On the permissibility of impredicative comprehension

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Abstract

Which comprehension axioms of higher-order logic are acceptable? It is well known that unrestricted higher-order comprehension is incompatible with unrestricted reification of higher-order entities. In search of a response to this conflict, an argument against all forms of impredicative comprehension is formulated. Although the argument is ultimately rejected, a careful analysis of it points the way to some milder logical restrictions, which suffice to resolve the conflict.

The aim of this paper is to develop and connect some topics that figure in Bob Hale's research: higher-order logic, the nature of the fundamental logico-mathematical notions of collection and generality, and—most of all—the permissibility of impredicative comprehension. While some of my claims will be congenial to Hale, there will also be points of disagreement.[1]

1 The question of impredicative comprehension

Is impredicative comprehension permissible? In order to understand the question, we first need to understand two forms of higher-order reasoning in logic.

First, there is *second-order logic*, which extends ordinary first-order logic by permitting quantification into predicate position. This enables us to quantify over what Frege called *concepts*, that is, over the semantic values of predicates. Consider for example the statement that Socrates thinks, formalized as:

(1) $$\textsc{Think}(\text{Socrates})$$

First-order logic allows us to generalize into the noun position to conclude:

(2) $$\exists x\, \textsc{Think}(x)$$

[1] I believe this is the best and most appropriate way to honor a philosopher I deeply respect. Ever since I first met Hale, shortly before completing my PhD, he has been a great source of inspiration and a wonderful interlocutor: a good listener, always thoughtful and constructive in discussions, and impressively generous with his time.

Only in second-order logic, however, are we allowed to generalize into the predicate position to conclude that there is a concept F under which Socrates falls:

(3) $$\exists F\, F(\text{Socrates})$$

For the purposes of this article, I shall simply assume—with Frege and against Quine—that quantification into predicate position is legitimate.[2]

Then, there is *plural logic*, which permits generalization into plural noun phrase position.[3] For example, plural logic permits us to conclude from the fact that Plato and Aristotle disagreed that there are some things that disagreed, which we formalize as '$\exists xx\, \textsc{Disagreed}(xx)$'. As is now fairly common, we use double letters as plural variables and '$x \prec yy$' to state that x is among the objects yy. I shall here follow the prevailing view that plural logic is different from second-order logic. After all, the two extensions of first-order logic are based on generalization into completely different kinds of position: plural noun phrase and predicate, respectively.

What is the correct logic for the resulting second-order or plural quantification? Let us begin with the former. We particularly want to know what is the correct *deductive system* for second-order logic. The introduction and elimination rules governing the second-order quantifiers can be formulated in a way that makes them fairly uncontroversial.[4] If second-order logic is permissible at all—Quine notwithstanding—these rules are no more problematic than the corresponding rules governing the first-order quantifiers. Far and away the most interesting and controversial part of the standard deductive system for second-order logic is its *unrestricted comprehension scheme*, which says that any open formula ϕ that doesn't contain 'F' free can be used to define a concept F:[5]

(2-Comp) $$\exists F \forall x (Fx \leftrightarrow \phi(x))$$

The axiomatization of plural logic is analogous, except that its comprehension scheme carries

[2] See e.g. (Quine, 1986).

[3] See (Boolos, 1984) and (for a survey) (Linnebo, 2012).

[4] More precisely, we formulate the rules such that they allow bound variables to be instantiated only by constants and free variables, not directly by open formulas. To get the effect of instantiation by an open formula, we need to go via the corresponding comprehension axiom—of which more anon.

[5] This syntactic restriction, which is needed to avoid a clash of variables, will henceforth be left implicit.

an existential presupposition, as there is no empty plurality:[6]

$$\text{(P-Comp)} \qquad \exists x\, \phi(x) \rightarrow \exists xx \forall x (x \prec xx \leftrightarrow \phi(x))$$

An instance of either of these two schemes is known as a *comprehension axiom*. A second-order (or plural) comprehension axiom is said to be *impredicative* if ϕ contains second-order (or plural) quantifiers, and *predicative* if not.

There is a rich history of controversy surrounding the legitimacy of impredicative comprehension.[7] Here is a simple predicativist argument. (A more sophisticated version of the argument will be developed below.) A second-order comprehension axiom can be regarded as a definition of a concept. The axiom specifies *what it is* for an object to be F, namely, to satisfy the condition ϕ. Moreover, we have been taught that definitions should not be circular. It follows that ϕ must not presuppose the concept F to be defined or any other concepts defined in terms of F. This prompts the question of when a formula presupposes a concept. Suppose the formula contains predicate constants or free predicate variables. Then clearly the formula certainly presupposes any concept for which one of the mentioned expressions stands. The significance of *bound* predicate variables is far less clear. Skeptics about impredicativity take a hard line and argue that the formula presupposes not only the values of its predicate constants and free variables but also every concept over which its bound predicate variables range. If this hard line is warranted, it follows that impredicative comprehension axioms are circular definitions. An axiom of this sort defines a concept in a way that presupposes this very concept. And this calls into question the legitimacy of impredicative comprehension axioms.

Hale has on several occasions opposed predicativity restrictions on second-order comprehension.[8] I believe Hale should reconsider this opposition. To show this, I discuss a triad of philosophical claims to which Hale is attracted. As Hale is well aware, however, the triad is inconsistent; so at least one of the claims will have to be restricted. Since one of the claims is that the unrestricted impredicative comprehension scheme is valid, we have reason to take a closer look at this part of the standard deductive system. I therefore develop a version of the argument against impredicative comprehension that was canvassed above. The remainder of

[6] I shall sometimes use the expression ‘plurality’ for ease of communication, although a plural locution or formalization would have been more appropriate.

[7] See (Feferman, 2005) for a survey.

[8] See in particular (Hale, 2013, ch. 8).

the article is a critical assessment of this argument. While I ultimately reject the argument, we shall see that there is much to be learnt from scrutinizing its two main premises. Each premise can be resisted—but only at the price of some other revision to the classical deductive system: either a different, but less severe, restriction on comprehension principles; or a retreat to intuitionistic logic to govern quantification over absolutely everything. As evidence that the price is worth paying, I show that the two revisions pave the way for a satisfying response to the problem posed by Hale's inconsistent triad.

2 The paradox of reification

I now state the three claims to which Hale is either committed or at least attracted, yet which are jointly inconsistent. First:

> Unrestricted comprehension. No restriction is needed on the second-order comprehension scheme.

Of course, had we operated with a sparse conception of properties, then restrictions would have been required. We are, however, operating with an abundant conception of properties—or concepts, as we are calling them—according to which a concept is merely the semantic value of a predicate. On this conception, Unrestricted comprehension enjoys very strong support. Consider any formula $\phi(x)$ in our language. Since we are assuming the language to be meaningful, every object in the domain either satisfies the formula or not. That is, the formula is true or false of any object in the domain. This ensures that the formula defines a function from the domain to truth-values. And on the abundant conception of properties or concepts, nothing more than such a function is required for a predicate to have a referent.[9] Moreover, as Frege taught us and subsequent semantic theorizing has confirmed, there is much to be gained by ascribing semantic values to predicates in this way.

In fact, there is pressure to be more liberal yet and allow second-order comprehension even on formulas that fail to give a definitive verdict on every object in the domain. For example, why not allow comprehension on vague predicates, which cannot be assumed to be determinately true or determinately false of every object? The resulting functions from the domain of objects to truth-values would of course be partial (that is, not defined on the entire

[9]Hale and Wright call this *minimalism* about predicate reference. See (Hale and Wright, 2009, §9) and (for a fuller discussion) (Hale, 2013, ch. 1.12).

domain). For now, it suffices to observe that this further liberalization could only worsen, not alleviate, the difficulties that we shall encounter.

Next:

> Concepts are things. Every concept can also figure as the value of a first-order variable.

This claim is attractive because it enables us to express in a proper way the kinds of claim that Frege famously made about the ontological categories and their relation to one another. According to Frege, proper names denote objects, while predicates denote concepts. And these two kinds of entity or thing are fundamentally different. Here is a characteristic example:

> *Functions with two arguments* are just as fundamentally distinct from *functions with one argument* as the latter are from objects. For, while the latter are fully *saturated*, functions with two arguments are less saturated than those with one argument, which are already *unsaturated*. (Frege, 2013, §21)

In order to properly express such claims, we need to be able to compare objects and concepts (or, more generally, functions); in particular, to say that one has characteristics that the other one lacks. This requires a form of expressibility that Frege himself cannot allow.

A natural way to achieve the desired expressibility is by adopting a nominalization operator ν, which applies to any second-order constant or variable to yield a first-order term, and where $\nu(F)$ is to be read as "the property of being F".[10] The availability of this operator means that we can now ask questions about the identity of properties. Under what conditions is the property of being F identical with the property of being G? At the very least, identical properties must be coextensional:

$$\nu(F) = \nu(G) \rightarrow \forall x(Fx \leftrightarrow Gx) \tag{4}$$

Readers will recognize this as the problematic direction of Frege's infamous Basic Law V. I shall understand Concepts are things to involve an individuation of properties that is at least fine-grained enough to validate (4). It is important to realize that none of the difficulties that we shall encounter would be alleviated by adopting a more fine-grained individuation of properties.

Finally:

[10] See (Hale and Linnebo, 2015).

Absolute generality. It is possible to generalize over absolutely all things.

Hale holds this view even when the domain in question is "indefinitely extensible", in something like Dummett's sense.[11]

Unfortunately, the three claims just stated are inconsistent. We show this by means of a straightforward property-theoretic version of Russell's paradox. Let $x \in y$ express that x falls under some concept whose extension is y, which is formalized as $\exists F(y = \nu(F) \wedge Fx)$. Let R be such that:

$$\forall x(Rx \leftrightarrow x \notin x) \tag{5}$$

(Notice that this uses impredicative comprehension.) Let $r = \nu(R)$, and ask whether $r \in r$. By instantiating the universal quantifier in (5) with respect to r, we get $Rr \leftrightarrow r \notin r$, from which it is routine to use (4) to derive the inconsistency $r \in r \leftrightarrow r \notin r$. Thus, Unrestricted comprehension and Concepts are things entail a contradiction.

The third claim, Absolute generality, serves merely to close a possible escape hatch that some philosophers have attempted to use.[12] Might it be that, by the time we have defined the property r, the range of the quantifiers has expanded, with the result that r lies beyond the range of the universal quantifier in (5)? If so, the instantiation of this quantifier with respect to r—which is a crucial step of the above argument—is blocked. Absolute generality enables us to stipulate that the mentioned quantifier has absolutely general range, which firmly closes this possible escape hatch.

The inconsistent triad is an instance of a far more general problem, which I shall call *the paradox of reification*. In the presence of Absolute generality, it is inconsistent to make both of the following assumptions:

1. Unrestricted second-order (or plural) comprehension

2. There is a one-to-one mapping from concepts (or pluralities) to things

This inconsistency is problematic because there are many plausible candidates for such one-to-one mappings. Hale's inconsistent triad provides one good example. Other examples include mapping some things to the set of these things, or to a proposition that is about

[11] See e.g. (Hale, 2013, Section 8.6).
[12] See e.g. (Parsons, 1977) and (Glanzberg, 2004).

precisely these things (say, the proposition that these things exist), or to a fact that involves precisely these things (say, the fact that these things exist).[13] The great generality of the paradox of reification means that even readers who are not attracted to the claim CONCEPTS ARE THINGS have reason to be interested in a well-motivated way to resist UNRESTRICTED COMPREHENSION.

The challenge posed by the paradox of reification is to balance the strength of our comprehension principles against the forms of reification that we permit. The dominant approach in the contemporary literature is to accept UNRESTRICTED COMPREHENSION at the expense of severe restrictions on the permissible forms of reification.[14] I wish to take a different tack and explore what reason we may have to restrict the comprehension scheme.[15]

3 An argument against impredicative comprehension

The classic source of opposition to UNRESTRICTED COMPREHENSION is Russell's vicious circle principle, which in one of its more famous formulations states that:

> If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total. (Russell, 1908, p. 30)

The principle has generated much controversy. The response that commands the greatest contemporary support is probably Gödel's, according to which the predicativity restriction inherent in Russell's principle is well motivated only on a constructivist conception of the collections in question.

I believe that Gödel's analysis is incorrect, both in its own terms and as an account of Russell's vicious circle principle. For present purposes, there is no need to pursue the scholarly half of this claim.[16] I shall focus instead on setting out a simple argument against UNRESTRICTED COMPREHENSION that requires no constructivist assumptions and is compatible

[13] See, respectively, (Linnebo, 2010), (McGee and Rayo, 2000), and (Hossack, 2014). An early version of the paradox is found in Appendix B of (Russell, 1903).

[14] See e.g. (Williamson, 2003).

[15] In recent work, Hale too seeks to curtail UNRESTRICTED COMPREHENSION so as to be able to hold on to the other two claims from the inconsistent triad; see (Hale and Linnebo, 2015, §9). His proposed restriction requires more stage setting than can be provided here. It involves a ban on comprehension on formulas involving certain semantic predicates.

[16] Let me simply state, without argument, that it is tempting to interpret the vicious circle principle as concerned, not with nominal definition, but with something more like real definition; after all, other formulations of the principle feature the notions of "involving" or "presupposing" instead of "only definable in terms of". This interpretation suggests an argument not altogether different from the one I am about to develop. See (Goldfarb, 1989) and (Jung, 1999) for interpretations along these lines.

with a robustly realist outlook. To be perfectly clear, let me repeat that I do not endorse the argument but develop it only as a source of inspiration for a way out of Hale's inconsistent triad and a response to the paradox of reification more generally.

The argument relies on a metaphysical notion of presupposition. I shall therefore refer to it as *the presupposition argument*. We assume that there is a metaphysical ordering of truths. Each non-fundamental truth obtains in virtue of one or more other truths, which thus provide a metaphysical explanation of the former truth. Given this assumption, we say that one truth *presupposes* another just in case the correct metaphysical explanation of the former proceeds via the latter. This conception of presupposition can be fleshed out in different ways.[17] For present purposes, I shall not take a stand on these options. We can afford to leave the notion of presupposition somewhat schematic.

Let me now present the promised argument. Consider an impredicative comprehension axiom, which asserts that there is a concept F defined by $\forall x(Fx \leftrightarrow \phi(x))$. First:

(i) Fa presupposes $\phi(a)$, for any object a.

The idea is that, when a concept is defined by some condition, then this concept applies to an object in virtue of this object's satisfying the defining condition.[18] (If you find this premise problematic, don't worry. I shall return to ways in which it might be resisted.)

Second:

(ii) A universal generalization presupposes each of its instances.

In virtue of what is everything so-and-so? It is natural to answer that this holds in virtue of *this* being so-and-so, *that* being so-and-so, and so on through all the instances of the generalization.[19] (Again, I shall return to ways in which this answer might be resisted.)

[17] One option is to start with the metaphysical notion of grounding, which has recently received much attention from philosophers (see e.g. (Fine, 2012) and (Rosen, 2010)). We can then use this notion to define presupposition. Specifically, we can say that p presupposes q when q is a so-called 'strict partial ground' of p; that is, roughly, when q is explanatorily prior to p and part of a complete explanation of p. A second option is to replace the non-monotonic notion of grounding with a monotonic notion of sufficiency, which records that some truths suffice to provide a metaphysical explanation of some other truth. Let a *sufficiency tree* for p be a tree whose root is p and such that each of its nodes n is either an explanatorily fundamental truth or has as its "children" some nodes that suffice to explain n. We can then define that p presupposes q just in case q is distinct from p but occurs in any sufficiency tree for p. This definition captures our central idea that any complete account of p needs to proceed via q. A third option is to adopt a primitive notion of presupposition.

[18] Similar ideas are found in (Fine, 2012, §9) and (Rosen, 2010, §10).

[19] This view is endorsed in important parts of the emerging literature on the logic of ground; see e.g. (Fine, 2010), (Fine, 2012, §7), and (Schnieder, 2011, p. 461). By contrast, (Rosen, 2010, §8) holds a view closer to the one to be defended below, namely that some, but not all, generalizations are grounded (in part) in their instances.

Finally, we lay down three further premises:

(iii) Presupposition is transitive.

(iv) Presupposition is irreflexive.

(v) A true disjunction only one of whose disjuncts is true presupposes this disjunct.

These further premises are very plausible.[20] So for the purposes of the present article, they will simply be assumed. Our critical discussion will focus on ways to resist one or both of the first two premises.

Our five premises entail that many instances of impredicative comprehension are impermissible. Consider for example Frege's famous definition of the concept of a natural number. Let $\mathrm{Her}_S(X)$ state that X is hereditary along the successor relation S, that is, $\forall x \forall y(Xx \wedge Sxy \rightarrow Xy)$. Then Frege defines $\mathbb{N}x$ as:

$$\forall X(X0 \wedge \mathrm{Her}_S(X) \rightarrow Xx)$$

That is, x is a natural number just in case x has every hereditary property that is had by 0. We may assume that the embedded conditional is shorthand for $\neg X0 \vee \neg \mathrm{Her}_S(X) \vee Xx$. It is now straightforward to derive a contradiction from the premises. By (i), (ii), and (iii), it follows that $\mathbb{N}0$ presupposes $\neg \mathbb{N}0 \vee \neg \mathrm{Her}_S(\mathbb{N}) \vee \mathbb{N}0$. By arithmetic, (v), and another appeal to (iii), it follows that $\mathbb{N}0$ presupposes $\mathbb{N}0$, which violates (iv). Thus, the celebrated Fregean instance of impredicative comprehension is impermissible.

Readers should have no problem extending the argument to show that various other impredicative comprehension axioms are impermissible, perhaps making use of yet further plausible premises such as:

(vi) A true conjunction presupposes each of its conjuncts.

4 Two conceptions of collection

Can the presupposition argument be resisted? There are two premises that can be challenged, namely (i) and (ii). These will be scrutinized in this section and Section 6, respectively.

[20]If presupposition is defined in terms of grounding, as outlined in the previous footnote, then these premises follow from analogous and widely held assumptions concerning grounding. See e.g. (Fine, 2012) and (Rosen, 2010, §10).

Premise (i), we recall, considers a concept F defined by comprehension on a condition $\phi(x)$ and asserts that Fa presupposes $\phi(a)$, for any a. Is this assertion true? The answer depends entirely on how we understand the generalization at the heart of the comprehension axiom, namely $\forall x(Fx \leftrightarrow \phi(x))$. Suppose the generalization is used merely to provide an extensionally correct characterization of the concept F. Then the assertion loses all plausibility. If there is one condition that provides an extensionally correct characterization of F, then there are many. Why, then, should Fa presuppose $\phi(a)$ rather than a's satisfaction of any of the alternative extensionally correct characterizations?

Things look very different when the generalization is understood as *individuating* the concept F or providing a *real* definition of it. On this understanding, the concept is explained as the concept that applies to an object a just in case a satisfies the defining condition $\phi(x)$. Thus, whenever the question arises whether the concept F applies to some object a, the concept is "unpacked" and gives way to its defining condition, which must accordingly be antecedently available. This view is particularly natural when concepts are understood as fine-grained intensional entities, structured in a way that mirrors the syntactic structure of their defining conditions. It is important to notice, however, that this richly structured conception is not required by the view under discussion. It suffices to assume that each concept has at least one real definition, and that each such definition is available antecedently to the concept itself, such that Fa presupposes $\phi(a)$ whenever $\phi(x)$ provides a real definition of F. To see that this suffices, consider a concept F which has no extensionally correct predicative characterization. By the mentioned assumption, F has a real definition for which premise (i) is true. It thus follows that there is an impredicative definition of F for which (i) is true.

Consider now the analogue of premise (i) for pluralities. Suppose that xx are all and only the ϕ's:

$$\forall y(y \prec xx \leftrightarrow \phi(y))$$

The analogue of premise (i) is then:

(i′) $a \prec xx$ presupposes $\phi(a)$.

While I believe that many instances of (i) are true, I shall now argue that (i′) is always false, because a plurality is individuated in terms of its members, not by any membership condition. If my argument succeeds, it will show that the presupposition argument does not apply to pluralities and thus poses no threat to the legitimacy of impredicative *plural* comprehension.

What, then, is the relation between (i) and (i′)? Both theses concern the relative priority of a claim about what we may loosely think of as "membership" in a "collection" and another claim about an object's satisfying a "membership" condition. (I shall use the words "collection" and "membership" in a deliberately unspecific way.) Does a truth about membership presuppose the corresponding truth about satisfaction of the membership condition? The answer, I claim, depends on the type of collection in question. An intensionally individuated collection, such as a concept, can be tied to a membership condition in a way that an extensionally individuated collection, such as a plurality, never is. The reason is that a concept can be individuated by a membership condition, while a plurality cannot.

Let me develop this theme a bit further. My objection to (i′) is based on the intuitive idea that a plurality is nothing over and above its members, and that a plurality is thus fully specified by circumscribing the things that are its members. A plurality is, as I shall put it, a *pure extension*.[21] I propose to explicate this intuitive idea by means of three precise claims. First, I claim that the properties of a plurality are fully determined by its members. This claim is encapsulated in the following analogue of Leibniz's Law:

(Indisc) $$xx \equiv yy \rightarrow (\phi(xx) \leftrightarrow \phi(yy))$$

where $xx \equiv yy$ abbreviates $\forall u(u \prec xx \leftrightarrow u \prec yy)$.

The second claim arising from my explication of the idea of pluralities as pure extensions concerns the "modal profile" of pluralities. Since a plurality is nothing over and above its members, it consists of the very same members at every world at which the plurality exists. There is no material available that might underwrite a non-trivial tracking across possible worlds. All we have to go on are the members. By contrast, a group—such as a football team or a philosophy department—*is* something over and above its members. It has some principle of organization that enables the group to gain or lose members and to be tracked in a non-trivial way across possible worlds. For example, membership in a philosophy department changes over time and across possible worlds.

A third and final component of the intuitive idea of pluralities as pure extensions concerns a form of canonical specification by means of a (perhaps infinite) disjunction of identities. Assume, for example, that xx has two members a and b. Then xx are specified by means of

[21]See (Linnebo, 2016b) for a more detailed development of the view outlined in this paragraph and the next.

the disjunction $x = a \vee x = b$. Let us say that the specifying disjunction provides a *traversal* of the plurality. What about infinite pluralities? Provided that we allow infinitary disjunctions and are willing to introduce a name $\bar{a}$ for each object a, then infinite pluralities too can be assumed to have traversals of the following form:[22]

$$\forall x(x \prec xx \leftrightarrow \bigvee_{a \prec xx} x = \bar{a})$$

Of course, these traversals rest on substantial idealizations. But such idealizations are customary in modern mathematics. So we shall assume that every plurality has a traversal.

Equipped with this analysis of pluralities as pure extensions, let us return to premise (i$'$), which I claimed is implausible. There may well be an extensionally correct criterion for membership in a pure extension, but unlike an intension, a pure extension is never inherently tied to this criterion. Assume, for example, that xx are all and only the ϕ's. This provides no reason to think that $a \prec xx$ presupposes $\phi(a)$. After all, the things in question may also admit of other definitions, say as all and only the ψ's. There is no more of a reason to take $a \prec xx$ to presuppose $\phi(a)$ than $\psi(a)$. Since $\phi(a)$ and $\psi(a)$ may be entirely different truths, it is not an option to accept both presupposition claims. Consequently, we should accept neither. This means that all instances of premise (i$'$) must be rejected.[23]

Taking stock, I have argued that the presupposition argument can be resisted. Since premise (i$'$) is always false, the argument fails to support a ban on impredicative *plural* comprehension. In fact, this observation makes available an indirect defense of impredicative *second-order* comprehension as well: for any plurality xx can be used to give an unproblematic definition of a Fregean concept F by letting $\forall u(Fu \leftrightarrow u \prec xx)$.

5 When is the extensional conception available?

It is the pure extensionality of pluralities that enables us to reject (i$'$) and thus to resist the demand for predicativity restrictions. I shall now argue that the pure extensionality of pluralities motivates a different kind of restriction on plural comprehension. Thus, while the pure extensionality of pluralities allows us to resist the presupposition argument against

[22]Compare (Rumfitt, 2005).

[23]What, then, might $a \prec xx$ presuppose? To answer the question, consider a traversal of xx. This traversal must have a disjunct, say $x = b$, such that $a = b$. Then it is plausible to take $a \prec xx$ to presuppose whatever $a = b$ presupposes, if anything.

impredicative comprehension, there is a price to pay. The dominant view of plural logic accepts unrestricted plural comprehension. Provided there is at least one ϕ, it is assumed that there are some things that are all and only the ϕ's. I have challenged this assumption in earlier work.[24] So here I shall be brief and content myself with conveying the intuitive idea underlying my opposition to the dominant view.

Suppose we are given some collection of web pages and instructed to design a new web page that links to all and only the members of this collection. (Remember that we are using the word "collection" in a deliberately unspecific way!) For which collections can the instruction to be carried out? In particular, might we design a new web page to link to the collection of web pages that don't link to themselves? It is easy to see that the answer to such questions depends on how the target collection is understood. Suppose the target is specified intensionally by the concept *web page that doesn't link to itself.* Then it is logically impossible to design a web page that links to all and only members of that collection. For the new page would have to link to itself just in case it does not link to itself. Suppose instead that the target collection is specified in a purely extensional manner as the plurality of each and every web page that in fact doesn't link to itself. Then there is no logical or conceptual obstacle to designing a new web page that links to all and only the members of this plurality. Since pluralities have a rigid modal profile, a plural target stays fixed when we consider counterfactual circumstances with more web pages, unlike a conceptual target, whose extension shifts with the circumstances. It is this fixity of the plural target that makes it possible to reach it.

My challenge to the now-dominant plural logic can now be explained by comparing sets with web pages. According to the influential iterative conception, sets are formed successively.[25] We start with zero or more non-sets. Then we use every "collection" of these objects to form a set, namely the set with precisely these objects as its members. Now more objects are available. So we repeat the operation of using every collection of available objects to form a set. We keep going in this way, *ad infinitum.* A question now arises which is analogous to the one we considered in connection with web pages. Which collections of objects can be used to form sets? Consider for example the collection of objects that are not elements of themselves. Can we form a set whose elements are the members of this collection? Again, the answer to such questions depends on how the target collection is specified. When the

[24] See (Linnebo, 2010) and (Linnebo, 2013), as well as (Yablo, 1993) and (Hossack, 2014) for kindred views.
[25] (Boolos, 1971) and (Parsons, 1977).

target is specified intensionally, by means of the condition $x \notin x$, the characterization of the desired set is logically incoherent. The desired set r would be such that $\forall x(x \in r \leftrightarrow x \notin x)$, and it is a theorem of first-order logic that this desire cannot be satisfied. Suppose instead that the target collection is specified in a purely extensional manner as the plurality of sets that are in fact not elements of themselves. Then there is no logical or conceptual obstacle to the formation of a set whose elements are precisely the members of this collection. All that the paradoxical reasoning shows is that the set has to lie outside of the target collection. But since this collection is specified in a purely extensional way as a plurality, there is no logical or conceptual obstacle to the formation of new objects outside of it.[26]

If I am right, then every plurality defines a set. Then, on pain of paradox, there must be many conditions that fail to define a plurality. The condition '$x \notin x$' provides an example. If this condition defined a plurality, then this plurality would form a set, which would lead to a contradiction. The same goes for a variety of other conditions as well, such as '$x = x$' or 'x is a set'. More generally, every condition that defines a plurality also defines a set. Since we know that many conditions don't define sets, it follows that many conditions don't define a plurality. So the plural comprehension scheme must be restricted. More specifically, we need *size restrictions* on plural comprehension, analogous to those that the iterative conception imposes on set existence. This means that it is harder to define a plurality than one naively might have thought. Although initially surprising, the resulting view has some attractive features. The only "collections" one could reasonably expect to form sets are the ones that are specified in a purely extensional way by means of a plurality. And all of these "collections" *do* form sets. Moreover, the view solves many instances of the paradox of reification. By motivating a restriction on the plural comprehension scheme, the view removes any obstacle to the reification of pluralities, whether as sets or in any other way.

Let us take a step back and reflect on what has been achieved in this section and the previous. We began by observing that the presupposition argument against impredicative comprehension can be resisted in the case of plurals. Since a plurality is a pure extension, it is not inherently tied to any membership condition in the way that intensional entities such as concepts are. This means that the plural analogue of premise (i) is always false. However, we have just seen that there is a price to pay. The pure extensionality of plural-

[26] This view of sets traces its roots back to Cantor, whose distinction between "consistent" and "inconsistent multiplicities" mirrors mine between purely extensional pluralities and intensional Fregean concepts. See (Linnebo, 2013) for discussion.

ities has the surprising consequence that the ordinary plural comprehension scheme needs to be restricted—based on considerations of size rather than predicativity. On the positive side, this restriction enables unrestricted reification of pluralities. Thus, by reflecting on the presupposition argument, we have arrived at a response to a large class of instances of the paradox of reification, just as we had hoped.

6 Two conceptions of generality

I turn now to second-order logic. Since concepts are intensional entities, the above considerations about pure extensionality do not apply. This observation has both good and bad effects. On the positive side, it removes any reason to impose size restrictions on second-order comprehension. The concept of being self-identical, for example, applies to absolutely everything, including things to be introduced later in the process of forming sets or other mathematical objects. On the negative side, the intensional nature of concepts deprives us of the response to (the plural analogue of) premise (i) that was developed in Section 4. We therefore need to investigate the other potentially controversial premise of the presupposition argument, namely (ii), which states that a universal generalization presupposes each of its instances. Is this premise true? I shall argue that the answer depends on how the generalization is understood. We need to distinguish between an *instance-based* and a *generic* conception of generality. While (ii) holds for the former, it fails for the latter.

Consider a true generalization of the form $\forall x\, \phi(x)$. What explains its truth? That is, in virtue of what is the generalization true? (This is a metaphysical question, not an epistemic one.) On the instance-based conception, the truth is explained (in this metaphysical sense) by each of its instances: $\phi(a)$, $\phi(b)$, and so on through the entire domain of quantification.[27] This analysis requires that the domain be traversable, in the sense of Section 4; that is, that it make sense to run through the domain, conjoining all of the resulting instances of the matrix $\phi(x)$. And when a domain is traversable, we can use this traversal to show that the domain can be given as a plurality. So suppose that the domain is specified as an plurality xx. On the instance-based conception, the generalization $\forall x\, \phi(x)$ over this domain is regarded as

[27] Compare (Fine, 2012, §7), and (Rosen, 2010, §8c), who agree that each instance is a partial ground of the generalization but make the further claim that all the instances, taken together, fail to provide a full ground: we additionally need a "totality fact" that ensures that a, b, etc. are all the objects there are. On what I am calling "the instance-based conception of generality", there is no need for any such totality fact; for as we shall see, the domain is given by a plurality, which essentially consists of a, b, etc. and no further objects. By contrast, we shall see that totality facts are required on the generic conception of generality.

equivalent to the conjunction $\bigwedge_{a \prec xx} \phi(\bar{a})$.

This conception of generality bears on premise (ii), which concerns the presuppositions of universal generalizations. This premise now follows from the uncontroversial assumption that a true conjunction presupposes each of its conjuncts. Thus, on the instance-based conception of generality, premise (ii) enjoys strong support. It is important to realize that this observation does not revive the presupposition argument, however. For the instance-based conception of generality requires that the domain be specified as a plurality, which in turn—as argued in Section 4—enables us to resist (the plural analogue of) premise (i). The following table summarizes our conclusions thus far:

type of domain	premise (i)	premise (ii)
pure extension	✗	✓

where '✗' and '✓' mean rejected and accepted, respectively.

While the instance-based conception of generality is familiar and well understood, its obvious drawback is its limited scope. As noted, the conception is only available when the domain can be specified as a plurality. Let us therefore investigate the alternative generic conception. Consider the following true universal generalizations:

Every atom of gold consists of 79 protons.

Every red object is colored.

Everything is self-identical.

These truths can be explained without citing any individual instance or mentioning any particular atoms, red objects, or things. The generalizations are true, not in virtue of their instances, but in virtue of what it is to be red, colored, a thing, and self-identical.[28]

Similar ideas arise in the philosophy of mathematics. A passage from Hermann Weyl provides a wonderful example. Weyl is interested in the idea that the natural numbers are potentially, but not actually, infinite. That is, however many natural numbers have been generated—say by producing a sequence of Hilbert strokes that represent numbers—it is always possible to generate more. One more Hilbert stroke can always be added. As

[28]This view is endorsed by others as well. For example, (Hale, 2013, §6.4.2) argues that universal generalizations over the natural numbers can "be taken to be true in virtue of a single nature, viz. the nature of the natural numbers in general", not in virtue of the infinitely many natural numbers. See also (Rosen, 2010, §8a–b).

Weyl realizes, this conception of the natural numbers has important consequences for our understanding of numerical generalizations. Consider the question of whether there is a natural number that has some decidable property P. Weyl writes:

> Only the finding *that has actually occurred* of a determinate number with the property P can give a justification for the answer 'Yes', and—since I cannot run a test through all numbers—only the insight, that it lies in the essence of number to have the property not-P, can give a justification for the answer 'No'; Even for God no other ground for decision is available.

As is well known, Brouwer and other intuitionists have sought to understand the truth of universal generalizations over the numbers in terms of our having produced an acceptable proof. This results in a radical and controversial form of anti-realism, which equates truth with our possession of a proof. But does an advocate of potential infinity have to follow Brouwer down this problematic path?

The passage from Weyl suggests an alternative.[29] A universal generalization over the natural numbers is true, not because we possess a proof, but because "it lies in the essence of number" to have the relevant property. This is a brilliant proposal. If the natural numbers are merely potentially infinite, it is impossible to complete their generation. This means that universal generalizations over the natural numbers do not admit of an instance-based explanation: for there can be no stage at which all of these instances are available. The generic conception of generality provides the needed alternative—without any reliance on the problematic and radical anti-realism. A generalization can be made true, not by its instances, but by general facts involving the concepts that figure in the generalization.

Of course, most theorists now accept that the natural numbers can be completed. But analogous considerations arise with the iterative conception of sets. This conception allows the natural numbers to be completed but not the hierarchy of sets: for however many sets have been formed, it is possible to go on and form more. So there is no stage at which all sets are available, which means that a universal generalization over sets cannot be given an instance-based explanation. So any such generalization must be given a generic explanation.

As should by now be clear, the generic conception of generality points to a second strategy for countering the presupposition argument. A generic explanation does not proceed via

[29] Though as mentioned in the previous footnote, Hale has formulated the same idea, as far as I can tell independently of Weyl.

the instances of the target generalization. So when a generalization admits of a generic explanation, it does not presuppose any of its instances, thus undermining premise (ii).[30] In fact, this second strategy promises to be more robust than the first, since it does not require that the domain can be specified in a purely extensional way as a plurality. In order to redeem the promise, I have recently explored the generic conception of generality.[31]

The single most important observation about the generic conception is that it makes universal generalizations very strong. Consider a domain—such as that of all sets—which can only be specified in an intensional way, not as a plurality. On the generic conception, a universal generalization over this domain applies not only to the objects available at some particular stage of the process of forming ever more sets but also to all objects that this process *might ever produce*. By contrast, an existential generalization must, in order to be true at some particular stage of the process, have a witness that is *available at this stage*. Thus, while a universal generalization is concerned with everything we might come to introduce into our domain, an existential generalization is only concerned with what has already been introduced. This "asymmetry of concern" means that the two quantifiers are not dual to one another, as they are on the instance-based conception and in the usual Tarskian semantics for classical logic. Since the universal quantifier is very strong, its dual is correspondingly weak; in particular, it is weaker than the existential quantifier. I show that this non-dual interpretation of the two quantifiers validates intuitionistic but not classical logic. It is particularly noteworthy that this observation turns entirely on structural considerations and thus remains valid when the talk about sets being "formed" in "stages" is understood as picturesque shorthand for claims about metaphysical dependencies.[32] So this talk need not be understood literally, as in the case of traditional constructivism.

It also merits mention that the departure from classical logic is very modest. After all, the instance-based conception of generality remains available for any domain specified as a plurality, and this conception poses no challenge to classical logic. So provided that classical logic is valid for each instance of the form $\phi(a)$, it is also valid for each instance-based generalization

[30]This second strategy is also relevant to the paradoxes of ground discussed in (Fine, 2010). As Fine observes, these paradoxes rely on impredicative definitions of facts and propositions. Are the generalizations involved in these definitions understood in the instance-based or generic way? If the former, a universal generalization is (partially) grounded in each of its instances (as assumed in the more stubborn paradoxes), while the impredicative definitions are problematic—for reasons much like those identified in our presupposition argument from Section 3. If the latter, the concern about the impredicative definitions dissipates, while instead the claim about the grounding of universal generalizations loses its motivation.

[31]See (Linnebo, 2016a).

[32]See (Parsons, 1977) for this way of explicating the iterative conception.

of the form $\bigwedge_{a \prec xx} \phi(\bar{a})$. It follows that classical logic remains valid for any domain specified as a plurality. In fact, even when the domain cannot be so specified, we can show that quantification restricted to any plurality behaves classically, in the following sense. Suppose we work in intuitionistic logic. As usual, we define the restricted quantifier '$\forall x \prec yy$' by letting '$(\forall x \prec yy)$ $\phi(x)$' abbreviate '$\forall x(x \prec yy \to \phi(x))$'. Next, we say that a formula $\phi(x)$ (which may have further free variables) is *decidable with respect to* x just in case $\forall x(\phi(x) \vee \neg\phi(x))$. This ensures that the formula behaves classically with respect to the argument x. When I claim that quantification restricted to a plurality "behaves classically", I mean that this form of quantification preserves classical behavior. Assuming that $\phi(x)$ behaves classically—in the sense described—then '$(\forall x \prec yy)\phi(x)$' too behaves classically—in the sense of being decidable. This validates the following logical principle:[33]

$$\forall x(\phi(x) \vee \neg\phi(x)) \;\to\; (\forall x \prec xx)\phi(x) \vee \neg(\forall x \prec xx)\phi(x)$$

The result is a very minor retreat from classical logic, known as "semi-intuitionistic" logic.

Once again, we have seen that a premise of the presupposition argument can be resisted, but only at the price of revising another aspect of the now-dominant higher-order logic—in this case, the applicability of classical logic to quantification over absolutely everything. The following table (which extends the previous) summarizes our conclusions:

type of domain	premise (i)	premise (ii)
pure extension	✗	✓
intension	✓	✗

Provided we restrict ourselves to domains that can be specified in a purely extensional way as a plurality, we can reject premise (i), while retaining classical logic by means of the instance-based conception of generality, which means upholding premise (ii). Other times, however, we wish to consider domains that can only be specified in an intensional way, say when giving

[33] *Proof sketch.* The traversability of any plurality enables us to rewrite (6) as:

$$\bigwedge_{a \prec xx} \phi(\bar{a}) \vee \neg \bigwedge_{a \prec xx} \phi(\bar{a})$$

And this formula is an infinitary intuitionistic consequence of the decidability of ϕ with respect to x, as can be seen by using the decidability of ϕ with respect to x to prove $\big(\phi(\bar{a}) \vee \neg\phi(\bar{a})\big) \wedge \big(\phi(\bar{b}) \vee \neg\phi(\bar{b})\big) \wedge \ldots$, which in turn implies $\big(\phi(\bar{a}) \wedge \phi(\bar{b}) \wedge \ldots\big) \vee \big(\neg\phi(\bar{a}) \vee \neg\phi(\bar{b}) \vee \ldots\big)$.

our quantifiers absolutely general range. Then (i) is inescapable, but we can reject (ii)—at the price of letting the logic become semi-intuitionistic.

7 How to balance comprehension and reification

I wish to end by returning to the hope that an analysis of the presupposition argument may suggest a response to the the paradox of reification, including Hale's inconsistent triad. So we need to ask whether my view provides an appropriate way to balance comprehension and reification. Let us consider the mentioned forms of logic in turn.

The case of plural logic was discussed in Sections 4 and 5, where I defended the permissibility of classical reasoning—that is, classical logic and full impredicative comprehension—on any domain specified by a plurality. As classical mathematics demonstrates, such reasoning is both intuitive and tremendously powerful when applied to infinite domains. Moreover, we observed that this entitlement to classical reasoning is compatible with unrestricted reification of pluralities. The only price is the imposition of size restrictions on plural comprehension. For most ordinary mathematical or scientific purposes, this restriction represents no loss at all; in particular, the pluralities may be as large as any of the sets that are postulated by higher set theory. So in the case of plurals, we have a very satisfactory balance of comprehension and reification.

The case of second-order logic has not yet been properly discussed in this article, other than to observe that this logic is indispensable when theorizing about domains that can only be specified in an intensional way. Here too we need to balance the permissible forms of comprehension against the permissible forms of reification. This is a difficult task with which I have struggled for some time.[34] My most developed proposal to date combines what one may think of as a loosely Kripke-inspired groundedness requirement on second-order comprehension with unrestricted reification of second-order concepts.[35] This proposal faces a serious challenge, however. The approach uses a property application predicate η in a way that requires the formula '$x\ \eta\ y$' (read as "x instantiates the property y") to have a determinate truth-value for any two arguments—but without allowing comprehension on this formula. This seems to violate the minimalist conception of predicate reference, discussed in Section 2, which says that a formula's having a determinate truth-value for any arguments

[34] See (Linnebo, 2006) and (Linnebo, 2009), as well as the closely related theory of (Fine, 2005).
[35] See (Linnebo, 2009).

suffices for it to have a semantic value and thus also define a Fregean concept.

I have all along wanted to respond to this challenge as follows. On my proposal, there is no fixed interpretation of η; rather, we build up ever larger interpretations as new properties are generated. On each of these "transitional" interpretations, the formula '$x\ \eta\ y$' can indeed be used in second-order comprehension, precisely as required by the minimalist conception. But there is no "final" or absolute interpretation of η, and for this reason, there is no failure of second-order comprehension on a formula with a determinate interpretation.

There is a worry, however, that we are just shifting the bump in the carpet. How it is possible to generalize over absolutely everything—which my theory very much aspires to permit—given that this range includes properties not yet generated and on which the crucial predicate η therefore doesn't yet have a determinate interpretation? Thankfully, the generic conception of universal generality enables us to give a pleasing answer to this follow-up question. *It is permissible to generalize over objects not yet generated so long as these generalizations can be understood as on the generic conception.* This is not the place to attempt to work out the technical details. I shall content myself with remarking that this proposal illustrates the potential philosophical value and utility of the generic conception.[36]

References

Benacerraf, P. and Putnam, H., editors (1983). *Philosophy of Mathematics: Selected Readings*, Cambridge. Cambridge University Press. Second edition.

Boolos, G. (1971). The Iterative Conception of Set. *Journal of Philosophy*, 68:215–32. Reprinted in (Boolos, 1998).

Boolos, G. (1984). To Be is to Be a Value of a Variable (or to Be Some Values of Some Variables). *Journal of Philosophy*, 81(8):430–449. Reprinted in (Boolos, 1998).

Boolos, G. (1998). *Logic, Logic, and Logic*. Harvard University Press, Cambridge, MA.

Feferman, S. (2005). Predicativity. In Shapiro, S., editor, *Oxford Handbook of the Philosophy of Mathematics and Logic*, pages 590–624. Oxford University Press, Oxford.

Fine, K. (2005). Class and membership. *Journal of Philosophy*, 102(11):547–572.

Fine, K. (2010). Some puzzles of ground. *Notre Dame Journal of Formal Logic*, 51(1):97–118.

Fine, K. (2012). Guide to ground. In Correia, F. and Schnieder, B., editors, *Metaphysical Grounding*, pages 37–80. Cambridge University Press.

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Frege, G. (2013). *Basic Laws of Arithmetic.* Oxford University Press, Oxford. Translated by Philip A. Ebert and Marcus Rossberg.

Glanzberg, M. (2004). Quantification and Realism. *Philosophy and Phenomenological Report*, 69:541–72.

Goldfarb, W. (1989). Russell's reasons for ramification. In Savage, C. W. and Anderson, C. A., editors, *Rereading Russell: Essays on Bertrand Russell's Metaphysics and Epistemology*, pages 24–40. University of Minnesota Pres, Minneapolis.

Hale, B. (2013). *Necessary Beings: An Essay on Ontology, Modality, and the Relations Between Them.* Oxford University Press, Oxford.

Hale, B. and Linnebo, Ø. (2015). Ontological categories and the problem of expressibility. Unpublished manuscript.

Hale, B. and Wright, C. (2009). The Metaontology of Abstraction. In et al., D. C., editor, *Metametaphysics: New Essays on the Foundations of Ontology*, pages 178–212. Oxford University Press, Oxford.

Hossack, K. (2014). Sets and plural comprehension. *Journal of Philosophical Logic*, 43(2-3):517–539.

Jung, D. (1999). Russell, presupposition, and the vicious-circle principle. *Notre Dame Journal of Formal Logic*, 40(1):55–80.

Linnebo, Ø. (2006). Sets, properties, and unrestricted quantification. In Rayo, A. and Uzquiano, G., editors, *Absolute Generality*, pages 149–178. Oxford University Press, Oxford.

Linnebo, Ø. (2009). Bad company tamed. *Synthese*, 170(3):371–391.

Linnebo, Ø. (2010). Pluralities and sets. *Journal of Philosophy*, 107(3):144–164.

Linnebo, Ø. (2012). Plural Quantification. In *Stanford Encyclopedia of Philosophy*, available at URL = <http://plato.stanford.edu/archives/fall2012/entries/plural-quant/>.

Linnebo, Ø. (2013). The potential hierarchy of sets. *Review of Symbolic Logic*, 6(2):205–228.

Linnebo, Ø. (2016a). Generalizations explained: A truth-maker semantics. Unpublished MS.

Linnebo, Ø. (2016b). Plurals and modals. *Canadian Journal of Philosophy*, 46:654–76.

McGee, V. and Rayo, A. (2000). A puzzle about de rebus beliefs. *Analysis*, 60(4):297–299.

Parsons, C. (1977). What Is the Iterative Conception of Set? In Butts, R. and Hintikka, J., editors, *Logic, Foundations of Mathematics, and Computability Theory*, pages 335–367. Reidel, Dordrecht. Reprinted in (Benacerraf and Putnam, 1983) and (Parsons, 1983).

Parsons, C. (1983). *Mathematics in Philosophy.* Cornell University Press, Ithaca, NY.

Quine, W. (1986). *Philosophy of Logic.* Harvard University Press, Cambridge, MA, second edition.

Rosen, G. (2010). Metaphysical dependence: Grounding and reduction. In Hale, B. and Hoffmann, A., editors, *Modality: Metaphysics, Logic, and Epistemology*, pages 109–36. Oxford University Press.

Rumfitt, I. (2005). Plural Terms: Another Variety of Reference. In Bermudez, J. L., editor, *Thought, Reference and Experience*, pages 84–123. Clarendon, Oxford.

Russell, B. (1903). *Principles of Mathematics.* Norton, New York.

Russell, B. (1908). Mathematical logic as based on a theory of types. *American Journal of Mathematics*, 30:222–262.

Schnieder, B. (2011). A logic for 'because'. *Review of Symbolic Logic*, 4(3):445–465.

Williamson, T. (2003). Everything. In Hawthorne, J. and Zimmerman, D., editors, *Philosophical Perspectives 17: Language and Philosophical Linguistics.* Blackwell, Boston and Oxford.

Yablo, S. (1993). Paradox without self-reference. *Analysis*, 53(4):251–2.