UiO : Department of Mathematics
University of Oslo

Tropical Poincaré duality spaces

Edvard Aksnes

Master's Thesis, Spring 2019

This master's thesis is submitted under the master's programme *Mathematics*, with programme option *Mathematics*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Certain polyhedral fans can be constructed from matroids, and these serve as the local model of tropical manifolds. Such matroidal fans satisfy a tropical version of Poincaré duality [JRS18]. In this thesis, we give conditions on pure polyhedral fans which are equivalent to this property. Moreover, we classify tropical Poincaré spaces of dimension two.

Furthermore, we develop the derived category of cellular sheaves on a polyhedral complex, based on work by Curry [Cur14], and use Verdier duality to prove a vanishing result on the compact support cohomology of the wave sheaf on a Cohen–Macaulay simplicial polyhedral fan.

Acknowledgments

First and foremost, I would like to thank my supervisor Kristin Shaw. The enthusiasm with which you've introduced me to tropical geometry is deeply contagious, and the patience you've shown as the project progressed has been deeply appreciated.

Furthermore, I would like to thank my parents for their caring support, guided yet unconditional, setting me on a path towards what I enjoy. Thank you to my father, Fredrik, for his encouragement in academic pursuits, and thank you to my mother, Ebba, for all the conversations that have helped me find myself.

Next, I am very grateful to Martin Helsø for sharing his boundless Latex expertise, considerably improving the aesthetic aspect of this text, and for gently insisting on our regular runs this past year, which have been deeply therapeutic.

Thank you to Luca Gazdag and all my fellow master students, I have reveled in our collective laments and celebrations. Knowing we were in the same boat certainly made the seas smoother. Thank you to Sverre Hartveit for enthusiastically eating the same Poke bowl with me every day, and carrying through with the lighthearted conversation from start to finish. Finally, thank you to all the friends that have made these past few years of studying joyful.

Edvard Aksnes,
May 2019.

Contents

CHAPTER 1

Introduction

1.1 A short introduction to Tropical Geometry

From a certain point of view, tropical geometry is the study of the geometry of the semi-ring $\mathbb{T}$, whose underlying set is $\mathbb{R} \cup \{-\infty\}$, with non-invertible addition $x \oplus y := \max\{x, y\}$ and $x \odot y := x + y$ as multiplication [Bru+15; MS15]. Since $\mathbb{T}$ is not a field, nor in fact a ring, one cannot simply apply the classical algebraic geometry theory.

Remarkably, one can still obtain a geometric theory by appropriately defining concepts from algebraic geometry. For instance, there are several equivalent definitions of *tropical varieties* [Bru+15]. For simplicity, we will restrict our attention to tropical geometry in $\mathbb{R}^N$. One can first define *tropical polynomials* as functions

$$f \colon \mathbb{R}^N \to \mathbb{R}, \tag{1.1}$$

$$x \mapsto \bigoplus_{i \in \mathbb{Z}^N} a_i x^{\otimes i}, \tag{1.2}$$

which is a polynomial using tropical operations. These tropical polynomials are piecewise linear functions, and one defines the *tropical hypersurface* $V(f)$ *associated to* f to be the set of points in $\mathbb{R}^N$ where f is not differentiable. This set is a finite union of convex rational polyhedrons, i.e. intersections of half-spaces $\{x \in \mathbb{R}^N \mid Ax \leq b,\ A \in \mathbb{Z}^{m \times n}, b \in \mathbb{R}^m\}$, which are equipped with weights obtained through the exponents of the polynomial.

More generally, a *tropical subvariety of* $\mathbb{R}^N$ is a pure dimensional rational finite polyhedral complex in $\mathbb{R}^N$, having positive integer weights on its maximal faces, satisfying a particular *balancing condition*, widespread in tropical geometry, for each of its codimension one polyhedrons [Bru+15; JRS18; MS15; MZ14]. One can generalize further to *tropical cycles in* $\mathbb{R}^N$ by allowing negative weights on the faces.

One way to obtain a tropical variety is through *tropicalization*. Let $(\mathcal{V}_t)_{t \in A}$ be a family of proper complex analytic subvarieties of dimension k in $(\mathbb{C}^\times)^N$, with $A \subset (1, \infty)$ a subset not bounded from above. Then the *componentwise base t logarithm maps*:

$$\begin{aligned} \operatorname{Log}_t \colon (\mathbb{C}^\times)^N &\to \mathbb{R}^N, \\ (z_1, \ldots, z_n) &\mapsto (\log_t |z_1|, \ldots, \log_t |z_n|), \end{aligned}$$

give *amoebas* $\mathrm{Log}_t(\mathcal{V}_t) \subset \mathbb{R}^N$, which are closed subsets [Bru+15]. Given such a family of amoebas, we may take the limit $\mathrm{limtrop}_{t\to\infty} \mathcal{V}_t := \lim_{t\to\infty} \mathrm{Log}_t(\mathcal{V}_t)$, which is a set admitting the structure of a rational finite polyhedral complex of dimension k in $\mathbb{R}^N$ [Bru+15]. Furthermore, one can obtain *weights* on the complex from the family $\mathcal{V}_t$ [Bru+15, Definition 5.13]. When these weights are well defined, we say that $V = \mathrm{limtrop}_{t\to\infty} \mathcal{V}_t$ is the *tropical limit* of the family $\mathcal{V}_t$. Moreover, the tropical limit $V = \mathrm{limtrop}_{t\to\infty} \mathcal{V}_t$ is a tropical subvariety of $\mathbb{R}^N$ [Bru+15; Ite+16], giving a relation between tropical varieties and complex varieties.

This connection between tropical geometry and complex geometry has spurred development of several results "computing invariants tropically". An early example is the proof by Mikhalkin that the *Gromov-Witten invariant* $N_{d,g}$, counting the number of algebraic curves of degree d and geometric genus g passing through $3d - 1 + g$ points in $\mathbb{CP}^2$, can be determined by counting the number of certain tropical curves in $\mathbb{R}^2$ with multiplicities [Mik05]. Moreover, Mikhalkin showed that a similar invariant for the real projective plane $\mathbb{RP}^2$, the *Welschinger invariant*, can also be computed tropically [Mik05]. Other instances of such connections between tropical and complex geometry include applications to the Gross–Siebert program in mirror symmetry [Gro11] and work on Brill–Noether theory [Coo+12; JP14; JP16].

A particular class of tropical subvarieties of $\mathbb{R}^N$ are the *tropical linear spaces* defined in [Spe04]. Speyer also formulated a tropical version of the f-vector conjecture, specifying exactly the number of polyhedrons of each dimension a tropical linear space can have. Not all such linear spaces arise from the tropicalization of a family of algebraic linear spaces, leading to a distinction between the *realizable tropical varieties*, which appear as tropical limits, and the *non-realizable tropical varieties*.

One way to obtain a tropical linear variety is through the use of *matroids* [Bru+15; Oxl11; Spe04]. One can associate a rational weighted finite polyhedral fan to a matroid (see Section 2.9). These satisfy the balancing condition and are tropical linear varieties. Any fan that can be constructed from a matroid is called *matroidal*. To generalize even further, one can define a *tropical manifold*, which is a topological space locally modeled on the Bergman fan of a matroid. An example of a tropical manifold is the *tropical projective space* $\mathbb{TP}^N$.

Tropical geometry also has an associated *tropical homology and cohomology* theory, giving groups $H_q(X, \mathcal{F}_p)$ and $H^q(X, \mathcal{F}^p)$, as well as Borel–Moore and compact support versions of these, defined in terms of the homology and cohomology of the p-th multi-tangent cosheaf $\mathcal{F}_p$ and p-th multi-cotangent sheaf $\mathcal{F}^p$ respectively. Tropical homology gives another characterization of the balancing condition: a k-dimensional tropical subvariety Z of $\mathbb{R}^N$ induces a class $[Z] \in H_k^{BM}(X, \mathcal{F}_k)$, called the *fundamental class of* Z, in the k-th Borel–Moore homology group of the k-th multi-tangent cosheaf.

A remarkable result from [Ite+16] states that tropical cohomology can be used to compute Hodge numbers of complex projective varieties: if a tropical submanifold X of $\mathbb{TP}^N$ is the tropical limit of complex analytic family $\mathcal{X}_t$ of projective varieties parametrized over the punctured disk $\mathcal{D}^*$, then for sufficiently large $|t|$, the complex variety $\mathcal{X}_t \subset \mathbb{CP}^N$ is smooth. Moreover, again for sufficiently large $|t|$, the (p, q)-th Hodge number $h^{p,q}(\mathcal{X}_t)$ of $\mathcal{X}_t$ is equal to the q-th Betti number $h^{p,q}(X) := \dim H^q(X, \mathcal{F}^p)$ of the p-th multi-cotangent sheaf $\mathcal{F}^p$, giving a connection between Hodge theory and tropical cohomology.

Jell, Rau and Shaw [JRS18] proved that, for any tropical manifold X, i.e. *tropical space* locally modeled on Bergman fans of matroids, the *cap product with the fundamental class* $\cap[X]$, which is a morphism:

$$\cap[X]\colon H^q(X, \mathcal{F}^p) \to H^{BM}_{n-q}(X, \mathcal{F}_{n-p}),$$

as defined in [JSS19], is an isomorphism, a property which has been called *tropical Poincaré duality* [JRS18; JSS19]. Similarly to Poincaré duality for smooth manifolds, this duality is proved locally, then lifted up to a global setting. One shows that tropical Poincaré duality holds for Bergman fans of matroids, then a Mayer–Vietoris argument is used to prove the result globally. However, it has been observed that there are polyhedral fans which satisfy tropical Poincaré duality, despite not being Bergman fans of matroids (see for instance Example 3.4.4).

A *rational polyhedral space* is a space locally modeled on polyhedral fans. This class then encompasses tropical manifolds. We will call any rational polyhedral space satisfying the tropical Poincaré duality a *tropical Poincaré space*. In this thesis, we will primarily be interested in determining which rational polyhedral spaces are tropical Poincaré spaces. To answer this question, in the same manner as for Poincaré duality in algebraic topology, one can work locally, which therefore reduces the question to:

Main question. *Which rational polyhedral fans are tropical Poincaré spaces?*

There are also several other connections from tropical homology to the study of matroids. In [AB14], Adiprassito and Björner showed that tropical manifolds satisfy an analogue of the Lefschetz hyperplane theorem. Furthermore, Adiprassito, Huh and Katz defined the *Chow ring of matroid*, which satisfies many properties of the cohomology ring of compact Kähler manifolds, such as Poincaré duality, the Hard Lefschetz theorem, and the Hodge–Riemann relations [AHK18]. These three properties, sometimes referred to collectively as the "Hodge package" or "Kähler package" [AHK18]. It is expected that the Chow ring of a matroid is isomorphic to the tropical homology of the compactification of the fan of a matroid [JSS19].

Another sheaf which is relevant in Tropical Geometry is the *p-th wave sheaf* $\mathcal{W}^p$, as introduced in [MZ14]. These sheaves induce corresponding cohomology theories, giving rise to the *wave groups* $H^q(X, \mathcal{W}^p)$. These groups act on tropical homology [MZ14]. In particular, Mikhalkin and Zharkov defined a *wave homomorphism*:

$$\hat{\phi}\colon H^{BM}_q(X, \mathcal{F}_p) \to H^{BM}_{q-1}(X, \mathcal{F}_{p+1}),$$

constructed using an element of the wave group $H^1(X, \mathcal{W}^1)$, which can be applied for any p and q. They observed that the fundamental class of any tropical cycle is in the kernel of $\hat{\phi}$. This observation leads to a "tropical Hodge conjecture": the kernel of $\hat{\phi}$ consists exactly of the classes of tropical cycles.

Using tropical Poincaré duality, Jell, Rau and Shaw [JRS18] proved that for any tropical manifold X, this statement holds true for the

$$\hat{\phi}\colon H^{BM}_{n-1}(X, \mathcal{F}_{n-1}) \to H^{BM}_{n-2}(X, \mathcal{F}_n).$$

The only property of tropical manifolds used in the proof is the tropical Poincaré duality, hence is expected that this will in fact hold for any tropical Poincaré space.

In investigating our main question, we develop several tools from the derived category of sheaves on a polyhedral complex. This allows us to consider the following question:

Question. *What is the compact support cohomology $H^q_c(X, \mathcal{W}^p)$ of the wave sheaves on a rational polyhedral fan?*

1.2 Main results

In the first part of this thesis, we aim to classify exactly which rational polyhedral fans are tropical Poincaré spaces. To that end, we first prove the following result:

Corollary 1.2.1 (Corollary 4.1.5)**.** *Let X be a rational polyhedral fan with no zero weights. Then there is an isomorphism $C^{p,n}_c(X) \cong C^{BM}_{n-p,n}(X)$.*

This corollary can be applied in the case of pure dimensional fans to give:

Theorem 1.2.2 (Theorem 4.3.1)**.** *Let X be a pure balanced rational polyhedral fan, with no zero weights. Then the cap with the fundamental class $\cap[X] \colon H^{p,q}(X) \to H^{BM}_{n-p,n-q}(X)$ is injective.*

This theorem can then be used to classify the pure rational polyhedral fans which are tropical Poincaré spaces in terms of some properties we will introduce in the main text:

Theorem 1.2.3 (Theorem 4.5.1)**.** *A rational balanced polyhedral fan of pure dimension n with no zero weights is a tropical Poincaré space, i.e.*

$$\cap[X] \colon H^q(X, \mathcal{F}^p) \to H^{BM}_{n-q}(X, \mathcal{F}_{n-p})$$

is an isomorphism for all $p, q = 0, \ldots, n$, if and only if

1. *X is uniquely p-balanced for all p, and*
2. *the dependence cosheaf $\mathcal{K}_p$ is acyclic in Borel–Moore homology for all p, that is, $H^{BM}_q(X, \mathcal{K}_p) = 0$ for $q \neq n-1$ and all p.*

However, there are non-pure fans which are tropical Poincaré spaces, see for instance Example 7.1.2. Moreover, we prove the two following conditions on

Corollary 1.2.4 (Corollary 4.6.1)**.** *Let X be a polyhedral fan of pure dimension n, with the cosheaf $\mathcal{K}_p$ acyclic in all degrees except $n-1$ for all p. Then $\cap[X]$ is an isomorphism if and only if*

$$(-1)^n \chi(C^{BM}_\bullet(X, \mathcal{F}_{n-p})) = \dim \mathcal{F}^p(v).$$

Corollary 1.2.5 (Corollary 4.6.2)**.** *Let X be a balanced polyhedral fan of pure dimension n. Suppose for each $\tau \in X$, with $\tau \neq v$, the tangent fan $T_\tau X$ is a tropical Poincaré space. Then $\cap[X] \colon H^q(X, \mathcal{F}^p) \to H^{BM}_{n-q}(X, \mathcal{F}_{n-p})$ is an isomorphism if and only if i$\cap[X] \colon H^q(X, \mathcal{F}^{n-p}) \to H^{BM}_{n-q}(X, \mathcal{F}_p)$ is an isomorphism.*

However, in the 2-dimensional case, it can be shown that purity is necessary (see Proposition 5.1.1), and we prove the following "geometric" classification theorem:

Theorem 1.2.6 (Theorem 5.6.1). *Let X be a rational 2-dimensional polyhedral fan. The cap product with the fundamental class* $\cap[X]\colon H^{p,q}(X) \to H^{BM}_{2-p,2-q}(X)$ *is an isomorphism if and only if*

1. *X is pure,*
2. *the boundary map $\partial_1\colon \oplus_{\tau\in X^1} \mathcal{K}_p(\tau) \to \mathcal{K}_p(v)$ is surjective,*
3. *X is uniquely balanced, and*
4. *X is uniquely balanced at each edge.*

We also show that each of the above conditions are independently necessary in a series of examples. Throughout the text, we have included examples of [`polymake`] scripts, which we use to compute the tropical homology and cohomology of several examples, by using the Cellular Sheaves package from [KSW17].

In the search for conditions guaranteeing tropical Poincaré duality, we extensively investigated Verdier duality for cellular complexes, as first introduced in [Cur14]. In Appendix A, we introduce the derived category of sheaves on a polyhedral complex, consisting of complexes of sheaves modulo quasi-isomorphisms as objects, and chain maps as morphisms. The Verdier dual functor, defined in terms of the derived functor of sheaf Hom of complexes between an object and the *dualizing complex* of a polyhedral complex, is as follows:

Definition 1.2.7 (Definition A.6.1). The *Verdier dual functor* is given by:

$$\begin{aligned} D\colon D^b(\mathbf{Shv}_X) &\to D^b(\mathbf{Shv}_X)^{\mathrm{op}}, \\ \mathcal{F}^\bullet &\mapsto \mathrm{R}\,\mathscr{H}om^\bullet(\mathcal{F}^\bullet, \omega_X^\bullet). \end{aligned}$$

The complex $D(\mathcal{F}^\bullet)$ is called *the Verdier dual complex of $\mathcal{F}^\bullet$*.

This functor can be used to compute the compact support cohomology of a sheaf by:

Theorem 1.2.8 (Theorem A.6.2). *Let $\mathcal{F}$ be a sheaf on a polyhedral complex X of dimension n. Then*

$$H^i_c(X,\mathcal{F}) \cong H^{n-i}(X, D(\mathcal{F}))^*,$$

where $H^q(X, D(\mathcal{F}))$ is the q-th cohomology of the complex $\{\Gamma(X, D(\mathcal{F})^i)\}_{i\in\mathbb{Z}}$, also called the hypercohomology of $D(\mathcal{F})$.

A possible approach to the question of classifying which polyhedral fans satisfy the tropical Poincaré duality then consists in determining exactly when $D(\mathcal{F}^p) = \mathcal{F}^{n-p}$. This approach has recently been carried through for tropical manifolds in [GS].

As an application of this theory, we partially resolve the following conjecture:

Conjecture 1.2.9 ([KSW17]). *Let $L \subset \mathbb{R}^n$ be a tropical linear space of dimension d. Then we have*

$$H^q(L, \mathcal{W}^p) = 0 \quad \text{if } p \neq q, \quad \text{and} \quad H^q_c(L, \mathcal{W}^p) = 0 \quad \text{if } p \neq d.$$

This conjecture can be related to the f-vector conjecture on tropical linear spaces from [Spe04].

We prove the following vanishing theorem on the compact-support cohomology of the wave sheaf for a particular class of polyhedral fans:

Theorem 1.2.10 (Theorem 6.3.2)**.** *If X is a Cohen–Macaulay simplicial polyhedral fan of dimension n, then $H_c^i(X, \mathcal{W}^p) = 0$ if $i \neq n$ for all p.*

Corollary 1.2.11 (Corollary 6.4.1)**.** *Let M be a matroid and $\mathcal{B}(M)$ its Bergman fan. Then $H_c^i(\mathcal{B}(M), \mathcal{W}^p) = 0$ if $i \neq \dim(\mathcal{B}(M))$ for all p.*

The theorem builds upon several results in the derived category of sheaves on a polyhedral complex, which have applications in a larger context. For instance, we prove the two following theorems:

Theorem 1.2.12 (Theorem 6.1.3)**.** *Let X be a polyhedral complex of dimension n. The dualizing complex $\omega_X^\bullet$ of X is concentrated in degree $-n$ if and only if X is Cohen–Macaulay.*

Theorem 1.2.13 (Theorem 6.3.1)**.** *A fan X is simplicial if and only if $\mathcal{W}^p$ is projective for all p.*

1.3 Outline of the thesis

The thesis is structured as follows:

Chapter 2 introduces polyhedral complexes and cellular sheaves, along with refinements of these. These are the basis for a cellular homology and cohomology theory. Moreover, we introduce tangent fans, a Cohen–Macaulay property and a class of polyhedral fans coming from matroids.

Chapter 3 introduces tropical theory. We introduce the $\mathcal{F}_p$ cosheaves, and define the balancing condition which determines when a polyhedral complex is a tropical variety. Furthermore, we introduce a cap product on tropical cohomology. Finally, we define what it means for this cap product to induce isomorphisms, giving a tropical Poincaré duality.

Chapter 4 develops tools to define which polyhedral fans are tropical Poincaré spaces. We prove a theorem giving necessary and sufficient criteria for the duality to hold in the pure-dimensional case. Finally, we reformulate these criteria in terms of Euler characteristics.

Chapter 5 is dedicated to the classification of fans of dimension 2 which are tropical Poincaré spaces.

Chapter 6 presents a vanishing result on a certain sheaf for Cohen–Macaulay simplicial polyhedral fans, partially confirming a conjecture of [KSW17].

Chapter 7 serves as a conclusion and discussion of possible future research directions.

Appendix A introduces the derived category of cellular sheaves on a polyhedral complex. We first develop the homological tools necessary to define the derived category, then we introduce derived functors. Using these tools, we derive a version of Verdier duality for cellular sheaves on polyhedral complexes.

Appendix B presents two results in commutative algebra which are used in the text.

PART I

Tropical Poincaré duality

CHAPTER 2

Polyhedral complexes and cellular sheaves

In this chapter, we introduce *polyhedral complexes*, give them the *Alexandrov topology*, and then define *cellular sheaves.* Next, we introduce several related concepts. Given a polyhedral complex, one can get a similar complex through *refinement.* This new polyhedral complex may be more amenable to computation, in particular if it is *simplicial.* Next, we introduce *tangent fans*, which encode the local structure of a polyhedral complex at a face. These are a construction used to build a polyhedral complex from a face of another one. Finally, we introduce the *Bergman fan of a matroid.* These provide a general class of examples, and serve as the local model of tropical manifolds.

2.1 Polyhedral complexes and Alexandrov topology

In this section, we follow the conventions of [KSW17] by defining *polyhedral complexes*, and then [Cur14] to give them a topology.

Definition 2.1.1. A *polyhedron* is a set of the form $\{x \in \mathbb{R}^N \mid Ax \leq b\}$, for some matrix $A \in \mathbb{R}^{m\times N}$ and a vector $b \in \mathbb{R}^m$. Given this definition, a polyhedron is always convex and closed. A polyhedron is *rational* if the coefficients of the matrix A are integral, i.e. $A \in \mathbb{Z}^{m\times N}$.

Definition 2.1.2. A *polyhedral complex* X is a collection of polyhedrons in $\mathbb{R}^N$ such that:

- Every face of each polyhedron of X is also in X, i.e. if $\sigma \in X$, and τ is a face of σ, then $\tau \in X$,
- The intersection of two polyhedrons in X is either empty or a face of both.

The *dimension of* X is the dimension of the polyhedron of X with greatest dimension. The elements of X are called *polyhedrons*, *cells* or *faces*, and the subset of elements of dimension i is denoted by X^i. A cell of maximal dimension is also sometimes called a *facet.* If a cell τ is a face of a cell σ, we write $\tau \leq \sigma$, which gives a partial ordering on X.

We say that a face $\sigma \in X$ is *maximal* if it is maximal with respect to the ordering $\leq$. A polyhedral complex is *pure dimensional*, or *pure* if all the maximal faces are of the same dimension.

We say that a polyhedral complex is a *polyhedral fan* if there is a non-empty polyhedron $\tau \in X$ such that $\tau \leq \sigma$ for all $\sigma \in X$, and it is a *pointed polyhedral fan* if the minimal polyhedron τ is a point, located at the origin.

A polyhedral complex is *rational* if all its polyhedrons are rational.

Remark 2.1.3. Note that our definition of polyhedral fan is somewhat more general than the standard definition, in which one thinks of polyhedral fans as a set of cones, corresponding to our pointed polyhedral fans. Our definition allows for polyhedral fans with non-empty lineality spaces.

Remark 2.1.4. Since the relation $\leq$ on X is a partial order, any polyhedral complex X can be made into a category, in the standard manner for a poset, defined by the data:

- The objects $\mathrm{Ob}(X)$ are the polyhedrons in the complex,
- There is a morphism $r_{\tau,\sigma} \in \mathrm{Mor}(X)$, with $r_{\tau,\sigma} \in \mathrm{Hom}_X(\tau, \sigma)$ if and only if $\tau \leq \sigma$.

Moreover, a poset can be graphically represented using a *Hasse diagram*:

Definition 2.1.5. Let $(X, \leq)$ be a partially-ordered set. The *Hasse diagram* of X is the directed graph where:

- the nodes are the elements of X, and
- there is an arrow from a node τ to a node σ when $\tau < \sigma$ and there is no element γ such that $\tau < \gamma < \sigma$.

Example 2.1.6. Consider the polyhedral complex X in $\mathbb{R}^3$ consisting of the cone $\mathrm{cone}\{e_1, e_2, e_3\}$ and all of its faces. The Hasse diagram of this complex is:

$$
\begin{tikzcd}
 & \mathrm{cone}\{e_1,e_2,e_3\} & \\
\mathrm{cone}\{e_1,e_2\} \arrow[ur] & \mathrm{cone}\{e_1,e_3\} \arrow[u] & \mathrm{cone}\{e_2,e_3\} \arrow[ul] \\
\mathrm{cone}\{e_1\} \arrow[u] \arrow[ur] & \mathrm{cone}\{e_2\} \arrow[ul] \arrow[ur] & \mathrm{cone}\{e_3\} \arrow[ul] \arrow[u] \\
 & 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

We wish to have a topology on each polyhedral complex, so that we may later speak of sheaves on such spaces. Since a polyhedral complex X can be considered as a partially ordered set using the face relation $\leq$, we may use the following topology:

Definition 2.1.7 ([Cur14, Definition 4.2.2])**.** Let $(X, \leq)$ be a partially ordered set. The *Alexandrov topology* on X is the topology whose open sets U are the sets such that

$$x \in U \text{ and } x \leq y \implies y \in U.$$

The open set $\mathrm{Star}(\sigma) := \{\rho \in X \mid \sigma \leq \rho\}$ is called the *star of σ*. The set of stars forms a basis for the topology.

Remark 2.1.8. Let X be a finite polyhedral complex in $\mathbb{R}^N$. Since this is a finite set, the Alexandrov topology is compact, and every subset of X is compact. In particular, for any polyhedron $\sigma \in X$, $\{\sigma\} \subset X$ is compact, even though the polyhedron σ is may not be compact as a subset of $\mathbb{R}^N$.

To make this distinction clear, we introduce the *support* of the complex:

Definition 2.1.9. Given a polyhedral complex X in $\mathbb{R}^N$, and any cell $\sigma \in X$, the *support of* σ is the set

$$|\sigma| := \{x \in \mathbb{R}^N | x \in \sigma\}.$$

The *support of* X is the set

$$|X| := \cup_{\sigma \in X} |\sigma| = \{x \in \mathbb{R}^N | \ \exists \sigma \in X, \text{ s.t. } x \in \sigma\}.$$

The support is equipped with the Euclidean topology inherited from $\mathbb{R}^N$. Moreover, each of the sets $|\sigma|$ is closed in $\mathbb{R}^N$, since our definition of polyhedrons (Definition 2.1.1) gives closed subsets of $\mathbb{R}^N$. If X is a finite polyhedral complex, then the support of X is a finite union of closed sets, hence is closed.

Definition 2.1.10. Given a polyhedral complex X of dimension n, the *f-vector* of X is the $n+1$-dimensional vector whose i-th component is the number of i-dimensional cells of X, and the *f^b-vector* is the f-vector of bounded faces.

Our interest in the Alexandrov topology primarily comes from the close relation between the category X of the polyhedral complex as a poset, and the category $\mathbf{Open}_X$ of open subsets of X, see (Remark 2.2.4).

Later, we will want to exploit the cellular nature of polyhedral complexes in computations of sheaf cohomology, such as in Definition 2.5.1. However, to be able to define boundary maps that form a complex, we need a notion of *orientation*. For each polyhedron of the complex, we can choose an ordered basis of the subspace of $\mathbb{R}^N$ parallel to the polyhedron. For each face of a polyhedron, one can compare the chosen orientation of the face with the orientation of the polyhedron. To detect whether these orientations are the same, we define a map, which will be essential in defining an appropriate boundary map for each polyhedral complex:

Definition 2.1.11. Given a polyhedral complex X with chosen orientations for each polyhedron, we define an *orientation map* $\mathcal{O}\colon X^{i-1} \times X^i \to \{0, 1, -1\}$ for each $i \leq \dim X$, by

$$\mathcal{O}(\tau, \sigma) := \begin{cases} 0 & \text{if } \tau \not\leq \sigma \\ 1 & \text{if the orientation of } \tau \subseteq \partial\sigma \text{ and } \tau \text{ coincide,} \\ -1 & \text{if the orientation of } \tau \subseteq \partial\sigma \text{ and } \tau \text{ do not coincide.} \end{cases}$$

We prove [Cur14, Lemma 6.1.8] in the case of polyhedral complexes:

Lemma 2.1.12. *If $\tau \in X$ is a face of a face $\sigma \in X$, such that $\tau \leq \gamma \leq \sigma$, then there are exactly two such γ.*

Proof. Since τ has codimension 2 in σ, there must be at least two distinct faces γ_1 and γ_2 of σ such that $\tau < \gamma_i$. Suppose there is a third γ_3, distinct from both others, with $\tau < \gamma_3$. Then, since σ is a polyhedron, the three faces must intersect in a codimension 3 face. But τ is in this intersection, which is impossible. Hence there can only be two faces of σ having τ as a face. ∎

2.2 Cellular sheaves of vector spaces

Using the Alexandrov topology (Definition 2.1.7), one can define *cellular sheaves* and *cosheaves* of vector spaces on a polyhedral complex.

Definition 2.2.1 ([Cur14, Corollary 4.2.13]). A *cellular sheaf* is a sheaf of vector spaces on a polyhedral complex with respect to Alexandrov topology.

A *cellular cosheaf* is a cosheaf of vector spaces on a polyhedral complex with respect to Alexandrov topology.

Remark 2.2.2. Note that we work with sheaves and cosheaves of vector spaces throughout this text. In particular, unless stated otherwise, we work with sheaves of $\mathbb{R}$ vector spaces. One could however define sheaves of sets or abelian groups to obtain a more general theory.

Proposition 2.2.3 ([Cur14, Theorem 4.2.10]). *Let X be a polyhedral complex, hence, by Remark 2.1.4, a category. Any cellular sheaf $\mathcal{G}$ is uniquely determined by a functor $\mathcal{F}\colon X \to \mathbf{Vect}_k$, where $\mathbf{Vect}_k$ is the category of vector spaces over a field k. By abuse of notation, the sheaf and functor are both denoted by $\mathcal{F}$. Similarly, any cellular cosheaf is uniquely determined by a functor $\mathcal{G}\colon X^{\mathrm{op}} \to \mathbf{Vect}_k$.*

Remark 2.2.4. This proposition allows us to define sheaves and cosheaves on a polyhedral complex merely by specifying corresponding functors. To understand fully the correspondence between sheaves and functors on Alexandrov spaces, we refer the reader to the discussion in [Cur14, Section 4.2.2]. A particularly useful consequence of this characterization of sheaves as functors is that, for any cell $\tau \in X$ and sheaf $\mathcal{F}$,

$$\mathcal{F}(\tau) = \mathcal{F}(\mathrm{Star}(\tau)),$$

where on the left $\mathcal{F}$ is thought of as a functor, and on the right, as a sheaf.

Remark 2.2.5. Note that the category of cosheaves $\mathbf{CoShv}_X$ is dual to the category of sheaves $\mathbf{Shv}_X$.

Proposition 2.2.6 ([Cur14, Remark 4.2.11]). *Let X be a polyhedral complex, $\mathcal{F}$ a cellular sheaf, and $\tau \in X$ a cell. Then the stalk $\mathcal{F}_\tau$ of $\mathcal{F}$ at τ is*

$$\mathcal{F}_\tau = \mathcal{F}(\tau).$$

Proof. By definition, the stalk $\mathcal{F}_\tau$ is the direct limit of the $\mathcal{F}(U)$ over open subsets U containing τ, i.e.

$$\mathcal{F}_\tau = \varinjlim_{U \ni \tau} \mathcal{F}(U).$$

Since the smallest open subset U containing τ is $\mathrm{Star}(\tau)$, we have that

$$\mathcal{F}_\tau = \mathcal{F}(\mathrm{Star}(\tau)) = \mathcal{F}(\tau),$$

where the last equality follows from the functor-interpretation of the cellular sheaf, as discussed in Remark 2.2.4. ■

We introduce some cellular sheaves as examples:

Definition 2.2.7. Let X be a polyhedral complex. The *constant sheaf* $\mathbb{R}_X$ *with values in* $\mathbb{R}$ is the sheaf defined as the functor $\mathbb{R}_X \colon X \to \mathbf{Vec}_{\mathbb{R}}$ taking values:

- For a cell $\sigma \in X$, one has $\mathbb{R}_X(\sigma) = \mathbb{R}$.
- For $\tau \leq \sigma$, the morphism $r \colon \tau \to \sigma$ is taken to $\mathbb{R}_X(r) = \mathrm{id}_{\mathbb{R}}$.

More generally, one can consider the constant sheaf of any vector space over any field:

Definition 2.2.8. Let X be a polyhedral complex. For a vector space V over a field k, the *constant sheaf* V_X *with values in* V is the sheaf defined as a functor $V_X \colon X \to \mathbf{Vec}_k$ taking values:

- For a cell $\sigma \in X$, one has $V_X(\sigma) = V$.
- For $\tau \leq \sigma$, the morphism $r \colon \tau \to \sigma$ is taken to $V_X(r) = \mathrm{id}_V$.

A similar definition gives constant cosheaves:

Definition 2.2.9. Let X be a polyhedral complex. For a vector space V over a field k, the *constant cosheaf* V_X *with values in* V is the sheaf defined as a functor $V_X \colon X^{\mathrm{op}} \to \mathbf{Vec}_k$ taking values:

- For a cell $\sigma \in X$, one has $V_X(\sigma) = V$.
- For $\tau \leq \sigma$, the morphism $s \colon \sigma \to \tau$ is taken to $V_X(r) = \mathrm{id}_V$.

2.3 Linear duality

Note that one can construct a cosheaf from a sheaf, and vice versa, through *linear duality*:

Definition 2.3.1 ([Cur14, Definition 6.2.8])**.** Given a sheaf $\mathcal{G}$, the *linear dual* $\hat{V}(\mathcal{G})$ is the cosheaf defined by dualizing all the vector spaces $\mathcal{G}(\sigma)$ for each $\sigma \in X$, and dualizing the restriction maps $\rho_{\tau\sigma} \colon \mathcal{G}(\tau) \to \mathcal{G}(\sigma)$ to extension maps $\iota_{\sigma\tau} \colon \mathcal{G}(\sigma)^* \to \mathcal{G}(\tau)^*$. Similarly, one can internally dualize a cosheaf $\mathcal{F}$ to obtain a *linear dual* sheaf $V(\mathcal{F})$. Note that linear duality is a contravariant functor $\hat{V} \colon \mathbf{Shv}(X)^{\mathrm{op}} \to \mathbf{CoShv}(X)$, with the linear dual inverse functor $V \colon \mathbf{CoShv}(X) \to \mathbf{Shv}(X)^{\mathrm{op}}$.

Example 2.3.2. The linear dual $V(k_X)$ of the constant cosheaf k_X, is just the sheaf k_X.

2.4 Elementary injective and projective sheaves and cosheaves

An important class of sheaves are the "injective" and "projective" sheaves and cosheaves:

Definition 2.4.1 ([Cur14, Definition 7.1.3])**.** Let X be a polyhedral complex and $\sigma \in X$ a face. We define $[\sigma]^W$, the *elementary injective cell sheaf concentrated at* σ, with values in the vector space W, given by:

$$[\sigma]^W(\tau) := \begin{cases} W & \text{if } \tau \leq \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

where the only non-zero restriction maps are the identity.

The use of "injective" in the name of these sheaves is explained by the following two results:

Proposition 2.4.2 ([Cur14, Lemma 7.1.5])**.** *For $\sigma \in X$ a face, the elementary injective sheaf $[\sigma]^W$ is injective.*

Proposition 2.4.3 ([Cur14, Lemma 7.1.6])**.** *A sheaf I on X is injective if and only if it is isomorphic to one of the form $\bigoplus_{\sigma \in X} [\sigma]^{V_\sigma}$.*

By duality, we also easily construct and classify projective sheaves.

Definition 2.4.4. Let X be a polyhedral complex and $\sigma \in X$ a face. We define $\{\sigma\}^W$, the *elementary projective cell sheaf concentrated at σ*, with values in the vector space W, given by:

$$\{\sigma\}^W(\tau) \coloneqq \begin{cases} W & \text{if } \tau \geq \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

where the only non-zero restriction maps are the identity.

Proposition 2.4.5. *For $\sigma \in X$ a face, the elementary projective sheaf $\{\sigma\}^W$ is projective.*

Proof. A projective object in a category $\mathcal{C}$ is, by duality, an injective object in $\mathcal{C}^{\mathrm{op}}$. Clearly each $\{\sigma\}^W$ is an injective object in the dual category $\mathbf{CoShv}_X$. ∎

Proposition 2.4.6. *A sheaf I on X is projective if and only if it is isomorphic to one of the form $\bigoplus_{\sigma \in X} \{\sigma\}^{V_\sigma}$.*

Proof. It suffices to look at I in the dual category X^{op}, then apply Proposition 2.4.3. ∎

In addition to the elementary sheaves, one can also define elementary cosheaves. The explicit version of the construction given in [Cur14] is:

Definition 2.4.7. Let X be a polyhedral complex and $\sigma \in X$ a face. We define $[\hat{\sigma}]^W$, the *elementary projective cell cosheaf concentrated at σ*, with values in the vector space W, given by:

$$[\hat{\sigma}]^W(\tau) \coloneqq \begin{cases} W & \text{if } \tau \leq \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

where the only non-zero extension maps are the identity.

Definition 2.4.8. Let X be a polyhedral complex and $\sigma \in X$ a face. We define $\{\sigma\}^W$, the *elementary injective cell cosheaf concentrated at σ*, with values in the vector space W, given by:

$$\{\hat{\sigma}\}^W(\tau) \coloneqq \begin{cases} W & \text{if } \tau \geq \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

where the only non-zero extension maps are the identity.

Example 2.4.9. The linear dual swaps elementary injectives and projectives: $\hat{V}([\sigma]^W) = [\hat{\sigma}]^{W^*}$ and $\hat{V}(\{\sigma\}^W) = \{\hat{\sigma}\}^{W^*}$.

2.5 Cellular homology and cohomology

Furthermore, cellular sheaves and cosheaves can be used to define cellular cochain and chain groups:

Definition 2.5.1. Given a polyhedral complex X of dimension n, an open subset $U \subset X$ and a cellular sheaf $\mathcal{G}$, define *cellular cochain groups* and *cellular cochain groups with compact support*, respectively by

$$C^q(U,\mathcal{G}) \coloneqq \bigoplus_{\substack{\sigma\in U \\ \dim\sigma=q \\ |\sigma| \text{ compact}}} \mathcal{G}(\sigma) \quad \text{and} \quad C^q_c(U,\mathcal{G}) \coloneqq \bigoplus_{\substack{\sigma\in U \\ \dim\sigma=q}} \mathcal{G}(\sigma),$$

for $q = 0,\dots,n$. The cellular cochain maps (usual or with compact support)

$$d\colon C^q(U,\mathcal{G}) \to C^{q+1}(U,\mathcal{G}) \quad \text{and} \quad d\colon C^q_c(U,\mathcal{G}) \to C^{q+1}_c(U,\mathcal{G})$$

are given componentwise for $\tau \in X^q$ and $\sigma \in X^{q+1}$ by $d_{\tau\sigma}\colon \mathcal{G}(\tau) \to \mathcal{G}(\sigma)$, where

$$d_{\tau\sigma} \coloneqq \mathcal{O}(\tau,\sigma)\mathcal{G}(r_{\tau,\sigma}).$$

Proposition 2.5.2 ([Cur14, Lemma 6.2.2])**.** *The morphism d is a boundary map, i.e. $d^2 = 0$ for each of the chain maps, hence $(C^\bullet_c(U,\mathcal{G}), d)$ and $(C^\bullet(U,\mathcal{G}), d)$ are complexes.*

Proof. The proof given in [Cur14, Lemma 6.2.2] can be carried through since the necessary argument [Cur14, Lemma 6.1.8] is Lemma 2.1.12 in the case of polyhedral complexes. ∎

Definition 2.5.3. Given a polyhedral complex X of dimension n, an open subset $U \subset X$ and a cellular cosheaf $\mathcal{F}$, define the *cellular chain group* and *Borel–Moore cellular chain groups*, respectively as

$$C_q(U,\mathcal{F}) \coloneqq \bigoplus_{\substack{\sigma\in U \\ \dim\sigma=q \\ |\sigma| \text{ compact}}} \mathcal{F}(\sigma) \quad \text{and} \quad C^{BM}_q(U,\mathcal{F}) \coloneqq \bigoplus_{\substack{\sigma\in U \\ \dim\sigma=q}} \mathcal{F}(\sigma),$$

for $q = 0,\dots,n$. The cellular chain maps (usual or with compact support)

$$\partial\colon C_q(U,\mathcal{F}) \to C_{q-1}(U,\mathcal{F}) \quad \text{and} \quad \partial\colon C^{BM}_q(U,\mathcal{F}) \to C^{BM}_{q-1}(U,\mathcal{F})$$

are given componentwise for $\sigma \in X^q$ and $\tau \in X^{q-1}$ by $\partial_{\sigma\tau}\colon \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$, where

$$\partial_{\sigma\tau} \coloneqq \mathcal{O}(\tau,\sigma)\mathcal{F}(s_{\sigma,\tau}).$$

where $s_{\sigma,\tau}\colon \sigma \to \tau$ is the morphism $r_{\tau,\sigma}\colon \tau \to \sigma$ reversed for the category X^{op}.

Proposition 2.5.4. *The morphism ∂ is a boundary map, i.e. $\partial^2 = 0$ for each of the chain maps, hence $(C^{BM}_\bullet(U,\mathcal{F}), \partial)$ and $(C_\bullet(U,\mathcal{F}), \partial)$ are complexes.*

Proof. This follows from dualizing the arguments from Proposition 2.5.2. ∎

Definition 2.5.5. Let $U \subset X$ be an open subset. The complexes from Definition 2.5.1 and Definition 2.5.3 give rise to cohomology and homology theories:

- The *cellular sheaf cohomology* $H^\bullet(U, \mathcal{G})$ of U with respect to $\mathcal{G}$ is the cohomology of the cellular cochain complex $(C^\bullet(U, \mathcal{G}), d)$ from Definition 2.5.1.

- The *cellular sheaf cohomology with compact support* $H_c^\bullet(U, \mathcal{G})$ of U with respect to $\mathcal{G}$ is the cohomology of the cellular cochain complex $(C_c^\bullet(U, \mathcal{G}), d)$ from Definition 2.5.1.

- The *cellular cosheaf homology* $H_\bullet(U, \mathcal{F})$ of U with respect to $\mathcal{F}$ is the homology of the cellular chain complex $(C_\bullet(U, \mathcal{F}), \partial)$ from Definition 2.5.3.

- The *cellular Borel–Moore cosheaf homology* $H_\bullet^{BM}(U, \mathcal{F})$ of U with respect to $\mathcal{F}$ is the homology of the cellular chain complex $(C_\bullet^{BM}(U, \mathcal{F}), \partial)$ from Definition 2.5.3.

Proposition 2.5.6. *Let X be a polyhedral fan, and $\mathcal{G}$ a cellular sheaf on X. Then, for $q > 0$,*

$$H^q(X, \mathcal{G}) = 0.$$

Proof. Since the only compact cell of a pointed polyhedral fan is the vertex, the only non-zero group $C^q(X; \mathcal{G})$ is for $q = 0$. If X is not pointed, there are no compact cells, hence all groups $C^q(X; \mathcal{G})$ are zero. ∎

Proposition 2.5.7. *Let X be a polyhedral fan, and $\mathcal{F}$ a cellular cosheaf on X. Then, for $q > 0$,*

$$H_q(X, \mathcal{F}) = 0.$$

Proof. The same argument from Proposition 2.5.6 applies. ∎

Remark 2.5.8. Note that all groups $C^q(X; \mathcal{G})$ being zero does not lead to an interesting cohomology theory for the non-pointed polyhedral fans. We will return to this point in Remark 2.6.7.

Proposition 2.5.9 ([Cur14, Lemma 6.2.9.])**.** *Linear duality (Definition 2.3.1) induces a relationship between homology and cohomology: $H_i(X, \hat{V}(\mathcal{G}))^* \cong H^i(X, \mathcal{G})$ and $H_i^{BM}(X, \hat{V}(\mathcal{G}))^* \cong H_c^i(X, \mathcal{G})$.*

Remark 2.5.10. Note that Curry does not write that the isomorphism is with the dual spaces of $H_i(X, \hat{V}(\mathcal{G}))$ and $H_i^{BM}(X, \hat{V}(\mathcal{G}))$, however this is necessary. Consider for instance the one-point space. Applying linear duality will dualize a sheaf, which is just a vector space, to its internal dual, which is just the dual vector space.

We now give two examples of cellular homology computations, which can also be adapted to example computations of cohomology:

Example 2.5.11. Let X be a pointed polyhedral fan of dimension n in $\mathbb{R}^N$. Let $\sigma \in X$, and consider the elementary projective cosheaf $[\hat{\sigma}]^{\mathbb{R}}$. We wish to compute the Borel–Moore homology of $[\hat{\sigma}]^{\mathbb{R}}$. This is given as the homology of the complex:

$$0 \longrightarrow C_n^{BM}(X, [\hat{\sigma}]^{\mathbb{R}}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0^{BM}(X, [\hat{\sigma}]^{\mathbb{R}}) \longrightarrow 0,$$

which is the complex:

$$0 \longrightarrow \oplus_{\gamma \in X^n} [\hat{\sigma}]^{\mathbb{R}}(\gamma) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \oplus_{v \in X^0} [\hat{\sigma}]^{\mathbb{R}}(v) \longrightarrow 0.$$

Since X is a pointed polyhedral fan, it has a unique vertex. Moreover, $[\hat{\sigma}]^{\mathbb{R}}(\gamma)$ is 0 if $\gamma \not\leq \sigma$, and $\mathbb{R}$ otherwise. This gives the complex:

$$0 \longrightarrow \bigoplus_{\substack{\gamma \in X^n \\ \gamma \leq \sigma}} \mathbb{R} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathbb{R} \longrightarrow 0.$$

The i-th component is $\mathbb{R}^{f_i^\sigma}$, where f_i^σ is the number of i-dimensional faces of σ. Furthermore, since σ is a face of a polyhedral fan, it is a cone over a polytope. Let P be the polytope such that $\operatorname{cone}(P) = \sigma$ (one can find such a P by intersecting σ transversely for instance). Then one observes that the f-vectors (Definition 2.1.10) of P and σ are related by the equality $f_k^P = f_{k+1}^\sigma$. We can then compare $(C_\bullet^{BM}(X, [\hat{\sigma}]^{\mathbb{R}}), \partial)$ with the reduced $\mathbb{R}$ coefficient homology of P in the complex $(\widetilde{C}_\bullet(P, \mathbb{R}), \partial_P)$. Since the cosheaf $[\hat{\sigma}]^{\mathbb{R}}$ has identity maps for extensions, the boundary operator ∂ of the first complex is equal to the boundary operator ∂_P. Hence, we have that the complex $(\widetilde{C}_\bullet(P, \mathbb{R}), \partial_P)$:

$$0 \longrightarrow \mathbb{R}^{f_n^P} \xrightarrow{\partial} \mathbb{R}^{f_{n-1}^P} \longrightarrow \cdots \xrightarrow{\partial} \mathbb{R} \longrightarrow 0,$$

also computes the homology of $(C_\bullet^{BM}(X, [\hat{\sigma}]^{\mathbb{R}}), \partial)$. Now since P is a bounded polytope, it is homotopy equivalent to a point. Therefore, its reduced homology is 0. Hence $H_q^{BM}(X, [\hat{\sigma}]^{\mathbb{R}}) = \widetilde{H}_q(P, \mathbb{R}) = 0$.

We can extend the computation for any direct sum of projective cosheaves:

Example 2.5.12. Let X be a pointed polyhedral fan of dimension n in $\mathbb{R}^N$. Let $Y \subset X$ be a collection of polyhedral, and consider the cosheaf $\bigoplus_{\sigma \in Y} [\hat{\sigma}]^{V_\sigma}$. We now wish to compute the Borel–Moore homology of $\bigoplus_{\sigma \in Y} [\hat{\sigma}]^{V_\sigma}$. This is given as the homology of the complex:

$$0 \longrightarrow C_n^{BM}(X, \bigoplus_{\sigma \in Y} [\hat{\sigma}]^{V_\sigma}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0^{BM}(X, \bigoplus_{\sigma \in Y} [\hat{\sigma}]^{V_\sigma}) \longrightarrow 0,$$

which now is the complex:

$$0 \longrightarrow \oplus_{\gamma \in X^n} \oplus_{\sigma \in Y} [\hat{\sigma}]^{V_\sigma}(\gamma) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \oplus_{\sigma \in Y} [\hat{\sigma}]^{V_\sigma}(v) \longrightarrow 0.$$

Since X is a pointed polyhedral fan, it has a unique vertex. Now we have:

$$\bigoplus_{\sigma \in Y} [\hat{\sigma}]^{V_\sigma}(\tau) = \bigoplus_{\substack{\sigma \in Y \\ \sigma \geq \tau}} V_\sigma.$$

We note now that the complex above splits over the direct sum $\oplus_{\sigma \in Y}$. Therefore, the homology is:

$$H_q^{BM}(X, \bigoplus_{\sigma \in Y} [\hat{\sigma}]^{V_\sigma}) = \bigoplus_{\sigma \in Y} H_q^{BM}(X, [\hat{\sigma}]^{V_\sigma}).$$

Now finally, the homology $H_q^{BM}(X, [\hat{\sigma}]^{V_\sigma})$ of the complex

$$0 \longrightarrow C_n^{BM}(X, [\hat{\sigma}]^{V_\sigma}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0^{BM}(X, [\hat{\sigma}]^{V_\sigma}) \longrightarrow 0,$$

is just the homology with coefficients in V_σ of the complex $(\widetilde{C}_\bullet(P, \mathbb{R}), \partial_P)$, by using the same arguments as in the previous example. Using the universal coefficient theorem for homology [Hat02, Theorem 3A.3], we have

$$\widetilde{H}_k(P, \mathbb{R}) \otimes V_\sigma \cong \widetilde{H}_k(P, V_\sigma)$$

since we work over $\mathbb{R}$, there is no torsion. Using the previous example, this implies that $\widetilde{H}_k(P, V_\sigma) = 0$, hence $H_k^{BM}(X, [\hat{\sigma}]^{V_\sigma}) = 0$, which then gives that $H_q^{BM}(X, \bigoplus_{\sigma \in Y} [\hat{\sigma}]^{V_\sigma}) = 0$.

2.6 Refinements and simplicial polyhedral complexes

Any polyhedral complex X can be considered as a poset, whose elements are the polyhedrons of X, ordered using the relation $\leq$, which is the inclusion of a polyhedron as a face of other polyhedron (see Remark 2.1.4). Using this structure, we define *refinements* of a polyhedral complex, which are new polyhedral complexes. These should be thought of as subdivisions of the complex, for instance as in barycentric subdivision, which preserve the overall structure of the complex.

Definition 2.6.1. Let X be a polyhedral complex in $\mathbb{R}^N$. A polyhedral complex $\widetilde{X}$ in $\mathbb{R}^N$ is a *refinement* of X if both complexes share the same support, i.e. $|X| = |\widetilde{X}|$, and there is a surjective order-preserving poset map:

$$\mathrm{P} \colon \widetilde{X} \to X.$$

Definition 2.6.2. Let X be a polyhedral complex and $\widetilde{X}$ a refinement of X, with the refinement map $\mathrm{P} \colon \widetilde{X} \to X$. Then there is a *sheaf refinement functor*:

$$\begin{aligned} \widetilde{\mathrm{P}} \colon \mathbf{Shv}(X) &\to \mathbf{Shv}(\widetilde{X}) \\ \mathcal{F} &\mapsto \widetilde{\mathcal{F}} \end{aligned}$$

where $\widetilde{\mathcal{F}}$ is the cellular sheaf given as a functor by:

$$\begin{aligned} \widetilde{\mathcal{F}} \colon \widetilde{X} &\to \mathbf{Vect}_k \\ \sigma &\mapsto \mathcal{F}(\mathrm{P}(\sigma)) \end{aligned}$$

and sending a restriction map $\rho_{\tau\sigma} \colon \tau \to \sigma \ \in \widetilde{X}$ to $\mathcal{F}(\mathrm{P}(\rho_{\tau\sigma}))$, where $\mathrm{P}(\rho_{\tau\sigma}) \colon \mathrm{P}(\tau) \to \mathrm{P}(\sigma) \ \in X$.

A particular kind of refinement we will be interested in are the ones into *simplicial* complexes:

Definition 2.6.3. A polyhedral complex X is a *simplicial polyhedral complex* if

- each bounded cell of X is a simplex, and.
- each unbounded cell of X is a cone over a simplex.

Remark 2.6.4. Hence for a simplicial polyhedral complex, the number of faces of a polyhedron is known exactly:

- for each bounded n-dimensional cell $\sigma \in X^n$, there are exactly $\binom{n}{k}$ cells $\tau \in X^k$ of dimension k such that $\tau \leq \sigma$, for all $n \leq \dim X$, and
- for each unbounded n-dimensional cell $\sigma \in X^n$, there are exactly n cells $\tau \in X^{n-1}$ of dimension $n-1$ such that $\tau \leq \sigma$, for all $n \leq \dim X$.

Remark 2.6.5. Note that this definition slightly generalizes the classical definition of simplicial complexes, since it defines what it means for an unbounded polyhedron to be simplicial. If a polyhedral complex consists only of bounded polyhedrons, then a simplicial refinement is a classical simplicial complex.

We will show that any polyhedral complex has a refinement that is simplicial:

Theorem 2.6.6. *Let X be a polyhedral complex in $\mathbb{R}^N$. Then X has a refinement which is a simplicial polyhedral complex.*

Proof. For any bounded cell $\sigma \in X$, one can simply take the barycentric subdivision of the cell to obtain a new complex $\widetilde{X}$, in which the cell σ is removed, replaced by several cells of equal dimension, all of which are simplicial, which themselves share the face of σ, and have own faces, which add "roots" to the Hasse diagram. The refinement map P takes any such root back to the face σ.

For an unbounded cell, intersect it with a sufficiently large simplex, then take the barycentric subdivision of this new bounded cell. Finally, add faces on the subdivided face corresponding to the bounded face. ■

Remark 2.6.7. Now we can define the cellular homology and cohomology of a sheaf $\mathcal{G}$ on a non-pointed fan to be the homology and cohomology of the sheaf $\widetilde{\mathcal{G}}$ on a simplicial refinement which has a vertex as a minimal face.

2.7 Tangent fans

Definition 2.7.1. Given a polyhedral complex X and a face $\tau \in X$, we define the *tangent fan $T_\tau X$ of X at τ* as the set

$$T_\tau X \coloneqq \{v \in \mathbb{R}^N \mid \exists \epsilon > 0 \text{ s.t. } \forall \delta < \epsilon,\ x + \delta v \in |X| \text{ for some } x \in \tau\},$$

where $|X|$ is the support of X (Definition 2.1.9).

Proposition 2.7.2. *For a polyhedral complex X in $\mathbb{R}^N$ and a face $\tau \in X$, the tangent fan $T_\tau X$ has the structure of a polyhedral fan, through an isomorphism as posets to the* $\operatorname{Star}(\tau)$.

Proof. Let $x \in \tau$ be an interior point. Let $v_1, \ldots, v_k$ be vectors spanning the space $L(\tau)$ parallel to τ, where $k \coloneqq \dim(\tau)$. For each face $\sigma \geq \tau$, with $\dim(\sigma) = \dim(\tau) + 1$, pick $v_\sigma \in \sigma$. Then the cells of $T_\tau X$ are the polyhedrons $C_\rho \coloneqq \operatorname{cone}\{v_\sigma \mid \tau \leq \sigma \leq \rho\} + \operatorname{span}\{v_1, \ldots, v_k\} \subset \mathbb{R}^N$ for all $\rho \geq \tau$, and the unique minimal cell $C_\tau = \operatorname{span}\{v_1, \ldots, v_k\} \subset \mathbb{R}^N$. Now any face of each of the C_ρ is included in $T_\tau X$, since each cone comes from a cell of X, and for any two such cones $C_\rho, C_{\rho'}$ intersect either in a C_σ, for some σ with $\tau \leq \sigma \leq \rho, \rho'$, or in C_τ. Since C_τ is the unique minimal non-empty cell, $T_\tau X$ is a polyhedral fan. ■

We will always be using this polyhedral structure on the tangent fans.

The cellular Borel–Moore homology of the tangent fan

The cellular Borel–Moore homology of the tangent fan at a face is related to the cellular Borel–Moore homology of the original complex by the following result.

Proposition 2.7.3. *Let X be a polyhedral complex of dimension n, $\tau \in X$ a face of dimension k and $T_\tau X$ the tangent fan. Then*

$$H_i^{BM}(\mathrm{Star}(\tau), \mathbb{R}_X) = H_i^{BM}(T_\tau X, \mathbb{R}_X).$$

Proof. This follows from comparing the two complexes. The cellular Borel–Moore homology of $T_\tau X$ is the homology of the complex

$$0 \longrightarrow C_n^{BM}(T_\tau X, \mathbb{R}_X) \longrightarrow \cdots \longrightarrow C_k^{BM}(T_\tau X, \mathbb{R}_X) \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0,$$

which yields summands $\mathbb{R}^{|A_i|}$ in position $-i$, where

$$A_i := \{\sigma \in X \mid \sigma \geq \tau \text{ and } \sigma \in X^i\}$$

is the set of i-cells containing X, and the boundary maps merely record the relative orientations. The cellular Borel–Moore homology at $\mathrm{Star}(\tau)$ is the homology of the complex:

$$0 \longrightarrow C_n^{BM}(\mathrm{Star}(\tau), \mathbb{R}_X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0^{BM}(\mathrm{Star}(\tau), \mathbb{R}_X) \longrightarrow 0,$$

which again yields summands $\mathbb{R}^{|A_i|}$ in position $-i$, with the same boundary map. Hence the homology groups $H_i^{BM}(\mathrm{Star}(\tau), \mathbb{R}_X)$ and $H_i^{BM}(T_\tau X, \mathbb{R}_X)$ agree for all i. ■

2.8 Cohen–Macaulay polyhedral complexes

Definition 2.8.1. Let X be a polyhedral complex, and k a field. We say that X is *Cohen–Macaulay over* k if

$$H_q^{BM}(T_\sigma X, k_X) = 0 \quad \text{for } q \neq \dim X,$$

for all faces $\sigma \in X$.

Remark 2.8.2. The name "Cohen–Macaulay" of this property comes from its relation to Cohen–Macaulay property of simplicial complexes, which itself comes from the Cohen–Macaulay property of modules over a ring. The connection is through the following criterion:

Theorem 2.8.3 ([MS05, Theorem 5.53], Reisner's criterion)**.** *The Stanley–Reisner ring S/I_Δ is Cohen–Macaulay over a field k if and only if, for every face $\sigma \in \Delta$, the link satisfies*

$$\widetilde{H}^i(\mathrm{link}_\Delta(\sigma), k_X) = 0 \quad \textit{for} \quad i \neq \dim(\Delta) - \dim(\sigma),$$

where $\operatorname{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta$ *and* $\tau \cap \sigma = \emptyset\}$*, where* $\tau \cup \sigma$ *is the simplex generated by the vertices in of both simplicies, and where* $\widetilde{H}^\bullet$ *indicates the reduced homology of the complex. We say that* Δ *is a* Cohen–Macaulay simplicial complex.

Another property motivating the "Cohen–Macaulay" name can be found in Remark 6.1.4.

Remark 2.8.4. By Proposition 2.5.9, and Example 2.3.2, one can also characterize Cohen–Macaulayness over k by

$$H_c^q(T_\sigma X, k_X) = 0 \quad \text{for } q \neq \dim X,$$

for all faces $\sigma \in X$.

Remark 2.8.5. Note that the Cohen–Macaulay property is only dependent on the poset of the polyhedral complex.

Lemma 2.8.6. *If* X *is a Cohen–Macaulay polyhedral fan, then* $H_q^{BM}(X, \mathbb{R}_X) = 0$ *for all* $q \neq \dim X$.

Proof. Since X is Cohen–Macaulay, we have that $0 = H_q^{BM}(T_\tau X, \mathbb{R}_X)$ for the minimal cell τ, hence

$$0 = H_q^{BM}(T_\tau X, \mathbb{R}_X) = H_q^{BM}(X, \mathbb{R}_X) = H_q^{BM}(X, \mathcal{F}_0).$$

■

Lemma 2.8.7. *If* X *is a Cohen–Macaulay polyhedral fan, then* X *is pure dimensional.*

Proof. Suppose τ is a maximal face with respect to inclusion, such that $q = \dim \tau < \dim X = n$. Then since τ is maximal, the tangent fan $T_\tau X$ is merely the linear space $L(\tau)$ parallel to τ and therefore:

$$H_k^{BM}(T_\tau X, \mathbb{R}_X) = \begin{cases} 0 & \text{for } k \neq q, \\ \mathbb{R} & \text{for } k = q, \end{cases}$$

which contradicts the Cohen–Macaulay condition $H_k^{BM}(T_\sigma X, \mathbb{R}) = 0$ for all $\sigma \in X$ and $k \neq \dim X$. ■

2.9 The Bergman fan of a matroid

In this section, we describe a class of examples of polyhedral fans which will reappear in Chapter 6. A *matroid* is a combinatorial generalization of the concept of linear independence. We refer to [Kat16] for a survey, and [Oxl11] for a textbook. There are many non-obviously equivalent definitions of matroids. To construct polyhedral fans from matroids, it is most convenient to introduce matroids as a set of *flats*:

Definition 2.9.1 ([Kat16, Definition 3.5])**.** Let E be a finite non-empty set. A *matroid* is a collection of subsets $\mathcal{F}$ of E that satisfy the following conditions

1. $E \in \mathcal{F}$,

2. if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$, and
3. if $F \in \mathcal{F}$ and $\{F_1, F_2, \ldots, F_k\}$ is the set of minimal members of $\mathcal{F}$ properly containing F then the sets $F_1 \setminus F, F_2 \setminus F, \ldots, F_k \setminus F$ partition $E \setminus F$.

The elements of $\mathcal{F}$ are called *flats*.

Remark 2.9.2 ([Kat16]). There is an equivalent definition of matroids in terms of *rank functions* $r\colon \mathcal{P}(E) \to \mathbb{Z}$. Using this definition, one can define a *loop* in a matroid. A matroid is said to be *loopless* if it has no loops. Not all matroids are loopless.

Definition 2.9.3 ([Kat16, Definition 3.5]). Let M be a loopless matroid on a finite set E of cardinality n. For a subset $S \subseteq E$, let

$$e_S \coloneqq \sum_{i \in S} e_i \in \mathbb{R}^n.$$

Given the collection of flats $\mathcal{F}$ of M, a *k-step flag of proper flats* is a sequence of proper flats ordered by containment:

$$F_\bullet \coloneqq \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E\}.$$

The *cone associated to $F_\bullet$* is the non-negative span

$$\sigma_{F_\bullet} \coloneqq \operatorname{span}_{\geq 0}\{e_{F_1}, \ldots, e_{F_k}\}.$$

The *Bergman fan* of M is the simplicial fan Δ_M consisting of the cones $\sigma_{F_\bullet}$ for all flags of flats $\mathcal{F}_\bullet$. Whenever we speak of the Bergman fan of a matroid, we implicitly assume that the matroid is loopless.

Theorem 2.9.4. *The Bergman fan of a matroid is Cohen–Macaulay.*

Proof. Recall that the Cohen–Macaulay property only depends on the poset of the polyhedral complex (Remark 2.8.5). The Bergman fan of a matroid is the cone over the *Bergman complex* of the matroid, which is a shellable simplicial complex, hence is Cohen–Macaulay [AK06]. Since the fan is the cone over the complex, it must also be Cohen–Macaulay. ∎

CHAPTER 3

Tropical homology

In this chapter, we introduce *tropical homology and cohomology*, as a set of invariants one can compute for any polyhedral complex. We then introduce the *balancing condition* on a polyhedral complex, which is the property defining *tropical cycles*. Moreover, we show that this balancing condition can be verified by the (n, n) tropical Borel–Moore homology group. Next, we show that, for the balanced polyhedral complexes, the (n, n) tropical Borel–Moore homology and $(0, 0)$ tropical cohomology groups are of equal dimension if the balancing of the complex is unique. This will lead us to examine whether there is a more general correspondence between the tropical Borel–Moore homology and tropical cohomology groups. We define the *tropical cap product* from [JRS18], which defines a map $\cap[X]\colon H^q(X, \mathcal{F}^p) \to H^{BM}_{n-q}(X, \mathcal{F}_{n-p})$ when X is balanced, reminiscent of the cap product for manifolds in algebraic topology. We will say that a polyhedral fan where this map is an isomorphism is a tropical Poincaré space. We give several examples of fans using the "Cellular Sheaves" package ([KSW17]) for [polymake] to compute several examples of fans, and check whether they are tropical Poincaré spaces. Finally, we classify fans of dimension 1 having this property.

3.1 Tropical sheaves

Definition 3.1.1 ([MZ14, Section 3.1])**.** Let X be a polyhedral complex of dimension n in $\mathbb{R}^N$. For $p = 0, \dots, n$, the *tropical wave sheaf* $\mathcal{W}^p$ is the cellular sheaf defined by the data:

- For $\sigma \in X$, $\mathcal{W}_p(\sigma) \coloneqq \bigwedge^p L(\sigma) \subseteq \bigwedge^p \mathbb{R}^N$, where $L(\sigma) \subset \mathbb{R}^N$ is the linear space parallel to the face σ.
- For $\tau \leq \sigma$, we have a morphism $(r\colon \tau \to \sigma) \in \mathrm{Mor}(X)$, and we define $\mathcal{W}_p(r) \coloneqq \iota_{\tau\sigma}$, where $\iota_{\tau\sigma}\colon \mathcal{W}_p(\tau) \to \mathcal{W}_p(\sigma)$ is the wedge of the inclusion $L(\tau) \to L(\sigma)$.

Definition 3.1.2 ([Ite+16, Definition 13, Definition 16])**.** Let X be a polyhedral complex of dimension n in $\mathbb{R}^N$. For $p = 0, \dots, n$, the *p-th multi-tangent space* $\mathcal{F}_p$ is the cellular cosheaf defined by the data:

- For $\sigma \in X$, $\mathcal{F}_p(\sigma) \coloneqq \sum_{\sigma \leq \gamma} \bigwedge^p L(\gamma) \subseteq \bigwedge^p \mathbb{R}^N$, where $L(\gamma) \subset \mathbb{R}^N$ is the linear space parallel to the face γ.

- For $\tau \leq \sigma$, we have a morphism $(r\colon \sigma \to \tau) \in \mathrm{Mor}(X^{\mathrm{op}})$, and we define $\mathcal{F}_p(r) \coloneqq \iota_{\sigma\tau}$, where $\iota_{\sigma\tau}\colon \mathcal{F}_p(\sigma) \to \mathcal{F}_p(\tau)$ is the wedge of the natural inclusion that follows from the ordering on the faces: if $\tau \leq \sigma$ then

$$\{\gamma \in X \mid \sigma \leq \gamma\} \subset \{\gamma \in X \mid \tau \leq \gamma\}.$$

Furthermore, the cosheaf $\mathcal{F}_p$ also gives rise to a sheaf $\mathcal{F}^p$ which is defined by $\mathcal{F}^p(\sigma) \coloneqq (\mathcal{F}_p(\sigma))^*$, with morphisms $\rho_{\tau\sigma}\colon \mathcal{F}_p(\tau) \to \mathcal{F}_p(\sigma)$ defined by dualizing $\iota_{\sigma\tau}\colon \mathcal{F}_p(\sigma) \to \mathcal{F}_p(\tau)$. Equivalently, the sheaf $\mathcal{F}^p$ is just the linear dual $D(\mathcal{F}^p)$ of the cosheaf $\mathcal{F}_p$ (see Definition 2.3.1).

Remark 3.1.3. Note that in [Ite+16, Definition 13], the cosheaves ${}^{\mathbb{Z}}\mathcal{F}_p$ are defined in terms of subgroups of $\bigwedge^p \mathbb{Z}^N$. Our definition is comparable to [Ite+16, Definition 16], by defining $\mathcal{F}_p = {}^{\mathbb{Z}}\mathcal{F}_p \otimes \mathbb{R}$. We do similarly for the sheaves $\mathcal{F}^p$.

Definition 3.1.4. Let X be a polyhedral complex. The cellular cohomology groups and cellular cohomology with compact support groups

$$H^q(X, \mathcal{F}^p), \quad \text{and} \quad H^q_c(X, \mathcal{F}^p),$$

are respectively called the *tropical cohomology groups* and *tropical cohomology with compact support groups* of X.

Similarly, the cellular homology groups and cellular Borel–Moore homology groups

$$H_q(X, \mathcal{F}_p), \quad \text{and} \quad H^{BM}_q(X, \mathcal{F}_p),$$

are respectively called the *tropical homology groups* and *tropical Borel–Moore homology groups* of X.

Proposition 3.1.5 ([JRS18, Remark 2.8])**.** *Let X be a polyhedral complex covered by stars of vertices. Let $\widetilde{X}$ be a simplicial refinement of X (see Theorem 2.6.6). Then the tropical homology and cohomology of X and $\widetilde{X}$, are equal. Explicitly we have:*

$$H^q(X, \mathcal{F}^p) \cong H^q(\widetilde{X}, \mathcal{F}^p), \quad \textit{and} \quad H^q_c(X, \mathcal{F}^p) \cong H^q_c(\widetilde{X}, \mathcal{F}^p),$$

while in cohomology we have:

$$H_q(X, \mathcal{F}_p) \cong H_q(\widetilde{X}, \mathcal{F}_p), \quad \textit{and} \quad H^{BM}_q(X, \mathcal{F}_p) \cong H^{BM}_q(\widetilde{X}, \mathcal{F}_p).$$

Remark 3.1.6. By the above proposition, we can always subdivide a complex to compute it's tropical cohomology. Hence, for our purposes, we can always replace a polyhedral complex X by one that is simplicial.

3.2 Balancing in tropical geometry

Definition 3.2.1. Given a polyhedral complex X of dimension n, an *integral weight function* is a function

$$\omega\colon X^n \to \mathbb{Z} \setminus \{0\},$$

assigning weights to each of the n-dimensional cells σ of X. A polyhedral complex with an integral weight function is said to be $\mathbb{Z}$*-weighted.* As a matter of notation, we write $\omega_\sigma \coloneqq \omega(\sigma)$ for the weight of a face $\sigma \in X^n$.

Definition 3.2.2 ([Bru+15, Definition 5.7])**.** Let X be a $\mathbb{Z}$-weighted rational finite polyhedral complex of dimension n in $\mathbb{R}^N$, and let $\tau \in X$ be a codimension 1 face of X. Let $\sigma_1, \dots, \sigma_s$ be the facets adjacent to τ, and let $\Gamma_{\sigma_i} \subset \mathbb{Z}^N$ denote the lattice parallel to σ, (analogously for Γ_τ). Let v_i be a primitive integer vector such that, together v_i and Γ_τ generate Γ_{σ_i}, and for any $x \in \tau$, one has $x + \epsilon v_i \in \sigma_i$ for $1 >> \epsilon > 0$. We say that X is *balanced along* τ if the vector

$$\sum_{i=1}^{s} \omega_{\sigma_i} v_i$$

is in Γ_τ, where ω_{σ_i} is the weight of the facet σ_i.

Remark 3.2.3. When the polyhedral complex X of dimension n is rational, the rank of the lattice Γ_σ parallel to each top dimensional face σ is n. Since the rank of $\Lambda^n \Gamma_\sigma = \Lambda^n \mathbb{Z}^n$ is 1 ([Eis95, Corollary A3.2]), one can choose the unique generator Λ_σ compatible with the orientation of the cell.

Definition 3.2.4 ([Bru+15, Definition 5.8])**.** A $\mathbb{Z}$-weighted rational finite polyhedral complex in $\mathbb{R}^N$ is said to be *balanced* if it is balanced along every codimension 1 face. Such a polyhedral complex is also called a *tropical cycle* in $\mathbb{R}^N$. If all the weights are non-negative, the tropical cycle is called *effective* or also a *tropical subvariety of* $\mathbb{R}^N$.

Definition 3.2.5 ([JRS18, Definition 4.8])**.** Let X be a $\mathbb{Z}$-weighted rational finite polyhedral complex of dimension n. The *fundamental chain of* X is

$$\mathrm{ch}(X) := (\omega_\sigma \Lambda_\sigma)_{\sigma \in X^n} \in C_n^{BM}(X, \mathcal{F}_n),$$

where ω_σ and Λ_σ are as in Definition 3.2.2. In [JRS18, Definition 4.8], the notation $\mathrm{ch}(X) = \sum_{\sigma \in X^n} \omega_\sigma \Lambda_\sigma \otimes \sigma := (\omega_\sigma \Lambda_\sigma)_{\sigma \in X^n}$ is used.

Proposition 3.2.6. *Let X be a weighted rational finite polyhedral complex. The fundamental chain of X is closed, in the sense that $\partial(\mathrm{ch}(X)) = 0$, if and only if X is balanced.*

Proof. Let X be a weighted rational finite polyhedral complex of dimension n in $\mathbb{R}^N$. Let $\sigma \in X^n$ be a top dimensional face, and let $\tau \in X^{n-1}$ be a codimension 1 face such that $\tau \leq \sigma$. Then either $\mathcal{O}(\tau, \sigma) = 1$ or $\mathcal{O}(\tau, \sigma) = -1$.

When $\mathcal{O}(\tau, \sigma) = 1$, we choose a vector v_σ as in Definition 3.2.2, and letting $v_\tau^1, \dots, v_\tau^{n-1}$ be generators of the lattice $\Gamma_\tau \subset \mathbb{Z}^N$ parallel to τ, we have the generator

$$\Lambda_\sigma := v_\sigma \wedge v_\tau := v_\sigma \wedge v_\tau^1 \wedge \dots \wedge v_\tau^{n-1}$$

of $\bigwedge^n \Gamma_\sigma$ of positive orientation. Then $\omega_\sigma \Lambda_\sigma$ can be chosen as the σ component of $\mathrm{ch}(X)$. Similarly when $\mathcal{O}(e, f) = -1$, we make the same choices, noting that now

$$\Lambda_\sigma := v_\sigma \wedge -v_\tau := v_\sigma \wedge -v_\tau^1 \wedge \dots \wedge -v_\tau^{n-1}$$

is the generator of $\bigwedge^n \Gamma_\sigma$ of positive orientation.

Now let τ be a face of codimension 1 of X. We compute the τ-component of $\partial(\mathrm{ch}(X))$:

$$\begin{aligned}(\partial(\mathrm{ch}(X)))_\tau &= \sum_{\tau\leq\sigma} \mathcal{O}(\tau,\sigma)\omega_\sigma\Lambda_\sigma \\ &= \sum_{\substack{\tau\leq\sigma \\ \mathcal{O}(\tau,\sigma)=1}} \mathcal{O}(\tau,\sigma)\omega_\sigma v_\sigma\wedge v_\tau + \sum_{\substack{\tau\leq\sigma \\ \mathcal{O}(\tau,\sigma)=-1}} \mathcal{O}(\tau,\sigma)\omega_\sigma v_\sigma\wedge -v_\tau \\ &= \sum_{\tau\leq\sigma}\omega_\sigma v_\sigma\wedge v_\tau\end{aligned}$$

Thus if the τ-component is zero, that is $\partial(\mathrm{ch}(X))_\tau = 0$, we have

$$\sum_{\tau\leq\sigma}\omega_\sigma v_\sigma\wedge v_\tau = 0.$$

This is equation holds if and only if $\sum_{\tau\leq\sigma}\omega_\sigma v_\sigma$ is in the span of the vectors $v_\tau^1,\dots,v_\tau^{n-1}$, i.e. $\sum_{\tau\leq\sigma}\omega_\sigma v_\sigma$ is in the lattice Γ_τ. Hence X is balanced at τ. By reversing the argument, we also see that if X is balanced at τ, then $(\partial(\mathrm{ch}(X)))_\tau = 0$.

We conclude that, if $\partial(\mathrm{ch}(X)) = 0$, this means that X is balanced along each codimension 1 face, hence X is balanced. Similarly, if X is balanced along each codimension 1 face, hence is balanced, then $\partial(\mathrm{ch}(X))_\tau = 0$, hence $\partial(\mathrm{ch}(X)) = 0$. Therefore, X is balanced if and only if $\mathrm{ch}(X)$ is a closed.

■

Remark 3.2.7. Notice that the element $\mathrm{ch}(X)\in C_n^{BM}(X,\mathcal{F}_n)$ could in fact be considered as an element of $C_n^{BM}(X,{}^{\mathbb{Z}}\mathcal{F}_p)$. This suggests one could consider more general weights:

Definition 3.2.8. Given a polyhedral complex X of dimension n, a *real weight function* is a function

$$\omega\colon X^n\to\mathbb{R}\setminus\{0\},$$

assigning weights to each of the n-dimensional cells σ of X. A polyhedral complex with a real weight function is said to be $\mathbb{R}$-*weighted*. As a matter of notation, again write $\omega_\sigma := \omega(\sigma)$ for the weight of a face $\sigma\in X^n$.

Remark 3.2.9. Using these generalized weights, we define the balancing conditions in the same manner as before. Going forward, we will assume all weights to be $\mathbb{R}$-weights, and balancing to mean $\mathbb{R}$-balancing.

Definition 3.2.10. If the fundamental chain $\mathrm{ch}(X)$ is closed, i.e. X is balanced by Proposition 3.2.6, the element $[X] := [\mathrm{ch}(X)]\in H_n^{BM}(X,\mathcal{F}_n)$ is called the *fundamental class of* X.

Definition 3.2.11. A rational finite polyhedral complex X of dimension n is *uniquely balanced* if there exists a unique real weight function (or alternatively integral) such that X is balanced, up to the equivalence relation: $\omega\sim\omega'$ if and only if there exists a $\lambda\in\mathbb{Z}$ such that $\omega(\sigma) = \lambda\omega'(\sigma)$ for all $\sigma\in X$.

Proposition 3.2.12. *Suppose X is a balanced rational finite polyhedral complex of dimension n. Then $H_n^{BM}(X, \mathcal{F}_n) = \langle [X] \rangle = \mathbb{R}$ if and only if X is uniquely balanced.*

Proof. Let $\omega\colon X^n \to \mathbb{N}$ be a weight function on X such that X is balanced, inducing a fundamental class $[X] \in H_n^{BM}(X, \mathcal{F}_n)$ by Proposition 3.2.6.

First suppose that $H_n^{BM}(X, \mathcal{F}_n) = \langle [X] \rangle = \mathbb{R}$. Suppose there is another weight function $\omega'\colon X^n \to \mathbb{R} \setminus \{0\}$, with $\omega' \neq \lambda\omega$ for all $\lambda \in \mathbb{R}$, such that the X is balanced with these new weights. By Proposition 3.2.6, the chain

$$x := (\omega'_\sigma \Lambda_\sigma)_{\sigma \in X^n} \in C_n^{BM}(X, \mathcal{F}_n)$$

is closed, i.e. $\partial_n(x) = 0$. But then $x \in H_n^{BM}(X, \mathcal{F}_n)$ but not in $\langle [X] \rangle$, hence $\dim H_n^{BM}(X, \mathcal{F}_n) > 1$, which contradicts the initial assumption. Hence any other weight function $\omega'\colon X^n \to \mathbb{R} \setminus \{0\}$ is such that $\omega' = \lambda\omega$ for some $\lambda \in \mathbb{R}$. Thus X is uniquely balanced.

Next suppose that X is uniquely balanced. Let

$$x = (\omega'_\sigma \Lambda_\sigma)_{\sigma \in X^n} \in C_n^{BM}(X, \mathcal{F}_n)$$

be an element such that for all $\lambda \in \mathbb{R}$, there is a $\sigma \in X^n$ such that $\omega'_\sigma \neq \lambda\omega_\sigma$. If $\partial_n(x) = 0$, then X induces a class in $H_n^{BM}(X, \mathcal{F}_n)$. By Proposition 3.2.6,

$$\begin{aligned} \omega'\colon X^n &\to \mathbb{Z} \setminus \{0\}, \\ \sigma &\mapsto \omega'_\sigma, \end{aligned}$$

is then another weight function satisfying the balancing condition, hence X would not be uniquely balanced, contradicting the initial assumption. ∎

Inspired by this proposition, we can define the following unique balancing for general fans, not necessarily being rational:

Definition 3.2.13. A polyhedral fan X of dimension n is *uniquely balanced* if $H_n^{BM}(X, \mathcal{F}_n) \cong \mathbb{R}$.

3.3 The tropical cap product

We will be using a contraction map from multilinear algebra, as developed in [Bou48, Section 5.6] and [FH91, Appendix B.3]. The following definitions will be sufficient for our needs:

Definition 3.3.1. Let V be a vector space over $\mathbb{R}$. Let $f \in V^*$ be an element of the dual vector space. For each p, there is a *contraction* map $c_f\colon \bigwedge^p V \to \bigwedge^{p-1} V$, such that for $v_1, \ldots, v_p \in V$,

$$c_f(v_1 \wedge \ldots \wedge v_p) = \sum_{i=1}^{p} (-1)^{i-1} f(v_i) v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_p,$$

where the $\widehat{v_i}$ indicates that v_i has been removed from the wedge.

We observe that this is an $\mathbb{R}$-linear map in f, so in fact it is giving a bilinear map $V^* \times \bigwedge^p V \to \bigwedge^{p-1} V$, defined by $(f, v) \mapsto c_f(v)$. We extend this then to define a multilinear map:

Definition 3.3.2. Let V be a vector space over $\mathbb{R}$. Let $v_1, \dots, v_p \in V$ and $f_1, \dots, f_{p'} \in V^*$, For each p, p' with $p \leq p'$, there is a *contraction* map:

$$\langle - \,;- \,\rangle \colon \bigwedge^{p} V^* \times \bigwedge^{p'} V \to \bigwedge^{p'-p} M,$$

given by composition of regular contractions:

$$\left(\; f_1 \wedge \dots \wedge f_{p'}, v_1 \wedge \dots \wedge v_p \; \right) \mapsto c_{f_{p'}} \circ c_{f_{p'-1}} \circ \dots \circ c_{f_1} \left(v_1 \wedge \dots \wedge v_p \right).$$

We can use contraction to define an operation on the tropical cosheaves:

Definition 3.3.3 ([JRS18, Definition 4.10])**.** Given $l \in \mathcal{F}^p(\sigma)$ and $v \in \mathcal{F}_{p'}(\sigma)$ with $p \leq p'$, the *contraction* $\langle l; v\rangle \in \mathcal{F}_{p'-p}(\sigma)$ is induced by the contraction map:

$$\langle -;-\rangle \colon \bigwedge^{p} (\mathbb{R}^r)^* \times \bigwedge^{p'} \mathbb{R}^r \to \bigwedge^{p'-p} \mathbb{R}^r.$$

More generally, given $\tau, \tau' \leq \sigma$ and $l \in \mathcal{F}^p(\tau), v \in \mathcal{F}_{p'}(\sigma)$, the contraction $\langle l; v\rangle$ is given by

$$\langle l; v\rangle = i_{\tau',\sigma} \left(\langle \rho_{\tau,\sigma}(l); v\rangle \right) \in \mathcal{F}_{p'-p}(\tau').$$

Recall that when computing tropical homology and cohomology, can replace a polyhedral complex X by a simplicial subdivision $\widetilde{X}$ (Remark 3.1.6). Given a balanced polyhedral fan X of dimension n, we can define a *cap product with the fundamental class of* X, following [JRS18]:

Definition 3.3.4 ([JRS18, Definition 4.11])**.** The *cap product with the fundamental class of* X is the map

$$\begin{aligned} \cap[X] \colon C^q(X, \mathcal{F}^p) &\to C^{BM}_{n-q}(X, \mathcal{F}_{n-p}), \\ \alpha &\mapsto \left(\sum_{[i_0, \dots, i_n] \in \widetilde{X}^n} \omega_\sigma \langle \alpha \,|_{[i_0, \dots, i_q]}; \Lambda_\sigma\rangle \right)_{[i_q, \dots i_n]}. \end{aligned}$$

Remark 3.3.5. Note that this is the definition stated in [JRS18], which uses a refinement of the polyhedral complex into a simplicial complex, using infinitely many simplices to subdivide the unbounded polyhedrons of the complex. We will only use this definition in the case of polyhedral fans, where this construction is not needed (Proposition 3.4.1).

Furthermore, as is noted in [JRS18], the Leibniz formula holds for the cap product, such that $\partial(\alpha \cap [X]) = (-1)^{q+1}(\partial(\alpha) \cap [X] - \alpha \cap \partial([X]))$.

Proposition 3.3.6. *If X is balanced, the map $\cap[X]$ descends to a map*

$$\cap[X] \colon H^q(X, \mathcal{F}^p) \to H^{BM}_{n-q}(X, \mathcal{F}_{n-p}).$$

Proof. If X is balanced, the fundamental class $[X]$ is closed by Proposition 3.2.6. Therefore,

$$\partial(\alpha \cap [X]) = -(\partial(\alpha) \cap [X] - \alpha \cap \partial([X])) = -(\partial(\alpha) \cap [X]),$$

hence if $\alpha \in H^q(X, \mathcal{F}^p)$, then $\alpha \cap [X] \in H^{BM}_n(X, \mathcal{F}_{n-p})$, meaning that the map can be restricted to homology. ∎

Remark 3.3.7. As observed in [JRS18], this definition can be applied for any class $\alpha \in H_n^{BM}(X, \mathcal{F}_n)$ to give a map

$$\cap\alpha \colon H^q(X, \mathcal{F}^p) \to H_{n-q}^{BM}(X, \mathcal{F}_{n-p}).$$

3.4 Tropical Poincaré duality

Observe that the cap product with the fundamental class in homology is particularly easy to formulate in the case of rational balanced polyhedral fans:

Proposition 3.4.1. *Let X be a rational balanced polyhedral fan of dimension n. The cap product with the fundamental class is then given by:*

- *For $q > 0$, the cap product with the fundamental class $\cap[X] \colon H^q(X, \mathcal{F}^p) \to H_{n-q}^{BM}(X, \mathcal{F}_{n-p})$ is the zero map and*
- *For $q = 0$, the cap product with the fundamental class $\cap[X] \colon H^0(X, \mathcal{F}^p) \to H_{n-p,n}^{BM}(X)$ is given by the map:*

$$\alpha \mapsto \big(\omega_\sigma \langle \alpha \mid_{L(\sigma)}; \Lambda_\sigma \rangle\big)_{\sigma \in X^n},$$

where $\langle\ ;\ \rangle$ is the contraction map $\langle\ ;\ \rangle \colon \mathcal{F}^p(\sigma) \times \mathcal{F}_n(\sigma) \to \mathcal{F}_{n-p}(\sigma)$ as introduced in Definition 3.3.3.

Proof. For $q > 0$, by Proposition 2.5.6, the group $H^q(X, \mathcal{F}^p)$ is zero, so the map $\cap[X]$ is also necessarily zero.

For $q = 0$, this expression is simply the definition given in Definition 3.3.4. ∎

Whenever we use this cap product, it will be assumed that the fan is pointed. Any non-pointed fan can be subdivided to this case.s

Using this cap product, we formulate the "tropical Poincaré duality for polyhedral fans" as follows:

Definition 3.4.2 ([JRS18, Definition 5.2])**.** We say that a balanced rational finite polyhedral fan X of dimension n is a *tropical Poincaré space* if the cap product with the fundamental class

$$\cap[X] \colon H^q(X, \mathcal{F}^p) \to H_{n-q}^{BM}(X, \mathcal{F}_{n-p})$$

is an isomorphism for all $p, q = 0, \ldots, n$. We will sometimes call such fans *tropical Poincaré spaces.*

A class of fans satisfying this condition has already been identified:

Theorem 3.4.3 ([JSS19, Proposition 4.27])**.** *Let M be a matroid, and $\mathcal{B}(M)$ its Bergman fan Definition 2.9.3. Then $\mathcal{B}(M)$ is a tropical Poincaré space.*

However, this is not exhaustive since there are examples of fans which are tropical Poincaré spaces even though they are not Bergman fans of matroids, which shows that matroidality is not a necessary condition as was noted in [JRS18].

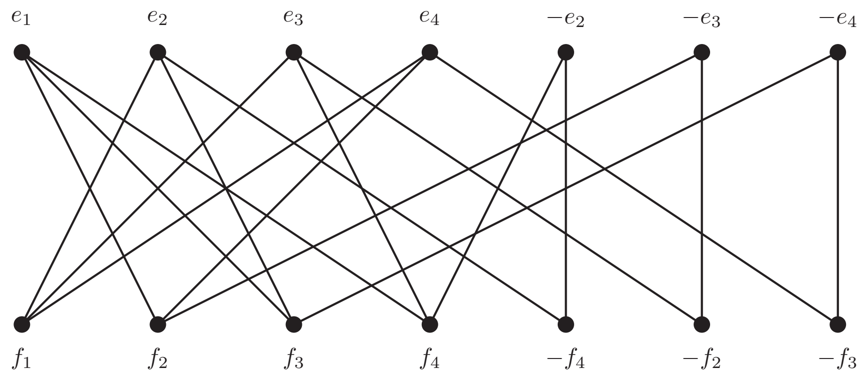

Figure 3.1: The graph of cones for Example 3.4.4.

Example 3.4.4. Let $f_1 \coloneqq (0,1,1,1)$, $f_2 \coloneqq (1,0,-1,1)$, $f_3 \coloneqq (1,1,0,-1)$ and $f_4 \coloneqq (1,-1,1,0)$ be vectors in $\mathbb{R}^4$ and let e_1, e_2, e_3 and e_4 be the standard basis. Consider the polyhedral fan generated by the cones of vertices connected by an edge in Figure 3.1, so that for instance the cone of e_1 and f_2 is included.

This fan, coming from [BH17] is not matroidal, since it does not satisfy the Hard Lefschetz property of [AHK18], however if we compute its cellular homology using the cellular sheaves package [KSW17] for [`polymake`]:

```
$fan = new PolyhedralFan(INPUT_RAYS=>[

[1,0,0,0,0],   #0 Projection point
[0,1,0,0,0],   #1 e_1
[0,0,1,0,0],   #2 e_2
[0,0,0,1,0],   #3 e_3
[0,0,0,0,1],   #4 e_4
[0,0,1,1,1],   #5 f_1
[0,1,0,-1,1],  #6 f_2
[0,1,1,0,-1],  #7 f_3
[0,1,-1,1,0],  #8 f_4
[0,0,-1,0,0],  #9 -e_2
[0,0,0,-1,0],  #10 -e_3
[0,0,0,0,-1],  #11 -e_4
[0,-1,0,1,-1], #12 -f_2
[0,-1,-1,0,1], #13 -f_3
[0,-1,1,-1,0], #14 -f_4
],

INPUT_CONES =>
new Array<Set<Int>>(
# faces spanning cones with e_1
        new Set<Int>(0,1,6),
        new Set<Int>(0,1,7),
        new Set<Int>(0,1,8),
# faces spanning cones with e_2
        new Set<Int>(0,2,5),
        new Set<Int>(0,2,7),
        new Set<Int>(0,2,14),
# faces spanning cones with e_3
        new Set<Int>(0,3,5),
        new Set<Int>(0,3,8),
        new Set<Int>(0,3,12),
# faces spanning cones with e_4
```

```
        new Set<Int>(0,4,5),
        new Set<Int>(0,4,6),
        new Set<Int>(0,4,13),
# faces spanning cones with -e_2
        new Set<Int>(0,9,8),
        new Set<Int>(0,9,14),
# faces spanning cones with -e_3
        new Set<Int>(0,10,6),
        new Set<Int>(0,10,12),
# faces spanning cones with -e_4
        new Set<Int>(0,11,7),
        new Set<Int>(0,11,13)
));

$complex = new PolyhedralComplex($fan);

@betti_usual = ();
@betti_bm = ();
for(my $i=0; $i<4; $i++){
   my $f = $complex->fcosheaf($i);
   my $s = $complex->usual_chain_complex($f);
   my $bm = $complex->borel_moore_complex($f);
   push @betti_usual, $s->BETTI_NUMBERS;
   push @betti_bm, $bm->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_bm);
```

Which gives the following output:

```
fan > print new Matrix(@betti_usual);
1 0 0
4 0 0
5 0 0
0 0 0

fan > print new Matrix(@betti_bm);
0 0 5
0 0 4
0 0 1
0 0 0
```

Hence this fan is a tropical Poincaré space.

Not all fans satisfy are tropical Poincaré spaces however:

Example 3.4.5. Let e_1, e_2 and e_3 be the standard basis on $\mathbb{R}^3$. Consider the polyhedral fan Σ whose center vertex lies at the origin, where the cones of Σ are given by cones of vertices connected by an edge in Figure 3.2, so that for instance the cone between $-e_1$ and $-e_3$ is part of the fan.

This fan is the tropical hypersurface associated to the Newton polytope of a pyramid with vertices $e_1, e_2, e_3, 0$ and $e_1 + e_2$, with summit in e_3. Since this Newton polytope is not a simplex, the fan cannot come from a matroid. Using the following [polymake] script, we compute its cellular homology using the cellular sheaves package [KSW17] for:

```
application 'fan';
$fan = new fan::PolyhedralFan(INPUT_RAYS=>[
[1,0,0,0],
[0,-1,0,0],
```

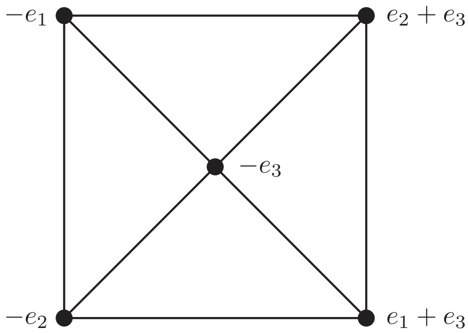

Figure 3.2: The graph of cones for Example 3.4.5.

```
[0,0,-1,0],
[0,0,0,-1],
[0,0,1,1],
[0,1,0,1]],
INPUT_CONES =>
[[0,1,2],
[0,1,3],
[0,2,3],
[0,1,4],
[0,3,4],
[0,3,5],
[0,4,5],
[0,2,5]]
);

$complex = new fan::PolyhedralComplex($fan);

@betti_usual = ();
@betti_bm = ();
for(my $i=0; $i<4; $i++){
   my $f = $complex->fcosheaf($i);
   my $s = $complex->usual_chain_complex($f);
   my $bm = $complex->borel_moore_complex($f);
   push @betti_usual, $s->BETTI_NUMBERS;
   push @betti_bm, $bm->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_bm);
```

We get the following output:

```
fan > print new Matrix(@betti_usual);
1 0 0
3 0 0
3 0 0
0 0 0

fan > print new Matrix(@betti_bm);
0 0 4
0 0 4
0 0 1
0 0 0
```

Hence we see that this fan is a not tropical Poincaré space.

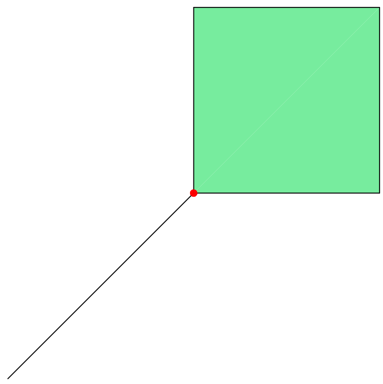

Figure 3.3: Picture of the fan

Example 3.4.6. Consider the fan in Figure 3.3, which is not pure dimensional. We can compute the usual cohomology and the Borel–Moore homology of the $\mathcal{F}^p$ sheaves of the fan using the cellular sheaves package [KSW17] for [polymake]:

```
application "fan";
$fan = new PolyhedralFan(
        INPUT_RAYS=>[[1,0,0],[0,-1,-1],[0,1,0],[0,0,1]],
        INPUT_CONES => [[0,1],[0,2,3]]);

$complex = new PolyhedralComplex($fan);

$complex -> VISUAL;

@betti_usual = ();
@betti_bm = ();
for(my $i=0; $i<3; $i++){
   my $f = $complex->fcosheaf($i);
   my $u = $complex->usual_chain_complex($f);
   my $bm = $complex->borel_moore_complex($f);
   push @betti_usual, $u->BETTI_NUMBERS;
   push @betti_bm, $bm->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_bm);
```

The Betti numbers are then output as follows:

```
fan > print new Matrix(@betti_usual);
1 0 0
2 0 0
1 0 0

fan > print new Matrix(@betti_bm);
0 1 0
```

```
0 1 0
0 0 0
```

Since the Betti numbers $\dim H_1^{BM}(X, \mathcal{F}_1)$ and $\dim H_1^{BM}(X, \mathcal{F}_0)$ are not 0, this fan does not satisfy tropical Poincaré duality.

3.5 Classification of tropical Poincaré fans in dimension 1

In this short section, we classify the tropical Poincaré fans of dimension 1.

Theorem 3.5.1. *Let X be a weighted pointed polyhedral fan of dimension 1. Then X is a tropical Poincaré space if and only if X is uniquely balanced.*

Proof. Let X be a weighted polyhedral fan of dimension 1, let v be the vertex of X, and let $\tau_1, \ldots, \tau_n \in X^1$ be the rays.

We have already seen that $\cap[X]\colon H^0(X, \mathcal{F}^0) \to H_1^{BM}(X, \mathcal{F}_1)$ is an isomorphism if and only if X is uniquely balanced by Proposition 3.2.12. It remains to determine when $\cap[X]\colon H^0(X, \mathcal{F}^1) \to H_1^{BM}(X, \mathcal{F}_0)$ is an isomorphism. Assume that $\operatorname{span}\{\tau_1, \ldots, \tau_n\} = \mathbb{R}^M$, for some M. Then $H^0(X, \mathcal{F}^1) = \mathbb{R}^M$. Since X must be uniquely balanced we have that $M = n-1$. Then $H_1^{BM}(X, \mathcal{F}_0) = \ker(C_1^{BM}(X, \mathcal{F}_0) \to H_0^{BM}(X, \mathcal{F}_0))$, which has dimension $n-1$. Therefore when X is uniquely balanced, $\cap[X]\colon H^0(X, \mathcal{F}^1) \to H_1^{BM}(X, \mathcal{F}_0)$ is an isomorphism. ∎

CHAPTER 4

Tropical Poincaré duality

In this chapter, we start the search for necessary and sufficient conditions for a polyhedral fan to to be a tropical Poincaré space. It was shown in [JSS19] that the Bergman fan of a matroid are tropical Poincaré spaces, but there are examples of fans which are not Bergman fans of matroids for which this duality still holds (Example 3.4.4). To examine further, we first introduce an isomorphism $\psi\colon C_c^n(X, \mathcal{F}^p) \to C_{n-p,n}^{BM}(X)$. Next, we exploit the structure of fans to construct a commutative diagram around the isomorphism. Then we use this diagram to determine equivalent conditions for the fan to satisfy the duality $\cap[X]\colon H^{p,0}(X) \to H_{n-p,0}^{BM}(X)$ in the case where X is a pure dimensional fan, identifying the following property:

Definition 4.0.1 (Definition 4.3.2)**.** Given a weighted polyhedral fan X of dimension n, we say that X is *uniquely p-balanced* if, given a cochain $b = (b_\sigma)_{\sigma \in X^n} \in C_c^n(X, \mathcal{F}^p)$, one has

$$\sum_{\tau \leq \sigma} \mathcal{O}(\tau, \sigma)\omega_\sigma \langle b_\sigma; \Lambda_\sigma \rangle = 0$$

for all $\tau \in X^{n-1}$, where ω_σ is the weight of the σ face, if and only if there is an $a \in \mathcal{F}^p(v)$ such that $b_\sigma = (s^*(a))_\sigma$ for all $\sigma \in X^n$.

This property is sufficient to determine when $\cap[X]\colon H^{p,0}(X) \to H_{n-p,0}^{BM}(X)$ is an isomorphism:

Theorem 4.0.2 (Theorem 4.3.5)**.** *Let X be a pure balanced rational polyhedral fan of dimension n in $\mathbb{R}^N$. The cap morphism $\cap[X] : H^{p,0}(X) \to H_n^{BM}(X, \mathcal{F}_{n-p})$ is an isomorphism if and only if X is uniquely p-balanced.*

Finally, we determine for a general polyhedral fan when $\cap[X]\colon H^{p,q}(X) \to H_{n-p,n-q}^{BM}(X)$ is an isomorphism, using a *dependence relation cosheaf $\mathcal{K}_p$* (Definition 4.4.1). We have:

Corollary 4.0.3 (Corollary 4.4.3)**.** *Let X be a polyhedral fan of dimension n. Then*

$$H_{q-1}^{BM}(X, \mathcal{K}_p) \cong H_q^{BM}(X, \mathcal{F}_p)$$

for $q = 1, \ldots, n$ and $p = 0, \ldots, n$, and

$$H_n^{BM}(X, \mathcal{K}_p) = 0 = H_0^{BM}(X, \mathcal{F}_p)$$

Both these results can then be combined to get equivalent conditions for a pure fans:

Theorem 4.0.4 (Theorem 4.5.1)**.** *A rational balanced polyhedral fan of pure dimension n is a tropical Poincaré space, i.e.*

$$\frown[X]\colon H^q(X, \mathcal{F}^p) \to H^{BM}_{n-q}(X, \mathcal{F}_{n-p})$$

is an isomorphism for all $p, q = 0, \dots, n$, if and only if

1. *X is uniquely p-balanced for all p, and*
2. *the dependence cosheaf $\mathcal{K}_p$ is acyclic in Borel–Moore homology for all p, that is, $H^{BM}_q(X, \mathcal{K}_p) = 0$ for $q \neq n-1$ and all p.*

4.1 An isomorphism between compact support cohomology and Borel–Moore homology

Let X be a weighted polyhedral fan of dimension n. We will first show that:

$$\bigoplus_{\sigma \in X^n} \left(\bigwedge^p L(\sigma) \right)^* = C^n_c(X, \mathcal{F}^p) \cong C^{BM}_{n-p,n}(X) = \bigoplus_{\sigma \in X^n} \bigwedge^{n-p} L(\sigma),$$

using an isomorphism closely related to the cap product $\frown[X]$. We build a componentwise map:

Definition 4.1.1. Let X be a weighted rational polyhedral complex of dimension n, and $\sigma \in X^n$ a face. Let Λ_σ be a generator of the p-th wedge of the $\mathbb{Z}$-lattice in $L(\sigma)$, so that $\Lambda_\sigma \in \bigwedge^n L(\sigma)$. Moreover, recall the definition of the contraction map from Definition 3.3.3. Then we have a linear map:

$$\begin{aligned} \psi_\sigma \colon \bigwedge^p L(\sigma)^* &\to \bigwedge^{n-p} L(\sigma), \\ \alpha &\mapsto \omega_\sigma \langle \alpha ; \Lambda_\sigma \rangle, \end{aligned}$$

since the contraction map $\langle - ; - \rangle \colon \bigwedge^p L(\sigma)^* \times \bigwedge^n L(\sigma) \to \bigwedge^{n-p} L(\sigma)$ is bilinear.

Remark 4.1.2. Note that, since $\bigwedge^p L(\sigma)^* \cong \left(\bigwedge^p L(\sigma)\right)^*$, this in fact defines a map from the σ-component of $C^n_c(X, \mathcal{F}^p)$ to the σ-component of $C^{BM}_n(X, \mathcal{F}_{n-p})$.

Theorem 4.1.3. *The morphism $\psi_\sigma \colon \bigwedge^p L(\sigma)^* \to \bigwedge^{n-p} L(\sigma)$ is an isomorphism for all σ where $\omega_\sigma \neq 0$.*

Proof. First note that, we can choose a basis $e^\sigma_1, \dots, e^\sigma_n$ for the parallel space $L(\sigma)$. Then, for each p, a basis for $\bigwedge^p L(\sigma)$ is

$$\langle e^\sigma_I \mid I \coloneqq (i_1, \dots, i_p) \subseteq \{1, \dots, n\}, |I| = p \text{ and } i_1 < \dots < i_p \rangle.$$

Using this, we can write $\Lambda_\sigma = v_\sigma e^\sigma_1 \wedge \ldots \wedge e^\sigma_n$, where v_σ is some coefficient. We can also find bases for the dual spaces $\left(\bigwedge^p L(\sigma)\right)^* \cong \bigwedge^p L(\sigma)^*$:

$$\langle (e^\sigma_I)^* \mid I \coloneqq (i_1, \dots, i_p) \subseteq \{1, \dots, n\}, |I| = p \text{ and } i_1 < \dots < i_p \rangle,$$

where $(e^\sigma_I)^*$ is the p-covector such that:

$$(e^\sigma_I)^*(e^\sigma_J) = \begin{cases} 1 & \text{if } J = I, \\ 0 & \text{otherwise.} \end{cases}$$

Note that since $(e_1^\sigma)^*, \dots, (e_n^\sigma)^*$ is a basis for $L(\sigma)^*$, hence for $I = (i_1, \dots, i_n)$, we have that:

$$(e_{i_1}^\sigma)^* \wedge \dots \wedge (e_{i_p}^\sigma)^* = (e_I^\sigma)^*$$

Since:

$$\dim \bigwedge^p L(\sigma)^* = \binom{n}{p} = \binom{n}{n-p} = \dim \bigwedge^{n-p} L(\sigma),$$

it is sufficient to show that ψ_σ is injective.

Assume therefore that $\psi_\sigma(y) = 0$ for some $y = \sum_I a_I (e_I^\sigma)^* \in \bigwedge^p L(\sigma)^*$. Then we have the following:

$$\begin{aligned} 0 &= \psi_\sigma(y), \\ &= \omega_\sigma \left\langle \sum_I a_I (e_I^\sigma)^*; \Lambda_\sigma \right\rangle, \\ &= \omega_\sigma \sum_I a_I \langle (e_I^\sigma)^*; \Lambda_\sigma \rangle, \end{aligned}$$

and using the observation from earlier that $\Lambda_\sigma = v_\sigma e_1^\sigma \wedge \dots \wedge e_n^\sigma$, we get:

$$\begin{aligned} 0 &= \omega_\sigma \sum_I a_I \langle (e_I^\sigma)^*; \Lambda_\sigma \rangle, \\ &= \omega_\sigma \sum_I a_I \langle (e_I^\sigma)^*; v_\sigma e_1^\sigma \wedge \dots \wedge e_n^\sigma \rangle, \\ &= \omega_\sigma \sum_I a_I v_\sigma \langle (e_I^\sigma)^*; e_1^\sigma \wedge \dots \wedge e_n^\sigma \rangle. \end{aligned}$$

Here to analyze the contraction mapping, we use the following lemma:

Lemma 4.1.4. *Contraction by $(e_I^\sigma)^*$ gives $\langle (e_I^\sigma)^*; e_1^\sigma \wedge \dots \wedge e_n^\sigma \rangle = \pm e_{I^c}^\sigma$, where $I^c = \{i \in [n] \mid i \notin I\}$ in strictly ascending order.*

Proof. We can compute the first step of the contraction to be:

$$\begin{aligned} \langle (e_I^\sigma)^*; e_1^\sigma \wedge \dots \wedge e_n^\sigma \rangle &= \langle (e_{i_1}^\sigma)^* \wedge \dots \wedge (e_{i_p}^\sigma)^*; e_1^\sigma \wedge \dots \wedge e_n^\sigma \rangle, \\ &= c_{\left(e_{i_p}^\sigma\right)^*} \circ \dots \circ c_{\left(e_{i_1}^\sigma\right)^*} (e_1^\sigma \wedge \dots \wedge e_n^\sigma). \end{aligned}$$

Now since

$$\left(e_{i_j}^\sigma\right)^* (e_k^\sigma) = \begin{cases} 1 & \text{if } k = i_j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} c_{\left(e_{i_1}^\sigma\right)^*}(e_1^\sigma \wedge \dots \wedge e_n^\sigma) &= \sum_{j=1}^n (-1)^j \left[\left(e_{i_1}^\sigma\right)^* (e_j^\sigma) \right] (e_1^\sigma \wedge \dots \wedge \widehat{e_j^\sigma} \wedge \dots \wedge e_n^\sigma), \\ &= (-1)^{i_1} e_1^\sigma \wedge \dots \wedge \widehat{e_{i_1}^\sigma} \wedge \dots \wedge e_n^\sigma, \end{aligned}$$

we have that

$$\langle (e_I^\sigma)^*; e_1^\sigma \wedge \dots \wedge e_n^\sigma \rangle = c_{\left(e_{i_p}^\sigma\right)^*} \circ \dots \circ c_{\left(e_{i_2}^\sigma\right)^*} ((-1)^{i_1} e_1^\sigma \wedge \dots \wedge \widehat{e_{i_1}^\sigma} \wedge \dots \wedge e_n^\sigma).$$

Now if we repeat this process for each i_k, we see that we remove the e_{i_k} term from the original wedge and possibly make a sign change. Therefore, in the end we have

$$\langle (e_I^\sigma)^*; e_1^\sigma \wedge \ldots \wedge e_n^\sigma \rangle = \pm e_{I^c}^\sigma,$$

which is the $(n-p)$-multivector composed of the wedge of the elements in the original basis with indices $0, \ldots, n \notin I$. ■

Lemma 4.1.4 can then be used in our situation, giving:

$$\begin{aligned} 0 &= \omega_\sigma \sum_I a_I v_\sigma \langle (e_I^\sigma)^*; e_1^\sigma \wedge \ldots \wedge e_n^\sigma \rangle, \\ &= \pm \omega_\sigma v_\sigma \sum_I a_I e_{I^c}^\sigma. \end{aligned}$$

Finally, since the $e_{I^c}^\sigma$ are elements of the basis for $\bigwedge^{n-p} L(\sigma)$, they must be linearly independent, hence for this equality to hold, we must have $a_I = 0$ for all I. This then finally gives:

$$y = \sum_I a_I e_I^\sigma = 0,$$

hence ψ_σ is injective. ■

Since these maps then induce isomorphisms on each component σ, we have:

Corollary 4.1.5. *Let X be a weighted polyhedral fan of dimension n. The map*

$$\begin{aligned} \psi \colon C_c^n(X, \mathcal{F}^p) &\to C_{n-p,n}^{BM}(X), \\ (y_\sigma)_{\sigma \in X^n} &\mapsto (\psi_\sigma(y_\sigma))_{\sigma \in X^n}, \end{aligned}$$

is an isomorphism.

Proof. Recall that we defined weight functions to be nowhere zero (Definition 3.2.8). Then by Theorem 4.1.3, this follows from being a direct sum of isomorphisms in each component since $\mathcal{F}_p(\sigma) = \bigwedge^p L(\sigma)$ and $\mathcal{F}^p(\sigma) = \left(\bigwedge^p L(\sigma)\right)^* \cong \bigwedge^p L(\sigma)^*$ for all $\sigma \in X^n$. ■

Explicitly, the isomorphism ψ is

$$\begin{aligned} \psi \colon C_c^n(X, \mathcal{F}^p) &\to C_{n-p,n}^{BM}(X), \\ (y_\sigma)_{\sigma \in X^n} &\mapsto (\omega_\sigma \langle y_\sigma; \Lambda_\sigma \rangle)_{\sigma \in X^n}, \end{aligned}$$

which can be compared to the very similar cap product with the fundamental class of X from Definition 3.3.4.

4.2 A commutative diagram for pure fans

Given a polyhedral fan X of dimension n, and any collection $P \subset X$ there is a summation map:

$$s\colon \bigoplus_{\gamma \in P} \mathcal{F}_p(\gamma) \to \mathcal{F}_p(v), \tag{4.1}$$

$$(x_\gamma)_{\gamma \in P} \mapsto \sum_{\gamma \in P} x_\gamma. \tag{4.2}$$

In particular, we will use this map when the chosen collection of X is a set of cells of equal dimension q:

$$\begin{aligned} s\colon C_q^{BM}(X, \mathcal{F}_p) &\to \mathcal{F}_p(v), \\ (x_\sigma)_{\sigma \in X^q} &\mapsto \sum_{\sigma \in X^n} x_\sigma. \end{aligned}$$

For any fan X, the $\mathcal{F}_p$ sheaf at the vertex can be taken over the maximal by inclusion faces of X, i.e.

$$\mathcal{F}_p(v) = \sum_{v \le \sigma} \bigwedge^p L(\sigma) = \sum_{\substack{\sigma \in X \\ \sigma \text{ maximal}}} \bigwedge^p L(\sigma),$$

If X is pure dimensional (Definition 2.1.2), the only maximal by inclusion faces are the top-dimensional ones. Therefore, $\mathcal{F}_p(v) = \sum_{\sigma \in X^n} \bigwedge^p L(\sigma)$, which guarantees that the map $s\colon C_n^{BM}(X, \mathcal{F}_p) \to \mathcal{F}_p(v)$ is surjective. Letting $\mathcal{K}_p(v)$ be the kernel of this map (this notation is explained in Definition 4.4.1, but can safely be ignored here), we have the exact sequence,

$$0 \longrightarrow \mathcal{K}_p(v) \xrightarrow{\ i\ } C_n^{BM}(X, \mathcal{F}_p) \xrightarrow{\ s\ } \mathcal{F}_p(v) \longrightarrow 0.$$

By dualizing this, we get the exact sequence

$$0 \longrightarrow \mathcal{F}^p(v) \xrightarrow{\ s^*\ } C_c^n(X, \mathcal{F}_p) \xrightarrow{\ i^*\ } \mathcal{K}^p(v) \longrightarrow 0,$$

where $\mathcal{K}^p(v) \coloneqq (\mathcal{K}^p(v))^*$Additionally, since $H_n^{BM}(X, \mathcal{F}_{n-p}) = \ker(\partial_n)$, we have the exact sequence

$$0 \longrightarrow H_n^{BM}(X, \mathcal{F}_{n-p}) \xrightarrow{\ j\ } C_n^{BM}(X, \mathcal{F}_{n-p}) \xrightarrow{\ \partial_n\ } B_{n-1}^{BM}(X, \mathcal{F}_{n-p}) \xrightarrow{\ \partial_{n-1}\ } 0,$$

where $B_{n-1}^{BM}(X, \mathcal{F}_{n-p}) \subseteq C_{n-1}^{BM}(X, \mathcal{F}_{n-p})$ is the set of boundaries.

Using these two exact rows, and the morphisms $\cap[X]$ and ψ (see Proposition 3.3.6 and Corollary 4.1.5 respectively), we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^p(v) \arrow[r, "s^*"] \arrow[d, "\cap[X]"] & C_c^n(X, \mathcal{F}^p) \arrow[r, "i^*"] \arrow[d, "\psi"] & \mathcal{K}^p(v) \arrow[r] & 0 \\
0 \arrow[r] & H_n^{BM}(X, \mathcal{F}_{n-p}) \arrow[r, "j"] & C_n^{BM}(X, \mathcal{F}_{n-p}) \arrow[r, "\partial_n"] & B_{n-1}^{BM}(X, \mathcal{F}_{n-p}) \arrow[r] & 0
\end{tikzcd}$$

Proposition 4.2.1. *The square in the diagram is commutative.*

Proof. We wish to show that for $y = \sum_I a_I^\sigma e_I^* \in \mathcal{F}^p(v)$, we have

$$(j \circ \cap[X])(y) = (\psi \circ s^*)(y).$$

First we use the definitions to compute the left side:

$$\begin{aligned}(j \circ \cap[X])(y) &= j(y \cap [X]),\\ &= j\left(\omega_\sigma \langle y; \Lambda_\sigma \rangle\right)_{\sigma \in X^n}.\end{aligned}$$

To compute the right side, recall that the map s^* comes from dualizing s, hence by examining the diagram

$$\begin{tikzcd} C_{p,n}^{BM}(X) \arrow[r, "s"] & \mathcal{F}_p(v) \arrow[r] \arrow[d, "y"] & 0, \\ & \mathbb{R} & \end{tikzcd}$$

we see that $s^*(y) = y \circ s =: ys$. Therefore, we can write the right hand side as

$$\begin{aligned}(\psi \circ s^*)(y) &= \psi(ys),\\ &= \left(\omega_\sigma \langle (ys)_\sigma; \Lambda_\sigma \rangle\right)_{\sigma \in X^n}.\end{aligned}$$

Now we note that, when restricting $s\colon C_n^{BM}(X, \mathcal{F}_p) \to \mathcal{F}_p(v)$ to an individual component, the sum is merely applying the identity map. Hence $(ys)_\sigma$ is the restriction $y|_{\mathcal{F}_p(\sigma)}$ of y to the multivectors in $\mathcal{F}_p(\sigma)$, so that $(ys)_\sigma = \rho_{v,\sigma}(y)$, where ρ is the sheaf restriction map. This gives:

$$\begin{aligned}(\psi \circ s^*)(y) &= \left(\omega_\sigma \langle (ys)_\sigma; \Lambda_\sigma \rangle\right)_{\sigma \in X^n},\\ &= \left(\omega_\sigma \langle \rho_{v,\sigma}(y); \Lambda_\sigma \rangle\right)_{\sigma \in X^n},\\ &= \left(\omega_\sigma \langle y; \Lambda_\sigma \rangle\right)_{\sigma \in X^n},\end{aligned}$$

where we simply clarify the contraction using the notation from Definition 3.3.3. Finally, since j is an injection, we can naturally identify

$$j\left(\omega_\sigma \langle y; \Lambda_\sigma \rangle\right)_{\sigma \in X^n} = \left(\omega_\sigma \langle y; \Lambda_\sigma \rangle\right)_{\sigma \in X^n},$$

which finally gives

$$(j \circ \cap[X])(y) = (\psi \circ s^*)(y).$$

■

Theorem 4.2.2. *Let X be a pure rational polyhedral fan of dimension n. Then the following diagram is commutative:*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}^p(v) \arrow[r, "s^*"] \arrow[d, "\cap[X]"] & C_c^n(X, \mathcal{F}^p) \arrow[r, "i^*"] \arrow[d, "\psi"] & \mathcal{K}^p(v) \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & H_n^{BM}(X, \mathcal{F}_{n-p}) \arrow[r, "j"] & C_n^{BM}(X, \mathcal{F}_{n-p}) \arrow[r, "\partial_n"] & B_{n-1}^{BM}(X, \mathcal{F}_{n-p}) \arrow[r] & 0 \end{tikzcd}$$

Proof. Since the left square of the diagram is commutative by Proposition 4.2.1, we know that we can use the technique from the right completion of diagrams (Lemma B.1.1) to give us a map from $\mathcal{K}^p(v)$ to $B^{BM}_{n-p,n}$, which will help us characterize exactly what conditions on polyhedral fans give the isomorphism $H^{p,0}_c(X) \cong H^{BM}_n(X, \mathcal{F}_{n-p})$.

Following the previous construction from right completion, and using the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^p(v) \arrow[r, "s^*"] \arrow[d, "\cap[X]"] & C^n_c(X, \mathcal{F}^p) \arrow[r, "i^*"] \arrow[d, "\psi"] & \mathcal{K}^p(v) \arrow[r] & 0, \\
0 \arrow[r] & H^{BM}_n(X, \mathcal{F}_{n-p}) \arrow[r, "j"] & C^{BM}_n(X, \mathcal{F}_{n-p}) \arrow[r, "\partial_n"] & B^{BM}_{n-1}(X, \mathcal{F}_{n-p}) \arrow[r] & 0,
\end{tikzcd}$$

we define a morphism:

$$\begin{aligned}
\phi \colon \mathcal{K}^p(v) &\to B^{BM}_{n-1}(X, \mathcal{F}_{n-p}), \\
y &\mapsto \partial_n(\psi(b)),
\end{aligned}$$

for $b \in C^n_c(X, \mathcal{F}^p)$ some element such that $i^*(b) = y$. By Lemma B.1.1, this map makes both squares of the diagram commutative. ∎

Next we will use a consequence of the snake lemma to determine when $\cap[X]$ and ϕ are isomorphisms.

4.3 Equivalence of isomorphisms

Theorem 4.3.1. *Let X be a pure balanced rational polyhedral fan. Then the cap with the fundamental class $\cap[X] \colon H^{p,q}(X) \to H^{BM}_{n-p,n-q}(X)$ is injective.*

Proof. By using Proposition B.2.1, for a pure dimensional balanced polyhedral fan X, the cap map $\cap[X]$ is always injective, the morphism ϕ is always surjective, and if one is an isomorphism, so is the other. ∎

Using this theorem, it is sufficient to find conditions on X so that ϕ is injective to determine when $\cap[X] \colon H^{p,0}_c(X) \to H^{BM}_n(X, \mathcal{F}_{n-p})$ is an isomorphism. With this goal in mind, we should investigate ϕ. Choose $y \in \mathcal{K}^p(v)$ and $b = (b_\sigma)_{\sigma \in X^n} \in C^n_c(X, \mathcal{F}^p)$ such that $i^*(b) = y$. Then we can write out $\phi(y)$:

$$\begin{aligned}
\phi(y) &= \partial_n(\psi(b)), \\
&= \partial_n\left((\omega_\sigma \langle b_\sigma; \Lambda_\sigma\rangle)_{\sigma \in X^n}\right), \\
&= \left(\sum_{\tau \leq \sigma} \mathcal{O}(\tau, \sigma)\omega_\sigma \langle b_\sigma; \Lambda_\sigma\rangle\right)_{\tau \in X^{n-1}}.
\end{aligned}$$

Now we wish to see what condition on X would give that ϕ is injective. Suppose

$$\begin{aligned}
0 &= \psi(y), \\
&= \left(\sum_{\tau \leq \sigma} \mathcal{O}(\tau, \sigma)\omega_\sigma \langle b_\sigma; \Lambda_\sigma\rangle\right)_{\tau \in X^{n-1}}.
\end{aligned}$$

We would wish for this to mean that $y = 0$, so that $b \in \ker(i^*) = \operatorname{im}(s^*)$, i.e. that $b_\sigma = a \circ s$ for some $a \in \mathcal{F}^p(v)$. This motivates the following definition:

Definition 4.3.2. Given a weighted polyhedral fan X of dimension n, we say that X is *uniquely p-balanced* if, given $b = (b_\sigma)_{\sigma \in X^n} \in C_c^n(X, \mathcal{F}^p)$,

$$\sum_{\tau \leq \sigma} \mathcal{O}(\tau, \sigma)\omega_\sigma \langle b_\sigma; \Lambda_\sigma \rangle = 0$$

for all $\tau \in X^{n-1}$, where ω_σ is the weight of the σ face, if and only if there is an $a \in \mathcal{F}^p(v)$ such that $b_\sigma = (s^*(a))_\sigma$ for all $\sigma \in X^n$.

Remark 4.3.3. By Proposition 3.3.6, given that the fan X is balanced, Definition 4.3.2 generalizes the definition of X being uniquely balanced (Definition 3.2.11) to higher codimensions. Indeed suppose X is uniquely balanced, let $p = 0$, and pick $b \in C_c^{0,n}(X) = \oplus_{\sigma \in X^n} \mathbb{R}$, so that $b_\sigma \in \mathbb{R}$ for all σ. Then,

$$\sum_{\tau \leq \sigma} \mathcal{O}(\tau, \sigma)\omega_\sigma \langle b_\sigma; \Lambda_\sigma \rangle = \sum_{\tau \leq \sigma} \mathcal{O}(\tau, \sigma)\omega_\sigma b_\sigma \Lambda_\sigma = 0$$

holds if and only if all the b_σ are the same, since otherwise $\omega'_\sigma = b_\sigma \omega_\sigma$ would be other weights balancing the fan, contradicting unique balancing. But then taking $a = b_\sigma$ for some σ, one has $b = (b_\sigma)_{\sigma \in X^n} = s^*(a)$. Hence X is 0-balanced. Moreover, by the same argument, we see that if X is uniquely 0-balanced, it is uniquely balanced.

Theorem 4.3.4. *The map ϕ is injective if and only if X is uniquely p-balanced.*

Proof. Suppose ϕ is not injective, so that we can find some $y \in \mathcal{K}^p(v)$ with $y \neq 0$, such that $\phi(y) = 0$. Since i^* is surjective, there is a $b \in C_c^n(X, \mathcal{F}^p)$, with $i^*(b) = y$, hence $b \notin \ker(i^*) = \operatorname{im}(s^*)$ such that

$$\begin{aligned} 0 &= \psi(y), \\ &= \partial_n(\phi(b)), \\ &= \left(\sum_{\tau \leq \sigma} \mathcal{O}(\tau, \sigma)\omega_\sigma \langle b_\sigma; \Lambda_\sigma \rangle \right)_{\tau \in X^{n-1}}. \end{aligned}$$

Since b is not in the image of s^*, there is no $a \in \mathcal{F}^p(v)$ such that $b = s^*(a)$, thus X is not uniquely p-balanced.

Next suppose X is not uniquely p-balanced, such that we can find some $b \in C_c^n(X, \mathcal{F}^p)$ with

$$\sum_{\tau \leq \sigma} \mathcal{O}(\tau, \sigma)\omega_\sigma \langle b_\sigma; \Lambda_\sigma \rangle = 0$$

for all faces $\tau \in X^{n-1}$, yet there is no $a \in \mathcal{F}^p(v)$ such that $b = s^*(a)$. Then since $\partial_n(\psi(b)) = 0$ hence $\phi(b) = 0$, so that $i^*(b) \in \ker(\phi)$, but $b \notin \operatorname{im}(s^*) = \ker(i^*)$. Therefore $i^*(b) \in \mathcal{K}^p(v)$ is not 0, hence ϕ is not injective. ■

Theorem 4.3.5. *Let X be a pure balanced rational polyhedral fan of dimension n in $\mathbb{R}^N$. The cap morphism $\cap[X] : H^{p,0}(X) \to H_n^{BM}(X, \mathcal{F}_{n-p})$ is an isomorphism if and only if X is uniquely p-balanced.*

Proof. This follows from Proposition B.2.1 and Theorem 4.3.4. ■

Corollary 4.3.6. *The Bergman fan of a matroid is uniquely p-balanced for all p.*

Proof. Since Bergman fans are pure balanced rational polyhedral fans, this follows from Theorem 4.3.5. ■

4.4 The dependence relation cosheaf

Next, we seek to establish when $H_q^{BM}(X, \mathcal{F}_p) = 0$ for $q = 0, \ldots, n-1$ and all p. Observe that we can resolve the $\mathcal{F}_p$ cosheaf using elementary projective cosheaves concentrated on maximal cells . Indeed, let

$$\mathcal{S}_p \coloneqq \bigoplus_{\sigma \text{ maximal}} [\hat{\sigma}]^{\mathcal{F}^p(\sigma)}.$$

Then the sequence

$$0 \longrightarrow \mathcal{K}_p \longrightarrow \mathcal{S}_p \xrightarrow{s} \mathcal{F}_p \longrightarrow 0$$

is exact, where s is the summation map from Equation (4.1), and $\mathcal{K}_p$ is the kernel cosheaf.

Definition 4.4.1. The cosheaf $\mathcal{K}_p$ as defined above is called the *dependence relation cosheaf.*

The exact sequence induces a long exact sequence in homology:

$$\begin{array}{l} 0 \longrightarrow H_n^{BM}(X, \mathcal{K}_p) \longrightarrow H_n^{BM}(X, \mathcal{S}_p) \longrightarrow H_n^{BM}(X, \mathcal{F}_p) \\ \hookrightarrow H_{n-1}^{BM}(X, \mathcal{K}_p) \longrightarrow \cdots \longrightarrow H_1^{BM}(X, \mathcal{F}_p) \\ \hookrightarrow H_0^{BM}(X, \mathcal{K}_p) \longrightarrow H_0^{BM}(X, \mathcal{S}_p) \longrightarrow H_0^{BM}(X, \mathcal{F}_p) \longrightarrow 0. \end{array}$$

Next, we have the following:

Proposition 4.4.2. *Given a polyhedral fan X of dimension n, we have*

$$H_q^{BM}(X, \mathcal{S}_p) = 0,$$

for $q = 0, \ldots, n$ and $p = 0, \ldots, n$.

Proof. Note that we have already proven this statement in greater generality in Example 2.5.12. ■

Alternatively, one can prove the proposition as follows:

Proof. The linear dual sheaf $\mathcal{S}^p := V(\mathcal{S}_p)$ of $\mathcal{S}_p$ (Definition 2.3.1), is an injective sheaf by Example 2.4.9. Since injective sheaves are flabby [Har77, Lemma III.2.4], hence soft, and finite polyhedral complexes with Alexandrov topology are compact, we can use [Ive86, Theorem III.2.7] to give:

$$H^i_c(X, \mathcal{S}^p) = 0, \quad \text{for } i \geq 1.$$

By applying Proposition 2.5.9, this gives:

$$H_i^{BM}(X, \mathcal{S}_p)^* \cong H^i_c(X, \mathcal{S}^p) = 0, \quad \text{for } i \geq 1.$$

Moreover, $H_0^{BM}(X, \mathcal{S}_p) = 0$ for any p. Looking at the last terms of the Borel–Moore homology of $\mathcal{S}_p$, we have:

$$\cdots \longrightarrow C_1^{BM}(X, \mathcal{S}_p) \xrightarrow{\partial_1} C_0^{BM}(X, \mathcal{S}_p) \longrightarrow 0,$$

which is just:

$$\cdots \longrightarrow \bigoplus_{\tau \in X^1} \bigoplus_{\substack{\sigma \geq \tau \\ \sigma \text{ maximal}}} \bigwedge^p L(\sigma) \xrightarrow{\partial_1} \bigoplus_{\sigma \text{ maximal}} \bigwedge^p L(\sigma) \longrightarrow 0.$$

For any maximal by inclusion cell σ of the fan, there is at least one edge $\tau \in X^1$ such that $\tau \leq \sigma$, hence the term $\bigwedge^p L(\sigma)$ appears at least once in the degree 1 term of the complex. Hence ∂_1 is surjective and so $H_0^{BM}(X, \mathcal{S}_p) = 0$. ■

Corollary 4.4.3. *Let X be a polyhedral fan of dimension n. Then*

$$H_{q-1}^{BM}(X, \mathcal{K}_p) \cong H_q^{BM}(X, \mathcal{F}_p)$$

for $q = 1, \ldots, n$ and $p = 0, \ldots, n$, and

$$H_n^{BM}(X, \mathcal{K}_p) = 0 = H_0^{BM}(X, \mathcal{F}_p)$$

Proof. This follows from Proposition 4.4.2, since $H_q^{BM}(X, \mathcal{S}_p) = 0$, for $q = 0, \ldots, n$, hence all the connecting homomorphisms must be isomorphisms, for $p = 0, \ldots, n$. Moreover, from the long exact sequence, we see that $H_n^{BM}(X, \mathcal{S}_p) = 0$ gives $H_n^{BM}(X, \mathcal{K}_p) = 0$, and $H_0^{BM}(X, \mathcal{S}_p) = 0$ gives $H_0^{BM}(X, \mathcal{F}_p) = 0$. ■

4.5 A classification theorem

We can use unique p-balancing and Section 4.4 to give a complete classification of pure fans which are tropical Poincaré spaces:

Theorem 4.5.1. *A rational balanced polyhedral fan of pure dimension n is a tropical Poincaré space, i.e.*

$$\frown [X] \colon H^q(X, \mathcal{F}^p) \to H_{n-q}^{BM}(X, \mathcal{F}_{n-p})$$

is an isomorphism for all $p, q = 0, \ldots, n$, if and only if

1. *X is uniquely p-balanced for all p, and*

2. *the dependence cosheaf* $\mathcal{K}_p$ *is acyclic in Borel–Moore homology in degrees other than* $n-1$*, that is,* $H_q^{BM}(X, \mathcal{K}_p) = 0$ *for* $q \neq n-1$*, for all* p*.*

Proof. By Proposition 2.5.6, we have that $H^q(X, \mathcal{F}^p) = 0$ for $q > 0$, for all p. Hence

$$\cap[X] \colon H^q(X, \mathcal{F}^p) \to H^{BM}_{n-q}(X, \mathcal{F}_{n-p})$$

is an isomorphism for $q > 0$ if and only if $H^{BM}_{n-q}(X, \mathcal{F}_{n-p}) = 0$ for $q > 0$. By Corollary 4.4.3, this is equivalent to $H^{BM}_{q-1}(X, \mathcal{K}_p) = 0$ for $q < n$.

Next, we need to determine when $\cap[X] : H^{p,0}(X) \to H^{BM}_n(X, \mathcal{F}_{n-p})$ is an isomorphism. By Theorem 4.3.5, for a pure dimensional fan, this is equivalent to X being uniquely p-balanced. ■

One might hope that the second condition guarantees that X is pure, making the criteria complete, however this is not the case:

Example 4.5.2. The following fan can be thought of as the Bergman fan of the uniform matroid U_4^6, placed in a hyperplane of $\mathbb{R}^8$, to which three rays have been appended in the orthogonal complement of the hyperplane. A pathological configuration between these three rays and the Bergman fan is then constructed. Concretely, one can read of this construction from the following the cellular sheaves package [KSW17] for [polymake], which we also use to compute the homology of the fan:

```
application "fan";
$fan = new fan::PolyhedralFan(INPUT_RAYS=>[
[1,0,0,0,0,0,0,0,0],
[0,-1,0,0,0,0,0,0,0],
[0,0,-1,0,0,0,0,0,0],
[0,0,0,-1,0,0,0,0,0],
[0,0,0,0,-1,0,0,0,0],
[0,0,0,0,0,-1,0,0,0],
[0, 1,1,1,1,1,0,0,0],
[0, 0,0,0,0,0,1,0,0],
[0, 0,0,0,0,0,0,1,0],
[0, 0,0,0,0,0,0,0,1]
],
INPUT_CONES =>
[
[6,1,2,0],[1,2,3,0],[6,2,3,0],[6,1,3,0],[6,1,4,0],[1,3,4,0],[6,3,4,0],
[6,1,5,0],[1,3,5,0],[6,3,5,0],[6,2,4,0],[2,3,4,0],[6,2,5,0],[2,3,5,0],
[6,4,5,0],[3,4,5,0],[1,2,4,0],[1,2,5,0],[1,4,5,0],[2,4,5,0],
[0,7,8],
[0,7,9],
[0,1,7,8],
]

);

$complex = new fan::PolyhedralComplex($fan);

@betti_usual = ();
@betti_borel_moore = ();
for(my $i=0; $i<5; $i++){
   my $f = $complex->fcosheaf($i);
   my $u = $complex->usual_chain_complex($f);
   my $bm = $complex->borel_moore_complex($f);
   push @betti_usual, $u->BETTI_NUMBERS;
   push @betti_borel_moore, $bm->BETTI_NUMBERS;
```

```
}

print new Matrix(@betti_usual);
print new Matrix(@betti_borel_moore);
```

The Betti numbers are then output as follows:

```
fan > print new Matrix(@betti_usual);
1 0 0 0
8 0 0 0
14 0 0 0
11 0 0 0
0 0 0 0

fan > print new Matrix(@betti_borel_moore);
0 0 0 10
0 0 0 10
0 0 0 5
0 0 0 1
0 0 0 0
```

Which means that this fan is not a tropical Poincaré space, and as can be seen from its input cones, it is not pure, despite having $H_q^{BM}(X, \mathcal{F}_p) = 0$, i.e. $H_{q-1}^{BM}(X, \mathcal{K}_p) = 0$, for $q < n$.

4.6 Euler characteristic conditions

Using the above theorem Theorem 4.5.1, one can also see when $\cap[X]$ is an isomorphism using the Euler characteristic.

Corollary 4.6.1. *Let X be a polyhedral fan of pure dimension n, with the cosheaf $\mathcal{K}_p$ acyclic in all degrees except $n-1$ for all p. Then $\cap[X]$ is an isomorphism if and only if*

$$(-1)^n \chi(C_\bullet^{BM}(X, \mathcal{F}_{n-p})) = \dim \mathcal{F}^p(v).$$

Proof. When $\mathcal{K}_{n-p}$ is acyclic, by Corollary 4.4.3, we have $0 = H_{q-1}^{BM}(X, \mathcal{K}_{n-p}) \cong H_q^{BM}(X, \mathcal{F}_{n-p})$ for $q \neq n$, hence the Euler characteristic is given by:

$$\chi(C_\bullet^{BM}(X, \mathcal{F}_{n-p})) = (-1)^n H_q^{BM}(X, \mathcal{F}_{n-p}).$$

Moreover, since X is pure, the cap with the fundamental class $\cap[X]$ is injective, hence an isomorphism if and only if $\dim \mathcal{F}^p(v) = \dim H_q^{BM}(X, \mathcal{F}_{n-p})$. We can now use the above equation to see that $\cap[X]$ is an isomorphism if and only if:

$$(-1)^n \chi(C_\bullet^{BM}(X, \mathcal{F}_{n-p})) = \dim \mathcal{F}^p(v).$$

■

Moreover, given a polyhedral fan X of dimension n, one can consider what happens in the case where every tangent fan $T_\tau X$ of a face $\tau \in X$ is a tropical Poincaré space.

$$(-1)^n \dim \mathcal{F}^p(\tau) = \chi(C_\bullet^{BM}(T_\tau X, \mathcal{F}_{n-p})) := \sum_{\tau \leq \sigma} (-1)^{\dim \sigma} \mathcal{F}_{n-p}(\sigma).$$

In this case, there is some form of duality relating the cap product $\cap[X]$ for p to the $\cap[X]$ for $n-p$:

Corollary 4.6.2. *Let X be a balanced polyhedral fan of pure dimension n. Suppose for each $\tau \in X$, with $\tau \neq v$, the tangent fan $T_\tau X$ is a tropical Poincaré space. Then $\cap[X]\colon H^q(X, \mathcal{F}^p) \to H^{BM}_{n-q}(X, \mathcal{F}_{n-p})$ is an isomorphism if and only if $i\cap[X]\colon H^q(X, \mathcal{F}^{n-p}) \to H^{BM}_{n-q}(X, \mathcal{F}_p)$ is an isomorphism.*

Proof. We know that X is a tropical Poincaré space if and only if:

$$\chi(C^{BM}_\bullet(X, \mathcal{F}_{n-p})) = (-1)^n \dim \mathcal{F}^p(v).$$

Using our assumptions on the tangent fans, we have that the left hand side of this equation is:

$$\begin{aligned}
\chi(C^{BM}_\bullet(X, \mathcal{F}_{n-p})) &= \sum_{\sigma \in X} (-1)^{\dim \sigma} \dim \mathcal{F}_{n-p}(\sigma) \\
&= \dim \mathcal{F}_{n-p}(v) + \sum_{\tau \neq v} (-1)^n \sum_{\tau \leq \sigma} (-1)^{\dim \sigma} \dim \mathcal{F}_p(\sigma) \\
&= \dim \mathcal{F}_{n-p}(v) + (-1)^n \sum_{\sigma \neq v} (-1)^{\dim \sigma} \dim \mathcal{F}_p(\sigma) \sum_{\substack{\tau \neq v \\ \tau \leq \sigma}} (-1)^{\dim \tau}
\end{aligned}$$

Now note that, for any given σ, one can compute the Euler characteristic of the polytope over which σ is a cone using $\sum_{\substack{\tau \neq v \\ \tau \leq \sigma}} (-1)^{\dim \tau - 1}$, which gives:

$$\sum_{\substack{\tau \neq v \\ \tau \leq \sigma}} (-1)^{\dim \tau} = -1.$$

This can then be used further to give:

$$\chi(C^{BM}_\bullet(X, \mathcal{F}_{n-p})) = \dim \mathcal{F}_{n-p}(v) + (-1)^n \sum_{\sigma \neq v} (-1)^{\dim \sigma + 1} \dim \mathcal{F}_p(\sigma).$$

Therefore,

$$(-1)^n \chi(C^{BM}_\bullet(X, \mathcal{F}_{n-p})) = \dim \mathcal{F}^p(v)$$

if and only if

$$(-1)^n \chi(C^{BM}_\bullet(X, \mathcal{F}_p)) = \dim \mathcal{F}^{n-p}(v),$$

i.e. by Corollary 4.6.1, X is a tropical Poincaré space for p if and only if it is for $n - p$. ■

CHAPTER 5

Classification of tropical Poincaré fans of dimension 2

In this chapter, we classify which polyhedral fans of dimension 2 are tropical Poincaré spaces, i.e. have the property

$$H^q(X, \mathcal{F}^p) \cong H^{BM}_{2-q}(X, \mathcal{F}_{2-p}),$$

for $q = 0, 1, 2$ and $p = 0, 1, 2$, where the isomorphism is given by the cap with the fundamental class $\cap[X]$. We first prove that any such fan is necessarily of pure dimension. Then we work with the dependence relation cosheaf $\mathcal{K}_p$ from Section 4.4 to obtain criteria for when $H^{BM}_q(X, \mathcal{F}_p) = 0$ for $q < 2$, which is necessary since $H^q(X, \mathcal{F}^p) = 0$ for $q > 0$ by Proposition 2.5.6. Next we utilize these relations along with the unique p-balancing from Definition 4.3.2 to determine when $H^0(X, \mathcal{F}^p) \cong H^{BM}_2(X, \mathcal{F}_{2-p})$, which completes the classification. The classification result can be stated as follows:

Theorem 5.0.1 (Theorem 5.6.1)**.** *Let X be a rational 2-dimensional polyhedral fan. The cap product with the fundamental class $\cap[X]\colon H^q(X, \mathcal{F}^p) \to H^{BM}_{2-q}(X, \mathcal{F}_{2-p})$ is an isomorphism if and only if*

1. *X is pure,*
2. *the boundary map $\partial_1\colon \oplus_{\tau \in X^1} \mathcal{K}_p(\tau) \to \mathcal{K}_p(v)$ is surjective,*
3. *X is uniquely balanced, and*
4. *X is uniquely balanced at each edge.*

5.1 Tropical Poincaré fans of dimension 2 are pure

We first observe that purity (see Definition 2.1.2) is necessary:

Proposition 5.1.1. *If a polyhedral fan of dimension 2 is a tropical Poincaré space, it is pure dimensional.*

Proof. Let X be a 2-dimensional polyhedral fan, and suppose X is not pure. Then there is a ray $\tau \in X$ which is maximal by inclusion of cells. Take any other ray τ' in X. Then consider the element $x = (x_\gamma)_{\gamma \in X^1} \in C_1^{BM}(X, \mathcal{F}_0)$ given componentwise:

$$x_\gamma = \begin{cases} 1 & \text{if } \gamma = \tau', \\ -1 & \text{if } \gamma = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\partial(x) = 1 - 1 = 0$, hence $[x] \in H_1^{BM}(X, \mathcal{F}_0)$, and since τ is maximal by inclusion, there is no $y \in C_2^{BM}(X, \mathcal{F}_0)$ such that $\partial(y) = x$, hence $[x] \neq 0 \in H_1^{BM}(X, \mathcal{F}_0)$. Therefore $\dim\big(H_1^{BM}(X, \mathcal{F}_0)\big) > 0 = \dim\big(H^1(X, \mathcal{F}^2)\big)$, so the fan cannot be a tropical Poincaré space. ■

Since purity is necessary, Theorem 4.3.5 can be used to settle part of the requirements on X:

Proposition 5.1.2. *Let X be a balanced rational polyhedral fan of dimension 2 in $\mathbb{R}^N$. For any p, the cap morphism $\cap[X] : H^0(X, \mathcal{F}^p) \to H_2^{BM}(X, \mathcal{F}_{2-p})$ is an isomorphism if and only if X is uniquely p-balanced.*

Proof. In Theorem 4.3.5, the equivalence stated holds only for pure fans. Since purity is necessary in the 2-dimensional case by Proposition 5.1.1, the theorem provides the equivalence in all cases. ■

5.2 The dependence relation cosheaf

We saw in Corollary 4.4.3 that we can use the dependence relation cosheaf $\mathcal{K}_p$ to determine when $H_q^{BM}(X, \mathcal{F}_p) = 0$ for $q < \dim(X)$. Briefly recapitulating, we have:

$$\mathcal{S}_p \coloneqq \bigoplus_{\sigma \text{ maximal}} [\hat{\sigma}]^{\mathcal{F}^p(\sigma)}.$$

Then the sequence

$$0 \longrightarrow \mathcal{K}_p \longrightarrow \mathcal{S}_p \longrightarrow \mathcal{F}_p \longrightarrow 0$$

is exact, where $\mathcal{K}_p$ is the kernel cosheaf, called the dependence relation cosheaf. This induces a long exact sequence in homology:

$$\begin{array}{l} 0 \longrightarrow H_2^{BM}(X, \mathcal{K}_p) \longrightarrow H_2^{BM}(X, \mathcal{S}_p) \longrightarrow H_2^{BM}(X, \mathcal{F}_p) \longrightarrow \\ \quad \longrightarrow H_1^{BM}(X, \mathcal{K}_p) \longrightarrow H_1^{BM}(X, \mathcal{S}_p) \longrightarrow H_1^{BM}(X, \mathcal{F}_p) \longrightarrow \\ \quad \longrightarrow H_0^{BM}(X, \mathcal{K}_p) \longrightarrow H_0^{BM}(X, \mathcal{S}_p) \longrightarrow H_0^{BM}(X, \mathcal{F}_p) \longrightarrow 0. \end{array}$$

We then have the following:

Corollary 5.2.1. *For a polyhedral fan X of dimension 2, we have*

$$H_2^{BM}(X, \mathcal{F}_p) \cong H_1^{BM}(X, \mathcal{K}_p)$$
$$H_1^{BM}(X, \mathcal{F}_p) \cong H_0^{BM}(X, \mathcal{K}_p)$$

and $H_0^{BM}(X, \mathcal{F}_p) = 0 = H_2^{BM}(X, \mathcal{K}_p)$.

Proof. This is just Corollary 4.4.3 in the 2-dimensional case. ■

Therefore, for a 2-dimensional fan, to determine when $H_q^{BM}(X, \mathcal{F}_p) = 0$ for $q = 0, 1$ and $p = 0, 1, 2$, it suffices to determine when $H_0^{BM}(X, \mathcal{K}_p) = 0$:

Theorem 5.2.2. *Let X be a 2-dimensional polyhedral fan. Then $H_q^{BM}(X, \mathcal{F}_p) = 0$ for $q = 0, 1$ if and only if the boundary map*

$$\partial_1 \colon \bigoplus_{\tau \in X^1} \mathcal{K}_p(\tau) \to \mathcal{K}_p(v)$$

is surjective, for each $p = 0, 1, 2$.

Proof. By Corollary 5.2.1, we have that $H_0^{BM}(X, \mathcal{F}_p) = 0$ for any p, and that $H_1^{BM}(X, \mathcal{F}_p) \cong H_0^{BM}(X, \mathcal{K}_p)$. Now $H_0^{BM}(X, \mathcal{K}_p) = 0$ if and only if the boundary map

$$\partial_1 \colon \bigoplus_{\tau \in X^1} \mathcal{K}_p(\tau) \to K_p(v)$$

is surjective. Hence $H_1^{BM}(X, \mathcal{F}_p) = 0$ if and only if ∂_1 is surjective. ■

5.3 Algebraic classification theorem

Using the two previous sections, we can give an "algebraic" classification of 2-dimensional polyhedral fans which are tropical Poincaré spaces:

Corollary 5.3.1. *Let X be a rational 2-dimensional polyhedral fan. The cap product with the fundamental class $\frown[X] \colon H^q(X, \mathcal{F}^p) \to H_{2-q}^{BM}(X, \mathcal{F}_{2-p})$ is an isomorphism if and only if*

1. *X is pure,*
2. *the boundary map $\partial_1 \colon \oplus_{\tau \in X^1} \mathcal{K}_p(\tau) \to \mathcal{K}_p(v)$ is surjective,*
3. *X is uniquely p-balanced for $p = 0, 1, 2$.*

Proof. This follows from applying Proposition 5.1.2 and Theorem 5.2.2. ■

This theorem provides a classification, but it is not straightforward to understand the geometry of such fans from the provided criteria. We now seek to clarify the geometric meaning of unique p-balancing. We have already seen in Remark 4.3.3 that unique 0-balancing is equivalent to the fan being uniquely balanced Definition 3.2.11. It remains to clarify unique 1- and 2-balancing for the fans of Corollary 5.3.1.

5.4 Characterizing unique 1-balancing

Proposition 5.4.1. *Let X be a rational balanced polyhedral fan of pure dimension 2, such that the boundary map $\partial_1 \colon \oplus_{\tau \in X^1} \mathcal{K}_1(\tau) \to \mathcal{K}_1(v)$ is surjective. Then the map $\frown[X] \colon H^0(X, \mathcal{F}^1) \to H_2^{BM}(X, \mathcal{F}_1)$ is an isomorphism if and only if X is uniquely balanced along each edge $\tau \in X^1$.*

Proof. By Theorem 5.2.2, since $\partial_1 \colon \oplus_{\tau \in X^1} \mathcal{K}_1(\tau) \to \mathcal{K}_1(v)$ is surjective, we have that $H_q^{BM}(X, \mathcal{F}_1) = 0$ for $q = 0, 1$. We can again work with the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^1(v) \arrow[r, "s^*"] \arrow[d, "\cap[X]"] & C_c^2(X, \mathcal{F}^1) \arrow[r, "i^*"] \arrow[d, "\psi"] & \mathcal{K}^1(v) \arrow[r] \arrow[d, "\phi"] & 0, \\
0 \arrow[r] & H_2^{BM}(X, \mathcal{F}_1) \arrow[r, "j"] & C_2^{BM}(X, \mathcal{F}_1) \arrow[r, "\partial_2"] & B_1^{BM}(X, \mathcal{F}_1) \arrow[r] & 0.
\end{tikzcd}$$

Working with the Euler characteristic of the complex $C_{1,\bullet}^{BM}(X)$, we can use that $H_q^{BM}(X, \mathcal{F}_1) = 0$ for $q = 0, 1$ to get:

$$\begin{aligned}
\chi\big(C_\bullet^{BM}(X, \mathcal{F}_1)\big) &= \sum_{i=0}^{2} (-1)^i \dim \big(C_i^{BM}(X, \mathcal{F}_1)\big), \\
\sum_{i=0}^{2} (-1)^i \dim \big(H_i^{BM}(X, \mathcal{F}_1)\big) &= \sum_{i=0}^{2} (-1)^i \sum_{\sigma \in X^i} \dim \big(\mathcal{F}_1(\sigma)\big), \\
\dim \big(H_2^{BM}(X, \mathcal{F}_1)\big) &= 2f_2 - \sum_{\tau \in X^1} \dim \big(\mathcal{F}_1(\tau)\big) + \dim \big(\mathcal{F}_1(v)\big).
\end{aligned}$$

Since $\cap[X]$ is an isomorphism if and only if

$$\begin{aligned}
\dim(\mathcal{F}^1(v)) &= \dim \big(H_2^{BM}(X, \mathcal{F}_1)\big), \\
&= 2f_2 - \sum_{\tau \in X^1} \dim \big(\mathcal{F}_1(\tau)\big) + \dim \big(\mathcal{F}_1(v)\big),
\end{aligned}$$

we must have:

$$2f_2 = \sum_{\tau \in X^1} \dim(\mathcal{F}_1(\tau)).$$

We claim that $\dim(\mathcal{F}_1(\tau)) = \mathrm{val}(\tau) + 1 - \beta_\tau$ for each τ, where $\mathrm{val}(\tau)$ is the number of faces containing τ, and β_τ is the number of ways to balance the fan tangent fan $T_\tau X$ at τ.

Indeed, suppose we add a face containing τ to the fan. If this face increases the dimension of $\dim(\mathcal{F}_1(\tau))$, then this new face could not have been obtained from a balancing of the others, hence β_τ is unchanged. Therefore the increase in dimension is accounted for by $\mathrm{val}(\tau)$. If however this new face does not increase the dimension, then it must be dependent on the other faces, which gives a new way to balance the fan, hence the positive contribution from $\mathrm{val}(\tau)$ is negated by the increase of β_τ. Moreover, since $L(\tau) \subset \mathcal{F}_1(\tau)$, we always have at least one dimension of freedom along the face τ.

Putting this formula to use, and noting that by counting the number of faces meeting at each edge, we count each face twice, so that $2f_2 = \sum_{\sigma \in X^1} \mathrm{val}(\sigma)$, we get the following

$$\begin{aligned} 2f_2 &= \sum_{\sigma\in X^1} \dim(\mathcal{F}_1(\sigma)), \\ 2f_2 &= \sum_{\sigma\in X^1} (\mathrm{val}(\sigma)+1-\beta_\sigma), \\ 2f_2 - \sum_{\sigma\in X^1} \mathrm{val}(\sigma) &= f_1 - \sum_{\sigma\in X^1} \beta_\sigma, \\ \sum_{\sigma\in X^1} \beta_\sigma &= f_1. \end{aligned}$$

Now, since the fan is balanced, so that there is at least one way to balance each tangent fan $T_\tau X$, this equality holds if and only if there is exactly one way to balance each edge of the fan, which is what we wanted to show. ■

5.5 Characterizing unique 2-balancing

Proposition 5.5.1. *Let X be a rational balanced polyhedral fan of pure dimension 2, such that the boundary map $\partial_1 \colon \oplus_{\tau\in X^1}\mathcal{K}_2(\tau) \to K_2(v)$ is surjective. Then $\cap[X]\colon H^{2,0}(X) \to H^{BM}_{0,2}(X)$ is an isomorphism if and only if X is uniquely balanced along each edge.*

Proof. By Theorem 5.2.2, the surjectivity of ∂_1 on X is equivalent to $H_q^{BM}(X,\mathcal{F}_2) = 0$ for $q = 0, 1$, so that we can yet again work with the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}^2(v) \arrow[r, "s^*"] \arrow[d, "\cap[X]"] & C_c^2(X,\mathcal{F}^2) \arrow[r, "i^*"] \arrow[d, "\psi"] & \mathcal{K}^2(v) \arrow[r] \arrow[d, "\phi"] & 0, \\ 0 \arrow[r] & H_2^{BM}(X,\mathcal{F}_0) \arrow[r, "j"] & C_2^{BM}(X,\mathcal{F}_0) \arrow[r, "\partial_2"] & B_1^{BM}(X,\mathcal{F}_0) \arrow[r] & 0. \end{tikzcd}$$

Working with the Euler characteristic of the complex $C_\bullet^{BM}(X,\mathcal{F}_0)$, we can use that $H_q^{BM}(X,\mathcal{F}_0) = 0$ for $q = 0, 1$ to get:

$$\begin{aligned} \chi\big(C_\bullet^{BM}(X,\mathcal{F}_0)\big) &= \sum_{i=0}^{2}(-1)^i \dim\big(C_i^{BM}(X,\mathcal{F}_0)\big), \\ \sum_{i=0}^{2}(-1)^i \dim\big(H_i^{BM}(X,\mathcal{F}_0)\big) &= \sum_{i=0}^{2}(-1)^i \sum_{\sigma\in X^i} \dim\big(\mathcal{F}_0(\sigma)\big), \\ \dim\big(H_2^{BM}(X,\mathcal{F}_0)\big) &= f_2 - f_1 + f_0 = f_2 - f_1 + 1. \end{aligned}$$

We have that $\cap[X]$ is an isomorphism if and only if

$$\begin{aligned} \dim(\mathcal{F}^2(v)) &= \dim\big(H_2^{BM}(X,\mathcal{F}_0)\big), \\ &= f_2 - f_1 + 1, \end{aligned}$$

and since $\dim(\mathcal{F}^2(v)) = \dim\big(C_c^2(X,\mathcal{F}^2)\big) - \dim(\mathcal{K}^2(v))$, the isomorphism is equivalent to $\dim(\mathcal{K}^2(v)) = f_1 - 1$.

Since $\phi\colon \mathcal{K}^2(v) \to B_1^{BM}(X, \mathcal{F}_0)$ is a surjection, and $\dim B_1^{BM}(X, \mathcal{F}_0) = \dim C_2^{BM}(X, \mathcal{F}^0) - \dim H_2^{BM}(X, \mathcal{F}^0) = f_1 - 1$, we have that:

$$\dim(\mathcal{K}^2(v)) \geq f_1 - 1.$$

Considering the dual of the first row of the diagram,

$$0 \longrightarrow \mathcal{K}_2(v) \xrightarrow{\ i\ } C_2^{BM}(X, \mathcal{F}_2) \xrightarrow{\ s\ } \mathcal{F}_2(v) \longrightarrow 0,$$

we note that, since X is a balanced fan, for each codimension 1 face $\tau \in X^1$, we have $\sum_{\tau \leq \sigma} \omega_\sigma \Lambda_\sigma = 0$, which gives at least one subgroup $\langle \beta(\tau) \rangle := (\omega_\sigma \Lambda_\sigma)_{\sigma \geq \tau} \subset \mathcal{K}_2(v)$. Furthermore, if we assume all the edges $\tau \in X^1$ are such that $\mathcal{O}(v, \tau)$, we can fix a given Λ_σ for each face $\sigma \in X^2$, so that for the two edges $\tau, \tau' \leq \sigma$, we have $\mathcal{O}(\tau, \sigma) + \mathcal{O}(\tau', \sigma) = 0$. This then gives a relation among all the $\beta(\tau)$, specifically $\sum_{\tau \in X^1} \beta(\tau) = 0$.

Moreover, this is the only possible relation among the $\beta(\tau)$. Indeed consider the map

$$\begin{aligned} \langle \beta(\tau) \rangle_{\tau \in X^1} &\to C_2^{BM}(X, \mathcal{F}_2), \\ (\alpha_\tau \beta(\tau))_\tau &\mapsto \left(\sum_{\tau \in X^1} \alpha_\tau \mathcal{O}(\tau, \sigma) \omega_\sigma \Lambda_\sigma \right)_{\sigma \in X^2}, \end{aligned}$$

where we note that a summand is 0 on all components where $\sigma \not\geq \tau$. Any relations among the $\beta(\tau)$ would then appear as the kernel of this map. However, since X is 2-dimensional, there are only two $\tau, \tau' \leq \sigma$ for any given σ, hence for any given component to be 0, we must have $\alpha_\tau = \alpha_{\tau'}$. Hence the only relation among the $\beta(\tau)$ is $\sum_{\tau \in X^1} \beta(\tau) = 0$.

Furthermore, note that there is a one-to-one correspondence between dependent sets of generators for the wedges of the faces and elements of $\mathcal{K}^2(v)$. Indeed let $v^* \in \mathcal{K}^2(v)$. Then by dualizing, v^* corresponds to an element $v \in \mathcal{K}_2(v)$. However $v \in \mathcal{K}_2(v)$ if and only if $i(v) := (v_\sigma \Lambda_\sigma)_{\sigma \in X^2} \in C_2^{BM}(X, \mathcal{F}_2)$ is an element of the kernel of the summation map $s\colon C_2^{BM}(X, \mathcal{F}_2) \to \mathcal{F}_2(v)$. This happens if and only if $\sum_{\sigma \in X^2} v_\sigma \Lambda_\sigma = 0$, so that v^* corresponds to a dependence relation among the generators of the spaces $\bigwedge^2 L(\sigma)$. Clearly, any such relation also induces a v^*, by taking the coefficients of each Λ_σ to the σ-component of $C_2^{BM}(X, \mathcal{F}_2)$, observing that this element must be in the kernel of s, and hence in $\mathcal{K}_2(v)$, and then dualizing.

We have seen that each set of balancing conditions $(\beta(\tau))_{\tau \in X^1}$ induces a subgroup of dimension $f_1 - 1$ of $\mathcal{K}^2(v)$. Any other balancing $\beta'(\tau)$ along any edge would induce a new independent subgroup $\langle \beta'(\tau) \rangle$. Hence $\dim(\mathcal{K}^2(v)) = f_1 - 1$ if and only if there is a unique set of balancing conditions $(\beta(\tau))_{\tau \in X^1}$, i.e. X is uniquely balanced along each edge $\tau \in X^1$, and every dependent set of generators Λ_σ of faces is a linear combination of the balancing relations of some edges.

Since X is pure, $\mathcal{K}_p$ fits in the exact sequence:

$$0 \longrightarrow \mathcal{K}_p \longrightarrow \bigoplus_{\sigma \in X^2} [\hat{\sigma}]^{\mathcal{F}^p(\sigma)} \longrightarrow \mathcal{F}_p \longrightarrow 0.$$

Taking the stalk at the vertex v, this gives:

$$0 \longrightarrow \mathcal{K}_p(v) \longrightarrow \bigoplus_{\sigma \in X^2} \mathcal{F}^p(\sigma) \longrightarrow \mathcal{F}_p(v) \longrightarrow 0,$$

Hence all the dependent set of generators Λ_σ of faces are recorded in $\mathcal{K}_p(v)$. Compare this with talking stalks at an edge τ:

$$0 \longrightarrow \mathcal{K}_p(\tau) \longrightarrow \bigoplus_{\substack{\sigma\in X^2\\ \sigma\geq\tau}} \mathcal{F}^p(\sigma) \longrightarrow \mathcal{F}_p(\tau) \longrightarrow 0,$$

meaning that $\mathcal{K}_p(\tau)$ records the dependencies among the Λ_σ on edges. Since $\partial_1\colon \oplus_{\tau\in X^1}\mathcal{K}_p(\tau) \to \mathcal{K}_p(v)$ is surjective, this means that any relation on all the Λ_σ comes from a relation on edges.

Hence we have that $\dim(\mathcal{K}^2(v)) = f_1 \; - 1$ if and only if there is a unique set of balancing conditions $(\beta(\tau))_{\tau\in X^1}$, i.e. X is uniquely balanced along each edge $\tau \in X^1$.

■

5.6 Geometric classification theorem

Using the work from the two previous sections, we can now provide a classification theorem with a more geometric flavor:

Theorem 5.6.1. *Let X be a rational 2-dimensional polyhedral fan. The cap product with the fundamental class $\cap[X]\colon H^{p,q}(X) \to H^{BM}_{2-p,2-q}(X)$ is an isomorphism if and only if*

1. *X is pure,*
2. *the boundary map $\partial_1\colon \oplus_{\tau\in X^1}\mathcal{K}_p(\tau) \to \mathcal{K}_p(v)$ is surjective,*
3. *X is uniquely balanced, and*
4. *X is uniquely balanced at each edge.*

Proof. We saw in Proposition 5.1.1 that pure dimensionality is necessary. Assuming purity, we have by Theorem 5.2.2 that $\dim H^{BM}_q(X,\mathcal{F}_p) = 0 = \dim H^{2-q}(X,\mathcal{F}^{2-p}) = 0$ for $q = 0,1$ and $p = 0,1,2$ if and only if $\partial_1\colon \oplus_{\tau\in X^1}\mathcal{K}_p(\tau) \to \mathcal{K}_p(v)$ is surjective. Moreover, by Proposition 5.4.1, since $\dim H^{BM}_q(X,\mathcal{F}_1) = 0$ for $q = 0,1$, the cap map $\cap[X]\colon H^0(X,\mathcal{F}^1) \to H^{BM}_2(X,\mathcal{F}_1)$ is an isomorphism if and only if X is uniquely balanced along each edge $\tau \in X^1$. Similarly, we may use Proposition 5.5.1 to see that $\cap[X]\colon H^{2,0}(X) \to H^{BM}_{0,2}(X)$ is an isomorphism if and only if X is uniquely balanced along each edge $\tau \in X^1$. Finally, we recall that $\cap[X]\colon H^{0,0}(X) \to H^{BM}_{2,2}(X)$ is an isomorphism if and only if X is uniquely balanced Proposition 3.2.12. ■

Moreover, examples have shown that each of these conditions is independently needed. Purity is necessary by Proposition 5.1.1. Unique balancing at each edge is necessary by Example 3.4.5. Unique balancing is necessary by Definition 3.2.11. For an example of a fan which is uniquely balanced along each edge but not uniquely balanced, see Example 5.6.3. Finally, the surjectivity of ∂_1 can be seen in Example 5.6.2.

One might for instance hope that the Cohen–Macaulay property is sufficient to guarantee that the groups $H^{BM}_q(X,\mathcal{F}_p)$ vanish for $q > 0$. However, as the following example shows, it is not:

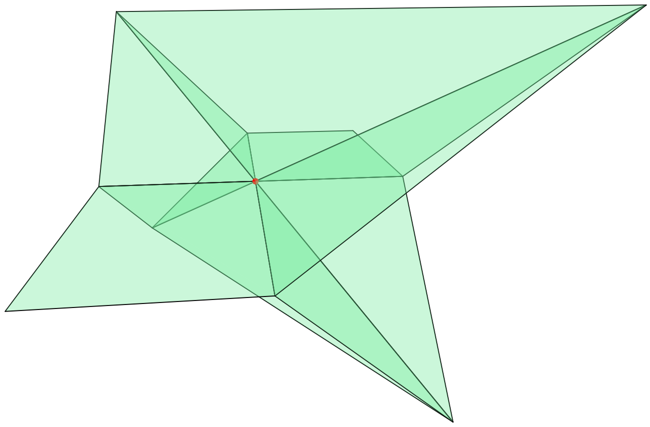

Figure 5.1: Picture of the fan

Example 5.6.2. Consider the polyhedral fan X from Figure 5.1. We see that, since there are exactly 3 faces meeting at each edge, none of which are parallel, each edge is uniquely balanced. Moreover, one sees that it is Cohen–Macaulay since the tangent fan at each face has only one cell, the tangent fan at an edge only has 4 cells, of which the top 3 surject onto the cell corresponding to the edge, and the tangent fan at the vertex, i.e. all of X, has only top dimensional Borel–Moore homology by the computation at the end of this example. The same computation shows that X is uniquely balanced.

However, not every dependent set of generators Λ_σ of faces is a linear combination of the balancing relations of some edges. Indeed, since there are faces which are parallel, such as the 2-dimensional faces corresponding to $\sigma = \operatorname{cone}\{(1,0,0),(0,1,0)\}$ and $\sigma' = \operatorname{cone}\{(-1,0,0),(0,-1,0)\}$, their generators are dependent, i.e. $\Lambda_\sigma + \Lambda_{\sigma'} = 0$ or $\Lambda_\sigma - \Lambda_{\sigma'} = 0$, since they both generate the same subspace of $\bigwedge^2 \mathbb{R}^3$.

We compute the homology groups of this fan using the cellular sheaves package [KSW17] for [polymake]:

```
application 'fan';
$fan = new fan::PolyhedralFan(INPUT_RAYS=>[
[1,0,0,0],
[0,1,0,0],
[0,0,1,0],
[0,-1,0,0],
[0,0,-1,0],
[0,1,-1,1],
[0,1,-1,-1],
[0,-1,1,1],
[0,-1,1,-1]
],
INPUT_CONES =>
[[0,1,2],[0,3,4],
[0,4,5],[0,1,5],[0,4,6],[0,1,6],
[0,3,7],[0,2,7],[0,3,8],[0,2,8],
[0,5,7],[0,6,8]
```

```
]
);

$complex = new fan::PolyhedralComplex($fan);
$complex -> VISUAL;

@betti_usual = ();
@betti_bm = ();
for(my $i=0; $i<5; $i++){
   my $f = $complex->fcosheaf($i);
   my $s = $complex->usual_chain_complex($f);
   my $bm = $complex->borel_moore_complex($f);
   push @betti_usual, $s->BETTI_NUMBERS;
   push @betti_bm, $bm->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_bm);
```

The Betti numbers are then output as follows:

```
fan > print new Matrix(@betti_usual);
1 0 0
3 0 0
3 0 0
0 0 0
0 0 0

fan > print new Matrix(@betti_bm);
0 0 5
0 0 3
0 2 1
0 0 0
0 0 0
```

Which means that this fan is not a tropical Poincaré space.

Example 5.6.3. Consider the polyhedral fan X from Figure 5.2. We see that there are exactly 3 faces meeting at each edge, so that it is uniquely balanced along each edge. However, as the following computation from the cellular sheaves package [KSW17] for [polymake], it is not uniquely balanced:

```
application "fan";
$fan = new fan::PolyhedralFan(INPUT_RAYS=>[
[1,0,0,0],
[0,1,0,0],
[0,0,1,0],
[0,0,0,1],
[0,1,1,1],
[0,-1,0,0],
[0,0,-1,0],
[0,0,0,-1],
[0,-1,-1,-1],
],
INPUT_CONES =>
[
[0,1,2],[0,1,3],[0,1,4],
[0,2,3],[0,2,4],
[0,3,4],
[0,5,6],[0,5,7],[0,5,8],
[0,6,7],[0,6,8],
[0,7,8]
```

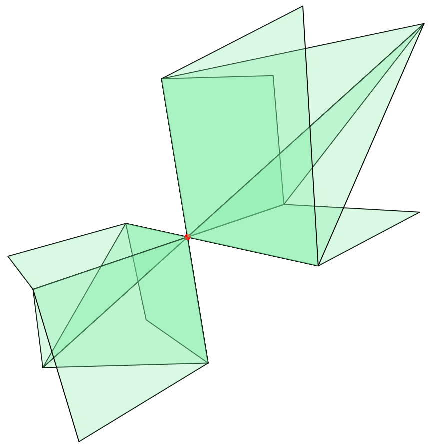

Figure 5.2: Picture of the fan

```
]
);

$complex = new fan::PolyhedralComplex($fan);
$complex ->VISUAL;

@betti_usual = ();
@betti_borel_moore = ();
for(my $i=0; $i<4; $i++){
   my $f = $complex->fcosheaf($i);
   my $u = $complex->usual_chain_complex($f);
   my $bm = $complex->borel_moore_complex($f);
   push @betti_usual, $u->BETTI_NUMBERS;
   push @betti_borel_moore, $bm->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_borel_moore);
```

The Betti numbers are then output as follows:

```
fan > print new Matrix(@betti_usual);
1 0 0
3 0 0
3 0 0
0 0 0


fan > print new Matrix(@betti_borel_moore);
0 1 6
0 3 6
0 3 2
0 0 0
```

PART II

Verdier duality on polyhedral complexes

CHAPTER 6

An application of Verdier duality on simplicial fans

In this chapter, we apply techniques from the derived category of sheaves of vector spaces, as developed in Appendix A, to questions about cellular sheaves. An important consequence of the theory developed there is the following theorem:

Theorem 6.0.1 ([Cur14, Theorem 12.1.2],Theorem A.6.2)**.** *Let $\mathcal{F}$ be a sheaf on a polyhedral complex X of dimension n. Then if the Verdier dual $D(\mathcal{F})$ of $\mathcal{F}$ is a sheaf:*

$$H_c^i(X, \mathcal{F}) \cong H^{n-i}(X, D(\mathcal{F}))^*.$$

In this thesis, the primary goal in developing this theory was to relate this to the tropical Poincaré duality, aiming to find conditions such that $D(\mathcal{F}^p) = \mathcal{F}^{n-p}$. This approach has been used to give a new proof of tropical Poincaré duality in the matroidal case [GS].

We will apply this theory to the *tropical f-vector conjecture*. In [Spe04], Speyer introduces *tropical linear spaces*, which are particular polyhedral complexes, and formulates the following conjecture about their f-vectors (see Definition 2.1.10):

Conjecture 6.0.2 (The f-vector conjecture)**.** *The number of i-dimensional faces of a tropical linear space of dimension d in n-space which become bounded after being mapped to $\mathbb{R}^n/(1, \ldots, 1)$ is at most $\binom{n-2i}{d-i}\binom{n-i-1}{i-1}$*

Recall the definition of the $\mathcal{W}^p$ sheaves:

Definition 6.0.3 (Definition 3.1.1)**.** Let X be a polyhedral complex of dimension n in $\mathbb{R}^N$. For $p = 0, \ldots, n$, the *tropical wave sheaf* $\mathcal{W}^p$ is the cellular sheaf defined by the data:

- For $\sigma \in X$, $\mathcal{W}_p(\sigma) := \bigwedge^p L(\sigma) \subseteq \bigwedge^p \mathbb{R}^N$, where $L(\sigma) \subset \mathbb{R}^N$ is the linear space parallel to the face σ.
- For $\tau \leq \sigma$, we have a morphism $(r\colon \tau \to \sigma) \in \mathrm{Mor}(X)$, and we define $\mathcal{W}_p(r) := \iota_{\tau\sigma}$, where $\iota_{\tau\sigma}\colon \mathcal{W}_p(\tau) \to \mathcal{W}_p(\sigma)$ is the wedge of the inclusion $L(\tau) \to L(\sigma)$.

As an approach to the f-vector conjecture, the following conjecture is introduced in [KSW17]:

Conjecture 6.0.4 ([KSW17]). *Let $L \subset \mathbb{R}^n$ be a tropical linear space of dimension d. Then we have*

$$H^q(L, \mathcal{W}^p) = 0 \quad \text{if } p \neq q, \quad \text{and} \quad H^q_c(L, \mathcal{W}^p) = 0 \quad \text{if } p \neq d.$$

If this conjecture holds, one can understand the f-vector by computing the Euler characteristics of the complexes $C^\bullet(L, \mathcal{W}^p)$ and $C^\bullet_c(L, \mathcal{W}^p)$, giving:

$$(-1)^p H^p(L, \mathcal{W}^p) = \sum_{q=0}^{d} (-1)^q \binom{q}{p} f^b_q,$$
$$(-1)^d H^d_c(L, \mathcal{W}^p) = \sum_{q=0}^{d} (-1)^q \binom{q}{p} f_q,$$

which would mean that understanding the f and f^b-vectors of a tropical linear space comes down to understanding the possible dimensions of $H^p(L, \mathcal{W}^p)$ and $H^d_c(L, \mathcal{W}^p)$. For example, it is possible to bound the f^b-vector by bounding $H^p(X, \mathcal{W}^p)$. This would give an approach to the f-vector conjecture for tropical linear space similar to the proof to the upper bound conjecture for polytopes.

A particular subset of tropical linear spaces are the following:

Definition 6.0.5 ([Bru+15, Definition 2.23]). A *tropical linear space $L \subset \mathbb{R}^n$* is a tropical subvariety of $\mathbb{R}^n$ which is the Bergman fan of a valuated matroid Definition 2.9.3.

We partially solve this conjecture by proving the following:

Theorem 6.0.6 (Theorem 6.3.2). *If X is a Cohen–Macaulay simplicial polyhedral fan of dimension n, then $H^i_c(X, \mathcal{W}^p) = 0$ if $i \neq n$ for all p.*

In particular this means that the compact-support part of Conjecture 6.0.4 holds for the Bergman fan of a matroid:

Corollary 6.0.7 (Corollary 6.4.1). *Let M be a matroid and $\mathcal{B}(M)$ its Bergman fan. Then $H^i_c(\mathcal{B}(M), \mathcal{W}^p) = 0$ if $i \neq \dim(\mathcal{B}(M))$ for all p.*

Moreover, since Bergman fans are fans, the vertex is the only compact cell, meaning that the f^b-vector is only $(1, 0, 0, \dots)$.

To prove this result, we use methods from the derived category of cellular sheaves on a polyhedral complex, which are developed in Appendix A.

6.1 The dualizing complex of a polyhedral complex

In this section we introduce a particular complex of sheaves, which will serve to build a duality relation among the cellular sheaves.

Definition 6.1.1 ([Cur14, Definition 12.2.4]). Let X be an n-dimensional polyhedral complex, and k a field. We define the *dualizing complex* $\omega^\bullet_X$ of X with respect to k to be the complex whose $(-i)$-th component is the sum over the elementary injective sheaves concentrated at cells of dimension i with values in k, which is

$$0 \longrightarrow \bigoplus\nolimits_{\sigma \in X^n} [\sigma]^k \xrightarrow{\partial} \bigoplus\nolimits_{\tau \in X^{n-1}} [\tau]^k \xrightarrow{\partial} \cdots \longrightarrow \bigoplus\nolimits_{v \in X^0} [v]^k \longrightarrow 0,$$

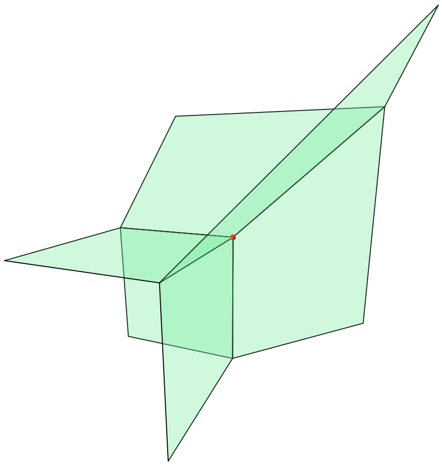

Figure 6.1: The polyhedral fan X

where ∂ is the map given componentwise on opens by $\partial_{\sigma\tau}(U) = \mathcal{O}(\tau,\sigma)\,\mathrm{id}_k$, guaranteeing that this is a complex. Note that we mostly use $k = \mathbb{R}$ for the field.

Example 6.1.2. Let X be the polyhedral fan in $\mathbb{R}^3$ with a vertex v at the origin, four edges in the directions $\tau_1 = (1,1,1)$, $\tau_2 = (-1,0,0)$, $\tau_3 = (0,-1,0)$ and $\tau_4 = (0,0,-1)$ respectively, and a face σ_{12} which is the positive cone of τ_1 and τ_2, σ_{13} the positive cone of τ_1 and τ_3, carrying on in the same manner to get σ_{14}, σ_{34}, σ_{23} and σ_{24} also. We compute the dualizing complex $\omega_X^\bullet$ with respect to k of X to be:

$$0 \longrightarrow [\sigma_{12}]^k \oplus [\sigma_{13}]^k \oplus \cdots \oplus [\sigma_{24}]^k \longrightarrow [\tau_1]^k \oplus \cdots \oplus [\tau_4]^k \longrightarrow [v]^k \longrightarrow 0.$$

Localizing at stalks preserves exactness, so we can take the stalk at v:

$$0 \longrightarrow k^6 \longrightarrow k^4 \longrightarrow k \longrightarrow 0$$

which we recognize as the complex $C_c^{-\bullet}(X, k_X)$.

We could have localized at one of the edges τ_i, obtaining:

$$0 \longrightarrow k^3 \longrightarrow k \longrightarrow 0 \longrightarrow 0$$

which in each case we can recognize as the complex $C_c^{-\bullet}(\mathrm{Star}(\tau_i), k_X)$, and furthermore localizing at one of the faces σ gives a complex:

$$0 \longrightarrow k \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

which again is the complex $C_c^{-\bullet}(\mathrm{Star}(\sigma), k_X)$.

The relation between the dualizing complex $\omega_X^\bullet$ and the compact cohomology can be used in general to characterize the Cohen–Macaulay property.

Theorem 6.1.3. *Let X be a polyhedral complex of dimension n. The dualizing complex $\omega_X^\bullet$ of X is concentrated in degree $-n$ if and only if X is Cohen–Macaulay.*

Proof. We wish to check that all but one of the cohomology sheaves $\mathcal{H}^i(\omega_X^\bullet)$ are the zero sheaf, for $i = -n+1, \ldots, 0$, where the indices start at $-n+1$ due to the position of $\omega_X^\bullet$. A sheaf of abelian groups is isomorphic to the zero sheaf if and only if the stalks at each point are zero, since the map to the zero sheaf is an isomorphism if and only if the zero maps on stalks are isomorphisms, which happens if and only if each stalk is zero. Therefore, for a polyhedral complex X, we have that $\omega_X^\bullet$ is concentrated in degree $-n$ if and only if $\mathcal{H}^i(\omega_X^\bullet)_x$ is 0 for $i = -n+1, \ldots, 0$, for all cells $x \in X$.

By Proposition A.1.10, we have that the stalk of the cohomology sheaf at x is the cohomology of the complex of stalks, i.e.

$$\mathcal{H}^i(\omega_X^\bullet)_x = H^i(\omega_{X,x}^\bullet).$$

Now since X is a polyhedral complex, we know that the stalk of a sheaf $\mathcal{F}$ at x is merely $\mathcal{F}(x)$ Proposition 2.2.6, hence for any given cell x, the complex $\omega_{X,x}^\bullet$ becomes

$$0 \longrightarrow \bigoplus_{\sigma \in X^n} [\sigma]^k(x) \xrightarrow{\partial} \bigoplus_{\tau \in X^{n-1}} [\tau]^k(x) \xrightarrow{\partial} \cdots \longrightarrow \bigoplus_{v \in X^0} [v]^k(x) \longrightarrow 0.$$

This is exactly the complex $C_c^{-i}(\mathrm{Star}(x), k_X)$, so that the cohomology at i is the cohomology $H_c^{-i}(\mathrm{Star}(x), k_X)$. Thus we have

$$\mathcal{H}^{-i}(\omega_X^\bullet)_x = H^i(\omega_{X,x}^\bullet) = H_c^i(\mathrm{Star}(x), k_X)$$

for all $x \in X$ and $i = 0, \ldots, n-1$. By Remark 2.8.4, if $H_c^i(\mathrm{Star}(x), k_X) = 0$ for all $i \neq n$ and all $x \in X$, then X is Cohen–Macaulay. Hence $\mathcal{H}^{-i}(\omega_X^\bullet)_x = 0$ for all $x \in X$ and $i = 0, \ldots, n-1$ if and only if X is Cohen–Macaulay. Hence $\mathcal{H}^{-i}(\omega_X^\bullet) = 0$ for $i \neq n$ if and only if X is Cohen–Macaulay. Finally this gives that the dualizing complex $\omega_X^\bullet$ of X is concentrated in degree $-n$ if and only if X is Cohen–Macaulay. ∎

Remark 6.1.4. Note the similarity of this result to algebraic geometry. One can define a *Cohen–Macaulay scheme*, and observe the same property of its dualizing complex:

Definition 6.1.5 ([Stacks, Definition 27.8.1])**.** A scheme X is *Cohen–Macaulay* if for every $x \in X$ there exists an affine open neighborhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.

Mirroring our situation, one then has:

Lemma 6.1.6 ([Stacks, Lemma 46.23.1])**.** *Let X be a locally Noetherian scheme with dualizing complex $\omega_X^\bullet$. Then X is Cohen-Macaulay if and only if $\omega_X^\bullet$ locally has a unique nonzero cohomology sheaf. Moreover, if X is connected and Cohen-Macaulay, then there is an integer n and a coherent Cohen-Macaulay $\mathcal{O}_X$-module ω_X such that $\omega_X^\bullet = \omega_X[-n]$.*

This provides another motivation for calling a polyhedral complex "Cohen–Macaulay" when it has the property from Definition 2.8.1.

6.2 Verdier dual of cellular sheaves

The dualizing complex of a polyhedral complex is then used to define the following dual functor:

Definition 6.2.1 ([Cur14, p. 238])**.** Let X be a polyhedral complex of dimension n in $\mathbb{R}^N$. The *Verdier dual functor* D is the functor taking a sheaf $\mathcal{F}$ to the complex

$$0 \longrightarrow \mathscr{Hom}(\mathcal{F}, \omega_x^{-n}) \longrightarrow \ldots \longrightarrow \mathscr{Hom}(\mathcal{F}, \omega_x^0) \longrightarrow 0,$$

where $\mathscr{Hom}(\bullet, \bullet)$ is the *sheaf of local morphisms*, also called *sheaf hom* (Definition A.3.3). The complex $D(\mathcal{F})$ is called *the Verdier dual complex of* $\mathcal{F}^\bullet$.

Note that this definition is only correct in the stated form. Attempting to "replace" $\omega_X^\bullet$ with an other complex having the same cohomology will most likely change the complex $D(\mathcal{F})$. To properly define this functor, we refer to Appendix A. However, using this dualizing functor, one has the very useful theorem:

Theorem 6.2.2 (Theorem A.6.2)**.** *Let $\mathcal{F}$ be a sheaf on a polyhedral complex X of dimension n. Then if $D(\mathcal{F})$ is a sheaf:*

$$H_c^i(X, \mathcal{F}) \cong H^{n-i}(X, D(\mathcal{F}))^*.$$

6.3 A duality result on simplicial polyhedral fans

In this section, we first characterize simplicial polyhedral fans (Definition 2.6.3) in term of the $\mathcal{W}^p$ sheaves (see Definition 3.1.1), and use this to prove that the most of the compact support cohomology groups with respect to $\mathcal{W}^p$ vanish. Recall that all projective (see Definition A.1.7) sheaves can be decomposed in terms of elementary projective sheaves (see Proposition 2.4.6).

Theorem 6.3.1. *A fan X is simplicial if and only if $\mathcal{W}^p$ is projective for all p, with $\mathcal{W}^p \cong \bigoplus_{\tau \in X^p} \{\tau\}^{\mathbb{R}}$.*

Proof. Observe that the restriction map $\rho_{\tau,\sigma} \colon \mathcal{W}^p(\tau) \to \mathcal{W}^p(\sigma)$ is an inclusion, which is the identity on $\mathcal{W}^p(\tau) \subset \bigwedge^p \mathbb{R}^N$ as a subspace.

Since X is simplicial, the projective sheaf $\oplus_{\tau \in X^p} \{\tau\}^{\mathbb{R}}$ at a k-dimensional cell σ is

$$\bigoplus_{\tau \in X^p} \{\tau\}^{\mathbb{R}}(\sigma) = \mathbb{R}^{\binom{k}{p}},$$

which is exactly the dimension of $\mathcal{W}^p(\sigma)$. For each $\sigma \in X$, we therefore define $\phi_\sigma \colon \mathcal{W}^p(\sigma) \to \bigoplus_{\tau \in X^p} \{\tau\}^{\mathbb{R}}(\sigma)$ by $\phi = \mathrm{id}$. Clearly

$$\rho_{\tau,\sigma}^{\oplus_\tau \{\tau\}^{\mathbb{R}}} \circ \phi_\tau = \phi_\sigma \circ \rho_{\tau,\sigma}^{\mathcal{W}^p},$$

since both sides of the equation are just the identity mapping, this is a sheaf morphism, which is also clearly an isomorphism.

Next suppose $\mathcal{W}^p$ is a projective sheaf, so that by Proposition 2.4.6, we have $\mathcal{W}^p \cong \bigoplus_{\sigma \in X} \{\sigma\}^{V_\sigma}$ for some cells σ and vector spaces V_σ.

Since $\dim \mathcal{W}^p(\tau) = \binom{\dim \tau}{p}$ for all $\tau \in X$, there are no $\{\gamma\}^{V_\gamma} \subset \bigoplus_{\sigma \in X} \{\sigma\}^{V_\sigma}$ with $\dim \gamma \leq p$, and the elementary projective sheaves on p-dimensional cells are all part of the sum, i.e. $\bigoplus_{\tau \in X^p} \{\tau\}^{\mathbb{R}} \subset \bigoplus_{\sigma \in X} \{\sigma\}^{V_\sigma}$. Moreover, each cell $\rho \in X^k$ has at least k faces $\sigma \in X^{k-1}$, therefore

$$\dim \mathcal{W}^p(\rho) = \dim \bigoplus_{\sigma \in X} \{\sigma\}^{V_\sigma}(\rho) \geq \dim \bigoplus_{\tau \in X^p} \{\tau\}^{\mathbb{R}}(\rho) \geq \binom{\dim \rho}{p} = \mathcal{W}^p(\rho).$$

Hence, we must have that $\mathcal{W}^p \cong \bigoplus_{\tau \in X^p} \{\tau\}^{\mathbb{R}}$ and that each cell $\rho \in X^k$ has exactly k faces $\gamma \in X^{k-1}$. Since $\mathcal{W}^p$ is projective for each p, this holds for all cells in X. Hence X is simplicial. ∎

Theorem 6.3.2. *If X is a Cohen–Macaulay simplicial polyhedral fan of dimension n, then $H^i_c(X, \mathcal{W}^p) = 0$ if $i \neq n$ for all p.*

Proof. Since X is simplicial, $\mathcal{W}^p$ is projective, hence the Verdier dual complex $D(\mathcal{W}^p)$ is given by the complex:

$$0 \longrightarrow \mathscr{H}\!om(\mathcal{W}^p, \mathcal{H}^{-n}(\omega^\bullet_X)) \longrightarrow \cdots \longrightarrow \mathscr{H}\!om(\mathcal{W}^p, \mathcal{H}^0(\omega^\bullet_X)) \longrightarrow 0.$$

To understand why one can replace $\omega^\bullet_X$ by its complex of cohomology sheaves $\mathcal{H}^\bullet(\omega^\bullet_X)$ in the case where $\mathcal{W}^p$ is projective, see Appendix A. Moreover, since X is Cohen–Macaulay, $\mathcal{H}^{-k}(\omega^\bullet_X) = 0$ for $k \neq n$, and therefore the only non-zero term in the complex $D(\mathcal{W}^p)$ is the term $\mathscr{H}\!om(\mathcal{W}^p, \mathcal{H}^{-n}(\omega^\bullet_X))$ in degree $-n$. Hence $D(\mathcal{W}^p) \cong \mathscr{H}\!om(\mathcal{W}^p, \mathcal{H}^{-n}(\omega^\bullet_X))[n]$ is a sheaf, and thus by Theorem 6.2.2 we have:

$$\begin{aligned} H^i_c(X, \mathcal{W}^p) &\cong H^{-i}(X, \mathscr{H}\!om(\mathcal{W}^p, \mathcal{H}^{-n}(\omega^\bullet_X))[n]), \\ &\cong H^{n-i}(X, \mathscr{H}\!om(\mathcal{W}^p, \mathcal{H}^{-n}(\omega^\bullet_X))). \end{aligned}$$

Since X is a polyhedral fan, the only compact cell is the vertex, so we have $H^{n-i}(X, \mathscr{H}\!om(\mathcal{W}^p, \mathcal{H}^{-n}(\omega^\bullet_X))) = 0$ for $i \neq n$. Therefore, this gives that $H^i_c(X, \mathcal{W}^p) = 0$ if $i \neq n$. ∎

6.4 Examples and counterexamples

In this section, we present some example applications of this result, and then show some examples of fans either failing to satisfy the criteria of Theorem 6.3.2 yet satisfying $H^i_c(X, \mathcal{W}^p) = 0$ if $i \neq n$ for all p, and also some cases where both criteria and conclusion fail.

Corollary 6.4.1. *Let M be a matroid and $\mathcal{B}(M)$ its Bergman fan. Then $H^i_c(\mathcal{B}(M), \mathcal{W}^p) = 0$ if $i \neq \dim(\mathcal{B}(M))$ for all p.*

Proof. We saw in Section 2.9 that the Bergman fan of a matroid is simplicial and Cohen–Macaulay. Hence Theorem 6.3.2 can be applied directly. ∎

Example 6.4.2. Consider [KSW17, Example 5], which is the Bergman fan of the matroid of the complete graph on four vertices. The Betti numbers of the usual cohomology and the cohomology with compact support of the sheaves $\mathcal{W}^p$ for $p = 0, 1, 2$ of this simplicial polyhedral fan can be computed with the cellular sheaves package [KSW17] for [`polymake`]:

```
application "graph";
$g = complete(4);
application "matroid";
$m = matroid_from_graph($g);
application "tropical";
$t = matroid_fan<Max>($m);
$t->VERTICES;
application "fan";
$berg = new PolyhedralComplex($t);


$complex = $berg;
@betti_usual = ();
@betti_compact = ();
for(my $i=0; $i<4; $i++){
   my $w = $complex->wsheaf($i);
   my $usual = $complex->usual_cochain_complex($w);
   my $comp = $complex->compact_support_complex($w);
   push @betti_usual, $usual->BETTI_NUMBERS;
   push @betti_compact, $comp->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_compact);
```

The Betti numbers are then output as follows:

```
fan > print new Matrix(@betti_usual);
1 0 0
0 0 0
0 0 0
0 0 0

fan > print new Matrix(@betti_compact);
0 0 6
0 0 20
0 0 15
0 0 0
```

which agrees with our result that for a Cohen–Macaulay simplicial polyhedral fan of dimension 2, then $H^i_c(X, \mathcal{W}^p) = 0$ if $i \neq 2$.

A simplicial polyhedral fan can have $H^i_c(X, \mathcal{W}^p) = 0$ for $i \neq \dim(X)$ even if it fails to be Cohen–Macaulay:

Example 6.4.3. Consider the fan Figure 6.2. This fan is not Cohen–Macaulay since there is an edge where 4 faces meet, which gives a class in $H^{BM}_{1,1}$ of the tangent fan of that edge. For each p, we compute the $\mathcal{W}^p$ cohomology of this fan using the cellular sheaves package [KSW17] for [`polymake`]:

```
application "fan";
$fan = new PolyhedralFan(
        INPUT_RAYS=>[[1,0,0,0],[0,-1,0,0],[0,0,-1,0],
                                [0,0,0,-1],[0,0,1,1],[0,1,0,1]
                               ],
        INPUT_CONES => [[0,1,2], [0,1,3], [0,2,3],
                                       [0,3,4], [0,3,5], [0,1,4],
                                       [0,2,5], [0,4,5]]);

$complex = new PolyhedralComplex($fan);
```

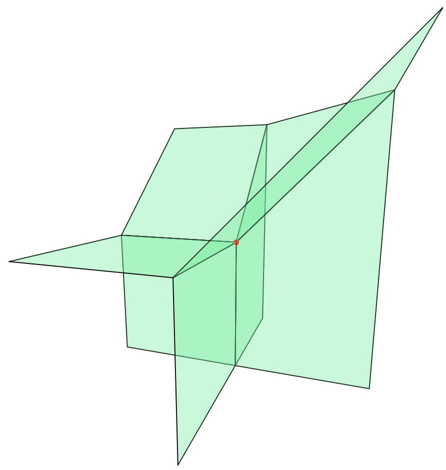

Figure 6.2: Picture of the fan

```
$complex -> VISUAL;

@betti_usual = ();
@betti_compact = ();
for(my $i=0; $i<4; $i++){
   my $w = $complex->wsheaf($i);
   my $u = $complex->usual_cochain_complex($w);
   my $c = $complex->compact_support_complex($w);
   push @betti_usual, $u->BETTI_NUMBERS;
   push @betti_compact, $c->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_compact);
```

The Betti numbers are then output as follows:

```
fan > print new Matrix(@betti_usual);
1 0 0
0 0 0
0 0 0
0 0 0

fan > print new Matrix(@betti_compact);
0 0 4
0 0 11
0 0 8
0 0 0
```

However, simplicial polyhedral fans can fail to satisfy the property if they are not Cohen–Macaulay:

Example 6.4.4. Consider the fan Figure 6.3, which is not Cohen–Macaulay since its link is not connected. For each p, we compute the $\mathcal{W}^p$ cohomology of this fan using the cellular sheaves package [KSW17] for [polymake]:

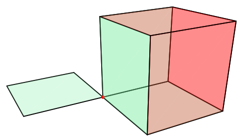

Figure 6.3: Picture of the fan

```
application "fan";
$fan = new fan::PolyhedralFan(INPUT_RAYS=>[
[1,0,0,0],
[0,1,0,0],
[0,0,1,0],
[0,0,0,1],
[0,-1,0,0],
[0,0,-1,0]],
INPUT_CONES =>
[[0,1,2,3], [0,4,5]]
);

$complex = new fan::PolyhedralComplex($fan);

@betti_usual = ();
@betti_compact = ();
for(my $i=0; $i<5; $i++){
   my $w = $complex->wsheaf($i);
   my $u = $complex->usual_cochain_complex($w);
   my $c = $complex->compact_support_complex($w);
   push @betti_usual, $u->BETTI_NUMBERS;
   push @betti_compact, $c->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_compact);
```

The Betti numbers are then output as follows:

```
fan > print new Matrix(@betti_usual);
1 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0

fan > print new Matrix(@betti_compact);
0 1 0 0
0 0 0 0
```

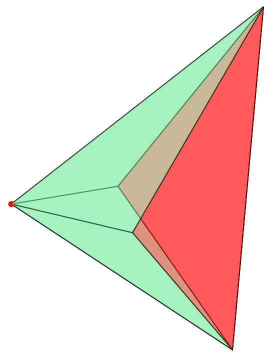

Figure 6.4: Picture of the fan

```
0 0 1 0
0 0 0 1
0 0 0 0
```

Finally, note that Cohen–Macaulayness without simpliciality is not sufficient:

Example 6.4.5. Consider the fan Figure 6.4 For each p, we compute the $\mathcal{W}^p$ cohomology of this fan using the cellular sheaves package [KSW17] for [polymake]:

```
application "fan";
$fan = new fan::PolyhedralFan(INPUT_RAYS=>[
[1,0,0,0],
[0,1,0,0],
[0,0,1,0],
[0,1,1,1],
[0,1,1,-1]],
INPUT_CONES =>
[[0,1,2,3,4]]
);

$complex = new fan::PolyhedralComplex($fan);

@betti_usual = ();
@betti_compact = ();
for(my $i=0; $i<5; $i++){
   my $w = $complex->wsheaf($i);
   my $u = $complex->usual_cochain_complex($w);
   my $c = $complex->compact_support_complex($w);
   push @betti_usual, $u->BETTI_NUMBERS;
   push @betti_compact, $c->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_compact);
```

The Betti numbers are then output as follows:

```
fan > print new Matrix(@betti_usual);
1 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0

fan > print new Matrix(@betti_compact);
0 0 0 0
0 0 1 0
0 0 1 0
0 0 0 1
0 0 0 0
```

CHAPTER 7

Discussion

In this chapter, we briefly recall the results developed in the text, and discuss possible future research directions.

7.1 On tropical Poincaré duality

Recall the classification result from Chapter 4:

Theorem 7.1.1 (Theorem 4.5.1)**.** *A rational balanced polyhedral fan of pure dimension n is a tropical Poincaré space, i.e.*

$$\cap[X]\colon H^q(X, \mathcal{F}^p) \to H^{BM}_{n-q}(X, \mathcal{F}_{n-p})$$

is an isomorphism for all $p, q = 0, \ldots, n$, if and only if

1. *X is uniquely p-balanced for all p, and*
2. *the dependence cosheaf $\mathcal{K}_p$ is acyclic in Borel–Moore homology for all p, that is, $H^{BM}_q(X, \mathcal{K}_p) = 0$ for $q \neq n-1$ and all p.*

There are several ways one might hope to extend this result. First, we note that purity seems not to be necessary, as the following example shows:

Example 7.1.2. Consider the Bergman fan of the uniform matroid U^3_4, to which we add a “flap” consisting of the two dimensional cone cone $-e_1, -e_1 + e - 2$. We compute the Betti numbers of tropical homology and cohomology using the following [polymake] script with the Cellular Sheaves package [KSW17]:

```
application "fan";
$fan = new fan::PolyhedralFan(INPUT_RAYS=>[
[0,0,-1,0,0,0],[0,0,0,-1,0,0],[0,0,0,0,-1,0],
[0,0,1,1,1,1],[0,0,0,0,0,-1],[1,0,0,0,0,0],
[0,0,-1,1,0,0],
],
INPUT_CONES =>
[
[0,1,2,5],[1,2,3,5],[0,2,3,5],[0,1,3,5],
[0,1,4,5],[1,3,4,5],[0,3,4,5],[0,2,4,5],
[2,3,4,5],[1,2,4,5],
[5,6,0]
]
);

$complex = new fan::PolyhedralComplex($fan);
```

```
@betti_usual = ();
@betti_borel_moore = ();
for(my $i=0; $i<5; $i++){
   my $f = $complex->fcosheaf($i);
   my $u = $complex->usual_chain_complex($f);
   my $bm = $complex->borel_moore_complex($f);
   push @betti_usual, $u->BETTI_NUMBERS;
   push @betti_borel_moore, $bm->BETTI_NUMBERS;
}

print new Matrix(@betti_usual);
print new Matrix(@betti_borel_moore);
```

This produces the following output:

```
fan > print new Matrix(@betti_usual);
1 0 0 0
4 0 0 0
6 0 0 0
4 0 0 0
0 0 0 0

fan > print new Matrix(@betti_borel_moore);
0 0 0 4
0 0 0 6
0 0 0 4
0 0 0 1
0 0 0 0
```

Which shows that this fan is a tropical Poincaré space.

Next, we should recall that the weight functions, as defined in this thesis, are not allowed to take zero values on any faces. However, it might be conceivable to allow such weight functions. Then one could hope to construct "virtual tropical cycles" living in polyhedral fans. One could then investigate how the cap product with the fundamental class of such cycles behaves, allowing a form of tropical Poincaré duality in more pathological fans.

Another possible extension is to lift this theorem to the "global" setting of rational polyhedral spaces, constructing "tropical Poincaré spaces", i.e. determining the class of rational polyhedral spaces for which tropical Poincaré duality holds. Preliminary investigations suggest that the Mayer–Vietoris argument applied for tropical manifolds can be applied in this situation also.

It is possible to formulate to take the multi-tangent cosheaves $\mathcal{F}_p$ with coefficients on integers, giving sheaves of abelian groups $\mathcal{F}_p^{\mathbb{Z}}$. One may then ask which fans are tropical Poincaré spaces in this setting. We believe that the arguments in Chapter 4 can be applied, when restricting the weight functions to take values in $\{1, -1\}$.

7.2 On Verdier duality for tropical sheaves

We introduced many technical tools about derived categories to investigate tropical homology. In particular, we clarified the meaning of the Verdier dual functor:

Definition 7.2.1 (Definition A.6.1)**.** The *Verdier dual functor* is given by:

$$\begin{aligned} D\colon D^b(\mathbf{Shv}_X) &\to D^b(\mathbf{Shv}_X)^{\mathrm{op}}, \\ \mathcal{F}^\bullet &\mapsto \mathrm{R}\,\mathscr{H}\!om^\bullet(\mathcal{F}^\bullet, \omega_X^\bullet). \end{aligned}$$

The complex $D(\mathcal{F}^\bullet)$ is called *the Verdier dual complex of* $\mathcal{F}^\bullet$.

This functor can be used to compute the compact support cohomology of a sheaf by:

Theorem 7.2.2 (Theorem A.6.2)**.** *Let $\mathcal{F}$ be a sheaf on a polyhedral complex X of dimension n. Then*

$$H_c^i(X, \mathcal{F}) \cong H^{n-i}(X, D(\mathcal{F}))^*,$$

where $H^q(X, D(\mathcal{F}))$ is the q-th cohomology of the complex $\{\Gamma(X, D(\mathcal{F})^i)\}_{i\in\mathbb{Z}}$, also called the hypercohomology *of $D(\mathcal{F})$.*

In principle, we can consider answering questions about sheaf cohomology on rational polyhedral spaces using the following lemma:

Lemma 7.2.3 ([Mur06, Lemma 1])**.** *Let $(X, \mathcal{O}_X)$ be a ringed space and $\psi\colon F \to G$ a morphism in $D(X)$. If $\{V_i\}_{i\in I}$ is a nonempty open cover of X then ψ is an isomorphism in $D(X)$ if and only if $\psi|_{V_i}$ is an isomorphism in $D(V_i)$ for every $i \in I$.*

Considering a rational polyhedral space X as the ringed space $(X, \mathbb{Z})$, we may directly apply the lemma, using the open cover provided by the charts.

For instance, if one defines Cohen–Macaulayness for rational polyhedral spaces to be the property that the fans in the charts are Cohen–Macaulay polyhedral fans, one should be able to prove that the dualizing complex of such rational polyhedral spaces is concentrated in one degree, by using the above lemma with the theorem:

Theorem 7.2.4 (Theorem 6.1.3)**.** *Let X be a polyhedral complex of dimension n. The dualizing complex $\omega_X^\bullet$ of X is concentrated in degree $-n$ if and only if X is Cohen–Macaulay.*

Assuming that the dualizing complex is concentrated in one degree, one might hope for a *tropical Serre duality* for Cohen–Macaulay rational polyhedral spaces.

Another possible line of inquiry, motivated by the above lemma, is to consider how to extend the following theorem to a global setting:

Theorem 7.2.5 (Theorem 6.3.2)**.** *If X is a Cohen–Macaulay simplicial polyhedral fan of dimension n, then $H_c^i(X, \mathcal{W}^p) = 0$ if $i \neq n$ for all p.*

The lemma suggests that one might be able to prove the same vanishing property for any space locally looking like a Cohen–Macaulay simplicial polyhedral fan. If this is true, one might be able to prove the conjecture from [KSW17] in general by checking whether there is an open cover of a tropical linear space by the stars of Cohen–Macaulay simplicial fans at vertices.

Appendices

APPENDIX A

Derived categories and cellular sheaves

In this section, we seek to develop a duality theory of cellular sheaves. The main goal of this development is a variation on the result [Cur14, Theorem 12.1.2], which in our case will be the following:

Theorem A.0.1. *Let $\mathcal{F}$ be a cellular sheaf on a polyhedral complex X of dimension n, and let $D(\mathcal{F})$ be its (to be defined in Definition A.6.1) dual complex of sheaves. Then, for all $i \in \mathbb{Z}$,*

$$H^i_c(X, \mathcal{F}) \cong H^{n-i}(X, D(\mathcal{F}))^*.$$

This is be proved in Theorem A.6.2. In particular, in view of our interest in polyhedral fans, we are particularly motivated by the following corollary:

Corollary A.0.2. *Let $\mathcal{F}$ be a cellular sheaf on a polyhedral fan X of dimension n, and suppose $D(\mathcal{F})$ is a sheaf. Then $H^i_c(X, \mathcal{F}) = 0$ for all $i \neq n$, and $H^n_c(X, \mathcal{F}) \cong H^0(X, D(\mathcal{F}))^*$.*

To get these results, we will need to develop several results from homological algebra, in particular theory with regards to *Verdier duality.* Our presentation borrows from [Dim04; GM03; Stacks; Tho00], with some extensions to exploit the features of cellular sheaves.

For a reader already familiar with some notions in derived categories, it may be sufficient to look at Definition A.3.4, then Definition A.5.6, before moving on to the Verdier dualizing functor of Definition A.6.1, which is the used in Theorem A.6.2.

A.1 The derived category

Definition A.1.1. Let $\mathcal{A}$ be an abelian category. We denote by $C(\mathcal{A})$ the *category of chain complexes in* $\mathcal{A}$, whose objects $(A^\bullet, d^\bullet)$ are complexes of objects from $\mathcal{A}$, given by a pair of objects $A^\bullet = \{A^i \in \mathcal{A} \mid i \in \mathbb{Z}\}$ and of morphisms between them $d^\bullet_A = \{d^i_A \colon A^i \to A^{i+1} \mid i \in \mathbb{Z}\}$. These data are usually presented as

$$(A^\bullet, d^\bullet_A)\colon \qquad \cdots \longrightarrow A^{-1} \xrightarrow{d_A^{-1}} A^0 \xrightarrow{d_A^0} A^1 \xrightarrow{d_A^1} \cdots,$$

where the differential maps satisfy $d^i_A \circ d^{i-1}_A = 0$ for all $i \in \mathbb{Z}$. We usually omit the subscript A when the context is clear. Morphisms in $C(\mathcal{A})$ are *chain* *maps*, so that a morphism between objects $f^\bullet \colon (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ is a set of morphisms in $\mathcal{A}$ between elements of the complexes $\{f^i \colon A^i \to B^i \mid i \in \mathbb{Z}\}$ commuting with the differential maps, i.e. $d_B^{i-1} \circ f^{i-1} = f^i \circ d_A^{i-1}$.

Definition A.1.2. There are several subcategories of $C(\mathcal{A})$:

- The *lower bounded category of chain complexes* $C^+(A)$ is the subcategory of $C(\mathcal{A})$ where objects are bounded below, i.e. for all $A^\bullet \in C^+(\mathcal{A})$, there is an $i(A^\bullet) \in \mathbb{Z}$ such that $A^i = 0$ for $i \leq i(A^\bullet)$.
- The *upper bounded category of chain complexes* $C^-(A)$ is the subcategory of $C(\mathcal{A})$ where objects are bounded above, i.e. for all $A^\bullet \in C^-(\mathcal{A})$, there is an $i(A^\bullet) \in \mathbb{Z}$ such that $A^i = 0$ for $i \geq i(A^\bullet)$.
- The *bounded category of chain complexes* $C^b(A)$ is the subcategory of $C(\mathcal{A})$ where objects are bounded both below and above, i.e. for all $A^\bullet \in C^b(\mathcal{A})$, there are $i_1(A^\bullet), i_2(A^\bullet) \in \mathbb{Z}$ such that $A^i = 0$ for $i \geq i_1(A^\bullet)$ and $A^i = 0$ for $i \leq i_2(A^\bullet)$.

Example A.1.3. For a topological space X, the *category* $\mathbf{Shv}_X$ *of sheaves of abelian groups on* X is an abelian category, and we denote by $C(\mathbf{Shv}_X)$ the category of complexes of sheaves of abelian groups.

Definition A.1.4 ([Dim04, p. 7])**.** For any complex $A^\bullet \in C(\mathcal{A})$ and integer $n \in \mathbb{Z}$, there is a complex $A^\bullet[n] \in A^\bullet$, defined by $(A^\bullet[n])^i = A^{i+n}$.

Definition A.1.5. Let $f^\bullet, g^\bullet \colon A^\bullet \to B^\bullet$ be two chain morphisms. If there is a chain morphism $h^\bullet A^\bullet \to B^\bullet[1]$ such that $f - g = d_B \circ h + h \circ d_A$, we say that $f^\bullet$ and $g^\bullet$ are *homotopic*, and we write $f^\bullet \sim g^\bullet$, noting that this is an equivalence relation on $\mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)$.

Definition A.1.6. We denote by $K(\mathcal{A})$ the *homotopy category of chain complexes in* $\mathcal{A}$, where objects are the same as in $C(\mathcal{A})$, but morphisms are instead equivalence classes of homotopic chain morphisms, i.e.

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \frac{\mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)}{\sim}.$$

Moreover, the restricted subcategories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ are defined by restricting $K(\mathcal{A})$ to objects from the categories $C^+(\mathcal{A})$, $C^-(\mathcal{A})$ and $C^b(\mathcal{A})$ respectively.

Definition A.1.7. Let $\mathcal{A}$ be an abelian category. We say that

- $I \in \mathcal{A}$ is *injective* if $\mathrm{Hom}(-, I)$ is exact,
- $P \in \mathcal{A}$ is *projective* if $\mathrm{Hom}(P, -)$ is exact.

Definition A.1.8. For a complex $(X^\bullet, d^\bullet) \in C(\mathcal{A})$, we define the *cohomology objects* $H^i(X^\bullet) \in \mathcal{A}$ for $i \in \mathbb{Z}$ by

$$H^i(X^\bullet) \coloneqq \frac{\ker(d^{i+1})}{\mathrm{im}(d^i)}.$$

We say that a complex morphism $f \colon X^\bullet \to Y^\bullet$ is a *quasi-isomorphism* if the induced map $H^i(f) \colon H^i(X^\bullet) \to H^i(Y^\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$. Note that $(X^\bullet, d^\bullet)$ is always quasi-isomorphic to $(H^i(X^\bullet), 0)$.

Example A.1.9. Given a complex of sheaves $\mathcal{F}^\bullet \in C(\mathbf{Shv}_X)$, the *cohomology sheaves* $\mathcal{H}^i(\mathcal{F}^\bullet) \in \mathbf{Shv}_X$ are the cohomology objects of the complex.

Proposition A.1.10. *The stalks $\mathcal{H}^i(\mathcal{F}^\bullet)_x$ of the cohomology sheaves are equal to the cohomology groups $H^i(\mathcal{F}^\bullet_x)$ of the complex of stalks $\mathcal{F}^\bullet_x \in C(\mathbf{Ab})$.*

Proof. This follows from the fact that taking the stalk commutes with the kernel and image, giving

$$\mathcal{H}^i(\mathcal{F}^\bullet)_x = \left(\frac{\ker(d^{i+1})}{\operatorname{im}(d^i)}\right)_x \cong \frac{\ker(d^{i+1})_x}{\operatorname{im}(d^i)_x} \cong \frac{\ker(d_x^{i+1})}{\operatorname{im}(d_x^i)} = H^i(\mathcal{F}^\bullet_x),$$

where we used that the stalk of a quotient presheaf is the quotient of the stalks, and that the stalk of the quotient presheaf is the stalk of the associated sheaf. ∎

Definition A.1.11 ([GM03, Theorem III.2.1])**.** Let $\mathcal{A}$ be an abelian category, $C(\mathcal{A})$ the category of complexes over $\mathcal{A}$. There exists a category $D(\mathcal{A})$ and a functor $Q\colon C(\mathcal{A}) \to D(\mathcal{A})$ such that $Q(f)$ is an isomorphism for any quasi-isomorphism f. The category $D(\mathcal{A})$ is called the *derived category* of $\mathcal{A}$.

By restricting to any of the categories $C^+(\mathcal{A})$, $C^-(\mathcal{A})$ or $C^b(\mathcal{A})$, we obtain correspondingly restricted categories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ or $D^b(\mathcal{A})$ respectively.

The construction of the derived category is quite technical, and we refer the interested reader to [GM03, Chapter III] for the required constructions.

Example A.1.12. Given a topological space X, there is a *derived category of sheaves of abelian groups on X*, denoted by $D(\mathbf{Shv}_X)$ or simply $D(X)$.

Note that the derived category of an abelian category is not abelian itself, except in trivial cases [GM03, Section III.3.1]. Therefore, the concept of an exact sequence is not well-defined. Instead one has *distinguished triangles*.

Definition A.1.13 ([GM03, Definition III.3.4])**.** In any category of complexes ($D(\mathcal{A})$, $K(\mathcal{A})$, $C(\mathcal{A})$ or any of their bounded subcategories), we define the following notions:

a) A *triangle* is a diagram of the form

$$K^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow K[1]^\bullet.$$

b) A *morphism of triangles* is a commutative diagram of the form

$$\begin{tikzcd} K^\bullet \arrow[r] \arrow[d, "f"] & L^\bullet \arrow[r] \arrow[d, "g"] & M^\bullet \arrow[r] \arrow[d, "h"] & K[1]^\bullet, \arrow[d, "f[1]"] \\ K^\bullet \arrow[r] & L^\bullet \arrow[r] & M^\bullet \arrow[r] & K[1]^\bullet. \end{tikzcd}$$

Such a morphism is called an *isomorphism* if f, g, h are quasi-isomorphisms.

c) A triangle is said to be *distinguished* if it is isomorphic to some triangle

$$K^\bullet \xrightarrow{\overline{f}} \mathrm{Cyl}(f) \xrightarrow{\pi} \mathrm{C}(f) \xrightarrow{\delta} K[1]^\bullet,$$

which is defined in [GM03, Section III.3.2].

Proposition A.1.14 ([GM03, Proposition III.3.5]). *An exact triple of complexes in $C(\mathcal{A})$ is quasi-isomorphic to a distinguished triangle.*

A useful cohomological property of distinguished triangles is the following:

Theorem A.1.15 ([GM03, Theorem III.3.6]). *Let*

$$K^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow K[1]^\bullet.$$

be a distinguished triangle in $D(\mathcal{A})$. Then the sequence

$$\cdots \longrightarrow H^i(K^\bullet) \longrightarrow H^i(L^\bullet) \longrightarrow H^i(M^\bullet) \longrightarrow H^{i+1}(K^\bullet) \longrightarrow \cdots$$

is exact.

A.2 Derived functors

Our main motivation for constructing the derived category $D(\mathcal{A})$ of $\mathcal{A}$ is to define the *right derived functors* $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ for left exact functors $F\colon \mathcal{A} \to \mathcal{B}$. This derived functor RF will extend F in the sense that it "repairs" the exactness on the right. The derived functor is constructed essentially as follows:

- Find a class of objects $\mathcal{R} \subset \mathcal{A}$ for which F is acyclic when applied componentwise in $C(\mathcal{R})$.
- Show that any object of $\mathcal{A}$ is a sub-object of an object of $\mathcal{R}$.
- Compute $RF(A^\bullet)$ for any object in $D(\mathcal{A})$ by replacing $A^\bullet$ with a quasi-isomorphic complex of objects in $\mathcal{R}$, on which one applies the functor F componentwise.

We refer the interested reader to [GM03, Section III.6] for a comprehensive explanation, or to [Tho00, Section 6] for a shorter one.

Definition A.2.1 ([Tho00, Definition 6.1]). Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A class of objects $\mathcal{R} \subset \mathcal{A}$ is *adapted* to a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ if

- $\mathcal{R}$ is stable under direct sums,
- F applied to an acyclic complex in $\mathcal{R}$ (that is, a complex with vanishing cohomology) is acyclic,
- any $A \in \mathcal{A}$ injects $0 \to A \to R$ into some $R \in \mathcal{R}$.

The following proposition states an equivalence of categories between the *bounded below homotopy category* $K^+(\mathcal{R})$ and the bounded below derived category $D^+(\mathcal{A})$, using a construction from [GM03, Chapter III.2]. For our purposes, this will mean that any $\mathcal{A}$-complex can be functorially replaced by a quasi-isomorphic $\mathcal{R}$-complex [Tho00, Section 6].

Proposition A.2.2 ([GM03, Proposition III.6.4]). *Let $\mathcal{R}$ be a class of objects adapted to a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$, and $S_{\mathcal{R}}$ a class of quasi-isomorphisms* *in $K^+(\mathcal{R})$. Then $S_{\mathcal{R}}$ is a localizing class of morphisms in $K^+(\mathcal{R})$ and the canonical functor*

$$K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^+(\mathcal{A})$$

is an equivalence of categories.

Since the derived category is usually not abelian, we need to define what it means for a functor between derived categories to be exact.

Definition A.2.3. A functor $D\colon D(\mathcal{A}) \to D(\mathcal{B})$ is *exact* if it maps distinguished triangles to distinguished triangles.

Using Proposition A.1.14 in conjunction with Theorem A.1.15, this then tells us that such exact functors induce long exact sequences

$$\cdots \to H^i(D(K^\bullet)) \to H^i(D(L^\bullet)) \to H^i(D(M^\bullet)) \to H^{i+1}(D(K^\bullet)) \to \cdots$$

in cohomology for all exact triples.

For any functor $F\colon \mathcal{A} \to \mathcal{B}$, one can define a functor on the bounded below homotopy categories $K^+(F)\colon K^+(\mathcal{A}) \to K^+(\mathcal{B})$ by applying F componentwise on complexes, which transforms homotopic morphisms into homotopic ones.

Definition A.2.4 ([GM03, Definition III.6.6])**.** The *derived functor* of an additive left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ is a pair consisting of an exact functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and a morphism of functors $\epsilon_F\colon Q_{\mathcal{B}} \circ K^+(F) \to RF \circ Q_{\mathcal{A}}$ satisfying the following universal property: For any exact functor between the bounded below derived categories $G\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and any morphism of functors $\epsilon\colon Q_{\mathcal{B}} \circ K^+(F) \to G \circ Q_{\mathcal{A}}$, there is exists a unique morphism of functors $\eta\colon RF \to G$ making the diagram

$$\begin{tikzcd}
 & Q_{\mathcal{B}} \circ K^+(F) \arrow[dl, "\epsilon_F"'] \arrow[dr, "\epsilon"] & \\
RF \circ Q_{\mathcal{A}} \arrow[rr, "\eta \circ Q_{\mathcal{A}}"] & & G \circ Q_{\mathcal{A}}
\end{tikzcd}$$

commute.

Theorem A.2.5 ([GM03, Theorem III.6.8])**.** *Assume that a left exact functor F admits an adapted class of objects $\mathcal{R}$. Then the derived functor RF exists.*

Finally we mention a useful result about injective objects.

Theorem A.2.6 ([GM03, Theorem III.6.12])**.** *If $\mathcal{A}$ contains sufficiently many injective objects, then the class of all injective objects is adapted to any left exact functor F.*

A.3 Hom of complexes

Definition A.3.1 ([GM03, p. 195])**.** Let $\mathcal{A}$ be an abelian category. For any two $M^\bullet, N^\bullet \in C(\mathcal{A})$, the chain complex $\operatorname{Hom}^\bullet(M^\bullet, N^\bullet)$ is defined by

$$\operatorname{Hom}^n(M^\bullet, N^\bullet) \coloneqq \prod_{i\in\mathbb{Z}} \operatorname{Hom}(M^i, N^{i+n}),$$

with chain map

$$df \coloneqq d_N \circ f - (-1)^n f \circ d_M,$$

for $f \in \mathrm{Hom}^n(M^\bullet, N^\bullet)$.

Proposition A.3.2 ([Stacks, Lemmas 13.18.8 and 13.19.8]). *Let $\mathcal{A}$ be an abelian category.*

- *If $I^\bullet \in C^b(\mathcal{A})$ is a complex of injectives, then for any complex $A^\bullet$, the natural homomorphism*

  $$\mathrm{Hom}_{K^b(\mathcal{A})}(A^\bullet, I^\bullet) \to \mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, I^\bullet)$$

  is an isomorphism.

- *If $P^\bullet \in C^b(\mathcal{A})$ is a complex of projectives, then for any complex $A^\bullet$, the natural homomorphism*

  $$\mathrm{Hom}_{K^b(\mathcal{A})}(P^\bullet, A^\bullet) \to \mathrm{Hom}_{D^b(\mathcal{A})}(P^\bullet, A^\bullet)$$

  is an isomorphism.

Definition A.3.3. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X. The sheaf $\mathscr{H}\!om(\mathcal{F}, \mathcal{G})$, defined by

$$U \mapsto \mathrm{Hom}_{\mathbf{Shv}(X)}(\mathcal{F}|_U, \mathcal{G}|_U),$$

is called the *sheaf of local morphisms* of $\mathcal{F}$ into $\mathcal{G}$, or *sheaf Hom*.

Definition A.3.4 ([Stacks, Section 20.35]). Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of sheaves on a topological space X. The complex of sheaves $\mathscr{H}\!om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is defined by

$$\mathscr{H}\!om^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \coloneqq \prod_{i \in \mathbb{Z}} \mathscr{H}\!om(\mathcal{F}^i, \mathcal{G}^{i+n}),$$

with chain map

$$df \coloneqq d_{\mathcal{G}^\bullet} \circ f - (-1)^n f \circ d_{\mathcal{F}^\bullet},$$

for $f \in \mathscr{H}\!om^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$.

Proposition A.3.5 ([Stacks, Equation 20.35.0.1]). *For any $n \in \mathbb{Z}$ and any open $U \subset X$,*

$$H^n(\Gamma(U, \mathscr{H}\!om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet))) = \mathrm{Hom}_{K(U)}(\mathcal{F}^\bullet|_U, \mathcal{G}^\bullet|_U[n]).$$

A.4 Properties of projective sheaves on a polyhedral complex

We develop some results about projective sheaves on cellular complexes, which will be used in the next section, and in particular in Proposition A.5.2.

Proposition A.4.1. *Let $\mathcal{P}$ be a projective sheaf on a polyhedral complex X, and $\sigma \in X$ a cell. Then $\mathcal{P}|_{\mathrm{Star}(\sigma)}$ is also projective.*

Proof. Since $\mathcal{P}$ is projective, by Proposition 2.4.6, it is a direct sum of elementary injectives $\bigoplus_{\tau \in X} \{\tau\}^{V_\tau}$. We will show that $\mathcal{P}|_{\mathrm{Star}(\sigma)}$ is also such a direct sum. For any $\rho \in X$, we have

$$\mathcal{P}|_{\mathrm{Star}(\sigma)}(\rho) = \begin{cases} \mathcal{P}(\rho) = \bigoplus_{\substack{\tau \in X \\ \tau \geq \rho}} V_\tau & \text{if } \rho \in \mathrm{Star}(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

For each $\tau \in X$, let $\tau_\sigma \in \mathrm{Star}(\sigma)$ be the cell

$$\tau_\sigma := \min\{\gamma \in X \mid \gamma \geq \tau, \sigma\},$$

where we let $\tau_\sigma = \varnothing$ if the set is empty. We claim that each such τ_σ is unique. Indeed suppose there are two such cells $\tau_\sigma, \tau_{\sigma'}$, both being minimal. Then $\tau_\sigma \cap \tau_{\sigma'}$ is a non-empty cell of X since $\sigma, \tau \leq \tau_\sigma \cap \tau_{\sigma'}$, and $\tau_\sigma \cap \tau_{\sigma'} \leq \tau_\sigma, \tau_{\sigma'}$, which is a contradiction. Now finally, observe that

$$\bigoplus_{\tau_\sigma \in X} \{\tau_\sigma\}^{V_\tau}(\rho) = \begin{cases} \bigoplus_{\substack{\tau \in X \\ \tau \geq \rho}} V_\tau & \text{if } \rho \in \mathrm{Star}(\sigma), \\ 0 & \text{otherwise,} \end{cases}$$

which means that $\mathcal{P}|_{\mathrm{Star}(\sigma)} = \bigoplus_{\tau_\sigma \in X} \{\tau_\sigma\}^{V_\tau}$, hence $\mathcal{P}|_{\mathrm{Star}(\sigma)}$ is projective. ■

Proposition A.4.2. *For a projective sheaf $\mathcal{P}$ on a polyhedral complex X, the functor $\mathscr{H}om(\mathcal{P}, -)$ is exact.*

Proof. Let $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ be sheaves on X such that the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is exact. We wish to see that the sequence

$$0 \longrightarrow \mathscr{H}om(\mathcal{P}, \mathcal{F}) \longrightarrow \mathscr{H}om(\mathcal{P}, \mathcal{G}) \longrightarrow \mathscr{H}om(\mathcal{P}, \mathcal{H}) \longrightarrow 0$$

is exact. It is sufficient to show this on stalks. Since the stalk of $\mathcal{F}$ at a point $\sigma \in X$ is merely $\mathcal{F}(\mathrm{Star}(\sigma))$ by Proposition 2.2.6, we let $U_\sigma = \mathrm{Star}(\sigma)$ and need to show that the sequence

$$0 \to \mathscr{H}om(\mathcal{P}, \mathcal{F})(U_\sigma) \to \mathscr{H}om(\mathcal{P}, \mathcal{G})(U_\sigma) \to \mathscr{H}om(\mathcal{P}, \mathcal{H})(U_\sigma) \to 0$$

is exact for all $\sigma \in X$. Since $\mathscr{H}om(\mathcal{A}, \mathcal{B})(U) = \mathrm{Hom}_{\mathbf{Shv}(X)}(A|_U, B|_U)$, we need to examine the sequence

$$0 \to \mathrm{Hom}(\mathcal{P}|_{U_\sigma}, \mathcal{F}|_{U_\sigma}) \to \mathrm{Hom}(\mathcal{P}|_{U_\sigma}, \mathcal{G}|_{U_\sigma}) \to \mathrm{Hom}(\mathcal{P}|_{U_\sigma}, \mathcal{H}|_{U_\sigma}) \to 0,$$

which is exact since $\mathcal{P}|_{U_\sigma}$ is projective by Proposition A.4.1. ■

Corollary A.4.3. *For a projective sheaf $\mathcal{P}$ and an open set $U \subset X$, the restricted sheaf $\mathcal{P}|_U$ is also projective.*

Proof. Since $\mathscr{H}om(\mathcal{P}, -)$ is exact, for any open $U \subset X$, the sequence of sections

$$0 \to \mathrm{Hom}(\mathcal{P}|_U, B|_U) \to \mathrm{Hom}(\mathcal{P}|_U, B|_U) \to \mathrm{Hom}(\mathcal{P}|_U, B|_U) \to 0$$

is exact, so that the functor $\mathrm{Hom}_{\mathbf{Shv}(U)}(\mathcal{P}|_U, -)$ is exact, which means that $\mathcal{P}|_U$ is exact. ■

A.5 Derived sheaf Hom of complexes for polyhedral complexes

In this section and onward, we will consider the space X to be a polyhedral complex X with Alexandrov topology. Moreover, we will restrict ourselves to the bounded derived category of sheaves $D^b(X)$.

Proposition A.5.1. *Let $\mathcal{I}^\bullet$ be a bounded complex of injective sheaves, and $U \subset X$ open. Then $\mathcal{I}^\bullet|_U$ is also a complex of injective sheaves.*

Proof. Apply [Stacks, Lemma 20.30.1], since abelian sheaves are $\mathbb{Z}_X$-module. ■

Proposition A.5.2. *Let $\mathcal{P}^\bullet$ be a bounded complex of projective sheaves, and $U \subset X$ open. Then $\mathcal{P}^\bullet|_U$ is also a complex of projective sheaves.*

Proof. This follows by applying Corollary A.4.3 on each element of the complex. ■

Proposition A.5.3. *Let $\mathcal{I}^\bullet$ and $\mathcal{P}^\bullet$ be bounded complexes of sheaves on a polyhedral complex X. If either $\mathcal{I}^\bullet$ is a complex of injectives or $\mathcal{P}^\bullet$ is a complex of projectives, then*

$$H^n(\Gamma(U, \mathscr{H}om^n(\mathcal{P}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(U)}(\mathcal{P}^\bullet|_U, \mathcal{I}^\bullet|_U[n]).$$

Proof. This is an extension of [Stacks, Lemma 20.35.6] to include projective complexes. We have

$$\begin{aligned} H^n(\Gamma(U, \mathscr{H}om^n(\mathcal{P}^\bullet, \mathcal{I}^\bullet))) &= \mathrm{Hom}_{K(U)}(\mathcal{P}^\bullet|_U, \mathcal{I}^\bullet|_U[n]), \\ &= \mathrm{Hom}_{D(U)}(\mathcal{P}^\bullet|_U, \mathcal{I}^\bullet|_U[n]), \end{aligned}$$

where the first equality follows from Proposition A.3.5, and the second equality holds since either $\mathcal{I}^\bullet$ is a complex of injectives, hence so is $\mathcal{I}^\bullet|_U$ by Proposition A.5.1, or since $\mathcal{P}^\bullet$ is a complex of projectives, hence so is $\mathcal{P}^\bullet|_U$ by Proposition A.5.2. Either way, one can apply Proposition A.3.2 to get the second equality. ■

Theorem A.5.4. *Let $\mathcal{P}^\bullet \to \mathcal{F}^\bullet$ and $\mathcal{I}^\bullet \to \mathcal{G}^\bullet$ be quasi-isomorphisms of bounded complexes of sheaves on a polyhedral complex X. If either $\mathcal{I}^\bullet$ is a complex of injectives or $\mathcal{P}^\bullet$ is a complex of projectives, then*

$$\mathscr{H}om^\bullet(\mathcal{P}^\bullet, \mathcal{G}^\bullet) \to \mathscr{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$$

is a quasi-isomorphism.

Proof. This is an extension of [Stacks, Lemma 20.35.7] to include projective complexes. Let I be the object in $D^b(X)$ represented by $\mathcal{I}^\bullet$ and $\mathcal{G}^\bullet$, and P be the object represented by $\mathcal{P}^\bullet$ and $\mathcal{F}^\bullet$. For ease of notation, $A^\bullet := \mathscr{H}om^\bullet(\mathcal{P}^\bullet, \mathcal{G}^\bullet)$ and $B^\bullet := \mathscr{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$. The sheaf $H^n(\mathscr{H}om^\bullet(\mathcal{P}^\bullet, \mathcal{G}^\bullet))$ is given by

$$U \mapsto \left(\frac{\ker\left(d^i_{A^\bullet}\right)}{\mathrm{im}\left(d^{i-1}_{A^\bullet}\right)}\right)(U),$$

which is the sheaf associated to the presheaf

$$U \mapsto \frac{\ker(d^i_{A^\bullet})(U)}{\operatorname{im}(d^{i-1}_{A^\bullet})(U)}.$$

By Proposition A.5.3, this is the presheaf

$$U \mapsto \frac{\ker\left(d^i_{A^\bullet}\right)(U)}{\operatorname{im}\left(d^{i-1}_{A^\bullet}\right)(U)} = H^n(\Gamma(U, \mathscr{H}\!om^\bullet(\mathcal{P}^\bullet, \mathcal{G}^\bullet))) = \operatorname{Hom}_{D(U)}(P|_U, I|_U[n]).$$

Similarly, $H^n(\mathscr{H}\!om^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet))$ is the sheaf associated to the presheaf

$$U \mapsto \frac{\ker(d^i_{B^\bullet})(U)}{\operatorname{im}(d^{i-1}_{B^\bullet})(U)} = H^n(\Gamma(U, \mathscr{H}\!om^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet))) = \operatorname{Hom}_{D(U)}(P|_U, I|_U[n]).$$

Therefore, $H^n(\mathscr{H}\!om^\bullet(\mathcal{P}^\bullet, \mathcal{G}^\bullet))$ and $H^n(\mathscr{H}\!om^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet))$ are sheaves associated to the same presheaf, hence are the same sheaf. ■

Proposition A.5.5. *Let X be a polyhedral complex, and $\mathcal{F}^\bullet \in C^b(X)$ be a bounded complex. Then $\mathcal{F}^\bullet$ has a* projective resolution*, that is, there exists a complex of projective sheaves $\mathcal{P}^\bullet \in C^b(X)$ along with a quasi-isomorphism $\alpha\colon \mathcal{P}^\bullet \to \mathcal{F}^\bullet$. Moreover, $\mathcal{F}^\bullet$ also has an* injective resolution*, i.e. there exists a complex of injective sheaves $\mathcal{I}^\bullet \in C^b(X)$ along with a quasi-isomorphism $\alpha\colon \mathcal{F}^\bullet \to \mathcal{I}^\bullet$.*

Proof. By [Cur14, Definition 7.4.2], the category of sheaves on X has enough projectives. Furthermore, the category of abelian sheaves always has enough projectives [Har77, Corollary III.2.3]. Then one can apply [Stacks, Lemma 13.19.3] to obtain projective resolutions and [Stacks, Lemma 13.18.3] to obtain injective resolutions. Finiteness of the resolution is a consequence of the inductive proofs terminating after finitely many steps, since $\mathcal{F}^\bullet$ only has finitely many non-zero terms. ■

Definition A.5.6. Let X be a polyhedral complex. The *right derived functor of sheaf hom*, $\mathrm{R}\,\mathscr{H}\!om^\bullet$, is defined on two objects $F, G \in D^b(X)$ by taking a bounded complex of injectives $\mathcal{I}^\bullet$ representing G and any bounded complex $\mathcal{F}^\bullet$ representing F, and setting

$$\mathrm{R}\,\mathscr{H}\!om^\bullet(F, G) \coloneqq \mathscr{H}\!om^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet),$$

which is well-defined by Theorem A.5.4. Equivalently, by Theorem A.5.4, we can choose a bounded complex of projectives $\mathcal{P}^\bullet$ of F and any representative $\mathcal{G}^\bullet$ of G, and set $\mathrm{R}\,\mathscr{H}\!om^\bullet(P, I) = \mathscr{H}\!om^\bullet(\mathcal{P}^\bullet, \mathcal{I}^\bullet)$. Either way to compute $\mathrm{R}\,\mathscr{H}\!om^\bullet(F, G)$ can be chosen in any case by Proposition A.5.5.

A.6 Verdier duality

The dualizing complex (Definition 6.1.1) is classically used to define a "dualizing functor" on the derived category of coherent sheaves on a scheme, see for instance [Har66, Chapter V]. In this spirit, Curry [Cur14] defines a *Verdier dual functor* for sheaves on polyhedral complexes with Alexandrov topology:

Definition A.6.1 ([Cur14, p. 238])**.** The *Verdier dual functor* is given by:

$$\begin{aligned} D\colon D^b(\mathbf{Shv}_X) &\to D^b(\mathbf{Shv}_X)^{\mathrm{op}}, \\ \mathcal{F}^\bullet &\mapsto \mathrm{R}\,\mathscr{H}om^\bullet(\mathcal{F}^\bullet, \omega_X^\bullet). \end{aligned}$$

The complex $D(\mathcal{F}^\bullet)$ is called *the Verdier dual complex of* $\mathcal{F}^\bullet$.

The following theorem, adapted from [Cur14, Theorem 12.1.2] and [Cur14, Lemma 6b.2.9], gives us the duality of cohomology we have been working towards.

Theorem A.6.2. *Let $\mathcal{F}$ be a sheaf on a polyhedral complex X of dimension n. Then*

$$H_c^i(X, \mathcal{F}) \cong H^{n-i}(X, D(\mathcal{F}))^*,$$

where $H^q(X, D(\mathcal{F}))$ is the q-th cohomology of the complex $\{\Gamma(X, D(\mathcal{F})^i)\}_{i\in\mathbb{Z}}$, also called the hypercohomology of $D(\mathcal{F})$.

Proof. Since $\omega_X^\bullet$ is a complex of injective objects, $D(\mathcal{F}) = \mathrm{R}\,\mathscr{H}om^\bullet(\mathcal{F}, \omega_X^\bullet)$ is merely the complex

$$0 \longrightarrow \mathscr{H}om\left(\mathcal{F}, \omega_X^{-n}\right) \xrightarrow{\partial} \mathscr{H}om\left(\mathcal{F}, \omega_X^{-n+1}\right) \xrightarrow{\partial} \cdots \longrightarrow \mathscr{H}om\left(\mathcal{F}, \omega_X^{0}\right) \longrightarrow 0,$$

where the leftmost non-zero term is placed in degree $-n$. At each term, we have that

$$\begin{aligned} \mathscr{H}om\left(\mathcal{F}, \omega_X^{-i}\right) &= \mathscr{H}om\left(\mathcal{F}, \oplus_{\sigma\in X^i}[\sigma]^k\right), \\ &= \bigoplus_{\sigma\in X^i} \mathscr{H}om\left(\mathcal{F}, [\sigma]^k\right), \\ &= \bigoplus_{\sigma\in X^i} [\sigma]^{\mathcal{F}(\sigma)^*}, \end{aligned}$$

so that the complex $D(\mathcal{F})$ becomes

$$0 \to \bigoplus\nolimits_{\sigma\in X^n}[\sigma]^{\mathcal{F}(\sigma)^*} \xrightarrow{\partial_{\mathcal{F}}^*} \bigoplus\nolimits_{\tau\in X^{n-1}}[\tau]^{\mathcal{F}(\tau)^*} \xrightarrow{\partial_{\mathcal{F}}^*} \cdots \to \bigoplus\nolimits_{v\in X^0}[v]^{\mathcal{F}(v)^*} \to 0,$$

where the differential maps are now given componentwise by $\partial_{\tau,\sigma} = \mathcal{O}(\tau,\sigma)\rho^{\mathcal{F}}_{\tau,\sigma}$. Now to compute the cohomology of this complex of injective sheaves, it suffices to pushforward componentwise to a point, giving

$$0 \longrightarrow \bigoplus\nolimits_{\sigma\in X^n}\mathcal{F}(\sigma)^* \xrightarrow{\partial_{\mathcal{F}}^*} \bigoplus\nolimits_{\tau\in X^{n-1}}\mathcal{F}(\tau)^* \xrightarrow{\partial_{\mathcal{F}}^*} \cdots \longrightarrow \bigoplus\nolimits_{v\in X^0}\mathcal{F}(v)^* \longrightarrow 0.$$

This complex has the cohomology of the dual of the compactly supported cohomology of $\mathcal{F}$, shifted by n in negative degree, which is what we wanted to show. ∎

Theorem A.6.3. *Let X be a polyhedral fan. If $\mathcal{F}^n \cong \mathcal{H}^{-n}(\omega_X^\bullet)$ and X is Cohen–Macaulay, then $D(\mathcal{F}^0) = \mathcal{F}^n[n]$.*

Proof. Since X is a fan, $\mathcal{F}^0 = \mathbb{R}_X$ is projective, and X is Cohen–Macaulay so that $\omega_X^\bullet$ is quasi-isomorphic to the pure complex $\mathcal{H}^{-n}(\omega_X^\bullet)[n]$ by Theorem 6.1.3, we have

$$D(\mathcal{F}^0) \cong \mathscr{H}\!om(\mathbb{R}, \mathcal{H}^{-n}(\omega_X^\bullet)[n]) \cong \mathcal{H}^{-n}(\omega_X^\bullet)[n] \cong \mathcal{F}^n[n]$$

by assumption. ∎

Corollary A.6.4. *Let X be a Cohen–Macaulay polyhedral fan. If the sheaves $\mathcal{F}^n$ and $\mathcal{H}^{-n}(\omega_X^\bullet)$ are isomorphic, then*

$$H^i(X, \mathcal{F}^0)^* \cong H_c^{n-i}(X, \mathcal{F}^n)$$

for all $i \in \mathbb{Z}$.

Proof. By applying Theorems A.6.2 and A.6.3, and shifting the cohomology by an increment of n, we get the desired isomorphisms. ∎

APPENDIX B

Some results in commutative algebra

In this appendix, we include two results from commutative algebra which are used in Chapter 4.

B.1 Completing diagrams

We give a way of "completing" a commutative diagram, i.e. adding a particular arrow canonically given certain morphisms. This is useful when considering how to relate the cokernels of the maps which inject the top Borel–Moore homology and cohomology into their respective chain and cochain groups.

Lemma B.1.1. *Given a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\rho"] & B \arrow[r, "h"] \arrow[d, "\psi"] & C \arrow[r] & 0, \\ 0 \arrow[r] & A' \arrow[r, "g"] & B' \arrow[r, "i"] & C' \arrow[r] & 0, \end{tikzcd}$$

of modules over a ring R, we can complete the diagram to:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\rho"] & B \arrow[r, "h"] \arrow[d, "\psi"] & C \arrow[r] \arrow[d, "\phi"] & 0, \\ 0 \arrow[r] & A' \arrow[r, "g"] & B' \arrow[r, "i"] & C' \arrow[r] & 0. \end{tikzcd}$$

Proof. Indeed since h is surjective, for all $c \in C$, we can find some $b \in B$ such that $h(b) = c$. Then we define:

$$\phi(c) = i(\psi(b))$$

To see that this is well-defined, suppose that $b' \in B$ is also such that $h(b') = c$. Then we have $h(b-b') = h(b) - h(b') = c - c = 0$, hence $b - b' \in \ker(h) = \mathrm{im}(f)$. Thus we can find $a \in A$ such that $f(a) = b - b'$. Then

$$\psi(b - b') = \psi(f(a)) = g(\rho(a)),$$

and so finally we have $i(\psi(b-b')) = i(g(\rho(a))) = 0$, since $\ker(i) = \mathrm{im}(g)$, which gives that $i(\psi(b)) = i(\psi(b'))$, so that ϕ is well-defined. Moreover, we see that by the definition of ϕ, the right square will be commutative. ■

B.2 Corollary of the snake lemma

Consider a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\rho"] & B \arrow[r] \arrow[d, "\psi"] & C \arrow[r] \arrow[d, "\phi"] & 0, \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0,
\end{tikzcd}$$

Then we have the following:

Proposition B.2.1. *Given a commutative diagram as above, with ψ an isomorphism. then ρ is injective and ϕ is surjective. Moreover, ρ is an isomorphism if and only if ϕ is an isomorphism.*

Proof. This is a standard application of the snake lemma. By exactness in the diagram, we have a long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \ker(\rho) \arrow[r] & \ker(\psi) \arrow[r] & \ker(\phi) \arrow[dll, rounded corners] & \\
 & \operatorname{coker}(\rho) \arrow[r] & \operatorname{coker}(\psi) \arrow[r] & \operatorname{coker}(\phi) \arrow[r] & 0,
\end{tikzcd}$$

which gives

$$\begin{tikzcd}
0 \arrow[r] & \ker(\rho) \arrow[r] & 0 \arrow[r] & \ker(\phi) \arrow[dll, rounded corners] & \\
 & \operatorname{coker}(\rho) \arrow[r] & 0 \arrow[r] & \operatorname{coker}(\phi) \arrow[r] & 0
\end{tikzcd}$$

From this sequence, we see that $\ker(\rho)$ and $\operatorname{coker}(\phi)$ are both 0, hence ρ and ϕ are respectively injective and surjective. Moreover, $\ker(\phi) \cong \operatorname{coker}(\rho)$, hence if one of the morphisms is an isomorphism, so is the other. ■

Bibliography

[polymake] Gawrilow, E. and Joswig, M. "polymake: a framework for analyzing convex polytopes". In: *Polytopes—combinatorics and computation (Oberwolfach, 1997)*. Vol. 29. DMV Sem. Birkhäuser, Basel, 2000, pp. 43–73.

[AB14] Adiprasito, K. A. and Björner, A. *Filtered geometric lattices and Lefschetz Section Theorems over the tropical semiring*. Jan. 2014. arXiv: 1401.7301 [math.AG].

[AHK18] Adiprasito, K., Huh, J., and Katz, E. "Hodge theory for combinatorial geometries". In: *Ann. of Math. (2)* vol. 188, no. 2 (2018), pp. 381–452.

[AK06] Ardila, F. and Klivans, C. J. "The Bergman complex of a matroid and phylogenetic trees". In: *J. Combin. Theory Ser. B* vol. 96, no. 1 (2006), pp. 38–49.

[BH17] Babaee, F. and Huh, J. "A tropical approach to a generalized Hodge conjecture for positive currents". In: *Duke Math. J.* Vol. 166, no. 14 (2017), pp. 2749–2813.

[Bou48] Bourbaki, N. *Éléments de mathématique. VII. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre III: Algèbre multilinéaire*. Actualités Sci. Ind., no. 1044. Hermann et Cie., Paris, 1948, pp. ii+157+ii.

[Bru+15] Brugallé, E. et al. "Brief introduction to tropical geometry". In: *Proceedings of the Gökova Geometry-Topology Conference 2014*. Gökova Geometry/Topology Conference (GGT), Gökova, 2015, pp. 1–75.

[Coo+12] Cools, F. et al. "A tropical proof of the Brill-Noether theorem". In: *Adv. Math.* Vol. 230, no. 2 (2012), pp. 759–776.

[Cur14] Curry, J. M. "Sheaves, cosheaves and applications". PhD thesis. 2014.

[Dim04] Dimca, A. *Sheaves in topology*. Universitext. Springer-Verlag, Berlin, 2004, pp. xvi+236.

[Eis95] Eisenbud, D. *Commutative algebra*. Vol. 150. Graduate Texts in Mathematics. With a view toward algebraic geometry. Springer-Verlag, New York, 1995, pp. xvi+785.

[FH91] Fulton, W. and Harris, J. *Representation theory.* Vol. 129. Graduate Texts in Mathematics. A first course, Readings in Mathematics. Springer-Verlag, New York, 1991, pp. xvi+551.

[GM03] Gelfand, S. I. and Manin, Y. I. *Methods of homological algebra.* Second. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003, pp. xx+372.

[Gro11] Gross, M. *Tropical geometry and mirror symmetry.* Vol. 114. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2011, pp. xvi+317.

[GS] Gross, A. and Shokrieh, F. “Sheafifying tropical homology”. In perparation.

[Har66] Hartshorne, R. *Residues and duality.* Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin-New York, 1966, pp. vii+423.

[Har77] Hartshorne, R. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496.

[Hat02] Hatcher, A. *Algebraic topology.* Cambridge University Press, Cambridge, 2002, pp. xii+544.

[Ite+16] Itenberg, I. et al. *Tropical Homology.* Apr. 2016. arXiv: 1604.01838.

[Ive86] Iversen, B. *Cohomology of sheaves.* Universitext. Springer-Verlag, Berlin, 1986, pp. xii+464.

[JP14] Jensen, D. and Payne, S. “Tropical independence I: Shapes of divisors and a proof of the Gieseker-Petri theorem”. In: *Algebra Number Theory* vol. 8, no. 9 (2014), pp. 2043–2066.

[JP16] Jensen, D. and Payne, S. “Tropical independence II: The maximal rank conjecture for quadrics”. In: *Algebra Number Theory* vol. 10, no. 8 (2016), pp. 1601–1640.

[JRS18] Jell, P., Rau, J., and Shaw, K. “Lefschetz $(1,1)$-theorem in tropical geometry”. In: *Épijournal Geom. Algébrique* vol. 2 (2018), Art. 11, 27.

[JSS19] Jell, P., Shaw, K., and Smacka, J. “Superforms, tropical cohomology, and Poincaré duality”. In: *Adv. Geom.* Vol. 19, no. 1 (2019), pp. 101–130.

[Kat16] Katz, E. “Matroid theory for algebraic geometers”. In: *Nonarchimedean and tropical geometry.* Simons Symp. Springer, [Cham], 2016, pp. 435–517.

[KSW17] Kastner, L., Shaw, K., and Winz, A.-L. “Cellular sheaf cohomology of polymake”. In: *Combinatorial algebraic geometry.* Vol. 80. Fields Inst. Commun. Fields Inst. Res. Math. Sci., Toronto, ON, 2017, pp. 369–385.

[Mik05] Mikhalkin, G. "Enumerative tropical algebraic geometry in $\mathbb{R}^2$". In: *J. Amer. Math. Soc.* Vol. 18, no. 2 (2005), pp. 313–377.

[MS05] Miller, E. and Sturmfels, B. *Combinatorial commutative algebra*. Vol. 227. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xiv+417.

[MS15] Maclagan, D. and Sturmfels, B. *Introduction to tropical geometry*. Vol. 161. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015, pp. xii+363.

[Mur06] Murfet, D. *Derived Categories Of Sheaves*. 2006. URL: http://therisingsea.org/notes/DerivedCategoriesOfSheaves.pdf (visited on 07/07/2019).

[MZ14] Mikhalkin, G. and Zharkov, I. "Tropical eigenwave and intermediate Jacobians". In: *Homological mirror symmetry and tropical geometry*. Vol. 15. Lect. Notes Unione Mat. Ital. Springer, Cham, 2014, pp. 309–349.

[Oxl11] Oxley, J. *Matroid theory*. Second. Vol. 21. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2011, pp. xiv+684.

[Spe04] Speyer, D. E. *Tropical Linear Spaces*. Sept. 2004. arXiv: math/0410455 [math.CO].

[Stacks] The Stacks project authors. *The Stacks project*. 2019. URL: https://stacks.math.columbia.edu (visited on 03/26/2019).

[Tho00] Thomas, R. P. *Derived categories for the working mathematician*. Jan. 2000. arXiv: math/0001045.