UiO : Department of Mathematics
University of Oslo

Counterparty Risk in Energy Markets

Daniel Årvik

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This master's thesis is submitted under the master's programme *Modelling and Data Analysis*, with programme option *Finance, Insurance and Risk*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Preface

The 1990's marked the beginning of the liberalisation of the electricity and gas markets. Before the liberalisation of the energy sector, prices were driven by the cost of production. The risks in energy markets include aspects such as price risks, volume risks and political risk. Due to ever increasing interrelation between states there is also political risk, an example of which can be a nation state nationalizing the energy resources. Some parts of the framework from traditional markets such as the bond market and stock market may be used in the energy markets as well. However, the energy markets do differ quite notably in some respects from the traditional markets. For one, electricity is rather difficult to store. Therefore, there must always be a balance in the generation and consumption of energy. To illustrate, take as an example a stock holder trading in a stock market. If the price of one of the stocks in the stockholders portfolio is low, the stockholder often has the opportunity to hold (or store) it until the price is acceptable. Contrarily, an owner of a wind turbine cannot easily, or rather cost-efficiently, store the electricity generated until the price is right. Nonetheless, in order to exploit many of the thoroughly researched concepts and results in traditional mathematical finance, assumptions are often made to make the theoretical foundation of energy markets as similar as possible. This thesis will try to uncover how counterparty risk affects the price of energy derivatives. More precisely, it considers the price of forward contracts where there exists a possibility that one of the parties have a risk of default. This possibility of default is known as *Counterparty risk*. For this purpose, an introduction of the framework established for stochastic modelling in energy markets is presented, before some of the most used mathematical models are stated. An introduction of credit risk modelling is given in the second chapter in order to establish a framework for the integration of risk management into energy markets modelling. Chapter 3 discusses the topic of *Monte Carlo simulations* of stochastic processes in general, and in addition a section is reserved for the consideration of a particular stochastic process which is widely used in energy market modelling. A special type of options, so-called *Quanto options*, are reviewed in chapter 4. Finally, in chapter 5, analytical solutions of forward prices and call options are given both in standard form, and with includement of credit risk management.

Acknowledgements

I would like to thank my supervisor and sensei, Professor Fred Espen Benth, for our weekly discussions on stochastic analysis and the energy markets. A huge thanks also goes to my family and friends who have supported me through my degree.

Daniel Årvik
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Contents

Chapter 1: Stochastic Modelling in Energy Markets

The theory in this chapter relies on results given in *Stochastic Modelling of Electricity and Related Markets*, by, Benth, Saltyte and Koekebakker, [BŠK08]. In this chapter, some of the classical models used for stochastic modelling in energy markets are presented. A main aspect of the models is that they have to account for prices driven by supply and demand, a dynamic of which for instance electricity prices are intuitively prone to follow. This is where the *Ornstein-Uhlenbeck process* (henceforth OU-process) comes into play. Another aspect is major fluctuations in price. Take as an example the oil crisis in 1973, when OAPEC proclaimed an oil embargo resulting in a quadrupling of the oil price. Before introducing the models, some of the theoretical framework is established.

1.1 Mathematical Framework

In this chapter, let $T* < \infty$ be some finite time date, and assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}\mathbb{P})$ is a filtered probability space. There are some elements in stochastical modelling in energy markets that differ from conventional models in mathematical finance. A key feature in traditional mathematical finance is the existence of *risk-neutral probability measure* or equivalent martingale measures. As a reminder to the reader, in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a probability measure $\mathbb{Q}$, is called a risk-neutral probability measure if it is equivalent to the objective probability measure P, and in addition discounted price dynamics are a martingales with respect to $\mathbb{Q}$, for any event $A \subset \Omega$. The existence of a risk-neutral probability measure is ensured by restricting the spot price models to the class of *semimartingale processes*, which informally can be defined on $(\Omega, \mathcal{F}, \mathbb{P})$ as a real-valued process $\tilde{M}$, where $\tilde{M}$ can be decomposed as

$$\tilde{M}(t) = M(t) + C(t),$$

where $M(t)$ is a (local) martingale and C is an adapted process with right continuous left limits and of finite variation. Furthermore, the existence of these risk-neutral probability measures is sufficient to ascertain that there are no arbitrage opportunities. In energy markets however, the assets in question may not be applicable for frictionless buying, selling or storing. In other words, the assets are not viewed as tradeable assets, and consequently any probability measure equivalent to the objective probability measure will be a risk-neutral probability measure. Despite this fact, it is often convenient to let the spot price dynamics be semimartingales as it is analytically advantageous. Contrarily, in the swap and futures market, transactions are made frictionlessly and arbitrage opportunity exists if the forward/futures price dynamics are not semimartingales. There are several ways of obtaining a risk-neutral probability measure from the objective probability measure. Whenever the price of a security or asset has a stochastic term in the form of a Brownian motion, the *Girsanov theorem* can be used to rewrite the objective price dynamics under $\mathbb{P}$ to risk-neutral price dynamics under $\mathbb{Q}$. Analogously, if the stochastic term is a Compound Poisson process (henceforth CPP-process,see section (1.4)), a generalization of the Girsanov theorem, the so-called *Esscher transform*, may be used to construct risk-neutral probabilities. The $\mathbb{P}$-dynamics are applied when modelling energy prices, while the $\mathbb{Q}$-dynamics are applied when pricing options. Both Girsanov's theorem and Esscher transform are structure preserving, meaning that a Brownian motion will still be a Brownian motion and the CPP-process will still be a CPP-process under the new probability measure. In most practical examples, the measure change will be part of the modelling work, and needs parameter estimation. In this text, the modelling work that involves measure change is not discussed to any great extent. The interested reader may be referred to Option Theory with Stochastic Analysis by Fred Espen Benth, [Ben04]. It will however be clearly stated whether the price dynamics at hand are given under the objective or risk-neutral probability measure. The original Esscher transform is given in the following definition.

Definition 1.1. Let f be a probability density and $\theta \in \mathbb{R}$. As long as

$$\int_{\mathbb{R}} e^{\theta y} f(y) dy$$

exists, there can be defined a transformed density

$$f(x;\theta) = \frac{e^{\theta x} f(x)}{\int_{\mathbb{R}} e^{\theta y} f(y) dy},$$

which is called the Esscher transform of f.

A version of the Girsanov theorem is given in what follows. For a more general version of the theorem, see e.g. [Øks03].

Theorem 1.2. *Let X be a continuous time stochastic process with dynamics given by*

$$dX(t) = \alpha(t)dt + \sigma dB(t), \qquad t < T \tag{1.1}$$

where B is a Brownian motion under $\mathbb{P}$. The coefficient functions $\alpha, \sigma \in \mathbb{R}$ are assumed to be integrable and Borel-measurable functions and $\alpha(t), \sigma(t)$ are $\mathcal{F}_t$-adapted. Assume there exist functions ζ, θ such that

$$\sigma(t)\theta(t) = \alpha(t) - \zeta(t),$$

where ζ, θ are integrable, F_t-adapted and Borel-measurable, while θ also satisfies

$$\exp\left(\frac{1}{2}\int_0^T |\theta(t)|^2 dt\right) < \infty. \tag{1.2}$$

Then (1.2) ensures that

$$M(t) = \exp\left(-\int_0^t \theta(u)dB(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right)$$

is a martingale on $[0, T]$. Further, define the Radon-Nikodym derivative by

$$\frac{dQ}{d\mathbb{P}}|_{\mathcal{F}_t} = M(t).$$

Then for $t < T$

$$\tilde{B}(t) = \int_0^t u(s)ds + B(t),$$

is a Brownian motion with respect to $\mathbb{Q}$, and Y has dynamics under Q given by

$$dY(t) = \zeta(t)dt + \theta(t)d\tilde{B}(t).$$

Remarks. The name equivalent martingale measure has a natural explanation. It is called equivalent since if $\mathbb{P}(A) > 0$ then $\mathbb{Q}(A) > 0$, and it is called a martingale since all price dynamics are semimartingales under $\mathbb{Q}$, and it is called a measure since $\mathbb{Q}$ is a probability measure. It is also worth mentioning that in some texts the objective probability measure $\mathbb{P}$ is called the *market probability*.

1.2 Spot Price Dynamics

The spot price dynamics have two main types, namely, arithmetic spot price and geomtric spot price. If it is assumed that the spot price is given by a price process P, then the arithmetic spot price S at time t is simply given by

$$S(t) = P(t), \tag{1.3}$$

and the geomtric spot price will be given by

$$S(t) = e^{P(t)}. \tag{1.4}$$

One of the noticeable differences between these two is that a geomtric spot prices can not be negative, while they can be in the arithmetic case. Intuitively, the restriction to non-negative spot prices may make sense, since a negative power price would imply the power producers to pay their clients when providing power. However, negative power prices may at times be seen on some power exchanges. One of the explanations for the occurence of negative power spot prices is that the producers may find themselves in a situation where paying to get rid of the power is cheaper than shutting down their production sites. Another difference is more convoluted, and reveals itself when modelling the forward price via the spot price. Details on this subject will be given in section (1.5). Independent of the choice of arithmetic or geometric dynamics, the following models may be used.

1.3 The Ornstein-Uhlenbeck Process

One of the canonical models lets the spot price of electricity follow an Ornstein-Uhlenbeck process.

Definition 1.3 (OU-process)**.** An OU-process is a stochastic process $X(t)$ satisfying the stochastic differential equation

$$dX(t) = (\mu(t) - \alpha(t)X(t))dt + \sigma(t)dB(t), \tag{1.5}$$

where α, σ, μ are assumed to be integrable and measurable functions, and $B(t)$ is a Brownian Motion.

The requirement for α, σ, μ to be integrable and measurable is to ensure that the stochastic differential equation (henceforth SDE), (1.5), describing the dynamics of the OU-process has a unique, strong, global solution. Moreover, the process has independent increments and it is stationary. Another essential aspect of the OU-process is that it is a so-called mean-reverting process, meaning that the stochastic process X is assumed to have given longterm mean-level described by $\mu(t)$. The rate at which the process tends to this level is determined by the coefficient $\alpha(t)$.

Proposition 1.4 (Explicit solution to Ornstein-Uhlenbeck)**.** *The stochastic process $X(s)$ for $s \geq t$ satisfying (1.5) is given by*

$$X(s) = X(t)e^{-\int_t^s \alpha(u)du} + \int_t^s \mu(u)(e^{-\int_u^s \alpha(v)dv})du + \sigma(u)\int_t^s e^{-\int_u^s \alpha(v)dv}dB(u). \tag{1.6}$$

If the functions μ, α, σ in (1.6) are assumed to be constants, the explicit solution (1.6) can be written as

$$\begin{aligned} X(s) &= X(t)e^{-\int_t^s \alpha du} + \int_t^s \mu(e^{-\int_u^s \alpha dv})du + \sigma \int_t^s e^{-\int_u^s \alpha dv} dB(u) \\ &= X(t)e^{-\alpha(s-t)} + \mu(1-e^{-\alpha(s-t)}) + \sigma \int_t^s e^{-\alpha(s-u)} dB(u) \end{aligned} \tag{1.7}$$

Proposition 1.5. *Let $X(s)$ be a stochastic process with explicit solution (1.7). Then for $s \geq t$*

$$\mathrm{E}[X(s)] = X(t)e^{-\alpha(s-t)} + \mu(1-e^{-\alpha(s-t)}) \tag{1.8}$$

$$\mathrm{Var}[X(s)] = \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha(s-t)}) \tag{1.9}$$

Proof. Due to the independent increments of the Brownian motion, it is clear that

$$\mathrm{E}\left[\sigma \int_t^s e^{-\alpha(s-u)} dB(u)\right] = 0,$$

and by the Itô isometry

$$\mathrm{Var}\left[\int_t^s e^{-\alpha(s-u)} dB(u)\right] = \int_t^s e^{-2\alpha(s-u)} du = \frac{1}{2\alpha}\left(1-e^{-2\alpha(s-t)}\right)$$

□

Furthermore, X has a limiting distribution, which is given by

$$\lim_{t\to\infty} \mathrm{E}[X(t)] = \lim_{t\to\infty} \left[X(0)e^{-\alpha t} + \mu(1-e^{-\alpha t})\right] = X(0) + \mu,$$

and

$$\lim_{t\to\infty} \mathrm{Var}[X(s)] = \lim_{t\to\infty} \left[\frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t})\right] = \frac{\sigma^2}{2\alpha}.$$

Figure (1.3) shows the simulation of five sample paths over 90 days of an OU-process of the form (1.7) with long-term mean level $\mu = 10$, speed of mean reversion $\alpha = 0.3$, and volatility $sigma = 0.05$.

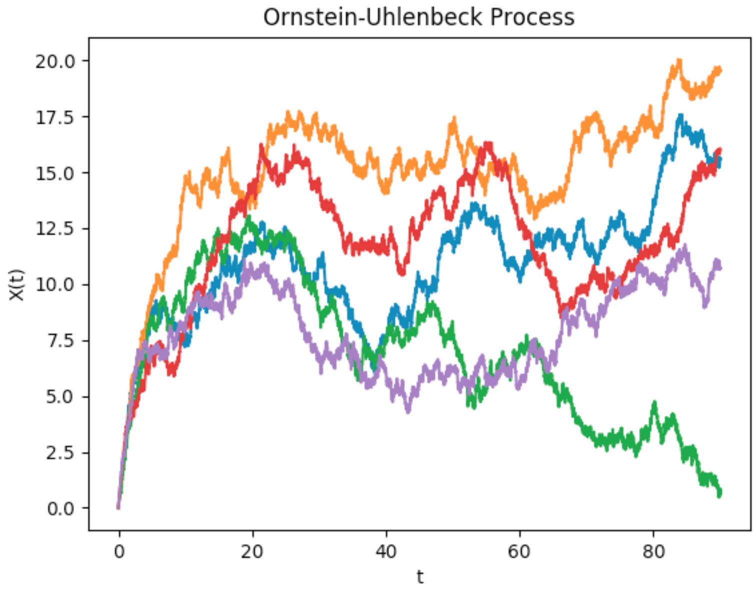

Figure 1: Simulation of five different sample paths of an OU-process over a period of 90 days with initial $X(0) = 0, \mu = 10, \alpha = 0.3, \sigma = 0.05$.

1.4 The Compound Poisson Process

Where the OU-process is included in energy spot price models to describe the long-term price development driven by supply & demand, the purpose of modelling with a Compound Poisson Process (henceforth CPP-process) is to simulate the greater fluctuations in price. The CPP-process will serve as the stochastic term in a stochastic differential equation, much like the form of a OU-process, though without the term including a Brownian motion.

Definition 1.6 (CPP-Process)**.** Let $N(t)$ be a Poisson process with intensity λ. A CPP-process is a stochastic process $I(t)$ on the form

$$I(t) = \sum_{i=1}^{N(t)} U_i, \tag{1.10}$$

where the U_i's are independent and identically distributed variables according to some distribution function F_U.

The CPP-process possesses, like the OU-process, stationarity and also has independent increments. In the figure below (1.4), the CPP-process is illustrated, where the U_i's are the jump sizes, the $t_i's$ are the jump times, and the jump times occure according to a Poisson distribution.

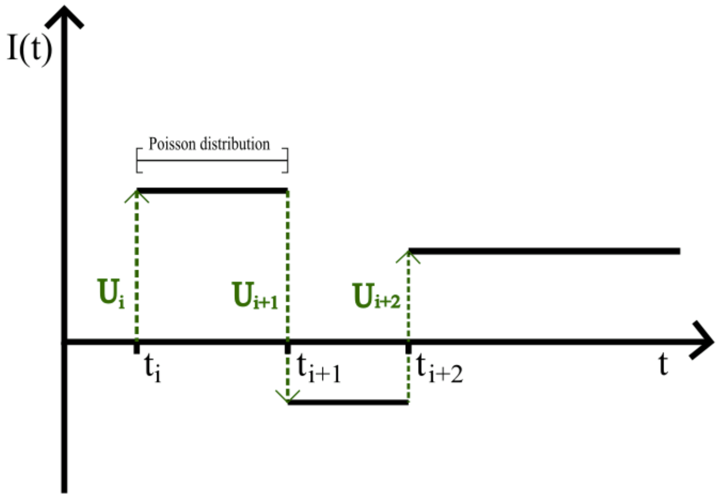

Figure 2: Illustration of a jump process. The U_i's represents the jump size and the t_i's are the jump times that occure according to a Poisson distribution.

In figure 1.4 below, three sample paths of a CPP-process of the form (1.10) are sample, with jump times distributed as $U_i \sim N(0, 3)$, and the jump times occur according to $t_i \sim \text{Poisson}(\lambda)$ and $\lambda = 0.5$.

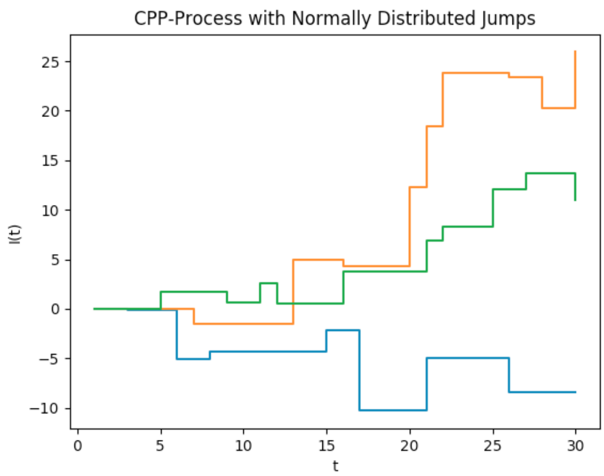

Figure 3: Simulation of three paths of a CPP-Process with $U_i \sim N(0, 3)$ and $\lambda = 0.5$.

Proposition 1.7. *The characteristic function of (1.10) is given by*

$$\mathrm{E}\left[e^{ixI(t)}\right] = e^{\psi(x)t}, \tag{1.11}$$

where

$$\psi(x) = \lambda \int_{\mathbb{R}} (e^{ixu} - 1) f(u) du, \tag{1.12}$$

and $F(u)$ is the distribution function of U.

Proof. This proof builds on a more general version of the proof which can be found in [BŠK08].
By the law of total expectation and the fact that the U_i's are independent indentically distributed variables

$$\begin{aligned}
\mathrm{E}\left[e^{ixI(t)}\right] &= \mathrm{E}\left[e^{ix\sum_{k=1}^{N(t)} U_k}\right] = \mathrm{E}\left[\mathrm{E}\left[e^{ix\sum_{k=1}^{n} U_k}\middle| N(t)=n\right]\right] \\
&= P(N(t)=n)\,\mathrm{E}[\Pi_{k=1}^{n} e^{ixU_k}] = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\left(\mathrm{E}[e^{ixU}]\right)^n \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\left(\int_{\mathbb{R}} e^{ixu}F(u)du\right)^n = e^{\lambda t\int_{\mathbb{R}}(e^{ixu}-1)f(u)du}.
\end{aligned}$$

$\square$

This latter result (1.12) will be useful when looking at the distribution of the SDE given in the following proposition.

Proposition 1.8. *Consider the stochastic differential equation*

$$dY(t) = (\delta - \beta Y(t))dt + \eta dI(t), \tag{1.13}$$

where δ, β and η are constants and $I(t)$ is a CPP-process as defined in (1.10) with $\mathrm{E}[U] < \infty$ *and* $\mathrm{Var}[U] < \infty$. *Then the explicit solution $Y(s)$ for $s \geq t$ is given by*

$$Y(s) = e^{-\beta(s-t)}Y(t) + \frac{\delta}{\beta}(1 - e^{-\beta(s-t)}) + \eta\int_t^s e^{-\beta(s-u)}dI(u). \tag{1.14}$$

The proof that (1.14) is a solution for (1.13) is shown for $\delta = 0$. Straighforward computation yields

Proof.

$$\begin{aligned}
-\beta\int_0^t Y(s)ds &= -\beta\int_0^t\left(e^{-\beta s}Y(0) + \eta\int_0^s e^{-\beta(s-u)}dI(u)\right)ds \\
&= -\beta\int_0^t e^{-\beta s}Y(0)ds - \beta\eta\int_0^t\int_0^s e^{-\beta(s-u)}dI(u)ds \\
&= -\beta Y(0)\left[-\frac{1}{\beta}e^{-\beta s}\right]_{s=0}^t - \beta\eta\int_0^t\int_0^s e^{-\beta(s-u)}dI(u)ds \\
&= e^{-bt}Y(0) - Y(0) - \beta\eta\int_0^t\int_0^s e^{-\beta(s-u)}dI(u)ds. \qquad (1.15)
\end{aligned}$$

Using the Fubini-Tonelli theorem on the last term in (1.15), the order of integration may be swapped. Hence

$$
\begin{aligned}
-\beta\eta \int_0^t \int_0^s e^{-\beta(s-u)} dI(u) ds &= -\beta\eta \int_0^t \int_u^t e^{-\beta(s-u)} ds dI(u) \\
&= -\beta\eta \int_0^t \left[-\frac{1}{\beta} e^{-\beta(s-u)} \right]_{s=u}^t dI(u) \\
&= \eta \int_0^t \left(e^{-\beta(t-u)} - 1 \right) dI(u) \\
&= \eta \int_0^t e^{-\beta(t-u)} - \eta I(t). \qquad (1.16)
\end{aligned}
$$

Combining (1.15) and (1.16) yields

$$
\begin{aligned}
-\beta \int_0^t Y(s) ds = e^{-bt} Y(0) - Y(0) + \eta \int_0^t e^{-\beta(t-u)} - \eta I(t) \qquad (1.17) \\
Y(t) - Y(0) - \eta I(t), \qquad (1.18)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
Y(t) &= Y(0) - \beta \int_0^t Y(s) ds + \eta I(t) \\
&= Y(0) - Y(0) + e^{-\beta t} Y(0) + \int_0^t (e^{-\beta(t-u)} - 1) dI(u) + \eta I(t) \\
&= e^{-\beta t} Y(0) + \int_0^t e^{-\beta(t-u)} dI(u).
\end{aligned}
$$

Thus (1.14) is indeed a solution for (1.13). □

Remarks. The coefficients α, β and η in (1.13) may be functions of t. If this is the case then $\alpha, beta$ and η must be assumed to integrable and measurable functions to ensure that (1.14) is a unique, strong and global solution. In this text however, these coefficients are assumed to be constants, though the results hold for certain functions as well.

Moving forward, the expected value and variance is given in the following proposition.

Proposition 1.9 (Expected value and variance of CPP). *Let the stochastic process $Y(t)$ be defined as in (1.10). Then for $s \geq t$*

$$
E[Y(s)] = e^{-\beta(s-t)} Y(t) + \frac{\delta}{\beta}(1 - e^{-\beta(s-t)}) + \lambda \, \mathrm{E}[U] \frac{\eta}{\beta}(1 - e^{-\beta(s-t)}) \qquad (1.19)
$$

and

$$
\mathrm{Var}[Y(s)] = \lambda \, \mathrm{E}\left[U^2\right] \frac{\eta}{2\beta}(1 - e^{-\beta(s-t)}) \qquad (1.20)
$$

Proof. The first two terms of $Y(s)$ are deterministic and thus expectation and variance are trivial. For the stochastic term, consider the characteristic function and observe that by (1.11) it follows that

$$\begin{aligned}\mathrm{E}\left[e^{ix\int_t^s e^{-\beta(s-u)}dI(u)}\right] =& e^{\psi\left(\int_t^s xe^{-\beta(s-u)}du\right)} \\ =& e^{\int_0^{s-t}\psi\left(xe^{-\beta u}\right)du\right)}.\end{aligned}$$

Furthermore, note that

$$\psi'(x) = \frac{\partial}{\partial x}\lambda\int_{\mathbb{R}}(e^{ixz}-1)F_U(dz) = \frac{\partial}{\partial x}\lambda\left(\mathrm{E}\left[e^{ixz}\right]-1\right) = i\lambda\,\mathrm{E}\left[ze^{ixz}\right], \quad (1.21)$$

and

$$\psi''(x) = \frac{\partial}{\partial x^2}\lambda\left(\mathrm{E}\left[e^{ixz}\right]-1\right) = -\lambda\,\mathrm{E}\left[z^2e^{ixz}\right]. \quad (1.22)$$

Expectation is the same as the first moment which is found by

$$\begin{aligned}\mathrm{E}\left[\int_t^s e^{-\beta(s-u)}dI(u)\right] =&(-i)\frac{\partial}{\partial x}\mathrm{E}\left[e^{ix\int_t^s e^{-\beta(s-u)}du}\right]\Bigg|_{x=0} \\ =&(-i)\frac{\partial}{\partial x}e^{\int_t^s\psi(xe^{-\beta u})du}\Bigg|_{x=0} \\ =&(-i)\int_0^{s-t}\psi'(xe^{-\beta u})e^{-\beta u}du\cdot e^{\int_t^s\psi(xe^{-\beta u})du}\Bigg|_{x=0} \\ =&(-i)\psi'(0)\int_0^{s-t}e^{-\beta u}du = \lambda\,\mathrm{E}\left[z\right]\frac{1}{\beta}(1-e^{-\beta(s-t)}),\end{aligned}$$

where the third equality is a result of the chain rule and fourth equality follows by $\psi(0)=0$.
Next, the second moment is found by obtaining the second derivative of the characteristic function

$$\begin{aligned}\mathrm{E}\left[\left(e^{ix\int_t^s e^{-\beta(s-u)}dI(u)}\right)^2\right] =& (-i)^2\frac{\partial}{\partial x^2}\mathrm{E}\left[e^{ix\int_t^s e^{-\beta(s-u)}du}\right]\Bigg|_{x=0} \\ =& -\frac{\partial}{\partial x^2}e^{\int_t^s\psi(xe^{-\beta u})du}\Bigg|_{x=0} \\ =& -\Bigg(\int_0^{s-t}\psi'(xe^{-\beta u})e^{-\beta u}e^{\int_t^s\psi(xe^{-\beta u})du}\cdot\int_0^{s-t}\psi'(xe^{-\beta u})e^{-\beta u}du \\ &+\int_0^{s-t}\psi''\left(xe^{-\beta u}\right)e^{-2\beta u}du\cdot e^{\int_0^{s-t}\psi(xe^{-\beta u})}du\Bigg)\Bigg|_{x=0} \\ =& -\left(\left(\psi'(0)\int_0^{s-t}e^{-\beta u}du\right)^2+\psi''(0)\int_0^{s-t}e^{-2\beta u}du\right),\end{aligned}$$

(1.22)

$$= -\left(i^2\left(\mathrm{E}\left[\int_t^s e^{-\beta(s-u)}dI(u)\right]\right)^2 - \lambda\,\mathrm{E}\left[z^2\right]\frac{1}{2\beta}\left(1-e^{-2\beta(s-t)}\right)\right)$$
$$= \left(\mathrm{E}\left[\int_t^s e^{-\beta(s-u)}dI(u)\right]\right)^2 + \lambda\,\mathrm{E}\left[z^2\right]\frac{1}{2\beta}\left(1-e^{-2\beta(s-t)}\right).$$

Finally, by the defintion of variance it is clear that

$$\begin{aligned}\mathrm{Var}\left[\int_t^s e^{-\beta(s-u)}dI(u)\right] &= \mathrm{E}\left[\left(e^{ix\int_t^s e^{-\beta(s-u)}dI(u)}\right)^2\right] - \left(\mathrm{E}\left[e^{ix\int_t^s e^{-\beta(s-u)}dI(u)}\right]\right)^2 \\ &= \lambda\,\mathrm{E}\left[z^2\right]\frac{1}{2\beta}\left(1-e^{-2\beta(s-t)}\right) \qquad (1.23)\end{aligned}$$

□

With the expected value and variance at hand, the limiting distribution is calculated below.

$$\begin{aligned}\lim_{t\to\infty}\mathrm{E}\left[Y(t)\right] &= \lim_{t\to\infty}\left[e^{-\beta t}Y(0) + \frac{\delta}{\beta}(1-e^{-\beta t}) + \lambda\,\mathrm{E}\left[U\right]\frac{\eta}{\beta}(1-e^{-\beta t})\right] \\ &= Y(0) + \frac{\delta}{\beta} + \lambda\,\mathrm{E}\left[U\right]\frac{\eta}{\beta},\end{aligned}$$

and

$$\lim_{t\to\infty}\mathrm{Var}\,Y(t) = \lim_{t\to\infty}\left[\lambda\,\mathrm{E}\left[U^2\right]\frac{\eta}{2\beta}(1-e^{-\beta t})\right] = \lambda\,\mathrm{E}\left[U^2\right]\frac{\eta}{2\beta}$$

1.5 Forward Pricing in Energy Markets

Forward pricing in energy markets is a well-studied field. In [**BSK08**], there is given a thorough mathematical framework as well as numerical examples, and this section will rely on results from this book. Let $T < T*$ be a finite delivery date of a forward contract, and suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space with corresponding filtration $\{\mathcal{F}_t\}_{0\le t\le T}$. In addition, let S be the stochastic process describing the dynamics of the spot price. Furthermore, let $F(t,T)$ denote the forward price agreed at time t with settlement T. Assumptions also include

- $F(t,T)$ is $\mathcal{F}_t$-measurable,
- $\mathrm{E}[|S(T)|] < \infty$,
- The existence of a risk-neutral pricing measure $\mathbb{Q}$ equivalent to the objective measure $\mathbb{P}$.

In the event that S is given as a geometric spot price, then of course $\mathrm{E}\left[S(T)\right] < \infty$ is sufficient. In practice, the assumption that the forward price $f(t,T)$ is assumed to be F_t-measurable is to say that it is based upon all available market information.
The payoff from paying forward price and receiving the spot price is then

$$S(T) - f(t,T). \tag{1.24}$$

In line with the no-arbitrage principle, the discounted expected payoff of any contingent claim should equal the price of entering the contract. Since there is no cost of entering a forward contract, it is clear that

$$e^{-r(T-t)}\,\mathrm{E}_{\mathbb{Q}}\left[S(T) - f(t,T)|\mathcal{F}_t\right] = 0. \tag{1.25}$$

Since $f(t,T)$ is $\mathcal{F}_t$-measurable and consequently adapted, it follows that the forward price is given by

$$F(t,T) = \mathrm{E}_{\mathbb{Q}}\left[S(T)|\mathcal{F}_t\right]. \tag{1.26}$$

Let it be noted that specifying the risk-netrual dynamics under $\mathbb{Q}$ is the same as defining the *risk premium* R defined by

$$R(t,T) = F(t,T) - \mathrm{E}\left[S(T)|\mathcal{F}_t\right] = \mathrm{E}_{\mathbb{Q}}\left[S(T)|\mathcal{F}_t\right] - \mathrm{E}\left[S(T)|\mathcal{F}_t\right]. \tag{1.27}$$

In most cases however, power is bought and delivered for a delivery period stretching over weeks or months.

Proposition 1.10. *Let τ_1, τ_2 be the starting time and end time of a delivery period for power, respectively. Denote by $F(t,\tau_1,\tau_2)$ the time t price of a forward contract with delivery period $[\tau_1, \tau_2]$ and settlement date $t < T < T^*$. Assume that the risk-free interest rate r is constant. Then*

$$F(t,\tau_1,\tau_2) = \frac{1}{\tau_2 - \tau_1}\,\mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(s)ds|\mathcal{F}_t\right].$$

Proof. The payoff at time of at time T

$$\int_{\tau_1}^{\tau_2} F(s,T) - S(s)ds,$$

and the price of entering the contract is zero, thus

$$e^{-r(T-t)}\,\mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} F(t,\tau_1,\tau_2) - S(s)ds|\mathcal{F}_t\right] = e^{-r(T-t)}\,\mathrm{E}\left[\int_{\tau_1}^{\tau_2} F(t,\tau_1,\tau_2)ds|\mathcal{F}_t\right] - e^{-r(T-t)}\,\mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(s)ds\right] = 0.$$

Hence

$$\mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} F(t,\tau_1,\tau_2)ds|\mathcal{F}_t\right] = \mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(s)ds|\mathcal{F}_t\right],$$

and since the forward price is based on information in $\mathcal{F}_t$, the forward price will be

$$\int_{\tau_1}^{\tau_2} F(t,\tau_1,\tau_2)ds = \mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(s)ds|\mathcal{F}_t\right],$$

or equivalently

$$F(t,\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(u)du|\mathcal{F}_t\right] \tag{1.28}$$

□

With the includement of a delivery period an auxiliary feature on whether utilizing the arithmetic (1.3) or geometric (1.4) model must be considered. This matter involves the distribution of

$$\int_{\tau_1}^{\tau_2} S(s)ds. \tag{1.29}$$

Every choice of stochastic spot price dynamics comes with a specified distribution, and the delivery period represented by the integral may alter this distribution. As an example, assume the spot price is given by an arithmetic OU-process, that is,

$$S(t) = X(t),$$

where X is defined as in (1.5). Then the distribution of S is normal due to the normality of X, and (1.29) can be written as the limit of a sum of independent, normally distributed random variables. S is therefore itself normally distributed. Choosing the geometric model on the on other hand, yields a sum of lognormally distributed random variables, which in general is not lognormally distributed.

Remarks. Throughout this text it will be assumed that the interest rate r is constant. Thus the theory on forward and futures prices are identical, and the results are therefore applicable for futures pricing as well.

1.6 Forward pricing with spot price dynamics given by OU-process

To differentiate between forward prices with arithmetic and geometric dynamics, the arithmetic case is denoted by F^A and the geometric case is denoted by F^G. Assume the spot price $S(t)$ of electricity follows the dynamics of an Ornstein-Uhlenbeck process where the mean-reverting level, speed of mean-reversion and volatility are constants i.e.,

$$dS(t) = (\mu - \alpha S(t))dt + \sigma d\tilde{B}(t), \tag{1.30}$$

where $\tilde{B}$ is a Brownian motion under a risk-neutral pricing measure $\mathbb{Q}$.
By (1.7) and (1.28), and assuming the initial value $S(t) = x$, $x \in \mathbb{R}$, the forward price becomes

$$\begin{aligned} F^A(t,T) = \mathrm{E}_{\mathbb{Q}}\left[S(T)|\mathcal{F}_t\right]_{S(t)=x} &= \mathrm{E}_{\mathbb{Q}}\Bigg[xe^{-\alpha(T-t)} + \mu(1 - e^{-\alpha(T-t)}) \\ &\qquad + \sigma \int_t^T e^{-\alpha(T-u)} d\tilde{B}(u) \Bigg| \mathcal{F}_t\Bigg] \\ &= S(t)e^{-\alpha(T-t)} + \mu(1 - e^{-\alpha(T-t)}) + \mathrm{E}_{\mathbb{Q}}\left[\sigma \int_t^T e^{-\alpha(T-u)} dW(u)\right], \end{aligned}$$

where the last equality is due to $S(t)$ being $\mathcal{F}_t$-measurable and the independent increment property of the Brownian motion $\tilde{B}$ makes $\tilde{B}(u)$ independent of $\mathcal{F}_t$ for all $t \leq u \leq T$. Additionally,

$$\mathrm{E}_{\mathbb{Q}}\left[\sigma \int_t^T e^{-\alpha(T-u)} d\tilde{B}(u)\right] = 0$$

since $\tilde{B}$ is Gaussian. Thus the forward price will be

$$F^A_{\mathrm{OU}}(t,T) = xe^{-\alpha(T-t)} + \mu(1 - e^{-\alpha(T-t)}).$$

This forward price has several features that in many instances makes it a reasonable choice. For one, as the time to maturity T increases, the forward price will converge to the mean since

$$\lim_{t\to\infty} F_{\mathrm{OU}}(t,T) = \lim_{t\to\infty}\left[xe^{-\alpha(T-t)} + \mu(1 - e^{-\alpha(T-t)})\right] = x + mu,$$

which on an intuitive level makes sense, because it is hard to make an educated guess on what the forward price will be 20 or 30 years from present time, making the mean the best estimate. A slightly more sophisticated model could even have a time dependent mean, which would make it possible to account for sustained increase(decrease) such as inflation(deflation). Another reasonable trait that comes with this forward price is that as the maturity time T approaches present time t it converges to

$$\lim_{T\downarrow t} F_{\mathrm{OU}}(t,T) = \lim_{T\downarrow t}\left[xe^{-\alpha(T-t)} + \mu(1 - e^{-\alpha(T-t)})\right] = x. \tag{1.31}$$

Thus the forward price converges to the initital value $S(t) = x$.
Moving on to the geometric model, the geometric spot price equivalent of (1.30) implies that for $T \geq t$

$$\ln S(T) = e^{-\alpha(T-t)}s(t) + \sigma \int_t^T e^{-\alpha(T-u)} d\tilde{B}(u),$$

making S lognormally distributed, and by the distribution properties of the OU-process given in (1.8), it is clear that

$$\begin{aligned} F_{OU}^{G}(t,T) = \mathrm{E}_{\mathbb{Q}}\left[S(T)|\mathcal{F}_t\right]_{X(t)=x} &= \mathrm{E}\left[\exp\left(e^{-\alpha(T-t)}X(t)\right)\right]_{S(t)=x} \\ &= \exp\left(e^{-\alpha(T-t)}x + \frac{1}{4\alpha}(1 - e^{-2\alpha(T-t)})\right) \end{aligned}$$

1.7 Forward pricing with spot price dynamics given by Compound Poisson process

As mentioned in section (1.1), attaining the risk-neutral dynamics of a spot price given by a CPP-process can be done by using the Esscher transform. The CPP-process will still be a CPP-process under the risk-neutral measure $\mathbb{Q}$, but will however have altered parameters.

Proposition 1.11. *Let the spot price have $\mathbb{P}$-dynamics as given by (1.13), and assume that an Esscher transform have given risk-neutral dynamics given by the SDE*

$$dS(t) = (\delta - \beta S(t))dt + \eta d\tilde{I}(t),$$

where $\tilde{I}$ is a CPP-process as defined in (1.10), under a risk-neutral measure $\mathbb{Q}$. Further, let the jumps U have mean and variance given by

$$E[U] = \mu_U < \infty, \qquad \mathrm{Var}[U] = \sigma_U < \infty.$$

Assume that S has time t value value $S(t) = x$, the forward price at time t will be given by

$$F_{CPP}(t,T) = xe^{-\alpha(T-t)} + \frac{\delta}{\beta}\left(1 - e^{-\beta(T-t)}\right) + \tilde{\lambda}\mu_U \frac{\eta}{\beta}(1 - e^{-\beta(T-t)}).$$

Proof. Again using (1.28), and (1.14)

$$\begin{aligned} F_{\mathrm{CPP}}(t,T) = E_{\mathbb{Q}}\left[S(T)|\mathcal{F}_t\right]_{S(t)=x} &= E_{\mathbb{Q}}\Bigg[S(t)e^{-\alpha(T-t)} + \frac{\delta}{\beta}\left(1 - e^{-\beta(T-t)}\right) \\ &\qquad + \eta \int_t^T e^{-\alpha(T-u)} d\tilde{I}(u)|\mathcal{F}_t\Bigg]_{S(t)=x} \\ &= xe^{-\alpha(T-t)} + \frac{\delta}{\beta}\left(1 - e^{-\beta(T-t)}\right) + \mathrm{E}_{\mathbb{Q}}\left[\eta \int_t^T e^{-\beta(T-u)} d\tilde{I}(u)|\mathcal{F}_t\right]. \end{aligned}$$

Since $\tilde{I}$ is a CPP-process, it has independent increments and thus $\tilde{I}(u)$ is indepenent of $\mathcal{F}_t$ for all $t \leq u \leq T$. Using independence and the result given in (1.19), it is clear that

$$\mathrm{E}_{\mathbb{Q}}\left[\eta \int_t^T e^{-\beta(T-u)} d\tilde{I}(u) | \mathcal{F}_t\right] = \mathrm{E}_{\mathbb{Q}}\left[\eta \int_t^T e^{-\beta(T-u)} d\tilde{I}(u)\right] = \lambda \mu_U \frac{\eta}{\beta}(1 - e^{-\beta(s-t)})$$

Thus

$$F_{\mathrm{CPP}}(t,T) = xe^{-\alpha(T-t)} + \frac{\delta}{\beta}\left(1 - e^{-\beta(T-t)}\right) + \tilde{\lambda}\mu_U \frac{\eta}{\beta}(1 - e^{-\beta(T-t)}).$$

□

This forward price will as well converge to its mean as $T \to \infty$, since

$$\begin{aligned} \lim_{T\to\infty} F_{\mathrm{CPP}}(t,T) &= \lim_{T\to\infty}\left[xe^{-\alpha(T-t)} + \frac{\delta}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right. \\ &\qquad \left. + \tilde{\lambda}\mu_U \frac{\eta}{\beta}(1 - e^{-\beta(T-t)})\right] \\ &= x + \frac{\delta}{\beta} + \tilde{\lambda}\mu_U \frac{\eta}{\beta}. \end{aligned}$$

As shown in the previous section, the forward price with spot price given by an OU-process, F_{OU}^G, converged to its initial value as T approached present time t. The same is the case for the forward price F_{CPP} since

$$\lim_{T\downarrow t} F_{\mathrm{CPP}}(t,T) = \lim_{T\downarrow t}\left[xe^{-\alpha(T-t)} + \frac{\delta}{\beta}\left(1 - e^{-\beta(T-t)}\right) + \tilde{\lambda}\mu_U \frac{\eta}{\beta}(1 - e^{-\beta(T-t)})\right] = x.$$

The price of a forward contract with a delivery period $[\tau_1, \tau_2]$ on the form (1.28) will be discussed in a chapter 5.

Chapter 2: Credit Risk Modelling

2.1 Introduction

This chapter aims to give an introduction to *credit risk modelling*. In *Credit Risk: Modelling, Valuation and Hedging* by Bielecki and Rutkowski, [BR02], there is given a thourough review which the theory in this thesis is based on. The discussion will evolve around a special type of credit risk, known as *counterparty risk*. Counterparty risk is the possibility of a counterparty in a contract not being able to meet its contractual obligations. In most instances counterparty risk is synonymous with *default risk*. In counterparty risk modelling there are two main approaches commonly used. The first model is called the *firm value model*, also known as *structural approach*. An important example of a firm value model is called Moody's KMV model which has been widely used in risk analysis in the financial industry. Secondly, there is the *hazard rate model*, also known as the *intensity-based approach*. This text starts with the mathematical framework for the firm value model, and further develops a generic risk-neutral pricing formula for defaultable claims, before an important model based on firm value methods is introduced and used in a pricing example. Next, some definitions and the framework for the hazard rate model is presented. Finally, the chapter provides an example of valuation via the hazard rate model. It can also be mentioned that there exists so-called *hybrid models* that involves using notions from both the firm value model and the hazard rate model, but such models are however not discussed in this text.

2.2 Modelling Corporate debt

The framework for this chapter is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ for some finite horizon date $T^* > 0$. Assume further that there exists a risk-neutral pricing measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. The first goal is to develop a generic risk-neutral valuation model for defaultable claims. For this purpose, assume that the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T^*}$ is sufficiently rich to support the following

(a) r: the short-term interest rate process,

(b) V: the firm's value process describing the total value of the firm,

(c) v: the barrier process which determines the trigger point for default,

(d) X: the promised contingent claim which is the amount the firm has to pay at maturity, given that default has not occured before maturity T,

(e) A: the process that models the promised dividends, i.e., the payments going to the holder of the claim at all times prior to maturity T,

(f) $\widehat{X}$: the recovery claim, i.e., the amount paid at time of maturity T if default happens prior to this time T,

(g) Z: the recovery process describing the recovery payoff at time of default, if default happens prior to maturity T.

Some technical assumptions must be made on these objects. First of all, V, Z, A and v are assumed to be progressively measurable with respect to the filtration $\mathbb{F}$. Furthermore, the random variables X and $\widehat{X}$ are $\mathcal{F}_t$-measurable. Regarding the dividend process A, it is assumed to have bounded variation as well as $A(0) = 0$. With these assumptions in place, only one object remains before a general risk-neutral valution formula for the structural approach can be stated, namely the time of default τ.

Definition 2.1 (Default Time Structural Approach)**.** Assume $\mathcal{T}$ to be a Borel measurable subset of the time interval $[0, T]$. Let V be the process describing the total value of the firm's assets and define v as the barrier process. The default time τ is then defined as

$$\tau = \inf\{t > 0 : t \in \mathcal{T}, V(t) < v(t)\}, \tag{2.1}$$

with the convention that $\inf\{\emptyset\} = +\infty$.

Note that for the setup in structural approach, τ is an $\mathcal{F}$-stopping time since for all $t \in [0, T]$, $\{\tau \leq t\} \in \mathcal{F}_t$. Moreover, if the underlying filtration $\mathbb{F}$ is generated by a standard Brownian motion, τ will be an $\mathbb{F}$-predictable stopping time and consequently the time of default may be predicted to a certain degree. This latter property is almost never present in the hazard rate model, where the default event may occure without any forewarnings.

2.3 Stochastic Differential Equations in the Structural Approach

Assume the financial market is represented by the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and that there exists a risk-neutral probability $\mathbb{Q}$ equivalent to

$\mathbb{P}$. To facilitate a generic risk-neutral valuation formula of a defaultable claim, suppose this underlying financial market is arbitrage-free. That is, the discounted price process of any tradeable security, which pays no coupons or dividends, is a $\mathbb{F}$-martingale under $\mathbb{Q}$.

The short-term term interest process r and the value process V of the firm's assets are defined as strong Markov diffusion models under a risk-neutral pricing measure $\mathbb{Q}$ in what follows.

Definition 2.2. Assume $\mu_r : \mathbb{R} \times [0, T] \to \mathbb{R}$ and $\sigma_r : \mathbb{R} \times [0, T] \to \mathbb{R}$ are measureable and integrable functions. The risk-neutral dynamics of the short-term interest rate process $r(t)$, $t \geq 0$, are given as

$$dr(t) = \mu_r(r(t), t)dt + \sigma_r(r(t), t)d\widehat{W}(t), \quad r(0) > 0, \tag{2.2}$$

where $\tilde{W}$ is a standard Brownian motion under $\mathbb{Q}$.

Note that a deterministic interest rate implies $\sigma_r = 0$, and a constant interest rate implies $\sigma_r = \mu_r = 0$.

Definition 2.3 (Value process of firm's assets)**.** Assume $\kappa : \mathbb{R} \times [0, T] \to \mathbb{R}$ and $\sigma_V : \mathbb{R} \times [0, T] \to \mathbb{R}$ are measureable and integrable functions. The risk-neutral dynamics of the process $V(t)$, $t \geq 0$, are given as

$$\frac{dV(t)}{V(t)} = (r(t) - \kappa(V(t), r(t), t))dt + \sigma_V(V(t), t)\widetilde{W}(t), \quad V(0) > 0, \tag{2.3}$$

where $\widetilde{W}(t)$ is a standard Brownian motion under $\mathbb{Q}$, and $r(t)$ is the short-term interest rate. A non-negative κ represents the payout ratio of the firm while any other value represents a capital inflow to the firm.

In the remainder of this chapter, let $X, A, \widetilde{X}$ and Z have objectives as given in (d),(e) (f), and (g) respectively. In addition, assume that $X, \widetilde{X}$ and Z satisfy

$$X = g(V(T, r(t))), \quad \widetilde{X} = h((V(T), r(T)), \quad Z(t) = z(V(t), r(t), t) \tag{2.4}$$

for all $t \in [0, T]$, and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,$h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $z : \mathbb{R}_+ \times \mathbb{R} \times [0, T] \to \mathbb{R}$ are measurable functions. Furthermore, A is defined as

$$A(t) = \int_0^t c(V(u), r(u), u)du, \tag{2.5}$$

for all $t \in [0, T]$, and some integrable function $c : \mathbb{R}_+ \times \mathbb{R} \times [0, T] \to \mathbb{R}$.

$$B(t) = e^{\int_0^t r(u)du}, \tag{2.6}$$

and thus the price of a unit default-free zero-coupon bond, by which the defaultable claims are discounted, is given by

$$B(t,T) = e^{-\int_t^T r(u)du}.$$

Let τ be defined as in (2.1) and introduce the right-continuous process H given by

$$H = \mathbb{1}_{\{\tau \le t\}}. \tag{2.7}$$

2.4 Generic Risk-Netrual Valuation Formula for the Structural Approach

This section tries to develop a risk-neutral valuation formula in order to set up a no-arbitrage pricing framework for defaultable claims using the structural approach. In the structural approach, the trigger for default is described by the barrier process v. In the simplest case it can be defined as $v = L$, where L is a constant giving the total liabilities or debt of a firm.

The process describing all the cashflows received by a holder of a defaultabe claim (dividend process) is defined in what follows.

Definition 2.4. Let D denote the process describing all the cashflows received by a holder of a defaultabe claim. Furthermore, let the firm's total value process V have dynamics (2.3). In addition, $X, \widetilde{X}, Z, A$ are assumed to satisfy (2.4), and (2.5), respectively. Let H be defined as (2.7), where τ is as defined in (2.1). Then the cashflows D may be written as

$$D(t) = X^d(T)\mathbb{1}_{[T,\infty)}(t) + \int_{(0,t]} (1 - H(u))dA(u) + \int_{(0,t]} Z(u)dH(u), \tag{2.8}$$

where $X^d(t) = X\mathbb{1}_{\{t>T\}} + \widetilde{X}\mathbb{1}_{\{t\le T\}}$.

This concludes the necessary framework for expressing a general valuation formula for a defaultable claim, which is given in what follows.

Definition 2.5. Let the money market account B be defined as in (2.6). Denote by DCT = $(X, \overline{X}, Z, \tau)$ a defaultable claim which has cashflows as described in (2.4). The price process $X^d(\cdot, T)$ of a such defaultable claim, with maturity at time T, is then given by

$$X^d(t,T) = B(t)\,\mathrm{E}_{\mathrm{Q}}\left[\int_{(t,T]} B^{-1}(u)\; dD(u) \middle| \mathcal{F}_t\right], \tag{2.9}$$

for any $t \in [0, T)$.

For modeling purposes, it is convenient to assume that only one of the payment recovery schemes $\widetilde{X}$ and Z are present at the same time. If $\widetilde{X} = 0$, it means that recovery payments start at time of default τ. If on the other hand $Z \equiv 0$, then the recovery payment is done in one transaction at time of maturity T. These different versions of defaultable claims lead to two special cases of the generic formula (2.5) which are summarized in the following definition.

Definition 2.6. Let $\mathrm{DCT}_1 = (X, A, \widetilde{X}, \tau)$ and $\mathrm{DCT}_2 = (X, A, Z, \tau)$ be two defaultable claims. The price process $X^{d,i}(\cdot, T)$ with maturity at time T, is then given by

$$X_i^d(t, T) = B(t)\,\mathrm{E}_Q\left[\int_{(t,T]} B^{-1}(u)dD_i(u)\middle|\mathcal{F}_t\right], \quad i = 1, 2\,,$$

for any $t \in [0, T]$.

2.5 Merton's Model

An important example of a firm value model is called *Merton's Model*. The framework for this model is established in an assumed complete market. This means that there are no transaction or bankruptcy costs, there are no restrictions on short-selling traded securities, and there are no limits for borrowing and lending at the same interest rate.

In the original Merton's model, the short-term interest rate r, the volatility coefficient of the firm's value process $\sigma_V = \sigma$, and the payout/infloat ratio κ are assumed to be constant. Consequenctly, the firm's value process V from (2.3) is a Geometric Brownian Motion (henceforth GBM) on the form

$$dV(t) = V(t)((r - \kappa)dt + \sigma d\widetilde{W}), \tag{2.10}$$

which has explicit solution for, $T \geq t$,

$$V(T) = V(t)e^{\sigma(\widetilde{W}(T) - \widetilde{W}(t)) + (\tilde{r} - \frac{1}{2}\sigma^2(T-t))}, \tag{2.11}$$

where $\tilde{r} = r - \kappa$.

Furthermore, due to the constant interest rate, the money market account from (2.6) will be on the form

$$B(t, T) = e^{-r(T-t)}. \tag{2.12}$$

Designate by L the amount giving the total liabilites of the firm, and let T be the maturity date of a contract. The time of default τ in Merton's model is then given by

$$\tau = T\mathbb{1}_{\{V(T)<L\}} + \infty\mathbb{1}_{\{V(T)\geq L\}}, \tag{2.13}$$

where $\infty \cdot 0 = 0$. Note that this means that default can only occure at time of maturity T, which is one of the major drawbacks in Merton's model. There does however exist extensions of Merton's model which allows for default to happen for any $t \in [0, T]$, but such models are not discussed in this text.

2.6 Pricing Example with Merton's Model

The part of this text concerning firm value models is concluded with a pricing example using Merton's model for default. Consider the special case of the general model $\mathrm{DCT}_1 = (X, A, \widetilde{X}, \tau)$ as defined in (2.6), and suppose that

(h) the promised contingent claim $X = L$,

(i) the dividends process $A \equiv 0$,

(j) the recovery claim paid at time of maturity if default occurs $\widehat{X} = V(T)$,

(k) the time of default $\tau = T\mathbb{1}_{\{V(T)<L\}} + \infty\mathbb{1}_{\{V(T)\geq L\}}$.

In fact, the fixed amount L can in this instance be viewed as the nominal value of a corporate zero-coupon bond. The following expression may be derived for the terminal payoff

Proposition 2.7. *The terminal payoff $X_1^d(T)$ of a defaultable claim $DCT_1 = (X, A, \widetilde{X}, \tau)$, with assumptions (h)-(k), is given by*

$$X_1^d(T) = L - \max\left(L - V(T)\right) \tag{2.14}$$

Proof. By the risk-neutral valuation proceeding, the terminal payoff X_1^d is given by

$$X_1^d(T) = X\mathbb{1}_{\{\tau>T\}} + \widetilde{X}\mathbb{1}_{\{\tau\leq T\}} = L\mathbb{1}_{\{V(T)\geq L\}} + V(T)\mathbb{1}_{\{V(T)<L\}}, \tag{2.15}$$

which is equivalent with

$$X_1^d(T) = \min(V(T), L)\mathbb{1}_{\{V(T)\geq L\}} + \min(V(T), L)\mathbb{1}_{\{V(T)<L\}} = \min\left(V(T), L\right),$$

and finally

$$\min\left(V(T), L\right) = L - \max\left(L - V(T), 0\right).$$

$\square$

Note that proposition (2.7) implies that the price process $X_1^d(t, T)$ of DCT_1 can be written as the difference between the nominal value of default-free zero-coupon bond L, and a European put-option on the total value of the firm's assets V, with exercise date T and strike price L. Furthermore, from (2.14) it is clear that

$$X_1^d(t,T) = LB(t,T) - P(t), \tag{2.16}$$

where $B(t,T)$ is the moneymarket account as defined in (2.12) and $p(t)$ is the time t price of a put-option on V with strike price L. This means that the price of the defaultable claim can be computed using the Black & Scholes method for put-option pricing. Details on Black & Scholes methodology may be found in [Ben04].

Proposition 2.8. *Let $D(t,T)$ denote the price of a defaultable claim $DCT_1(X, A, \widetilde{X}, \tau)$, with assumption (h)-(k). Then*

$$D(t,T) = LB(t,T)N(d_1) + V(t)e^{-\kappa(T-t)}N(d_2), \tag{2.17}$$

where N is the standard normal cumulative distribution function, and

$$d_1 = \frac{\ln\left(\frac{x}{L}\right) + (\tilde{r} - \frac{1}{2}\sigma^2(T-t))}{\sigma\sqrt{T-t}},$$

and

$$d_2 = \frac{\ln\left(\frac{L}{x}\right) - (\tilde{r} + \frac{1}{2}\sigma^2(T-t))}{\sigma\sqrt{T-t}}.$$

Proof. Though this can be proved by using (2.16) and the Black & Scholes formula, another approach using standard pricing methods is used in this proof. By the no-arbitrage principle, the price of a contingent claim is equal to the discounted expected value of the payoff function. The payoff function is given in (2.15), thus

$$\begin{aligned} D(t,T) =& B(t,T)\,\mathrm{E}_{\mathbb{Q}}\left[L\mathbb{1}_{\{V(T)\geq L\}} + V(T)\mathbb{1}_{\{V(T)<L\}}|\mathcal{F}_t\right] \\ =& B(t,T)\,\mathrm{E}_{\mathbb{Q}}\left[L\mathbb{1}_{\{V(T)\geq L\}}|\mathcal{F}_t\right] + B(t,T)\,\mathrm{E}_{\mathbb{Q}}\left[V(T)\mathbb{1}_{\{V(T)<L\}}|\mathcal{F}_t\right]. \\ =& B(t,T)L\mathbb{Q}\left(V(T)\geq L|\mathcal{F}_t\right) + B(t,T)\,\mathrm{E}_{\mathbb{Q}}\left[V(T)\mathbb{1}_{\{V(T)<L\}}|\mathcal{F}_t\right] \end{aligned}$$

Each term is evaluated individually in the following calculation. The normal distribution of the Browninan motion implies

$$\sigma(\widetilde{W}(T) - \widetilde{W}(t)) \sim N(0, \sigma^2(T-t)). \tag{2.18}$$

Furthermore, the explicit solution $V(T)$ is given by (2.11), and observe that $V(t) = x$ is $\mathcal{F}_t$-measurable and $(W(T) - W(t))$ is independent of $\mathcal{F}_t$. Hence

$$\begin{aligned}
\mathbb{Q}\left(V(T) \geq L|\mathcal{F}_t\right) &= \mathbb{Q}\left(V(t)e^{\sigma(\widetilde{W}(T)-\widetilde{W}(t))+(\tilde{r}-\frac{1}{2}\sigma^2(T-t))} \geq L \,|\mathcal{F}_t\right) \\
&= \mathbb{Q}\left(\sigma(\widetilde{W}(T) - \widetilde{W}(t))) \geq \ln\left(\frac{L}{x}\right) - (\tilde{r} - \frac{1}{2}\sigma^2(T-t))|\mathcal{F}_t\right) \\
&= \mathbb{Q}\left(Z \geq \frac{\ln\left(\frac{L}{x}\right) - (\tilde{r} - \frac{1}{2}\sigma^2(T-t))}{\sigma^2\sqrt{T-t}}\right) \\
&= \mathbb{Q}\left(Z \leq \frac{\ln\left(\frac{x}{L}\right) + (\tilde{r} - \frac{1}{2}\sigma^2(T-t))}{\sigma\sqrt{T-t}}\right) = N(d_1),
\end{aligned}$$

when $Z \sim N(0,1)$.
For evaluation of the second term, define the probability

$$\widetilde{\mathbb{Q}}(A) = \mathrm{E}_{\mathbb{Q}}\left[\mathbb{1}_A M(T)\right], \tag{2.19}$$

for $A \subset \Omega$, and

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = e^{\sigma\widetilde{W}(T)-\frac{1}{2}\sigma^2 T} = M(T). \tag{2.20}$$

Denoting $A = \{V(T) < L\}$ yields

$$\mathrm{E}_{\mathbb{Q}}\left[V(T)\mathbb{1}_A|\mathcal{F}_t\right] = V(0)e^{\tilde{r}T}\,\mathrm{E}_{Q}\left[M(T)\mathbb{1}_A|\mathcal{F}_t\right],$$

and by Bayes' rule

$$\begin{aligned}
V(0)e^{\tilde{r}T}\,\mathrm{E}_{\mathbb{Q}}\left[M(T)\mathbb{1}_A|\mathcal{F}_t\right] =& V(0)e^{\tilde{r}T}\,\mathrm{E}_{\mathbb{Q}}\left[M(T)|\mathcal{F}_t\right]\mathrm{E}_{\widetilde{\mathbb{Q}}}\left[\mathbb{1}_A|\mathcal{F}_t\right] \\
=& V(0)e^{\tilde{r}T}M(t)\widetilde{\mathbb{Q}}\left(A|\mathcal{F}_t\right) \\
=& B^{-1}(t,T)V(t)e^{-\kappa(T-t)}\widetilde{\mathbb{Q}}\left(A|\mathcal{F}_t\right),
\end{aligned}$$

where the last equality follows from

$$V(0)e^{\tilde{r}T}M(t) = V(0)e^{\sigma\widehat{W}(t)-\frac{1}{2}\sigma^2 t+(r-\kappa)T} = V(t)e^{(r-\kappa)(T-t)}.$$

Girsanov's theorem ensures that $\overline{W}(t) = \widetilde{W}(t) - \sigma t$ is a standard Browninan motion under $\widetilde{\mathbb{Q}}$. The $\widetilde{\mathbb{Q}}$-dynamics of V is given by

$$dV(t) = V(t)(\tilde{r} + \sigma^2)dt + \sigma d\overline{W}(t),$$

which has explicit solution

$$V(T) = V(t)e^{\sigma(\overline{W}(T)-\overline{W}(t))+(\tilde{r}+\frac{1}{2}\sigma^2)(T-t)}.$$

In addition, $\overline{W}(T) - \overline{W}(t)$ is independent of $\mathcal{F}_t$ and $V(t) = x$ is $\mathcal{F}_t$-measurable. The distribution of $\overline{W}$ is the same as (2.18). Hence

$$\begin{aligned}
\widetilde{\mathbb{Q}}(A|\mathcal{F}_t) &= \widetilde{\mathbb{Q}}\left(V(t)e^{\sigma(\overline{W}(T)-\overline{W}(t))+(\tilde{r}+\frac{1}{2}\sigma^2(T-t))} < L\ |\mathcal{F}_t\right)\\
&= \widetilde{\mathbb{Q}}\left(\sigma(\overline{W}(T)-\overline{W}(t))) < \ln\left(\frac{L}{x}\right) - (\tilde{r}+\frac{1}{2}\sigma^2(T-t))|\mathcal{F}_t\right)\\
&= \widetilde{\mathbb{Q}}\left(\overline{Z} \geq \frac{\ln\left(\frac{L}{x}\right) - (\tilde{r}+\frac{1}{2}\sigma^2(T-t))}{\sigma^2\sqrt{T-t}}\right)\\
&= \mathbb{Q}\left(\overline{Z} < \frac{\ln\left(\frac{L}{x}\right) - (\tilde{r}+\frac{1}{2}\sigma^2(T-t))}{\sigma\sqrt{T-t}}\right) = N(d_2),
\end{aligned}$$

Thus the price of a the defaultable claim is given by

$$\begin{aligned}
D(t,T) &= B(t,T)L\mathbb{Q}\left(V(T) \geq L|\mathcal{F}_t\right) + B(t,T)\,\mathrm{E}_{\mathbb{Q}}\left[V(T)\mathbb{1}_{\{V(T)<L\}}|\mathcal{F}_t\right]\\
&= LB(t,T)N(d_1) + B^{-1}(t,T)V(t)e^{-\kappa(T-t)}\widetilde{\mathbb{Q}}\left(A|\mathcal{F}_t\right)\\
&= LB(t,T)N(d_1) + V(t)e^{-\kappa(T-t)}N(d_2).
\end{aligned}$$

□

2.7 Hazard Functions and Hazard Processes

In this chapter, hazard rate models are discussed. The framework involves the concept of a *random time*, which will serve as the time when the default event is triggered. To capture the information from the random time as well as the remaining market information, the filtration setup in this chapter needs to be defined in a slightly different way than what was done in the structural approach.

The probability space considered is denoted by $(\Omega, \mathcal{G}, \mathbb{P})$. The first goal is to find a filtration that captures all available information at time $t \in \mathbb{R}_+$. For now however, let $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$ be an arbitrary filtration on $(\Omega, \mathcal{G}, \mathbb{Q})$, where $\mathbb{Q}$ is risk-neutral pricing measure equivalent to $\mathbb{P}$. A random time can then be defined as follows.

Definition 2.9 (Random Time)**.** Denote by τ a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, satisfying

(1) $\mathbb{Q}\{\tau = 0\} = 0$,

(2) $\mathbb{Q}\{\tau > t\} > 0$,

for any $t \in \mathbb{R}_+$. Then τ is called a random time under $\mathbb{Q}$.

To obtain the natural filtration of the random time, denote by H the right-continuous process defined by $H = \mathbb{1}_{\{\tau<t\}}$. The natural filtration of τ is then the associated filtration $\mathbb{H} = \mathcal{H}_t = \{\mathcal{H}_t\}_{u\leq t}$. Finally, introduce an auxiliary filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t>0}$ (henceforth reference filtration) such that

$$\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t, \tag{2.21}$$

for any $t \in \mathbb{R}_+$. Both $\mathcal{H}_t$ and $\mathcal{G}_t$ are assumed to be right-continuous and complete.
In most credit risk models, the auxiliary filtration $\mathbb{F}$ is generated by a Brownian motion W under $\mathbb{Q}$.

Definition 2.10 ($\mathbb{F}$-Survival Process)**.** Let F be defined as

$$F(t) = Q(\tau \leq t \mid \mathcal{F}_t). \tag{2.22}$$

for any $t \in \mathbb{R}_+$. The $\mathbb{F}$-survival process G of τ under $\mathbb{Q}$ is then defined as

$$G(t) = 1 - F(t) = P(\tau > t \mid \mathcal{F}_t). \tag{2.23}$$

Definition 2.11 ($\mathbb{F}$-Hazard Process)**.** Let F and G be defined as in (2.22) and (2.23), respectively. An increasing function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Gamma(t) = -\ln G(t) = -\ln(1 - F(t))$$

for any $t \in \mathbb{R}_+$, is called the $\mathcal{F}$-hazard process of τ.

Definition 2.12 ($\mathbb{F}$-Hazard Rate)**.** Assume that the $\mathcal{F}$-hazard process Γ of τ satisfies

$$\Gamma(t) = \int_0^t \gamma(u)du, \tag{2.24}$$

for some non-negative, $\mathbb{F}$-progressively measurable process γ, then γ is called the $\mathbb{F}$-hazard rate of τ under $\mathbb{Q}$.

Note that (2.24) is equivalent with Γ having absolutely continous sample paths with regards to the Lebesgue measure on $\mathbb{R}_+$. It can also be mentioned that the hazard rate is sometimes called the $\mathbb{F}$-*intensity* of τ if γ is stochastic, or the intensity function of τ if γ is deterministic. If (2.24) holds, it invokes a useful property, namely a martingale representation of γ. The martingale representation is given in the following proposition.

Proposition 2.13. *Assume (2.24) holds where γ is the hazard rate of τ. Then the process $\widehat{M}$ given by*

$$\widehat{M} = H_t - \int_0^{t\wedge\tau} \gamma(u)du \tag{2.25}$$

follows an $\mathcal{H}$-martingale for any $t \in \mathbb{R}_+$.

One additional object is needed in order to define the time of default in the hazard rate model. Therefore, assume that the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is sufficiently rich to support a random variable ϵ, which has distribution

$$\epsilon \sim \text{unif}(0, 1). \tag{2.26}$$

In addition, it will be assumed that ϵ is independent of the filtration $\mathbb{F}$ under $\mathbb{Q}$. The time of default τ can then be defined as follows

Definition 2.14 (Random Time of Default)**.** Let ϵ be defined as in (2.26), and let Γ be the $\mathbb{F}$-hazard process of τ as defined in (2.24). The random time $\tau : \Omega \to \mathbb{R}$ given by

$$\tau = \inf\{t \in \mathbb{R}_+ : e^{-\Gamma(t)} \leq \epsilon\} = \inf\{t \in \mathbb{R}_+ : \Gamma(t) \geq \eta\},$$

where η is a random variable given by

$$\eta = \ln \epsilon,$$

and consequently η has a unit exponential distribution under $\mathbb{Q}$.

2.8 Generic Risk-Neutral Valuation Formula for the Intensity-based approach

In this section the price of defaultable claims using the hazard rate model is discussed. The relevant probability space is $(\Omega, \mathcal{G}, \mathbb{Q})$, supplied with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$. The filtration $\mathcal{G}$ is as defined in (2.21), where $\mathcal{H}_t$ is the filtration associated with the right-continuous process $H(t) = \mathbb{1}_{\{\tau \leq t\}}$, and τ is a random time as defined in (2.9). The short-term interest rate r follows an $\mathcal{F}$-progressively measurable process, such that the money-market account as defined in (2.6) is well-defined.

In order to describe the cashflows of a defaultable claim, some of the objects from Section 1 are needed. Therefore, in the remainder of this text, let X, A, Z and $\widehat{X}$ have objectives as in (d), (e), (g) and (f), respectively and that they satisfy suitable integrability conditions such that (2.8) is well-defined. Furthermore, Z and A are, as in Section 1, assumed to be $\mathbb{F}$-predictable, with A following a process of finite variation and $A(0) = 0$. The promised contingent claim X and the recovery claim $\widehat{X}$ are $\mathcal{F}_T$-measurable. Moreover, the sample paths of all processes are assumed to be right-continuous functions, with finite left-hand limits, almost surely. The dividend process D in this section is the same as defined in (2.4). A generic risk-neutral valuation formula for the intensity-based approach can now be stated.

Definition 2.15. Let the money market account B be defined as in (2.6). Denote a defaultable claim by DCT $= (X, A, \widetilde{X}, Z, \tau)$, where τ is as defined (2.14). The price process $X^d(\cdot, T)$ of a such defaultable claim, with maturity at time T, is then given by

$$X^d(t,T) = B(t)\,\mathrm{E}_{\mathbb{Q}}\left[\int_{(t,T]} B^{-1}(u)\; dD(u)\middle|\mathcal{G}_t\right], \tag{2.27}$$

for any $t \in [0, T)$.

Notice that the only difference from the price process in Section 1, (2.5), is the filtration used in the conditioned expectation.
Combining (2.8) and (2.27) yields a price process of $X^d(t, T)$ on the following form.

$$\begin{aligned} X^d(t,T) = B(t)\,\mathrm{E}_Q\Bigg[\int_{(t,T]} B^{-1}(u)(1-H(u))dA(u) + \int_{(t,T]} B^{-1}(u)Z(u)dH(u) \\ + B^{-1}(T)X^d(T)\bigg|\mathbb{G}_t\Bigg], \end{aligned} \tag{2.28}$$

where $X^d(T) = X^d(T,T) = \overline{X}H(T) + X(1 - H(T))$.

2.9 Valuation via the Hazard Process

As a final example, the pre-default value $D^0(t, T)$ of a corporate zero-coupon bond with recovery payment at time of maturity T is computed. Recovery payment at maturity implies $Z \equiv 0$, and zero-coupon implies $A \equiv 0$. In other words, the relevant defaultable claim is DCT$=(X, \overline{X}, \tau)$. In the remainder of this text it assumed that the reference filtration $\mathbb{F}$ in (2.21) is the trivial filtration ($\mathcal{F}_t = \mathcal{F}_0 = \{\emptyset, \Omega\}$), so that $\mathbb{G} = \mathbb{H}$. The trivial reference filtration and a deterministic γ yields the two following equalitites.

$$\mathbb{Q}(t < \tau < T|\mathcal{G}_t) = \mathbb{Q}(t < \tau < T|\mathcal{H}_t) = \mathbb{1}_{\{\tau>t\}}(1 - e^{-\int_t^T \gamma(u)du}), \tag{2.29}$$

and

$$\mathbb{Q}(\tau > T|\mathcal{G}_t) = \mathbb{Q}(\tau > T|\mathcal{H}_t) = \mathbb{1}_{\{\tau>t\}}e^{-\int_t^T \gamma(u)du}. \tag{2.30}$$

Proposition 2.16. *Let $D^0(t, T)$ denote the pre-default value a of defaultable claim DCT $= (X, \overline{X}, \tau)$. Assume the default time τ is given by (2.14), where Γ is the hazard process of τ satisfying (2.24) for a deterministic function γ. In addition, assume that the short-term interest rate r and the promised contingent claim X as well as the recovery claim $\overline{X}$ are deterministic. Then*

$$D^0(t,T) = B(t,T)\mathbb{1}_{\{\tau>t\}}\left(\overline{X}(1 - e^{-\int_t^T \gamma(u)du}) + Xe^{-\int_t^T \gamma(u)du}\right) \tag{2.31}$$

Proof. The value at maturity T for the defaultable claim DCT = $(X, \overline{X}, \tau)$ is

$$X^d(T) = \overline{X}H(T) + X(1 - H(T)),$$

thus by the filtration assumption $\mathbb{G} = \mathbb{H}$, along with (2.28), (2.29) and (2.30)

$$\begin{aligned}
D^0(t,T) &= B(t)\,\mathrm{E}_Q\left[B^{-1}(T)\left(\overline{X}H(T) + X(1-H(T))\right)|\mathcal{G}_t\right] \\
&= B(t)B^{-1}(T)\,\mathrm{E}_Q\left[\overline{X}H(T) + X(1-H(T))|\mathcal{H}_t\right] \\
&= B(t,T)\left(\overline{X}\left(\mathrm{E}_Q\left[H(T)|\mathcal{H}_t\right] + X\,\mathrm{E}_Q\left[(1-H(T))|\mathcal{H}_t\right]\right)\right) \\
&= B(t,T)\left(\overline{X}\mathbb{Q}(t < \tau \leq T|\mathcal{H}_t) + XQ(\tau > T|\mathcal{H}_t)\right) \\
&= B(t,T)\left(\mathbb{1}_{\{\tau>t\}}\overline{X}(1 - e^{-\int_t^T \gamma(u)du}) + X\mathbb{1}_{\mathbb{1}_{\{\tau>t\}}}e^{-\int_t^T \gamma(u)du}\right) \\
&= B(t,T)\mathbb{1}_{\{\tau>t\}}\left(\overline{X}(1 - e^{-\int_t^T \gamma(u)du}) + Xe^{-\int_t^T \gamma(u)du}\right).
\end{aligned}$$

□

Chapter 3: Simulations

In this chapter Monte Carlo simulations are explained and discussed. Monte Carlo simulation is a class of computational methods used to obtain a numerical result by repeated random samples. In [Monte Carlo methods in Finance], by Jackel, [Jäc02], there is given an overview of Monte Carlo methods in finance known to expert practitioners.

The purpose with Monte Carlo simulations in this text is to find prices of weather derivatives via numerical approximations. More specifically, the goal is to find the expected value of a function f given a specified distribution density F over $x \in A \subseteq \mathbb{R}$. Mathematically, the aim is to obtain a value p defined as

$$p = \mathrm{E}_{F(x)}[f(x)] = \int_A f(x)F(x)dx. \tag{3.1}$$

An integral of the above form can be simulated in the following manner.

Algorithm 1 Expected value of a function

```
procedure
    Make a function that draws variates x from specified distribution F(x)
    Define a variable to contain computed function values RunningSum = 0
    Define a counter i = 0
    Define a variable to for the average sum RunningAverage = 0
    Draw variate x_i and compute f_i = f(x_i)
    Add computed function value to RunningSum
    Add +1 counter i
    Set average sum variable as RunningAverage = RunningSum/i
```

The above algorithm yields a Monte Carlo estimator after n iterations given by

$$\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{3.2}$$

3.1 Error Estimation

In this text two ways of approximating or measuring the error for Monte Carlo methods are mentioned. One approach tries to track the variation of the numerical results, while the other method is to compare the numerical result with an analytical solution.

The first approach uses the *central limit theorem*, the Monte Carlo estimator $\hat{p}_N$ has characteristics approximate to a normal variate.

Proposition 3.1. *Let $\hat{x}_N$ be a Monte Carlo estimator of the form*

$$\hat{x}_n = \sum_{i=1}^{n} x_i, \tag{3.3}$$

where the x_i's are independent and identically distributed random variables drawn from a distribution with expected value μ and finite variation σ^2, respectively. Then $\hat{x}_N$ converges in distribution to the normal distribution, denoted as

$$\hat{x}_n \overset{i.d.}{\rightarrow} N\left(\mu, \frac{\sigma^2}{n}\right). \tag{3.4}$$

Proof. The proof is a direct result from the Lindeberg-Lévy Central limit theorem which states that given a sequence of identically distributed and inpendent random variables $\{x_1, x_2, \ldots, x_N\}$, the random variables $(\hat{x}_N - \mu)\sqrt{n}$ converge to the standard normal distribution, that is

$$(\hat{x}_N - \mu)\sqrt{n} \overset{i.d.}{\rightarrow} N\left(0, \sigma^2\right).$$

Thus

$$\hat{x}_n \overset{\text{i.d.}}{\rightarrow} N\left(\mu, \frac{\sigma^2}{n}\right).$$

□

the uncertainty in each simulation $\hat{x}_n$ may be quantified by a statistical measure which is the *standard deviation* of $\hat{x}_n$ given by

$$\sqrt{\text{Var}[\hat{x}_n]} = \frac{\sigma}{\sqrt{n}}.$$

The standard deviation σ is however usually not known when performing Monte Carlo simulations. That is, if the true standard deviation was known, then the true mean μ could be computed, which is exactly the value the Monte Carlo simulations are supposed approximate. With the convergence to a normal distribution as described (3.4) however, an estimate $\hat{\sigma}^2$ of σ^2 can be obtained by using the variance in each simulation, i.e., for each N

$$\hat{\sigma}_n = \sqrt{\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2},$$

which leads to a standard error given by

$$\epsilon_n = \frac{\hat{\sigma}_n}{\sqrt{n}}.$$

It is important to note that this measure of error is stochastic. To illustrate, any single simulation may deviate significantly more than one standard error, indeed, for each n, there is according to [Jäc02] only 68.3% probability that

$$\hat{x}_n \in (\mu - \epsilon_n, \mu + \epsilon_n) \,.$$

Before looking at the other method, the Monte Carlo simulation of an OU-process is discussed.

3.2 Ornstein Uhlenbeck process

In this section an OU-process of the form

$$dX(t) = (\mu - \alpha X(t))dt + \sigma dB(t)$$

with explicit solution

$$X(t) = e^{-\alpha t}X(0) + \mu(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-u)} dB(u) \tag{3.5}$$

In order to use Monte Carlo simulations on X, it is convenient to exploit the normal distribution of the Brownian Motion,i.e.,

$$W(t) \sim N(0, t).$$

Consequently

$$X(t) \overset{i.d.}{=} e^{-\alpha t}X(0) + \mu(1 - e^{-\alpha t}) + \sigma \sqrt{\frac{1}{2\alpha}(1 - e^{-2\alpha t})} Z, \tag{3.6}$$

for a standard normal variable Z. This means that speaking in terms of the algorithm (3), a Monte Carlo simulation of an OU-process can be done by defining

$$f(x_i) = e^{-\alpha t}X(0) + \mu(1 - e^{-\alpha t}) + \sigma \sqrt{\frac{1}{2\alpha}(1 - e^{-2\alpha t})} x_i, \tag{3.7}$$

where $x_i \sim N(0, 1)$.

Remarks. Determining the coefficients μ, α and σ will in many cases be a major part of the modelling work. Several different methods exist for estimating these coefficients, with one of the most widely used being linear regression. Another thing worth mentioning is that with slight modifications in the algorithm (3) and the objective function (3.7), there is also a possibility of making the coefficients functions of t. Defining the coefficients as functions of t rather than constants is in many instances more realistic. Consider for example a spot price for electricity over a year. In the months with stable weather conditions there will often be small changes in price and thus a low volatility, while in the months with unstable weather conditions prices may fluctuate more, implying a larger volatility. A volatility function $\sigma(t)$ will be able to take these seasonal variations into account, while a constant volatility will not.

Rather than assessing the validity of the numerical result by keeping track of the error for each simulation, the second method aims to argue at which number n of simulations the numerical result is satisfactory. A satisfactory result is ambiguous depending on the subject at hand, however in finance, an accuracy of two decimal points is considered by most practitioners to be sufficient. The way the method works is that the Monte Carlo simulation is compared to a closed form solution. For this purpose the pricing formula for a call-option on an asset with spot price dynamics given by a geometric OU-process is found in what follows.

3.3 Pricing Formula geometric OU-process

Let $S(t)$ be the spot price of some commodity at time t and let the spot price be given by

$$S(t) = e^{X(t)}, \tag{3.8}$$

where $X(t)$ is an OU-process of the form

$$dX(t) = -\alpha X(t)dt + \sigma dB(t), \tag{3.9}$$

where $B(t)$ is a brownian motion under a probability measure $\mathbb{P}$.
A spot price of the form (3.8) is a so-called geometric spot price, as opposed to the arithmetic version where the spot price is simply given by $S(t) = X(t)$. Let $p(t)$ be the time t price of a call option on this commodity with strike K and maturity T, i.e.,

$$p(t) = e^{r(T-t)} \,\mathrm{E}\left[(S(T) - K)|\mathcal{F}_t\right], \tag{3.10}$$

where the r is the interest rate and $\mathcal{F}_t$ is a filtration sufficiently rich to support S, consequently making $S(t)$ $\mathcal{F}_t$-measurable for all $t \leq T$. In the spirit of the Black & Scholes formula for call options on securities with spot price dynamics in the form of a GBM, one can derive a similar closed form solution on a call option with spot price dynamics given by an OU-process. The closed form solution is given in the proposition that follows.

Proposition 3.2. *Let S as defined in (3.8) be the spot price of some financial asset or security, where X is an OU-process of the form (3.9). Let T be the finite maturity date of a call-option with strike price K. Further assume that the risk-free interest rate is zero. The price $p(t)$ at time t of such a call-option is then given by*

$$p(t) = e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\Phi(M+\tilde{\sigma}) - K\Phi(M), \tag{3.11}$$

where

$$\mathrm{E}[X(t)] = \tilde{\mu}, \qquad \mathrm{Var}[X(t)] = \tilde{\sigma}^2, \tag{3.12}$$

and Φ is the cumulative distribution function for the normal distribution.

Proof. By the no-arbitrage principle, the price should be equal to the discounted expected value of the payoff function. Let $\mathcal{F}_t$ be the natural filtration of the Brownian motion B. With risk-free interest rate equal to zero the price at time t will be given by

$$\begin{aligned}
p(t) &= \mathrm{E}\left[(S(T)-K)|\mathcal{F}_t\right] \\
&= \mathrm{E}\left[(S(T)-K)\mathbb{1}_{\{S(T)>K\}}|\mathcal{F}_t\right] \\
&= \mathrm{E}\left[S(T)\mathbb{1}_{\{S(T)>K\}}|\mathcal{F}_t\right] - \mathrm{E}\left[K\mathbb{1}_{\{S(T)>K\}}|\mathcal{F}_t\right] \\
&= \mathrm{E}\left[S(T)\mathbb{1}_{\{S(T)>K\}}\right] - K\,\mathrm{E}\left[\mathbb{1}_{\{S(T)>K\}}\right],
\end{aligned} \tag{3.13}$$

where the last equality is a consequence of $S(t)$ being $\mathcal{F}_t$-measurable and the Brownian motion $B(u)$ for $u \in [t,T]$ is independent of $\mathcal{F}_t$.
Using the relation given in (3.6), it is clear that

$$S(T) \overset{i.d.}{=} e^{e^{-\alpha(T-t)}X(t)+\mu(1-e^{-\alpha(T-t)})+\sigma\sqrt{\frac{1}{2\alpha}(1-e^{-2\alpha(T-t)})}Z}, \qquad Z \sim N(0,1).$$

Using notation as in (3.12) yields

$$e^{-\alpha(T-t)}X(t) + \mu(1-e^{-\alpha(T-t)}) + \sigma\sqrt{\frac{1}{2\alpha}(1-e^{-2\alpha(T-t)})}Z = \tilde{\mu} + \tilde{\sigma}Z.$$

Since

$$\tilde{\mu} + \tilde{\sigma}Z \sim N\left(\tilde{\mu}, \tilde{\sigma}^2\right),$$

S will be lognormally distributed, i.e.

$$S(T) \sim N\left(e^{\tilde{\mu}+\frac{1}{2}\sigma^2}, e^{2\tilde{\mu}+\tilde{\sigma}^2}(e^{\tilde{\sigma}^2}-1)\right).$$

Furthermore, $\mathbb{1}_{\{S(T)>K\}} > 0$ whenever

$$\begin{aligned} S(T) - K &> 0 \\ e^{\tilde{\mu}+\tilde{\sigma}Z} &> K \\ \tilde{\mu} + \tilde{\sigma}Z &> \ln K \\ Z &> \frac{\ln K - \tilde{\mu}}{\tilde{\sigma}} = -M \end{aligned}$$

Then the first term in (3.13) can be rewritten as

$$\begin{aligned} \mathrm{E}\left[S(T)\mathbb{1}_{\{S(T)>K\}}\right] &= \mathrm{E}\left[e^{\tilde{\mu}+\tilde{\sigma}Z}\mathbb{1}_{\{Z>-M\}}\right] \\ &= e^{\tilde{\mu}}\,\mathrm{E}\left[e^{\tilde{\sigma}Z}\mathbb{1}_{\{Z<M\}}\right] \\ &= e^{\tilde{\mu}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{M} e^{\tilde{\sigma}z}e^{-\frac{1}{2}z^2}dz \\ &= e^{\tilde{\mu}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{M} e^{\tilde{\sigma}z-\frac{1}{2}z^2}dz \\ &= e^{\tilde{\mu}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{M} e^{-\frac{1}{2}(z-\tilde{\sigma})^2+\frac{1}{2}\tilde{\sigma}^2}dz \\ &= e^{\tilde{\mu}+\frac{1}{2}\sigma^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{M} e^{-\frac{1}{2}(z-\tilde{\sigma})^2}dz \\ &= e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{M+\sigma} e^{-\frac{1}{2}z^2}dz = e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\Phi(M+\tilde{\sigma}). \end{aligned}$$

Finally, the second term in (3.13) becomes

$$K\,\mathrm{E}\left[\mathbb{1}_{\{S(T)>K\}}\right] = KP(S(T) > K) = KP(Z < M) = K\Phi(M).$$

Thus

$$p(t) = e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}\Phi(M+\tilde{\sigma}) - K\Phi(M),$$

which concludes the proof. □

Example 3.3 (A geometric spot price comparison)**.** With the pricing formula (3.10), the exact price of a call option can be computed. Meanwhile, there also exists Monte Carlo methods for this price. Thus it is possible to investigate how many simulations is needed before the numerical result closes in on the exact price. To construct an example, let the parameters in (3.7) be $\alpha = 0.3, \sigma = 0.05, \mu = 5$. Let the maturity date be set 90 days in the future, that is $T = 90$, and let the strike price be 150 so that $K = 150$. Further, set the initial value of $X(0) = 4$. These values results in the following parameters

$$M = -0.165, \quad \tilde{\mu} = 4.999, \quad \tilde{\sigma} = 0.064$$

Inserting into (3.10) yields price p as

$$p = 3.24.$$

In table (3.3) the different prices are given, computed with an increasing number of simulations N.

N	MC price
10	1.34
100	3.97
1000	3.36
10000	3.17
100000	3.23
1000000	3.24

Table 1: Estimated Monte Carlo prices of a call option after N simulations.

As one can see from the table, there is needed $N = 10^6$ simulations before an accuracy of two desimal points is achieved. There is of course no guarantee that this N gives the right numerical solution in every case. Nonetheless, it does give an indication of how many simulations are needed before a Monte Carlo simulation of a geometric OU-process reaches an exact numerical result. In this thesis, arithmetic OU-processes are used to describe spot price dynamics, which intuitively will converge on an even less number of simulations than that of the geometric case.

Chapter 4: Quanto options

Traditionally, quanto options were incorporated in the pricing framework of stocks and bonds as a mean to hedge against currency risk. In later years there has been an increase of the usage of quanto options in energy markets as well. In energy markets a quanto option may be used to hedge against volume risk or price risk. In *Pricing and Hedging Quanto Options in Energy Markets* [BLM15], both pricing and hedging of quanto options in energy markets is considered. Amongst other topics, there is developed a general framework for pricing of quanto options in the form of put/call combinations on energy and temperature derivatives. There is also given a closed form option pricing formula via the Heath Jarrow Morton approach. The mathematical framework includes a pricing measure $\mathbb{Q}$ and a filtration $\mathcal{F}_t$ containing all the market information. Furthermore, a general model is introduced where the forwards price dynamics under a pricing measure $\mathbb{Q}$ of energy price and the number of heating degree days (HDD) over a measurement period $[\tau_1, \tau_2]$ is given by

$$F_E(T, \tau_1, \tau_2) = F_E(t, \tau_1, \tau_2) e^{\mu_E + X}, \tag{4.1}$$

and

$$F_I(T, \tau_1, \tau_2) = F_I(t, \tau_1, \tau_2) e^{\mu_I + Y}, \tag{4.2}$$

for $T \geq t$ where the random variables X and Y are bivariate normally distributed, which is to say that

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathrm{Var}(X) & \rho_{XY} \\ \rho_{XY} & \mathrm{Var}(Y) \end{bmatrix} \right). \tag{4.3}$$

where $\rho_{XY} = \mathrm{corr}(X, Y)$, and the covariance structure is dependend on t, T and τ_1, τ_2. See (A) for theory on the bivariate normal distribution. The forward price dynamic as given in (4.1) falls under the Heath-Jarrow-Morton (henceforth HJM) approach, which traditionally has been framework constructed to model interest rate curves. As one can see from (4.1), the forward price dynamics in the HJM approach is given directly, rather than finding a relation to the spot price and the specification of a risk-neutral pricing measure.

An interesting example presented in [BLM15] is to use bivariate geometric Brownian motions. Then the futures price dynamics takes the form

$$F_E(T,\tau_1,\tau_2) = F_E(t,\tau_1,\tau_2)e^{-\frac{1}{2}\sigma_E^2(T-t)+\sigma_E(W(T)-W(t))},$$

and

$$F_I(T,\tau_1,\tau_2) = F_I(t,\tau_1,\tau_2)e^{-\frac{1}{2}\sigma_I^2(T-t)+\sigma_I(B(T)-B(t))},$$

,where B and W are Brownian motions with correlation $\rho_{X,Y}$. Pricing is done subject to the payoff function p given by

$$p = \max(F_E(\tau_2,\tau_1,\tau_2) - K_E, 0) \cdot \max(F_E(\tau_2,\tau_1,\tau_2) - K_I), 0). \tag{4.4}$$

K_E and K_I are the strikes for European call-options on energy and temperature, respectively. In this text, the combining of weather derivatives and counterparty risk will appear as a quasi-quanto option. As a link between these two concepts a quanto option with payoff function (4.4) is considered. The futures prices however will have a dynamics based on the spot prices of energy and temperature. Se om du finner notater

4.1 Correlated prices

Multiple studies suggest a correlation between prices of different commodities in the energy markets. For instance in the book Managing Energy Risk by Markus Burger, Bernhard Graeber and Gero Schindlmayr, [BGS08], a linear regression analysis was done on the relation between electricity, coal, oil and carbon emission prices, (henceforth EUA). Specifically, there was performed regression on electricity forward prices based on the forward prices of the mentioned remaining. That is, electricity forward prices were fitted to the linear equation

$$F_{el} = c_0 + c_1 t + c_{coal}F_{coal} + c_{oil}F_{oil} + c_{co2}F_{co2}.$$

Using forward prices from the European Energy Exchange from 2007, the most significant regressors were time and EUAs. Coal turned out to be the only regressor not significant at a 99% confidence level. Even though this linear regression was done using forward prices, it is a natural implication that the spot prices will be correlated as well. The correlation for spot prices will however not be as profound due the high volatility of spot prices. Inspired by this correlation, there will be given a pricing example of a quanto option in the following section.

4.2 Quanto Option Pricing Example

Denote by S_1 and S_2 the spot price of electrity and EUA's, respectively. Further assume the prices have dynamics according to OU-processes of the form

$$dS_i(t) = -\alpha_i S_i(t)dt + \sigma_i d\tilde{W}_i(t) \tag{4.5}$$

for $i = 1, 2$. B_1 is a standard Brownian motion under a risk-neutral pricing measure $\mathbb{Q}$, and B_2 is defined as

$$\tilde{W}_2(t) = \rho\tilde{W}_1(t) + \sqrt{1-\rho^2}\tilde{U}(t), \quad \rho \in [-1, 1], \tag{4.6}$$

where $U(t)$ is again a standard Brownian motion under $\mathbb{Q}$, and independent of B_1.
Note that (4.5) is an OU-process with mean μ equal to zero. The affect on the explicit solution given in (1.6) is that the deterministic term $\mu(1 - e^{-\alpha t})$ vanishes.

Proposition 4.1. *The correlation between two standard Brownian motion $\tilde{W}_1$ and the Brownian motion $\tilde{W}_2$ as described in (4.6), is given by*

$$corr(\tilde{W}_1(t), \tilde{W}_2(t)) = \rho t. \tag{4.7}$$

Proof. By the defintion of correlation

$$\begin{aligned}
\operatorname{corr}(\tilde{W}_1(t), \tilde{W}_2(t)) = \frac{\operatorname{cov}(\tilde{W}_1(t), \tilde{W}_2(t))}{\sigma_1\sigma_2} &= \mathrm{E}_{\mathbb{Q}}[\tilde{W}_1(t)\tilde{W}_2(t)] - \mathrm{E}_{\mathbb{Q}}[\tilde{W}_1(t)]\,\mathrm{E}_{\mathbb{Q}}[\tilde{W}_2(t)] \\
&= \mathrm{E}_{\mathbb{Q}}\left[\tilde{W}_1(t)\left(\rho\tilde{W}_1(t) + \sqrt{1-\rho^2}\tilde{U}(t)\right)\right] = \mathrm{E}_{\mathbb{Q}}\left[\rho\tilde{W}^2(t)\right] = \rho t,
\end{aligned}$$

since the product expectation due to independence is equal to

$$\mathrm{E}_{\mathbb{Q}}[\tilde{W}\tilde{U}] = \mathrm{E}_{\mathbb{Q}}[\tilde{W}]\,\mathrm{E}_{\mathbb{Q}}[\tilde{U}] = 0.$$

□

In the following quanto option example, the payoff function is defined as

$$f(S_1, S_2, K_1, K_2) = \max\left(K_1 - D_1(\tau_1, \tau_2), 0\right) \cdot \max\left(D_2(\tau_1, \tau_2) - K_2, 0\right), \tag{4.8}$$

where D_i is the average spot price of electricity and EUA's over a delivery period $[\tau_1, \tau_2]$, respectively, i.e.,

$$D_i(\tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} S_i(s)ds, \tag{4.9}$$

for $i = 1, 2$ and S_i has dynamics as in (4.5).

Proposition 4.2. *The average spot price D_i as defined in (4.9) is given by*

$$
\begin{aligned}
D_1(\tau_1,\tau_2) = \frac{1}{\alpha_1(\tau_2-\tau_1)}\Bigg(& S_1(t)\left(e^{-\alpha_1(\tau_1-t)} - e^{-\alpha_1(\tau_2-t)}\right) \\
& + \sigma_1 \int_t^{\tau_1} \left(e^{-\alpha_1(\tau_1-u)} - e^{-\alpha_1(\tau_2-u)}\right) d\tilde{W}_1(u) \\
& + \int_{\tau_1}^{\tau_2} \left(1 - e^{-\alpha_1(\tau_2-u)}\right) d\tilde{W}_1(u)\Bigg),
\end{aligned} \tag{4.10}
$$

and

$$
\begin{aligned}
D_2(\tau_1,\tau_2) = \frac{1}{\alpha_2(\tau_2-\tau_1)}\Bigg(& S_2(t)\left(e^{-\alpha_2(\tau_1-t)} - e^{-\alpha_2(\tau_2-t)}\right) \\
& + \sigma_2\rho\Bigg(\int_t^{\tau_1} \left(e^{-\alpha_2(\tau_1-u)} - e^{-\alpha_2(\tau_2-u)}\right) d\tilde{W}_1(u) \\
& + \int_{\tau_1}^{\tau_2} \left(1 - e^{-\alpha_2(\tau_2-u)}\right) d\tilde{W}_1(u)\Bigg)\Bigg) \\
& + \sigma_2\sqrt{1-\rho^2}\Bigg(\int_t^{\tau_1} \left(e^{-\alpha_2(\tau_1-u)} - e^{-\alpha_2(\tau_2-u)}\right) d\tilde{U}(u) \\
& + \int_{\tau_1}^{\tau_2} \left(1 - e^{-\alpha_2(\tau_2-u)}\right) d\tilde{W}_1(u)\Bigg)\Bigg)
\end{aligned} \tag{4.11}
$$

Proof. Proof is only given for $D_1(\tau_1,\tau_2)$ since the procedure is very similar for $D_2(\tau_1,\tau_2)$.
The explicit solution for the SDE giving spot price dynamis, (4.5), for $s \geq t$, is given by

$$
S_1(s) = S_1(t)e^{-\alpha_i(s-t)} + \sigma_1 \int_t^s e^{-\alpha_1(s-u)} d\tilde{W}_1(u), \tag{4.12}
$$

which implies

$$
\int_{\tau_1}^{\tau_2} S_1(s)ds = \int_{\tau_1}^{\tau_2} S_1(t)e^{-\alpha_1(s-t)}ds + \sigma_1 \int_{\tau_1}^{\tau_2}\int_t^s e^{-\alpha_1(s-u)} d\tilde{W}_1(u)ds.
$$

The order of integration in the second term can be changed by the *Fubini-Tonelli* theorem and computing

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_t^s e^{-\alpha_2(s-u)} d\tilde{W}_1(u)ds &= \int_{\tau_1}^{\tau_2}\int_t^{\tau_2} \mathbb{1}_{[t,s]}(u) e^{-\alpha_2(s-u)} d\tilde{W}_1(u)ds \\
&= \int_t^{\tau_2}\int_{\tau_1}^{\tau_2} \mathbb{1}_{[t,s]}(u) e^{-\alpha_1(s-u)} ds d\tilde{W}_1(u) \\
&= -\frac{1}{\alpha_2}\left(\int_t^{\tau_1} \left(e^{-\alpha_2(\tau_2-u)} - e^{-\alpha_2(\tau_1-u)} d\tilde{W}_1(u)\right)\right).
\end{aligned}
$$

Straightforward integral computation gives the result. □

Proposition 4.3. *Let the average spot price over a delivery period $[\tau_1, \tau_2]$ be given by D_i as described in (4.9). Assume the interest rate r is zero, and that the correlation between D_1 and D_2 is given by ρ, where $|\rho| < 1$. Let T the maturity date be of a quanto option with strike price K_1 and K_2. The time t price p of a quanto option with payoff function (4.4) is given by*

$$\begin{aligned}
p(t, T, \tau_1, \tau_2) = {} & K_1 \left(\tilde{\alpha}_2(t, T) S_2(t) C(d_1, d_2, \rho) + \tilde{\sigma}_2(t, T)\rho\phi(m)\right) \\
& - \left(K_1 K_2 C(d_1, d_2, \rho)\right) \\
& + \tilde{\sigma}_1(t, T)\tilde{\sigma}_2(t, T) \left(m\phi(m) - \Phi(m)\right) \\
& K_2 \tilde{\alpha}_1 S_1(t) C(d_1, d_2, \rho) + \tilde{\sigma}_1(t, s)\rho\phi(m),
\end{aligned} \tag{4.13}$$

where C is the standard bivariate normal CDF as defined in (A.4), ϕ is the standard normal PDF, and Φ is the standard normal CDF, and $m = \min(d_1, d_2)$.

Proof. According to the no-arbitrage principle, the price p of the quanto option is equal to the discounted expected value, (with regards to a risk-neutral measure) of the payoff function, i.e.,

$$p(t) = \mathrm{E}_Q\left[f(S_1, S_2, K_1, K_2)|\mathcal{F}_t\right] \tag{4.14}$$

Let $D_i(\tau_1, \tau_2) = D_i$ for $i = 1, 2$.. Considering the payoff function by itself, the function can be rewritten as follows

$$\begin{aligned}
f(S_1, S_2, K_1, K_2) &= \max\left(K_1 - D_1), 0\right) \cdot \max\left(D_2 - K_2, 0\right) \\
&= (K_1 - D_1)\, \mathbb{1}_{\{D_1 < K_1\}} \cdot (D_2 - K_2)\, \mathbb{1}_{\{D_2 > K_2\}} \\
&= \left(K_1 \mathbb{1}_{\{D_1 < K_1\}} - D_1 \mathbb{1}_{\{D_1 < K_1\}}\right) \cdot \left(D_2 \mathbb{1}_{\{D_2 > K_2\}} - K_2 \mathbb{1}_{\{D_2 > K_2\}}\right) \\
&= K_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} - K_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} \\
&\quad - D_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} + D_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}}.
\end{aligned}$$

Inserting this last expression into the price (4.14) yields

$$\begin{aligned}
p(t) = {} & \mathrm{E}_Q\left[K_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} - K_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}}\right. \\
& \left. - D_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} + D_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} |\mathcal{F}_t\right] \\
= {} & \Bigg(\mathrm{E}_Q\left[K_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} |\mathcal{F}_t\right] - \mathrm{E}_Q\left[K_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} |\mathcal{F}_t\right] \\
& - \mathrm{E}_Q\left[D_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} |\mathcal{F}_t\right] + \mathrm{E}_Q\left[D_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} |\mathcal{F}_t\right] \Bigg) \\
= {} & \Bigg(\mathrm{E}_Q\left[K_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}}\right] - \mathrm{E}_Q\left[K_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}}\right] \\
& - \mathrm{E}_Q\left[D_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}}\right] + \mathrm{E}_Q\left[D_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}}\right] \Bigg),
\end{aligned} \tag{4.15}$$

where the last equality is a consequence of the Brownian motions $B_1(u)$ and $B_2(u)$ being independent of $\mathcal{F}_t$ for $u \geq t$, and $S_1(t)$,$S_2(t)$ are assumed to be $\mathcal{F}_t$-measurable.
By (4.2), the following relation holds.

$$
\begin{aligned}
D_1(\tau_1, \tau_2) &\stackrel{i.d.}{=} \frac{1}{\alpha_1(\tau_2 - \tau_1)} S_1(t)(e^{-\alpha_1(\tau_1 - t)} - e^{\alpha_1(\tau_2 - t)})) \\
&+ Z\sigma_1 \frac{1}{\sqrt{2}\alpha_1^2}(e^{-2\alpha_1(\tau_1 - s)} - e^{-2\alpha_1(\tau_1 - t)} - e^{-2\alpha_1(\tau_2 - s)} + e^{-2\alpha_1(\tau_2 - t)})^{\frac{1}{2}} \\
D_2(\tau_1, \tau_2) &\stackrel{i.d.}{=} \frac{1}{\alpha_2(\tau_2 - \tau_1)} S_2(t)(e^{-\alpha_2(\tau_1 - t)} - e^{\alpha_2(\tau_2 - t)}) \\
&+ (\rho + \sqrt{1 - \rho^2} U(s))\sigma_2 \frac{1}{\sqrt{2}\alpha_2^2} \\
&\cdot (e^{-2\alpha_2(\tau_1 - s)} - e^{-2\alpha_2(\tau_1 - t)} - e^{-2\alpha_2(\tau_2 - s)} + e^{-2\alpha_2(\tau_2 - t)})^{\frac{1}{2}}
\end{aligned}
$$

To ease notation, denote $\tilde{\alpha}_i$ and $\tilde{\sigma}_i$ by

$$
\tilde{\alpha}_i(t, s) = \frac{1}{\alpha_i(\tau_2 - \tau_1)} \left(e^{-\alpha_i(\tau_1 - t)} - e^{-\alpha_i(\tau_2 - t)}\right), \quad (4.16)
$$

and

$$
\tilde{\sigma}_i(t, s) = \frac{\sigma_i}{\sqrt{2}\alpha_i^2} \left(e^{-2\alpha_i(\tau_1 - s)} - e^{-2\alpha_i(\tau_1 - t)} - e^{-2\alpha_i(\tau_2 - s)} + e^{-\alpha_i(\tau_2 - t)}\right), \quad (4.17)
$$

which yields

$$
D_1(\tau_1, \tau_1) = \tilde{\alpha}_1(t, s) S_1(t) + Z\tilde{\sigma}_1(t, s) \quad (4.18)
$$

and

$$
D_2(\tau_1, \tau_2) = \tilde{\alpha}_2(t, s) S_2(t) + (\rho Z + \sqrt{1 - \rho^2} U)\tilde{\sigma}_2(t, s). \quad (4.19)
$$

The two random variables at hand are $X_1 = Z$ and $X_2 = \rho Z + \sqrt{1 - \rho^2} U$, both of which separately are standard normal random variables, but they are however not independent of each other, making them bivariate normally distributed, that is

$$
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) \quad (4.20)
$$

$$
\begin{aligned}
\mathrm{E}_Q\left[K_1K_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &= K_1K_2\,\mathrm{E}_Q\left[\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] \\
&= K_1K_2\mathbb{Q}\left(D_1 < K_1 \bigcap D_2 > K_2\right) \\
&= K_1K_2\mathbb{Q}\Big(\tilde{\alpha}_1(t)S_1(t) + Z\tilde{\sigma}_1(t,s) < K_1 \\
&\qquad \bigcap \tilde{\alpha}_2 S_2(t) + (\rho Z + \sqrt{1-\rho^2}U)\tilde{\sigma}_2(t,s) > K_2\Big) \\
&= K_1K_2\mathbb{Q}\Big(Z < \frac{K_1 - S_1(t)\tilde{\alpha}_1(t)}{\tilde{\sigma}_1(t,s)} \\
&\qquad \bigcap \rho Z + \sqrt{1-\rho^2}U > \frac{K_2 - S_2(t)\tilde{\alpha}_2(t)}{\tilde{\sigma}_2(t,s)}\Big) \\
&= K_1K_2\mathbb{Q}\left(X_1 < \frac{K_1 - S_1(t)\tilde{\alpha}_1(t)}{\tilde{\sigma}_1(t,s)} \bigcap X_2 > \frac{K_2 - S_2(t)\tilde{\alpha}_2(t)}{\tilde{\sigma}_2(t,s)}\right) \\
&= K_1K_2\mathbb{Q}\left(X_1 < \frac{K_1 - S_1(t)\tilde{\alpha}_1(t)}{\tilde{\sigma}_1(t,s)} \bigcap X_2 < \frac{S_2(t)\tilde{\alpha}_2(t) - K_2}{\tilde{\sigma}_2(t,s)}\right) \\
&= K_1K_2C(d_1, d_2, \rho), \qquad (4.21)
\end{aligned}
$$

where C is the bivariate normal cumulative distribution function for X_1, X_2 with covariance ρ, and

$$d_1 = \frac{K_1 - S_1(t)\tilde{\alpha}_1(t)}{\tilde{\sigma}_1(t,s)}, \qquad (4.22)$$

$$d_2 = \frac{S_2(t)\tilde{\alpha}_2(t) - K_2}{\tilde{\sigma}_2(t,s)}. \qquad (4.23)$$

Movin on to the next term in (4.15), it is clear that

$$\mathrm{E}_{\mathbb{Q}}\left[D_1K_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}|\mathcal{F}_t\right] = K_2E_{\mathbb{Q}}\left[D_1\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right].$$

Furthermore, inserting the expression for D_1 yields

$$
\begin{aligned}
K_2E_{\mathbb{Q}}\left[D_1\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &= K_2E_{\mathbb{Q}}\left[(\tilde{\alpha}_1(t)S_1(t) + Z\tilde{\sigma}_1(t,s))\,\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] \\
&= K_2E_{\mathbb{Q}}\Big[\tilde{\alpha}_1 S_1(t)\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}} \\
&\qquad + X_1\tilde{\sigma}_1(t,s)\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\Big] \\
&= K_2\tilde{\alpha}_1 S_1(t)E_{\mathbb{Q}}\left[\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] \\
&\qquad + \tilde{\sigma}_1(t,s)E_{\mathbb{Q}}\left[X_1\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] \\
&= K_2\tilde{\alpha}_1 S_1(t)C(d_1, d_2, \rho) + \\
&\qquad + \tilde{\sigma}_1(t,s)E_{\mathbb{Q}}\left[X_1\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right].
\end{aligned}
$$

(4.24)

Note that from the equality (4.19), $D_2 > K_2$ is equivalent with

$$\begin{aligned}
\tilde{\alpha}_2(t,s)S_2(t) + (\rho Z + \sqrt{1-\rho^2}U)\tilde{\sigma}_2(t,s) &> K_2 \\
\tilde{\alpha}_2(t,s)S_2(t) + X_2\tilde{\sigma}_2(t,s) &> K_2 \\
X_2\tilde{\sigma}_2(t,s) &> K_2 - \tilde{\alpha}_2(t,s)S_2(t) \\
X_2 &> \frac{K_2 - \tilde{\alpha}_2(t,s)S_2(t)}{\tilde{\sigma}_2(t,s)} \\
X_2 &< \frac{\tilde{\alpha}_2(t,s)S_2(t) - K_2}{\tilde{\sigma}_2(t,s)} = d_2, \qquad (4.25)
\end{aligned}$$

and from (4.18), $D_2 > K_2$ is equivalent with

$$\begin{aligned}
\tilde{\alpha}_1(t,s)S_1(t) + Z\tilde{\sigma}_1(t,s) &< K_1 \\
X_1\tilde{\sigma}_1(t,s) &< K_1 - \tilde{\alpha}_1(t,s)S_1(t) \\
X_1 &< \frac{K_1 - \tilde{\alpha}_1(t,s)S_1(t)}{\tilde{\sigma}_1(t,s)} = d_1.
\end{aligned}$$

Furthermore, the relation (A.5) implies that

$$(X_1|X_2 = x_2) \sim N\left(\rho x_2, (1-\rho^2)\right). \qquad (4.26)$$

Thus

$$\begin{aligned}
\mathrm{E}_{\mathbb{Q}}\left[Z\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &= \mathrm{E}_{\mathbb{Q}}\left[X_1\mathbb{1}_{\{X_1<d_1\}}\mathbb{1}_{\{X_2<d_2\}}\right] \\
&= \mathrm{E}_{\mathbb{Q}}\left[\mathrm{E}_{\mathbb{Q}}\left[X_1\mathbb{1}_{\{X_1<d_1\}}\mathbb{1}_{\{X_2<d_2\}}\middle| X_2 = x_2\right]\right] \\
&= \int_{-\infty}^{\infty} \rho x_2 \mathbb{1}_{\{x_2<d_1\}}\mathbb{1}_{\{x_2<d_2\}}\phi(x_2)dx_2 \\
&= \rho\int_{-\infty}^{\min(d_1,d_2)} x_2\phi(x_2)dx_2. \\
\rho phi(m),
\end{aligned}$$

where $m = \min(d_1, d_2)$.
Hence

$$\mathrm{E}_{\mathbb{Q}}\left[K_2 D_1\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] = K_2\tilde{\alpha}_1 S_1(t)C(d_1,d_2,\rho) + \tilde{\sigma}_1(t,s)\rho\phi(m) \qquad (4.27)$$

Similarly, the next term in (4.15) can be computed as follows.

$$
\begin{aligned}
\mathrm{E}_Q\left[K_1 D_2 \mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}|\mathcal{F}_t\right] =& K_1\,\mathrm{E}_Q\left[D_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]\\
=& K_1\,\mathrm{E}_Q\Big[(\tilde{\alpha}_2 S_2(t) + (\rho Z + \sqrt{1-\rho^2}U)\tilde{\sigma}_2(t,s))\\
& \left(\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right)\Big]\\
=& K_1\,\mathrm{E}_Q\left[\tilde{\alpha}_2 S_2(t)\mathbb{1}_{\{X_1<d_1\}}\mathbb{1}_{\{X_2<D_2\}}\right] \qquad (4.28)\\
& + K_1\,\mathrm{E}_{\mathbb{Q}}\left[X_2\tilde{\sigma}_2(t,s)\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] \qquad (4.29)
\end{aligned}
$$

Observe that (4.28) is similar to what was computed in (4.21), thus

$$
K_1\,\mathrm{E}_Q\left[\tilde{\alpha}_2 S_2(t)\mathbb{1}_{\{X_1<d_1\}}\mathbb{1}_{\{X_2<D_2\}}\right] = K_1\tilde{\alpha}_2 S_2(t)C(d_1,d_2,\rho). \tag{4.30}
$$

Regarding (4.29), remember that $D_1 < K_1$ is equivalent with $X < d_1$, where d_1 is defined in (4.22), and $D_2 > K$ is equivalent with $X_2 < d_2$ where d_2 is defined in (4.23). Consequently

$$
\begin{aligned}
K_1\,\mathrm{E}_{\mathbb{Q}}\left[X_2\tilde{\sigma}_2(t,s)\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &= K_1\tilde{\sigma}_2(t,s)\,\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{X_1<d_1\}}\mathbb{1}_{\{X_2>d_2\}}\right]\\
&= K_1\tilde{\sigma}_2(t,s)\,\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]. \qquad (4.31)
\end{aligned}
$$

Note that the (4.26) also implies

$$
(X_2|X_1 = x_1) \sim N\left(\rho x_1, (1-\rho^2)\right). \tag{4.32}
$$

Using this last relation, and again the law of total expectation, (4.31) becomes

$$
\begin{aligned}
\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{Z<d_1\}}\mathbb{1}_{\{U<\hat{d}_2\}}\right] &= \left[\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{X_1<d_1\}}\mathbb{1}_{\{X_2<d_2\}}|X_2 = x_2\right]\right]\\
&= \int_{-\infty}^{\infty}\mathbb{1}_{\{x_1<d_1\}}\rho x_1\mathbb{1}_{\{x_1<d_2\}}\phi(x_1)dx_1\\
&=\rho\int_{-\infty}^{\min(d_1,d_2)} x_1\phi(x_1)dx_1\\
&=\rho\int_{-\infty}^{m} x_1\phi(x_1)dx_1\\
&= \phi(m), \qquad (4.33)
\end{aligned}
$$

where the last equality follows from (4.27). Combining (4.33) and (4.31) yields

$$
K_1\tilde{\sigma}_2(t,s)\,\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{X_1<d_1\}}\mathbb{1}_{\{X_2<d_2\}}\right] = K_1\tilde{\sigma}_2(t,T)\rho\phi(m). \tag{4.34}
$$

Thus

$$\mathrm{E}_Q\left[K_1 D_2 \mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}|\mathcal{F}_t\right] = K_1\tilde{\alpha}_2(t,T)S_2(t)C(d_1,d_2,\rho) + K_1\tilde{\sigma}_2(t,T)\rho\phi(m) \tag{4.35}$$

To compute $\mathrm{E}_Q\left[D_1 D_2 \mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]$, observe that inserting expressions (4.18) and (4.19) for D_1 and D_2 respectively, D_1D_2 can be rewritten as

$$\begin{aligned}
D_1D_2 &= (\tilde{\alpha}_1(t,T)S_1(t)+Z\tilde{\sigma}_1(t,T))\left(\tilde{\alpha}_2(t,T)S_2(t)+(\rho Z+\sqrt{1-\rho^2}U)\tilde{\sigma}_2(t,T)\right)\\
&= (\tilde{\alpha}_1(t,T)S_1(t)+X_1\tilde{\sigma}_1(t,T))\,(\tilde{\alpha}_2(t,T)S_2(t)+X_2\tilde{\sigma}_2(t,T))\\
&= \tilde{\alpha}_1(t,T)S_1(t)\tilde{\alpha}_2(t,T)S_2(t)\\
&\quad+\tilde{\alpha}_1(t,T)S_1(t)X_2\tilde{\sigma}_2(t,T)\\
&\quad+X_1\tilde{\sigma}_1(t,T)\tilde{\alpha}_2(t,T)S_2(t)\\
&\quad+X_1\tilde{\sigma}_1(t,T)X_2\tilde{\sigma}_2(t,T)
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{E}_{\mathbb{Q}}\left[D_1D_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &= \mathrm{E}_{\mathbb{Q}}\left[\tilde{\alpha}_1(t,T)S_1(t)\tilde{\alpha}_2(t,T)S_2(t)\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]\\
&\quad+\mathrm{E}_{\mathbb{Q}}\left[\tilde{\alpha}_1(t,T)S_1(t)X_2\tilde{\sigma}_2(t,T)\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]\\
&\quad+\mathrm{E}_{\mathbb{Q}}\left[X_1\tilde{\sigma}_1(t,T)\tilde{\alpha}_2(t,T)S_2(t)\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]\\
&\quad+\mathrm{E}_{\mathbb{Q}}\left[X_1\tilde{\sigma}_1(t,T)X_2\tilde{\sigma}_2(t,T)\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]\\
&=\tilde{\alpha}_1(t,T)S_1(t)\tilde{\alpha}_2(t,T)S_2(t)\,\mathrm{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &\text{(4.36)}\\
&\quad+\tilde{\alpha}_1(t,T)S_1(t)\tilde{\sigma}_2(t,T)\,\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &\text{(4.37)}\\
&\quad+\tilde{\sigma}_1(t,T)\tilde{\alpha}_2(t,T)S_2(t)\,\mathrm{E}_{\mathbb{Q}}\left[X_1\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &\text{(4.38)}\\
&\quad+\tilde{\sigma}_1(t,T)\tilde{\sigma}_2(t,T)\,\mathrm{E}_{\mathbb{Q}}\left[X_1X_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &\text{(4.39)}
\end{aligned}$$

Starting with (4.36), which is equal to (4.21), it is clear that

$$\begin{aligned}
&\tilde{\alpha}_1(t,T)S_1(t)\tilde{\alpha}_2(t,T)S_2(t)E_Q\left[\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]\\
&\qquad=\tilde{\alpha}_1(t,T)S_1(t)\tilde{\alpha}_2(t,T)S_2(t)C(d_1,d_2,\rho).
\end{aligned}\tag{4.40}$$

where C is the bivariate normal cumulative distribution function for X_1, X_2. Moving on to (4.37), the expectation is the same as the one computed in (4.34), thus

$$\tilde{\alpha}_1(t,T)S_1(t)\tilde{\sigma}_2(t,T)\,\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] = \tilde{\alpha}_1(t,T)S_1(t)\tilde{\sigma}_2(t,T)\rho\phi(m). \tag{4.41}$$

Regarding (4.38), the expectation has been computed in (4.27), hence

$$\begin{aligned}\tilde{\sigma}_1(t,T)\tilde{\alpha}_2(t,T)S_2(t)\,\mathrm{E}_{\mathbb{Q}}&\left[X_1\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]\\ &= \tilde{\sigma}_1(t,T)\tilde{\alpha}_2(t,T)S_2(t)\rho\phi(m)\end{aligned} \tag{4.42}$$

Finally, the expectation term in (4.39) can be computed the following way.

$$\begin{aligned}\mathrm{E}_{\mathbb{Q}}\left[X_1X_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] &= \mathrm{E}_{\mathbb{Q}}\left[\mathrm{E}_{\mathbb{Q}}\left[X_1X_2\mathbb{1}_{\{X_1<d_1\}}\mathbb{1}_{\{X_2<d_2\}}|X_2=x_2\right]\right]\\ &= \int_{-\infty}^{\infty} x_2\mathbb{1}_{\{x_2<d_2\}}\,\mathrm{E}_Q\left[X_1\mathbb{1}_{\{X_1<d_2\}}|X_2=x_2\right]\phi(x_2)dx_2\\ &= \int_{-\infty}^{\infty} x_2\mathbb{1}_{\{x_2<d_2\}}\rho x_2\mathbb{1}_{\{x_2<d_1\}}\phi(x_2)dx_2\\ &= \rho\int_{-\infty}^{\min(d_1,d_2)} x_2^2\phi(x_2)dx_2\end{aligned} \tag{4.43}$$

Let $m=\min(d_1,d_2)$.
Looking at the integral, it can be rewritten as

$$\int_{-\infty}^{m} x^2\phi(x_2)dx_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{m} x_2^2e^{-\frac{1}{2}x_2^2}dx_2 = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{m} x_2(-x_2e^{-\frac{1}{2}x_2^2})dx_2.$$

Denote $u(x_2)=x_2, v(x_2)=e^{-\frac{1}{2}x_2^2}$, then $v'(x_2)=-x_2e^{-\frac{1}{2}x_2^2}$ and $u'(x_2)=1$, so that

$$\begin{aligned}-\frac{1}{2\pi}\int_{-\infty}^{m} u(x_2)v'(x_2)dz &= -\frac{1}{\sqrt{2\pi}}\left[u(x_2)v(x_2)\right]_{x_2=-\infty}^{m} + \frac{1}{\sqrt{2\pi}}\int_{infty}^{m} v(x_2)u'(x_2)dx_2\\ &= -\frac{1}{\sqrt{2\pi}}\left[x_2e^{-\frac{1}{2}x_2^2}\right]_{x_2=-\infty}^{\min(d_1,d_2)} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{1}{2}x_2^2}dx_2\\ &= -m\phi(m)+\Phi(m).\end{aligned} \tag{4.44}$$

So (4.39) is given from (4.43) and (4.44) by

$$\begin{aligned}\tilde{\sigma}_1(t,T)\tilde{\sigma}_2(t,T)\,\mathrm{E}_{\mathbb{Q}}&\left[X_1X_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right]\\ &= \tilde{\sigma}_1(t,T)\tilde{\sigma}_2(t,T)\left(\Phi(m)-m\phi(m)\right).\end{aligned} \tag{4.45}$$

Summing up (4.40), (4.41), (4.42), (4.45) gives the equality

$$\begin{aligned}\mathrm{E}_{\mathbb{Q}}\left[D_1D_2\mathbb{1}_{\{D_1<K_1\}}\mathbb{1}_{\{D_2>K_2\}}\right] =&\tilde{\alpha}_1(t,T)S_1(t)\tilde{\alpha}_2(t,T)S_2(t)C(d_1,d_2,\rho)\\ &+\tilde{\alpha}_1(t,T)S_1(t)\tilde{\sigma}_2(t,T)\rho\phi(m)\\ &+\tilde{\sigma}_1(t,T)\tilde{\alpha}_2(t,T)S_2(t)\rho\phi(m)+ &(4.46)\\ &+\tilde{\sigma}_1(t,T)\tilde{\sigma}_2(t,T)\left(\Phi(m)-m\phi(m)\right) &(4.47)\end{aligned}$$

Ultimately, the price is given by (4.21), (4.27), (4.35), , (4.47), , thus

$$\begin{aligned}
p(t) = \Big(& \mathrm{E}_Q \left[K_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} \right] - \mathrm{E}_Q \left[K_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} \right] \\
& - \mathrm{E}_Q \left[D_1 D_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} \right] + \mathrm{E}_Q \left[D_1 K_2 \mathbb{1}_{\{D_1 < K_1\}} \mathbb{1}_{\{D_2 > K_2\}} \right] \Big) \\
& = K_1 \left(\tilde{\alpha}_2(t,T) S_2(t) C(d_1, d_2, \rho) + \tilde{\sigma}_2(t,T) \rho \phi(m) \right) \\
& - \left(K_1 K_2 C(d_1, d_2, \rho) \right) \\
& + \tilde{\sigma}_1(t,T) \tilde{\sigma}_2(t,T) \left(m\phi(m) - \Phi(m) \right) \\
& K_2 \tilde{\alpha}_1 S_1(t) C(d_1, d_2, \rho) + \tilde{\sigma}_1(t,s) \rho \phi(m),
\end{aligned} \tag{4.48}$$

which corresponds with the price in (4.13).

□

The payoff function in this example is just one of many different put/call-paritys one can use for a energy quanto option. Another combination could have been a put option on the spot price of wind power and a call option on the number of windy days in a month. If the number of windy days is low, the profit from the long position on wind power spot will increase. Simultanously, the call option on the number of windy days will make a hegde against the event that the number of windy days should be numerous. Other combinations could be coal price and coal power price, or electricity price and the number of heating degree days over a month.

Figure (4.2) below, shows six simulations of two OU-processes with dynamics as given in (4.5). As is clearly illustrated by the plots, the value fluctuations correspond increasingly as ρ increases. It is worth noting that even though ρ is close to 1, the values of the correlated processes may differ substantially, but increases or decreases in process value will correspond in both processes.

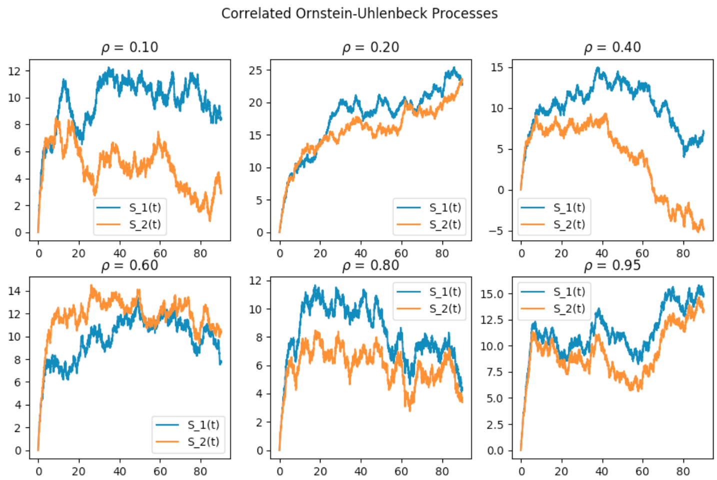

Figure 4: Simulations of correlated OU-processes with six different ρ's.

Chapter 5: Energy derivatives with counterparty risk

5.1 Forward price with counterparty risk

Consider two parties A and B, where A is default-free while B has a risk of default. Assume the risk-free interest rate r is zero. Let the total value of firm B be given by a value process V which follows the dynamics of a GBM on the form (2.10), that is,

$$dV(t) = -\kappa V(t)dt + \sigma_V d\widetilde{W}(t), \tag{5.1}$$

and let the time of default τ be given by

$$\tau = \tau_2 \mathbb{1}_{V(\tau_2)<L} + \infty \mathbb{1}_{V(\tau_2)>L},$$

where L is the liabilities of company B, and with the usual convention $0 \cdot \infty = 0$. Furthermore, assume that the spot price of hydro power is given by S which has risk-neutral dynamics

$$dS(t) = (\mu - \alpha S(t))dt + \sigma d\widetilde{B}(t), \tag{5.2}$$

and $\widetilde{B}, \widetilde{W}$ are Brownian motions under $\mathbb{Q}$. The dependency or independency of $\widetilde{B}, \widetilde{W}$ will differ in the following examples, it will however be clearly stated whether they are indedependent or dependent. Let $T* < \infty$ be some finite time date, and suppose $(\Omega, \mathcal{F}, \mathcal{Q})$ is a probability space endowed with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$. The filtration is assumed to be sufficiently rich to support the value process V and the spot price process S.

Proposition 5.1. *Let the spot price of hydro power be assumed to have dynamics as given in (5.2). Denote by $F_D(t, \tau_1, \tau_2)$ the time t price of a power forward contract with delivery period $[\tau_1, \tau_2]$ and settlement at τ_2, for $t \leq \tau_1 \leq \tau_2$. Let the party selling the forward contract have a total asset value given by V, which has dynamics given by (5.1). The forward price is then given by*

$$F_D(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \mathrm{E}\left[\int_{\tau_1}^{\tau_2} S(u)du\right] \Phi(\hat{d}_1), \tag{5.3}$$

where Φ is the standard normal cumulative function, and

$$\hat{d}_1 = \frac{\ln\left(\frac{V(t)}{L}\right) - \frac{1}{2}\sigma_V^2(T-t)}{\sigma_V\sqrt{T-t}}$$

Proof. Let firm A enter a short position in a financial power forward contract.
The payoff for party A of entering such a forward contract will be

$$\mathbb{1}_{\{V(T)\geq L\}}\int_{\tau_1}^{\tau_2} S(u)du - D(t,\tau_1,\tau_2).$$

Since the risk-free interest rate r is zero, the arbitrage-free price of the forward contract will be given by the equation

$$\mathrm{E}_{\mathbb{Q}}\left[\left(\mathbb{1}_{\{V(\tau_2)\geq L\}}\int_{\tau_1}^{\tau_2} S(u)du - D(t,\tau_1,\tau_2)\right)|_{\mathcal{F}_t}\right] = 0.$$

Due to the independent increments of $\tilde{B}$ and $\tilde{W}$, they are independent of $\mathcal{F}_t$, which leads to the forward price

$$F_D(t,\tau_1,\tau_2) = E_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2}\mathbb{1}_{\{V(\tau_2)\geq L\}}S(u)du\right]. \tag{5.4}$$

Consider first the indicator function in (5.4). It can be shown that the explicit solution to the GBM (5.1) for $\tau_2 \geq t$ is

$$V(\tau_2) = V(t)e^{\sigma_V(W(\tau_2)-W(t))-\frac{1}{2}\sigma_V^2)(\tau_2-t))},$$

The indicator function is nonzero when $V(\leq) \geq L$, i.e.

$$V(t)e^{\sigma_V(W(\tau_2)-W(t))+(r-\frac{1}{2})\sigma_V^2)(\tau_2-t))} \geq L,$$

which is equivalent to

$$\sigma_V(W(\tau_2)-W(t)) \geq \ln\left(\frac{L}{V(t)}\right) + \frac{1}{2}\sigma_V^2(T-t).$$

Exploiting the normal distribution of the brownian motion, it is clear that the left-hand side of the equation is equal in distribution to

$$\sigma_V\sqrt{\tau_2-t}X_1, \tag{5.5}$$

where $X_1 \sim N(0,1)$.
Thus the indicator is nonzero when

$$X_1 \leq \frac{\ln\left(\frac{V(t)}{L}\right) - \frac{1}{2}\sigma_V^2(\tau_2-t)}{\sigma_V\sqrt{T-t}} = \hat{d}_1. \tag{5.6}$$

Since $\widetilde{W}$ and $\widetilde{B}$ are independent the forward price (5.4) becomes

$$\begin{aligned} F_D(t,\tau_1,\tau_2) &= \mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(u)du \mathbb{1}_{\{V(\tau_2)\geq L\}}\right] \\ &= \mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(u)du\right] E_{\mathbb{Q}}\left[\mathbb{1}_{\{V(\tau_2)\geq L\}}\right] \\ &= \mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(u)du\right] \mathbb{Q}\left(V(\tau_2)\geq L\right) \\ &= \frac{1}{\tau_2-\tau_1}\mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(u)du\right]\Phi(\hat{d}_1), \end{aligned} \tag{5.7}$$

where $\hat{d}_1 = \frac{\ln\left(\frac{V(t)}{L}\right)-\frac{1}{2}\sigma_V^2(T-t)}{\sigma_V\sqrt{T-t}}$, and Φ is the standard normal cumulative distribution function.

□

Comparing the forward price (1.28) in section 1.5 with (5.3), it is evident that with the appearence of a default risk, the forward price is discounted by a factor of $\Phi(\hat{d}_1)$, that is, discounted by the probability that the counterparty company does not default. Figure (5.1) shows the plot of the probability $(\mathbb{P}(V(T) > L))$ as a function of the volatility σ_V, and thus shows may be seen as a plot of the discount $\Phi(\hat{d}_1)$ of a forward contract as well.

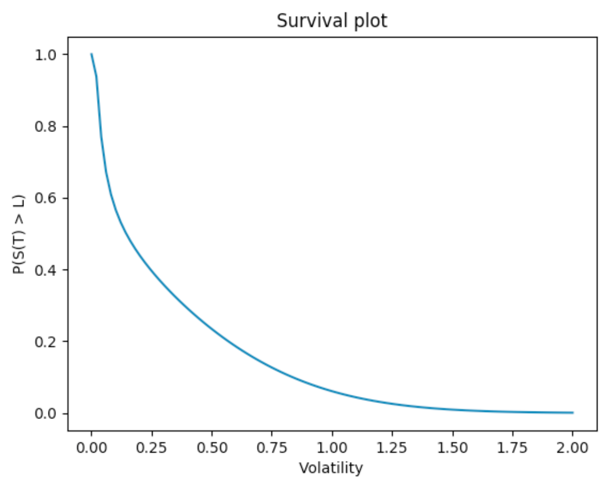

Figure 5: Plot of the survival probability of a firm as a function of the volatility of the firm's value process V, with $x = 100, L = 90, T = 10$.

As one can see in figure (5.1), the probability that a company defaults has a positive correlation with the volaltility of the firm's value process. Something that makes sense intuitively as well, since value processes with high volatility have increased risk compared to a value process with modest volatility.

The assumption that $\widetilde{W}$ and $\widetilde{B}$ from (5.1) and (5.2) are independent implies that the total value of the company is independent of the spot price of the power the company is selling. The next proposition states the forward price if $\widetilde{W}$ and $\widetilde{B}$ are correlated.

Proposition 5.2. *Denote by $F_D(t,\tau_1,\tau_2)$ the time t price of a financial power forward contract where the spot price of power has dynamics given by (5.2). Assume that the seller of the forward contract has a default risk, where time of default The forward price is then given by*

$$F_D(t,\tau_1,\tau_2) = \rho\phi(\hat{d}_1) + \tilde{\alpha}_1 S(t)\Phi(\hat{d}_1), \tag{5.8}$$

where $\hat{d}_1 = \frac{\ln\left(\frac{V(t)}{L}\right) - \frac{1}{2}\sigma_V^2(\tau_2 - t)}{\sigma_V\sqrt{T-t}} = \hat{d}_1.$

Proof. Using notation from chapter 4, the average spot price over a delivery period of $[\tau_1,\tau_2]$ is given as

$$\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} S(u)du = D(\tau_1,\tau_2),$$

and

$$D(\tau_1,\tau_2) \overset{i.d.}{=} \tilde{\alpha}S(t) + X_2\tilde{\sigma},$$

where $X_2 \sim N(0,1)$.
Combining this last relation with the forward price equation (5.7), the forward price is given by

$$\begin{aligned} F_D(t,\tau_1,\tau_2) &= \mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(u)du\mathbb{1}_{\{V(\tau_2)\geq L\}}|\mathcal{F}_t\right] \\ &= \mathrm{E}_{\mathbb{Q}}\left[D(\tau_1,\tau_2)\mathbb{1}_{\{V(\tau_2)\geq L\}}\right] \\ &= \mathrm{E}_{\mathbb{Q}}\left[(\tilde{\alpha}_1 S(t) + X_1\tilde{\sigma}_1)\,\mathbb{1}_{\{V(\tau_2)\geq L\}}\right]. \end{aligned} \tag{5.9}$$

The assumption that $\widetilde{W}$ and $\widetilde{B}$ are correlated can satisfied by defining

$$X_1 = \rho Z + \sqrt{1-\rho^2}U,$$

and

$$X_2 = Z,$$

where Z and U are independent standard normal variables and $|\rho| < 1$. The correlation of X_1 and X_2 will be

$$\begin{aligned} \mathrm{E}_Q[X_1X_2] = \mathrm{E}_Q[(\rho Z + \sqrt{1-\rho^2})Z] =& \mathrm{E}_Q[\rho Z^2] + \mathrm{E}_{\mathbb{Q}}[\sqrt{1-\rho^2}ZU] \\ =& \rho + \sqrt{1-\rho^2}\,\mathrm{E}_{\mathbb{Q}}[U]\,\mathrm{E}_{\mathbb{Q}}[Z] = \rho. \end{aligned} \tag{5.10}$$

By (5.6), $V(\tau_2) > L$ is equivalent with $X_1 < \hat{d}_1$, thus the last forward price equation (5.9) becomes

$$\begin{aligned}
\mathrm{E}_{\mathbb{Q}}\left[(\tilde{\alpha}_1 S(t) + X_2\tilde{\sigma}_1)\,\mathbb{1}_{\{V(\tau_2)\geq L\}}\right] &= \mathrm{E}_{\mathbb{Q}}\left[(\tilde{\alpha}_1 S(t) + X_2\tilde{\sigma}_1)\,\mathbb{1}_{\{X_1<\hat{d}_1\}}\right] \\
&= \mathrm{E}_{\mathbb{Q}}\left[\tilde{\alpha_1} S(t)\mathbb{1}_{\{X_1<\hat{d}_1\}}\right] \\
&\quad + \mathrm{E}_{\mathbb{Q}}\left[X_2\tilde{\sigma}_1\mathbb{1}_{\{X_1<\hat{d}_1\}}\right] \\
&= \tilde{\alpha}_1 S(t)\,\mathrm{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{X_1<\hat{d}_1\}}\right] \qquad (5.11) \\
&\quad + \tilde{\sigma}_1\,\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{X_1<\hat{d}_1\}}\right]. \qquad (5.12)
\end{aligned}$$

The term (5.11) can be straigtforwardly computed as

$$\tilde{\alpha}_1 S(t)\,\mathrm{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{X_1<\hat{d}_1\}}\right] = \tilde{\alpha}_1 S(t)\Phi(\hat{d}_1). \tag{5.13}$$

While (5.12) can be computed using the law of total expectaton in the following manner.

$$\begin{aligned}
\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{X_1<\hat{d}_1\}}\right] &= \mathrm{E}_{\mathbb{Q}}\left[\mathrm{E}_{\mathbb{Q}}\left[X_2\mathbb{1}_{\{X_1<\hat{d}_1\}}|X_1 = x_1\right]\right] \\
&= \int_{-\infty}^{\infty} \mathrm{E}_{\mathbb{Q}}\left[X_2|X_1 = x_1\right]\mathbb{1}_{\{X_1<\hat{d}_1\}}\phi(x_1)dx_1 \\
&= \int_{-\infty}^{\infty} \rho x_1\mathbb{1}_{\{x_1<\hat{d}_1\}}\phi(x_1)dx_1 \\
&= \rho\int_{-\infty}^{\hat{d}_1} x_1\phi(x_1)dx_1 \\
&= \rho\phi(\hat{d}_1). \qquad (5.14)
\end{aligned}$$

Combining (5.11) and (5.14) yields the forward price

$$F_D(t,\tau_1,\tau_2) = \rho\phi(\hat{d}_1) + \tilde{\alpha}_1 S(t)\Phi(\hat{d}_1),$$

which is what was to be shown. □

5.2 Call Option with Counterparty Risk

In this section, a closed form solution of a call-option with the presence of counterparty is derived. Before looking at a call option involving counterparty risk, a call option without counterparty risk is priced. Remember that the time t price of a forward contract over a delivery period $[\tau_1, \tau_2]$ may be given as

$$F(t,\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\,\mathrm{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(u)du|\mathcal{F}_t\right]. \tag{5.15}$$

The following proposition will give an analytical price on a call option with no counterparty risk.

Proposition 5.3. *Let the time t price of a hydro power forward contract with delivery period $[\tau_1, \tau_2]$, $F(t, \tau_1, \tau_2)$, be given by (5.15), where S is a continuous stochastic process given by (5.1). Let $T < \tau_1$ be the maturity date of a call option with strike price K, and settlement date τ_2. Suppose the risk-free interest rate r is zero. The time t price of such a call option is then given by*

$$p(t, T, \tau_1, \tau_2) = \Phi(c_2)(\tilde{\alpha}S(t) - K) + \tilde{\sigma}\phi(c_2) \tag{5.16}$$

Proof. The payoff function f is given by

$$f = \max\left(F(t, \tau_1, \tau_2) - K, 0\right). \tag{5.17}$$

and chapter 4 grants the relation

$$\frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} S(s)ds \overset{i.d.}{=} \tilde{\alpha}S(t) + X_2\tilde{\sigma}, \tag{5.18}$$

for $X_2 \sim N(0, 1)$, and $\tilde{\alpha}$, $\tilde{\sigma}$ on the form (4.16), (4.17), respectively.
Thus the forward price may be given as

$$F(t, \tau_1, \tau_2) = \mathrm{E}_{\mathbb{Q}}\left[\tilde{\alpha}S(t) + X_2\tilde{\sigma}|\mathcal{F}_t\right]. \tag{5.19}$$

Solving $\tilde{\alpha}S(t) + X_2\tilde{\sigma} > K$ for X_2, yields

$$X_2 < \frac{\tilde{\alpha}S(t) - K}{\tilde{\sigma}} = c_2. \tag{5.20}$$

Since the risk-free interest rate r is zero, the arbitrage-free price is given by

$$\begin{aligned}
P(t, T, \tau_1, \tau_2) &= E_{\mathbb{Q}}\left[\max\left(F(t, \tau_1, \tau_2) - K, 0\right)|\mathcal{F}_t\right] \\
&= \mathrm{E}_{\mathbb{Q}}\left[\left(F(t, \tau_1, \tau_2) - K\right)\mathbb{1}_{\{F(t,\tau_1,\tau_2) > K\}}\right] \\
&= E_{\mathbb{Q}}\left[\left((\tilde{\alpha}S(t) + X_2\tilde{\sigma}) - K\right)\mathbb{1}_{\{X_2 < c_2\}}\right] \\
&= E_{\mathbb{Q}}\left[\tilde{\alpha}S(t)\mathbb{1}_{\{X_2 < c_2\}}\right] - E_{\mathbb{Q}}\left[X_2\tilde{\sigma}\mathbb{1}_{\{X_2 < c_2\}}\right] - K\mathbb{1}_{\{X_2 < c_2\}} \\
&= \tilde{\alpha}S(t)\mathbb{Q}(X_2 < c_2) + \tilde{\sigma}\int_{c_2}^{-\infty} x_2\phi(x_2)dx_2 - K\mathbb{Q}(X_2 < c_2) \\
&= \Phi(c_2)(\tilde{\alpha}S(t) - K) + \tilde{\sigma}\phi(c_2),
\end{aligned}$$

which corresponds with (5.16). ∎

The following proposition gives the analytical price of a defaultable call option.

Proposition 5.4. *Let the time t price of a hydro power forward contract with delivery period $[\tau_1, \tau_2]$, $F(t, \tau_1, \tau_2)$, be given by (5.15), where S is a continuous stochastic process given by (5.1). Let $T < \tau_1$ be the maturity date of a call option with strike price K, and settlement date τ_2. Assume that the option seller has a default risk and that the default time τ is given by*

risk-free interest rate r is equal to zero, and that The time t price, p_D, of such a call option is then given by

$$p(t, T, \tau_1, \tau_2) = \tilde{\alpha} S(t) C(c_2, \hat{d}_1, \rho) + \tilde{\sigma}_1 \phi(\min(c_2, \hat{d}_2)) - K C(c_2, \hat{d}_1, \rho). \quad (5.21)$$

Proof. The payoff function at time τ_2 is given by

$$f = \max\left(F(t, \tau_1, \tau_2) - K, 0\right) \mathbb{1}_{\{V(\tau_2) > L\}}$$

Remember that (5.6) implies $V(\tau_2) > L$ is equivalent with $X_1 < \hat{d}_1$, and relying on (5.18), (5.19), (5.20) above, the arbitrage-free price is given by

$$\begin{aligned}
p_D(t, T, \tau_1, \tau_2) &= E_{\mathbb{Q}}\left[\max\left(F(t, \tau_1, \tau_2) - K, 0\right) \mathbb{1}_{\{V(T) > L\}} | \mathcal{F}_t\right] \\
&= \mathrm{E}_{\mathbb{Q}}\left[\left(F(t, \tau_1, \tau_2) - K\right) \mathbb{1}_{\{F(t, \tau_1, \tau_2) > K\}} \mathbb{1}_{\{V(\tau_2) > L\}}\right] \\
&= E_{\mathbb{Q}}\left[\left((\tilde{\alpha} S(t) + X_2 \tilde{\sigma}) - K\right) \mathbb{1}_{\{X_2 < c_2\}} \mathbb{1}_{\{X_1 < \hat{d}_1\}}\right] \\
&= E_{\mathbb{Q}}\left[\tilde{\alpha} S(t) \mathbb{1}_{\{X_2 < c_2\}} \mathbb{1}_{\{X_1 < \hat{d}_1\}}\right] \\
&\quad + E_{\mathbb{Q}}\left[X_2 \tilde{\sigma} \mathbb{1}_{\{X_2 < c_2\}} \mathbb{1}_{\{X_1 < \hat{d}_1\}}\right] \\
&\quad - E_{\mathbb{Q}}\left[K \mathbb{1}_{\{X_2 < c_2\}} \mathbb{1}_{\{X_1 < \hat{d}_1\}}\right] \\
&= C(c_2, \hat{d}_1, \rho)(\tilde{\alpha} S(t) - K) + \tilde{\sigma}_1 \phi(\min(c_2, \hat{d}_1)),
\end{aligned}$$

where the last equality relies on results from (4.21) and (4.35). □

Depending on whether $c_2 < \hat{d}_1$ or $c_2 > \hat{d}_1$, the difference in price d for a non-defaultable call option (5.16) and a defaultable call option (5.21) will be

$$\begin{aligned}
d = p_D(t, T, \tau_1) - p(t, T, \tau_1, \tau_2) &= C(c_2, \hat{d}_1, \rho)(\tilde{\alpha} S(t) - K) + \tilde{\sigma}_1 \phi(c_2,) \\
&\quad - (\Phi(c_2)(\tilde{\alpha} S(t) - K) + \tilde{\sigma} \phi(c_2)) \\
&= (\tilde{\alpha} S(t) - K)(C(c_2, \hat{d}_1, \rho) - \Phi(c_2)), \quad (5.22)
\end{aligned}$$

for $c_2 < \hat{d}_1$, and

$$d = (\tilde{\alpha} S(t) - K)(C(c_2, \hat{d}_1, \rho) - \Phi(c_2)) + phi(c_2) - phi(d_1), \quad (5.23)$$

for $c_2 > \hat{d}_1$.

Appendix A: The Bivariate Normal Distribution

This theory is derived from *The Mulitvariate Normal Distribution* by *Y.L. Tong* given in [Ton90]. In order to see that the bivariate normal distribution is a natural extension of the univariate normal distribution, and as a reminder to the reader, the definition of a normally distributed random variable is defined in what follows.

Definition A.1. A random variable X is said to have a normal distribution with mean μ and variance $\sigma^2 > 0$ if its probability density function f is of the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\left(\frac{x-\mu}{2\sigma^2}\right)}, \tag{A.1}$$

where $\mu \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$.

The bivariate normal distribution on the other hand is defined as follows.

Definition A.2. Let $\mathbf{X}$ be a 2-dimensional random variable of the form

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{bmatrix}\right) \tag{A.2}$$

, where $\mu_i \in \mathbb{R}$ and $\sigma_i^2 \in (0, \infty)$ for $i = 1, 2$, and ρ is the correlation of X_1 and X_2 given by

$$\rho = \text{corr}(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sigma_1 \sigma_2}. \tag{A.3}$$

The random variable $\mathbf{X}$ is said to have a bivariate normal distribution if its probability density function f is of the form

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)}.$$

A.1 Properties of the Bivariate Normal Distribution

If $\mathbf{X}$ is defined as in (A.2), then the following properties hold. Let C is the cumulative distribution function of $\mathbf{X}$ and $\mathbb{P}$ is a probability measure, then for $\mathbf{x} \in \mathbb{R}^2$

$$P(\mathbf{X} < \mathbf{x}) = C(x_1, x_2, \rho) = \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(x_1, x_2) dx_1 dx_2. \tag{A.4}$$

For $x_2 \in \mathbb{R}$ the following conditional expectation will have distribution

$$(X_1 | X_2 = x_2) \sim N\left(\mu_1 + \frac{\sigma_1}{\sigma_2}\rho(x_2 - \mu_2), (1-\rho^2)\sigma_1^2\right). \tag{A.5}$$

Appendix B: Codes

B.1 Ornstein-Uhlenbeck Process

```
import matplotlib.pyplot as plt
import numpy as np
# dX(t) = (mu- alpha X(t))dt + sigma dB(t)
n = 5                              # number of processes
mu = 10
alpha = 0.3
sigma = 0.05
T = 90                              # total time
x = 0                               # initial value X
dt = 0.01                           # length of time steps
N = int(T/dt)                       # number of time steps
t = np.linspace(0,T,N)              # time points

#mean
def m(x,alpha,mu,t):
    return x*np.exp(-alpha*t) + mu*(1-np.exp(-alpha*t))
#variance
def v(alpha,sigma,t):
return sigma*np.sqrt((1-np.exp(-2*alpha*t))/(2*alpha))

for i in range(n):
    normalvariates = np.random.normal(0,1,N)    # N standard normal var
        B = np.cumsum(normalvariates)           # cumulative sum of nor
        X =  m(x,alpha,mu,t)+ v(alpha,sigma,t)*B
        plt.plot(t, X)
        plt.title('Ornstein-Uhlenbeck Process')
        plt.xlabel('t')
        plt.ylabel('X(t)')

plt.show()
```

B.2 Compound Poisson Process

```
import numpy as np
import matplotlib.pyplot as plt


N = 3                          # number of processes
n = 30                         # number of steps
Lambda = 0.5                   # parameter for jumptimes Poisson(Lambda)
mu_U = 0                       # mean of the jump
sigma_U = 3                    # variance of the jump
poiss = np.random.poisson(Lambda, size = n)
time = np.linspace(1,n,n)
value = np.zeros(n)
for i in range(N):
    poiss = np.random.poisson(Lambda, size = n)
    value = np.zeros(n)
    for t in range(1,n):
        value[t] = value[t-1] + np.sum(np.random.normal(mu_U, sigma_U,
        plt.step(time,value, where = 'post')
        plt.title('CPP-Process with Normally Distributed Jumps')
        plt.xlabel('t')
        plt.ylabel('I(t)')
plt.show()
```

B.3 Survival Plot

```
import numpy as np
from math import sqrt, log
import scipy.stats as si
import matplotlib.pyplot as plt


x_0 = 100.0    # initial value of company
L = 90.0       # trigger value for deafult
T = 10.0       # maturity

def default_constant(x_0, T, sigma, L):
return (log(x_0/L)-0.5*(sigma**2*T))/(sigma*sqrt(T))


volatilites = np.linspace(0.001, 2, 100)

ds = []       # list for constants
for i in volatilites:
d = si.norm.cdf(default_constant(x_0, T, i, L), 0.0, 1.0)
ds.append(d)


plt.plot(volatilites, ds)
plt.xlabel('Volatility of V')
plt.ylabel('P(S(T) > L)')
plt.show()
```

B.4 Monte Carlo estimation call option

```
#OU-process parameters
x = 4
mu = 5
alpha = 0.3
sigma = 0.05
K = 150              # strike price
T = 90               # maturity date
N = 1000000          # number of simulations

#E[X(t)]
tildeMu = x*np.exp(-alpha*T) + mu*(1-np.exp(-alpha*(T)))
#V[X(t)]
tildeSigma = sigma*np.sqrt((1-np.exp(-2*alpha*T))/(2*alpha))

M = (tildeMu - np.log(K))/tildeSigma

def assetprice(m,v):
    return np.exp(tildeMu + tildeSigma*np.random.normal(0.0,1.0))

def payoff(S, K):
return max(0.0, S-K)

def exact_price(tildeMu, tildeSigma, M):
    return np.exp(tildeMu + 0.5*tildeSigma**2)*
            si.norm.cdf(M+tildeSigma,0.0,1.0) - K*si.norm.cdf(M, 0.0,

prices = []
for i in range(N):
    S_T = assetprice(tildeMu, tildeSigma)
    prices.append(payoff(S_T,K))

MCprice = sum(prices)/float(N)

exact_price = exact_price(tildeMu, tildeSigma, M)
print(exact_price, MCprice)
```

B.5 Simulation of Correlated OU-processes

```
import matplotlib.pyplot as plt
import numpy as np
# dX_1(t) = (mu- alpha X(t))dt + sigma dB(t)
# dX_2(2) = (mu-alpha X(t))dt + sigma d( rhoB(t) + sqrt(1-rho^2)U(t))
n = 6
mu = 10
alpha = 0.3
sigma = 0.05
T = 90
x = 0
dt = 0.01
N = int(T/dt)
t = np.linspace(0,T,N)


def m(x,alpha,mu,t):
    return x*np.exp(-alpha*t) + mu*(1-np.exp(-alpha*t))

def v(alpha,sigma,t):
return sigma*np.sqrt((1-np.exp(-2*alpha*t))/(2*alpha))
    rho = [0,0.1,0.2,0.4,0.6,0.8,0.95]

for i in range(1,n+1):
    normalvariates1 = np.random.normal(0,1,N)
    normalvariates2 = np.random.normal(0,1,N)
    B = np.cumsum(normalvariates1)                # cumulative sum of
    U = np.cumsum(normalvariates2)
    X = m(x,alpha,mu,t) + v(alpha,sigma,t)*B
    Y = m(x,alpha,mu,t) + v(alpha,sigma,t)*(rho[i]*B + np.sqrt(1-rho[i
    plt.subplot(2,3,i)
    plt.plot(t,X)
    plt.hold(True)
    plt.plot(t,Y)
    plt.legend(['S_1(t)', 'S_2(t)'])
    plt.title(r'$\rho$ = %1.2f' %rho[i])
    plt.hold(False)
    plt.suptitle('Correlated Ornstein-Uhlenbeck Processes')

plt.show()
```

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