Mechanics of columns in sway frames – Derivation of key characteristics

Jostein Hellesland

Professor, Mechanics Division, Department of Mathematics,
University of Oslo, P. O. Box 1053 – Blindern, NO-0316 Oslo, Norway

ABSTRACT

Columns in sway frames may be divided into laterally "supporting (bracing) sway columns", which can resist applied lateral shears, and "supported (braced) sway columns", which must have negative shears from the rest of the frame to support them. The supported sway column acts like a braced column with an initial displacement of one end relative to the other. The maximum moment will occur at the end of a supporting sway column, but may occur between the ends of a supported sway column. Mechanics of such columns, single or part of panels, are studied using elastic second-order theory. The main objective is to establish simple, explicit expressions for characteristic points in the moment versus axial load space. Such results may be useful in teaching, in design practice to assist in the assessment of the often complicated column response, and also useful as a supplement to full, second-order structural analyses.

KEYWORDS

Buckling, columns (supports), design, elasticity, frames, stability, second-order analysis, structural engineering

1

1 Introduction

Design procedures for slender concrete and steel columns are most often based on elastic analyses. Studies of elastic behaviour of individual columns are of interest for several reasons: Elastic mechanics serve to demonstrate many aspects of the mechanics of inelastic behaviour; The complexity of exact second-order analytical methods often obscures the understanding of the mechanics of individual subassemblages or members; Design procedures in structural codes for slender columns are most often based on elastic concepts.

This paper examines columns in elastic frames that are able to displace laterally under load. Columns in such frames with sideways displacements (sidesway), may be divided into laterally "supporting sway columns" ("bracing columns", "leaned-to columns"), which can resist applied lateral shears, and "supported sway columns" ("braced", "leaning columns"), which must have negative shears from the rest of the frame to support them. A special case of the latter is the column pinned at both ends. In cases without transverse loading, the maximum moment will always occur at the end of a supporting sway column in a frame, but may occur between the ends of a supported sway column. Supported sway columns act like braced columns with an imposed relative end displacement (of one end relative to the other). Whether a specific restrained column is to be classified as one or the other, depends on the column's flexural stiffness and axial load.

The major objective of the paper is to study the mechanics of the response of rotationally restrained columns with sidesway, over the full range of axial loads, thus considering both supporting and supported sway columns. A one-level frame, or a storey of a multistorey frame, may have both kinds of columns on the same level. There is a vast literature covering various aspects of this theme. In this paper, the emphasis is on identifying and establishing simple, novel closed form expressions defining characteristic points, or behavioural "landmarks", in the axial load-moment solution space. Apart from being of importance for the general understanding of column mechanics, such results may be very useful in the teaching of the subject, and helpful in design practice for correct understanding and assessment of the often complicated column response, and, finally, useful as a supplement to full second-order structural analysis.

Towards this goal, and for deriving and assessing approximate results, exact results are obtained according to second-order (small rotations) theory for an isolated, restrained column, and for a panel with two columns, interacting through

connecting beams. The columns considered are initially straight before loading and are rotationally restrained at column ends by beams (surrounding structure). They have constant cross-sectional properties (EI) and constant axial load (normal force N) along their lengths. No transverse loads are applied between columns ends. Shearing and axial deformations are neglected, and torsional and out-of-plane bending phenomena are not considered.

2 Pseudo-critical loads and load indices

In describing the response of "framed" columns, that may be part of a larger frame, specific, individual member load indices are often helpful when considering the columns in isolation and will be used in this paper. These are primarily those defined below:

$$\alpha_{cr} = \frac{N}{N_{cr}} \ ; \ \alpha_s = \frac{N}{N_{cs}} \ ; \ \alpha_b = \frac{N}{N_{cb}} \ ; \ \alpha_E = \frac{N}{N_E} \qquad (1\,\text{a}-\text{d})$$

where N is the axial force in the column and N_{cr} is system (storey) critical load of the column at system instability. N_{cs} and N_{cb} are critical loads of the column when considered in isolation from the rest of the frame, but with rotational restraints (to be assumed) corresponding to the respective bending mode (sway or braced) of the real frame. N_{cs} is determined for the column considered completely free to sway, and N_{cb} for the column considered fully braced. Except when the frame consists of a single column, these are strictly pseudo-critical loads. They can be very useful in column characterization and discussion.

For an elastic, framed member of length L, uniform axial load level and uniform sectional stiffness EI along the length, the critical loads above can in the conventional manner be defined by

$$N_{cr} = \frac{N_E}{\beta^2} \quad ; \quad N_{cs} = \frac{N_E}{\beta_s^2} \quad ; \quad N_{cb} = \frac{N_E}{\beta_b^2} \quad ; \quad N_E = \frac{\pi^2 EI}{L^2} \qquad (2\,\text{a}-\text{d})$$

Here, N_E is the socalled Euler load (critical load of a pin-ended column), and is a convenient reference load parameter in several contexts. The effective length factor, β, is equal to β_s and β_b for the free-to-sway and the braced case, respectively.

As defined, the load indices are interrelated. For instance, $\alpha_E = \alpha_s/\beta_s^2$ or $\alpha_E = \alpha_b/\beta_b^2$. The free-sway critical load is defined by $\alpha_s = 1.0$ or, for instance, $\alpha_E = 1/\beta_s^2$, and the braced critical load by $\alpha_b = 1.0$ or $\alpha_E = 1/\beta_b^2$.

3 Supporting an supported sway columns

Fig. 1(a) shows a laterally loaded frame that is not braced against sidesway, i.e, it is free to sway. In the absence of axial forces in the columns, the lateral load (H) gives rise to a first-order sway displacement Δ_0, equal at all column tops when axial beam deformations are neglected. The lateral load is resisted by column shears (V_0) that are proportional to the relative lateral stiffness of the columns.

(a) Laterally loaded sway frame –– 1st order analysis

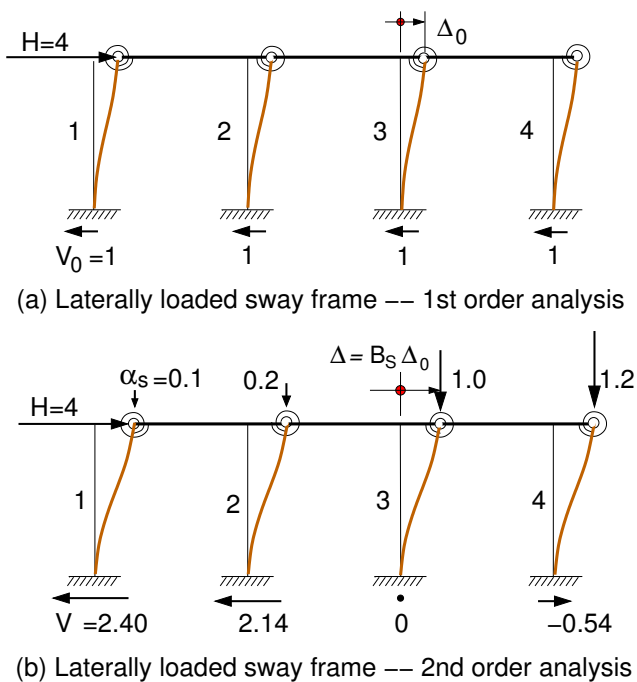(b) Laterally loaded sway frame –– 2nd order analysis

Figure 1: Effects of axial loads on column shears in multibay sway frame: (a) First and (b) second-order analysis ($B_s = 2.67$).

Under the additional action of axial forces, caused by vertical loads on the frame, Fig. 1(b), the displacement increases to the total displacement $\Delta = B_s\Delta_0$. Here B_s is a sway displacement magnification factor that reflects the global (storey, or system) second-order effects of axial loads. Had the individual columns, having different axial load levels, been free to sway independent of each other, some would sway more (the more flexible ones) and some less (the more stiff ones) than the the overall frame. Since the interconnected columns are forced to act together, a redistribution of shears will consequently have to take place, as seen in Fig. 1(b).

To illustrate, consider for simplicity the lateral stiffness of all the columns in Fig. 1 to be equal, and that axial deformations in the connecting beams are negligible. Then, a lateral load $H = 4.0$ causes a (first-order) shear of $V_0 = H/4 = 1.0$ in

each of the columns (Fig. 1(a)). If vertical loads now are applied, in Fig. 1(b) given nondimensionally in terms of the free-sway load index $\alpha_s(= N/N_{cs})$ of each column, the lateral displacement is increased and a redistribution of shears take place between the columns, to approximately $V_1 = 0.9B_s$, $V_2 = 0.8B_s$, $V_3 = 0$, $V_4 = -0.2B_s$ (calculated using Eq. (35), presented later). Since the sum of these shears ($\sum V = 1.5B_s$) must equal the applied load $H = 4$, the displacement magnifier becomes $B_s = H/\sum V = 2.67$, and the final shears as those given in the figure.

Columns 3 and 4 with α_s=1.0 and 1.2, respectively, would fail sideways if they acted independently. However, sidesway instability of these columns is prevented by the interaction with the other columns since all the interconnected columns must undergo an equal lateral displacement, Δ. In this process, the redistribution of shears will be from the less stiff columns 3 and 4 to the stiffer columns 1 and 2.

In column 4, the shear has reversed direction. This column requires in other words a negative shear to hold it in the position given by the frame. Rather than resist the external lateral loads, this column adds to the lateral shears in columns 1 and 2.

In this paper a distinction will be made, as implied above, between
(a) "supporting columns" (bracing, leaned-to columns) which contribute to resisting lateral loads and to bracing of other columns, and
(b) "supported columns"(partially braced, leaning columns) which do not contribute to the lateral resistance and which will need lateral support from the supporting columns. Such a column may be visualized as being allowed to undergo a given sidesway and then braced against further sway.

In terms of the total shears, V, supporting sway columns are those in which $V > 0$ and supported sway columns are those for which $V < 0$.

4   Second-order analysis

4.1   General

Theoretical results will be obtained both for a panel frame, and for individual columns isolated from a frame. For the latter case, it is of interest to derive

explicit expressions. A tailor made analysis for this purpose is presented below. For the panel frame, results are computed using a stiffness formulation of the second-order theory that incorporates the the slope-deflection equations given below (Section 4.3). Axial forces in beams are neglected.

4.2   Isolated column with given end displacement

The column studied is shown in Fig. 2(a). It is initially straight and has a length L, uniform section stiffness EI and rotational end restraint stiffnesses (equal to the moments required to give a unit rotation of one) labeled k_1 and k_2, respectively, at ends 1 (top) and 2 (bottom). The rotational end restraints are illustrated in the figure by beams rotationally restrained at the far ends. Only lateral loading and vertical loading causing an axial load N are considered. The combined effect of these loads is to give a relative lateral displacement $\Delta = B_s\Delta_0$. No gravity load induced moments (such as from loading on beams) are included. V is the shear (lateral loading) required to give a displacement Δ.
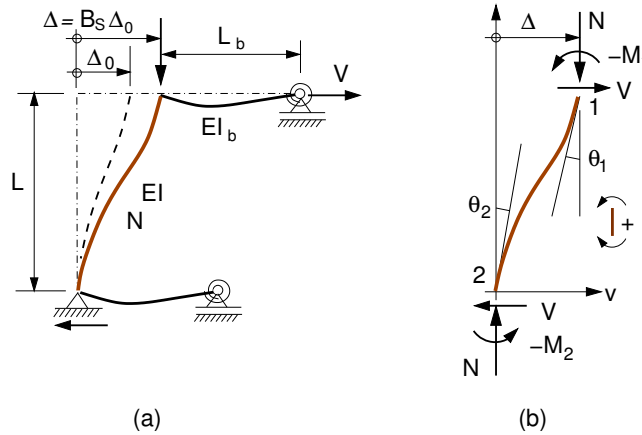

Figure 2: Column definition and sign convention.

The column can be considered in isolation from a greater frame. If so, the rotational end restraints should reflect the rotational interaction with restraining beams ("horizontal interaction") at the joints, and, in multistorey frames, also with other columns framing into the considered joint ("vertical interaction").

A discussion on the determination of what end restraints are appropriate for an isolate column study, is given in Hellesland (2009a) for analyses aimed at computing moments and shears, and in Hellesland and Bjorhovde (1996a, 1996b) for critical load analyses. For a brief discussion of the approach adopted in conjunction with the panel studies in this paper, see Section 7.2

6

The rotational restraints can conveniently be represented by nondimensional restraint stiffnesses κ, at end j defined by

$$\kappa_j = \frac{k_j}{(EI/L)} \qquad j = 1, 2 \tag{3}$$

or by other similar factors, such as the well known G factors. Unlike the κ factors, that are relative stiffness factors, the G factors are relative, nondimensional, *scaled* flexibility factors. In their generalized forms (Hellesland and Bjorhovde, 1996a, 1996b) they can be defined by

$$G_j = b_o \frac{(EI/L)}{k_j} \quad (= \frac{b_o}{\kappa_j}) \qquad j = 1, 2 \tag{4}$$

where b_o is simply a scaling (reference, datum) factor by which the relative restraint flexibilities are scaled. If the rotational restraints are provided by beams, such as assumed here, the restraint stiffness can be written

$$k = k_b = \sum bEI_b/L_b$$

where the summation is over all beams that are restraining the considered column end (joint), and b is the bending stiffness coefficient. In such cases, the G factor in Eq. (4) becomes

$$G_j = \frac{(EI/L)}{(\sum (b/b_0)EI_b/L_b)_j} \tag{5}$$

Typically, for beams with negligible axial forces, the familiar values of $b = 3$, $b = 4$, $b = 2$ and $b = 6$ are obtained for beams pinned at the far end, fixed at the far end, bent into symmetrical, single curvature bending, and for beams bent into antisymmetrical, double curvature bending, respectively. Conventional datum values, as adopted for instance in AISC (2016), ACI (2014), are $b_o = 6$ and $b_o = 2$ for unbraced and braced frames, respectively.

It should be noted that, if not otherwise stated, the reference factor $b_o = 6$ is used throughout this paper in presentations in terms of G factors. Thus, for cases with beams bent into double, antisymmetrical curvature, $b/b_o = 1$ in Eq. (5), giving the well known, conventional G factor expression.

If several columns frame into a joint, such as in multistorey frames, k_b must be "divided between" the columns at the joint ("vertical interaction"). Conventionally this is done by assigning the restraining beam stiffness at a joint to the attached columns in proportion to their EI/L values . Doing this, a G factor definition is obtained that also includes a summation sign in the numerator of Eq. (5). For further discussion, see for instance Hellesland and Bjorhovde (1996a, 1996b) or Hellesland (2009a).

4.3 Basic slope-deflection equations

For completeness, the major elements of the second-order (small rotations) analysis for the single column in Fig. 2 are developed in sufficient detail to allow the establishment of closed form solutions in suitable forms for use and discussion later in the paper.

The adopted sign convention is shown in Fig. 2. Clockwise acting moments and rotations are defined to be positive. Consequently, the moments shown in Fig. 2, acting counter-clockwise, are negative ($-M_1$ and $-M_2$).

The member stiffness relationship, expressed by the slope-deflection equations for a compression member, can be derived from the column's differential equation, and can be given by the matrix equation below.

$$
\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} C & S & -(C+S)/L \\ S & C & -(C+S)/L \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \Delta \end{bmatrix} \tag{6}
$$

Here, the upper case C and S are the socalled stability functions defined by

$$
C = \frac{c^2}{c^2 - s^2} \quad ; \quad S = \frac{s^2}{c^2 - s^2} \tag{7 a, b}
$$

where

$$
c = \frac{1}{pL^2}\left(1 - \frac{pL}{\tan pL}\right) \quad ; \quad s = \frac{1}{pL^2}\left(\frac{pL}{\sin pL} - 1\right) \tag{8 a, b}
$$

and

$$
pL = L\sqrt{N/EI} \quad (= \pi\sqrt{\alpha_E}) \tag{9}
$$

The functions C and S incorporate the second-order effects of axial forces acting on the deflection away from the secant through the column ends. In first-order theory, when the axial force is zero and this effect is not present, and they take on the familiar values of 4 and 2, respectively.

The formulation above was used by Galambos (1968) and later by Galambos and Surovek (2008). In the literature, other formulations can be found (e.g., Bleich (1952), Timoshenko and Gere (1961), Chen and Lui (1991), Livesley (1956, 1975). The lower case c and s functions are flexibility factors and are equal to 1/3rd of the corresponding functions (ϕ and ψ) in Timoshenko and Gere (1961).

4.4 End moments and shears

From the moment equilibrium requirements at the joints,

$$M_1 + k_1\theta_1 = 0 \quad ; \quad M_2 + k_2\theta_2 = 0 \tag{10 a, b}$$

where k_1 and k_2 are the rotational stiffnesses of the end restraints (beams), the end rotations can be expressed by the end moments (e.g., $\theta_1 = -M_1/k_1$). Substitution of these into Eq. (6) gives

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} -(C+S)\,EI\Delta/L^2 \\ -(C+S)\,EI\Delta/L^2 \end{bmatrix} \tag{11}$$

from which the total end moments according to second-order theory can be solved for and expressed by

$$M_1 = -\frac{(C+S)\,(b_{22}-b_{12})}{b_{11}b_{22}-b_{12}b_{21}} \cdot \frac{EI\Delta}{L^2} \tag{12}$$

$$M_2 = -\frac{(C+S)\,(b_{11}-b_{21})}{b_{11}b_{22}-b_{12}b_{21}} \cdot \frac{EI\Delta}{L^2} \tag{13}$$

Here, with the rotational stiffness of the end restraints taken as $k = 6(EI/L)/G$ (Eq. (4) with $b_o = 6$),

$$b_{11} = 1 + \frac{CG_1}{6} \quad ; \quad b_{12} = \frac{SG_2}{6} \quad ; \quad b_{21} = \frac{SG_1}{6} \quad ; \quad b_{22} = 1 + \frac{CG_2}{6}$$

When the determinant $D = b_{11}b_{22} - b_{12}b_{21}$ approaches zero, the end moments M_1 and M_2 approach infinity. This happens when the axial load approaches the critical (instability) load. This limit is independent of the lateral displacement .

The shear is found from statics of the displaced column ($VL = -(M_1 + M_2 + N\,\Delta)$). In non-dimensional form it can be written

$$\frac{V}{EI\,\Delta/L^3} = -\frac{M_1 + M_2}{EI\,\Delta/L^2} - (pL)^2 \tag{14}$$

First-order results.
It is convenient to present results in terms of first-order results. Such results are found by substituting $C = 4$ and $S = 2$ into Eqs. (12) and (13). The resulting sway magnified first-order moments (due to the magnified first-order sidesway $B_s\Delta_{0s}$) become

$$B_s M_{02} = -\frac{6(G_1 + 3)}{2(G_1 + G_2) + G_1 G_2 + 3} \cdot \frac{EI\Delta}{L^2} \tag{15}$$

9

and

$$B_s M_{01} = B_s M_{02} \frac{G_2 + 3}{G_1 + 3} \tag{16}$$

respectively at end 2 and 1. The corresponding sway modified first-order shear becomes, from statics,

$$B_s V_0 = -B_s \left(M_{01} + M_{02}\right)/L \tag{17}$$

4.5   Location and magnitude of maximum column moment

The moment expression of the axially loaded column subjected to the total end moments M_1 and M_2, can be found from the differential equation $(M = -EIv'')$ and expressed in the familiar form given by

$$M(x) = M_2 \left[\frac{\mu_t - \cos pL}{\sin pL} \sin px + \cos px \right] \tag{18}$$

where

$$\mu_t = -M_1/M_2 \tag{19}$$

This end moment ratio, between total end moments (from the second-order analysis), becomes positive when the end moments at the two ends act in opposite directions ("single curvature").

From the maximum moment condition, $dM(x)/dx = 0$, the location x_m of the maximum moment between ends can be obtained as

$$\tan px_m = \frac{\mu_t - \cos pL}{\sin pL} \tag{20}$$

Now, by substituting Eq. (20) into Eq. (18), the maximum moment can be solved for and expressed as

$$M_{max} = B_{tmax} M_2 \tag{21}$$

where

$$B_{tmax} = \frac{1}{\cos px_m} \qquad for \quad \mu_t > \cos pL \tag{22 a}$$

$$B_{tmax} = 1.0 \qquad for \quad \mu_t \leq \cos pL \tag{22 b}$$

The maximum moment may become positive or negative. In practice, it is the absolute value that is of interest. For a restrained column with $N > N_E$ (i.e., $\alpha_E > 1$), maximum moment will always be between ends (see also Eq. (42), Section 6.4).
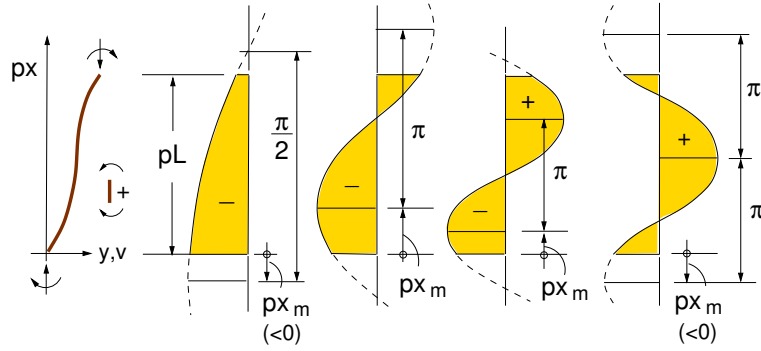
Figure 3: Formation of maximum moments at different axial load levels.

Sometimes, details of location(s) and sign of maximum moments are of interest. Noting that the period of Eq. (20) is π, the maximum moment location(s) can be obtained from

$$px_m = \arctan \frac{\mu_t - \cos pL}{\sin pL} + n\pi \qquad n = 0, 1, ..\qquad (23)$$

If for any value of n (in practice $n=0$ or 1 for a restrained column, and $n=0$ for a pin-ended column),

$$0 < px_m < pL \qquad (24)$$

then maximum moment is located between ends. If no px_m value can be found that satisfies Eq. (24), the maximum moment is located at one column end and is then equal to the largest end moment.

Fig. 3 shows different moment diagrams that may result. The column length is represented by $pL = \pi\sqrt{\alpha_E}$. In the first case, with $px_m < 0$, the maximum moment is at the lower end. In the other three cases, maximum moments occur between ends. In the third case, two maxima occur, one on either side of the column. This may occur when end moments are equal or nearly equal and acting in opposite directions.

Apart from the more general criteria set up above for maximum moment to form between ends of the column, the above derivation follows that by Galambos (1968) for pin-ended members, for which $\alpha_E \le 1$, and consequently $N \le N_E$. For such columns, taking M_2 as the largest end moment, the maximum moment will occur between ends when $\mu_t > \cos pL$ and at end 2 otherwise.

An alternative form of Eq. (21) can be obtained by some trigonometric considerations (see e.g., Galambos 1968). By letting the numerator and denominator in Eq. (20) be the sides in a right angled triangle, and denoting the hypotenuse of the triangle by h, then $\cos px_m = \cos pL/h$. Calculating h (by Pythagoras),

11

and substituting the resulting $\cos px_m$ into Eq. (21), the absolute value of B_{tmax} may be rewritten in the well known form below.

$$B_{tmax} = \left| \frac{\sqrt{1 + \mu_t^2 - 2\mu_t \cos pL}}{\sin pL} \right| \qquad (25)$$

4.6 Presentation of moments and shears

The analysis presented above can be used to compute moments and shears in a column for different axial loads and restraints as a function of the total relative lateral displacement Δ between column ends.

The global (system) second-order effects ("$N\Delta$" effects) are here represented by the sway displacement magnifier

$$B_s = \frac{\Delta}{\Delta_0} \qquad (26)$$

This magnifier is a function of the axial loads of all members in the frame system. Other notations for B_s is for instance δ in ACI 318 Building Code (ACI 2014) and B_2 in AISC 360 Specifications (AISC 2016). A review of available B_s expressions in the literature, incl. codes, and a general B_s proposal that cover both lateral supporting and lateral supported columns, are given in Hellesland (2009a, 2009b).

The local column second-order ($N\delta$) effect in an individual column with a specified sidesway $\Delta = B_s\Delta_0$, can be quantified by the ratios of results for a given axial load N and results for $N = 0$ (first-order). These ratios, or magnifiers, for maximum moment, end moments at column end 1 and 2, and shear force, are here denoted as

$$B_{max} = \frac{M_{max}(N)}{M_2(N = 0)} = \frac{M_{max}(N)}{B_s M_{02}} \qquad (27)$$

$$B_1 = \frac{M_1(N)}{M_1(N = 0)} = \frac{M_1(N)}{B_s M_{01}} \qquad (28)$$

$$B_2 = \frac{M_2(N)}{M_2(N = 0)} = \frac{M_2(N)}{B_s M_{02}} \qquad (29)$$

$$B_v = \frac{V(N)}{V(N = 0)} = \frac{V(N)}{B_s V_0} \qquad (30)$$

Above, M_{max} is expressed as a function of the first-order moment at end 2, which per definition is taken as the end with the larger first-order end moment (absolute

value). Note that B_{max} above is different from the B_{tmax} in Eq. (21) (and there applied to M_2 obtained from the second-order analysis).

Since the load effects above are all explicit functions of the lateral displacement Δ (Eqs. (12), (13), (14)), Δ will cancel out in the ratios. As a result, B_{max}, B_1, B_2 and B_v are independent of the magnitude of Δ.

Thus, for frames with sway due to lateral and axial loading only, these coefficients depend only on (a) the end restraints, which then uniquely define the first-order moment gradient (Eq. (16)), and on (b) the axial load level defined for instance by the nondimensional load parameters α_E or α_s (Eq. (1)).

$$M_{max} = B_{max} B_s M_{02} \tag{31}$$

$$M_1 = B_1 B_s M_{01} \quad (= (B_1 M_{01}/M_{02}) B_s M_{02}) \tag{32}$$

$$M_2 = B_2 B_s M_{02} \tag{33}$$

$$V = B_v B_s V_0 \tag{34}$$

5 Elastic columns in frames with sway

Typical moment and shear response versus increasing axial load of columns with stationary end restraints, computed with the presented second-order analysis, are shown in Fig. 4. The column is illustrated by the insert in the upper, left hand part of the figure, and has flexible rotational end restraints. The column top is initially displaced laterally by an amount $B_s \Delta_0$ ("sway-magnified first-order displacement"), and then kept constant at this value (in a real case, by the action of the overall frame of which the column may be considered isolated from).

Similar results are shown in Figs. 5 and in Fig. 6 for columns with stiff end restraints at the top, and very stiff at the base ($G = 0.6$ and ∞ (fully fixed) at the base). Fig. 7 gives typical results for a column with very stiff rotational end restraints.

The moments and shear are shown nondimensionally in terms of the respective B factors, Eqs. (27) to (30)), and the axial forces are given nondimensionally in terms of load indices α_s and α_E (Eq. (1))). All results in the figure are given in terms of $B_s M_{02}$. Therefore, the B_1 response is represented by $B_1 \cdot M_{01}/M_{02}$
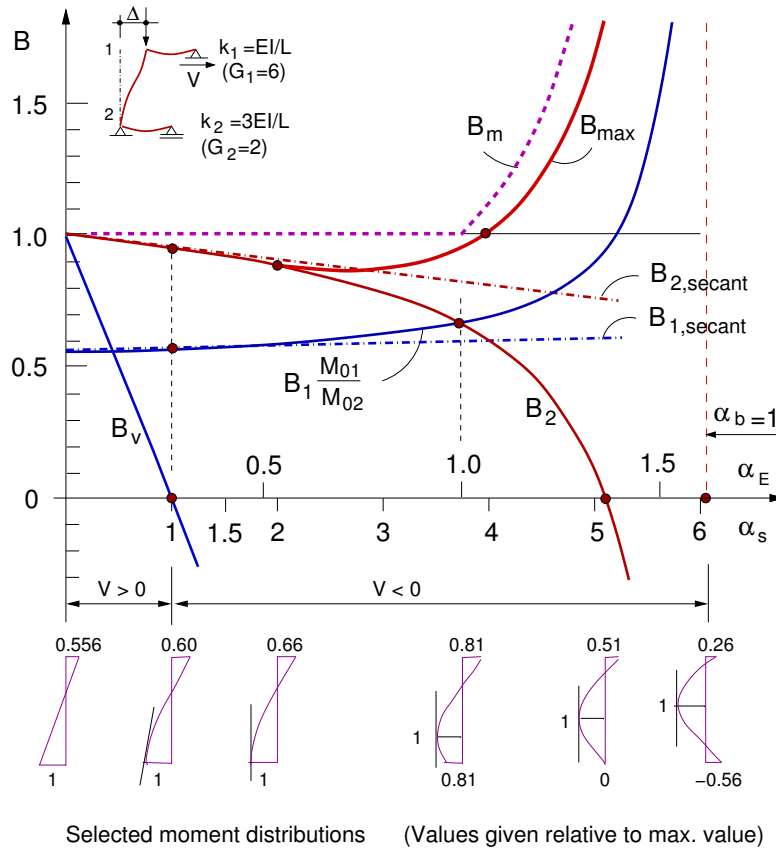
Figure 4: Moments and shear versus axial load level in column with relative flexible end restraints ($\beta_s = 1.932$, $\beta_b = 0.785$, $\alpha_E = 0.268\alpha_s$)

The curves labeled $B_{2,secant}$ are secant approximations, discussed in Section 6.9, to the end moment curves. The curves labeled B_m, are maximum moment predictions in accordance with present procedures in major design codes, and are discussed later in Section 8.

It is seen in the figures that both moments and shears approach infinity (in either the positive or negative direction) as the axial load approaches the elastic critical load N_{cb} of the fully braced column. The corresponding load index at that instance is $\alpha_b = 1$, or $\alpha_E = 1/\beta_b^2$. For an elastic column, the braced critical load is independent of whether the column is fully braced at zero or at a non-zero end displacement.

If the considered column was a part of a larger, unbraced frame, it should be emphasized that system instability of the frame may be reached for an axial load (N_{cr}) in the column that is well below its fully braced value N_{cb}. Also, lateral sway magnification (B_s) will most often, but not always, reach large, unacceptable values well before this load level.
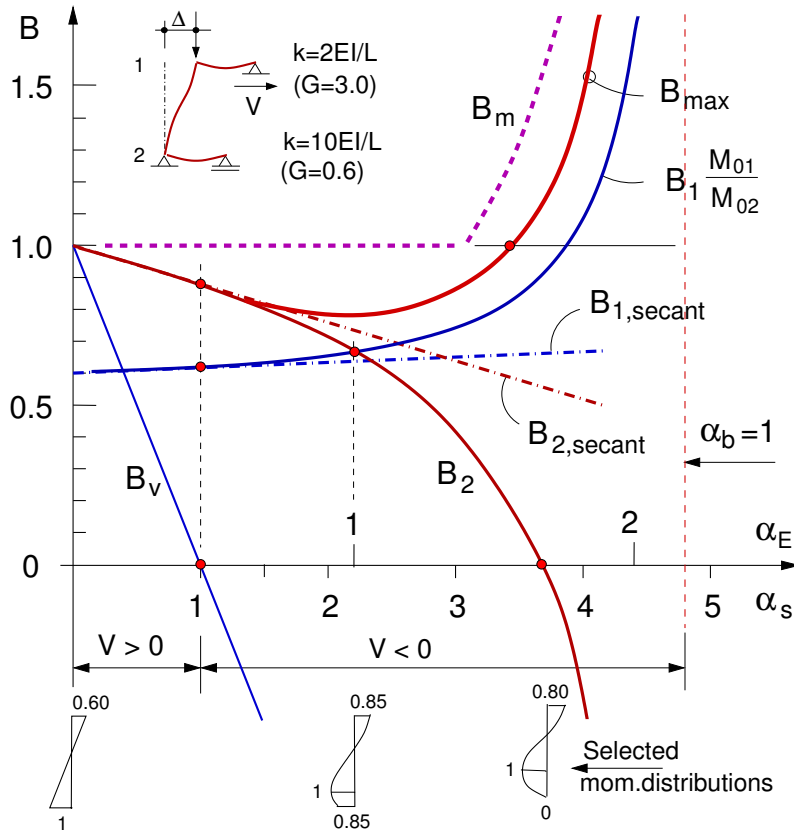
14

Figure 5: Moments and shear versus axial load level in column with stiff end restraints, $G_1 = 3$, $G_2 = 0.6$ ($\beta_s = 1.483$, $\beta_b = 0.678$, $\alpha_E = 0.455\alpha_s$))

Shears.

In the absence of axial load, a shear $V = B_s V_0$ is required to give a sway column a displacement $\Delta = B_s \Delta_0$. When an axial load is applied, the initial overturning moment is increased. This tends to further increase the sway (if it had been free to sway). To maintain the displacement at the original specified value, V must decrease. This is seen by the sloping line labeled B_v in the figures.

The value of B_v is 1.0 for zero axial load ($\alpha_s = 0$, or $\alpha_E = 0$). With increasing load, it decreases to zero at $\alpha_s = 1$ (or $\alpha_E = 1/\beta_s^2$) and becomes negative as α_s increases further. Zero shear is the instability criterion for a column that is free to sway. When free sway of the column is not permitted by interaction with other columns or bracing elements, the column will still be able to resist end moments and axial loads beyond the free sway critical load if the rest of the structure provides lateral support, as indicated by change of direction of the shear force.

At $\alpha_s = 1$, the function of the column changes. Columns with $\alpha_s < 1$ are capable of resisting external lateral loads, and are capable of supporting, or bracing, columns with higher axial, in excess of $\alpha_s = 1$. As stated earlier, columns with
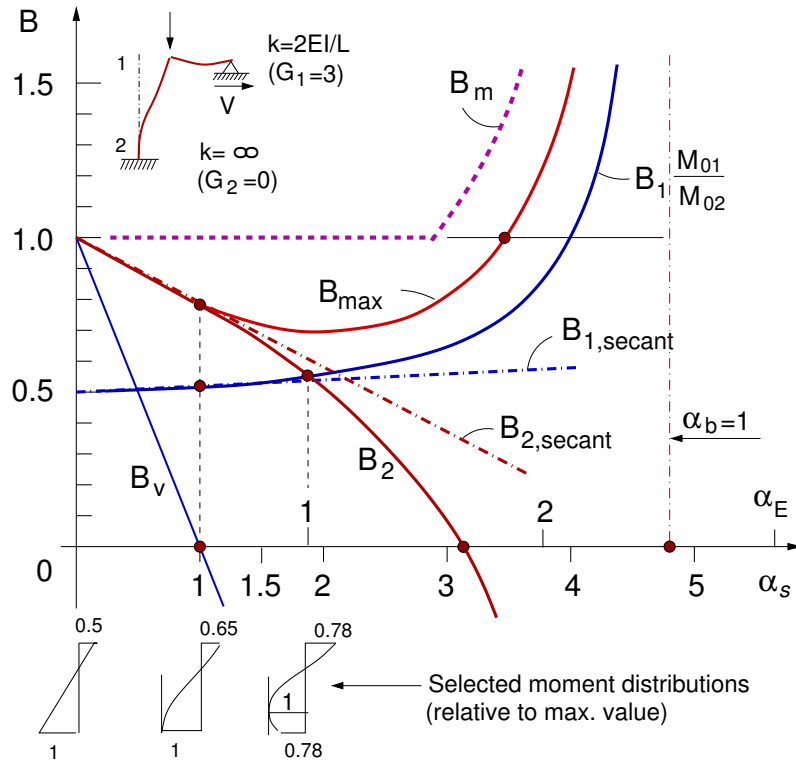
15

Figure 6: Moments and shear versus axial load level in column with stiff end restraints, $G_1 = 3$, $G_2 = 0$ ($\beta_s = 1.373$, $\beta_b = 0.626$)

$\alpha_s < 1$ will be referred to as laterally "supporting sway columns" while those with $\alpha_s > 1$ will be referred to as laterally "supported sway columns".

Shear variation.
As seen in the figures, the variation of B_v is almost linear up to and somewhat beyond $\alpha_s = 1$. In this range, a linear shear variation given by $V = B_v B_s V_0$ with

$$B_v = 1 - \alpha_s \tag{35}$$

provides an excellent approximation. It becomes increasingly inaccurate (giving too small negative values) for larger α_s values (Hellesland 2009a).

Moment at end 2 (with the stiffer end restraint).
For the laterally loaded columns with given, stationary (invariant) end restraints considered here, the largest first-order end moment in terms of absolute value will always develop at the end with the largest rotational restraint stiffness. This can be seen in the figures, and also in the first-order moment diagram (for $N = 0$ ($\alpha_E = 0$)) at the bottom Fig. 4.

This end, with the initially largest first-order end moment, is conventionally denoted end 2 and the moment M_2. So also here.
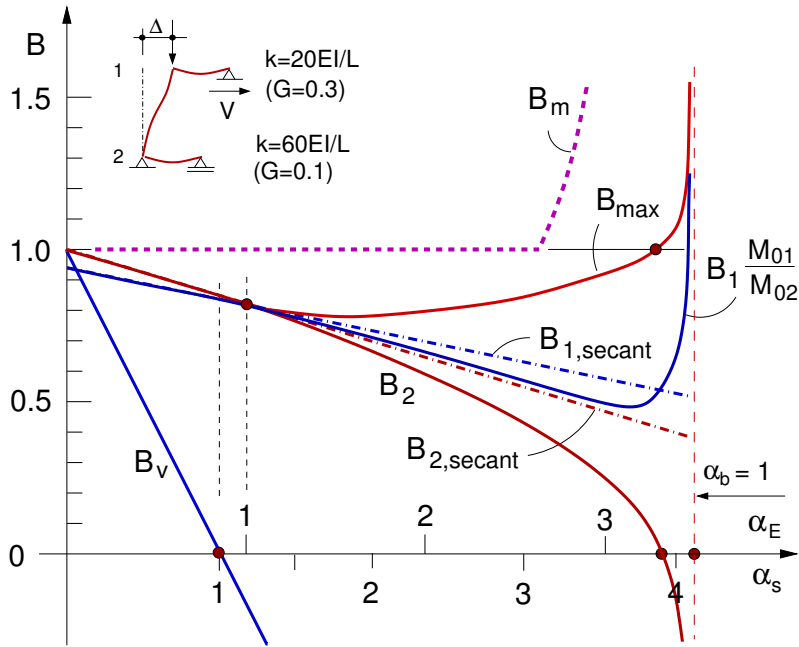
16

Figure 7: Moments and shear versus axial load level in column with very stiff end restraints ($\beta_s = 1.065$, $\beta_b = 0.532$, $\alpha_E = 0.882\alpha_s$)

M_2 (or B_2) decreases continuously with increasing load level. At some point, it becomes zero and changes direction (at about $\alpha_s = 5.1$, or $\alpha_E = 1.37$, in Fig. 4), and eventually approaches minus infinity (at $\alpha_b = 1$). The response of the stiff restraint case, Fig. 7, is similar.

As the moment at end 2 changes sign, the column deflection shape changes from double to triple curvature bending. The last moment diagram at the bottom of Fig. 4 is for an axial load of 0.998 times the braced critical load. At this stage, which is of theoretical interest only, the magnitude of the initial, first order end moments are small compared to the total moments.

Moment at end 1 (with the smaller end restraint).
The M_1 response represented by $B_1 \cdot M_{01}/M_{02}$ in Fig. 4, is typical for cases with relatively low restraint stiffness at end 1. The end moment at end 1 stays fairly constant, or increases slowly, until the end moments at the two ends become equal at $\alpha_E = 1$. From there, it increases more sharply towards infinity at $\alpha_b = 1$.

In the case with unequal, stiff restraints, such as in Fig. 7, both end moments decrease markedly at first with increasing axial load. The moment factor B_1 decreases at a slower rate than B_2 till it reaches a minimum. Since the end restraints at the two ends are not too different, B_1 and B_2 follow each other fairly closely up till rather high load levels at which B_1 reaches a minimum and then

starts increasing sharply towards plus infinity as $\alpha_b = 1$ is approached.

Maximum moment.
The maximum moment (M_{max}, B_{max}) is initially equal to the end moment at end 2. It stays at end 2 until $\alpha_s \geq 1$ (see later discussion Section 6.4). For the column in Fig. 4, this is so until $\alpha_s = 1.99$. The moment diagram for this load level is seen (bottom, Fig. 4) to have a vertical tangent at the end 2. As the axial load is increased further, the tangent at end 2 changes direction and the maximum moment forms between ends. It should be noted that the displayed moment diagrams are approximate and have been scaled so that the the maximum moment has a value of unity.

Following an initial decrease, B_{max} starts increasing with increasing load level, and approach plus infinity for axial loads approaching the braced critical load.

Moments in columns with equal or nearly equal end restraints.
When end restraints at the two ends are exactly equal, the end moments will also be equal. With increasing axial load, the end moments decrease continuously to zero at the braced critical load ($\alpha_b = 1$). Up to this point, the column deflection mode is one in antisymmetrical double curvature bending. At $\alpha_b = 1$, the smallest disturbance will cause the deflected shape to change suddenly from double curvature bending into the braced buckling mode. In the case of a column pinned at both ends and subjected to equal and opposite end moments, it will buckle from double into the lower single curvature bending mode. A restrained column with equal end restraints, will deflect from perfect antisymmetrical quadruple bending. At $\alpha_b = 1$, it will, for the smallest disturbance, buckle into the lower triple curvature buckling mode, causing moments to approach infinity.

This phenomenon is often referred to as *unwrapping* (Baker et al. 1956) or *unwinding* (Ketter 1961). Differences in end restraints will always be present in practical cases. Thus, for a column with stationary (constant) end restraints and with moments induced by lateral displacement, the problem of unwinding is mainly of academic interest only.

Columns with non-stationary restraints Columns in multicolumn-frames will typically not have stationary end restraints. Such cases are studied in Section 7.

18

6 "Landmarks" in single column response

6.1 Characteristics to be considered

Definition of a number of key characteristics, or "landmarks" in the moment versus axial load "map", may be useful for establishing a good understanding of the mechanics of laterally displaced columns, and for enabling a quick establishment of moment-axial load relationship. Only restrained, single columns are considered. Markers (bullets) can be identified with reference to Fig. 4. They are:
(a) When is $V = 0$ ($B_v = 0$)?
(b) When does moments approach infinity?
(c) When does M_{max} (B_{max}) leave the column end?
(d) When does M_{max} (B_{max}) exceed $B_s M_{02}$ ($B_2 > 1$)?
(e) When does M_1 and M_2 become equal and what are their values at that point?
(f) When does M_2 (B_2) at end 2 become zero?
(g) Values of end moments at $\alpha_s = 1$ (supporting column limit)?

6.2 When does the shear force become zero?

The shear resistance of a supporting sway columns becomes zero at $\alpha_s = N/N_{cs} = 1$. For $\alpha_s > 1$, the column requires lateral support.

The effective length factor β_s, defining N_{cs} (Eq. (2b)) can be determined exactly, graphically from diagrams such as the well known alignment charts (in terms of G factors), or from one of the many approximate methods available. A summary and evaluation of such methods is given in Hellesland (2012). One of these, derived directly from column mechanics (Hellesland 2007), is defined in a simple manner by

$$\beta_s = \frac{2\sqrt{R_1 + R_2 - R_1 R_2}}{R_1 + R_2} \tag{36}$$

where R_1 and R_2 are degree-of-rotational-fixity factors at ends $j = 1$ and $j = 2$, respectively, and are equal to zero at a pinned end and 1 at a fully fixed end. They are defined by

$$R_j = \frac{k_j}{k_j + c\,EI/L} \qquad \left(= \frac{1}{1 + (c/b_0)G_j} \right) \tag{37}$$

where the c factor is a constant.

In conjunction with Eq. (36), it is to be taken as $c = 2.4$. In the second, alternative formulation above, in terms of G factors, the ratio in the denominator becomes $c/b_0 = 2.4/6 = 0.4$. Prediction accuracies of Eq. (36) for positive end restraints ($0 \leq R \leq 1$) is within 0% and +1.7% of exact results.

Application to the case in Fig. 4, with degree-of-end-fixities $R_1 = 0.294$ and $R_2 = 0.556$, $\beta_s = 1.950$ is obtained. This corresponds well with the exact $\beta_s = 1.932$ (0.9 % greater). For the column in Fig. 7, with end fixities $R_1 = 0.556$ and $R_2 = 0.962$, Eq. (36) gives $\beta_s = 1.076$ (exact 1.065, Eq. (38)).

An alternative effective length factor expression is given in Hellesland (2012) in three equivalent forms: either in terms of end moments, G or R factors. In terms of R factors defined by Eq. (37) with $c = 2$, it is given by

$$\beta_s = \left[\frac{\gamma_s \pi^2}{12} \left(\frac{6}{(R_1 + R_2)} - 2 \right) \right]^{1/2} \tag{38}$$

where

$$\gamma_s = 1 + 0.216 \cdot \frac{R_1 R_2 + 4(R_1 - R_2)^2}{(R_1 + R_2 - 3)^2} \tag{39}$$

Eq. (39) gives results within about 0.1 % of exact results in cases with positive restraint combinations.

6.3   When does moments approach infinity?

The outer limit of the $M - \alpha$ relationship is defined by $\alpha_b = 1$, representing braced instability. Similarly to free-sway instability, there is a number of tools available for determining corresponding effective length factors β_b (Eq. (2c)). One of several alternative formulations (Hellesland 2007, 2012) for this case is

$$\beta_b = 0.5\sqrt{(2 - R_1)(2 - R_2)} \tag{40}$$

where the degree-of-rotational-fixity factors R are identical to those defined by Eq. (37) with $c = 2.4$. This expression is accurate to within -1% and +1.5% for positive end restraints.

For the case in Fig. 4, $\beta_b = 0.785$ is obtained from Eq. (40). This is equal to the exact value. Instability ($\alpha_b = 1$) results at $\alpha_E = 1/\beta_b^2 = 1.62$. For the case in Fig. 7, $\beta_b = 0.536$ results (exact 0.532).

6.4 Maximum moment leaves end 2 of the column

The stage at which the maximum moment leaves the end 2 of a column can be discussed in terms of the moment gradient at the column end. For the column shown in Fig. 2 it is given by

$$dM/dx = N\,\theta + V \tag{41}$$

A positive gradient indicates that the maximum moment (absolute value) is at the end as shown by the first two moment diagrams in Fig. 4. For zero moment gradient, the maximum moment is on the verge of leaving the end as shown in the third moment diagram in Fig.4. A negative gradient indicates that the maximum moment is away from the end.

Initially, the maximum moment will always be at the end 2 with stiffest restraint (smallest G). For $\alpha_s \leq 1$ it will remain at the end since both V and θ are postive in this case. For the special case with full fixity at the end ($G = 0$), θ is zero and the maximum moment leaves the end of the column when V becomes negative, i.e., as α_s exceeds 1.0.

If neither end is fully fixed, the maximimum moment will move away when $N\theta = -V$ at a higher load index. For the column in Fig. 4, this occurs at $\alpha_E = 1.99$. The exact value can be found from Eq. (20) as the load at which x_m becomes zero. I.e., for $cospL = \mu_t$ when the origin is taken at the end with the stiffer restraint. The minimum value is obtained for $\mu_t = 0$ (for a column pinned at one end) as $pL = \pi/2$ and $\alpha_E = 0.25$. Similarly, the maximum is obtained for a column with equal end restraints, i.e., with $\mu_t = \text{-}1$ (antisymmetrical curvature), as $pL = \pi$ and $\alpha_E = 1.0$ (or $\alpha_s = \beta_s^2$). These limits are independent of the end restraints.

In summary, the load index at which the maximum moment starts to form away from the end of a column will be within the alternative (and equivalent) limits given by

$$0.25 \leq \alpha_E \leq 1.0 \qquad \text{and} \qquad 1.0 \leq \alpha_s \leq \beta_s^2 \tag{42}$$

respectively, in terms of the column's α_E value and its free-sway stability index α_s.

Comments, non-stationary restraints. These results also provide a reasonably good description of the panel columns. See Section 7.
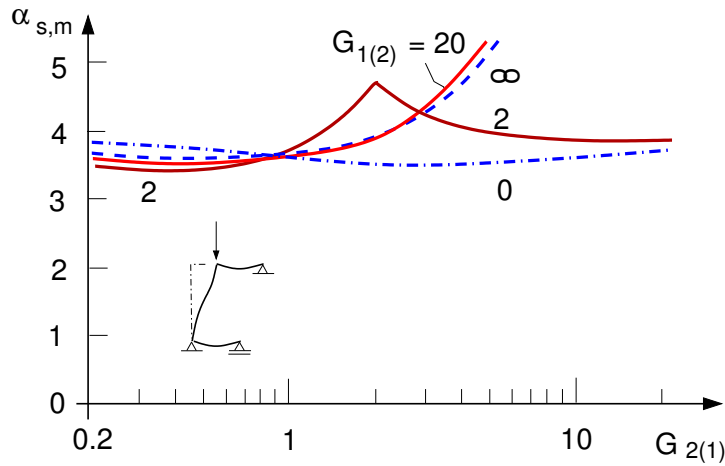
Figure 8: Axial load level in terms of $\alpha_s = N/N_{cs}$ at which the maximum moment between ends is equal to the larger sway modified end moment.
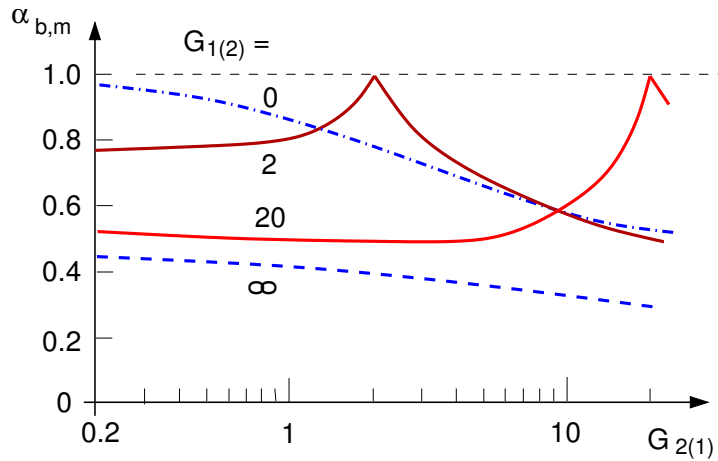

Figure 9: Axial load level in terms of $\alpha_b = N/N_{cb}$ at which the maximum moment between ends is equal to the larger sway modified end moment.

6.5 Maximum moment exceeds the larger sway-magnified first-order moment

The axial load level $\alpha = \alpha_m$ at which the maximum moment M_{max} exceeds the largest sway magnified end moment $B_s M_{02}$, i.e., when B_{max} exceeds 1.0, is of considerable interest in design. The maximum moment may conservatively be taken as $B_s M_{02}$ ($B_{max} = 1$) below this load level. Computed second-order analysis results are given in Fig. 8 and Fig. 9.

Fig. 8 gives free-sway load indices $\alpha_{s,m}$ at which the maximum moment exceeds the larger magnified first-order moment. Results are given for end restraints at the two ends varying from $G = 0$ (fixed end) to about ∞ (pinned end). The same

22

results are reproduced in Fig. 9 for the braced load index $\alpha_{b,m}$.

The peaks in the figures are obtained when end restraints are equal, in which cases moments remain less than $B_s M_{02}$ (B_{max} remains less than 1.0) until unwinding take place at the braced critical load level ($\alpha_b = 1$).

It should be noted that also the G factors in Fig. 9 are defined by Eq. (4) with $b_0 = 6$, rather than with the more customary $b_0 = 2$ for braced cases (for instance in conjunction with the use of the alignment chart for effective length determination. If defined with $b_0 = 2$, the G factors would be one third of those in Fig. 9.

In practice, it is difficult to obtain perfectly pinned and fully fixed connections. Practical values of G of 10 and 1, respectively for these cases, are often suggested (AISC Commentary, 2016). Within such a practical range of G values, it can be seen that the $\alpha_{s,m}$ curves are gathered within a reasonably narrow band, between about 3.5 to 5. A conservative lower value would be about $\alpha_{s,m} = 3.5$. The $\alpha_{b,m}$ curves (Fig. 9) are more spread out, but with a conservative lower value of $\alpha_{b,m}$ between 0.5 and 0.6 for G values between 10 and 1.

Based on these results for single columns with stationary (invariant) end restraints, it seems reasonable to conclude that

$$B_{max} < 1 \qquad \text{for} \qquad \alpha_s < 3.5 \qquad \text{or} \qquad \alpha_b < 0.5 \qquad (43)$$

Many codes accept that individual column slenderness effects can be neglected when the maximum moment does not exceed the larger first-order end moment, or in the present case, $B_s M_{02}$, by more than 5% to 10%. With such criteria, the load index limits above would be greater.

Comments, non-stationary restraints. For the panel columns in Section 7 it is found that

$$B_{max} < 1 \qquad \text{for} \qquad \alpha_s < 3 \qquad \text{or} \qquad \alpha_b < 0.8 \qquad (44)$$

Although these limits are based on a rather limited number of panel columns, they are believed to be quite representative. It should be noted, that the α_s and α_b values in Eq. (44) are computed with the critical loads N_{cs} and N_{cb}, respectively, of the panel, rather than those of the isolated, single columns with stationary end restraints (Eq. (43)).

The free-to-sway critical loads N_{cs} for the two cases are found to be almost identical. On the other hand, the braced critical loads N_{cb} of the panel columns

are lowered considerably below those of the single columns, thereby pressing the rising, maximum moment branch upwards towards larger values, at lower load levels. The reason for the reduced critical loads N_{cb} of the panel columns is the softening of the beam restraints as the critical load is approached. This is due to beams that unwinds from double curvature towards the more flexible, single curvature type bending (Fig. 11) for axial loads approaching the braced instability load.

6.6   Equal end moments, $M_1 = M_2$

The load index at which moments at the two column ends become equal, and the value of these end moments, are of interest. For M_1, Eq. (12), and M_2, Eq. (13), to become equal, it can be seen from the moment equations that the stability functions C and S must become equal, and thus $c = s$. Rewriting of the latter results in $2 \sin pL - pL \cos pL = pL$, the solution of which is $pL = \pi$. Consequently,

$$M_1 = M_2 \quad \text{for} \quad \alpha_E = 1 \tag{45}$$

or in terms of the free-sway critical load index, for $\alpha_s = \alpha_E \beta_s^2 = \beta_s^2$.

By evaluating Eq. (13) for $pL = \pi$, an expression for M_2 is obtained in terms of the end restraints and $EI\Delta/L$. It is given by

$$M_2 = -\frac{12}{G_1 + G_2 + 2 \cdot 1.216} \cdot \frac{EI\Delta}{L^2} \tag{46}$$

The corresponding magnified first-order end moment for $\Delta = B_s \Delta_0$ is given by Eq. (15). The B_2 and B_1 factors at $\alpha_E = 1$ then become

$$B_2 = \frac{M_2}{B_s M_{02}} = \frac{4(G_1 + G_2) + 2G_1 G_2 + 6}{(G_1 + G_2 + 2.432)(G_1 + 3)} \tag{47}$$

$$B_1 = \frac{M_1}{B_s M_{01}} = B_2 \frac{G_1 + 3}{G_2 + 3} \tag{48}$$

Comments, non-stationary restraints. The results above are found to apply also to the panel columns, except for a rather unrealistic case of a panel with very large differences in column stiffnesses, in which the critical loading was reached prior to $\alpha_E = 1$. See Section 7 (Fig. 15).

6.7 Axial load giving zero end moment, $B_2 = 0$

In order to obtain $B_2 = 0$, or $M_2 = 0$ in Eq. (13), it is required that $C - S = -6/G_1$. Thus,

$$\frac{1 - \cos pL}{pL \sin pL} = -\frac{G_1}{6} \tag{49}$$

Since $M_2 = 0$, there is as expected no interaction with G_2 at end 2.

If accurate solutions are required, it is necessary to solve for $pL(= \pi\sqrt{\alpha_E})$ by iteration. In order to establish an expression that allows direct establishment of the axial load level, plots of Eq. (49) were studied. Based on these, a reasonably simple approximation, that gives results within about ± 2 percent, has been found to be given by

$$\alpha_E = \frac{4 + 1.1\, G_1}{1 + 1.1\, G_1} \tag{50}$$

For the columns in Figs. 4, 6 and 7, B_2 can be seen to become zero at about $\alpha_E = 1.37$, 1.67 and 3.4, respectively. These corresponds well with the comparative values by Eq. (50) of 1.40, 1.67 and 3.3.

Comments, non-stationary restraints. For the panel columns, zero end moments are obtained at somewhat smaller α_E values. See Section 7.

6.8 End moments at $\alpha_s = 1$ (supporting column limit)

The value of end moments at $\alpha_s = 1$ is of significant interest, in particular for developing approximate moment relationships for supporting sway columns. Values of B_1 and B_2 at $\alpha_s = 1$ are labeled B_{1s} and B_{2s}, respectively. Computed results for a wide range of end restraint combinations are plotted in Fig. 10.

B_{2s} coincide with B_{1s} in the case with equal end restraints. Results for this case are shown by the dash-dot borderline (labeled $G_1 = G_2$). B_{1s} results, shown by dashed lines in the figure, and B_{2s} results, shown by solid lines, are located above and below the borderline, respectively. G_2 is by definition taken to represent the end with the stiffer restraint (with the smaller G value). Corresponding B_{1s} and B_{2s} curves terminates therefore at the dash-dot curve.

To explain the figure, consider a column with $G_1 = 2$. As G_2 increases (along the abscissa) from zero to 2, B_{1s} is seen to decrease along the dashed line (labeled $G_1 = 2$) from about 1.02 to 0.98, and B_{2s} increases along the solid line (labeled
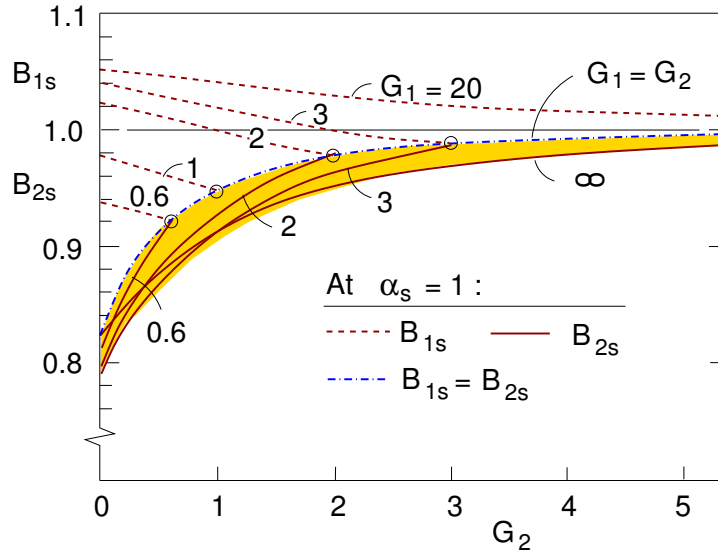
Figure 10: End moment factors at $\alpha_s = 1$ versus end restraints.

$G_2 = 2$) from about 0.8 to 0.98, i.e., to the same value as B_{1s} terminates at (as the restraints become equal at the borderline curve). At $G_2 = 0$ (fixed end), B_{2s} may have values between 0.79 and 0.82, and B_{1s} between about 0.82 and 1.05.

Comments, non-stationary restraints. The results above, and the secant expressions given below, are also reasonably applicable to the panel columns in Section 7.

6.9   Secant approximations to column end moments

The results above are informative and can for instance be used to establish useful secant approximations to the end moment curves at load levels in the range $\alpha_s = 0$ to 1.0, and also somewhat beyond 1.0. The secants through the moment points at $\alpha_s = 0$ and $\alpha_s = 1$ are given by

$$B_{1,secant} = \frac{M_1}{B_s M_{01}} = 1 - (1 - B_{1s}) \, \alpha_s \tag{51}$$

and

$$B_{2,secant} = \frac{M_2}{B_s M_{02}} = 1 - (1 - B_{2s}) \, \alpha_s \tag{52}$$

where, $B_{1,s}$ and $B_{2,s}$ are the moment factor values at $\alpha_s = 1$ shown in Fig. 10 for a wide combination of end restraints.

For instance, for the restraint combinations $(G_1, G_2) = (6, 2)$, $(3, 0.6)$, $(3, 0)$ and $(0.3, 0.1)$, representative for the columns in Figs. 4, 5, 6 and 7, pairs of B_{1s} and
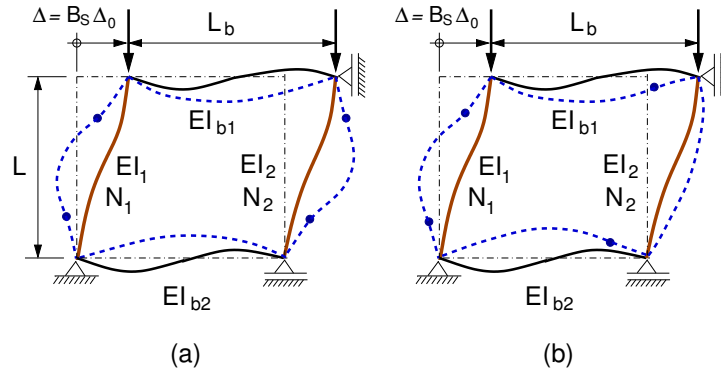
Figure 11: Initial and final deflection shapes of panel frames: (a) Col. 2 is slightly stiffer than Col. 1; (b) Col. 2 is significantly stiffer than Col. 1

B_{2s} can be read from Fig. 10 approximately as (1.02, 0.955), (1.03, 0.78), (1.04, 0.79) and (0.89, 0.85), respectively.

Such secants, labeled $B_{2,secant}$ and $B_{1,secant}$ are shown in Figs. 4, 5, 6 and 7, and are seen to provide close approximations to the end moment curves well beyond $\alpha_s = 1$.

7 Columns with non-stationary restraints

7.1 General

In the preceding section, end restraint stiffnesses of the single columns were given as constant (stationary, invariant) values. For frames with more than one column, the restraints may vary with the axial load level. To illustrate this, panel frames with two columns are considered, Fig. 11. The two columns are interconnected rigidly at the top and bottom by beams. The panels are initially given an imposed lateral displacement, $\Delta = B_s \Delta_0$, and then subjected to increasing vertical loading. The initial deflection shapes are the typical sidesway shapes, as shown by the solid lines in the figures. As the axial column loads increase, the deflection shapes change gradually. Slowly at first, and more rapidly towards deflection shapes approaching the braced buckling modes as the critical loading is approached.

The purpose of the panel analyses is in part to study the general mechanics of behaviour and interaction between the panel columns, and to clarify to what extent the column behaviour in the panels will affect conclusions reached earlier

from the single column studies.

Two panels are investigated, labeled Panel 1 and Panel 2. The axial column loads are the same in the columns of both panels: $N_1 = N_2 = N$. Member stiffnesses for Panel 1 are $EI_1/L = EI/L$, $EI_2/L = 1.1EI/L$, $EI_{b1}/L_b = 0.333EI/L$ and $EI_{b2}/L_b = 1.667EI/L$. Since the bottom beam is considerably (5 times) stiffer than the upper beam, it will attract the larger first-order end moments due to the sidesway. End 2 of each column is consequently taken to be at the base (bottom).

Compared to Panel 1, Panel 2 has a considerable stiffer Column 2, with $EI_2/L = 2EI/L$, but it has otherwise the same properties as Panel 1.

7.2   End restraints - Isolated column analyses

Column 1 (left hand) and Column 2 (right hand) of each panel will also be considered in isolation, with appropriate, stationary end restraints (Fig. 2), and analysed by the theory in Section 4.

In sidesway analyses, the most common approach (e.g., AISC, ACI) in isolated column analysis is probably to choose beam stiffnesses corresponding to perfect antisymmetrical bending (giving $k = 6EI_b/L_b$). This is equivalent to assuming a hinged support at midspan of the beams. With this assumption, $k_1 = 6EI_b/L_b = 2EI/L$ and $k_2 = 10EI/L$ are obtained. Corresponding G values (Eq. (4)) become (G_1, G_2)=(3, 0.6) for Column 1 of Panel 1, (3.3, 0.66) for Column 2 of Panel 1, (3, 0.6) for Column 1 of Panel 2 and (6, 1.2) for Column 2 of Panel 2. However, instead of these, the chosen end restraints were determined by assuming a hinged support at the real first-order inflection point locations in the beams. Measured from the top and bottom joint of Column 1, these were $0.498L_b$ and $0.491L_b$, respectively, for Panel 1, and $0.491L_b$ and $0.443L_b$, respectively, for Panel 2. The corresponding G values are given in the figures. For the columns of Panel 1, they are almost identical to those above. For the columns of Panel 2, the difference is somewhat larger, but of little practical importance.

Finally, the most rational approach would have been to compute isolated column end restraints according to first-order theory, e.g. from $\kappa = k/(EI_b/L_b) = 6/(2 - (M_f/M_n))$, where M_n and M_f are the first-order beam moments at the end considered and at the far end, respectively (Hellesland 2009). This gives, for instance, $(k_1, k_2) = (6.05, 6.23)$ and $(G_1, G_2) = (2.98, 0.58)$ for Column 1 of Panel 1. In the considered cases, resulting differences in computed B values will be minor, and will not alter the conclusions drawn below based on inflection-point
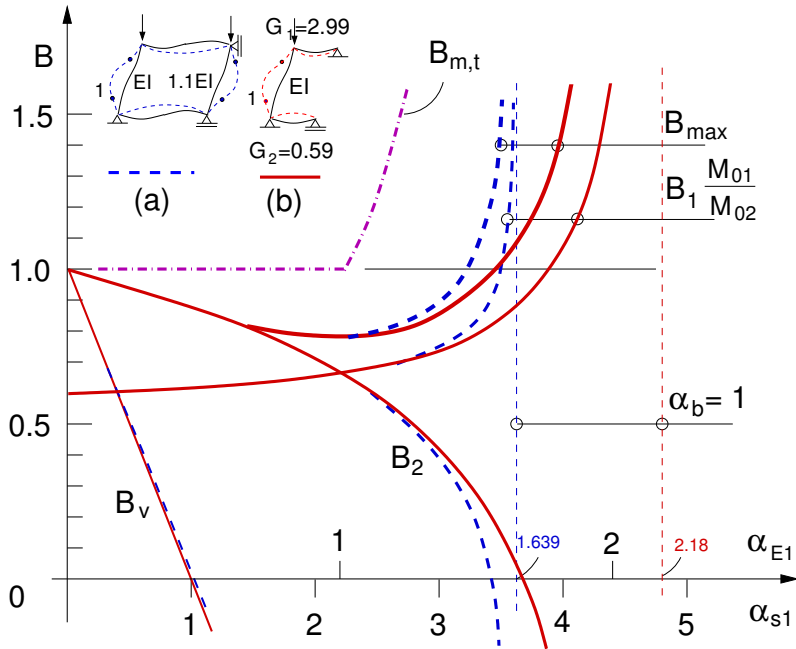
28

Figure 12: Moments and shear versus axial load level for two cases: (a) Column 1 (left hand) of Panel 1. (b) Column 1 in the panel considered in isolation with approximate restraints.

restraints.

7.3 Results

System instability of the panels will always be initiated in the most flexible column. This is Column 1, with the highest load index α_E of the two columns. Panel 1, with reasonably close load indices, $(\alpha_{E1} = 1.1\alpha_{E2})$, represents a more practical case than Panel 2, with rather big difference in load indices $(\alpha_{E1} = 2\alpha_{E2})$.

Results for both columns in the two panels are shown in Figs. 12, 13, 14 and 15. Panel column results are shown by dashed (blue) lines and the isolated column results are shown by solid (red) lines. The two cases are shown by inserts in the upper left corner of the figures. The dash-dot curves labeled $B_{m,t}$, representing present design maximum moment magnifiers, will be discussed later (Section 8).

Results are plotted versus the nominal load indices α_E. In addition, abscissas in terms of the free-sway index α_s are added for the convenience of reading and interpretation of results. The α_s abscissa is in each case computed with the free-sway critical load of the respective isolated columns $(\alpha_s = \alpha_E \beta_s{}^2)$. Thus, zero
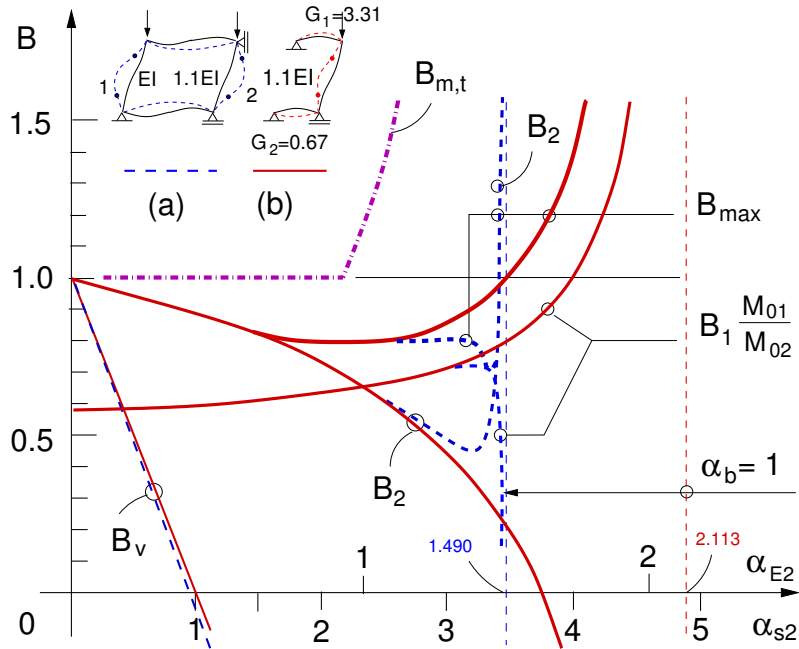
Figure 13: Moments and shear versus axial load level for two cases: (a) Column 2 (right hand) of Panel 1. (b) Column 2 in the panel considered in isolation with approximate restraints.

shear of the isolated columns will always be obtained for $\alpha_s = 1$ in the figures.

Effective lengths of isolated columns:
Panel 1 Column 1: $\beta_s = 1.483$ and $\beta_b = 0.678$ (for $G_1 = 2.99$ and $G_2 = 0.59$).
Panel 1 Column 2: $\beta_s = 1.522$ and $\beta_b = 0.688$ (for $G_1 = 3.31$ and $G_2 = 0.67$).
Panel 2 Column 1: $\beta_s = 1.466$ and $\beta_b = 0.672$ (for $G_1 = 2.95$ and $G_2 = 0.53$).
Panel 2 Column 2: $\beta_s = 1.824$ and $\beta_b = 0.757$ (for $G_1 = 6.11$ and $G_2 = 1.34$).

7.4 Response characteristics

The shears in the panel columns and the isolated single columns are seen to become zero at almost the same load index. The difference is about $\pm\,1\%$ for the columns of Panel 1, and about $\pm\,3.8\%$ for Panel 2 at $\alpha_s = 1$.

Also, the moments of the panel columns are seen to initially follow the isolated column moments quite closely. Consequently, the panel columns respond initially with almost stationary end restraints that are nearly equal to the restraints employed in the isolated column analyses. For the columns of Panel 1 $(EI_2 = 1.1EI_1))$, the difference in maximum moment values is less than 5% for
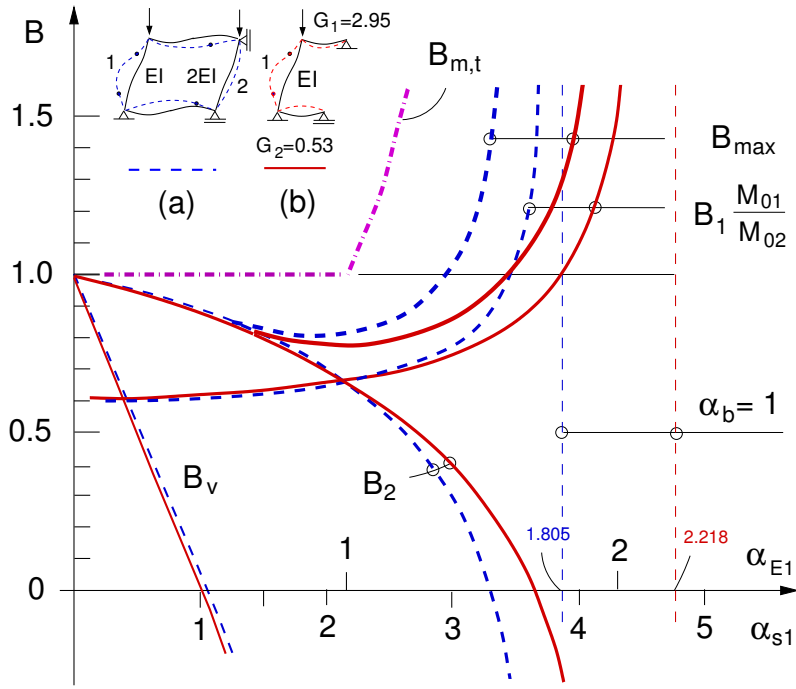
Figure 14: Moments and shear versus axial load level for two cases: (a) Column 1 (left hand) of Panel 2. (b) Column 2 in the panel considered in isolation with approximate restraints.

axial force levels $\alpha_b < 0.85$, i.e. for force levels less than about 85% of the axial force at panel instability (Figs. 12, 13).

For the columns of Panel 2, with large difference in column stiffnesses ($EI_2 = 2EI_1$), the load level giving 5% difference in moment values is now about 65% of the panel instability loading (Figs. 14 and 15). This is still a rather high load level. In design practice, load levels above about 60 % of the system critical load is believed to be rare, and not to be recommended due to the sensitivity of compression members to second-order effects at such high load levels. Consequently, the isolated column results can be considered to be representative for framed columns with practical load levels.

The critical loads of the panel columns are lower than those of the single columns (with stationary restraints). This is due to the "softening" of the end restraints provided by the beams with increasing axial column loads. This "softening" is due to the beams becoming gradually more flexible when they "unwind" from double curvature towards more single curvature type bending, as illustrated in Fig. 11.

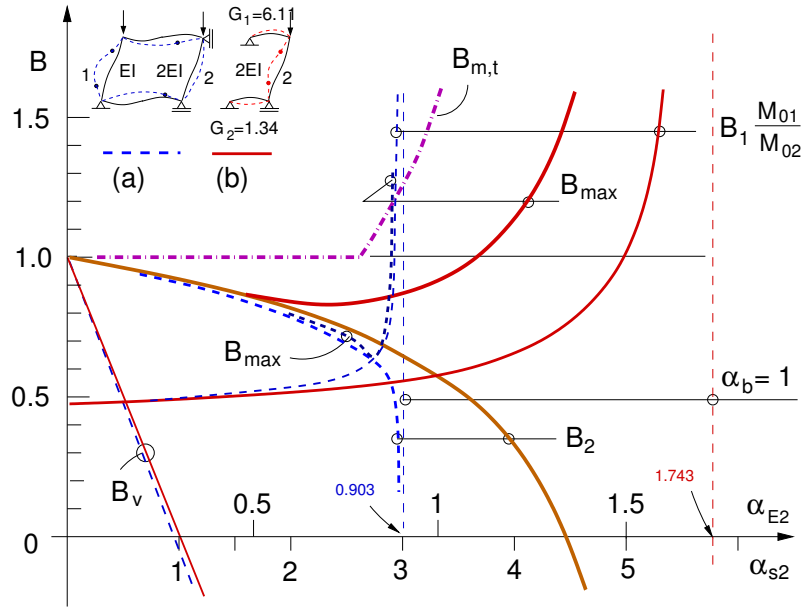In the case of Panel 1, Fig. 12, the beam unwinding to single curvature implies a

Figure 15: Moments and shear versus axial load level for two cases: (a) Column 2 (right hand) of Panel 2. (b) Column 2 in the panel considered in isolation with approximate restraints.

reduction in restraint stiffness to about one third of the initial (double curvature) stiffness. The effect of this is that Column 1, the most flexible of the two columns, initiates system instability ($\alpha_b = 1$) at a lower load index (1.639 α_{E1}) than that of the isolated single Column 1 (2.18 α_{E1}). Furthermore, associated with the unwinding of the beams is a rather sudden reversal (unwinding) of end moments in the stiffer Column 2. This is seen in Fig. 13, for loads close to the panel critical load.

In other cases, only a partially unwinding of beams may take place (Fig. 11(b)). This is the case for Panel 2, with Column 2 being significantly stiffer than the other ($\alpha_{E2} = \alpha_{E2}/2$). For axial loads close to panel instability, the stiff Column 2 unwinds from double to single curvature bending (Fig. 15). Such a bending reflects a braced effective length factor of Column 2 that is greater than 1.0, which again implies that Column 2 contributes (together with the beams) to the restraint of the flexible Column 1. By comparing Fig. 14 to Fig. 12, this contribution can be seen to have increased the panel critical load, in terms of α_{E1}, from $1.639\alpha_{E1}$ for Panel 1 to $1.805\alpha_{E1}$ for Panel 2.

8 Maximum design moment - Present practice

The maximum moment factor, B_{max}, for in-plane bending is commonly approximated by a factor, here denoted B_m, that can be given by

$$B_m = \frac{C_m}{1 - \alpha_b} \geq 1.0 \tag{53 a}$$

where

$$C_m = 0.6 + 0.4\mu_0 \tag{53 b}$$

$$\mu_0 = -\frac{B_s M_{01}}{B_s M_{02}} = -\frac{M_{01}}{M_{02}} \tag{53 c}$$

Above, α_b is the critical braced load index (Eq. (1c)), C_m is a first-order moment gradient factor (that accounts for other than uniform first-order moments), and μ_0 is the first-order end moment ratio taken to be positive when the member is bent single curvature, and negative otherwise.

This approximation is adopted by a number of major structural design codes such as for instance ACI 318 (ACI 2014) for concrete structures and AISC 360 (AISC 2016) for steel structures, and there denoted δ_{ns} and B_1, respectively. The same approximations are given in the European code EC2 (CEN 2004) and is implicit in EC3 (CEN 2005). These two codes limit C_m to 0.4.

Single restrained columns.
B_m predictions using Eq. (53) are shown in Figs. 4, 5, 6 and 7. These are based on $\alpha_b = N/N_{cb}$ with N_{cb} computed for end restraints given in the respective figures (i.e., with the same restraints used in the laterally displaced column analyses). The first-order end moment ratios are given by $\mu_0 = -(G_2 + 3)/(G_1 + 3)$ (from Eq. (16)).

As seen, the approximate B_m curves are rather conservative, and particularly so for columns with nearly equal end restraints (Fig. 7). This is primarily due to the approximate nature of the moment gradient factor C_m, that tends to be conservative (too large) in double curvature bending cases, such as the present cases.

Panel columns.
For a regular multicolumn frame, it is common design practice, and in accordance with most codes of practice (such as AISC, ACI etc.), to assume that beams bend into symmetrical, single curvature at braced frame instability. This implies that the beam deflection has a horizontal tangent at beam midlength, and that the rotational beam stiffnesses are $k = 2EI_b/L_b$ (1/3rd of stiffnesses for beams in

antisymmetrical curvature bending). B_m factors (Eq. (53)) computed with such restraints will for the sake of distinction and clarity be denoted $B_{m,t}$ (subscript t for tangent.

Such $B_{m,t}$ are shown in Figs. 12, 13, 14 and 15. The first-order moment ratios used in the calculations are those obtained from the first-order panel analyses (μ_0 = -0.600, -0.585, -0.585, -0.483, respectively, for Column 1 and 2 of Panel 1, and Column 1 and 2 of Panel 2). These $B_{m,t}$ predictions are even more conservative than those found above for the single columns with stationary restraints.

Column 2 of Panel 2 (Fig. 15) represents a case in which the assumption of beam restraint stiffness $k = 2EI_b/L_b$ gives a N_{cb} that is greater than the braced critical load of the panel. If B_m had been computed with the latter load, which is an accepted alternative by the codes, the rising branch would move towards the left, thus giving even more conservative predictions for this case.

Conclusions.
It is clear that B_m given by Eq. (53) is very conservative for columns in frames with moments caused by sidesway, in particular in the rising branch region ($B_m > 1$). Until better design moment formulations become available, and for axial load indices within the limits given in Section 6.5, the maximum moments can be taken equal to the larger sway magnified first-order end moment $B_s M_{02}$ (implying $B_m = 1$).

Efforts to derive improved moment formulations are under way, and will be presented in a separate report (Hellesland 2019).

9   Summary and conclusions

Columns in sway frames can be divided into "supporting sway columns" with axial loads below the free-sway load index ($\alpha_s < 1$), which resist lateral shears, and "supported sway columns" ($\alpha_s > 1$), which require lateral support in the form of a negative shear force. A supported sway column acts like a braced column with an initial displacement of one end relative to the other end.

Second-order theory for such columns with sidesway have been derived in a form suitable for studying the general member mechanics for increasing axial loads.

Theoretically derived, closed form expressions have been presented for a number

of member response characteristics that enable quick establishment of typical moment-axial load curves. These will be useful in education towards providing a general understanding of the often complicated column response, and also helpful in practical design work, and as a complement to full second-order analyses.

Laterally displaced single column models, isolated from laterally displaced panels, and thus with stationary restraints, are found to describe the response of panel columns very closely up axial load levels close to (0.6-0.85 of)the critical panel loading.

End moments described by secants through moment points at zero axial load and the free-sway critical load ($\alpha_s = 1$) provide good end moment approximations over a wide axial force range.

The maximum column moments due to sidesway may develop between ends, but will be less than the sway-magnified first-order moment ($B_s M_{02}$) for a wide axial force range, and particularly so for columns restrained by stiff beams.

For practical frames it is found that the maximum moment is less than $B_s M_{02}$ for axial loads up to 50% of the fully braced critical load ($\alpha_b < 0.5$) for single columns (with stationary end restraints), and up to about 80% of the fully braced critical load of columns with non-stationary restraints (such as in panels, or multibay frames). Axial forces will most often be lower than these limits. Thus, design for the sway-magnified first-order moments ($B_s M_{02}$) will thus be conservative in most practical cases.

The rising branch approximations of the maximum moment curve in present structural design codes are found to be quite conservative for columns with moments solely due to sidesway.

ACKNOWLEDGMENTS

NOTATION

B_m = approximate maximum moment magnification factor;
B_{max} = maximum moment magnification factor;
B_s = sway magnification factor;

B_1, B_2 = end moment factors, at end 1 and 2;
EI, EI_b = cross-sectional stiffness of columns, and beams;
G_j = relative rotational restraint flexibility at member end j;
H = applied lateral storey load (sum of column shears and bracing force);
L, L_b = lengths of considered column and of restraining beam(s);
N = axial (normal) force;
N_{cr} = critical load in general $(= \pi^2 EI/(\beta L)^2)$
N_{cb}, N_{cs} = critical load of columns considered fully braced, and free-to-sway;
N_E = Euler buckling load of a pin-ended column $(= \pi^2 EI/L^2)$
R_j = rotational degree of fixity at member end j;
V_0, V = first-order, and total (first+second-order) shear force in a column;
k_j = rotational restraint stiffness (spring stiffness) at end j
α_{cr} = member (system) stability index $(= N/N_{cr})$;
α_b, α_s = load index of column considered fully braced, and free-to-sway;
α_E = nominal load index of a column $(= N/N_E)$;
β = effective length factor (at system instability);
β_b, β_s = effective length factor corresponding to N_{cb} and N_{cs};
Δ_0, Δ = first-order, and total lateral displacement;
κ_j = relative rotational restraint stiffness at end j $(=k_j/(EI/L))$.

REFERENCES

ACI (2014). *"Building code requirements for structural concrete (ACI 318-M14) and Commentary (ACI 318RM-14)."* American Concrete Institute, Farmington Hills, Mich.

AISC (2016). *"ANSI/AISC 360-16 Specification for structural steel buildings, and Commentary."* American Institute of Steel Construction, Chicago, IL.

Baker, J.F., Horne, M.R., Heyman, J. (1956), "The steel skeleton", Vol. II, University Press, Cambridge, England.

Bleich, F. (1952). *"Buckling strength of metal structures."* McGraw-Hill Book Co., Inc., New York, N.Y.

CEN (2004). *"Eurocode 2: Design of concrete structures – Part 1-1: General rules and rules for buildings (ENV 1992-1-1:2004:E)."* European standard, European Committee for Standardization (CEN), December, Belgium.

CEN (2005). *"Eurocode 3: Design of steel structures – Part 1-1: General rules and rules for buildings (EN 1993-1-1:2005:E)"*, European standard, European Committee for Standardization (CEN), December, Belgium.

Chen, W.F. and and Lui, E. M. (1991). *"Stability design of steel frames."* CRC Press, Boca Raton, Fl., USA.

Galambos, T. V. (1968). *"Structural members and frames."* Prentice Hall, Inc., Englewood Cliffs, N.J., USA.

Galambos, T. V., and Surovek, A. E. (2008). *"Structural stability of Steel: concepts and applications for structural engineers."* John Wiley & Sons, Inc., Hoboken, N.J., USA.

Hellesland, J. (1976). "Approximate second order analysis of unbraced frames." Technical Report, Dr. ing. Aas-Jakobsen Inc., Oslo, Norway, 43 pp.

Hellesland, J. and MacGregor, J.G. (1981). "Mechanics and design of columns in sway frames", Preliminary draft (unpublished), Dept. of Civ. Engineering, Univ. of Alberta, Edmonton, Canada.

Hellesland, J. and Bjorhovde, R. (1996a). "Restraint demand factors and effective lengths of braced columns." *J. Struct. Eng.*, ASCE, **122**(10), 1216–1224.

Hellesland, J. and Bjorhovde, R. (1996b). "Improved frame stability analysis with effective lengths." *J. Struct. Eng.*, ASCE, **122**(11), 1275–1283.

Hellesland, J. (2007). "Mechanics and effective lengths of columns with positive and negative end restraints." *Engineering Structures*, **29**(12), 3464–3474.

Hellesland, J. (2009a). "Extended second order approximate analysis of frames with sway-braced column interaction." *Journal of Constructional Steel Research*, **65**(5), 1075–1086.

Hellesland, J. (2009b). "Second order approximate analysis of unbraced multistorey frames with single curvature regions." *Engineering Structures*, **31**(8), 1734–1744.

Hellesland, J. (2012). "Evaluation of effective length formulas and application in system instability analysis." *Engineering Structures*, **45**(12), 405–420.

Hellesland, J. (2019). "Approximate end moment and maximum moment formulations for slender columns in frames with sway." *Research Report in Mechanics*, No.1, March, Dept. of Math., Univ. of Oslo, Norway, 44 pp.

Ketter, R.L. (1961). "Further studies of the strength of beam-columns", *Proceedings*, ASCE (ST.6).

Livesley, R.K. (1956). "The application of an electronic digital computer to some problems of structural analysis". *The Structural Engineer*, January.

Livesley, R.K. (1975). "*Matrix methods of structural analysis.*" 2nd ed., Pergamon Press, New York, N.Y., USA.

Timoshenko, S.P. and Gere, J.M. (1961). "*Theory of elastic stability*", 2nd ed., McGraw-Hill Book Co.