Propositional Quantification in Bimodal **S5***

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Final draft

Abstract

Propositional quantifiers are added to a propositional modal language with two modal operators. The resulting language is interpreted over so-called products of Kripke frames whose accessibility relations are equivalence relations, letting propositional quantifiers range over the powerset of the set of worlds of the frame. It is first shown that full second-order logic can be recursively embedded in the resulting logic, which entails that the two logics are recursively isomorphic. The embedding is then extended to all sublogics containing the logic of so-called fusions of frames with equivalence relations. This generalizes a result due to Antonelli and Thomason, who construct such an embedding for the logic of such fusions.

1 Introduction

Propositional modal logics allow the formulation of general laws for a given modality. E.g., claiming $\Box p \to p$ to be a theorem of the logic of metaphysical necessity means claiming that what is necessary is the case. But not all general statements of interest to philosophers have this simple universal form. An example is the claim that every possible proposition is strictly implied by some possible proposition which strictly implies, for every proposition, either it or its negation. To express such statements, it is natural to introduce quantifiers binding the propositional variables of standard modal logic, with which it can be formulated as follows:

$$\text{(At)} \quad \forall p(\Diamond p \to \exists q(\Diamond q \land \Box(q \to p) \land \forall r(\Box(q \to r) \lor \Box(q \to \neg r))))$$

Such quantifiers have a long history in modal logic, having been employed by pioneers such as Lewis and Langford (1932, p. 179) and Kripke (1959, p. 12). (At) is also of great philosophical interest: A proposition which strictly implies, for each proposition, either it or its negation, can be thought of as playing the role of a possible world, so the question whether (At) is true is crucial to the philosophical debate about the nature of possible worlds, and in particular the proposal that worlds can be taken to be propositions of a special kind. Such matters have been, and continue to be, discussed at length in philosophy and logic; see, e.g., Gallin (1975, pp. 79–89), Stalnaker (1976), Fine (1977), Williamson (2013, pp. 102–108), and Holliday (forthcoming). It is clear that the

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philosophical investigation of such matters will benefit greatly from a systematic formal investigation of the role which (At) plays in theorizing about propositions and modality; this motivates the formal study of modal logics with propositional quantifiers.

The study of propositionally quantified modal logics received some attention around 1970, with contributions by Bull (1969), Fine (1970), Kaplan (1970), and Gabbay (1971). The most important strand of investigation interprets logics with propositional quantifiers over Kripke frames. Taking the elements of the set W in a Kripke frame $\langle W, R\rangle$ to represent possible worlds, and taking a proposition to correspond to a collection of worlds (the worlds in which it is true), propositional quantifiers are interpreted as ranging over the powerset of W in a Kripke frame $\langle W, R\rangle$. Each class of Kripke frames therefore determines a propositionally quantified modal logic as usual, namely as the set of formulas true in each world of each such frame on any valuation function mapping propositional variables to sets of worlds. As shown by Fine (1970), many of the classes of frames which give rise to interesting propositional modal logics lead to propositionally quantified modal logics which are not recursively axiomatizable. These results were later strengthened and unified by Kaminski and Tiomkin (1996), who show for every class of frames containing all frames validating **S4.2**, i.e, all reflexive, transitive and convergent frames, that the set of sentences of second-order predicate logic valid on all standard models (henceforth **SOL**) can be recursively embedded in the propositionally quantified logic of this class of frames. It follows by routine considerations that the two sets are recursively isomorphic; see Kremer (1993) for a careful presentation of the relevant complexity-theoretic matters. The fact that adding propositional quantifiers to propositional modal logic often gives rise to such a complex logic may be one reason why the study of propositional quantifiers has been comparatively marginal within modal logic.

Although propositional quantifiers often lead to highly complex modal logics, this is not always the case. The most important result in this regard concerns the propositionally quantified modal logic of frames whose relations are equivalence relations. Kaplan (1970) and Fine (1970) show it to be decidable, and provide recursive axiomatizations. Such results are of particular interest to philosophers interested in the role of (At) in thinking about metaphysical necessity, a modality which is often considered to be governed by the modal logic **S5**, the set of formulas valid on all and only those frames whose relation is an equivalence relation. Fine (1977) suggests that not just worlds, but also instants (of time) can be understood as special propositions, relying on a variant of (At) using a temporal operator for *always*, for which **S5** is plausible as well. This proposal suggests the question whether a propositionally quantified modal logic with two operators, one formalizing *necessarily* and one formalizing *always*, each of which individually is governed by the theorems of **S5**, is recursively axiomatizable.

Such a logic is considered by Antonelli and Thomason (2002), who investigate the propositionally quantified bimodal logic of the class of frames which contain two equivalence relations interpreting the two modalities. They show that this logic is not recursively axiomatizable by embedding **SOL** in it, a result for which Kuhn (2004) provides a simpler proof. While this is interesting, it is a somewhat limited result, since there is a range of classes of bimodal frames which give rise to bimodal logics whose two unimodal fragments are both **S5**, but which differ in the interaction principles between the two modalities. This paper strengthens the result, showing how to embed **SOL** in any logic of an interval of propositionally quantified bimodal logics whose lower bound is the logic considered by Antonelli and Thomason. The interval contains many of the most plausible candidates of a combined modal-temporal logic, and so shows that in investigating principles like (At) in such a setting, it may well be impossible to characterize the logical truths using a recursive axiom system.

2 Frames and Fusions

Let the formal language to be used be based on a countable set of propositional variables Φ, using the Boolean operators $\neg$ for negation and $\wedge$ for conjunction, two unary modal operators $\boxminus$ and $\boxbar$, and a universal quantifier $\forall$ binding propositional variables. Other Boolean operators such as $\vee$, $\rightarrow$ and $\leftrightarrow$ are understood as syntactic abbreviations, and similarly for $\diamondminus$, $\diamondbar$, and $\exists$, where $\diamondminus := \neg \boxminus \neg$ and $\diamondbar := \neg \boxbar \neg$.

Let a *unimodal frame* be a pair $\mathfrak{F} = \langle W, R \rangle$ such that W is a set, called the *set of worlds*, and R is a binary relation on W, called the *accessibility relation*. Let an *equivalence frame* be a frame $\langle W, R \rangle$ such that R is an equivalence relation on W. A *bimodal frame* is a triple $\mathfrak{F} = \langle W, R_{-}, R_{|} \rangle$ such that W is a set, and R_{-} and $R_{|}$ are binary relations on W. Relative to a valuation function $V : \Phi \rightarrow \mathcal{P}(W)$ and a world $w \in W$, truth is defined inductively as usual:

$$
\begin{aligned}
&\mathfrak{F}, V, w \vDash p \text{ iff } w \in V(p)\\
&\mathfrak{F}, V, w \vDash \neg\varphi \text{ iff not } \mathfrak{F}, V, w \vDash \varphi\\
&\mathfrak{F}, V, w \vDash \varphi \wedge \psi \text{ iff } \mathfrak{F}, V, w \vDash \varphi \text{ and } \mathfrak{F}, V, w \vDash \psi\\
&\mathfrak{F}, V, w \vDash \boxminus\varphi \text{ iff } \mathfrak{F}, V, v \vDash \varphi \text{ for all } v \in W \text{ such that } R_{-}wv\\
&\mathfrak{F}, V, w \vDash \boxbar\varphi \text{ iff } \mathfrak{F}, V, v \vDash \varphi \text{ for all } v \in W \text{ such that } R_{|}wv\\
&\mathfrak{F}, V, w \vDash \forall p\varphi \text{ iff } \mathfrak{F}, V[P/p], w \vDash \varphi \text{ for all } P \subseteq W
\end{aligned}
$$

where $V[P/p](p) = P$ and $V[P/p](q) = V(q)$ for all $q \in \Phi \backslash \{p\}$.

For any bimodal frame $\mathfrak{F} = \langle W, R_{-}, R_{|} \rangle$, let a formula φ be *valid on* $\mathfrak{F}$ if $\mathfrak{F}, V, w \vDash \varphi$ for all $V : \Phi \rightarrow \mathcal{P}(W)$ and $w \in W$. For any class C of bimodal frames, let φ be *valid on* C if φ is valid on all frames in C. Let the *quantified logic of* C, written Λ_{C}, be the set of formulas φ valid on C.

Antonelli and Thomason (2002) and Kuhn (2004) consider the class FE of *fusions of equivalence frames*: the class of frames $\mathfrak{F} = \langle W, R_{-}, R_{|} \rangle$ such that R_{-} and $R_{|}$ are both equivalence relations on W. They establish that Λ_{FE} is recursively isomorphic to **SOL**. FE is minimally restrictive among classes of bimodal frames validating the theorems of **S5** for both unimodal fragments: in any bimodal frame which is not a fusion of equivalence frames, at least one accessibility relation is not an equivalence relation, and so some theorem of **S5** for one of the two modalities is not valid on this frame. To a first approximation, one can therefore think of FE as appropriate for bimodal systems containing two **S5** modalities which are completely independent.

3 Products

In many applications of bimodal logics, the two modal operators are related in non-trivial ways. E.g., consider a combined spatio-temporal modal logic of two operators *everywhere* and *always*. In this case, one might want to require the two operators to commute:

$$\boxminus \boxbar p \leftrightarrow \boxbar \boxminus p$$

How might one construct bimodal frames for a logic of *everywhere* and *always* which validate this formula? A natural idea is to start with two sets of indices, one for locations and one for times, associated with equivalence relations of being spatially connected and being temporally connected. The set of worlds is then the set of location-time pairs. Two such pairs stand in the accessibility relation for *everywhere* if they contain the same time index and the two location indices are spatially connected; they stand in the accessibility relation for *always* if they contain the same location index and the two time indices are temporally connected.

Thinking of the two sets of indices with their associated equivalence relations as equivalence frames, this suggests a more general procedure, which allows one to combine any two unimodal frames into a bimodal frame; this is known as forming the product of two unimodal frames. Formally, for uni-modal frames $\mathfrak{F}_- = \langle W_-, R_- \rangle$ and $\mathfrak{F}_| = \langle W_|, R_| \rangle$, let the *product of* $\mathfrak{F}_-$ *and* $\mathfrak{F}_|$ be the bimodal frame $\mathfrak{F}_- \times \mathfrak{F}_| = \langle W_- \times W_|, R'_-, R'_| \rangle$, where

$$R'_-\langle x_-, x_| \rangle \langle y_-, y_| \rangle \text{ iff } R_- x_- y_- \text{ and } x_| = y_|, \text{ and}$$
$$R'_|\langle x_-, x_| \rangle \langle y_-, y_| \rangle \text{ iff } x_- = y_- \text{ and } R_| x_| y_|.$$

Let PE be the class of products of equivalence frames.

Bimodal frames along these lines were already explored in (Segerberg, 1973), and it follows from Segerberg's results that the unquantified bimodal logic of PE is decidable. Many applications of bimodal logic can be understood as involving versions of such products. For more on the general theory of products and their applications, see Gabbay and Shehtman (1998), Gabbay et al. (2003) and Kurucz (2007). In fact, the case which motivates the present study, the modal-temporal logic of *necessarily* and *always*, is often investigated using such products, e.g., in Kaplan (1978).

The rest of this section proves that just like Λ_{FE}, the propositionally quantified logic of *fusions* of equivalence frames, Λ_{PE}, the propositionally quantified logic of *products* of equivalence frames, is recursively isomorphic to **SOL**.

Theorem 1. Λ_{PE} *is recursively isomorphic to* **SOL**.

As noted above, it suffices to construct a recursive embedding of **SOL** in Λ_{PE}. This will be done in several stages; first, it will be noted that **SOL** can be recursively embedded in one of its fragments, then it will be shown how to recursively embed the logic of a subclass of PE in Λ_{PE}, and finally, a recursive embedding of the fragment of **SOL** in the logic of the relevant subclass of PE will be specified.

The restricted fragment of **SOL** does not contain any non-logical constants, no logical identity connective, and only binary second-order variables. Call this fragment $\mathbf{SOL}'$; it can be obtained by intersecting $\mathbf{SOL}$ with the set of sentences built up from first-order variables and binary second-order variables (both assumed to form countably infinite sets), allowing only atomic predications, Boolean combinations (for simplicity, using only $\neg$ and $\wedge$) and first- and second-order quantifiers. Montague (1965, p. 257) already notes that every second-order sentence is equivalent, on standard models, to one containing only binary second-order variables, attributing the result to Kaplan. Although no explicit mapping is specified, the existence of a recursive embedding of $\mathbf{SOL}$ in $\mathbf{SOL}'$ follows, e.g., from a result due to Rabin and Scott, presented in Nerode and Shore (1980, section 1). In contrast to frames, where the set W may be empty, it will be assumed that only non-empty sets serve as standard models for second-order logic; this choice is immaterial, but simplifies the presentation of the results.

Turning to the required subclass of PE, define first a *universal frame* to be a unimodal frame $\mathfrak{F} = \langle W, R\rangle$ such that R is the universal relation on W, i.e., $R = W^2$. Let a *square frame* be the product of a universal frame with itself. Let PU be the class of products of universal frames, and S the class of square frames. It will now be shown how to recursively embed Λ_{S} in Λ_{PE}.

For any bimodal frame $\mathfrak{F} = \langle W, R_{-}, R_{|}\rangle$ and $w \in W$, let $\mathfrak{F}_w$ be the *subframe generated by* w, defined as usual (see, e.g., Blackburn et al. (2001, p. 138)). The following lemma extends a standard observation on subframes to the present language with propositional quantifiers:

Lemma 2. *A formula φ is valid on a bimodal frame $\mathfrak{F} = \langle W, R_{-}, R_{|}\rangle$ iff for all $w \in W$, φ is valid on $\mathfrak{F}_w$.*

Proof. Let $w \in W$, and let $\mathfrak{F}_w = \langle W', R'_{-}, R'_{|}\rangle$ be the subframe generated by w. For each $V : \Phi \to \mathcal{P}(W)$, let $V' : \Phi \to \mathcal{P}(W')$ be the mapping $p \mapsto V(p) \cap W'$. An induction on the complexity of φ shows that for all $V : \Phi \to \mathcal{P}(W)$ and $v \in W'$, $\mathfrak{F}, V, v \vDash \varphi$ iff $\mathfrak{F}_w, V', v \vDash \varphi$. As all the other cases are routine, consider the induction step for propositional quantifiers:

$$\begin{aligned}
\mathfrak{F}, V, v \vDash \forall p\varphi \text{ iff } & \mathfrak{F}, V[P/p], v \vDash \varphi \text{ for all } P \subseteq W \\
\text{iff } & \mathfrak{F}_w, (V[P/p])', v \vDash \varphi \text{ for all } P \subseteq W \text{ (by IH)} \\
\text{iff } & \mathfrak{F}_w, V'[P/p], v \vDash \varphi \text{ for all } P \subseteq W' \\
\text{iff } & \mathfrak{F}_w, V', v \vDash \forall p\varphi
\end{aligned}$$

Thus a formula is falsifiable at a world v in $\mathfrak{F}_w$ just in case it is falsifiable at v in $\mathfrak{F}$, from which the claim follows. □

Lemma 3. $\Lambda_{\mathsf{PE}} = \Lambda_{\mathsf{PU}}$.

Proof. The products of universal frames are the subframes generated by worlds of products of equivalence frames, so the claim follows from Lemma 2. □

Two syntactic abbreviations in the bimodal language are needed to construct the embedding of Λ_{S} in Λ_{PE}. It is easy to see that the following formula expresses – over products of universal frames – that p is an atomic (singleton) proposition:

$$\mathrm{atom}(p) := \diamondminus \diamondbar\, p \wedge \forall q(\boxminus \boxbar\,(p \to q) \vee \boxminus \boxbar\,(p \to \neg q))$$

Fact 4. *For any frame $\mathfrak{F}$ in PU, valuation function V and world w: $\mathfrak{F}, V, w \vDash \mathrm{atom}(p)$ iff $V(p)$ is a singleton set.*

Propositions in products of universal frames can be understood as relations between the two underlying sets of worlds. It is easy to see that the following construction expresses – over products of universal frames – that p is a bijection:

$$
\begin{aligned}
\Delta(p) \coloneqq {} & \boxminus \diamondbar(p \wedge \exists q(\mathrm{atom}(q) \wedge \boxbar(p \to q))) \wedge \\
& \boxbar \diamondminus(p \wedge \exists q(\mathrm{atom}(q) \wedge \boxminus(p \to q)))
\end{aligned}
$$

Fact 5. *For any product $\mathfrak{F}$ of universal frames on sets W_{-} and $W_{|}$, valuation function V and world w: $\mathfrak{F}, V, w \vDash \Delta(p)$ iff $V(p)$ is a bijection from W_{-} to $W_{|}$.*

Thus, $\exists p\Delta(p)$ expresses – over products of universal frames – the property of being isomorphic to a square frame; with this, Λ_{S} can be embedded in $\Lambda_{\mathsf{PU}} = \Lambda_{\mathsf{PE}}$:

Lemma 6. *The mapping $\varphi \mapsto (\exists p\Delta(p) \to \varphi)$ recursively embeds Λ_{S} in $\Lambda_{\mathsf{PU}} = \Lambda_{\mathsf{PE}}$.*

Proof. If $\varphi \notin \Lambda_{\mathsf{S}}$, then there is a square frame on which φ is not valid. This is also a product of universal frames, on which, by Fact 5, $\exists p\Delta(p)$ is valid. Thus this frame shows that $\exists p\Delta(p) \to \varphi$ is not valid on products of universal frames, and so $\exists p\Delta(p) \to \varphi \notin \Lambda_{\mathsf{PU}}$.

If $\exists p\Delta(p) \to \varphi \notin \Lambda_{\mathsf{PU}}$, then there is a universal frame on which $\exists p\Delta(p) \to \varphi$ is not valid. Thus relative to some valuation function V and world w, $\exists p\Delta(p)$ is true and φ is false. By Fact 5, it follows that the frame is isomorphic to a square frame. Since validity is invariant under isomorphisms, φ is invalid on some square frame, and so $\varphi \notin \Lambda_{\mathsf{S}}$. □

It therefore only remains to recursively embed **SOL′** in Λ_{S}. The idea is to map each **SOL′**-sentence to a sentence of the bimodal language such that the first is true when interpreted on a set non-empty D just in case the second is true when interpreted on the square frame $\mathfrak{F}_D$ which is the product of the universal frame on D with itself. The task is thus to use the bimodal language to simulate, on any such square frame, quantification over elements of D and binary relations on D.

To simulate first-order quantification, any $d \in D$ will be uniquely associated with a proposition of $\mathfrak{F}_D$. There are many options; a natural one is given by the function $\gamma_D : d \mapsto \{\langle d, e\rangle : e \in D\}$. Call a proposition a *column in $\mathfrak{F}_D$* if it is in the range of γ_D. This is easily seen to be expressible in the bimodal language over square frames as follows:

$$
\mathrm{column}(p) \coloneqq \diamondminus \boxbar\, (p \wedge \forall q(\boxminus(p \to q) \vee \boxminus(p \to \neg q)))
$$

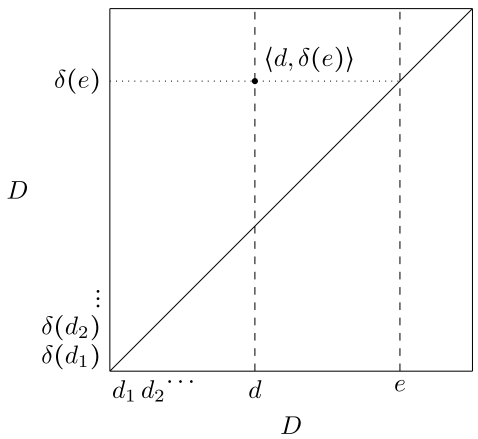
Fact 7. *For any square frame $\mathfrak{F}_D$, valuation function V and world w: $\mathfrak{F}, V, w \vDash \mathrm{column}(p)$ iff $V(p)$ is a column in $\mathfrak{F}_D$.*

First-order quantification over D is thus easily simulated using propositional quantification restricted to columns.

Quantification over binary relations on D may seem straightforward to simulate, since the propositions in $\mathfrak{F}_D$ are simply subsets of D^2, i.e., binary relations on D. One might therefore try to simulate such second-order quantifiers directly using propositional quantifiers. The difficulty with this proposal is simulating predications: a bimodal formula is needed which, given an arbitrary proposition $P \subseteq D^2$ and columns $\gamma_D(d)$ and $\gamma_D(e)$, expresses that $\langle d, e\rangle \in P$. No formula suggests itself.

Instead, it is helpful to fix an arbitrary permutation δ on D; that a proposition is a permutation can be expressed using $\Delta(p)$. Let a binary relation R not be represented by itself, taken as a proposition, but by the proposition $\delta.R = \{\langle d, \delta(e)\rangle : \langle d, e\rangle \in R\}$. Since the map $R \mapsto \delta.R$ is a permutation of $\mathcal{P}(D^2)$, unrestricted propositional quantification can still be used to simulate binary second-order quantification. And it is now possible to express in the bimodal language that $\langle d, e\rangle \in R$ using the propositions $\gamma_D(d)$, $\gamma_D(e)$ and $\delta.R$: this is the case iff $\langle d, \delta(e)\rangle \in \delta.R$, and so iff there is a world in which $\delta.R$ and $\gamma_D(d)$ are true, and from which a world is $\boxminus$-accessible where both δ and $\gamma_D(e)$ are true.

The following diagram illustrates this idea. The elements of D are depicted as arranged horizontally in some arbitrary order, and vertically in the corresponding order obtained by applying δ to each element. The columns of d and e are indicated using dashed lines.



To make this idea precise, associate, injectively, each first-order variable x with a propositional variable p_x and each binary second-order variable X with a propositional variable p_X. A further variable q will be used (to refer to a permutation of the domain as required for the simulation). With these propositional variables, a mapping from the second-order language to the bimodal language is defined recursively:

$$\begin{aligned}
(Xxy)^* &:= \diamondplus \diamondminus (p_X \wedge p_x \wedge \diamondminus(q \wedge p_y)) \\
(\neg A)^* &:= \neg A^* \\
(A \wedge B)^* &:= A^* \wedge B^* \\
(\exists x A)^* &:= \exists p_x(\text{column}(p_x) \wedge A^*) \\
(\exists X A)^* &:= \exists p_X A^*
\end{aligned}$$

Consider a non-empty set D and permutation δ of D. For any assignment function a of first- and second-order variables to elements of D and binary relations on D, let $a^\delta : \Phi \to \mathcal{P}(D^2)$ such that $a^\delta(q) = \delta$, $a^\delta(p_x) = \gamma_D(a(x))$ and $a^\delta(p_X) = \delta.a(X)$ (as defined above), for all first- and second-order variables x/X. Writing $D, a \vDash A$ for A being true on D under the assignment function a, the next lemma shows that $\cdot^*$ successfully simulates second-order logic, in the following sense:

Lemma 8. *For any non-empty set D, permutation δ of D, second-order assignment function a on D, second-order formula A, and $w \in D^2$,*

$$D, a \vDash A \text{ iff } \mathfrak{F}_D, a^\delta, w \vDash A^*.$$

Proof. By induction on the complexity of A. The most interesting case of the induction is the base case; the other cases are routine:

$\mathfrak{F}_D, a^\delta, w \vDash (Xxy)^*$
iff $\mathfrak{F}_D, a^\delta, w \vDash \diamondsuit \diamond (p_X \wedge p_x \wedge \diamond(q \wedge p_y))$
iff there is a $\langle d, e\rangle \in a^\delta(p_X) \cap a^\delta(p_x)$ such that for some $d' \in D, \langle d', e\rangle \in a^\delta(q) \cap a^\delta(p_y)$
iff there is a $\langle d, e\rangle \in \delta.a(X) \cap \gamma_D(a(x))$ such that for some $d' \in D, \langle d', e\rangle \in \delta \cap \gamma_D(a(y))$
iff there is a $\langle d, e\rangle \in \delta.a(X)$ such that $d = a(x)$ and $\delta^{-1}(e) = a(y)$
iff $\langle a(x), \delta(a(y))\rangle \in \delta.a(X)$
iff $\langle a(x), a(y)\rangle \in a(X)$
iff $D, a \vDash Xxy$

□

As a last step, a variant $\cdot^\dagger$ of the mapping $\cdot^*$ is defined which guarantees that q is interpreted as a permutation:

$$A^\dagger := \forall q(\Delta(q) \to A^*)$$

The next lemma shows that this has the intended effect when applied to *sentences* of the second-order language:

Lemma 9. *For any second-order sentence A, $A^\dagger \in \Lambda_{\mathsf{S}}$ iff $A \in$ **SOL**$'$.*

Proof. If $A^\dagger \notin \Lambda_{\mathsf{S}}$, then there is a set D, valuation function V and $w \in D^2$ such that $\mathfrak{F}_D, V, w \nvDash \Delta(q) \to A^*$. Thus $V(q)$ is a permutation of D. Since only q may be free in A^*, for any second-order assignment function a, $\mathfrak{F}_D, a^{V(q)}, w \nvDash A^*$. So by Lemma 8, $D, a \nvDash A$, whence $A \notin$ **SOL**$'$.

Conversely, if $A \notin$ **SOL**$'$, then $D, a \nvDash A$ for some non-empty set D and second-order assignment function a. Let δ be a permutation of D and $w \in D^2$; by Lemma 8, $\mathfrak{F}_D, a^\delta, w \nvDash A^*$. Since δ is a permutation, $\mathfrak{F}_D, a^\delta, w \vDash \Delta(q)$, and so $\mathfrak{F}_D, a^\delta, w \nvDash \forall q(\Delta(q) \to A^*)$, whence $A^\dagger \notin \Lambda_{\mathsf{S}}$. □

Proof of Theorem 1. As noted above, **SOL** can be recursively embedded in **SOL**$'$. Lemma 9 recursively embeds **SOL**$'$ in Λ_{S}. Lemma 6 recursively embeds Λ_{S} in Λ_{PE}. Composing the three embeddings produces a recursive embedding of **SOL** in Λ_{PE}, which, as remarked earlier, shows that the two logics are recursively isomorphic. □

4 In-Between

Fusions and products are among the best-studied ways of constructing a class of bimodal frames from two classes of unimodal frames. As fusions impose in a certain sense minimal constraints on the interactions of the two modal operators and products impose quite strong constraints on their interactions, it is of interest to consider combinations in strength between these two – compare the combinations surveyed in Kurucz (2007). In this section, the embedding of second-order logic in propositionally quantified bimodal logic is generalized to any set of formulas between the logic of fusions of equivalence frames and the logic of products of equivalence frames.

All that is required for the proof is to express – over fusions of equivalence frames – the property of being a world whose generated subframe is isomorphic to a product of universal frames. This can be done using the conjunction of the following sentences:

$$
\begin{aligned}
\mathrm{com} &:= \forall p \boxminus \boxbar (\boxminus \boxbar p \leftrightarrow \boxbar \boxminus p)\\
\mathrm{sing} &:= \forall p \boxminus \boxbar (\forall q \forall r ((\boxminus q \wedge \boxbar r) \rightarrow \boxminus \boxbar (p \rightarrow (q \wedge r))) \rightarrow\\
&\qquad\qquad \forall q (\boxminus \boxbar (p \rightarrow q) \vee \boxminus \boxbar (p \rightarrow \neg q)))
\end{aligned}
$$

The first guarantees a strong form of *com*mutativity between $\boxbar$ and $\boxminus$; adding the second guarantees that every non-empty set of worlds which are all related to each other by both relations is a *sing*leton set.

Lemma 10. *For any frame $\mathfrak{F}$ in FE, valuation function V, and world w:*

$$\mathfrak{F}, V, w \vDash \mathrm{com} \wedge \mathrm{sing} \textit{ iff } \mathfrak{F}_w \textit{ is isomorphic to a frame in } \mathsf{PU}.$$

Proof. The *if* direction is straightforward to verify; assume therefore that $\mathfrak{F}, V, w \vDash \mathrm{com} \wedge \mathrm{sing}$. Let $\mathfrak{F} = \langle W, R_{-}, R_{|} \rangle$ and $\mathfrak{F}_w = \langle W', R'_{-}, R'_{|} \rangle$. For a binary relation R and set X, write $R[X]$ for $\{y : \text{there is an } x \in X \text{ such that } Rxy\}$.

Claim 1: $W' = R_{-}[R_{|}[\{w\}]]$. Since R_{-} and $R_{|}$ are equivalence relations, it suffices to show that (i) $R_{-}[R_{-}[R_{|}[\{w\}]]] \subseteq R_{-}[R_{|}[\{w\}]]$ and (ii) $R_{|}[R_{-}[R_{|}[\{w\}]]] \subseteq R_{-}[R_{|}[\{w\}]]$. (i) follows from the transitivity of R_{-}. For (ii), assume $wR_{|}vR_{-}uR_{|}t$. Then by com, $wR_{|}vR_{|}sR_{-}t$ for some $s \in W$. So by transitivity of $R_{|}$, $wR_{|}sR_{-}t$.

Claim 2: For any $v, u \in W'$, $[v]_{R'_{-}} \cap [u]_{R'_{|}} \neq \varnothing$. Let $v, u \in W'$; by claim 1, there are $t, s \in W'$ such that $vR'_{-}tR'_{|}wR'_{|}sR'_{-}u$. So $vR'_{-}tR'_{|}sR'_{-}u$, and thus by com, $vR'_{-}tR'_{-}rR'_{|}u$ for some $r \in W'$. So $vR'_{-}rR'_{|}u$, as required.

Claim 3: For any $v, u \in W'$, $|[v]_{R'_{-}} \cap [u]_{R'_{|}}| \leq 1$. Assume for contradiction that there are distinct $t, s \in [v]_{R'_{-}} \cap [u]_{R'_{|}}$. Consider a valuation which maps p to $\{t, s\}$. By claim 1, the main conditional of sing must be true at t and s under this valuation. However, while the antecedent is true, the consequent is false, as can be seen by interpreting q as $\{t\}$ or $\{s\}$, contradicting the truth of the conditional.

Define $\mathfrak{F}^*$ to be the product of the universal frames based on the quotient sets $W'/R'_{|}$ and W'/R'_{-} (the sets of equivalence classes of the relevant equivalence relation). It now suffices to show that the map $f : v \mapsto \langle [v]_{R'_{|}}, [v]_{R'_{-}} \rangle$ on W' is an isomorphism from $\mathfrak{F}_w$ to $\mathfrak{F}^*$. Surjectivity follows by claim 2, injectivity by claim 3, and that f respects the accessibility relations is straightforward. □

Theorem 11. SOL *can be recursively embedded in every set Λ such that $\Lambda_{\mathsf{FE}} \subseteq \Lambda \subseteq \Lambda_{\mathsf{PE}}$.*

Proof. Assume $\Lambda_{\mathsf{FE}} \subseteq \Lambda \subseteq \Lambda_{\mathsf{PE}}$. Define a mapping $\tau : \varphi \mapsto ((\mathrm{com} \wedge \mathrm{sing}) \rightarrow \varphi)$ on the modal language. τ recursively embeds Λ_{PU} in Λ, as will now be shown:

If $\varphi \notin \Lambda_{\mathsf{PU}}$, then φ is not valid on some product of universal frames. By Lemma 10, $\mathrm{com} \wedge \mathrm{sing}$ is valid on this frame, so $\tau(\varphi)$ is not valid on it. Thus $\tau(\varphi) \notin \Lambda_{\mathsf{PE}}$, and since $\Lambda \subseteq \Lambda_{\mathsf{PE}}$, $\tau(\varphi) \notin \Lambda$.

If $\tau(\varphi) \notin \Lambda$, then as $\Lambda_{\mathsf{FE}} \subseteq \Lambda$, $\tau(\varphi) \notin \Lambda_{\mathsf{FE}}$. So there is a fusion of equivalence frames $\mathfrak{F}$ on which $(\mathrm{com} \wedge \mathrm{sing}) \to \varphi$ is not valid. Thus $\mathrm{com} \wedge \mathrm{sing}$ is true and φ is false relative to some valuation function and world w. By Lemma 10, it follows that $\mathfrak{F}_w$ is isomorphic to a product of universal frames. Since truth is invariant under taking generated subframes (see the proof of Lemma 2) and isomorphisms, it follows that there is a product of universal frames isomorphic to $\mathfrak{F}_w$ on which φ is not valid. Thus $\varphi \notin \Lambda_{\mathsf{PU}}$.

As shown in Lemma 3, $\Lambda_{\mathsf{PU}} = \Lambda_{\mathsf{PE}}$, so τ recursively embeds Λ_{PE} in Λ. Since Theorem 1 recursively embeds **SOL** in Λ_{PE}, composing the embeddings produces a recursive embedding of **SOL** in Λ. $\square$

5 Conclusion

The propositionally quantified modal logic of equivalence frames is decidable, but this does not extend to the propositionally quantified bimodal logic of frames with two equivalence relations: Antonelli and Thomason (2002) show that full second-order logic can be recursively embedded in the latter. Here, it is shown that this holds as well for a much stronger propositionally quantified bimodal logic, the logic of products of equivalence frames, as well as all logics in the interval between these two logics. In particular, this shows that many of the most natural candidates for the propositionally quantified modal-temporal logic of *necessarily* and *always* are not recursively axiomatizable. Such logics are of particular interest for a proposal of Fine (1977) that worlds and instants of time can be understood as special propositions. In systematically investigating this proposal using a propositionally quantified modal-temporal logic, it may well not be possible to use a recursive axiomatization to characterize the logical truths.

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