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Halvor Mehlum

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Department of Economics
University of Oslo

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P. O.Box 1095 Blindern
N-0317 OSLO Norway
Telephone: + 47 22855127
Fax: + 47 22855035
Internet: http://www.sv.uio.no/econ
e-mail: econdep@econ.uio.no

In co-operation with
The Frisch Centre for Economic Research

Gaustadalleén 21
N-0371 OSLO Norway
Telephone: +47 22 95 88 20
Fax: +47 22 95 88 25
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A polar confidence curve applied to Fieller's ratios

Halvor Mehlum

Abstract

I derive the polar representation of Fieller's estimation of confidence sets for ratios and construct a polar plot of the test statistics for all angles associated with the ratios. This procedure helps in visualizing and clarifying, but also systematizing, the features of the Fieller solution. In conclusion I discuss, using Ramanujan, the case where Fieller's method yields a confidence set covering the entire real line.

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Keywords: Confidence curve, Fieller solution, test

1 Introduction

Fieller (1932,1940) derived the solution to the confidence sets for a ratio of the two means of a bivariate normal distributed variable. This confidence set may be a finite interval, two disjoint unbounded intervals or the non informative set given by the entire real line. The fact that the entire real line may be the confidence set also for α-levels strictly positive is puzzling and also led to debate (e.g. the symposium contributions by Fieller 1954 and Creasy 1954 and the discussions therein.) After Fieller, several subsequent works have all provided new insights and fresh perspectives. One recent example containing a generalization is Broda and Kan (2016). Another definitive reference is Koschat (1987) who is one of the many who confirms Fieller's argument and he concludes "[...] there is no procedure that gives bounded α-level confidence intervals with probability 1." and "[...] within a large class of solutions the Fieller solution is the only one that gives exact coverage probability for all parameters."(p.462). More recently, the insights from Fieller's method has also proved its relevance for a non-linear combination of regression coefficients (e.g Hirschberg et al. 2008) and in instrumental variable estimation (e.g Dufour 1997). With regards to the Fieller solution, Dufour concludes: "Accepting the possibility of an unbounded confidence set for a structural coefficient is simply a matter of logic and scientific rigour: the data may simply be uninformative about such coefficients." (p.1383.) The main contribution of the paper,is the introduction of a polar confidence curve. In particular I consider the angle associated with Fieller's ratio and present the problem in polar coordinates. The test statistics associated with the confidence ellipses of the two regression coefficients is then offset directly onto a polar confidence curve for the angle associated with the ratio.[1] A confidence curve in Cartesian coordinates is an idea that goes back to Birnbaum (1961)[2] and confidence curves in Cartesian coordinates appear regularly in the literature and are used by Blaker and Spjøtvoll (2000) in the case of Fieller's solution.[3] Confidence curves in *polar* coordinates is, I believe, novel in the literature. With such a polar confidence curve, the discontinuities of the Fieller solution disappear and the puzzling features become more intuitive.

2 Estimating confidence intervals for a ratio

2.1 The problem

Consider the regression equation

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \epsilon_i, \quad i = 1 \dots N \tag{1}$$

[1]A translation of the Fieller ratio into polar coordinates is also done in James et al (1974), Koschat (1987), Guiard, V. (1989), Von Luxburg and Franz (2009) and in Hirschberg and Lye (2010). These authors do not, however, derive the corresponding confidence curve.

[2]But where he plots α-levels I plot test statistics.

[3]Four different confidence curves with corresponding confidence intervals (the blue lines) are shown in Figure 3.

When the variables satisfy the standard assumptions, OLS can be used to arrive at estimates for the slope parameters $\hat{\beta}_1$ and $\hat{\beta}_2$. Using the estimated residuals the variance of the error term s^2 can also be estimated. In turn, yielding estimates for the variances, V_1 and V_2, and the covariance, CV, of $\hat{\beta}_1$ and $\hat{\beta}_2$.

In order to create a confidence interval for $\gamma = \beta_2/\beta_1$, Fieller's solution starts from the linear constraint $\beta_2 = \gamma\beta_1$ versus the alternative $\beta_2 \neq \gamma\beta_1$. The test statistic for this test is

$$t = \frac{\hat{\beta}_2 - \gamma\hat{\beta}_1}{\sqrt{V_1\gamma^2 + V_2 - 2CV\gamma}} \tag{2}$$

which is t-distributed with $df = N - 3$ degrees of freedom. An alternative but equivalent approach starts from the confidence ellipse for a test of a joint hypothesis for β_1 and β_2. Such a test yields the F-statistic

$$F = \frac{1}{(V_1V_2 - CV^2)}\left(V_2(\hat{\beta}_1 - \beta_1)^2 + V_1(\hat{\beta}_2 - \beta_2)^2 - 2CV(\hat{\beta}_1 - \beta_1)(\hat{\beta}_2 - \beta_2)\right) \geq 0 \tag{3}$$

with 2 and df degrees of freedom. For a given value of $F = F_0$, the quadratic form (3) describes an ellipse with centre in $(\hat{\beta}_1, \hat{\beta}_2)$. All points inside this ellipse are associated with $F < F_0$. Hence, at the confidence level corresponding to a particular $F = F_\alpha$ all (β_1, β_2)- hypotheses outside the $F = F_\alpha$ ellipse can be rejected. In the following its informative to consider the square root of F.

$$f(\hat{\beta}_1 - \beta_1, \hat{\beta}_2 - \beta_2) \equiv \sqrt{F} \tag{4}$$

which can be called the $f-$*distance* from the point $(\hat{\beta}_1, \hat{\beta}_2)$ to a point (β_1, β_2). The $f-$distance is in effect the Mahalanobis distance from $(\hat{\beta}_1, \hat{\beta}_2)$.[4] The $f-$distance is homogeneous of degree 1 in its arguments. Hence, along a given ray starting in $(\hat{\beta}_1, \hat{\beta}_2)$ the Cartesian distance to the ellipse with value f is proportional to f.

In order to use f to test an hypothesis about $\gamma = \beta_2/\beta_1$, f should first be minimized with respect to β_1, β_2 under the constraint $\beta_2 = \gamma\beta_1$. This minimization is equivalent to minimizing the Lagrangean

$$L = f(\hat{\beta}_1 - \beta_1, \hat{\beta}_2 - \beta_2) - \lambda\left(\gamma(\hat{\beta}_1 - \beta_1) - (\hat{\beta}_2 - \beta_2) - \gamma\hat{\beta}_1 + \hat{\beta}_2\right) \tag{5}$$

with respect to $(\hat{\beta}_1 - \beta_1)$ and $(\hat{\beta}_2 - \beta_2)$. The solution is straightforward and the resulting f is equal to

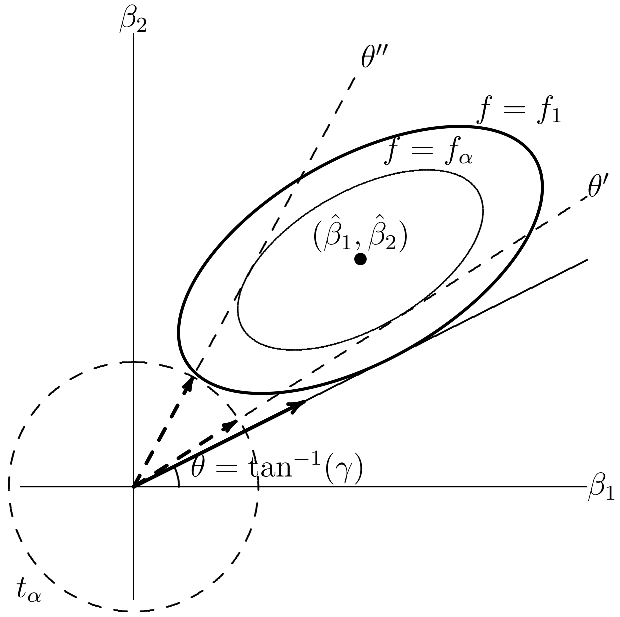
$$f = \frac{\hat{\beta}_1\gamma - \hat{\beta}_2}{\sqrt{V_1\gamma^2 + V_2 - 2CV\gamma}} = t \tag{6}$$

Hence, the constrained minimization of $f = \sqrt{F}$ corresponds to t-value derived in (2) . Thus the t-value is the minimized f-distance between the line $\beta_2 = \gamma\beta_1$ and the point $(\hat{\beta}_1, \hat{\beta}_2)$.

[4]When using the unconditional estimates V_1, V_2, and CV.

2.2 The polar confidence curve

Figure 1: Construction of polar confidence curve.



The geometry of this result is shown in Figure 1, where the slope of the solid line is γ. It shows that f_1 is the minimal f-value for the given γ. The figure also illustrates that the slope γ is associated with an angle θ. The polar confidence curve is constructed by offsetting the t-values as the radius for each θ. For the particular θ in the figure the length of the arrow reflects this t-value. The longer the arrow, the higher the t-value. Using the same procedure, such polar representation of the t-value can be constructed for all angles $\theta \in [0, 2\pi]$. A confidence set for θ should then contain all angles who's radius is less than a chosen critical cut of value t_α. The figure contains a dashed circle reflecting the t_α following from a chosen significance level α. The figure shows that the corresponding confidence set for θ is the interval $[\theta', \theta'']$. Both the dashed rays are tangent to $f = f_\alpha = t_\alpha$. The radius for both θ' and for θ'' are therefore exactly equal to t_α

A further question is for what values of γ the radius t reaches its extreme values. The minimum for the absolute value of f is zero corresponding to the numerator in (6) being zero. The maximum is found by differentiation yielding the other extreme point for γ

$$\gamma = \frac{\hat\beta_2}{\hat\beta_1} \text{ and } \gamma = -\frac{V_2\hat\beta_1 - CV\hat\beta_2}{V_1\hat\beta_2 - CV\hat\beta_1} \tag{7}$$

The first solution obviously is the slope given by the point estimate itself yielding a minimum point with $t = 0$. Inserting the maximum point in f yields

$$f = \sqrt{\frac{V_2\hat\beta_1^2 + V_1\hat\beta_2^2 - 2CV\hat\beta_1\hat\beta_2}{(V_1V_2 - CV^2)}} = f(\hat\beta_1, \hat\beta_2) = f_0 \tag{8}$$

Therefore f_0 is the largest t-value for any γ. It corresponds to the square root of the basic F statistics for joint significance of the two parameters (H_0: $\beta_1 = \beta_2 = 0$ vs the complement of H_0). The associated value of γ corresponds to the slope of the f-contour passing through the origin.[5] The result that f (and t) has a finite maximum value leads to the case of Fieller's non informative confidence set. Already at the α-level corresponding to $t = f_0$ all hypotheses about γ has to be accepted. At a sufficiently strict significance level the data does not contain sufficient information to reject *any* hypothesis about γ. When not able to reject H_0: $\beta_1 = \beta_2 = 0$ no real number $\gamma = \beta_2/\beta_1$ can be rejected either. This basic insight was exactly what the Dufour citation in the introduction reflected.

This property is contrary to the standard property in hypothesis testing where, at any strictly positive α-level, it's generally possible to formulate an hypothesis so extreme that it will be rejected. In the case of a ratio it is simply not possible to formulate an hypothesis that is more extreme than the hypothesis that both the numerator and denominator are zero.

In addition to producing the entire real line as a confidence interval for γ, Fieller's solution may generate closed intervals and disjoint unbounded intervals. The intervals are generally never symmetric around the point estimate $\hat{\beta}_2/\hat{\beta}_1$. These results follow naturally when mapping the confidence interval for the angle θ onto $\gamma = \tan(\theta)$. The disjoint property follows when $\pi/2$ or $-\pi/2$ is inside the confidence interval, while the asymmetry is a result of the many non-linear transformations when first going from the f-contours to the t-values and then from angles to tangents. Hence, several of the non-standard (and to some non-intuitive) properties are simply a result of mapping a closed circle of angles θ onto a infinite line of real numbers γ. The polar confidence curve is a direct visual way to appreciate the qualitative properties of the first step of these mappings and transformations.

3 Discussion

Based on the analysis above, Figure 2 contains four archetypal configurations showing the relationship between the f-contours and the confidence curve as per construction described above. In each of the panels, the $f = f_0$ contour is centred around the estimate $(\hat{\beta}_1, \hat{\beta}_2)$. In the first panel neither the major nor the major axis of the quadratic form goes through the origin. In the second and third panel the minor and major axis respectively go through the origin. In the fourth the contour is a circle and hence it has no axes. In each panel, the resulting confidence curves are the pairs of ovals contained within the radius of $t = f_0$.

[5] Using F and taking the differential yields

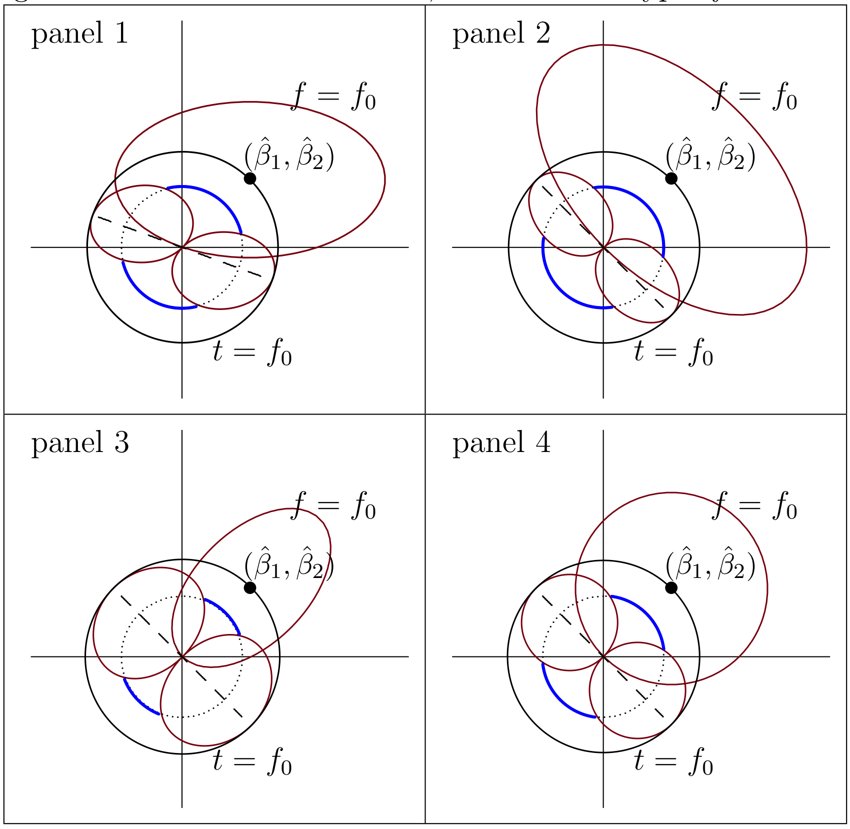
$$-2V_2(\hat{\beta}_1 - \beta_1)\mathrm{d}\beta_1 - 2V_1(\hat{\beta}_2 - \beta_2)\mathrm{d}\beta_2 + 2CV(\hat{\beta}_2 - \beta_2)\mathrm{d}\beta_1 + 2CV(\hat{\beta}_1 - \beta_1)\mathrm{d}\beta_2 = 0 \tag{9}$$

inserting for $\beta_1 = \beta_2 = 0$ yields

$$-V_2\hat{\beta}_1\mathrm{d}\beta_1 - V_1\hat{\beta}_2\mathrm{d}\beta_2 + CV\hat{\beta}_2\mathrm{d}\beta_1 + CV\hat{\beta}_1\mathrm{d}\beta_2 = 0 \tag{10}$$

$$\frac{\mathrm{d}\beta_2}{\mathrm{d}\beta_1} = -\frac{V_2\hat{\beta}_1 - CV\hat{\beta}_2}{V_1\hat{\beta}_2 - CV\hat{\beta}_1} \tag{11}$$

Figure 2: Polar confidence curve, for four archetypal f-contours



These four cases spans out the possibilities of features and all other configurations share the qualitative features displayed, or a mix of features. First, the displayed relationships are invariant to rotating the axes.[6] Second, as each polar plot is normalized with respect to f_0 the displayed relationships are invariant to the actual size of the estimates, their variances and their covariance.[7] Third, having the major axis increasing relative to the minor axis accentuates the displayed features. Fourth, having the major axis decreasing relative to the minor axis brings the features closer to those of the circular case.

The archetypal features can be summarized as follows.

- The polar confidence plot inherits the general shape of the f-contour.[8].
- The maximal radius of the polar confidence plot coincides with the tangent of $f = f_0$ through the origin.
- When the major axis goes through (or close to) the origin (panel 3) a large fraction of angles will have a t-value close to f_0.

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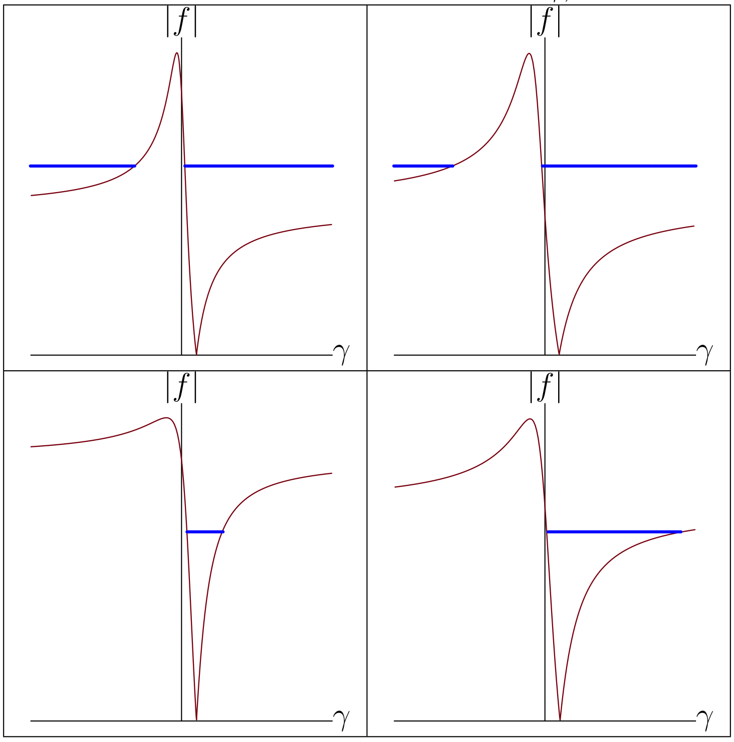
[6] The actual position of the axis only matters when taking the tangent.

[7] The actual estimates will matter for how close to the origin the t_α circle is located.

[8] This is only exactly true in the case of a circle as in panel 4. In Appendix A this result is shown to reflect Thales' semi circle theorem

Figure 3: Confidence curves and intervals for ratio γ, derived from Figure 2



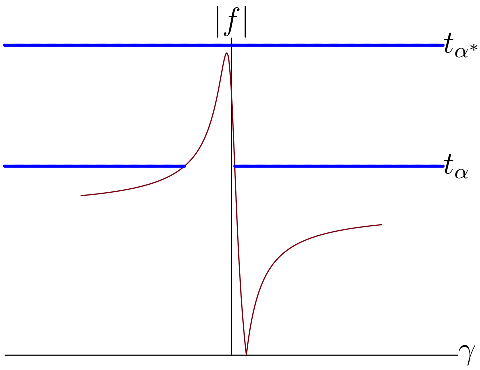
- When the minor axis goes through (or close to) the origin (panel 2) a only a minor fraction of angles will have a t-value close to f_0.

All these features has consequences for the confidence intervals for the angle. In order to compare across panels assume that the value f_0 is the same in all panels. Then the dotted circle reflect a chosen common t_α. The confidence interval for θ, as indicated by the blue circle segments, covers the angles with t-value inside t_α. Panel 2 has the longest interval, panel 3 the shortest, followed by panel 4, with panel 1 thereafter. Panel 1 has an asymmetric interval and all the others have symmetric ones.[9]

The disjoint feature of Fieller's solution follows only after the transformation into ratios. Then, as panel 1 and panel 2 has $\pi/2$ within the confidence interval for θ, they will both deliver disjoint intervals for γ of the sort $\{\gamma \in \mathrm{R} | \gamma \notin [\gamma_1, \gamma_2]\}$, where $[\gamma_1, \gamma_2]$ is the (in this case) closed set of ratios associated with $|f| = t > t_\alpha$. This is shown in Figure 3 where the confidence curves with respect to $\gamma = \tan(\theta)$ are shown in cartesian coordinates. These are the traditional confidence curves presented in the literature. Each of the four panels in Figure 3 corresponds the ones in Figure 2 and the blue lines are the confidence intervals. Panel 1 and 2 exhibits disjoint intervals, with the blue lines extending both to minus and plus infinity. Finally, if t_α was chosen to be larger than f_0, all angles would be inside the confidence set. Then, by transformation into ratio the result would the entire real line. This is shown in Figure 4 where the left panel show the derived interval when using the same t_α as in Figure 3 while the right panel shows an interval covering the entire real line when using $t_{\alpha^*} > f_0$.

[9]A symmetry with respect to θ. This vanishes after transforming to ratios.

Figure 4: Confidence interval covering real line when $t_{\alpha^*} > f_0$



4 Concluding remarks inspired by Ramanujan.

It might be disappointing to get an uninformative confidence set but then it may be comforting to know that the result is a glimpse of Ramanujans theory of reality and pursuit after God. The Fieller solution speaks directly to one of Ramanujan's favorite themes: What is "0/0"? One of his students (Sastri 1960) tells the following story

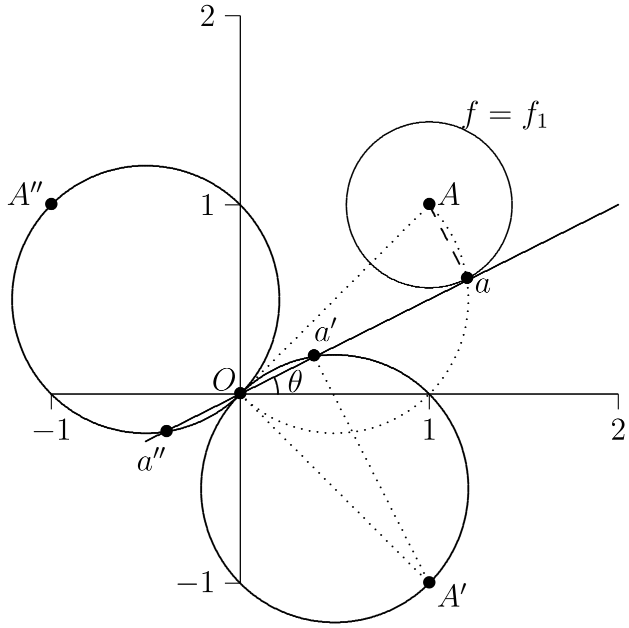
> The one notable feature about Ramanujan was that his pursuit of mathematics was a pursuit after God. He very often used to say that in Mathematics alone, one can have a concrete realisation of God. " $\frac{0}{0}$ ", he used to ask, "what is its value ?" His answer was: It may be anything. The zero of the numerator may be several times the zero of the denominator and vice versa. The value cannot be determined. (p.92)

One interpretation is that Ramanujan saw " 0/0 " as encompassing the limiting value of the slope of all rays starting in the origin, as their lengths tended to zero. His speculations went way beyond mathematical limits, however, and the reference to God in relation to zero is more elaborate in the following memoir from 1913 by his friend in Cambridge the statistician P.C Mahalanobis [10](as referred in S. R. Ranganathan, 1967)

> He was eager to work out a theory of reality which would be based on the fundamental concepts of "zero", "infinity" and the set of finite numbers. I used to follow in a general way but I never clearly understood what he had in mind. He sometimes spoke of "zero" as the symbol of the absolute (Nirguna-Brahman) of the extreme monistic school of Hindu philosophy, that is, the reality to which no qualities can be attributed, which cannot be defined or described by words, and which is completely beyond the reach of the human mind. According to Ramanujan, the appropriate symbol was the number "zero", which is the absolute negation of all attributes. He looked on the number "infinity" as the totality of all possibilities, which was capable of becoming manifest in reality

[10] In fact the same Mahalanobis as in the distance used above.

Figure 5: Thales' semi circle theorem with circular f-contours.



> and which was inexhaustible. According to Ramanujan, the product of infinity and zero would supply the whole set of finite numbers.(p.82)

Ramanujan had apparently no problem with a calculation delivering the entire real line, rather the contrary. So those accepting Fieller's solution are in good company and they may in fact get spiritual inspiration.

Appendix: Polar confidence curve using Thales' semi circle theorem.

When the estimated covariance matrix is $\mathrm{I}_2 s^2$, the f-contours will be circular. In that case the f-value is everywhere proportional to the Cartesian distance to the centre $(\hat{\beta}_1, \hat{\beta}_2)$. Such a case is illustrated in Figure 3, where $(\hat{\beta}_1, \hat{\beta}_2) = (1, 1)$ is the estimate. Thales' theorem states that any triangle, with the diameter of a semi circle as the hypotenuse and one corner at the circumference, will have a right angle. Hence, when constructing the unique norm to (1,1) from the θ-ray through the origin the norm will start where the ray intersect the dotted semi circle at a. Moreover, the length of the norm will be proportional to the value f_1. When, offsetting the length of the norm as the radius along the ray, the radius is ending in a'. It follows that triangle $A'Oa'$ is congruent to OAa and to $A''Oa''$. Therefore, the polar confidence curve will be two circles both with a tangent in O going through A and the diameter starting in O ending in A' and A'' respectively.

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