Gödel's Realism

How Intuition of Concepts Leads to Mathematical Knowledge

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ABSTRACT

In this thesis I investigate Kurt Gödel’s realist views and his postulation of mathematical intuition to acquire knowledge of mathematical objects and concepts. I argue that a correct interpretation of Gödel, what I call his theory of reason, must reflect and emphasize certain aspects of his views. His view on mathematical intuition, for example, must be understood in light of its strong connection to his conceptual realism. Also, his views that mathematics is a science, his rationalistic optimism concerning reason and the difference between intuition of truths and intuition of objects must be rightfully considered.

In the first chapter I explore the analogy between mathematics and the empirical sciences and see how the existence of mathematical objects is necessary for explaining well-formed mathematical theories, in the same way as physical objects are necessary for explaining our well-formed physical theories.

In the second chapter, I begin by pointing out some of the criticisms Gödel has faced. These are often quite ungenerous readings of Gödel, as Gödel is sometimes used as the epitome of the craziest and far out version of platonism there is. Then, I take on the notion of mathematical intuition and tracks the development of this view, which culminates in its full form in “What is Cantor’s Continuum Problem?” (1964). Here, I argue that there is an interplay of our formal concept also in our relationship to physical reality, so that our knowledge of concepts become formative also of our knowledge of physical reality.

Then, I present a Husserlian reading of Gödel, where I explain three Husserlian notions. By applying the third notion, the concept of *Fundierung*, I argue that there is a reciprocal dependence relation between intuition of objects and intuition of truths. I conclude this chapter by rejecting the Husserlian reading due to too many discrepancies with Gödel’s views in important ways, e.g. the lack of importance placed on the connection between mathematical intuition and his conceptual realism, that a Husserlian reading is too object-oriented and that it does not fully consider intuition of concepts and intuition of truths.

In the third chapter I present two considerations that supports my reading of Gödel as putting forth a theory of reason rather than a theory of intuition. The first is his belief in the power of reason and reason’s capabilities as to abstract reasoning. The second is his belief that mathematics is a descriptive science. I then argue that the rejection of the Husserlian reading lead us to push Gödel in a more rationalistic and Kantian direction.

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CONTENTS

INTRODUCTION

> [Mathematical data] may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality.
>
> Kurt Gödel in "What is Cantor's Continuum Problem?" (1964)

TWO PRINCIPAL CLAIMS

In this thesis I argue for two principal claims. The first claim concerns one of the more popular readings of Gödel, namely a Husserlian reading, while the second claim puts forth an alternative reading.

1. The Faulty Interpretation Claim: A Husserlian reading of Gödel fails to account for Gödel's notion of mathematical intuition, because it undervalues the emphasis Gödel puts on the connection between mathematical intuition and his conceptual realism.

The Husserlian reading of Gödel tries to render mathematical intuition less mysterious by founding it on phenomenology. The Husserlian equivalent to Gödelian intuition, *Wesensschau*, allows for intuition of general notions, i.e. general notions that are aspects of physical objects, in addition to abstract objects. The theoretical consequence of such a founding is that mathematical intuition cannot do the work Gödel meant it to do, i.e. it cannot reach higher-order structures and thus further our mathematical knowledge. This is because a Husserlian reading is too object-oriented and lacks attention towards intuition *of* concepts and intuition *that* axioms are true (in which the concepts are constitutive parts).

2. The Theory of Reason Claim: Gödel's notion of mathematical intuition should instead be understood in connection with his conceptual realism. Intuition *of* concepts and intuition *that* axioms are true are formative also of our relationship to physical reality, and as such there is an interplay of formal concepts in our empirical ideas.

In interpreting Gödel's work as a theory of reason, instead of as a theory only of intuition, I strengthen the connection between mathematical intuition and conceptual realism. Also, by interpreting Gödel in light of his broader philosophical influences and views (such as his view that mathematics is a descriptive science and the importance placed on the power of reason), I avoid some, but not all, of the criticisms raised against Gödel.

THESIS OUTLINE

In the first chapter I introduce Gödel's realist view, namely that mathematical reality exists as independently and objectively as physical reality. I also track how his position develops over his academic career. I explore the analogy to physics, where the existence of mathematical objects and concepts are deemed equally necessary for our mathematical theories as physical objects are for physical theories.

In the second chapter I take on Gödel's notion of mathematical intuition. In section 1, I will make some preliminary remarks on intuition as such, e.g. how the word "intuition" has been used, its role in early modern philosophy and the difference between intuition of truths and intuition of objects. In section 2 I investigate Gödel's notion of mathematical intuition and see how he discusses intuition before and up till the culmination of his view on intuition in "What is Cantor's Continuum Problem?". I also argue that Gödel's arguments for the existence of mathematical reality and mathematical intuition largely rest on the analogy between mathematics and the empirical sciences, and the principle of epistemological parity. The principle of epistemological parity is the view that if you have physical objects on one hand and mathematical objects on the other, then from what we know about them, there is no reason to be more or less committed to the existence of one than of the other (Kennedy 2014:6). I argue that Gödel sets forth a view where our knowledge of concepts is formative also of our relationship to physical reality, and where there is a partial cooperation between the physical senses and mathematical intuition.

In section 3 of the second chapter I give a general outline of a Husserlian reading of Gödel. I begin by commenting on some textual evidence for Husserl's influence on Gödel, by following Dagfinn Føllesdal's work (1992; 1995; 2016). I explain three Husserlian notions, *intentionality*, *Wesensschau*, and *Fundierung*. I then apply the concept of *Fundierung* to Gödel's position and argue that there is a difference between epistemological (intuition) and ontological (mathematical reality) dependence relations. I argue that there is a reciprocal dependence relation between intuition *of* concepts and intuition *that* axioms are true, due to the importance of his conceptual realism. I conclude this section with an assessment of the Husserlian reading of Gödel. I argue that it does not square with Gödel's views in important ways, and should therefore be rejected.

In the third chapter I present the alternative reading of Gödel, where his is interpreted as developing a theory of reason. In section 1 I present Gödel's view on the power of reason and the role of reason in Gödel's view on absolutely undecidable propositions and justification for axioms. In section 2 I argue that his view that mathematics is a descriptive science has two especially important features: 1) the style of argument he uses, and 2) the fallibility of intuition might lead to the view that mathematics is more revisable than desired. In section 3, I argue that the rejection of the Husserlian reading pushes Gödel in a more rationalistic and Kantian direction. I conclude by suggesting a possible interpretation of Gödel in light of both Husserl and Kant.

Chapter 1: Gödel's Realism

The overall reception of Gödel's contributions to philosophy has not been altogether favourable. While his discoveries in mathematics are lauded, his philosophical efforts are known for being "extreme platonist views", as Donald A. Martin puts it (2005:207). More damning is the assessment of Charles Chihara, who deems Gödel's arguments for platonism and his notion of mathematical intuition to be like arguments from theology, that is, not particularly strong (1990:21). In this chapter I will give an interpretation of Gödel's realist views in philosophy of mathematics, especially the more controversial part regarding his conceptual realism. In the first section I give an account of mathematical platonism and also point out two challenges that this position faces. Section II treats Gödel's version of platonism. Firstly, I shall deal with his view on mathematical objects (sets, numbers, etc.); secondly, with his conceptual realism (the existence of, say, 'concept of set'); thirdly, with his view on discovering new axioms; and fourthly, I shall compare his view to those of Plato and Frege.

Section I – Mathematical Platonism

1.1 Introduction

Platonism in the philosophy of mathematics is a cluster of slightly different views about the ontological status of mathematical objects and the truth values of mathematical propositions. All these views, however, share three commitments: 1) there *exist* mathematical objects, 2) mathematical objects are *independent* of us and our language, thought, etc., and 3) mathematical objects are *abstract*. Mathematical objects such as sets and numbers exist, and the axioms and theorems which refer to these objects are true or false independently of our actions or mental processes. On a general platonist view, the example of the natural number 2 is thus an abstract object which exists independently of human thought, understanding and language. All natural numbers, the real numbers, sets, etc. have the same sort of independent existence, which is why we can discover their properties and relations, and also why we can express our knowledge of them in our language.

This means that mathematical knowledge can only be discovered, as opposed to being constructed or extended by our minds. This is not to say that our knowledge of mathematics cannot be extended (as it most certainly can), but rather that the domain of true mathematical propositions is, and always has been, exhaustive.

So, why is mathematical platonism[1] a tempting route to take? As Penelope Maddy (1990) points out, mathematical platonism conforms with a sort of pre-philosophic attitude amongst mathematicians and laymen alike (Maddy 1990:ch. 1). The belief in the existence of abstract mathematical objects fits with how mathematicians operate when they are doing mathematics. This is also the case for elementary mathematical operations, such as division. When the number 1 is divided by 3, the result, the fraction 1/3, can be rounded down to the number 0.33, but it is obvious that this is not a perfect representation of what the fraction 1/3 really is. 1/3 is as perfect as a circle, impossible to accurately draw in the physical world. Platonism conforms with this attitude, and it tallies with how mathematicians consider the modules they are given in their field. When mathematicians do mathematics and discover, say, a new theorem or law, they use exactly the word "discover". They do not use verbs like "invent" or "construct". This, again, shows how ingrained the belief that you discover something *that is already there* when you further mathematical knowledge. Implied in the word "discover," is the thought that you found something that has existed before you came to know of its existence. And this *discovering* is in line with the pre-philosophic attitude mathematical platonism exhibits.

Moreover, platonism provides an explanation for why our mathematical theories are well-functioning. If mathematical entities have an abstract and independent existence, that is, if there is an objective mathematical reality, then this fact largely explains why we find an internal order in the domain of mathematical entities and relations and why mathematical theories are consistent. It also explains the truth of our mathematical theorems and propositions – they are true because the entities and relations that figure in them exist and because they describe their relationship correctly. By accepting platonism, we are provided with the most straight-forward explanation for why this is the case.

However, internal well-ordering of entities and relations, consistency and truth are all points that can be made for other positions in philosophy of mathematics as well. Some would argue that, say, formalism, which is one of the dominant schools within philosophy of

[1] Mathematical platonism and platonism will be used interchangeably in relation to the discussion on Gödel. Also, in some quotes platonism will be written with a capital P.

mathematics, and which roughly says that mathematics is a formal game and consists of mere manipulation of symbols, accomplishes the same thing. That is, by considering mathematics to be a formal game where the rules of engagement are determined by humans, one can also explain why we have truth, consistency and well-defined prescribed roles for the entities in play (Maddy 1989:1123). Where does this leave us then? While platonism offers the simplest and most direct explanation, simplicity in itself is not enough to favour the position above any other that accommodates the same mathematical attributes, though perhaps more clumsily or intricately. This argument for platonism is therefore left open-ended and does not hold much sway.[2]

An alternative case for platonism was made by Frege, arguing against formalism. One difficulty of formalism, Frege argues (*Grundgesetze der Arithmetik Volume 2* of 1903), is to account for the applicability of mathematics to increase our understanding of the world, when mathematics is understood as the consequences of a game of symbols with man-made rules of manipulation (1903:§91). Platonism, in contrast, provides a solid case for why mathematics can so successfully be applied in the empirical sciences. Is it not plausible that the reason for why the laws of nature are written in the mathematical language is exactly because mathematical reality has a real existence just as physical reality? It does indeed provide a neat explanation. If, however, mathematics is a mere manipulation of symbols, we end up with an ontological gulf between our scientific theories (given that we do endorse some version of realism in philosophy of science) and the language in which they are written, namely our mathematical theorems and propositions.

However, it is a common conception that platonism entails unnecessarily many consequences that are difficult to defend. What is the need for postulating a platonic realm consisting of causally inert, abstract objects that exist independently of us, when it is easy to imagine how our minds could make the abstraction from two trees visible in front of us to the number 2? The number is, after all, such an everyday concept. Is it not possible that this is how arithmetic has gradually been understood and developed as a discipline, the notion of infinity marking the leap from countably many twigs to a heap of twigs? Thus one has argued against mathematical platonism. This line of reasoning gives rise to a more general argument against platonism, namely that it leads to seemingly unnecessarily many metaphysical commitments.

[2] Unless, perhaps, you adhere to simplicity as being the number one criterion for accepting one scientific theory over another, and if you allow mathematical platonism, along with other positions in philosophy of mathematics, to count as a scientific theory.

The totality of these metaphysical commitments breaches the limit for what many are willing to accept.

However, we can also flip the argument around, so that a platonist position seems more easily imaginable when it comes to simple arithmetic or geometry. The way that one imagines the universal properties of, for instance, a triangle can indicate this. It is not the triangle drawn on the blackboard one has in mind, mathematicians rather act on and think of that triangle as if its properties exist in some idealized, general way. And, it is quite obvious to them that the drawn triangle is a mere representation of the one that truly exists. As soon as we speak of higher-order abstract concepts in set theory, for instance, the position suddenly demands greater effort in connecting such a representation to the represented. We cannot "see" such properties in the same immediate way, and in order to grasp these concepts they must be the target of some reflection. By "see" here, I mean that process that takes place when you have come to realize that some true mathematical claim is indeed the case, and how it is impossible to *un-see* the truth of it.[3] As we shall see in chapter 2, Gödel makes a similar point relating to mathematical intuition.

Further, one can also argue that the need for mathematical platonism first arises in exactly the cases of infinity and complex structures. That is, mathematical reality seems too wondrous to exist simply because of our own construction. That we, as finite human beings, should have created such a thing seems more unlikely than that we somehow found these truths and learnt to understand more from them. If we do not have the existence of the complex structures we are trying to describe to lean on, what do our dealings with them really amount to? Still, it is admittedly rather difficult to imagine that all mathematical objects and concepts exist in this realm. Such things as different sizes of infinity, for example, – simply seems too vast a realm to exist.[4]

[3] A point Gödel himself makes when he talks of evidence for axioms, where the "the axioms force themselves upon us as being true" (1964:268). This is a point I will return to, both in chapter 2 on mathematical intuition (specifically on the notion of intuition *that*), and in chapter 3 on justification for axioms.
[4] This can be said for the physical universe as well. Perhaps you can argue that it might seem less daunting in some way, since we all have "perceived" the infinite in the physical world, as in the case of a horizon, where you can, supposedly, see an infinite limit.

1.2 CHALLENGES

There are two especially challenging objections to platonism, both presented by the philosopher Paul Benacerraf in his famous articles “What Numbers Could Not Be” (1965) and “Mathematical Truth” (1973).

The first objection (1965) revolves around how mathematical objects on a platonist view are metaphysically challenging. What do we take the properties of numbers to be? Are they, as Benacerraf argues, merely structural? Should we, for instance, in axiomatic set theory, define the natural numbers following Zermelo or von Neumann? If we claim that natural numbers do have more than structural properties, this is problematic, as we have two (equally well defined) definitions of the number 2. What properties does the first express that the second does not? These are all questions that make the existence of mathematical abstract objects difficult to explain.

The second is an epistemological objection (1973) which questions how we can have knowledge of abstract objects. If we take some mathematical propositions from the domain of accepted mathematics today to be true (as most mathematicians do), then it becomes a problem for platonism to explain why these propositions are reliably justified when they purport to describe and quantify over entities that exist in some platonic realm that is causally closed off from our own. If platonism is true, Benacerraf says, then this reliable justification problem makes it extremely difficult to explain how and why we believe in mathematical propositions.

In his argument, Benacerraf takes as a premise that our best theories of knowledge are *causal* theories – where our direct or indirect causal relation to the matters of fact is how we obtain knowledge.[5] Our knowledge of the objects of science, whether it is the natural sciences, the social sciences, etc., depends upon our causal relationship to the relevant phenomena, whether it is the observations we make in physics or the study of our social structures – we can always retrace the intricate causal relation which resulted in us having knowledge of these phenomena. This is not the case with mathematics. If mathematical objects, their relations and properties all exist independently of us in an abstract way, and since all mathematicians exist in time and space, we cannot explain how mathematicians are justified in believing in mathematical propositions, where we do not have any causal connection to the objects described and quantified over.

[5] Even if a causal theory of knowledge is not taken as a premise, Benacerraf’s argument can be improved so it goes for other epistemological theories as well. Hartry Field (1989) puts forth such an improvement, where it is a reliabilist theory that is assumed (Field 1989:67–69).

Section II – Gödel's Realism

2.1 On Objects

Where, in this landscape of arguments and objections, does Gödel position himself? Gödel held realist views regarding mathematical objects and concepts and proposed that we can have knowledge of these objects. In "Russell's Mathematical Logic" (1944) he compares the existence of mathematical entities to the existence of physical bodies:

> It seems to me that the assumption of such objects [classes and concepts] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data", i.e., in the latter case the actually occurring sense perceptions. (137)

This seems to suggest a sort of indispensability argument, where the need for postulating mathematical objects is directly linked to our having successful theories in mathematics. By this I mean a general sort of indispensability argument (that is, not the specific Quine-Putnam argument on the applicability of mathematics in science), where the truth of a claim is to be established on the basis that its truth is indispensable for certain ends. In this case, Gödel appeals to our belief in our best mathematical theories, and so tries to establish the metaphysical claim that the mathematical objects that figure in these theories therefore must exist, analogous to the role of physical objects in theories of sense perception. This is therefore an appeal to abductive reasoning, where our belief in mathematical objects is justified because it best explains how and why our mathematical theories are successful.[6]

This is also in line with how the practice of mathematics is conducted, and so captures a kind of pre-philosophical attitude. However, this is not enough to ensure the existence of mathematical objects, and they remain postulated in a way that does not answer the reliable justification problem posed by Benacerraf in section 1.2. Neither does this line of argument justify Gödel's belief that sets, numbers, etc., exist in a realm wholly (causally) unconnected

[6] This is also a type of argument Gödel employs in his justification for the search of new axioms, and it was also one of the main arguments used by Zermelo for accepting the Axiom of Choice – its indispensability for set theory as a discipline.

with us. However, as we shall see in chapter 2, Gödel's response to this challenge lies in his conception of mathematical intuition, which is closely connected to his overall realism.

As to the metaphysical challenge, Gödel would probably not consider it as a damning problem for his position. There are two ways of defining the natural numbers, either following Zermelo or von Neumann, and therefore it becomes a question as to which one captures the nature of the natural numbers. However, Gödel could simply answer that they both capture some feature of the nature of a natural number, and that while we do not have a definite answer as to which is the true definition, his position does not claim to have such an answer. Another possible route is to refute both of them, i.e. Gödel does not need to choose either of them. Why should he? Also, Gödel is a set theorist, and whether the natural numbers have this or that set theoretic foundation is not really important. For Gödel, both structures of sets are well-defined, and the one does not exclude the other in any way. He is quite convinced that we have not exhausted the domain of possible mathematical knowledge, and he also questions our understanding of the primitive terms and axioms. In his Gibbs Lecture of 1951 he says:

> For, our knowledge of the world of concepts may be as limited and incomplete as that of [[the]] [7] world of things. It is certainly undeniable that this knowledge, in certain cases, not only is incomplete, but even indistinct. This occurs in the paradoxes of set theory, which are frequently alleged as a disproof of Platonism, but, I think, quite unjustly. Our visual perceptions sometimes contradict our tactile perceptions, for example, in the case of a rod immersed in water, but nobody in his right mind will conclude from this fact that the outer world does not exist. (*1951:321)

Our knowledge of mathematics is fallible, and Gödel likens our difficulties in determining a mathematical object completely and distinctly to our fallible sense perceptions. Our knowledge of mathematical objects is incomplete, yes, but it is also *indistinct*. What does this mean? Well, not only are we uncertain whether there is some fundamental part of the mathematical reality we have not so far been able to describe, but also, and far more dire for the status of our present knowledge in mathematics, we are not even sure that the knowledge we do purport to have of mathematical objects is distinct. This means that even our most primitive concepts in set theory may be blurred and misunderstood.

What becomes clear is that Gödel's position is not refuted by this metaphysical challenge. That our *knowledge* of mathematical objects is faulty is not a decisive argument

[7] The double square brackets indicate the editor's amendments, and they will appear in later quotes as well.

against the metaphysical claim that mathematical objects exist. The challenge to explain how we choose to define the set theoretic foundation of the natural numbers remains, of course, but it does not seem to be fatal to platonism.

For now, however, I will continue to sketch out Gödel's realism: What does he commit himself to?

In the Gibbs Lecture, Gödel describes platonism as the view that "mathematical objects and facts (or at least *something* in them) exist objectively and independently of our mental acts and decisions", further: "Thereby I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind" (*1951:311; 323). In the Supplement to "What is Cantor's Continuum Problem?" (1964) he writes that the question of the "objective existence of the objects of mathematical intuition" is said to be "an exact replica of the question of the objective existence of the outer world" (1964:268).

What do these two quotes tell us about Gödel's realism? For one, that he adheres to the independence component of mathematical platonism. And that, for Gödel, this independence lies in its non-constructivism. By this I mean that mathematical objects are not dependent on us, in that we construct them, in the same way that the outer world does not depend upon our sense-experiences of it. It also tells us that if the question of the existence of mathematical objects is a "replica" of the question of the objective existence of the outer world, this means that Gödel takes the existence component to be as certain as with our everyday physical objects. The objects of mathematics are as indubitable as, for example, this table before me, and cannot be doubted unless one is a global sceptic.

2.2 Conceptual Realism

A most noteworthy component of Gödel's platonism is the fact that he is a *conceptual realist* in mathematics, and that he considered one of the basic problems of philosophy to be the question of the objective reality of concepts. By *concepts* (relations and properties in set theory), Gödel means abstract *objects* that are picked out by predicates, and which are not necessarily reducible to *sets*. For instance, properties that cannot have sets as extensions, and so are primitive notions of set theory, e.g. "property of set" or "concept of set itself" (Parsons 1995:48). The primitive notions of membership (denoted by "$\in$") and the concept of set itself are central to Gödel's discussion of axioms, where the question whether some of the axioms *fully express* the concept of set is central, but also how it is *from* our concept of set we are able to grasp higher and higher infinities.

This conceptual realism reveals that Gödel's realism is of a very strong kind. Not only does he believe in the independent existence of abstract objects (set, classes, numbers, etc.), but he also believes in the independent existence of mathematical concepts. If mathematical concepts *belong* to an objective reality, this means that they are part of the world in a way that does not immediately *affect* us or vice versa, which, in this case, means that the mathematical reality is in no way steered by the human mind.

> What is wrong, however, is that the meaning of the terms (that is, the concepts they denote) is asserted to be something man-made and consisting merely in semantical conventions. The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe. (*1951:320)

This quote from the Gibbs Lecture shows us that our choices and mental acts cannot affect the realm of mathematical concepts. The *meaning* of a mathematical term is not a construct made by the human mind. Rather, the meaning of a mathematical term is objective and unchanging, whether or not we humans have grasped it – as it is the *referent* of the term. And so, the meaning of a term does neither conform to our thinking nor to our *knowledge* of it, i.e. the generally accepted and consistent mathematical framework in which the term figures and has a prescribed role. Even if we alter our definitions of mathematical terms (usually in the belief that we are correcting some mistake or imprecision, e.g. the case of naïve set theory and the resultant Russell's paradox), the concept the term denotes has not changed at all. The only thing that has changed is *how* we think about the concept – as its role in our mathematical language has changed – and *what* we think we know about it.

Here, an analogy with physics is useful. When we chose to change our definition of, say light, and accepted it as electromagnetic waves, as explained by Maxwell's equations, the physical phenomenon of light had not itself changed, only what *we* thought of it had (and thus an accepted scientific paradigm had been replaced with another) (Kuhn 1962:ch. 9). In the Gibbs Lecture, Gödel is claiming that this is also the case with mathematical concepts.

> I have purposely spoken of two separate worlds (the world of things and of concepts), because I do not think that Aristotelian realism (according to which concepts are parts or aspects of things) is tenable. (*1951:321)

Even if we chose to define the membership predicate differently, and we change which mathematical objects it can and cannot hold between, this would not change the very real phenomenon that is the membership relation. Such a change of the definition of a term would only amount to, as Gödel puts it, a change in our semantical conventions. It would, of course, be a change in the language of our formal system, but the concept itself and the mathematical reality it is part of, would not have undergone any sort of change.

It would, however, change our supposed *knowledge* of the concept. As its prescribed *role* in the mathematical framework had changed, so would our knowledge of it and, furthermore, so would the particular set theoretic theory. However, this is not to say that changing a mathematical term's definition and role in a given system can make parts of mathematics that were once true, subsequently false. That is, the mathematical reality itself does not change. For example, when the distinction between sets and classes was introduced, in order to prevent sets being too big and thus leading to paradoxes, our understanding of the concept of set itself changed, as the predicate "set" no longer was used to pick out the same mathematical objects it had before – as some of them did no longer earn that name. And so, when the change had not yet come about, and a mathematical proposition (mistakenly) identified an object as a set, we accepted the proposition as true. When we later revised our position and claimed the sentence to express a falsehood, it was not the case that something once true in mathematical reality now had *become* false, as a change in our semantical conventions cannot have such substantial, reality-altering consequences. According to Gödel, the proposition expressed had never been true (and would certainly not become so, should we choose to redefine the term once more), as it relied on a faulty concept of set. Rather, this result is the perfect example of Gödel's claim that our knowledge of mathematics as a whole and of the discipline's most fundamental concepts and objects is fallible, incomplete and even indistinct.

Again, an analogy to science might help clarify the point. In the same way as a scientific realist believes that science more or less accurately describes the physical world, so does a realist in the philosophy of mathematics believe that mathematics accurately describes the mathematical reality. When reviewing the history of science, it is a fact that science has seen immense progress, but the progress is unavoidably linked to the downfall of so many, now realised to be false, theories. However, the realist never fails to believe that her quest for true physical theories is fruitful, nor that it does lead her closer to her goal: To produce a correct description of the physical world that will not be thrown out and replaced. Similarly, Gödel believes that with the correction of our understanding of the concept of set, we can continue to pursue our goal of accurately describing mathematical reality.

So, the concept denoted is independent of human thought, choice, and what we, at a given time, accept as knowledge. That is, the concept exists in some platonic realm. This objective reality of concepts may strike us as odd. If not even *concepts* are formed by human thought, what is? What this means is that in our dealings with mathematics, there are two different levels which stand in a one-way dependence to the other. Our mathematical language cannot influence the concepts and objects which it attempts to describe, i.e. the linguistic level cannot influence the non-linguistic level, viz. the realm of real mathematical facts. On the other hand, the non-linguistic level does determine the linguistic level, which is to say that we adapt our mathematical language and how we use it to how things really are; to the real mathematical facts. On this point, Gödel is satisfied in our development of a well-functioning mathematical language.

> The first part of the problem [of giving a foundation for mathematics] has been solved in a perfectly satisfactory way, the solution consisting in so-called "formalization" of mathematics, which means that a perfectly precise language has been invented, by which it is possible to express any mathematical proposition by a formula. (Gödel *1933o:45)

So, make no mistake, if our mathematical theories are faulty, this is not caused by some inconsistency in the mathematical realm, but is only due to our misconstruction *of* that non-linguistic realm. The fault lies, however, not with our understanding, as our reason and potential for mathematical understanding is, as far as we know, quite unparalleled in any other living being. Rather, it is the considerable gap between mathematical reality and ourselves that is the problem. It is the one-way dependence between the non-linguistic level of mathematical objects and the linguistic, constructed level that is the root to the epistemological difficulty we find ourselves in.[8] This is often called the *access problem*: How do our minds engage with this realm and extract knowledge of the concepts within it? How can we be certain that our accepted mathematical sentences express true propositions, and thus qualify as knowledge? And how is it, when we cannot influence mathematical objects

[8] This is not to say, however, that our linguistic level is any less real, only that it is constructed by humans, and not discovered as an unchanging and true entity, a claim easily exemplified by the fact that our natural languages evolve and that words that once held a certain meaning now have another.

and concepts in any way, that we have the ability to direct our mathematical language, terms and theories onto the world of mathematics, and that we are doing this quite successfully?

Even though the realm of mathematical concepts is an objective reality and determines our knowledge of mathematics, the question as to how this is done remains unanswered. The realm is, after all, causally closed off from our own physical world. It is not from some causal effect that we adjust our mathematical language and our use of it to the real mathematical facts. There are two things, then, that make the access problem especially difficult: 1) The one-way dependence relation between the non-linguistic level and the linguistic level, where the non-linguistic level determines the linguistic level, but not the other way around and, 2) that there is no causal interaction between the two levels as they are causally closed off from each other. If one of these claims had been false, the epistemological gap would shrink drastically. I will first examine the second claim, before I turn to the first.

I cannot really see that Gödel would reject the second claim, as that would mean that mathematical objects are either: i) not abstract and thus exist in the world of physical things, or ii) exist as aspects or parts of things. However, accepting one of these alternatives would leave us in a radically different position, a position that faces quite different problems.[9] It is quite evident that Gödel rejects the first alternative. That he also rejects the second needs to be explained. Is there a possibility that Gödel could accept mathematical concepts as being aspects of physical things, that is, that he would accept an Aristotelian picture? In "Is Mathematics Syntax of Language?" (*1953/9–III & *1953/9–V) Gödel claims that physical things are determined without any reference to formal concepts.

> I even think this comes pretty close to the state of affairs, except that this additional sense (i.e. reason) is not counted as a sense, because its objects are quite different from those of all other senses. For while through sense perception we know particular objects and their properties and relations, with mathematical reason we perceive the most general (namely the 'formal') concepts and their relations, which are separated from space-time reality insofar as the latter is completely determined by the totality of particularities without any reference to the formal concepts. (*1953/9–III:354)

[9] Such as, how is it that something seemingly abstract is to be found in the physical world? Does each physical object have some mathematical component? How do we know which mathematical component exists in which physical body? And how do we extend our mathematical knowledge and discover which relations hold between which mathematical entities? That is, the first alternative is an empiricist view on mathematics, like that of John Stuart Mill. Frege argued persuasively in *The Foundations of Arithmetic* (1884) how Mill's account of mathematics fails (§6).

> For while with that latter [the senses] we perceive particular things, with reason we perceive concepts (above all primitive concepts) and their relations. (from version IV of the Syntax paper, quoted from Parsons 1995:63)

This quote puts forth a very strong claim, and it goes further than what is needed to show that Gödel does not support an Aristotelian picture. In the introductory note to the **1953/9* text, the philosopher Warren Goldfarb writes that Gödel later supports the existence of an interplay of formal concepts in our knowledge of the physical world (1995:333). Notably, in the *1964* version of "What is Cantor's Continuum Problem?" there *is* reference to formal concepts, e.g. the 'idea of object itself'. In *1964*, he also claims there are "abstract elements contained in our empirical ideas" as "our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations" (1964:268). This suggests that his position developed. Does the admission that there are abstract elements contained in our empirical ideas open for the possibility of the aforementioned Aristotelian alternative? The two are absolutely compatible, but it does not follow from *1964* that mathematical concepts *are* mere aspects of physical things, as Gödel makes clear. In particular, they are not to be reduced to aspects of physical things.

> I have purposely spoken of two separate worlds (the world of things and of concepts), because I do not think that Aristotelian realism (according to which concepts are parts or aspects of things) is tenable. (*1951:321)

This quote from the Gibbs Lecture is written two years earlier than "Is mathematics syntax of language?" of 1953. What we must determine, then, is whether he rejected this position later on. While he did think that the world of physical things was determined without any reference to formal concepts (as in **1953/9*) and that Aristotelian realism was untenable (**1951*), my claim is that he later abandoned the view of **1953/9* as improbable, while he still stood by his rejection of Aristotelian realism in **1951*. The reason behind this is that **1951* and **1953/9* say two different things about two different philosophical concerns. In *1964* he concedes only that there is a role for formal concepts to play in our *knowledge* of the physical world, specifically in our ideas referring to physical objects, and so this is an epistemological and conceptual claim. It has to do with *how* we are able to think of a physical object.

The claims of **1953/9*, on the other hand, is twofold: First, Gödel says that it is only through sense-perception we know particular objects, their properties and relations, i.e. he makes an epistemological claim about how sense-perception provides us with knowledge of the physical world. Secondly, **1953/9* separates the "most general (namely the 'formal') concepts and their relations" from space-time, which leads to the claim that physical objects are "completely determined by the totality of particularities without any reference to the formal concepts". This is the decisive part. What does it really mean? There seems to be two interpretations: Either that physical objects are completely determined metaphysically by the totality of particularities, i.e. that there is a metaphysical dichotomy between the world of physical things and the world of concepts. Or, that physical objects are completely determined by the totality of particularities also in our thinking of them, i.e. that they are determined epistemically and semantically *for us* without reference to formal concepts. If we take into account the rest of the passage, I find the second interpretation better supported by the text, as its subject is exactly how we perceive and how we come to know formal concepts and physical objects. However, I do not find it the most probable view to have. It paints sense perception as a faculty that is something more than just immediate registration of sensations and seems to disregard how the mind forms ideas of our sensations.

Let us consider the 'idea of an object itself', which also figures in Gödel's *1964*. It is not so that from a variety of physical objects we abstract the concept of objecthood. That is to say, we do not form our idea of an object itself in the same way as we do with an everyday concept. Let us use the example of the concept of dog. When we know the concept of dog, it is because we have perceived one dog or more dogs, often of different kinds, and because we are able to recognize that the different dogs all fall under the concept of dog. However, the idea of an object itself does not follow the same pattern. I do not think that our minds register a variety of vastly different physical objects, from a tea spoon to an ancient temple or to an organic physical being like a dog, and from these different physical objects draw the conclusion that they are all objects. That is, I do not think that we consider these as equally belonging to the set of objects, and from that fact get the idea of an object in itself. Rather, the concept of object itself is epistemologically primordial, and is something we know from the realm of concepts. It is only because we already have knowledge of the world of concepts that we recognize that a physical object partakes in the idea of an object itself. First, we have the concept of an object in itself, and only then do we consider each object as satisfying the concept of objecthood. Which is to say that we project our formal concepts onto the physical world, and that there is in fact an interplay of formal concepts in our knowledge of the physical world.

If, however, Gödel believed that physical objects are completely determined by the totality of particularities without any reference to the formal concepts, I would say that he later must have reconsidered this belief. If we choose to accept the first interpretation however, namely that there is a metaphysical dichotomy, it makes his position more consistent with his other writings. And, furthermore, we avoid complicating further the relation between physical reality, human thought and the objective existence of concepts.

Gödel does defend the separate existences of physical bodies and the existence of classes and concepts in *1944* (456), and also, in *1964*, he clearly states that "the objects of transfinite set-theory … clearly do not belong in the physical world and even their indirect connection with physical experience is very loose" (1964:267). However, in neither *1944* nor *1964* does he deny that our ideas of space-time reality have abstract constituents, but actually opens for some loose connection. And so, the interpretation that space-time reality "is completely determined by the totality of particularities without any reference to the formal concepts" in our minds is implausible (*1953/9-III:354). It is inconsistent with *1964*, specifically with his notion of an 'idea of an object itself', which is present in our understanding due to something other than the actual existence of physical objects. What this 'other' is supposed to be is something I will return to in chapter 2.

As we have seen, then, Gödel accepts the second claim: There is no causal interaction between the linguistic level and the non-linguistic level, i.e. between our thoughts and practices and the mathematical objects. This is because he rejects both alternatives, (i) formal concepts are not abstract and thus exist in the world of physical things, and (ii) they exist as aspects or parts of things, which a denial of the second claim entails.

Therefore, let us return to the first claim: 1) There is a one-way dependence relation between the non-linguistic level and the linguistic level. If this claim is false, it would narrow the epistemological gap we are facing. It would mean that the world of mathematical concepts is affected by our thoughts and choices, and that a change in our mathematical language leads to a corresponding change in mathematical reality. That is, mathematics would behave like a social construct, e.g. like our social conventions or legislation. When, say, our social conventions change, it is because our thoughts, choices, and actions change what is socially acceptable. Norms change due to a collective effort, whereby behaviours and lifestyles previously deemed unacceptable become acceptable. As to language, we literally change what is legal and illegal when laws are rewritten, removed from or added to our legislation. And performative speech acts, such as “I hereby pronounce you married”, provide a striking example of how our language actually change a small part of reality. These examples illustrate how there are areas where there is no epistemological gap.

Mathematics, on the other hand, is not as obvious a candidate for the same to be the case. It seems quite unimaginable that if all of human kind collectively chose to make ‘2 + 2 = 5’ true, that it would actually become true. It goes against every intuition we have, as mathematical truths consistently have been deemed the most eternal and necessary of all. What would happen, then, if the same was true of mathematics? And which consequences would this have? For example, our redefinitions of mathematical terms like “$\in$” would, then, change the membership relation itself. If our influence on the mathematical realm was a constant one, we would gradually alter it. However weak the influence, the two different worlds would become gradually closer and intertwine, until they became one. We would create a new mathematical reality, where seemingly eternal mathematical facts would have to yield to our choices. This potential endgame seems especially pressing for, say, geometry, where our mathematical truths are so visibly related to something we can also, in some sense, perceptually understand. Would, for instance, Pythagoras’ theorem no longer hold if we chose that it should not? This would lead us to a very different view on mathematics as a whole, and in addition, our physical theories would have to be altered, as the mathematics in play in our physical theories would have changed.

Also, would we develop different mathematical systems, or would we collectively choose to let our unison choices only affect the mathematical reality? Would we even control which choices could and which could not affect it? And if we could, figuring out the rules for choosing why and how this influence were to be implemented, seems an insurmountable challenge. I think the most pressing practical concern would be whether every layman could alter mathematical reality, which would result in us having a different mathematical reality for each person who thought about mathematics, or whether it was to be influenced only by professional mathematicians, thus still preserving some stability in the mathematical community and a continuance with mathematical knowledge and tradition.

On the other hand, we already have different logical systems, each used with great success in different fields. It is not considered a problem that in some systems quantification over properties is allowed and in others it is not. The different logical systems are used to draw interesting results in different fields and ways, and while first order logic and second order logic are more adapted for expressing axioms within set theory, no one can doubt the efficacy of propositional logic and the intuitive depiction of truth and necessity we get from a syllogism. And, on the face of it, it is we who have chosen to let some things be allowed in one logic and not in another, e.g. it was a decision made by humans not to allow the law of excluded middle in intuitionistic logic. If logic is, as Gottlob Frege said, "the science of the most general laws of truth" and that logical laws are the normative rules for human thought whose goal is 'truth', one would think that we did not invent different logical systems, but rather discovered them, as in discovering different methods to reach truth (1897:F139/228).[10] Could not this be the case for mathematical reality as well? However, if one inspects one extreme consequence of a mathematical reality conforming to our choices, it still seems utterly absurd that '2 + 2 = 5' can be true in one reality, '2 + 2 = 4' in another and '2 + 2 = 3' in yet another. I cannot see how this could all be true and that we still would have a well-functioning mathematical community, by which I mean that different mathematicians could communicate and conduct fruitful research.

If mathematics is supposed to resemble physics, in that it describes one true objective reality, what can this tell us about allowing different mathematical theories to exist side by side? One would think that the theories we have in physics would have something in common with the theories we have in mathematics. Not to say that the theories themselves would resemble each other, but rather that they would face the same type of problems. In the history of science this has usually meant that one theory was obviously (at least in retrospect) better and closer to capturing the truth of how the world is. For instance, there was for a period of time overlap between the Cartesian mechanistic worldview and Newtonian physics where both received roughly equal scientific recognition. Today, it seems obvious that Newtonian physics was the better choice of theory. But if we turn our attention to our best physical theories today, what do we see? The fact is that special relativity and quantum mechanics are not consistent with each other. But we still believe that they are both quite close to the truth, i.e. that they to a very high degree correctly describe the physical world. However, this inconsistency has not really led to that many crises, inasmuch as physics as a discipline has not crumbled.[11] They

[10] Here, it must be noted that Frege did not endorse different logics as being different ways of reaching truth. Frege was a logicist (until his eventual abandonment of the project after the discovery of Russell's paradox), and he held the belief that arithmetic could be reduced to logical laws. That there exist different and inconsistent logics, then, is not something he would have supported.

[11] Even though the discipline as such has not crumbled, it has fragmented into many sub-disciplines that no longer speak to each other.

are both extremely informative and well-functioning theories (in the sense that they explain an extremely large number of phenomena to a satisfying degree), and they have both spurred on fruitful research. How mathematics would face such a challenge, i.e. one that concerns inconsistencies between theories and how this relates to truth, is a different story and something I will return to below.

Could it not be, then, that this is also the case for different mathematical theories? Different theories, with inconsistencies between them, encouraging research and debate? Well yes, of course this can also be the case in mathematics, and to some extent, it already is. Look at the different positions regarding the Continuum Hypothesis[12], where the pluralists want to accept different theories where CH can be true in one system and ¬CH true in another, i.e. both CH and ¬CH are true relative to different, legitimate theories of sets, and Gödel's view (the monists), that we need stronger axioms in order to settle CH uniquely, i.e. as either true or false.

When it comes to truth, mathematicians behave differently than what physicists do. The empirical sciences have not, at least not in the same way, the same standard, i.e. the same absolute goal of truth. A high success rate or degree of probability will in many cases suffice, whereas this cannot be said for mathematics. Even though pluralists and monists disagree on what they should do with CH and how they should tackle the problem of its independence from ZFC, they both still want to reach the goal of truth. Mathematical truths once discovered are deemed necessary and eternal, and this belief is an ingrained part of what we think mathematics is really about and how we practice mathematics. We really do believe or feel in some sense that we have discovered something that is and always has been true. Consequently, if mathematical reality really is influenced by our choices, the very notion of truth in mathematics is challenged. It simply feels like there is no alternate universe where Pythagoras' theorem is not true, even before our conceiving of it. Contrary to our cases of performative speech acts and legislation, we cannot yield our intuitive conception of truth in

[12] The Continuum Hypothesis is a conjecture regarding the size of the continuum, i.e. whether there is an infinite set of reals that is neither in one-to-one correspondence with the set of natural numbers nor with the set of real numbers. The Continuum Hypothesis states that there is no such set. Gödel proved that CH is consistent with ZFC (Zermelo-Fraenkel axiomatization of set theory plus the Axiom of Choice, which is the standard axiomatization of set theory) in 1938, and the mathematician Paul Cohen proved that its negation is also consistent with ZFC in 1963, establishing its independence from or undecidability relative to the consistency of ZFC (Gödel 1964:269–270; Hallett 2006:117).

mathematics. If we could, truth as such would become inconstant and susceptible to change, which goes against the very nature of the concept of truth itself.

2.3 Finding Axioms?

While the evidence for Gödel's platonism is well-documented in his writings, both published and not, we can also find some evidence for the opposite position. "The Present Situation in the Foundations of Mathematics" is a lecture given at a meeting with the American Mathematical Society in Cambridge, Massachusetts 29–30 December 1933 (Feferman 1995:36). The article published in the *CW: III* is drawn from Gödel's handwritten notes for this lecture, and discusses the problem of giving a foundation for mathematics, which Gödel considers falling into two parts:

> At first these methods of proof have to be reduced to a minimum number of axioms and primitive rules of inference, which have to be stated as precisely as possible, and then secondly a justification in some sense or other has to be sought for these axioms, i.e., a theoretical foundation of the fact that they lead to results agreeing with each other and with empirical facts. (*1933o:45)

Here "the methods of proof" are the methods actually used by mathematicians and the language they are stated in. He argues that we have already found a satisfying solution to this first part of the problem (by the invention of the simple theory of types), which leads us to avoid the paradoxes that arose from Frege's early work (Feferman 1995:37).

The second part of the problem is then to give a justification for the axioms. This justification must be able to explain 1) the consistency of the axioms, the primitive rules of inference and the theorems deducible, and 2) the applicability of mathematics in the empirical sciences. Later in the article we find a quite startling quote:

> The result of the preceding discussion is that axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent. (*1933o:50)

This is extremely surprising, considering his later explicit platonism as in *1944* and *1964*. That a position he defends in other writings "cannot satisfy any critical mind" simply does not add up. In the introductory note to **1933o*, Solomon Feferman suggests that the attack on platonism in **1933o* might be due to a development in the strength of his platonism, meaning that he might have been a platonist regarding only integers at the time, but that he later also included sets (Feferman 1995:40).

Also, the first quote, where he wants a justification for the consistency of the axioms and for their agreeing with empirical facts, conforms to some sort of platonism. When the result of a discussion of how to justify the axioms is said to "necessarily presuppose a kind of Platonism" is where things get troublesome. According to Gödel, the problems arise when we do not consider our formalism to be a mere game of symbols, but rather attach a meaning to our symbols, i.e. when we consider mathematics to have a real content (*1933o:49). This is a notion which he clearly defends in *1944*, **1951*, **1953/9* and *1964*. It is therefore difficult to understand how his views in **1933o* fit with his overall position. What is certain is that platonism is a viable alternative that offers a straight-forward justification for the axioms. As I mentioned in section 1.1, the most direct explanation for why we have well-functioning mathematical theories that are consistent and agree with empirical facts is because there is a mathematical reality that we have successfully described with these theories. It might very well be that Gödel later, in failing to find an alternative that provides us with such a "theoretical foundation", he chose to accept platonism after all. Later, Gödel did affirm in his correspondence with Hao Wang from 1967 and 1968, reproduced in *Wang 1974*, that he was a mathematical and conceptual realist, and that he considered himself to have been so since 1925 (Feferman 1995:39).[13] And so, as Feferman notes, whether Gödel temporarily wavered in his belief, whether his platonism grew stronger or whether his statements somehow are consistent with his later writings, the interpretation of this particular quote remains uncertain (Feferman 1995:40).

However, even though platonism offers a justification for the axioms, it is not of the kind Gödel seeks in **1933o.* In **1933o* he goes on to say that the axioms (for which platonism fails to "produce the conviction that they are consistent") are in fact likely to be consistent as the consequences of the axioms "have been followed up in all directions … without ever reaching any inconsistency" (*1933o:50). The task of providing such a justification for the axioms is to find one that does not use objectionable methods, e.g. which does not use the law of the excluded middle on existence claims of, say, an integer (*1933o:52). He goes on to investigate Hilbert's program[14] for securing this kind of

[13] Also, in answering the Grandjean questionnaire, Gödel claims that realism had been his position since 1925 (Wang 1987:17–18).

[14] David Hilbert proposed to give a metamathematical proof, so that the meaningless infinitary statements can be seen "as a tool in deriving meaningful statements about the finite" (Maddy 1989:1123). Hilbert's program wanted to use only finitary methods in proofs, as Hilbert only allowed

theoretical foundation but comes up short as "the hope of succeeding along these lines has vanished entirely in view of some recently discovered facts", i.e. the Second Incompleteness Theorem[15] (*1933o:52). In other words, perhaps the wish to explain more (i.e. give a justification for the axioms and the primitive rules of inference) while committing to less (i.e. fewer existence claims and the wish to use constructive proofs without the law of the excluded middle) was something Gödel eventually gave up on. And so, he fully embraced full-fledged platonism in order to use the tools, i.e. the previously deemed "objectionable methods", in order to expand his view on mathematical reality and use the telescope which the law of the excluded middle provides.

Gödel thought that while we can discover new mathematical objects and theorems, they are, as he wrote in the Gibbs Lecture, "as objective and independent of our free choice and our creative acts as is the physical world" (*1951:312n.17). What is clear is that Gödel held strong realist views as to the existence of mathematical objects and concepts, and that he advocated our ability to have epistemic access to them, in a way similar to our perception of physical objects. As we saw, Gödel investigated the possibility of finding a justification for the axioms with Hilbert's program but abandoned it based on the "recently discovered facts". Where does he stand, then, on non-finitary reasoning? In a letter to Wang (December 7th 1967) Gödel writes:

> This blindness (or prejudice, or whatever you may call it) of logicians is indeed surprising. But I think the explanation is not hard to find. It lies in a widespread lack, at that time, of the required epistemological attitude toward metamathematics and toward nonfinitary reasoning. (quoted from Wang 1996a:240; 1996b:122)

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for the existence of finite objects. In "On the Infinite" (1926), Hilbert writes: "[O]ur principal result is that the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought – a remarkable harmony between being and thought…Operating with the infinite can be made certain only by the finitary. The role that remains for the infinite to play is solely that of an idea – …in Kant's terminology, a concept of reason which transcends all experience and which completes the concrete as a totality – that of an idea which we may unhesitatingly trust within the framework erected by our theory" (1926:201). Hilbert's program is often considered to have induced the rise of metamathematics, and is also considered to have motivated Gödel's Incompleteness Theorems, which, incidentally, had devastating consequences for Hilbert's program (Maddy 1990:24–25; Kleene 1986:138–39).

[15] The First Incompleteness Theorem states that for any consistent formal system *F*, containing a certain amount of number theory, there are statements expressed in the language of *F* which are neither be provable nor disprovable in *F* (Hallett 2006:116). According to the Second Incompleteness Theorem, a formal system *T*, no matter how powerful it is and assuming its consistence, *T* cannot prove its own consistency (Hallett 2006:116).

Furthermore, Gödel claimed that his ontological realism in mathematics actively contributed to his most important mathematical discoveries, namely both his Completeness Theorem for first-order logic (every sentence that is logically true, a semantic notion, can be proved by the system, a syntactic notion) and his Incompleteness Theorems. For example, the Completeness Theorem can be showed to be a rather easy consequence of a result obtained by mathematician Thoralf Skolem, but Skolem himself did not arrive at this conclusion (Wang 1996b:122). Gödel himself thought this was due to their different philosophical positions, where Gödel was a mathematical and conceptual realist, whereas Skolem, while not belonging to any definite school of philosophy of mathematics, was partial towards finitism.

In another letter from September 29th 1966, this time to the mathematician Alonzo Church, Gödel discusses the result by Paul Cohen on the Continuum Hypothesis. In 1963 Cohen proved that the negation of the Continuum Hypothesis was consistent with ZFC (the standard axiomatization of set theory, Zermelo-Fraenkel axioms plus the Axiom of Choice), and as Gödel had already proved that the Continuum Hypothesis was consistent with ZFC in 1938, CH was now proved to be independent from or undecidable relative to ZFC.

> I disagree about the philosophical consequences of Cohen's result. In particular I don't think realists need expect any permanent ramifications…as long as they are guided, in the search of axioms, by mathematical intuition and by other criteria of rationality. (Gödel 2003a:372)

The philosophical question that arises in light of this result is: (i) whether the axioms paint an exhaustive picture of mathematical reality, (ii) whether we need to find stronger axioms in order to settle CH, or (iii) whether we should accept a pluralist view, where different axiomatizations live side by side. As we can see from Gödel *1964*, the third possibility was ruled out on Gödel's part, as he believed in the one mathematical reality, i.e. the universe view, where mathematical reality is the subject of study in the descriptive science that is mathematics.[16] This leaves us with two alternatives. If the axioms we have do exhaust the mathematical reality, it means that there exist *absolutely* undecidable mathematical

[16] Gödel did earlier play around with the thought that there is some kind of pluralism in set theory. In a lecture in Göttingen from 1939, right after he had proved the consistency of CH relative to the consistency of ZFC, he writes: "[T]he consistency of the proposition *A* (that every set is constructible) is also of interest in its own right, especially because it is very plausible that with *A* one is dealing with an absolutely undecidable proposition, on which set theory bifurcates into two different systems, similar to Euclidian and non-Euclidian geometry" (*1939b:155). Of course, Gödel changed his mind later on, which is evident from *1946* and *1964*. I will come back to Gödel's view on absolute undecidability in section 1.2 in chapter 3.

propositions (it has already been proven that there are a lot of undecidable propositions *relative* to ZFC, such as CH). As the last part of the quote shows us, this was not Gödel's view.

This leaves us with the first alternative, with the need to find new axioms, doable if, as Gödel says, one is "guided…by mathematical intuition and by other criteria of rationality" (2003a:372). (I must add that is to be done not exclusively by "mathematical intuition and other criteria of rationality". In establishing new and stronger axioms, Gödel uses the notions of *intrinsic* and *extrinsic* evidence, a point I will return to in chapter 3.) So, where does this leave us? First of all, Gödel believes that we are able to find new axioms that will be able to settle the question of undecidable propositions relative to ZFC such as CH. Secondly, this is to be done by mathematical intuition and with the help from "other criteria of rationality" (Gödel 2003a:372). Also, it is the postulation of mathematical intuition as a psychological fact that "suffices to give meaning to the question of the truth or falsity of propositions such as Cantor's continuum hypothesis" (1964:268). Thus, evidently Gödel believed in the power of reason – it will be reason and the faculty of mathematical intuition that will enable us to make advances in set theory. And yes, such advances will be made, if we are to share Gödel's optimism. It is his notion of mathematical intuition that will be the main theme in chapter 2.

2.4 Gödel, Frege and Plato

What we have seen in the previous sections is that Gödel's conceptual realism is extensive. He believes that mathematics is a descriptive science, and the mathematician's job is to study the mathematical reality and express this in a mathematical language. Mathematics can thus be compared to physics, where there is a one-way dependence between the non-linguistic level (i.e. the real mathematical facts) and the linguistic level (i.e. our theories written in a formal language invented by us).

This conceptual realism leads to ideas about mathematics that quite clearly separates it from other fields of study, for instance from those fields of study that rely upon social construction, such as legislation and social norms. In these fields the truth of what is written or said depends on human subjects. Our interpretation of a law can be correct to a varying degree, but it cannot be true (in the Fregean sense, i.e. that it is unalterable and eternal). Also, the interpretation of a law is not fixed, as our practice of interpreting laws progresses in a certain way (say, in correlation with the progress of social norms). Mathematics also differs from the other extreme, the sciences that study the physical world, for instance physics. In section 2.2, I brought up the fact that the standard according to which a physical theory is deemed good enough is lower than that of mathematics. In physical theories it is a high degree of probability or success in explaining physical phenomena that is the rule. I argued that even though both the empirical sciences and mathematics have a subject matter that is not constructed by us (or so the platonist claims), mathematics strives for absolute truth with no room for error in any way. For example, if the proof for a mathematical theorem only has one small logical fallacy, this would not be acceptable at all to a mathematician.

Let us see, then, how Gödel's position can be compared to that of Plato and another famous platonist, Frege. Gödel's insistence on the objective reality of concepts, and that it constitutes another world from that of things is vital here. Like the form of justice exists for Plato, so does the concept of set exist for Gödel, and, more importantly, the concept of set needs not be instantiated, i.e. it exists independently of whether there exist any sets (although, sets do exist, according to Gödel). Here, however, it quickly becomes difficult. It is often unclear whether Gödel speaks of mathematical objects or mathematical concepts and what importance this difference really has. If we remember what he writes in "Russell's Mathematical Logic": "It seems to me that the assumption of such objects [*classes and concepts*] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence" (my emphasis, 1944:137). Here, it is clear that a mathematical concept is, in fact, an object in the same way that a set or class is. What sets it apart, though, is a concept's role in mathematical knowledge. As we shall see later on, it is first and foremost the mathematical intuition of concepts that Gödel describes as a source of mathematical knowledge, and not perception of the sets themselves.

But how does this position relate to Plato? Well, Plato would not make the juxtaposition of the existence of physical bodies and mathematical concepts because to him, they did not have the same value. Where the forms are part of the highest and most ideal world, the existence of physical bodies do not have an equally valuable existence. Also, there is a teleological aspect to it, where one strives to be part of the higher and more ideal reality. According to Plato, the mathematical reality is already on a level closer to the forms and therefore better than the physical world. Gödel's conceptual realism, on the other hand, does not defend such normativity. Mathematical concepts are not said to be better than the physical bodies in our everyday world, nor is a particular set less good than the concept of set. Mathematical objects and concepts do not play the same role in our acquiring mathematical knowledge, but their *worth* does not depend on this (I will return to this point in chapter 2). I think Plato's influence on Gödel is limited to providing an example, i.e. giving a version of conceptual realism, where the concepts exist independently of us. Also, Gödel says that the existence of concepts is as objective as physical objects, but they are not inherently better. In fact, any talk of inherent worth regarding physical and mathematical reality would probably be alien to Gödel.

Now that we have ruled out any normativity à la Plato on Gödel's part, let us see how he compares to Frege, who, along with his logicist project, is also largely known for positing a "third realm" of thoughts. How does Frege's account of thought in (the fittingly named) "Thought" (1918-19) relate to Gödel? Frege claims that a "sentence *expresses* a thought" and that "the only thing that raises the question of truth at all is the sense of sentences" (1918-19:328, 327). In the same way as a mathematical term has a meaning which is the concept, sentences have a sense, and it is this sense that belongs to the realm of thought. It is only thoughts that have a truth value, because thoughts are objective. This is why the Pythagorean theorem can be grasped by many, and is *discovered*. It belongs to the realm of thought, and so is independent of the people grasping it – it has an abstract, independent existence. In this respect, Gödel and Frege are quite similar. The domain of true mathematical propositions is already and always what it is. However, the role of these platonic realms is somewhat different. Frege's thoughts are sentences with truth values that can be grasped by anyone, and so ensure intersubjective understanding, both in mathematics and elsewhere. Gödel's concepts are notions that give rise to axioms and theorems, and it is from reflection on the concept of set and membership that we can extend our mathematical knowledge. It is by reflecting on the concepts we already know that we can be guided to new stages of abstraction.

There are some obvious ways in which Gödel's realism differs from that of Frege. Firstly, there is the notion of concept itself. A *concept* for Frege is different from that of Gödel. Frege's concept is extensional, which means that the concept is identified by the objects which fall under it. Gödel's concepts, on the other hand are intensional, which means that the concept is picked out by specifying the necessary and sufficient conditions for when it should be used, e.g. in the case of nouns where the properties are given for when the object is a referent of the term (Martin 2005:208).

Secondly, Frege's thoughts, which behave in much of the same way as Gödel's concepts, exist in what he called the "third realm" (Frege 1918-19:69). This third realm has similarities both to the realm of physical things and to the realm of ideas that consists of our mental activity. A thought, then, figures in the third realm, where it is the similarities with the two other realms that on the one hand secures intersubjective understanding (with its similarity to the objectiveness of the physical realm), and on the other enables a subject to grasp the thought (which is the similarity with the realm of our mental content). Gödel does not make this distinction. His mathematical concepts do figure in the objective mathematical reality, but the reality itself is neither marked by any affinity with a realm for our mental activity nor with the world of physical objects. If anything, Gödel’s concepts would have an affinity with the world of things, not in the way that a concept is similar to a physical object, but that they have the same sort of objective existence.

Perhaps the most important difference between Frege and Gödel is that Frege does not explain how the epistemological processes of how we grasp a thought work. The epistemic relations we have to these thoughts are not accounted for, except, perhaps, if it is exactly the thoughts’ similarity to our mental acts that is supposed to explain this. Gödel, on the other hand, proposes one of his most notorious philosophical contributions, namely his notion of *mathematical intuition*, which I will turn to next.

Chapter 2: Mathematical Intuition

Now that I have given an account of Gödel's realism in chapter 1, chapter 2 is devoted to his conception of mathematical intuition. Gödel's perception-like mathematical intuition is meant to explain how we can have knowledge of mathematical objects and concepts. If this intuition in some way can bridge the epistemological gulf between abstract objects and us, that would ensure the possibility of us having knowledge of them, and we would be well on our way in defending a platonist position. The key question is whether this intuition can be explained in some non-mystical way.

In this chapter I will give an interpretation of Gödel's mathematical intuition, in a more favourable light than what his severest critics have allowed. In the first section I will make some preliminary remarks on mathematical intuition as such. In section II I will delve into Gödel's position and present some of the criticism his postulation of mathematical intuition has met. In section III I will offer a Husserlian interpretation of Gödel's notion of mathematical intuition. I will give an interpretation that follows Dagfinn Føllesdal (1995), who tries to make this intuition less mysterious by linking it to Edmund Husserl's philosophical framework. I argue that a Husserlian reading does indeed render Gödel's intuition less mysterious, but that it should be rejected, as there are too many discrepancies between Gödel's view and the Husserlian reading of it. Also, I will suggest another possible interpretation, where it is seen as a method for recognizing and extending patterns, and thereby reach higher levels of abstraction.

Section I – Introduction

1.1 Intuition of Truths and Objects

So, what is intuition, and more specifically, mathematical intuition? Intuition has designated many different things, and the word has often worked as a conceptual umbrella for either different ways to acquire knowledge or for the different parts of reason involved in acquiring knowledge. It has mostly been used as pure, formal or categorial intuition, where what one intuits are ideal or abstract objects or truths.[17] This is also an important distinction: the

[17] There are exceptions though. Kant, for instance, distinguished between *empirical* and *a priori* intuition, where a priori intuition was how we could understand time and space, and where empirical intuition was reserved for the understanding of a concrete object in its totality (Kjosavik 1999:6).

difference between intuition of truths and intuition of objects, most often distinguished as intuition *that* and intuition *of*, respectively (Kjosavik 1999:3). As we shall see, this is also a distinction Gödel makes, even though it is not always clear which type of intuition is used when (Parsons 1995:59).

One further complication about intuition of truths is how it has been used in the philosophical tradition, as Charles Parsons (1995) and also Frode Kjosavik (1999) point out. The ambiguity is that the propositional kind of intuition does not make it obvious whether we speak of (a) *knowledge*, i.e. the truth of a proposition, or (b) the belief that we have on the outset of a philosophical investigation, i.e. what we believe to be true but is not necessarily so, and which can actually be non-reliable guides to truth (Parsons 1995:59). Traditionally, early modern philosophers have not made this distinction clearly, but have still championed intuition of truths, as in the case with Descartes' "immediate apprehension of true propositions, like '2 + 2 = 4'" (Kjosavik 1999:3).

Intuitive knowledge has been a very popular concept, often posited together with notions such as self-evidence, clearness or, to follow Descartes, the natural light of reason. Intuition has been considered one of the greatest, most reliable and truth-leading abilities humans can have. In this respect intuition has not been in any conflict with *reason* (i.e., if one makes the common-sense understanding of the word form a contrast to the philosophical traditional understanding of it). It is not the case that our pure understanding or logical abilities have been in any opposition to intuition, rather intuition itself has been seen as one of the clearest cases of an application of reason, and this is why intuition in the rationalistic tradition has had such a prominent role. This is something that still is very much the case for Gödel's concept of mathematical intuition. Chapter 3 will deal with this rationalistic aspect of Gödel's view.

To give an example, for a moral theory such as deontology one has to explain the problem of, say, a white lie. In this case one needs to see if and how the deontological theory can account for something that seems, intuitively, morally acceptable, but which is unacceptable according to the theory. With this example I want to illustrate the broad range of different things, situations or problems we have intuitions about. Sometimes we have intuitions of something being true or false, morally acceptable or unacceptable, and it is intuition that leads us on to make judgements about these things. This is why intuition has often figured as the abovementioned conceptual umbrella, where the processes or parts of reason described are clouded and where we cannot distinguish each constituent of the processes or parts with the rigour we desire.

1.2 Intuitive Knowledge

The notion of intuitive knowledge has also been considered a conceptual umbrella, where the necessary and sufficient conditions have not been properly delimited. In our updated language, intuitive knowledge are instances of intuition *that*. According to Leibniz knowledge is qualified as intuitive if:

> [I]t is *clear*, i.e., it gives the means for recognizing the object it concerns, *distinct*, i.e., one is in a position to enumerate the marks or features that distinguish an instance of one's concept, *adequate*, i.e., one's concept is completely analysed down to primitives, and finally one has an immediate grasp of all these elements. (quoted in Parsons 1995:45n.3)

The qualities necessary for intuitive knowledge – clearness, distinctness and adequateness – are all reminiscent of the "natural light of reason" described in Descartes' *Meditationes* (Kjosavik 1999:3). The way Parsons expands on it, however, gives a nuanced picture of what it takes to qualify as intuitive knowledge. As Gödel considered Leibniz one of his philosophical heroes, it is very possible that Leibniz's version of intuitive knowledge came to bear on Gödel's own understanding of intuition (Parsons 1995:59).

The first component, clearness, seems a somewhat vague requirement. If the knowledge is clear, it is supposed to provide "the means for recognizing the object it concerns" (Parsons 1995:45n.3). But what does this really mean? It seems like the clearness component captures what intuition itself is meant to do, to enable us to grasp something and see it for what it is. If our knowledge of the concept "∈" is clear, we would recognize it as the object of which we have knowledge. Is not this exactly what grasping a concept is? To recognize the thing we have knowledge about? If, on the other hand, one knows one thing about a person, say, that Elizabeth I was the daughter of Henry VIII, this does not guarantee that we recognize her in a painting picturing her. That is to say, in order to have knowledge about something, we do not have to know everything about it. This ability of identifying the object of which we have knowledge is what the first component, clearness, gives us.

The second feature, distinctness, resembles two features concepts can have: Extensionality and intensionality. As mentioned in section 2.4 in chapter 1, the notion of concepts for Gödel differs from that of Frege (Gödel's concepts are intensional, whilst Frege's are extensional). Distinctness here actually says two things: i) one should be able to name all the features of one's concept, and ii) these necessary and sufficient conditions must be satisfied in order for an object to fall under said concept. If we relate this to set theory, the first point concerns the set's intension while the second relates to its extension.

The third component, adequateness, ensures our having the rigorous step-by-step knowledge about a concept's parts, i.e. each constituent is accounted for and has a known and explained place and role. If we compared this rigorous step-by-step knowledge to a proof-structure for instance, it would mean that each logical step taken did not have any gaps, as Frege would say (1893:1). These conditions, that intuitive knowledge is clear, distinct and adequate, then, ring true for much of what Gödel writes as well, especially the importance placed on knowing the primitives, recognizing a concept for what it is, and defining a concept intensionally.

If we now consider the last part of this quote: "and finally one has an immediate grasp of all these elements," we can see how the bar is set quite high for knowledge to qualify as intuitive. However, what does it mean to have such an "immediate grasp"? It seems that in order to satisfy this last requirement, intuitive knowledge has something in common with perceptual knowledge, insofar as intuitive knowledge (in all its different elements and related conceptual constituents) is something that is thrusted upon you, and to which you respond right away, and this response is to possess the knowledge in question. By this I have in mind the process of acquiring knowledge in the way that Gödel describes our relation to the axioms. Here, when we realise the truth of the axioms, they "force themselves upon us as being true," and there is no way of avoiding or delaying this grasping process of something true (1964:268). If you have this immediate grasp of all the elements required by clearness, distinctness and adequateness – you have acquired intuitive knowledge.

Section II – Gödel's Intuition

Gödel compares how we come to have knowledge of mathematical objects to how we *perceive* physical objects. Gödel writes that we do have something like perception of the objects of set theory, and that this perceiving is due to the psychological fact he takes mathematical intuition to be (1964:267–68). On Gödel's view, then, mathematical intuition is not a "mysterious sixth sense" *per se*, but something that gives justification to our beliefs.

As mentioned in section 1.1 in this chapter, there is a distinction between intuition *of* and intuition *that*, i.e. between intuition of objects and intuition of propositions. According to Parsons (1995:59), Gödel uses both types of intuition. Parsons argues that Gödel primarily uses intuition *that*, as it is closely connected to his conceptual realism, especially the intuition of the concept of set (59, 65). Also, in describing the search for new axioms, Gödel claims that "the axioms force themselves upon us as being true", which is clearly an occurrence of intuition *that*. However, he also says we have "perception …of the objects of set theory", which is an occurrence of intuition *of*, and which is evident by the fact that we have intuition *that* the axioms are true. This, Parsons says seem to be a logical fallacy, as the intuition of existing objects of set theory does not follow from the fact that "the axioms force themselves upon us as being true". We cannot from our intuition *that* infer that we have intuition *of* such and such mathematical objects. However, one can think that an instance of intuition *that* some proposition is true would lead to intuition *of* the objects that appear in the proposition. That is, if some proposition stated that some particular set was such and such, it could lead to intuition *of* that particular set. If this was the case, then the proposition in question must be about a *particular* set, and not about some unidentified sets, i.e. a proposition about some sets in general. As Parsons argues, we could, perhaps, identify and individuate some sets by concepts, so that we could "perceive" a particular set by our perception of the concept that picks it out uniquely (1995:65n.43). This would also lead to possible perceptions of the natural numbers as well, but, as Parsons notes, Gödel neither affirms nor denies this in his work (1995:65n.43). However, in the cases where the intuition that some proposition is true is not about such particular sets, it is difficult to see how we could infer from our intuition *that* also an intuition *of*.

Here, we must also remember that Gödel considered *concepts* to be objects, or properties of objects, and so he primarily intends a perception of the primitive concepts (e.g., the membership relation) of set theory, and not only of sets and classes. Parsons argues that Gödel considers "rational evidence in general as involving perception of the concepts that are the constituents of the proposition in question" (1995:65). As we saw in section 2.1 and 2.2 in chapter 1, it is from his *conceptual* realism that mathematical knowledge is to be attained, and this through continued reflection on concepts and the presence of mathematical data in us, or, as Parsons would call it, rational evidence (which will be discussed in 2.4). The question whether Gödel primarily uses intuition *of* or intuition *that* is therefore a difficult one, but as far as our mathematical knowledge goes this is due to our perception of concepts and not of *sets* as such. In section 3.2.3, I will investigate further the relation between intuition *that* and intuition *of*, and how they possibly depend on each other.

As evidence for the existence of the faculty of mathematical intuition, Gödel gives two different lines of argument. The first, and also the one to appear in his earlier work, explores the analogy to physics, where the perceptual knowledge of physical objects is likened to us perceiving the objects of set theory. The second, which is expressed in its full form in *1964*, involves his view on our different senses and how we can perceive the world as *one*, consisting of both physical and mathematical reality, a view which leads him to affirm the existence of mathematical intuition as a psychological fact. In this section, I will firstly present some criticisms the postulation of mathematical intuition has met; secondly, I will introduce the physics analogy kind of argument; thirdly, I will discuss the principle of *epistemological parity*; and fourthly, I will turn to the view of *1964*, where his notion of mathematical intuition appears in its most developed form and where it is directly posited as a psychological fact.

2.1 Criticism

As previously noted, the criticism Gödel has faced regarding his philosophical contributions has been severe. It revolves around his two main claims: 1) his platonism as to mathematical objects and concepts, and 2) the postulation of mathematical intuition. The criticism he faces for his platonism usually revolves around it being crude and naïve, while the criticism regarding mathematical intuition tends to say that it is postulated as a "wholly mysterious sixth sense," as Potter notes (2001:331). It is especially the combination of realism and mathematical intuition that is attacked: "Naïve realism itself troubles many philosophers, but the addition of mathematical intuition – associated, as it is, with Kantian anti-realist views about mathematics – seems to take us to the edge of inconsistency" (Folina 2014:32). The criticism continues in the same vein:

> Gödel the *philosopher* – and indeed even today it is a matter of debate, whether Gödel can be regarded as a philosopher at all – has traditionally been seen as advocating a crude form of Platonism in his philosophical writings, one entangled with the views of Kant and Leibniz in a way which was seen as philosophically naïve and primarily historical. … as the antiquarian views of an old-fashioned, albeit great mathematician, untrained in philosophy and nostalgic for the days when the concept of mathematical truth was considered to be beyond criticism – an ironic development in the light of Gödel's actual discoveries. (Kennedy 2014:1–2)

As we see, the criticism usually rejects Gödel's whole view without really engaging with his work. It tends to be very brutal and it rejects the possibility that Gödel could have had any sophisticated philosophical understanding. When the general criticism involves such formulations that deem Gödel "a logician *par excellence*…but a philosophical fool," it is clear that one has not really tried to give a generous reading of Gödel's work (James 1992:131).

I do not mean to say that Gödel's work does not *deserve* criticism, as it certainly does. His published philosophical work is limited, and his more outré claims, such as the objective existence of mathematical objects and our "perception" of them, are not really examined in *enough* detail. His most explicit remarks on mathematical intuition are only included in the Supplement of 1964 version of "What Is Cantor's Continuum Problem?". Therefore, even though he tries, to some extent, to explain the parts that are in play in intuition, he does not give a systematic account of how it actually works. That there is something "given" underlying mathematics and that this also plays a role in our empirical ideas, i.e. ideas about physical reality, is left unexamined (but will be important in my discussion in section 2.4 and chapter 3). Also, as Gödel's arguments for realism largely rely on his analogy with the empirical sciences, it becomes a problem that he does not give a detailed account of how we perceive the physical world.

One of the most fervent criticisms come from Charles Chihara (1990) and (1982). His criticism revolves around exactly this: that we have perception of the objects of set theory *like* our perception of the physical world, but that Gödel does not explain the causal processes that lead us to gain knowledge of abstract objects.

> After all, there is supplied no description of a causal mechanism by which we humans are able to "perceive" objects that do not exist in the physical world. The appeal to mathematical intuition does not explain how we are able to "perceive" sets – it, essentially, only asserts that we do. (Chihara 1990:19)

Here, it must be noted that it seems clear that Chihara attacks the instances of intuition *of*, i.e. intuition of the sets. Chihara lets this be his final judgement of Gödel: "Gödel's appeal to mathematical perceptions to justify his belief in sets is strikingly similar to the appeal to mystical experiences that some philosophers have made to justify their belief in God" (1990:21). E.P. James calls Chihara's criticism simplistic and simply false (1992:131), while Michael Potter notes that it is "wide of the mark" (2001:331). What is true is that Gödel's account does have its shortcomings, especially regarding the details of the actual processes that take place and their relation to our perceptions of the physical world (to which, as mentioned, I will return to in section 2.4 and chapter 3). However, the naiveté of his platonism cannot merely be asserted without investigating further his conceptual realism and its relation to an intuition *that*, which, certainly, have more merit than Chihara and the general criticism allow for.

2.2 Analogy with the Natural Sciences

Gödel does not mention "intuition" much in his earlier work, for example, as Parsons notes, the word only occurs three places in *1944* (1995:56). Also, there it is not used as a faculty with which we perceive the objects of set theory, but rather, as described above in section 1.1, as the belief or inclination we hold at the outset of a philosophical investigation and where the belief's truth is not guaranteed (1995:56). So, do we find traces in *1944* of a notion resembling the later meaning of mathematical intuition, that is the mathematical intuition of *1964*? Well, yes, we do. First of all, the analogy between mathematics and the natural sciences is quite prominent in *1944*:

> The analogy between mathematics and a natural science is enlarged upon by Russell also in another respect (in one of his earlier writings). He compares the axioms of logic and mathematics with the laws of nature and logical evidence with sense perception, so that the axioms need not necessarily be evident in themselves, but rather their justification lies (exactly as in physics) in the fact that they make it possible for these "sense perceptions" to be deduced; which of course would not exclude that they also have a kind of intrinsic plausibility similar to that in physics. I think that (provided "evidence" is understood in a sufficiently strict sense) this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future. (1944:121)

Here, we can see that when Gödel is describing Russell's analogy, the axioms are likened to natural laws. The justification for the axioms is likened to the justification for the natural laws, i.e. logical evidence is compared to sense perceptions. Gödel uses 'logical evidence' and 'mathematical data' as synonyms, I will continue to use 'mathematical data'. Gödel points out that the justification for the axioms is exactly that they make it possible for the mathematical data to be deduced. This means that our being able to deduce mathematical theorems from the axioms increases the strength of the justification we have for the axioms.[18] He also describes this situation to be exactly the same as in physics, which means that he views the acceptance of a natural law to be more motivated if it explains many natural phenomena and sense perceptions. This kind of argument is something that we have seen Gödel give elsewhere as well and which I describe in section 2.1 in chapter 1 – when the

[18] However, Gödel also underlines the fact that it is very much possible that the axioms have *intrinsic* plausibility as well. The difference between *intrinsic* and *extrinsic* evidence for axioms is something I will come back to in chapter 3 in section 1.2.

legitimacy of the existence of mathematical objects is maintained because their existence would explain why our mathematical theories are successful. So, how does this passage relate to mathematical intuition? To answer this let us again consider the passage on this legitimacy from *1944*:

> It seems to me that the assumption of such objects [classes and concepts] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and *in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data", i.e., in the latter case the actually occurring sense perceptions.* (my emphasis, 1944:128)

If we concentrate on the lines that are italicized, we can begin to see where intuition comes into play. But we have to keep two things in mind:

1) Sense perceptions are likened to mathematical data, and their presence in us is due to the objectively existing physical and mathematical objects.
2) We have sense perceptions of physical objects, and similarly we "have" mathematical data, as in the case where these "data" strengthen our justification for natural laws on the one hand, and for axioms on the other.

When Gödel writes that it is impossible to interpret the propositions we want to make about physical and mathematical objects as propositions about the "data," what does he mean? It seems plausible that this cannot be because the propositions we want, i.e. about the objects, are in fact prospective true propositions, i.e. knowledge about the objects. And, we do not, when we form a physical theory believe that we have discovered something true only about our sense perceptions, but rather that we have discovered something true about the physical world – and that this knowledge explain why we have the sense perceptions that we do. If, as the analogy Gödel encourages, the same goes for mathematical objects and the mathematical data we get from them, then this should mean that we do have some ability to get these mathematical data. And, as we have sense perceptions because we are able to *perceive* the physical world, then we should also have the ability to *perceive* the mathematical reality. Later in *1944*, this kind of perception is, in fact, mentioned:

> The difficulty is only that we don't perceive the concepts of "concept" and of "class" with sufficient distinctness, as is shown by the paradoxes. … one should take a more conservative course, such as would consist in trying to make the meaning of the terms "class" and

> “concept” clearer, and to set up a consistent theory of classes and concepts as objectively existing entities. (1944:139–40)

Even though this is a negative statement, he still says something about perception of mathematical objects and concepts (Parsons 1995:56). Also, he marks out the course onwards. In order to perceive these objects clearer, one has to develop a theory where the objects do have an objective existence. What we have seen, then, is that even though Gödel’s line of argument in *1944* does not mention intuition in the strong sense of *1964*, he is laying the groundwork for such a faculty. And this is done by developing the analogy between the natural sciences and mathematics. By likening their subject matter, viz. the physical world and the mathematical reality, and suggesting that the two different kinds of objects cause these data that we have, he also motivates that we have something like perception of mathematical objects as well.

2.3 Epistemological Parity

Gödel puts the legitimacy of the existence of both physical and mathematical objects on an equal footing. The assumption of mathematical objects is said to be “quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence” (Gödel 1944:128). Here we can see the source of what came to be a pervasive aspect to Gödel’s position. According to the philosophers Mark van Atten and Juliette Kennedy, the principle of *epistemological parity* functioned as a regulative principle in Gödel’s thought (van Atten & Kennedy 2003:434; Kennedy 2014:6). The idea is that if you consider “physical objects on the one hand and abstract or mathematical objects on the other, from the point of view of what we know about them, there is no reason to be more (or less) committed to the existence of one than the other” (Kennedy 2014:6).

So, this principle is related to an epistemological juxtaposition of physical and mathematical objects and has a distinctive sceptical flair. First of all, it is meant as an argument in favour of mathematical objects. This much is clear. If we remember the above discussion of how our knowledge of mathematical objects is acquired on the basis of something else that is given to us, i.e. something akin to sense perceptions that are our *mathematical data*, we see that the sceptical attitude present in epistemological parity is not misplaced, but rather quite efficient. Because, as Bertrand Russell (1912) also notes, the possibility of global scepticism always remains, even though we do not have the slightest reason to believe that the external world does not exist. When Gödel puts the existence of the two kinds of objects on an equal footing, he involves the possibility that the physical world does not exist, and, as the thought of solipsism has never been especially tempting, this prompts us to accept the existence of mathematical objects as well as the physical ones. The point is this: If we are willing to look past the everlasting risk of scepticism about the external world, why should we not also accept this regarding the mathematical reality? As we can see from Gödel's own *1964*, the sceptical flair present in epistemological parity continues to ring true in his later work: "However, the question of the objective existence of the objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here" (1964:268).

However, putting these two existence questions on an equal footing does have some attendant results not entirely in favour of mathematical knowledge. As discussed in section 2.2 in chapter 1, mathematical facts, such as '2 + 2 = 4', have been deemed the most certain knowledge it is possible to have. If one possesses mathematical knowledge, it is supposed to be above revision (given, that the mathematical propositions you believe to be true are in fact true, as mathematicians can err in their practice as well). When, for example, you are overcome by doubts about the external world, mathematical knowledge, at least, has been necessary and eternal. But if you follow Gödel's principle of epistemological parity, mathematical facts are doubted in the same way as physical facts (an example being that the sun rises tomorrow). So, because of the epistemic strength of how we acquire knowledge of both physical and mathematical objects, mathematical knowledge can be said to be open to "new" doubts. With the principle of epistemological parity, mathematical knowledge becomes open to the same kind of sceptical doubts as knowledge of physical reality. For example, although Einstein's theory of gravitation, known as general relativity, is a highly successful and well-confirmed theory, one can, of course, doubt the truth of the theory. If the axioms of set theory have the same status as Einstein's theory of general relativity, then one could also doubt the truth of the Axiom of extensionality[19].

Furthermore, this is due to Gödel's belief that mathematical propositions purport to say something true of the mathematical objects, and not of the *mathematical data*. If we remember the passage from *1944* quoted above where Gödel compares sense perceptions to

[19] The Axiom of extensionality says that for every set, if they have the same elements, then they are the same set.

the assumption of the existence of mathematical objects: “[I]n both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the “data”, i.e., in the latter case the actually occurring sense perceptions” (1944:128). As we see, it is clear that while our theories – both physical and mathematical – explain the sense perceptions and our mathematical data (as mentioned earlier in *1944*), they are about the different entities belonging to the physical world and the mathematical reality, respectively. If our theories were merely about the physical and mathematical data, then we would be guarded against the sceptical threat.

Secondly, epistemological parity pulls, according to van Atten and Kennedy, in two different directions – realism and rationalism (2003:435). The pull towards realism is because one is encouraged to accept the existence of abstract, mathematical objects, and the pull towards rationalism is because the basis on which we are to accept the existence of these objects is abstract evidence (van Atten & Kennedy 2003:435). The most salient formulation of the principle of epistemological parity we find in the Syntax paper (*1953/9–III & *1953/9–V), where Gödel writes:

> The similarity between mathematical intuition and a physical sense is very striking. *It seems arbitrary to me to consider the proposition "This is red" an immediate datum, but not so to consider the proposition stating modus ponens or complete induction (or perhaps some simpler propositions from which the latter follows).* For the difference, as far as it is relevant here, consists solely in the fact that in the first case a relationship between a concept and a particular object is perceived, while in the second case it is a relationship between concepts. (my emphasis, *1953/9–V:359).

The italicized sentence shows us what Gödel thought of knowledge, what it can look like and how we can acquire it. If it is arbitrary to consider an empirical proposition more certain than a proposition expressing a logical rule of inference, what does this mean? Well, it means that he did not deem the epistemological route to empirical knowledge more reliable than the one to mathematical knowledge. The similarity between mathematical intuition and a physical sense is said to be “very striking,” and in combination with the formulation of the principle of epistemological parity, this must mean that it is mathematical intuition that is meant to provide this reliable route to knowledge (Gödel *1953/9–III:359). As van Atten and Kennedy point out, if one accepts the principle of epistemological parity, “then the question how the varieties of evidence and, correspondingly, the varieties of objects they are evidence for, are connected, becomes secondary” (2003:435). It does not, then, matter whether the immediate data are empirical or logical, they should both be considered as equally strong evidence for the objects they are meant to justify belief in.

Thirdly, let us turn to how concepts are related. In another version of the Syntax paper, the last sentence ends with “and that, moreover, the second case relates to concepts of a different kind” (*1953/9–III:347n.34). So, in this second case, where it is a relationship between concepts we are concerned with, the concepts involved are of a different kind. If this “other kind” merely points to the fact that they are part of the mathematical reality and not intertwined with the empirical world, is uncertain. If this is indeed the point, then we need not worry too much. As we saw above, if one adheres to the principle of epistemological parity, the differences in types of evidence and the objects they serve as evidence for, becomes secondary.

And fourthly, when it comes to a relationship between concepts, I think we must understand this as Parsons does, when he assumes that “perceiving a relation between concepts involves perceiving the concepts” (1995:62). This means that in perceiving a concept, it is implicit that when we perceive it fully or completely, we can also perceive how it relates to another concept (of the same type). Now, let us again look at adequateness, the third component of Leibniz’s notion of intuitive knowledge described in section 1.1, where “one’s concept is completely analysed down to primitives” (Parsons 1995:45n.3). So, by employing mathematical intuition and regarding the proposition of modus ponens to be an immediate datum, we have 1) secured a reliable route to mathematical knowledge, 2) this is done by perceiving the mathematical concepts completely, which 3) means that the concept is analysed down to primitives, which 4) secures our perceiving the relationship between the concepts.

2.4 Other Relationship

In the Supplement to “What is Cantor’s Continuum Problem?” (1964) Gödel makes explicit remarks on his view that abstract mathematical objects and concepts are part of an objective reality. Our ability to intuit these objects and concepts and our subsequent understanding of them is explained, according to Gödel, by how we deal with mathematical data. Even though mathematical data “cannot be associated with actions of certain things upon our sense organs,” they are not “something purely subjective, as Kant asserted” (1964:268). This means that the data that lead us to knowledge about the mathematical reality are not specific or subjective to any person. It is not so that the data I possess for justifying the Axiom of pairing are somehow inaccessible to others; rather mathematical data are supposed to ensure intersubjectivity, so that we can reach mathematical understanding and develop well-functioning mathematical theories. In this way, mathematical data are reminiscent of Frege's thoughts, in that they provide intersubjectivity. However, mathematical data are on a different level than thoughts. Thoughts are rather on the level with Gödel's concepts, i.e. eternal and objectively existing entities. Mathematical data, on the other hand, are akin to sense perceptions caused by the real existing physical objects. In the Syntax paper, Gödel writes:

> However, in truth, experiences are not the object of most other sciences either. E.g., animals seen in hallucinations are not objects of zoology…Hence, again, there is no substantial difference between mathematics and other sciences. (*1953/9–V:359)

Here Gödel emphasizes the different levels that we are dealing with: i) There is the non-linguistic level consisting of the real physical and mathematical facts, ii) there is the level of mathematical data and sense perceptions that are caused by these two kinds of facts, and iii) there is the linguistic level, consisting of our physical and mathematical theories of these facts. In the case of mathematical intuition, we are dealing with the non-linguistic level of mathematical facts and the mathematical data they cause.

Sense perceptions do vary from person to person, and it is not so that a physical object or event will cause the exact same sense perceptions. Inevitably there will be variations. But we do get (given normal conditions and us having well-functioning faculties) sense perceptions that are quite similar, and which generally cause us to form the same ideas and develop physical theories of what the world is like, i.e. they provide an epistemically justified way to gain knowledge of physical facts. This same point is what Gödel makes for our mathematical data and our mathematical intuition, where mathematical intuition allows us to process the data and form beliefs about what the mathematical reality is like, and, furthermore, that they lead us towards mathematical knowledge. In describing mathematical data, he continues:

> [T]hey, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality. (1964:268).

What does this other kind of relationship amount to? It is clear that Gödel intends this relationship to be different from the one we have with the physical world, but how so? The claim does have some intuitive merit. The objects in the physical world are, well, *physical*, and we thus have a causal relationship to these objects.

However, with the principle of epistemological parity comes the attendant rejection of causal theories of knowledge. It is clear that if one believes that abstract evidence is valid justification for knowledge of abstract objects, then we cannot also believe that we get knowledge of the world *only* through our causal relationship to the world, as there is knowledge of the purely abstract world as well, which, for Gödel, is mathematical reality.[20] But, as Gödel believes that the objects of mathematical intuition are abstract – "the objects of transfinite set theory … clearly do not belong to the physical world" – what he needs is an account of how we get knowledge of abstract objects (1964:267). In *1964*, his answer to this problem is the postulation of mathematical intuition. So, what does he say about this faculty in *1964*?

What we have gathered so far is that the presence of mathematical data in us (also a sort of first step for acquiring mathematical knowledge) is "due to another kind of relationship between ourselves and reality" (1964:268). This seems, at first glance, utterly puzzling. First of all, what is our *first* relationship to reality? The immediate response that springs to mind is that our first relationship to reality is, of course, our causal relationship to the physical world. But, are there more relationships between ourselves and reality that are different? Traditionally, philosophers have often answered yes to this question, often by positing that we have some soul or soul-like faculty descended from some god-like realm which enables us to reason and deal with abstract things.

However, one should think that – even if we grant Gödel the existence of abstract objects and some way of having knowledge of them – that our relationship to the world would remain the same. That is, one should think that we would only have *one* relationship to the

[20] Penelope Maddy tries to include the world of sets into our everyday, physical world, by talking of *impure* sets, i.e. sets that include at least one *urelement* in its transitive closure. An *urelement* is an object that, in a domain of set theory, is not itself a set. In *Realism in Mathematics*, she defends Gödel's intuition by claiming that when one sees three trees one also sees the set of whose elements are the trees, this is thus an impure set. In this way, the set of three trees adopt the spatio-temporal properties of the trees, and that is how we can perceive sets and classes. From these impure sets consisting of physical objects we can then construct the iterative hierarchy with V_O consisting of physical objects (if one runs into trouble with the empty set, as it cannot be spatio-temporally located, just omit it all together (1990:48, 156–57).

world, but that this can be *many-faceted*. We are, after all, firmly located in space and time, with no apparent floating into some mathematical reality, or into some third realm à la Frege. This means that Gödel also rejects the view of one of his philosophical heroes, Plato. In *Meno*, the soul is said to be immortal and to have already learnt whatever that can be learnt (*Meno* 81cd). The soul is described as floating in and out of bodies, i.e. constant reincarnation, which means that in "learning" mathematics, it is actually a remembering process, where the eternal travelling of the soul provides us with knowledge of a higher realm than our incarnate life.

Being situated in this way, this will limit how and in which ways we can deal with reality. Even though we are capable of abstract reasoning, this does not mean that our relationship to the mathematical world is something wholly unconnected to our everyday relationship to *all of reality*, both physical and mathematical. Our relationship to mathematical reality is rather a constituent of our already existing relationship to the world. So, why would this be the case for Gödel? He does not, as we must admit, strike us as the biggest adherent to either naturalizing the mathematical reality or our relationship to it.

First of all, perhaps this "other relationship" is not meant to be completely separated from our *normal* relationship, i.e. the causal one, but rather only point to the fact that it is indeed different. What do I mean by "completely separated" here? I mean the separation between our two tentative relationships to the world, and the separation between the physical senses and mathematical intuition. With *complete* separation, I mean that there would be no interplay at all, not between our relationships or between the senses. To put it differently, there is no cooperation in us and between our different kinds of faculties, and our relationship to the whole of reality is *forked*. If our relationship to reality was forked in this way, then the part of us that have knowledge of mathematical reality would not play a role, or have any function, in our experience of physical reality. The input from physical reality and the input from mathematical reality (if we allow, for the sake of the argument, the domain of true mathematical propositions to be so called here) would follow non-overlapping paths but become different outputs independently of each other. Here, I understand input as mathematical data and sense perception, and their respective output as instances of prospective empirical and mathematical knowledge. I say prospective here, as both faculties (viz. mathematical intuition and the physical senses) are fallible and so do not guarantee that the output we get, i.e. a set of justified beliefs, qualify as knowledge, i.e. that they are true.

If, on the other hand, our relationship to reality has only a partial separation between mathematical intuition and the physical senses, then the input-output process, or function, would behave differently from what we described above. Then, the non-overlapping paths would become overlapping, at least partially, and they would, to an extent, intertwine. Whether the input route of sense perceptions and mathematical data would coincide immediately or whether they would first "meet" in us and then cooperate to make output is uncertain. The input-output function could also be constructed so that, if we remember the 'idea of an object in itself' of *1964* and described in section 2.2 in chapter 1, we project our formal concepts onto the physical world. By this, I mean that these concepts are constitutive for our understanding of a physical object as such, i.e. the understanding that an object partakes in objecthood or the 'idea of an object itself'. If Gödel's one reality outlook, as I argue, leads to this sort of cooperation of the senses, then there would be a role for formal concepts also in our knowledge of the physical world, as there would be reference to formal concepts (Goldfarb 1995:333)

Let us now consider an example of how formal concepts could play such a role. If one has two bags *A* and *B* filled with marbles, where in bag *A* there are marbles that are both blue and yellow, and in bag *B* there are marbles that are both red and yellow. If someone asked you to pick the marbles that are partially yellow and put them into a third bag *C*, it would certainly contain all the marbles of *A* and *B*. In this case, our physical sense of sight has distinguished three colours from each other while still finding the feature shared by all the marbles, i.e. their partial yellowness. Could this not be seen as some rudimentary way for our physical senses and our mathematical intuition to work together? Could this not be a sort of mathematical data (as mentioned in 2.2) present in us which help justify the Axiom of union? Even if we would not grant this children's task as amounting to mathematical data, it is an instance of where we project different conceptual classifications onto our physical reality and where we organize reality accordingly. That is, we can see some cooperation between our intuition of formal concepts and our physical senses also in our empirical ideas, i.e. here, as a rudimentary understanding of the membership relation as in "belonging to a certain bag".

By saying that the two tentative relationships to reality are not completely separated, I want to say that there is in fact some interplay between our relationship to the world by way of our physical senses and our relationship to the world by way of mathematical intuition. The thing we now need to address is how this non-complete separation works. One way to do this has been to appeal to the notion of *founding* by Edmund Husserl, where a dependency between ideal content, or the *function*, and its facticity, i.e. its physical existence, can provide an explanation for our relationship to ideal and physical objects. In section 3.1 I will present a Husserlian reading of Gödel and investigate whether his "other relationship" can be established in this way.

Let us, again, consider the separation of mathematical intuition – with its resultant "other relationship" – and our physical senses. In the Syntax paper, Gödel touches upon this issue while discussing mathematical propositions and analyticity:

> … [mathematical propositions] are, in a sense, separable from other propositions, because no synthetic (empirical) propositions follow from them. Therefore, if we had a physical sense whose objects were of a similar regularity and similarly separated from those of the other senses, we could interpret also the propositions based on impressions of *this* sense to be syntactical conventions without content and associate no facts or objects with them or their constituents. (*1953/9–V:359)

Here, Gödel imagines another physical sense, whose objects are like the ones in mathematics. Mathematical intuition is said to be separated from the other senses, on the grounds that no empirical propositions follow from mathematical propositions. When the subject matter is abstract, it would indeed be odd if one could deduce a proposition about an empirical fact from a mathematical proposition. One could, of course, deduce propositions about multiplication *of* physical objects, but, we do not have any means to establish – from purely mathematical propositions – that any of the physical objects do in fact exist. However, Gödel writes that they are separable only "in a sense". Also, it is only one of the ways in which empirical and mathematical propositions can follow from each other that is excluded. The other way, mathematical propositions following from empirical propositions, is not touched upon. For now, I only want to suggest that Gödel does support a partial, but not a complete separation. And also, let us bear in mind that we now speak of a threefold separation, or three different levels of separation: 1) between our relationships to reality, 2) between mathematical intuition and the physical senses, and 3) between mathematical and empirical propositions.

Perhaps, what is meant is just the fact that, yes, we are capable of mathematical knowledge, and us perceiving the relationship between mathematical concepts is only a not so familiar aspect of our everyday situation. Our dealing with these concepts and acquiring knowledge of them is, after all, different from how we deal with physical objects. So, in light of Gödel's principle of epistemological parity, the fact that this relationship is different and 'other' might, then, be partially expected. Of course, it must be different, and so what if it is? Both van Atten and Kennedy (2003) and W. W. Tait (2005) point out that we should not strive for uniform accounts of both empirical and mathematical knowledge (2003:436; 2005:69–70). And, this other relationship between us and the mathematical reality is, after all, not an attempt to explain metaphysically how we are in this world and whether we are somewhere else when dealing with mathematics (as the above discussion might suggest), but rather to give an epistemological account of mathematical knowledge and how we obtain it. That the two accounts of empirical and mathematical knowledge need not be uniform is therefore an important point, and one we should do well to remember.

This is again the time to emphasize Gödel's view that mathematics is a descriptive science. The question as to how we procure mathematical knowledge is therefore closely connected to scientific knowledge as such, and therefore also connected to the different levels discussed in section 2.2 in chapter 1, on his conceptual realism. If there is a non-linguistic level consisting of the mathematical facts that determine the linguistic level (consisting of our theories about the mathematical facts), and if this two-level structure is analogous to the one we find in the natural sciences, and if the faculty of mathematical intuition is similar to a physical sense, then there is no reason to believe that our relationship to the mathematical reality should not be a constituent of how we deal with reality as a whole. That is, if the analogy with physics pans out and if one upholds the principle of epistemological parity, there can be no talk of reaching into *another* reality than our everyday reality. The point being that reality has many aspects, where the natural sciences explain some parts of it and mathematics others. We can have knowledge of the different parts by using our many senses, one being mathematical intuition, the existence of which he takes as a psychological fact (Gödel 1964:268).

Let us consider how, for instance, our many senses work together to produce more complete or more detailed sense perceptions, as in the case when we taste the strawberry, see its colour and feel its texture and weight. This cooperation of the physical senses results in us having a more complete idea of what a strawberry is. Could it not be, also, that such a cooperation can exist between the physical senses as a whole and our "sense" of mathematical intuition? Only that, in the case of this cooperation, it would result in us grasping both the idea of a physical object as such and also the object's particular qualities.

This is supported by the philosopher Janet Folina (2014), who emphasizes the point that Gödel's conceptual realism is very much in line with this one-reality view. The existence of concepts is something that correctly organizes the world, and "reflect real categories, or general features, of reality" (Folina 2014:54). In support of this, she brings forth Hao Wang (1987) where he describes Gödel as hypothesizing that mathematical intuition is a "physical organ" (2014:55).

Now I want to consider two quotes in which both the perception of concepts and our sense perceptions fail us. That is, Gödel argues by analogy how they are similar in their fallibility, but that this possibility of failing perceptions should be no more damning for the existence of mathematical reality than it is for physical reality. These two quotes on paradoxes support this view, the first from *1944*, and the second from the Gibbs Lecture of 1951:

> The difficulty is only that we don't perceive the concepts of "concept" and of "class" with sufficient distinctness, as is shown by the paradoxes. (1944:139–140)
>
> For, our knowledge of the world of concepts may be as limited and incomplete as that of [[the]] world of things. It is certainly undeniable that this knowledge, in certain cases, not only is incomplete, but even indistinct. This occurs in the paradoxes of set theory, which are frequently alleged as a disproof of Platonism, but, I think, quite unjustly. Our visual perceptions sometimes contradict our tactile perceptions, for example, in the case of a rod immersed in water, but nobody in his right mind will conclude from this fact that the outer world does not exist. (*1951:321)

This other organ can, then, grasp mathematical concepts, but, it is *fallible*. It is fallible in the exact same way as our other physical senses are fallible, but the results of these prospective failures are different. Let us consider the first possible failure, that of our physical senses. For example, if our usually reliable eyesight failed us, and we could not discern a figure some distance away, we would improve our situation and move closer to the figure observed. Hopefully, we would then see the figure clearly. And so, our eyesight would once more be a reliable route to gain knowledge of some empirical fact.

Let us turn to the other possible failure, that of mathematical intuition. If our mathematical intuition fails us, then set-theoretic paradoxes occur. To avoid this, the remedy is continued reflection on the concepts. This ensures that our mathematical data can be checked and proofed and that our understanding of mathematical concepts is more complete and distinct. This continued reflection on mathematical concepts help remedy our fallible perception of mathematical concepts. Also, this continued reflection, or, rather, these "continued appeals to mathematical intuition" are "necessary not only for obtaining unambiguous answers to the questions of transfinite set theory, but also for the solution of the problems of finitary number theory" (Gödel 1964:269). We shall see in chapter 3 how continued reflection on the concept of set can be decisive for finding new axioms.

As mentioned above, there is the question as to the separation in our relationship to reality between 1) the physical senses and mathematical intuition, and 2) between empirical and mathematical propositions. I have suggested that this separation is meant to be partial and not complete. The strongest evidence we find for this, is the fact that we do have the *idea of a physical object* as such. This idea of a physical object is separated from the actual instances of it. That is to say, as discussed in section 2.2 on Gödel's conceptual realism, we do not form our concept of a physical object on the basis of an abstraction of all the many different physical objects. The concept of an object in itself is different from our concept of, say, a dog, where we do in fact recognize dog-ness and form the concept of dog from the many physical dogs.

However, the idea of an object in itself is connected to our ability to form complex thoughts as to an object's general features, i.e. features that can be instantiated by any number of different objects. Let us see what Gödel writes on the idea of an object itself:

> It should be noted that mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else which *is* immediately given. Only this something else here is *not*, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g. the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. (1964:268)

The idea of an object in itself appears alongside Gödel's most detailed attempt to explain mathematical intuition. The quote is meant to capture what mathematical intuition is and how it actually works. So, before I turn to the idea of an object itself, let me first discuss three points that deal with parts *active in* or *about* mathematical intuition: (i) mathematical intuition does not need to provide immediate knowledge, (ii) however, there is something that *is* immediately "given" to us, and (iii) this something "given" is supposed to be qualitatively different from the sensations and combinations of sensations. So, what do these points tell us about mathematical intuition?

Let us start with the first point – that mathematical intuition does not need to be conceived of as a faculty which provides immediate knowledge of the objects it concerns. It is expected that Gödel specifies this point, it is, after all, this claim he laid the groundwork for in earlier work, with his description of "data" as seen in *1944*. It follows from his view on reality, i.e. that it consists of objectively existing objects – either mathematical or physical – that there is something we perceive which are not the objects described by our theories.

This contains the second point: that what is immediately given are our sense perceptions and our mathematical data as discussed above. So, Gödel emphasizes the point that mathematical intuition is not meant to be an epistemically stronger faculty than our physical senses. That is, our access to the mathematical facts is as limited as our access to the empirical facts. This follows from the principle of epistemological parity. The principle does not only secure an epistemological foothold for mathematical intuition as a reliable route to knowledge equal to our physical senses, but also, the principle limits the reach of mathematical intuition equal to the reach of our physical senses. In the same way as we cannot *perceive* physical phenomena as described in modern physics, e.g. we cannot distinguish the atoms making up some physical object from each other, our perception of mathematical reality is marked by our indistinct and incomplete perception of mathematical concepts, e.g. we cannot always immediately grasp the axioms that are behind our immediate understanding of '2 + 2 = 4' being true.

The third part – that this "given" is something that is qualitatively different from the sensations – is evident from the different natures of the physical and mathematical world, one being physical and the other abstract. Therefore, it would indeed be unexpected if the "given" of which we gain immediate knowledge through mathematical intuition were the sensations, i.e. sense perceptions. That combinations of such physical experiences are also excluded from mathematical intuition seems equally plausible, on the basis that such brute emergence of abstract content from combinations of physical experience, is unlikely.

Gödel writes that by our thinking we cannot "create any qualitatively new elements, but only reproduce and combine those that are given" (1964:268). He thus puts a quite strong limitation on our capacity for abstract thinking, and our reason's ability to create is weakened. He takes this limitation of creative power as evidence for the existence of something "given". That is because if sensations or combinations of sensations cannot account for the idea of object itself, and if our thinking cannot create such elements, but is limited to reproduce and combine elements already present in us, there must be something given underlying mathematics which causes such elements in us. Again, he appeals to a sort of indispensability argument, where he reaches the conclusion that such elements must exist on the basis that it is the best and most straight-forward explanation for our mathematical knowledge. He concludes with: "Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas" (1964:268). The idea of an object itself is thus deemed to be the ultimate evidence for there being something "given" underlying mathematics which is also contained in our empirical ideas. What does this say about our relationship to reality? It supports my suggestion that the threefold separation as discussed above is partial. It confirms that there is interplay of formal concepts in our empirical thinking as discussed in section 2.2 on conceptual realism, where Gödel appears to doubt this in **1953/9*. Also, it suggests that Gödel's conceptual realism in connection with his mathematical intuition puts forward a view that is rationalistic, in the sense that human thought and capability for abstract reasoning is formative also of our relationship to physical reality. This point will be the main theme of chapter 3, where I will defend Gödel's view as being a *theory of reason* rather than a theory of mathematical intuition.

SECTION III – ESTABLISHING INTUITION

Now that I have discussed different features of Gödel's mathematical intuition – the analogy with physics, the principle of epistemological parity and the "given" – we need to take stock of the situation. How far have we come and which problems remain? First of all, the different parts active in mathematical intuition have been somewhat illuminated. We do know that Gödel meant to liken mathematical data to sense perceptions, and so open for a sort of mathematical experience. The mathematical data present in us account for the immediacy of some of our mathematical knowledge, such as '2 + 2 = 4'. This is meant to limit mathematical intuition and its epistemic force.

Also, with the principle of epistemological parity the legitimacy of the existence of physical and mathematical objects is put on an equal footing. We are, with the juxtaposition of physical and abstract evidence for the two kinds of objects, justified in believing that they exist. In believing that they exist, we infer that the best explanation for us having knowledge of mathematical objects is our having a perception-like faculty with mathematical intuition.

Furthermore, it is on the basis of our understanding of mathematical *concepts* that we acquire mathematical knowledge and not from direct perception of sets. It is also from the certainty of the *existence* of mathematical intuition that we can hope to determine propositions that, for now, remain undecidable (relative to ZFC), as we saw in section 2.3 in chapter 1.

Where does this leave us? Well, there are certainly problems that remain. Aspects of mathematical intuition are still not examined, such as the link Gödel draws between our sense perceptions and mathematical data and what this means for our cognition as such. Do humans have such a partially separated relationship to reality, where abstract elements are formative also of our empirical ideas? The processes in mathematical intuition and how we cognize reality remain, therefore, underdeveloped on Gödel's account. Let us, then, try another route. One way to explain this partial separation has been to appeal to the philosophy of Edmund Husserl. Can Gödel's "other relationship" to reality be established by a Husserlian reading of Gödel and thus render mathematical intuition less mysterious?

Now we need to address how this non-complete separation works on a Husserlian interpretation of Gödel's intuition. First of all, we have to explain some key aspects of Husserl's philosophical framework. My intention here is not to give a full account of Husserl's philosophy, that task would be insurmountable to take on in this thesis, but I will look at some aspects of his philosophy that may help remedy Gödel's conception of mathematical intuition. Also, another reason is that this section is first and foremost meant as a contrastive reading to the one I will present in chapter 3.

In this section I will first make some introductory remarks on Husserl's influence on Gödel. Second, I will explain three different notions in Husserl's philosophy that will be central for the interpretation of Gödel: i) the notion of *intentionality*, ii) *Wesensschau*, the equivalent to Gödel's intuition, and iii) *Fundierung*, or founding. Third, I will assess whether a Husserlian interpretation of Gödel helps remedy his account of mathematical intuition and whether it is a plausible interpretation of how Gödel views our relationship to reality.

3.1 Husserl's Influence on Gödel

We do know that Gödel started to study Husserl in 1959, and, according to Wang, this was to look for a "deeper foundation of human knowledge in everyday life" (1982:658; 1987:12). What does this mean? If Wang's analysis of Gödel's motives is correct, this means that he turned to phenomenology in order to i) find a foundation for human knowledge, and ii) that his interest in knowledge was not confined to mathematical knowledge, but also "everyday" knowledge. The first point seems like an endeavour a philosopher such as Gödel, who investigates into how we acquire mathematical knowledge, would gladly undertake. The second point is more interesting. It agrees with the one reality outlook discussed in 2.4. If

Gödel did in fact study Husserl to look for a foundation of *all* human knowledge, this would support the claim in 2.4 that mathematical intuition is only partially separated from the physical senses, and that this bears on our relationship to all of reality. However, whether this actually was Gödel's motivation for studying Husserl remains somewhat uncertain, though it is clear from his writings that he did think that phenomenology could lead to a deeper understanding of concepts.

Also, we do know, from his comments in his copies of Husserl's work, that he regarded Husserl's transcendental phenomenology as being an impressive framework, and that he especially appreciated *Ideen* (1913), that is Husserl's work after his "idealist" conversion in 1906/07 (Føllesdal 1992:386). Dagfinn Føllesdal (1995) identifies some similarities between Husserl and Gödel, here regarding Gödel's conceptual realism.

> Mathematical propositions, it is true, do not express physical properties of the structures concerned [in physics], but rather properties of the *concepts* in which we describe those structures. But this only shows that the properties of those concepts are something quite as objective and independent of our choice as physical properties of matter. This is not surprising, since concepts are composed of primitive ones, which, as well as their properties, we can create as little as the primitive constituents of matter and their properties. (Gödel *1953/9–V:360)

Føllesdal notes that in this quote – which, by the way, is an exemplary formulation of his realism – Gödel does not say that concepts are straightforwardly objective, but argues by way of comparison (1995:369). The concepts are said to be objective *in the same way as* physical properties, a position that Føllesdal argues brings him closer to Husserl, as Husserl describes physical objects, concepts and mathematical objects as objective, but not in a direct, realist sense (1995:369).

Whether this brings Gödel closer to Husserl is uncertain; it can also be seen as evidence for the principle of epistemological parity as discussed in section 2.3. It does not have to mean that Gödel weakens the objective existence of physical objects towards a Husserlian conception of realist existence. As discussed in 2.3, it can rather imply that our belief in their objective existence is *as justified as* our belief in the existence of physical objects. That is, we can also understand this passage as bearing on the strength of the *justification* for postulating certain objects – physical and mathematical – and not on the degree of their real existence.

In **1961/?*, a draft for a lecture that was never given, Gödel explicitly praises phenomenology as a possible method for a clarification of meaning.

> …there exists today the beginning of a science which claims to possess a systematic method for such clarification of meaning, and that is the phenomenology of Husserl. Here clarification of meaning consists in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely, onto our own acts in the use of these concepts, onto our powers in carrying out our acts, etc. But one must keep clearly in mind that this phenomenology is not a science in the same sense as the other sciences. Rather it is [or in any case should be] a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us. I believe there is no reason at all to reject such a procedure at the outset as hopeless. (Gödel *1961/?:383)

So, Gödel does make two disclaimers regarding phenomenology: that it is not a science like the other sciences (empirical sciences and mathematics), and that there is no reason to reject it as hopeless in its very beginning. What he does find appealing with phenomenology is the use of it as a *method*, and something that can sharpen our view on concepts. Since Gödel considers a deeper understanding of concepts as what it takes to advance mathematical knowledge, this is quite the endorsement from his side. This is supported by Gödel's description of the questions in the foundations of mathematics. Here, Gödel writes that the "scarcity of results" is not only due to mathematical difficulties, but that "there are also deeper reasons involved and that a complete solution of these problems can be obtained only by a more profound analysis (than mathematics is accustomed to giving) of the meanings of the terms occurring in them" (1964:257). A clarification of concepts is deemed necessary, and perhaps we must step outside of mathematics to procure it.

Furthermore, as van Atten and Kennedy discovered (2003), one sentence present in an earlier draft of the Supplement to his *1964* was removed from the final version: "Perhaps a further development of phenomenology will, some day, make it possible to decide questions regarding the soundness of primitive terms and their axioms in a completely convincing manner" (van Atten & Kennedy 2003:466). In both **1961/?* and the sentence deleted from *1964*, Gödel makes it clear that phenomenology is not yet adept for providing the clarification of meaning that he seeks. But it is a promising method, and phenomenology may possibly be what is needed in order to gain the deeper foundation for human knowledge that Wang reports on (1987:87). However, the sentence in the Supplement was removed from the final version. Also, the draft for the lecture never did see the light of day. This should not count for too much though, as Gödel sometimes refrained from stating his position if he could not argue for it as he liked (Føllesdal 2016:409).[21]

3.2 Three Notions in Husserl's Philosophy

In this section I will explain three key notions in Husserl's philosophy that are relevant for a phenomenological interpretation of Gödel: *intentionality*, *Wesensschau* and *Fundierung*. These three notions have to do with three different aspects of Husserl's philosophy. *Intentionality* relates to Husserl's general view on consciousness and is therefore a good example of how Husserl presents an overall philosophical framework, where general features of consciousness have a place, and not only specific parts of it (as Gödel's intuition is an example of). The second, *Wesensschau* or '(in)sight into essences', is the Husserlian equivalent to Gödel's mathematical intuition. To see how *Wesensschau* is supposed to work can thus help us understand a Husserlian reading of Gödel. The third, *Fundierung* or 'founding', has to do with establishing mathematical intuition as providing mathematical knowledge. If mathematical reality and our knowledge of it can be *founded*, then we can more easily justify mathematical intuition as a psychological fact, instead of merely claiming that we have such a faculty. This can thus help us in answering Chihara's criticism, where the "appeal to mathematical intuition does not explain how we are able to "perceive" sets – it, essentially, only asserts that we do" (Chihara 1990:19). I will therefore apply the framework of *Fundierung*-relations onto Gödel's view, and see whether it fits.

3.2.1 Intentionality

Let us first start with the notion of intentionality. The notion of intentionality is what Husserl came to see as the "essential character" or "universal fundamental property" of our mental life (Moran 2005:5). In applying Franz Brentano's descriptive psychology to the clarification of mathematical concepts,[22] Husserl pursued the foundations of human knowledge, by inquiring

[21] Føllesdal notes that this point has been made by both Richard Tieszen (2011) and Solomon Feferman (1998). Also, this is supported by the fact that the Syntax paper **1953/9* never was published, though he worked on it for six years and completed several different versions. In the final letter to Paul Arthur Schilpp (who had invited him to write the paper to a volume on Rudolf Carnap), he wrote: "The fact is that I have completed several different versions, but none of them satisfies me. …[I]n view of widely held prejudices, it may do more harm than good to publish half done work" (Goldfarb 1995:324; Gödel 2003b:244).

[22] Franz Brentano was Husserl's teacher, and launched *descriptive psychology* as a classificatory science of mental acts and their contents relying on the "self-evidence of inner reflection," and

into the subjective side of the framework of cognitive acts, e.g. judgements (Moran 2005:4–5). The fundamental feature of human consciousness was then realized to be that all our mental acts are directed at an object. As Dermot Moran puts it: "Phenomenology, then, considers every object in so far as it is an object-for-a-subject. For Husserl, intentionality became the 'indispensable fundamental concept' for phenomenology" (2005:5).

To illuminate what this directedness consists of, Føllesdal gives the example of the duck/rabbit case. In this example, we see different things depending on what anticipations we have, because, as Føllesdal points out, the picture does not change according to each person, so the difference must come from us (1992:386). If one had never seen a duck before, one would perceive a rabbit in the picture, and the other way around. That is, we structure the world around us and this structuring differs from person to person. By structuring the world differently and thus having different anticipations to objects, each human consciousness is *directed at something*. This is how, according to Husserl, intentionality accounts for our constant structuring of the world and why we consider objects as object-for-a-subject (Føllesdal 1992:387; Moran 2005:5).

Let us consider intentionality in relation to the acquiring of mathematical knowledge. Say we have two persons, one with high school level education in mathematics and one whose mathematical knowledge is limited to subtraction and addition with natural numbers. Let us further say that all their other knowledge of the world is the same. One would still think that their consciousness would structure the world differently. Whereas the person with very limited mathematical knowledge would recognize objects as being one, many and so on, the other person could, when contemplating objects, detect geometrical features of objects, and so abstract the feature of being spherical from many, otherwise different, objects. This difference in structuring the world could, perhaps, be true of higher levels of mathematical knowledge, as well. Let us return to this point in section 3.2.3 when we go on to discuss *Fundierung*.

3.2.2 Wesensschau

Second, let us now consider the notion of *Wesensschau*. Husserl's transcendental phenomenology is a philosophical system in which the *eidetic sciences*, i.e. the sciences of essences ['*Wesen*'], hold a particular position (Føllesdal 1992:387). For Husserl, only

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adopted the concept of "directedness to an object," or *intentionality* (Moran 2005:17; Føllesdal 1992:386).

mathematics was considered to be a highly advanced eidetic science. These sciences of essences were accessible through what Husserl called *Wesensschau*, a somewhat abstruse notion, meant to designate the *intuition/(in)sight of essences*. According to Husserl, we have two kinds of intuition (‘*Anschauung*’): (1) *perception* of physical objects, and (2) *Wesensschau* or *eidetic intuition* (Føllesdal 1992:388). With *Wesensschau* the objects are essences, either abstract objects or general features of physical objects. As mathematical objects are abstract, they are thus intuited via *Wesensschau*. Husserl endorsed the view that the abstract objects of mathematics have the same ontological status as physical objects and claimed that mathematics was the only advanced eidetic science (though he is open to new eidetic sciences apart from mathematics) (Føllesdal 1992:388–89).

In eidetic intuition, the objects intuited, i.e. the *essences*, belong to an objective reality. It is an important feature of essences that they also include general notions. For instance, Føllesdal argues that the example of shape counts as an essence in this way (1992:388). If we perceive, say, something spherical in our perception of a physical object, e.g. a basketball, then we also intuit the general feature of being spherical. We can, then, switch between the physical object we first experienced and another that also instantiate this feature. If we then return to the bag of marbles example, we would recognize that each marble instantiates the feature of being spherical, i.e. we would have an intuition of an essence. If we had a third object, for example an American football (which is egg-shaped), then we could pick out a feature that the basketball and the American football share, namely that they are both used in sports or that they are both hollow. If we now asked which two things one would pick out that shared the same feature, the answer could go both ways. We could choose the two spherical objects, or the two objects used in sports. So, in this example, we get two different possibilities to recognize general features, i.e. to recognize two different sets of essences. This means that it is not only abstract objects like a set that are intuited with *Wesensschau*, but also general features of physical objects.

For Husserl, the structuring of the world that intentionality gives us does not distinguish between physical and abstract objects, i.e. we structure the world we meet, which consists of both kinds of objects. Here, we can again see the similarity with the one reality outlook from section 2.4. If we deal with reality as a whole, then our way of dealing with it, i.e. how we structure reality, should reflect this. With Husserl’s intentionality and intuition – both *perception* and *Wesensschau* – there is cooperation between our faculties that can be likened to the possible cooperation between Gödel's mathematical intuition and our physical senses. That is, adopting Husserl's larger philosophical framework does, on the face of it, fit quite well with the partial separation described above. Also, Husserl's intuition of essences has the advantage that it has a wider reach, so as to address a more general aspect of how we orient ourselves in the world.

3.2.3 Fundierung

Third, let us now turn to *Fundierung* or 'founding' as it is also called. This is a notion introduced in *Logische Untersuchungen* and refers to different dependence relations (Rota 1989:70). *Fundierung* aims at capturing how something is dependent on something else, either for its existence, essence or meaning, i.e. in order to be what it is. In describing *Fundierung*, Gian-Carlo Rota (1989) gives several examples hoping that "the underlying concept will eventually come through", similarly to how we learn certain operations in set theory such as union and intersection (70). According to Rota, *Fundierung*-relations, or dependence relations, can express the dissonance exhibited by, for example, what something is existence-wise, and the role a physical thing plays, which makes up its essence (Rota 1989:71). Rota claims that the concept of *Fundierung* should be understood as a logical concept (even if *Fundierung*-relations cannot be formalized) (1989:70). *Fundierung*-relations have two important features, *facticity* and *function* (Rota 1989:72–73). The facticity is what something *depends on*, e.g. my eyesight depends on my eyes. The function is that which *is dependent*, e.g. me having eyesight. In following Rota, let us consider a few more examples in order to get a better grasp of the concept, before we turn to its relation to intuition and mathematics.

Let us start with the example of *reading*.[23] We will apply the notions of a *Fundierung*-relation, function and facticity. With the example of reading, it is the content of the text that constitute the function, and the text itself that constitutes the facticity. Facticity can be understood as the "most real", in the sense that it is an existing physical object, i.e. the sheet of paper with symbols on it, and it is therefore called the *independent* ['*selbsständig*'] part in the *Fundierung*-relation. The content of the text, on the other hand, is the *dependent* ['*unselbsständig*'] part and is *ideal*, insofar as it does not have a physical existence. The

[23] The reading example is taken from Ludwig Wittgenstein's *Philosophical Investigations*, but, as Rota shows, it illuminates the difference between facticity and function very well (Rota 1989:71).

content cannot be *read* (i.e. understanding the symbols) unless it is written down in a text. That is, the content is *dependent on* the facticity of the text, i.e. the actual sheet of paper. However, it is the content that, as far as reading goes, is *relevant* (Rota 1989:73). While the content can never be read unless there is an actual text, the text can be swapped for another that displays the same content; that is why the text only plays a supporting role to the content (Rota 1989:73). By thus describing the *Fundierung*-relations present in reading, where the physical existence is deemed less important than the ideal content, the concept of *Fundierung* can be seen as a way to put less importance on an object's reality.

Roles are also typical examples of a function in a *Fundierung*-relation (Rota 1989:72). For instance, the "role" of a basketball in the game is dependent on the ball itself (facticity). The facticity of the basketball is merely that it is round, often made of leather and bounces. However, this is not the essence of what a basketball *is*; rather the basketball's *function* is the role it plays in the game of basketball. We cannot grasp the function of the basketball by studying its facticity, however detailed our knowledge is of its size, shape or bounciness; we cannot infer the role it holds in the game of basketball. Also, we could swap one basketball for another, and still be perfectly able to play the game.

Another example of a *Fundierung*-relation is given by Moran, where a "*judgement* of perception is founded on, but essentially different from, a *perceiving*" (2005:150). This is an example where a judgement of perception is, following Rota, seen as the *relevant* part, whereas the perceiving itself is not. How does this work? Well, it seems to lead to a view where our formulated beliefs are more important than the sense perceptions they stem from. But, of course, my belief about my sense perception is wholly dependent on that perception. And so, even though the belief is not ontologically prior, I still use the belief if I want to describe the world. Also, it is our set of beliefs that is the basis for prospective knowledge, or rather, that is *relevant*, as Rota writes (1989:73). There are also nested *Fundierung*-relations, as with Moran's example. One can imagine that the perceiving could be the function, and that it is founded on the object perceived. Here, the perceiving itself is dependent on there being an object which causes the perceiving, so that the physical object constitutes the facticity of the *Fundierung*-relation. Or, another possibility is to say that our physical senses constitute the facticity. The point is that there are *Fundierung*-relations all around us, and that we are aware of some of them, but definitely not all.

Now, let us consider the mathematical reality and how mathematics relates to the concept of *Fundierung*. Rota gives the example of seeing a triangle drawn on the blackboard and the *viewing* of a triangle, which is meant to capture the same difference as between seeing written symbols and reading the content of a text (Rota 1989:74–75). Here, the drawing of a triangle founds our *viewing* of a triangle, i.e. the viewing of a triangle constitutes the function, while the drawing constitutes the facticity. So, while our *viewing* of a triangle is dependent on the imperfect drawing, it is our *viewing* of the triangle that is relevant for mathematical knowledge. This is, incidentally, a way of explaining the ideal features a mathematician "sees" when looking at a triangle on the blackboard, as mentioned in section 1.1. in chapter 1. So far, so good.

More interestingly, how does Gödel's realism fit with *Fundierung*-relations? Let us apply the notions of *function* and *facticity* to Gödel's position. Can the concept of *Fundierung* help explain how we acquire knowledge of mathematical reality through mathematical intuition? If so, a Husserlian reading will have some serious advantages compared to other readings of Gödel. As we saw in section 2.2 in chapter 1, there is a one-way dependence between the linguistic level, i.e. theories, and the non-linguistic level, i.e. mathematical facts. In adopting the *Fundierung* framework, Gödel would say that it is the mathematical theories that constitute the functions, while the mathematical facts constitute the facticity. That is, the theories are *founded on* the facts.[24] This is due to Gödel's realism and his view that theories purport to say something true of an objective mathematical reality. Given the concept of *Fundierung* explained above, this would lead to the view that mathematical theories are relevant, while mathematical objects and concepts are not.

Relevant for *what* exactly? According to Rota, "All *what*'s whatsoever are functions in *Fundierung*-relations. All *what*'s "are" by the grace of some *Fundierung*-relation whose context-dependence cannot be shoved under the rug" (1989:76). That is, *Fundierung*-relations are complex and nested, and they are to be found wherever there is some context-dependence. The labels of facticity and function may vary depending on the case at hand. In the case of theories being founded on objects and concepts, the *relevant*-aspect could answer any number of *what*'s, such as: what is mathematics? what is mathematical practice? etc. How we label facticity and function would, then, depend on which questions *we want to answer*. This is a

[24] We can also consider Gödel's view on mathematical reality and knowledge as a situation where we have nested *Fundierung*-relations: where mathematical theories are founded on mathematical data and where the mathematical data are founded on the mathematical objects and concepts. For now, however, I choose to consider the case corresponding to the two levels described earlier, i.e. where theories are directly founded on objects and concepts. This, I think, will suffice for my present objective – to see whether the concept of *Fundierung* helps justify Gödel's view.

first discrepancy between *Fundierung*-relations and Gödel's view on mathematics. Since Gödel's believed in an objective mathematical reality that we can only discover and try to accurately describe, the context-dependence attendant to *Fundierung*-relations does not quite fit. That humans should decide what is relevant and not in mathematics based on our context and wishes, is not a route Gödel would approve of.

So, if mathematical theories are founded on mathematical objects and concepts, there is an asymmetrical relevance relation, as theories take precedence. At first, this priority of relevance of mathematical theories to objects and concepts seems alright. If we were to ask a mathematician if this view fits with how mathematics is practiced, she would certainly say that it does. After all, mathematical work is done by engaging with axioms and theorems, not by directly dealing with the abstract objects of mathematical reality. However, if we consider Gödel's view, it is not as straightforward. After all, for Gödel, it is mathematical reality that is relevant for advancing mathematical knowledge, i.e. the reflection on and clarification of the primitive concepts. The two questions that his writings revolve around – what mathematical objects are and how we acquire knowledge of them – do imply that making mathematical reality play only a supportive role to mathematical theories is difficult to square with Gödel's project. Our accepted mathematical theories show our best efforts in understanding mathematical reality, but, as Gödel writes: "[O]ur knowledge of the world of concepts may be as limited and incomplete as that of [[the]] world of things. It is certainly undeniable that this knowledge, in certain cases, not only is incomplete, but even indistinct" (*1951:321). Gödel's main concern *in philosophy* is not to question the truth of our accepted mathematical theories, i.e. our mathematical knowledge, but to inquire into the nature of mathematical reality and understand how we acquire such knowledge. Given these ontological and epistemological concerns, the precedence of theories and their *Fundierung*-relation to the facts becomes difficult.[25]

If we are to follow Gödel, and mathematical theories are founded on mathematical reality, what happens if we were to reverse the dependence relation? As mentioned above, Rota writes that how we look at *Fundierung*-relations is a matter of context and the questions we want answered. Which direction the dependence relation is fixed in a *Fundierung*-relation, i.e. which parts that constitute the facticity or the function, depends on the case at hand. Let us

[25] Unless, of course, Gödel's objective was not philosophical, and the main concern was mathematical knowledge and its practice.

now try to reverse the direction of Gödel's dependence relation, so that mathematical objects and concepts are founded on theories. Is there any way that Gödel could accept this direction of dependency? Ontologically, this seems unlikely. That mathematical objects and concepts are founded on theories goes against the platonist view. According to platonism, it is simply false.

Let us try anyway, but by considering *epistemological dependence*. If we consider an epistemological dependence relation, we entertain the idea that our intuition *of* mathematical concepts and objects are founded on our intuition *that* the axioms of our accepted theories are true. That is, I am invoking the difference between intuition *of* (objects and concepts) and intuition *that* (propositions, viz. axioms, theorems and theories), as discussed in section 1.1 and the introduction to section 2. The first direction of the dependence relation I want to investigate is thus that concepts and objects are intuited *only insofar* we intuit the axioms in which they appear, i.e. intuition *of* is founded on intuition *that*. Dependence between intuition *of* and intuition *that* is addressed in the Syntax paper.

> *For these axioms there exists no other rational…foundation except either that they…can directly be perceived to be true (owing to the meaning of the terms or by an intuition of the objects falling under them), or that they are assumed (like physical hypotheses) on the grounds of inductive arguments, e.g., their success in the applications.* (Gödel's emphasis, *1953/9–III:346–47)

Here, Gödel says that the only possible justification for axioms is an intuition *that* they are true, or if we assume them on the grounds of an inference to the best explanation. The latter alternative involves the "success" of an axiom, which I shall return to in chapter 3. The former alternative, this direct intuition *that* they are true, has two possible explanations: i) because of the concepts occurring in the axioms, or ii) that we have an intuition *of* the objects the axioms purport to quantify over. In the introduction to section 2 on the difference between intuition *of* and intuition *that*, I argued that we cannot infer an instance of intuition *of* a *particular* object, i.e. a particular set, from an instance of intuition *that*. This second variation – that we have an intuition *of* the objects the axioms purport to quantify over – seems to go against the view previously set forth, as the objects here are not particular but objects in general. However, in this case the direction of the dependence relation is not the same. Here, it is the intuition *of* objects that is taken to justify our intuition *that* the axioms are true, i.e. it is from our possible intuition *of* objects that we can infer an instance of intuition *that*, which is not a logical fallacy. What we have found so far, then, is that the direction of the dependence relation is the same as before, i.e. we cannot logically infer from intuition *that* an axiom is true to an instance of intuition *of* objects in general. (By 'objects in general' I mean that it is not a particular object that can be uniquely determined by a concept, as established in the introduction to section 2.) The direction of the dependence relation is, then, from intuition *of* to intuition *that*, so that it is parallel to how theories are founded on objects, and where intuition *of* constitutes the facticity and intuition *that* constitutes the function.

So far, the direction of the dependence relation has been as expected. Now, I want to present a possible reading where the dependence relation goes in both directions. That is, I will try to establish that the dependence relation can go in the other direction as well, so that intuition *of* is founded on the intuition *that* axioms are true. When we consider the epistemological *Fundierung*-relations, it does not seem possible that our intuition *of* objects and concepts, i.e. sets in general, can be founded on our intuition *that* axioms are true. However, we must separate the ontological *Fundierung*-relations from the epistemological. While mathematical objects and concepts are ontologically prior to theories in the *Fundierung*-relation, and the same direction of the dependence relation has been shown to be true for the epistemological account, there is a difference between the epistemological and ontological *Fundierung*-relations. As I see it, the difference is internal to one part of the relation, namely that there is a difference between concepts and objects. For the ontological dependence relation, this makes no difference. For the epistemological dependence relation, it does. When it comes to intuition *of* concepts, one can argue that there is a reciprocal dependence relation, i.e. that the direction of the dependence relation goes both ways. This way, intuition *of* concepts is founded on intuition *that* axioms are true, as the relationship between concepts is *intuited* in the axioms.[26] Mathematical concepts are constitutive for the axioms, so it seems plausible that the intuition *of* concepts and the intuition *that* axioms are true should also be intimately related, i.e. by a reciprocal dependence relation. This is, then, one way to see the difference between epistemological and ontological *Fundierung*-relations. Also, this is how intuition *that* axioms are true can constitute the facticity in the *Fundierung*-relation, while intuition *of* concepts constitutes the function.

Let us now sum up how Gödel's position would fit in the *Fundierung* framework:

[26] In saying this, I follow Parsons' assumption that intuiting the relationship between concepts involves intuition of the concepts (1995:62).

1) mathematical theories (axioms) are founded on mathematical facts (objects, concepts)
2) intuition *that* axioms are true is founded on intuition *of* objects and concepts
3) also, intuition *of* concepts is founded on intuition *that* axioms are true

It is 3) that is the surprising element. 3) ensures that our intuition *of* concepts depends upon our intuition *that* axioms are true, i.e. that the direction of the dependence relation goes both ways. So, what does this reciprocal dependence relation say about how the *Fundierung* framework fits with Gödel's view? And, do these dependence relations help justify how mathematical intuition is a source of mathematical knowledge? I can see how the ontological dependence relation fits with Gödel's two levels, where the linguistic level is dependent on the non-linguistic level. The priority of relevance this leads to, however, does not fit with Gödel's view. That is because it fails to recognize the importance placed on his conceptual realism.

As the epistemological dependence relation is reciprocal, Gödel's conceptual realism is not deemed less relevant. However, because it is reciprocal, it means that neither is really founded on the other, or rather, that both 2) and 3) hold, but in a non-interesting way. That is, they hold in a way that will help justify mathematical intuition *only to a certain degree*. Both 2) and 3) explain the relationship between intuition *of* and intuition *that*, but they do not establish the relation between intuition as such and our grasping the higher-order concepts of set theory. To illustrate this, let us return to the example from section 3.2.1 on intentionality, where one person can abstract general features from physical objects, and the other cannot. So, one person can identify the general notion of being spherical in physical objects, and therefore have the intuition *that* "All spherical objects have a centre" is true, and thus have the intuition *of* a spherical object. As understanding the concept of being spherical means to understand that it has a centre, the intuitions are here reciprocally dependent. *Fundierung* thus helps understand how intuition *that* and intuition *of* is meant to reveal the structure of intuition, but it does not help to justify our intuition *of* higher order concepts and our intuition *that* the axioms in which they appear are true. That is, a *Fundierung* based justification for intuition is still cut off from higher-order structures.

The founding of intuition must consider Husserl's *Wesensschau*. When we perceive essences and ideal objects this can also result in the recognition of certain patterns. That is, if we consider how different mathematical objects are ideal objects, and how some general notions are instantiated by physical objects (as discussed in section 3.2.2), the insight of essences that takes place in *Wesensschau* is thus both of aspects of physical objects and of mathematical objects as such. This certainly *founds* intuition, as it also concerns aspects of physical objects. However, it also leads to the difficulty of seeing how we can further our knowledge beyond the finitary and find answers to questions like "the truth or falsity of propositions like Cantor's continuum hypothesis" (Gödel 1964:268). Even to give meaning to such a question must involve a conception of intuition that is not founded in phenomenological perception, i.e. *Wesensschau* of aspects of physical objects. Mathematical intuition does, according to Gödel, "produce the axioms of set theory and an open series of extensions of them", and it is difficult to see how this can be done with our intuition founded on aspects of physical things. That is, *Fundierung*-relations, regarding intuition *of* and intuition *that*, are still grounded in the phenomenological intuition *of* general notions of physical objects, i.e. it exhibits an Aristotelian quality, which does not square with Gödel's view. If we look at the example of being spherical again, we see that the fact that the feature of being spherical is founded on the object, does not help us reach higher order structures. Gödel's intuition (both *of* and *that*) is meant to, for instance, advance our understanding of the size of the continuum, and *Fundierung*-relations do not provide us with such understanding. That the dependence relation internal to intuition is reciprocal only supports the idea that we are cut off from the higher order structures and does not help to establish them.

Before we conclude this section, let us take a brief look at how intuition can be founded as non-mysterious in another way. For instance, we can make it more palatable by considering intuition as a method for recognizing patterns. On Michael Potter's reading of Gödel (2001), mathematical intuition becomes a method (rather than a sixth sense sort of view) for reaching higher levels of abstraction of concepts. It is by reflecting on lower-level concepts we can grasp higher-order concepts, e.g. that you must start with the properties of the natural numbers before you can understand more advanced mathematical concepts. Potter suggests that Gödel's intuition should be understood as guiding us to more abstract concepts by progressive reflection, that is on reflecting on the mathematical knowledge we do have in order to reach new levels of abstraction (2001:331, 341–42). Also, Potter's reading can be seen in relation to a structuralist approach, where it is the recognizing of patterns and the reflection on these that allow us to transcend the finitary to the infinite.[27]

3.3 Assessment of the Husserlian Interpretation

Let us now assess how the Husserlian framework fits with Gödel's position and if it can render Gödel's intuition less mysterious. As we have already seen, the concept of *Fundierung* does not take us as far as we want to go. Even if our understanding of how intuition *of* and intuition *that* has improved, i.e. how concepts appear in axioms and their reciprocal dependence relation, we have not been able to go beyond the perceiving of essences in the physical world, which inherently must be limited.

Let us, then, consider some advantages with a Husserlian reading of Gödel. One advantage is how it might weaken the ontological commitment to the objective existence of objects and concepts. As Føllesdal suggested in section 3.1, saying that the existence of concepts is *as objective as* that of the physical objects, given that the existence of physical objects is not taken as a direct, realist existence, can weaken the ontological commitments. This is often seen as a serious drawback for platonism, which is why Føllesdal's suggestion is appealing. Another advantage is that by way of *Fundierung*-relations, we have now come to understand better how intuition *of* and intuition *that* is related, which is evident from the example of being spherical given above. Lastly, as we saw in 3.2.2, a Husserlian account captures the partial relationship to reality very well. When we include intuition of aspects of physical objects, we can explain better how we have the 'idea of object itself' and also how formal concepts are implicit in our understanding of the physical world.

However, there are arguments against a Husserlian reading other than not being able to justify higher order structures of mathematics. First of all, Husserl's *Wesensschau* is similar to Gödel's mathematical intuition, but the possible objects of intuition are different. While Gödel sees the difference between intuition *of* (objects and concepts) and intuition *that* (axioms and theories are true), *Wesensschau* is of abstract objects and general notions. This is problematic. The general notions are, as mentioned, aspects of physical objects, and have therefore an Aristotelian quality. This, of course, does not square with Gödel's mathematical

[27] Stewart Shapiro (2011) puts forth a recognizing of patterns in what he calls stratified epistemology. This is a structuralist approach, where the reflection on places in *ante rem* structures is supposed to deliver our understanding of the infinite. An important feature is that he rejects the need to find extra-mathematical justification for mathematical understanding, which is reminiscent of how Gödel allows abstract evidence for the existence of abstract objects, as discussed in section 2.3 in chapter 2.

intuition, as he does "not think that Aristotelian realism (according to which concepts are parts or aspects of things) is tenable" (*1951:321). (We saw a discussion on a possible Aristotelian interpretation in the section on his conceptual realism in section 2.2 in chapter 1, where it was firmly ruled out.)

Second, even if the possible objects of intuition are different for Gödel and Husserl, the mere lack of attention towards intuition *that* also poses a problem. For Gödel, the most important thing is 1) the objective existence of concepts, and 2) intuition *of* concepts and intuition *that* axioms (in which the concepts appear) are true. If we consider Husserl's broader intuition ['*Anschauung*'] – perception of physical objects and *Wesensschau* – we see that all intuition for Husserl is directed at objects. As we saw in section 3.2.1, intentionality is the fundamental property of consciousness, and attendant in this notion is the directedness at an object. For Gödel, this object-oriented view of consciousness does not ring true, as it is conceptual realism and the intuition *that* axioms (in which these concepts appear) are true that takes precedence. It is the axioms that "force themselves upon us as being true", and not our perceptions of aspects of objects (1964:268).

And third, there is a general idea in phenomenology of justifying something abstract by means of something that is concrete, which does not really sit well with Gödel's view. He did believe that a development of phenomenology could help clarify our understanding of concepts, and so perhaps further our mathematical knowledge. According to Føllesdal, Husserl's view on justification relies heavily on general features of phenomenology. Let me give four examples to illustrate what kind of ultimate ground of justification Føllesdal has in mind: 1) the directedness towards objects in the world, 2) anticipations that we are not aware of, i.e. that we do not actively discuss, such as the anticipation that a ball is round, also on the back that is out of our visual field, 3) beliefs that we hold but are not aware that we hold, i.e. sort of pre-judgements about how the world is and our place in it, and 4) unknown *Fundierung*-relations, e.g. that the perception of colour is founded on something being extended (unless you are a philosopher, of course, this is most probably an unknown *Fundierung*-relation). These structures, beliefs and relation are all something that we fall back on, in the sense that they constitute the ground on which we have thoughts, beliefs and anticipations about the world that we are aware of. A common feature is that there is this lack of awareness of them, and that are implicit in how we act and think about reality, i.e. they are *unthematized*, as Føllesdal calls it (1992:395). These constitute the world we live in and make up the "unthematized nature of the lifeworld… [which is] the ultimate ground of justification" (1992:395). That such underlying structures should, then, provide the ultimate justification for our *mathematical* theories as well, and not only our day-to-day life, is not a view that can be plausibly attributed to Gödel. His belief in a realm of mathematical facts does not seek this kind of extra-mathematical, ultimate justification. Rather, if anything, it is the concepts of mathematical reality that help structure and justify our empirical ideas.

The combination of these arguments against a Husserlian reading leads me to reject it. Although a Husserlian reading might render mathematical intuition less mysterious, it also leads to a conception of intuition where intuition cannot do the work it was intended to do, i.e. reach higher-order structures and further our mathematical knowledge. This, I think, is because a Husserlian conception of intuition is too object-oriented and does not square with the importance placed on the connection between mathematical intuition and Gödel's conceptual realism.

Chapter 3: Theory of Reason

We have now seen how Gödel's conceptual realism and mathematical intuition work together. The combination of the two helps Gödel avoid the label of a "crude and naïve" platonist. Moreover, in my application of the concept of *Fundierung* to Gödel's view, we have seen how intimately the relation between intuition *that* axioms are true and our intuition *of* concepts can be understood. In this chapter I want to further solidify the connection between his conceptual realism and mathematical intuition, and so defend the claim that Gödel puts forth a broader theory of reason. As I see it, in order to defend this, we must consider his wider philosophical remarks, on matters such as absolutely undecidable propositions, justification in set theory and mathematics as a science. This will hopefully lead us to entertain the idea of the partial separation of our relationship to reality as discussed in chapter 2 (section 2.4 and 3.2.2). Prevalent in his views on these matters is his belief in the power of reason, and how human thought and reasoning can transcend the finitude of physical reality and reach new insights in transfinite set theory. Attendant to this belief, I have argued, is the rejection of a phenomenological founding of intuition. Firstly, I shall deal with his view on reason and absolute undecidability; secondly, with his view on mathematics as a science; and thirdly, I shall consider a possible Kantian influence and how a conceptual framework can be projected onto physical reality.

Section I – Reason and Undecidability

1.1 Power of Reason

In this section I shall first present some general remarks on the power of reason, and how Gödel believed it can answer the open questions in mathematics. Second, I will present Gödel's view on absolute undecidability and *intrinsic* and *extrinsic* evidence for axioms.

According to Wang, Gödel had three philosophical heroes: Plato, Leibniz and Husserl (1996a:297). In the preceding chapter, I discussed some aspects of Husserl's influence on Gödel and presented a possible reading where mathematical intuition is founded on phenomenological perceiving of abstract objects or aspects of physical objects. I concluded section 3 by arguing that a Husserlian reading fails to recognize the importance Gödel places on conceptual realism and its connection with intuition *that* axioms are true. While Plato is cited as one of Gödel's "heroes", it seems Plato's influence on Gödel is restricted to offering a general model of conceptual realism (as mentioned in section 2.4 in chapter 1) (Parsons 2010:168). This leaves us with his third "hero", Leibniz. I suggested in section 2.4 in chapter 2 that the combination of conceptual realism and mathematical intuition puts forth a rationalistic view. This is where we find the Leibnizian influence. The power of human thought and its capability of finding answers to the questions it poses is brought up several times by Gödel.

> Either the human mind surpasses all machines (to be more precise: it can decide more number theoretical questions than any machine) or else there exist number theoretical questions undecidable for the human mind.
>
> Gödel thinks Hilbert was right in rejecting the second alternative. If it were true it would mean that human reason is utterly irrational by asking questions it cannot answer, while asserting emphatically that only reason can answer them. (Wang 1974:324–25)

This first disjunct is expressed by Gödel in the Gibbs Lecture of 1951. Parsons (2010:169) calls this Gödel's "rationalistic optimism", where, even in light of his own mathematical results, i.e. the Incompleteness Theorems, Gödel believes that reason can find new, stronger axioms to settle previously undecided propositions. Even to ask questions not determinable by human reason is deemed "utterly irrational". The human mind is thus conceived of as being more than a machine and able to reach beyond the finite. His belief in the power of reason is once again exhibited in *1961/?*: "[O]n the one hand, to safeguard for mathematics the certainty of its knowledge, and, on the other, to uphold the belief that for clear questions posed by reason, reason can also find clear answers" (*1961/?:381). This confirms the rationalistic optimism described by Parsons. And here, Gödel advocates how one should think about philosophy of mathematics and what its tasks are. This quote makes clear that he thoroughly believed that to the well-formed questions we pose, we will also find answers.

That Gödel favours the disjunct that "the human mind surpasses all machines" is again emphasized in two draft letters, commented on and discussed by van Atten (they do not appear in *Collected Works I–V*) (2006:259–60). This extract is from a letter annotated by Gödel as being from June 1963 and to Professor Tillich.[28]

> I said that in math[[ematical]] reasoning the non-comput[[ational]] (i.e. intuitive) element consists in intuitions of higher & higher infinities. This is quite true but [*sic.*] it this situation

[28] Van Atten notes that the letter can have been intended to Paul Tillich, professor of theology at the University of Chicago at that time (2006:260). However, it has not been confirmed that it was indeed to Paul Tillich, though Gödel wrote a letter to his mother in 1963 mentioning the use of his proofs in religion, which supports that it was (van Atten 2006:260).

> can be further analysed & then it turns out that they result (as becomes perfectly clear when these things are carried out in detail) from a deeper & deeper self knowledge of reason [to be more precise from a more & more complete rational knowledge of the <u>essence</u> of reason (of which essence the fac[[ulty]] of self knowledge is itself a constituent part)] [I believe that comput[[ational]] reason also results from self knowledge of reason but not from essential but factual knowledge] It seems to me that this is a verification (in the field of math[[ematics]]) of some tenets of idealistic philosophy. (van Atten 2006:259–60)

Here, we see that it is the *intuitive* element that is considered to transcend the computational and finite. Moreover, we have "intuitions of higher and higher infinities". Whether these intuitions are intuition *that* or intuition *of* is not made clear, but I think both alternatives are possible. If we uphold the interpretation I presented in section 3.2.3 in chapter 2, this would not have different consequences, as they have a reciprocal dependence relation, where the one involves the other and vice versa. However, when we have intuitions of higher and higher infinities, Gödel writes that these intuitions are a result from deeper and deeper *self-knowledge of reason*. What does this mean? It seems to be the case that Gödel thought reflection on reason itself can further our mathematical knowledge. If we take Gödel's view on reason to be exactly "more than a machine" as to its capabilities for understanding higher infinities, then we can interpret this self-knowledge of reason to at least include reflection on our mathematical knowledge, i.e. knowledge of concepts. This is one way to square self-knowledge with his conceptual realism.

This is made more plausible when he references the difference between *essential* and *factual* knowledge. Here, I take factual knowledge to be based on physical reality. Essential knowledge, on the other hand, seems to revolve around the possibility of self-knowledge of reason as such, i.e. knowledge of the essence of reason. In considering what the essence of reason is, I think that Gödel references the mere possibility of abstract reasoning as such, as well as our ability to reflect upon this faculty. However, this is, of course, difficult to say for certain. What is true is that reflection on our knowledge and reasoning is involved when we have intuitions of infinities. Van Atten takes the verification of some tenets of idealistic philosophy to be something along the lines of: "1. There are aspects of reality that cannot be reduced to material configurations; 2. These (abstracts) aspects are accessible to, and, moreover, constituted by, the mind" (2006:260). The first point seems probable enough, given what Gödel says about computational features of physical reality, and how mathematical objects "form a second plane of reality, which confronts us just as objectively and independently of our thinking of nature" (Gödel 2003a:505). The second point is perhaps more problematic. The first part, where the abstract aspects are said to be accessible to the mind seems alright. The second part, where the abstract aspects are said to be *constituted* by the mind, does not sit equally well. However, to be constituted by the mind does not necessarily mean to be created by the mind (as Gödel fervently denied this in **1951*), but rather that they are established by it. Thus, the grasping of aspects in the mind can perhaps explain how we further mathematical knowledge by self-reflection of reason.

In this draft letter, Gödel also addresses the finitude of physical reality and reasoning. We also have computational reasoning, he writes, but this is based on factual knowledge and not on essential knowledge, i.e. knowledge of reason. Computational reasoning is limited, as it is knowledge based on physical reality. This is reminiscent of what he says in *1964* where he also remarks on the finitude of our dealings with physical reality.

> That something besides the sensations actually is immediately given follows…from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations…whereas…by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. (1964:268)

Here, we can see that our reasoning based solely on physical reality is finite. By our thinking on physical reality alone, i.e. with no conceptual framework projected onto it, we can only "reproduce and combine". That we have the idea of object itself is, then, considered evidence for conceptual realism and reason's capability to reflect on concepts and the knowledge provided by these concepts.

1.2 Absolute Undecidability and Justification for Axioms

Let us take a look at Gödel's view on undecidable propositions. In "On the Question of Absolute Undecidability" (2006), Peter Koellner points out that Gödel's view on absolutely undecidable propositions underwent a development. From believing that there are some absolutely undecidable propositions, i.e. propositions that cannot be decided by any well-justified axiomatic theory (**1933o* and **1939b*), Gödel came to believe that there are not any absolutely undecidable propositions. In "Remarks Before the Princeton Bicentennial Conference on Problems in Mathematics" (1946), Gödel suggests that there are not any such absolutely undecidable propositions relative to large cardinal axioms (Koellner 2006:161). He even considers the possibility of a generalized completeness theorem:

> It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets. (1946:151)

Koellner notes that Gödel believed that any interesting mathematical statement that were thus far undecidable relative to ZFC could be decided if stronger axioms were found; this search for stronger axioms is known as Gödel's program (2006:162). In *1964*, Gödel proposes that reflection on concepts can help in this respect: "[T]here may exist, besides the usual axioms, …other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts" (1964:261). Here, we can see that it is our understanding of the concepts that will help decide the propositions that are undecidable relative to ZFC. However, even though this was a promising enterprise at the time, it later became clear that this kind of reflection on concepts and axioms in order to find stronger axioms to settle CH was not successful (Koellner 2006:167). The instances of intuition *of* concepts and *that* axioms are true are not enough to find the axioms Gödel wanted. This is why the "success" of a potential axiom candidate became more important as justification for accepting the axiom.

There is a difference in what type of justification we can have for axioms. In *1964* Gödel makes a distinction between what has become known as *intrinsic* and *extrinsic* evidence for axioms. It is an instance of *intrinsic* evidence when there is some self-evidential aspect to the axiom. For example, when we have a thorough understanding of the concepts underlying mathematics and have an intuition of them, this constitutes intrinsic evidence. For example, say that we have two sets *A* and *B*. Anyone with a thorough understanding of the concept of set will conclude that there is a third set *C* whose elements are *A* and *B*. Thus, the Axiom of Pairing, asserting that for any two sets there is a pair set of these two, is intrinsically evident. For Gödel, instances of mathematical intuition are considered as intrinsic evidence, as in the case where the "axioms force themselves upon us as being true" (1964:268).

Extrinsic evidence, on the other hand, is a type of abductive reasoning. One can accept an axiom as true if it is "successful", i.e. its consequences are useful, not only in discovering simpler proofs, but also by using the new axiom to contract many proofs in one (Gödel 1964:261). It might greatly simplify our theories. It could have independently verifiable consequences, such as the intuitively evident mathematical data. Gödel thought that such a new extrinsically evident axiom might both provide powerful methods and generally shed "so much light upon a whole field" that, even if the axiom is not intrinsically necessary, it "would have to be accepted at least in the same sense as any well-established physical theory" (1964:261). Here, Gödel once more draws upon the analogy between mathematics and physics, as we have seen throughout with the exposition of both the existence of mathematical objects and of mathematical intuition. An axiom should thus be accepted on the grounds that it is *as useful* in explaining mathematical phenomena *as* a physical theory is in explaining physical phenomena. Koellner writes that it is the combination of his belief in reason and the acceptance of such probabilistic arguments that "led Gödel to reject absolute undecidability and bifurcation in set theory" (2006:163).

SECTION II – MATHEMATICS AS SCIENCE

In this section I shall explore Gödel's view that mathematics is a descriptive science and the consequences this entails. As we have seen throughout the preceding chapters, Gödel believed that mathematical objects and concepts form a second plane of reality which is as objective and independent as that of physical objects. I do not intend here to determine the exact ways in which mathematics is the same, and the ways in which it is different from the empirical sciences. I will mention two main points however: 1) that the subject matter of mathematical theories is abstract, and 2) that mathematics has a higher standard for accepting mathematical theories as true, i.e. mathematics has a goal of absolute truth, a goal the empirical sciences do not share. By regarding mathematics as a descriptive science and drawing on an analogy between mathematics and the empirical sciences, we come across some interesting features attendant to this belief. In this section I will discuss two of them. Firstly, I shall discuss what this has to say for the kind of arguments Gödel uses; and secondly, I shall see if the fallibility of mathematical intuition leads to doubting the certainty of mathematics.

2.1 ARGUMENT STYLE

Gödel has faced criticism for drawing an analogy between physical and mathematical reality, especially when it comes to our epistemic access. The claim that we have "something like perception also of the objects of set theory" does not sit well with his critics (as we saw in section 2.1 in chapter 2). In particular, the fact that Gödel draws a comparison between sense perceptions and mathematical data is problematic as he does not give a detailed enough account of how we acquire knowledge of mathematical reality.[29] The lack of elaboration in this respect is undoubtedly a weakness. However, it is by positing the principle of epistemological parity while drawing the analogy between physical and mathematical reality, that his arguments can be more palatable. If we accept his view that mathematics is a science like the empirical sciences, then the forms of argument favoured by scientific realists should be made available to Gödel as well. By accepting the principle of epistemological parity, a kind of argument is made accessible to him, namely inference to the best explanation. His arguments for his realism and our access to it through mathematical intuition can be likened to arguments made for scientific realism. This kind of argument is pervasive throughout Gödel's thought: in his account of conceptual realism, in postulating the faculty of mathematical intuition and in his optimism about finding new axioms to settle statements that are undecidable relative to ZFC (as we saw in the preceding section).

With his arguments for realism, we have two slightly different cases, the argument for mathematical objects and the argument for mathematical concepts specifically. When it comes to objects they are said to be necessary for having well-functioning mathematical theories in the same sense as having well-functioning physical theories. And as the best explanation for why our physical theories are well-functioning is that there is an objective physical reality, the same should also go for mathematical reality. The arguments that are exclusively directed at the objective existence of *concepts* are a bit different. Here, Gödel makes the case that since: 1) we have the idea of object itself, 2) this is not due to our sensations or combinations of them, 3) we cannot create by our thinking, only reproduce and combine, 4) formal concepts play a role in our knowledge of the physical world, and thus 5) there must exist formal concepts. This argument has, then, the same kind of structure, where given certain premises, the best explanation for us having the idea of object itself is that there exists a world of concepts, and that concepts do not exist as aspects of physical things.

[29] We can also consider the comparison to go the other way, i.e. that the certainty of physical reality is likened to that of mathematical reality. Throughout his writings, Gödel wants to say that the existence of mathematical reality and mathematical intuition is analogous to physical reality and sense perceptions. Implicit in this argument is that physical reality and sense perceptions are considered to be indubitable and firmly established by science. Gödel's objective is to say that we should believe in mathematical reality as much as we believe in physical reality. One could, though, in principle, say that Gödel is transferring the uncertainty in our belief in mathematical reality to our belief in physical reality, so that it is actually the existence of physical reality that becomes uncertain. This is, though, of course, not what Gödel wanted.

When it comes to his arguments for mathematical intuition, we also see some variation. In *1964*, Gödel appeals to our ability to produce the axioms of set theory and all its possible extensions. He thus takes the success of mathematics as a discipline to affirm the existence of mathematical intuition as a psychological fact. Also, we see that our incomplete understanding of the concepts is said to be the reason why there are set-theoretic paradoxes (1944:139–140). He then likens these paradoxes to contradictory sense perceptions, such as *seeing* a rod immersed in water, which makes it appear to be bent, and our tactile perception of it, where it is not bent (*1951:321). Because we do not conclude from this that the physical world does not exist and that our perceptions are illusory, we should not conclude from the set-theoretic paradoxes that mathematical reality does not exist and that our intuition of concepts is illusory. This is thus an argument that first and foremost relies on the principle of epistemological parity, and where we are urged to accept mathematical intuition despite its imperfectness.

2.2 Fallibility of Intuition: Is Mathematics Revisable?

Let us turn to the fallibility exhibited by mathematical intuition, and what this leads to. As we saw in section 2 in chapter 1 and in section 2.4 in chapter 2, mathematical intuition is fallible. Mathematical intuition is meant as a *source* of knowledge, which means that our having mathematical intuition is not the same as having mathematical knowledge (Parsons 1995:60). For instance, Gödel writes that "mathematical intuition…produces the conviction" we can have about mathematical facts (*1953/9–III:340). As mentioned in the discussion on conceptual realism (section 2.2, chapter 1), Gödel considered mathematical knowledge to be fallible due to our imperfect understanding of the primitive concepts. Here, Gödel is flirting with a dangerous thought, considering his realist views. In introducing this fallibility aspect, Gödel also opens for a revisionary view on mathematics. To which extent should we consider mathematics to be revisable? In claiming that mathematical intuition is fallible, the certainty of our accepted mathematical theories is questioned. This captures the larger discussion of the relationship between reality and theory. How can we say that a theory is true? If mathematical knowledge is constituted by our best mathematical theories, are the theories true because they accurately describe mathematical reality? In dealing with an abstract subject-matter, the problem of describing the relationship between theory and reality becomes very complicated. For the purpose of this thesis, however, I will limit the discussion to Gödel's view.

While Gödel opens for doubt concerning our mathematical knowledge, he does not want to pursue this line of thought. If anything, Gödel believed that the sceptical tendencies in philosophy of mathematics had been taken too far. With the view that mathematics is a descriptive science comes the attendant questions faced by scientific realism, such as the fact that every theory in science have been rejected and replaced. But, the fact that every theory in the history of science has been rejected and replaced seems to have an exception when it comes to mathematics. At least we cannot observe the same pattern in the history of mathematics. We cannot therefore, as in the case of the empirical sciences, argue on the grounds of induction that the current accepted mathematical theories most likely are wrong. Mathematical truths are still considered to be necessary and eternal, in a way empirical truths are not. For instance, Pythagoras' theorem remains true and unaltered. That our mathematical theories can be faulty is because our understanding of the primitive concepts is lacking. This is also the cause of set-theoretical paradoxes.

In *1964*, Gödel writes: "It might seem at first that the set-theoretical paradoxes would doom to failure such an undertaking [of explaining the foundations of set theory], but closer examination shows that they cause no problem at all" (1964:268). Why would the paradoxes in set theory not pose a threat to the certainty of mathematical theories? A possible answer is found in **1961/?* where Gödel presents a schema for possible philosophical world-views ('*Weltanschauungen*'). According to Gödel one can define two groups, where "skepticism [*sic.*], materialism and positivism stand on one side, spiritualism, idealism and theology on the other" (*1961/?:375). Furthermore, regarding the possibility of knowledge, pessimism is said to belong to the former group and optimism to the latter, as scepticism is certainly pessimistic as to the acquiring of knowledge. Gödel claims that since the Renaissance, the tendency in philosophy has been to move from spiritualism, idealism and theology towards scepticism, materialism and positivism. This tendency has not really reached mathematics, due to its nature of being an a priori science. Longer than any other science, then, mathematics withstood this *Zeitgeist*.[30] However, with the paradoxes in set theory at the turn of the 20th century, "its hour struck" (*1961/?:377). As Gödel also touches upon in *1964*, Gödel writes that the paradoxes should not be considered a problem for mathematics.

[30] Despite the efforts of John Stuart Mill, who tried to establish an empiricist understanding of mathematics. This attempt, Gödel notes, was not very successful and did not become particularly popular (*1961/?:377).

> [I]n the first place, these contradictions did not appear within mathematics but near its outermost boundary toward philosophy, and secondly, they have been resolved in a manner that is completely satisfactory and, for everyone who understands the theory, nearly obvious. (*1961/?:377)

Because the paradoxes arise at the boundary towards philosophy, they should not lead to scepticism regarding the truth of mathematics. Gödel rejects the tendency where "mathematicians denied that mathematics, as it had been developed previously, represents a system of truths" (*1961/?:377). As the paradoxes are solved satisfactorily within mathematics, the fact that they did arise in the first place should not cause a wavering of belief in the certainty of mathematical knowledge. Had they not been satisfactorily solved, on the other hand, but continued to raise doubts about our most fundamental statements in set theory, mathematics would be in trouble.

The situation where we have undecidable statements relative to ZFC, as in the case of CH, does not present a similar challenge. That is because it is not a question of internal contradictions caused by our incomplete understanding of the concepts. As Gödel writes in *1964*: "For if the meanings of the primitive terms of set theory…are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false" (1964:260). Here, Gödel expresses a belief he is known for: that the axioms describing mathematical reality are incomplete in their description of that reality, and that the hitherto undecidable statements relative to ZFC do indeed have a determinate truth value. So, even though Gödel does allow for mathematical intuition to be fallible and our mathematical knowledge to be incomplete, he endorses the view that mathematics represents a system of truths. In that respect, the fallibility aspect is toned down, and the rationalistic optimism is gaining ground.

SECTION III – CONCEPTUAL FRAMEWORK ONTO PHYSICAL REALITY

In this section I return to the partially separated relationship to reality as introduced in section 2.4 in chapter 2. Here, I will argue that Gödel's notion of mathematical intuition is only understandable in light of what can be called Gödel's *theory of reason*, as Parsons suggests (1995:45), rather than a theory of intuition. I argue that his platonist position and notion of mathematical intuition must be seen in light of his larger philosophical influences and views, such as his belief in the power of reason with its ability to gain knowledge of itself (section 1) and his belief that mathematics is a descriptive science (section 2). This section, then, aims to combine what we have seen in the preceding chapters and presents an interpretation where Gödel's rationalistic optimism and view on mathematics as a science lead us to entertain this broader idea of a theory of reason. In order to do this, I will address how the rejection of a Husserlian reading of Gödel pulls towards a conception of intuition that has a Kantian streak. In doing this, I will consider how far our understanding of reality is formed by Gödel's combination of conceptual realism and mathematical intuition.

3.1 Kantian Streak?

In section 3 in the preceding chapter, I gave a Husserlian interpretation of Gödel, where the notions of intentionality, *Wesensschau* and *Fundierung* were central. By applying a Husserlian conceptual framework to see how Gödel's views fit together, we saw that Gödel's mathematical intuition could be explained as part of a wider philosophical system. However, there were some aspects of this reading that did not square with Gödel's position. Here, I refer to 1) how the founding of intuition in general phenomenological perceiving did not bring us very far as to higher-order structures, 2) allowing general notions, i.e. aspects of physical objects, to count as essences and thereby allowing them to be perceived by *Wesensschau*, 3) the lack of attention towards the intuition *that* axioms are true, and 4) how, in phenomenology, the ultimate justification for mathematical knowledge consists of our implicit anticipations and directedness towards an object. These four points led us to reject the Husserlian reading.

There are two places in the Supplement of *1964* where another philosopher is mentioned. Van Atten and Kennedy (2003) discovered a sentence (that was later removed) where phenomenology was posited as a possible way to clarify concepts and their axioms (2003:466). However, in the end, the philosopher that was included in the final version of *1964* was not Husserl, but Kant.

The four abovementioned considerations do not only lead us to reject the Husserlian reading, but also pull us towards another possible reading, where the combination of conceptual realism and intuition (*that* axioms are true and *of* concepts) is given more importance. Here, I am referring to a reading suggested earlier (section 2.4 in chapter 2), where our intuition of formal concepts is formative also of our relationship to physical reality. The direction in which we are pushed from rejecting the Husserlian reading, then, can be towards a more general form of intuition, i.e. an intuition that both enables and limits our dealings with all of reality.

This kind of intuition would be a way to deal with reality through our conceptual framework. That is, we experience reality through the lenses of our conceptual framework, as we project it onto physical reality. This might square with the partial separation of our relationship to reality, in a way that gives Gödel’s conceptual realism more emphasis. However, if we experience reality only through our understanding of concepts, sense perceptions of physical reality become determined by our formal concepts. This would be to push Gödel’s rationalism too far. In that case, we would go further than a *partial separation* between mathematical intuition and physical senses. Also, if we experience reality through these conceptual lenses, it would lead us to claim that all our dealings with reality is first and foremost determined by our conceptual understanding. I do not want to go this far. Gödel does not claim that sense perceptions are determined by conceptual understanding, and he writes only that something else is “given” in our empirical ideas, such as the idea of an object itself. Moreover, the possibility and limitations of human knowledge was not the main subject of inquiry in Gödel’s work, rather it was mathematical reality and mathematical knowledge.

Let us turn to what Gödel actually writes about Kant. Gödel mentions Kant in two places in the Supplement of *1964*. First, we have the rejection of Kant’s view that mathematical data are “something purely subjective” (1964:268). This is also commented on in a footnote (that was later crossed out) in the Gibbs Lecture: “Moreover, according to the Kritik of pure reason the mathematical concepts too are subjective since they are obtained by applying the purely subjective categories of thinking to the objects of intuition.” (quoted from Parsons 2010:171). Parsons argues that Gödel considered the categories of pure understanding to be subjective because they do not provide knowledge of things in themselves (which, Parsons argues, is not a very controversial interpretation) (2010:172). The data that are referenced in the quote of *1964*, are caused by mathematical objects and concepts, where it is the instances of intuition that explain “their presence in us” (1964:268). As we know, Gödel considered intuition *of* concepts and intuition *that* axioms are true to have objective validity. The “given” present in empirical ideas is something we all have access to, and it is not so that the idea of object itself is subjective to each person. Intuition is supposed to provide us with knowledge of concepts and axioms, and even if the axioms do not exhaust the description of mathematical reality, the project of finding new axioms to complete the description is considered necessary to better match our mathematical theories to mathematical reality. As we saw above, however, these instances of intrinsic justification for axioms have not been as successful as Gödel hoped for, thus leading to the growing importance placed on extrinsic justification.

The second mention of Kant in *1964* is in a footnote, where Gödel says there is a close relationship between the concept of set and Kant's categories of pure understanding, as their function is the same, namely that of "'synthesis', i.e., the generating of unities out of manifolds (e.g., in Kant, of the idea of *one* object out of its various aspects)" (1964:268n.40). The concept of set is thus considered by its operation "set of *x*'s", which, Gödel says, "has never led to any antinomy whatsoever…and has so far proved completely self-consistent" (1964:258–59). Here, Gödel notes that the set of all sets cannot exist, because by the concept of set's very nature, one can always perform another "set of" operation (1964:259n.15). Gödel's interpretation of Kant's synthesis as "the idea of *one* object out of its many aspects", can be linked to Husserl's notion of *Wesensschau*, where we can perceive aspects of objects and recognize the same feature in otherwise different objects (from section 3.2.2 in chapter 2). For Kant, "synthesis" is the idea of *one* out of its many aspects in mathematics, while Husserl would say that we are directed at an object so that we can always have more anticipations or perceive different general features of an object, e.g. in the example of the basketball, the marble and the American football. It seems, then, that the unity of an object is taken to be primary for Husserl.

Now, I will turn from Kantian ideas on the relationship between concepts and their application on reality, to remark on Kantian intuition specifically. According to Michael Hallett, "Gödel dismisses Hilbert's reliance on a version of Kantian intuition for elementary arithmetic, which he sees as a kind of quasi-spatio-temporal, concrete intuition" (2006:119). Hallett argues that Kant's use of intuition in mathematics is too weak and does not help Gödel's concept of intuition. The dismissal of Hilbert's reliance on a version of Kantian intuition is from 1972, where Gödel also underlines that:

> "*Concrete intuition*", "*concretely intuitive*" are used as translations of "Anschauung", "anschaulich". What Hilbert means by "Anschauung" is substantially Kant's space-time intuition confined, however, to configurations of a finite number of discrete objects…Note that it is Hilbert's insistence on *concrete* knowledge that makes finitary mathematics so surprisingly weak. (1972:272n.**b**)

However, it must also be noted that in **1961/?*, when Gödel also mentions Kantian intuition, the word used is "Intuitionen" and not the Kantian "Anschauung" (Føllesdal 1995:367n.**a**). As

Føllesdal emphasizes, the relation between Kant's and Gödel's conceptions of intuition is very complicated, and I will therefore leave it for now.

At this point, I will include a quote from Tyler Burge, and while he is writing on Frege's platonism in the quoted paper, I think it is applicable to Gödel as well. When writing on Frege's epistemology and its relation to Kant, Burge writes: "It would be incompatible with Platonism to regard [mathematical entities] as essentially part of an appearance or perspective for a thinker – as Kant would have – though they may impose constitutive conditions on such appearances or perspectives" (1992:637–38). The Husserlian inclusion of aspects of physical objects is here denied. But so is the possibility of experiencing reality through conceptual lenses (as sketched above). On the other hand, Burge allows for the mathematical entities, which, for my purpose, I take to be Gödel's mathematical concepts, to perhaps "impose constitutive conditions on such appearance or perspectives" (1992:638). This might capture how formal concepts are formative also of our relationship to physical reality. That is, the constitutive conditions might express how the "given" underlying mathematics is present in our empirical ideas. Even though these conditions do not completely determine our relationship to physical reality, they are part of how we come to have the idea of object itself.

What we have seen in the above discussion makes it clear that there is a tension as to how rationalistic Gödel's theory is, and how intuition can be founded in phenomenological perceiving. Since the Husserlian reading does not lead us to the higher-order structures we want, and since intuition *of* concepts and intuition *that* axioms are true cannot be prior to all experience of reality, Gödel's position appears to be somewhere in the middle. A further complication is, of course, that it is difficult to say what Gödel intended with the two mentions of Kant in *1964*.[31] It can, possibly, be seen as a pull towards a rationalistic version of Gödel's mathematical intuition, where intuition is thought to be "a pure, extra-conceptual source of information…[and that it] together with the most fundamental concepts, the categories, constitutes the framework or our experience" (Kjosavik 1999:6). If we can read all of this into the mentions of Kant (which is uncertain), it would certainly emphasize the power of reason, especially compared to Husserl's *Wesensschau*, which is founded on

[31] See Parsons (2010), Hallett (2006) and Folina (2014) for a further exploration of the relation between Kant and Gödel.

phenomenological perceiving. But, it can also relate to the emphasis put on objective validity (instead of data being subjective) and the broader sense of the concept of set as a "synthesis".

I will give two final considerations on this topic. The two philosophical giants – Kant and Husserl – were, as we know, studied carefully by Gödel. Gödel considered Kant's assertions on philosophy of mathematics to be false if understood literally, "but in a broader sense [to] contain deep truths" (*1961/?:385). Moreover, he gave Husserl the credit for providing the first correct interpretation of Kant, and ends **1961/?* with the following: "[I]f the misunderstood Kant has already led to so much that is interesting in philosophy, and also indirectly in science, how much more can we expect it from Kant understood correctly?" (*1961/?:387). The questions addressed in this section, how far Gödel can be interpreted in a Husserlian or Kantian way, remain very much open-ended. It would, however, be extremely interesting, and fruitful, I think, to continue this push and pull between Husserl and Kant while interpreting Gödel. Furthermore, it could possibly lead to a deeper understanding of how Gödel's intuition should be understood. Similarly to how we applied the concept of *Fundierung* on Gödel's intuition, I think that – by considering Gödel's work in light of both Husserl and Kant – we might find a way to make implicit structures in Gödel's work explicit.

CONCLUSION

I have argued that Gödel's arguments for realism largely rest on his principle of epistemological parity, thus putting the legitimacy of the existence of mathematical reality on an equal footing as the existence of physical reality. Arguing that mathematical reality forms a second plane to physical reality, definitely involves many metaphysical commitments. However, Gödel's view that mathematics is a descriptive science opens up the possibility for arguing by inference to the best explanation, similarly to how we argue for scientific realism. We have seen that the attempt at providing an epistemic access to mathematical reality results in the postulation of mathematical intuition. While the access problem remains, the epistemological gulf is shrunk if one allows abstract evidence to justify our belief in abstract objects.

I have argued that the faculty of mathematical intuition must be seen in connection to Gödel's conceptual realism, as it is the intuition *of* concepts and intuition *that* axioms are true that is supposed to provide mathematical knowledge. Moreover, to reject mathematical intuition without engaging with its internal structure – the *Fundierung*-relation between intuition *of* objects and *that* axioms are true – is to reject his position without proper examination. By applying the concept of *Fundierung*, I have argued that there is a reciprocal dependence relation between intuition *of* concepts and intuition *that* axioms are true, because the concepts are constitutive for the axioms.

This reciprocal dependence relation pushed the interpretation of Gödel in a more rationalistic direction, where his optimism regarding reason's ability to acquire knowledge of infinite structures is better accounted for. The combination of his conceptual realism, his intuition *of* concepts and *that* axioms are true, in addition to the emphasis placed on the power of reason leads me to conclude that Gödel's platonist position cannot be seen as crude or naïve. Also, the strong connection between his conceptual realism and mathematical intuition leads to the view that our conceptual framework is partially formative for our relationship to physical reality. This is what we have called Gödel's theory of reason, where we project our intuitive knowledge of concepts onto our prospective knowledge of physical reality.

The rejection of the Husserlian reading has led us to engage with a possible Kantian reading, and I think that, in order to do justice to Gödel's philosophical efforts, further interpretive work – where both the Husserlian and Kantian influence are attended to – is in order. I think that the Husserlian idea of structuring reality and the Kantian idea of demarcating the possibility and limitation of human knowledge would be an interesting subject of further study if applied to Gödel's notion of intuition. Then we could also explore the limits of Gödel's mathematical intuition, and thereby investigate at which point the potential for knowledge by intuition is exhausted, and when we have to turn to extrinsic justification for finding new axioms.

BIBLIOGRAPHY

van Atten, M., (2006) “Two Draft Letters from Gödel on Self-Knowledge of Reason”, in *Philosophica Mathematica* (III), vol. 14(2):255–261.

van Atten, M. & Kennedy, J., (2003) “On the Philosophical Development of Kurt Gödel”, in *The Bulletin of Symbolic Logic*, vol. 9(4):425–476.

Benacerraf, P., (1965) “What numbers could not be”, in *Philosophy of Mathematics: Selected Readings* (1983) eds. Benacerraf, P. & Putnam, H., 2nd edition, Cambridge University Press, Cambridge.

———. (1973) “Mathematical Truth”, in *Philosophy of Mathematics: Selected Readings* (1983) eds. Benacerraf, P. & Putnam, H., 2nd edition, Cambridge University Press, Cambridge.

Burge, T., (1992) “Frege on Knowing the Third Realm”, in *Mind*, Vol. 101(404):633–650.

Chihara, C.S., (1982) “A Gödelian Thesis Regarding Mathematical Objects: Do they Exist? and Can We Perceive Them?”, in *The Philosophical Review*, Vol. 91(2):211–227.

———. (1990) *Constructibility and Mathematical Existence*, Oxford University Press, Oxford.

Feferman, S., (1995) “Introductory Note to **1933o*”, in *Collected Works Vol. III*, pp. 36–44.

———. (1998) *In the Light of Logic*, Oxford University Press, New York.

Field, H., (1989) *Realism, Mathematics and Modality*, Basil Blackwell, Oxford.

Folina, J., (2014) “Gödel on how to have your mathematics and know it too”, in *Interpreting Gödel: Critical Essays*, (2014) ed. Kennedy, J., Cambridge University Press, Cambridge.

Frege, G., (1884) *The Foundations of Arithmetic*, in *The Frege Reader*, (1997) ed. Beaney, M., Blackwell, Oxford.

———. (1893) *Basic Laws of Arithmetic*, (2013) trans. & eds. Ebert, P.A. & Rossberg, M., Oxford University Press, Oxford.

———. (1897): “Logic”, in *The Frege Reader*, (1997) ed. Beaney, M., Blackwell, Oxford.

———. (1903) *Basic Laws of Arithmetic*, (2013) trans. & eds. Ebert, P.A. & Rossberg, M., Oxford University Press, Oxford.

———. (1918-19) “Thought”, in *The Frege Reader*, (1997) ed. Beaney, M. pp. 325–45, Blackwell, Oxford.

Føllesdal, D., (1992) "Gödel and Husserl", in *Naturalizing Phenomenology: Issues in Contemporary Phenomenology and Cognitive Science*, (1999) eds. Petitot, J., Varela, F.J., Pachoud, B. &Roy, J.-M., Stanford University Press, pp. 385–400.

———. (1995) "Introductory Note to **1961/?*", in *Collected Works Vol. III*, pp. 364–373.

———. (2016) "Richard Tieszen. *After Gödel. Platonism and Rationalism in Mathematics and Logic.*" in *Philosophica Mathematica* (III) Vol. 24(3):405–421.

Gödel, K., (*1933o) "The Present Situation in the Foundations of Mathematics", in *Collected Works Vol. III*, pp. 45–53.

———. (*1939b) "Lecture at Göttingen", in *Collected Works Vol. III*, pp. 126–155.

———. (1944) "Russell's Mathematical Logic", in *Collected Works Vol. II*, pp. 119–141.

———. (1946) "Remarks Before the Princeton Bicentennial Conference on Problems in Mathematics", in *Collected Works Vol. II*, pp. 150–153.

———. (*1951) "Some Basic Theorems on the Foundations of Mathematics and their Implications", Josiah Willard Gibbs Lecture, American Mathematical Society, 1951, in *Collected Works Vol. III*, pp. 304–323.

———. (*1953/9–III) "Is Mathematics Syntax of Language?", in *Collected Works Vol. III*, pp. 334–356.

———. (*1953/9–V) "Is Mathematics Syntax of Language?", in *Collected Works Vol. III*, pp. 356–362.

———. (*1961/?) "The Modern Development of the Foundations of Mathematics in the Light of Philosophy", in *Collected Works Vol. III*, pp. 374–387.

———. (1964) "Cantor's Continuum Problem?", in *Collected Works Vol. II*, pp. 254–270.

———. (1972) "On an Extension of Finitary Mathematics Which Has Not Yet Been Used", in *Collected Works Vol. II*, pp. 271–280.

———. (1986) *Collected Works, Volume I: Publications 1929–1936*. Oxford University Press, New York and Oxford.

———. (1990) *Collected Works, Volume II: Publications 1938–1974*. Oxford University Press, New York and Oxford.

———. (1995) *Collected Works, Volume III: Unpublished Essays and Lectures*. Oxford University Press, New York and Oxford.

———. (2003a) *Collected Works, Volume IV: Correspondence A–G*. Oxford University

Press, New York and Oxford.

———. (2003b) *Collected Works, Volume V: Correspondence H–Z*. Oxford University Press, New York and Oxford.

Goldfarb, W., (1995) “Introductory Note to **1953/9*”, in *Collected Works Vol. III*, pp. 324–334.

Hallett, M., (2006) “Gödel, Realism and Mathematical ‘Intuition’”, in *Intuition and the Axiomatic Method*, (2006) (eds.) Carson, E. & Huber, R., Springer, Dordrecht.

Hilbert, D., (1926) “On the Infinite”, in *Philosophy of Mathematics: Selected Readings* (1983) eds. Benacerraf, P. & Putnam, H., 2nd edition, Cambridge University Press, Cambridge.

James, E.P., (1992) “The Problem of Mathematical Existence”, in *Philosophical Books* vol. 33(3):129–138.

Kennedy, J., (2014) “Introduction. Gödel and Analytic Philosophy: How did we get here?”, in *Interpreting Gödel: Critical Essays*, (2014) ed. Kennedy, J., Cambridge University Press, Cambridge.

Kjosavik, F., (1999) *The Kantian Conception of “Anschauung”: A Phenomenological Interpretation and Defence*. Faculty of Arts, University of Oslo Unipub, Oslo.

Kleene, S.C., (1986) “Introductory Note to *1930b*, *1931* and *1932b*”, in *Collected Works Vol. I*, pp. 126–141.

Koellner, P., (2006) “On the Question of Absolute Undecidability”, in *Philosophica Mathematica*, Vol. 14(2):153–188.

Kuhn, T.S., (1962) *The Structure of Scientific Revolutions*. 4th edition, Chicago University Press, Chaicago.

Maddy, P., (1989) “The Roots of Contemporary Platonism”, in *The Journal of Symbolic Logic*, Vol. 54(4):1121–1144.

———. (1990) *Realism in Mathematics*, Clarendon Press, Oxford.

Martin, D.A., (2005) “Gödel’s Conceptual Realism”, in *The Bulletin of Symbolic Logic*, Vol. 11(2):207–224.

Moran, D., (2005) *Edmund Husserl: Founder of Phenomenology*. Polity, Cambridge.

Parsons, C., (1995) “Platonism and Mathematical Intuition in Kurt Gödel’s Thought”, in *The Bulletin of Symbolic Logic*, Vol. 1(1):44–74.

———. (2010) “Gödel and Philosophical Idealism”, in *Philosophica Mathematica*, Vol. 18(2):166–192.

Potter, M., (2001) “Was Gödel a Gödelian Platonist?”, in *Philosophica Mathematica*, Vol.

3(3):331–346.

Rota, G.-C., (1989) "Fundierung as a Logical Concept", in *The Monist*, Vol. 72(1):70–77.

Russell, B., (1912) *The Problems of Philosophy*. Henry, Holt & Company, New York/Williams & Norgate, London.

Shapiro, S., (2011) "Epistemology of Mathematics: What Are the Questions? What Count as Answers?" in *The Philosophical Quarterly*, vol. 61(242):130–150.

Tait, W.W., (2005) *The Provenance of Pure Reason: Essays in the Philosophy of Mathematics and its History*, Oxford University Press, Oxford.

Tieszen, R., (2011) *After Gödel: Platonism and Rationalism in Mathematics and Logic*. Oxford University Press, Oxford.

Wang, H., (1974) *From Mathematics to Philosophy*.

———. (1981) "Some Facts about Kurt Gödel", in *The Journal of Symbolic Logic*, Vol. 46(3):653–659.

———. (1987) *Reflections on Kurt Gödel*, MIT Press, Cambridge (Mass.).

———. (1996a) *A Logical Journey: From Gödel to Philosophy*, MIT Press, Cambridge (Mass.).

———. (1996b) "Skolem and Gödel", in *Nordic Journal of Philosophical Logic*, Vol. 1(2):119–132.