found to provide excellent agreement with exact results [13]. Below, attempts are made to derive closed form expressions for B_{1s} and B_{2s}.

Results for B_{2s} and B_{1s} coincide in the case with equal end restraints, and are shown by the dash-dot borderline (labelled $G_1 = G_2$). Results for B_{1s}, shown by dashed lines in the figure, and B_{2s}, shown by solid lines, are located above and below the borderline, respectively. G_2 is by definition taken to represent the end with the stiffer restraint (with the smaller G value). Corresponding B_{1s} and B_{2s} curves terminates therefore at the dash-dot curve. At $G_2 = 6/\kappa = 0$ (fixed end), B_{2s} may have values between 0.79 and 0.82, and B_{1s} between about 0.82 and 1.05.

7.2. B_{2s} factors for two special cases

For cases with only one unknown end moment (e.g., M_2) and linear shear ($\gamma_n = \gamma_s$), Eq. (16) transforms to

$$B_2 = \frac{M_2}{B_sM_{02}} = 1 - (1 - \frac{1}{\gamma_s})\alpha_s \tag{19}$$

This equation applies to the case of a column pinned at one end ($M_1 = 0$) and to a symmetrically restrained column ($M_1 = M_2$). Evaluated at $\alpha_s = 1$, B_2 becomes $B_{2s} = 1/\gamma_s$. For the two special cases considered, for which γ_s can be found in the literature, B_{2s} can be given as follows:

(1) *Column pinned at end 1*; $\gamma_s = \gamma_{s,pin1}$:

$$B_{2s} = \frac{1}{\gamma_{s,pin1}} \quad \text{with} \quad \gamma_{s,pin1} = 1 + \frac{0.216}{(1 + 0.5G_2)^2} \tag{20a,b}$$

(2) *Column with equal end restraints*; $\gamma_s = \gamma_{s,equal}$:

$$B_{2s} = \frac{1}{\gamma_{s,equal}} \quad \text{with} \quad \gamma_{s,equal} = 1 + \frac{0.216}{(1 + G_2)^2} \tag{21a,b}$$

The pin-ended γ_s (Eq. (20b)) was derived by Hellesland [9], and along different lines and in a different form ($C_L = 1 - \gamma_s$) by LeMessurier [11].

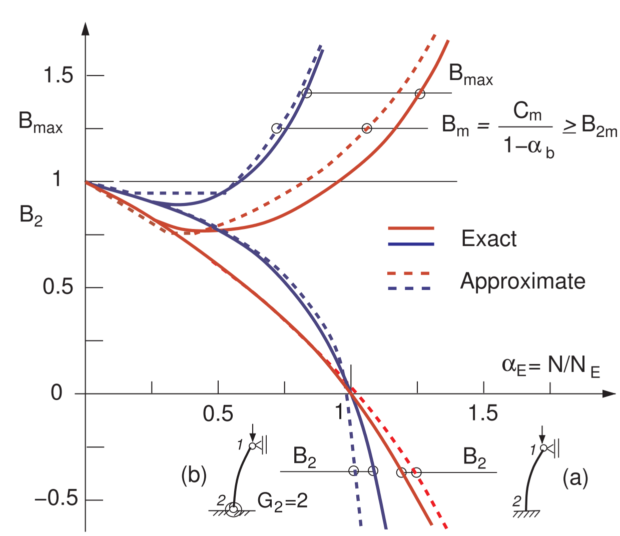

Fig. 9. Moments versus axial load level for two columns pinned at the top: (a) Fixed at the base ($\beta_s = 2$, $\beta_b = 0.7$); (b) Partially fixed at the base ($\beta_s = 2.635$, $\beta_b = 0.843$); $\alpha_s = \alpha_E\beta_s^2$.

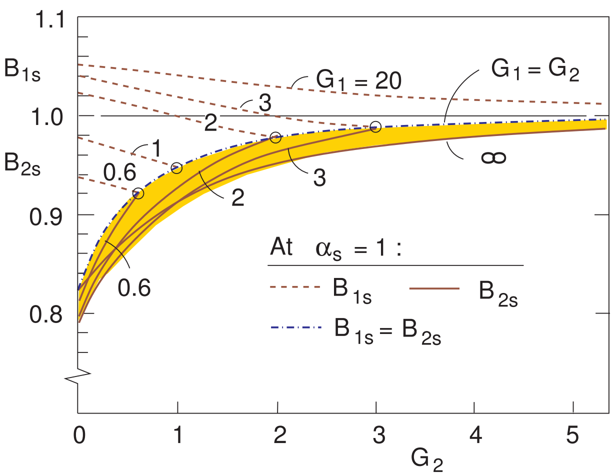

Fig. 10. End moment factors B_{1s} and B_{2s} at $\alpha_s = 1$ versus end restraints in terms of G factors (from Hellesland [13]).

Also, Eq. (14) breaks down into this expression in the pin-ended case. The equal restraint case (Eq. (21b)) can be obtained from Eq. (20b) by replacing L by $L/2$. The factor 0.216 may clearly be rounded off to 0.22, or even 0.2, in practical applications.

$B_{2s} = 1/\gamma_s$ for the two cases above can be identified in Fig. 10 by the solid line marked "∞" and by the dash-dot line marked "$G_1 = G_2$", respectively. The equal-ended case can be seen to provide an upper bound on the B_{2s} results, and the pin-ended case a practical lower bound, except at low G_2 values (less than about 1.5; $\kappa > 4$) where slightly, but insignificantly, smaller B_{2s} values result.

7.3. Approximate B_{2s} factor – general case

The equal-ended case (Eq. (21)) provides an upper bound on the B_{2s} values, and might be chosen to provide conservative estimates of the maximum end moment in the general case. However, since the B_{2s} results lie within a reasonably narrow band (shaded in Fig. 10), it is proposed to adopt an "average" B_{2s} value, located approximately in the middle of the shaded band, given by

$$B_{2s} = \frac{1}{\gamma_{s,aver}} \quad \text{with} \quad \gamma_{s,aver} = 1 + \frac{0.24}{(1 + 0.75G_2)^2} \tag{22a}$$

or, when rewriting, by

$$B_{2s} = 1 - \frac{0.24}{0.24 + (1 + 0.75G_2)^2} \tag{22b}$$

This expression is independent of G_1, reflecting the relative insensitivity to G_1, and gives B_{2s} values within ± 2.5% of correct values. This accuracy is generally quite acceptable. This B_{2s} is adopted in this paper for linearized B_{2lin} predictions by Eq. (18).

7.4. Approximate B_{1s} factor

When B_{2s} is known, the moment factor B_{1s} at column end 1 can be solved for from Eq. (15) for $\alpha_s = 1$ (at which $\gamma_n = \gamma_s$). It becomes

$$B_{1s} = \left(B_{2s} - \frac{1 - \mu_0}{\gamma_s}\right)\frac{1}{\mu_0} \tag{23}$$

where

$$\mu_0 = -\frac{M_{01}}{M_{02}} = -\frac{G_2 + 3}{G_1 + 3} \tag{24}$$

With the adopted sign convention for end moments (Fig. 3(a)), this ratio between the first-order end moments becomes positive for single first-order curvature bending and negative for double curvature bending. This ratio in terms of G_2 factors can easily be established [13]. Sufficiently accurate γ_s values above can be obtained from Eq. (14). An overestimation of B_{2s} will lead to an underestimation of B_{1s}, and vice verse. Prediction accuracies are found to be within about ± 2 to 2.5%.

Considering the lesser importance of obtaining very accurate B_{1s} predictions, a simpler approach may be justified. Based on Fig. 10, it is, alternatively, proposed to calculate B_{1lin} (Eq. (17)) with

$$B_{1s} = 1 \quad \text{for} \quad G_1 > 1.25 \tag{25a}$$

$$B_{1s} = 1 - \frac{0.22}{0.22 + (1 + G_1)^2} \quad \text{for} \quad G_1 \leqslant 1.25 \tag{25b}$$

Eq. (25b) is given by the curve labelled $G_1 = G_2$ in Fig. 10. The prediction accuracy of Eq. (25) is close to that of Eq. (23), and is considered sufficiently accurate. At one stage [19], G_1 greater or smaller than 1.0 was considered in Eq. (25).

7.5. B_{2lin} and B_{1lin} compared with exact results

Linear end moment predictions at end 2 by B_{2lin} (Eq. (18)), with B_{2s} given by Eq. (22), are shown in Figs. 4 and 5 for single restrained columns, and in Figs. 6 and 7 for the two columns of a panel. Predictions at end 1, B_{1lin} (Eq. (17)), computed with the simplified B_{1s} (Eq. (25)) are shown only in Figs. 4 and 5.

It can be seen that the linearized end moment approximations generally provide good B_2 and B_1 predictions, not only for supporting sway columns ($\alpha_s < 1$), but also well beyond this range in the cases considered.

8. Lower bound on maximum moments

In the figures (Figs. 4–7) it is seen that maximum moment factors, B_{max}, are less than 1.0 up to fairly high load indices, and that they are well below 1.0 in cases with relatively stiff end restraints.

An approximate lower bound on the maximum moment can be given by a B_{2m} factor defined by a bilinear curve that follows the descending linear approximation B_{2lin} (Eqs. (18), (22)) up to $\alpha_s = 1.25$, and then stays constant:

$$B_{2m} = B_{2lin} \geqslant B_{2in}(\alpha_s = 1.25) \tag{26a}$$

which transforms to

$$B_{2m} = 1 - g_2 \cdot \alpha_s \geqslant 1 - g_2 \cdot 1.25 \tag{26b}$$

$$g_2 = 1 - B_{2s} = \frac{0.24}{0.24 + (1 + 0.75G_2)^2} \tag{26c}$$

Predictions by this B_{2m} are included in Figs. 4–7. It can be seen to give good estimates of the maximum moment for α_s less than about $\alpha_s = 3$. A more detailed discussion is given in Hellesland [20], where also other cases are presented.

Bullet points are included on the B_{2m} curve at load indices of $\alpha_{b,t} = 0.7$, computed with single curvature beam bending (horizontal tangent at beam midlength). As a potential application limit indicator, $\alpha_{b,t}$ of 0.7–0.8 was considered. However, on the overall, $\alpha_s = 3$, computed with "sidesway restraints", represents a more consistent limit indicator.

In an initial phase, $\alpha_s = 1$ was considered instead of 1.25 in Eq. (26). A value of 1.5 has also been considered, and found to be acceptable in all cases except for those with very stiff restraints, such as in Fig. 5.

9. "Rising" moment branch

9.1. Approximate formulation

The exact maximum moment (Eq. (4)), is here approximated by

$$M_{max} = B_m B_s M_{02} \tag{27}$$

where the approximate member magnification factor is denoted B_m to distinguish it from the exact B_{max}. The "rising" moment branch, above a lower limit "limB_m", which will discussed later, is commonly computed using variations of B_m below:

$$B_m = B_b \geqslant \lim B_m \tag{28}$$

where

$$B_b = \frac{1 + A\,\alpha_b}{1 - \alpha_b} \cdot C_m \tag{29a}$$

$$C_m = 0.6 + 0.4\mu_0 \tag{29b}$$

$$\mu_0 = -\frac{B_s M_{01}}{B_s M_{02}} = -\frac{M_{01}}{M_{02}} \tag{29c}$$

Here, $\alpha_b = N/N_{cb}$ is the critical load index (Eq. (7b)) of the column considered braced, C_m is a moment gradient factor that accounts for non- uniform first-order bending [21], and μ_0 is the first-order end moment ratio, to be taken as positive when the member is bent in single first-order curvature, and negative otherwise. A is a factor, typically about 0.25 for pin-ended columns with uniform first-order bending ($C_m = 1$), but commonly neglected. Also in present computations, $A = 0$ is chosen.

With the lower limit taken as lim$B_m = 1$, and $A = 0$, the approximate B_m formulation above can be found in most structural design codes (such as ACI 318 and AISC 360 (limited to "braced moments"), Eurocodes 2 and 3). The approximation and its faults are discussed in considerable detail by Lai et al. [22]. In the literature, the C_m factor is extensively discussed for both elastic and inelastic applications (e.g. [23–29]). However, C_m in Eq. (29b) is the most widely adopted factor for elastic applications.

For regular frames, it is common design practice to assume that beams bend into symmetrical, single curvature (with rotational bending stiffness $2EI_b/L_b$) at braced frame instability if more correct values are not established, and to use such restraints in the calculation of α_b. This is considered a prudent approach, and is in accordance with most codes of practice. This assumption will also be adopted in the comparisons below.

9.2. Comparisons with exact results

Single column results. Approximate predictions B_m for single restrained columns according to Eq. (28) are shown in Figs. 4, 5.

The predictions labelled B_m are computed with the restraints given in the inserts of the figures. The curves labelled $B_{m,t}$ are included for comparative reasons. They are computed assuming that the columns have been isolated from a greater frame with rotational beam stiffnesses of $k_b = 2EI_b/L_b$ (single, symmetrical curvature bending). These are 1/3rd of the restraints given in the figures.

Panel column results. Two sets of predictions are shown for the panel columns in Figs. 6 and 7. The curves labelled $B_{m,t}$ are computed as described above (with beam stiffnesses $k_b = 2EI_b/L_b$). The $B_{m,c}$ curves are computed with α_b taken as the exact critical load indices of the two panel columns (in terms of α_E equal to 1.639 and 1.490, respectively).

The $B_{m,t}$ curves are seen to be very close to the $B_{m,c}$ curves. This implies that the assumed single, symmetrical curvature beam bending in the $B_{m,t}$ computations is close to the exact one for this panel. The difference will be more marked in cases with greater difference between the columns of the panel.

As seen, the B_m predictions (incl. $B_{m,t}$ and $B_{m,c}$) by Eq. (28) are rather conservative. This is primarily due to the C_m approximation, that tends to become very conservative (too large) for columns with significant double curvature bending, as in the present cases.

10. Maximum column moment proposals

10.1. Maximum moment proposal 1

It has proven difficult to develop reasonably accurate approximations for maximum moments in the general case. An attempt is made below to provide some improvements, with different degrees of conservativeness, to common regular design work procedures at present.

Alternative (1a) – Low to moderately high axial load levels:

For most practical load levels, it is acceptable for design purposes to compute the maximum moment according to $M_{max} = B_m B_s M_{02}$, Eq. (27), with the simplification

$$B_m = 1.0 \tag{30}$$

It has been found [13], from elastic second-order analyses of single columns with practical (and invariant) end restraints, that this B_m approximation is conservative for load indices given in terms of the free-sway and the braced critical load indices, respectively, by

$$\alpha_s < 3.5(3.0) \quad \text{or} \quad \alpha_b < 0.5(0.8) \tag{31}$$

If some 5–10% underestimation of moments were accepted, the limits above could be increased somewhat. For columns in sway frames, most columns will have load indices well below the values indicated above.

The numbers in parentheses were found for columns in panels (frames), in which, as discussed before, the critical loads are lower than those of isolated, single columns due to the reduced restraints offered by the panel beams as the axial column loading increases towards the braced critical load. As a consequence, the rising, maximum moment branch will be pressed upwards, towards larger values (eventually infinity) at smaller load indices than in the single column case.

The limit in terms of the free-sway load index, α_s, is affected much less than the corresponding limit in terms α_b by the changing of the end restraints described above. Consequently, α_s is the better suited parameter of the two, to indicate range of applicability of simplified maximum moment expressions.

Alternative (1b) – Present practice, for any load level:

Axial load indices are not likely to exceed those in Eq. (31), but may possibly do so in partially braced frames. To complement Alternative (1a) above, it is therefore prudent, at very high load levels, to include a rising moment branch. Common practice is, as mentioned before, to calculate B_m according to the conservative Eq. (28) defined with lim$B_m = 1$ and $A = 0$. Then, adopting the same,

$$B_m = B_b \geqslant 1.0 \tag{32}$$

This case is not specifically identified in Figs. 4–7, but it is clear that it is equal to the case below, exept that the lower bound on the rising branch (B_m) is taken as 1.0 rather than a value less than 1.0.

10.2. Maximum moment proposal 2

To simplify presentation and discussion, proposal 2 alternatives are illustrated for a specific restraint case in Fig. 11.

Alternative (2a) – Low to moderately high load levels:

A more economical design than that in Alternative 1(a) can be achieved for $\alpha_s < 3$ by taking B_m according to

$$B_m = B_{2m} \tag{33}$$

where B_{2m} is "maximum moment modified B_{2lin}" given by Eq. (26), and defined also in Fig. 11. The rotational end restraints to be used in calculating B_{2m} are those for the column considered free-to-sway. Typically, $k = k_b = 6EI_bL_b$, and $G_2 = 6(EI/L)/k$, corresponding to antisymmetrical beam bending.

For unbraced frames ($S_B = 0$ in Eq. (6b), individual α_s values will normally be at most about 1 to 2, and never likely to ever exceed 3.0 (Hellesland [20]). This estimate is based on storey sway magnifiers, B_s (Eq. (5)), in the range 1.5 to 2.0.

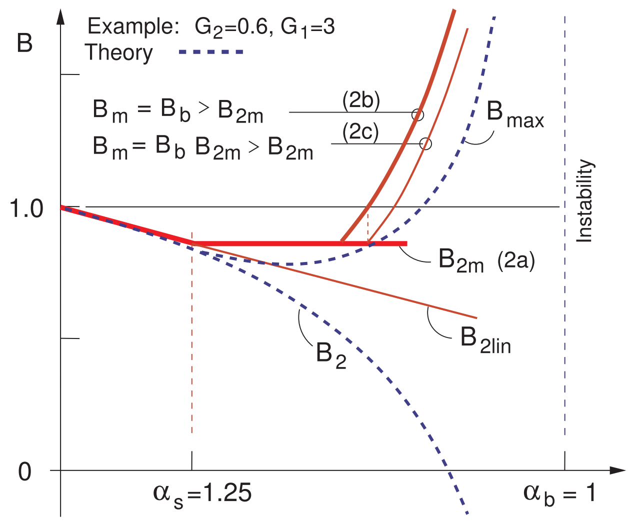

Fig. 11. Maximum moment proposal alternatives (2a), (2b) and (2c), as illustrated for a specific case ($G_1 = 3$ ($\kappa = 2$), $G_2 = 0.6$ ($\kappa = 10$)). Alternatives (1a) and the "present practice" alternative (1b) are not shown in the figure, but are equal to (2a) and (2b), respectively, when B_{2m} is taken as 1.0.

For partially braced frames with sway ($S_B > 0$), the column indices can be greater than in the unbraced case above without inflicting unacceptable B_s indices. However, there are many good reasons in design to limit axial loads to levels well below the critical braced column loads [20]. Column loads, in terms of critical loads for braced cases, or better in terms of the system critical loads, should probably be limited in regular design work to values below $0.7\alpha_b$, or so, in order to avoid column behaviour sensitive to uncertainties in restraint assessments, including possible effects of beam yielding, and unwinding type phenomena.

Alternative (2b) – Any load level:

Again, to cover very high load levels, a rising moment branch approximation may, in lieu of better formulations, be taken according to

$$B_m = B_b \geqslant B_{2m} \tag{34}$$

Alternative (2c) – Any load level:

A less conservative magnifier than that defined above by Eqs. (32) and (34) is illustrated in Fig. 11 by the curve marked (2c). It is given by a product defined by

$$B_m = B_b \cdot B_{2m} \geqslant B_{2m} \tag{35}$$

In this formulation, the rising branch of the proposal Alternative (2b) is lowered by the multiplication with B_{2m} (Eq. (26)).

Predictions by Eq. (35) are not included in the figures (Figs. 4–7), but the effect is that the conservativeness of the rising moment branch predictions is reduced, yet still quite conservative in most cases investigated.

An exception is found for the case of a column pinned at the top ($G_1 = \infty$) and fully fixed (clamped) at the base ($G_2 = 0$), for which B_m (Eq. (35)) gives somewhat unconservative predictions. However, two aspects of the fixed-pinned column case deserving attention are the i) theoretically full fixity at one end and ii) the theoretically perfect hinge connection to the adjoining structure. Both are difficult to achieve in practice. It is found (Hellesland [20]) that only a slight fixity reduction from $G_2 = 0$ to $G_2 = 0.25$ ($\kappa_2 = 24$) is sufficient to provide good prediction accuracy. Also, it is found that a "practical pinned connection" with a small restraint of about $G_1 = 10$ ($\kappa_1 = 0.6$) or so, is sufficient to compensate for the mentioned unconservativeness. It is felt that these considerations separately justify the use Eq. (35) also for practical pin-ended columns with very stiff base restraints.

Summary and conclusions on maximum moment proposals:

The maximum design moment proposals for columns with sway due to lateral loads are all considered feasible in regular design work.

From a practical design point of view, the proposal (1) alternatives (a and b) are believed to be the most suitable in early design phases, when details of column end restraints are yet to be determined.

The proposal (2) alternatives (a, b and c) will allow more economical designs than the proposal (1) alternatives, but are believed to be most suitable in final design phases, following preliminary design, and in possible design check situations.

The alternative (2c) formulation has the advantage over alternative (2b) that it is less conservative. Also, and conceptually more important, it can be applied in a more rational manner in cases with load combinations that also include moments from gravity loading.

11. Overview of possible application steps

1. **First-order load effects**. Establish end moments (M_{01}, M_{02}) and shears (V_0) from a conventional first-order analysis based on assumed sectional stiffnesses.
2. **Column end restraints**. For calculation of local second-order effects for columns considered in isolation, establish end restraint stiffnesses k_1 and k_2 (and corresponding nondimensional parameters as discussed in Section 4.2 (and in Section 4.3).
3. **Global sway magnification factor**. Establish B_s (Eq. (5)) for the sidesway action, at the level (floor) of the frame considered, using a storey stability index (α_{ss}) defined by Eq. (6a) or (6b), or by simplifications of these.
4. **Maximum design moment**. Establish $M_{max} = B_m B_s M_{02}$ (Eq. (27)).
   - Preliminary and early design phases: the maximum moment factor B_m may simplified be approximated by $B_m = 1$ (Eq. (30)) in most practical cases, or conservatively by the "present practice" factor $B_m = B_b \geqslant 1.0$ (Eq. (32)) in very high axial load level cases.
   - Final design checks: a more rational and economical design is obtained with $B_m = B_{2m}$ (Eq. (33)) in most practical cases, or more conservatively with $B_m = B_b \geqslant B_{2m}$ (Eq. (34)) in very high axial load level cases, or alternatively with $B_m = B_b \cdot B_{2m} \geqslant B_{2m}$ (Eq. (35)); see earlier discussion.
5. **End moments**. If end moments, given by $M_j = B_j B_s M_{0j}$ ($j = 1, 2$), are of interest for the design of a foundation or adjacent beams, they can be estimated with B_1 and B_2 approximated by B_{1lin} and B_{2lin}, respectively, as given in Section 7.
6. **Shear transfer**. Column shear at a foundation attached to a column may sometimes be required. The reduced shear $V = B_v B_s V_0$, can be computed with B_v given by Eq. (12a) or (11b), or simplified by the linear version $B_v = 1 - \alpha_s$.

12. Maximum moment in load combinations

A brief mention on how to deal with load combinations seems appropriate. For framed columns with first-order end moments from both lateral loading (M_{0s}) and gravity loading (M_{0b}), the sway modified first-order moment sum (not including local second-order effects) at the two ends, can in line with the principle of superposition be defined by

$$M_{01}^* = M_{01b} + B_s M_{01s} \quad \text{and} \quad M_{02}^* = M_{02b} + B_s M_{02s} \tag{36}$$

Here, M_{02}^* is taken, per definition, to be the moment at the end with the larger moment sum (absolute value), and M_{01}^* the moment at the end with the smaller end moment sum.

Proposal 2, alternative (2c) above, offers the most rational maximum column sway moment formulation. Adopting this alternative, it is proposed to calculate maximum column moments from

$$M_{max} = B_b (M_{0b} + B_{2m} B_s M_{0s})_2 \tag{37}$$

where

(i) B_{2m} is to be taken according to Eq. (26), when the larger moment sum occurs at the end with the larger sway moment M_{0s} (i.e., at the end with the stiffer end restraint), and

(ii) $B_{2m} = 1$ when the larger moment sum occurs at the end with the smaller sway moment M_{0s} (i.e., at the most flexible end restraint).

Furthermore, B_b, Eq. (29a), is now subject to the restriction $B_b \geqslant 1$, and C_m, Eq. (29b), is now defined with μ_0 in Eq. (29c) replaced by

$$\mu_0 = -\frac{M_{01}^*}{M_{02}^*} = -\frac{(M_{0b} + B_s M_{0s})_1}{(M_{0b} + B_s M_{0s})_2} \quad (38)$$

This moment ratio is taken positive when the member has single first-order curvature bending, and negative otherwise. Eq. (37) breaks down to $M_{max} = B_m B_s M_{02}$ (Eq. (27)) with B_m given by Eq. (35) when $M_{0b} = 0$, and into the conventional, common practice, formulation for gravity load moments when $M_{0s} = 0$ [30].

It is a drawback of the proposal that it is necessary to check if the larger end moment sum and the larger sway moment occur at the same or different column ends. The advantage is a more economical final design. A more extensive discussion, and comparisons with present code formulations, are beyond the scope of the present paper.

13. Conclusions

Development of shears, end moments and maximum moments between ends of framed columns with sidesway have been studied using elastic second-order theory. The full range of axial loads are covered, and therefore both "supporting sway columns" ($\alpha_s < 1$) and "supported (braced) sway columns" ($\alpha_s > 1$).

End moment and maximum moment expressions for columns with sway are derived. These represent novel contributions that are suitable in typical design code formats, and in practical design work. They will allow more rational column assessments, and will reduce the conservativeness of current procedures, thus allowing more economical designs than presently available structural code expressions.

A possible extension of the maximum moment formulation to the general case of columns with moments from both gravity and lateral loading, is briefly presented for the sake of completeness, but not discussed in any depth.

Declaration of Competing Interest

None.

Acknowledgments

During a research stay by the author at the Univ. of Alberta (UA), Edmonton, Canada, in 1981, a preliminary draft entitled "Mechanics and design of columns in sway frames", co-authored with the now deceased Professor J.G. MacGregor, was neither published nor completed. The topic is still of considerable interest, though. The present paper is significantly revised and extended. Early input by Prof. MacGregor is greatly appreciated. So is the running of the initial computer analyses of the panels by S.M.A. Lai, then a PhD student at UA. In the early phase of this study, support was provided to the author by the Research Council of Norway and a research associateship at UA.

Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.engstruct.2019.109804.

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