

# The interval structure of $(0, 1)$ -matrices

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## Abstract

Let  $A$  be an  $n \times n$   $(0, *)$ -matrix, so each entry is 0 or  $*$ . An  $A$ -interval matrix is a  $(0, 1)$ -matrix obtained from  $A$  by choosing some  $*$ 's so that in every interval of consecutive  $*$ 's, in a row or column of  $A$ , exactly one  $*$  is chosen and replaced with a 1, and every other  $*$  is replaced with a 0. We consider the existence questions for  $A$ -interval matrices, both in general, and for specific classes of such  $A$  defined by permutation matrices. Moreover, we discuss uniqueness and the number of  $A$ -permutation matrices, as well as properties of an associated graph.

**Key words.**  $(0, 1)$ -matrices, intervals, perfect matching.

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## 1 Introduction

This paper deals with the structure of intervals in a  $(0, *)$ -matrix, defined in a natural way, in rows and columns, as consecutive  $*$ 's separated by 0's in the matrix. The class of interval graphs are well-studied in graph theory, and one may view this paper as an investigation into a two-dimensional version of interval graphs.

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Let  $A$  be an  $n \times n$   $(0, *)$ -matrix, so each entry is 0 or \*. The 0's of  $A$  partition the \*'s (the positions thereof) of  $A$  into *horizontal intervals* (consecutive \*'s bounded by two 0's or the sides of  $A$ ) and *vertical intervals*, defined similarly. We shall refer to the zeros of  $A$  as the *barriers* of  $A$ .

For example, if

$$A = \begin{bmatrix} * & * & 0 & * & * \\ * & * & * & 0 & * \\ * & 0 & * & * & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{bmatrix}, \quad (1)$$

there are eight horizontal intervals (e.g. the three \*'s in the first three columns of row 2) and eight vertical intervals. In general, the number of horizontal and vertical intervals may be different.

An  $n \times n$   $(0, 1)$ -matrix  $P = [p_{ij}]$  is an  $A$ -interval matrix if it is obtained from  $A$  by replacing with a 1 one \* in every horizontal and vertical interval of \*'s, and then setting all other \*'s equal to 0 (these 0's are left blank in order to distinguish them from the original 0's). For example with  $A$  as in (1) we get

$$P = \begin{bmatrix} & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ 1 & 0 & & 1 & \\ 0 & 1 & & & \\ 1 & & & & \end{bmatrix}$$

which is actually the only  $A$ -interval matrix.

There are close connections between  $A$ -interval matrices and perfect matchings in certain graphs, as well as between  $A$ -interval matrices and alternating sign matrices (ASMs). We shall discuss these connections in detail later.

In general, if  $X$  is a  $(0, 1)$ -matrix,  $X^*$  denotes the  $(0, *)$ -matrix obtained by replacing each 1 with a \* resulting in a  $(0, *)$ -matrix.

**Example 1.1.** • Let  $A = J_n^*$ , the  $n \times n$  matrix of all \*'s. Then a  $J_n^*$ -interval matrix is just an ordinary permutation matrix.

- More generally, let  $A = [a_{ij}]$  be an  $n \times n$   $(0, *)$ -matrix whose \*'s have a *staircase pattern*. This means that there are integers  $1 \leq l_1 \leq l_2 \leq \dots \leq l_n \leq n$  and  $1 \leq r_1 \leq r_2 \leq \dots \leq r_n \leq n$  satisfying  $l_i \leq i \leq r_i$  ( $i \leq n$ ) and such that  $a_{ij} = *$  when  $l_i \leq j \leq r_i$  ( $i \leq n$ ), and  $a_{ij} = 0$  otherwise. Then, every  $A$ -interval matrix is a permutation matrix  $P$  such that all the 1's of  $P$  are within the staircase

pattern, e.g.,  $I_n$  is an  $A$ -interval matrix. In this case, the set of  $A$ -interval matrices determines a face of the Birkhoff polytope  $\Omega_n$ . For example, with

$$A = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ 0 & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix},$$

$I_6$  is an  $A$ -interval matrix.

- Let  $A = (J_n - I_n)^*$  be the  $(0, *)$ -matrix with 0's only on its main diagonal. Then an  $A$ -interval matrix  $P$  is unique and it has 1's in the superdiagonal and subdiagonal. For instance, if  $n = 5$ , we have

$$P = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

More generally, if  $Q = P_k \oplus I_{n-k}$  where  $P_k$  is the permutation matrix corresponding to the permutation cycle  $(2, 3, \dots, k, 1)$ , then for  $A = (J_n - Q)^*$ , an  $A$ -interval matrix is unique.

Note also that for a general  $(0, *)$ -matrix, the number of horizontal edges need not equal the number of vertical edges. For instance, with

$$A = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix},$$

the number of horizontal intervals is 2 and the number of vertical intervals is 3.

There is a close connection between  $A$ -interval matrices and alternating sign matrices (ASMs). Recall that an ASM is a square  $(0, \pm 1)$ -matrix such that in each row and column, ignoring 0's, the 1's and  $-1$ 's alternate beginning and ending with a 1. This is described in the next lemma, whose proof is immediate. We define two positions in a matrix to be *adjacent* if they are next to each other in the same row or column.

**Lemma 1.2.** *Let  $A$  be an  $n \times n$   $(0, 1)$ -matrix such that no zeros in  $A$  are adjacent and no zeros of  $A$  are in its first and last rows and columns. Let  $Q = J_n - A$ . If  $P$  is an  $A^*$ -interval matrix, then  $P - Q$  is an ASM. Moreover, every ASM arises in this way for a suitable matrix  $A$ .*

Thus, roughly speaking,  $A$ -interval matrices are the positive parts of ASMs. We shall later use this connection to ASMs. We refer to [4] for ASM and completion problems (see also later), and to [2] for our recent study of ASMs and related matrix classes and polyhedra.

We remark that *any*  $(0, 1)$ -matrix whose ones are not consecutive either in rows or columns may occur as an  $A$ -interval matrix, for some  $A$ . In fact, this condition is necessary as two adjacent ones have to lie in the same interval (horizontal or vertical). For the converse, if  $P$  is a  $(0, 1)$ -matrix with no consecutive ones, let  $A = P^*$ , and then  $P$  is an  $A$ -interval matrix.

## 2 Existence, interval covers and interchanges

In this section we give some general results on  $A$ -interval matrices, establishing and using a connection to perfect matching theory.

Let  $A = [a_{ij}]$  be an  $n \times n$   $(0, *)$ -matrix. Let  $\nu_h(A)$  and  $\nu_v(A)$  denote the number of horizontal and vertical intervals in  $A$ , respectively. Let  $\mathcal{G}_A = (\mathcal{I}, \mathcal{J}, E)$  be the bipartite graph whose color classes are  $\mathcal{I}$ , the set of horizontal intervals, and  $\mathcal{J}$ , the set of vertical intervals. There is an edge between a horizontal interval  $I \in \mathcal{I}$  and a vertical interval  $J \in \mathcal{J}$  whenever  $I \cap J$  is nonempty, i.e., there exists a (necessarily unique) position  $(i, j) \in I \cap J$ . The degrees of the vertices  $\mathcal{I} \cup \mathcal{J}$  are the cardinalities of the corresponding intervals. Let  $B_A$  be the  $\nu_h(A) \times \nu_v(A)$  biadjacency matrix of  $\mathcal{G}_A$ . There is a bijection between the set of  $A$ -interval matrices and the set of perfect matchings in the bipartite graph  $\mathcal{G}_A$  associated with  $A$ . The permanent of  $B_A$ ,  $\text{per}(B_A)$ , equals the number of  $A$ -interval matrices.

Let again  $A = [a_{ij}]$  be an  $n \times n$   $(0, *)$ -matrix. We say that an interval in  $A$  (vertical or horizontal) *covers* the positions that it contains. An *interval cover*  $\mathcal{I}$  of  $A$  is a set of intervals in  $A$  whose union equals the support of  $A$ , i.e., each position of a  $*$  in  $A$  is covered by at least one interval in  $\mathcal{I}$ . The *size* of an interval cover is its number of intervals. Let  $\tau(A)$  be the minimum size of an interval cover of  $A$ .

An example of an interval cover is the *horizontal interval cover*  $\mathcal{I}$  of  $A$ , consisting of all horizontal intervals, so  $|\mathcal{I}| = \nu_h(A)$ . Similarly, we have the *vertical interval cover* of size  $\nu_v(A)$ . In general, an interval cover may contain both horizontal and vertical intervals.

The minimum size  $\tau(A)$  of an interval cover of  $A$  is closely related to the existence of an  $A$ -interval matrix.

**Theorem 2.1.** *Let  $A$  be an  $n \times n$   $(0, *)$ -matrix. Then an  $A$ -interval matrix exists*

if and only if

$$\nu_h(A) = \nu_v(A) = \tau(A).$$

**Proof.** From our remark above, an  $A$ -interval matrix exists if and only if  $\mathcal{G}_A$  has a perfect matching. Thus the characterization follows from König's minmax theorem ([3]) which says that the minimum number of vertices that cover all edges of a bipartite graph equals the maximum cardinality of a matching.  $\square$

**Example 2.2.** Consider

$$A = \left[ \begin{array}{c|c|c} * & * & 0 \\ \hline 0 & 0 & * \\ \hline 0 & * & * \end{array} \right].$$

Then,  $\nu_h(A) = 3$  while  $\nu_v(A) = 4$ , so no  $A$ -interval matrix exists. Another way to see this is that a possible  $A$ -interval matrix must have ones in both position  $(1, 1)$  and  $(1, 2)$  (two column intervals), but then the row interval  $\{(1, 1), (1, 2)\}$  contains two ones, so an  $A$ -interval matrix does not exist.  $\square$

Theorem 2.1 also implies that, computationally, one can check efficiently, for given  $A$ , if an  $A$ -interval matrix exists by network flow techniques (a maximum matching or max-flow algorithm).

The following gives a necessary condition for the existence of an  $A$ -interval matrix.

**Lemma 2.3.** *Let  $A = [a_{ij}]$  be an  $m \times n$   $(0, *)$ -matrix such that for some  $k$  with  $1 \leq k \leq n - 2$  and some  $p$  and  $q$  with  $1 \leq p < q \leq m$ , we have that  $a_{pk} = a_{p,k+2} = a_{qk} = a_{q,k+2} = 0$ . Assume that there exists an  $A$ -interval matrix  $S = [s_{ij}]$ . Then there exists  $r$  with  $p < r < q$  such that  $a_{r,k+1} = 0$ .*

**Proof.** We must have  $s_{p,k+1} = s_{q,k+1} = 1$  and hence  $a_{r,k+1} = 0$  for some  $r$  with  $p < r < q$ .  $\square$

We now turn to a general result on differences of  $A$ -interval matrices. Let  $A$  be a  $(0, *)$ -matrix of order  $n$ . Let  $C = [c_{ij}]$  be an  $n \times n$   $(0, \pm 1)$ -matrix such that every row and column in  $C$  is either a zero line or it contains both one 1 and one  $-1$  in the same  $A$ -interval. We call  $C$  an  $A$ -cycle matrix. If  $C$  has exactly two nonzero rows and two nonzero columns, then  $C$  is called an  $A$ -interchange matrix.

**Theorem 2.4.** *Let  $A$  be an  $n \times n$   $(0, *)$ -matrix such that an  $A$ -interval matrix exists. Let  $P_1$  and  $P_2$  be two distinct  $A$ -interval matrices. Then there exist  $A$ -cycle matrices  $C_i$  ( $1 \leq i \leq t$ ) such that*

$$P_2 = P_1 + C_1 + C_2 + \cdots + C_t$$

and such that each matrix  $P_1 + C_1 + C_2 + \cdots + C_s$  ( $1 \leq s \leq t$ ) is an  $A$ -interval matrix.

**Proof.** Consider the interval-intersection bipartite graph  $\mathcal{G}_A$ . Then each of  $P_1$  and  $P_2$  corresponds to distinct perfect matchings  $M_1$  and  $M_2$  in  $\mathcal{G}_A$ . Let  $\Delta$  denote the subgraph of  $\mathcal{G}_A$  induced by the edges in the symmetric difference of  $M_1$  and  $M_2$ . Consider a non-isolated vertex  $v$  in  $\Delta$ . It corresponds to an interval in  $A$ , say horizontal (for vertical, similar arguments apply), in which the unique ones of  $P_1$  and  $P_1$  are placed in different positions. But then each of these positions belong to a vertical interval in which the ones of  $P_1$  and  $P_1$  are placed in different positions. This proves that the degree of the vertex  $v$  in  $\Delta$  equals 2. Thus all vertices in  $\Delta$  have degree 0 or 2, so  $\Delta$  is a union of vertex-disjoint cycles, say  $t$  of these, and isolated vertices. Thus, if we take  $M_1$  and complement edges in the first  $s$  of these cycles, we get another perfect matching, and for  $s = t$  we get  $M_2$ . In matrix language, these cycles correspond to  $A$ -cycle matrices, and the mentioned perfect matchings correspond to  $A$ -interval matrices, as desired.  $\square$

In Example 4.3 we show that  $A$ -interchanges do not suffice to move from one  $A$ -interval matrix to another.

To conclude this section we note that  $\nu_h(A) = \nu_v(A)$  whenever the zeros of  $A$  are nonadjacent and no zeros are in the first and last rows or columns of  $A$ . This holds because each of these numbers equals two times the number of zeros in  $A$ , and for each zero, there are two incident vertical intervals and two incident horizontal intervals. As we have seen, we may have  $\nu_h(A) = \nu_v(A)$  even when there are zeros in the first and last rows and columns.

### 3 Barriers in a permutation pattern

We consider the special case when  $A = (J_n - P)^*$  where  $P$  is a permutation matrix. Thus, the barriers, the zeros of  $A$ , are in a permutation pattern. We are then able to compute  $\tau(A)$  explicitly and to find minimum interval covers.

**Theorem 3.1.** *Let  $A = [a_{ij}]$  be an  $n \times n$   $(0, *)$ -matrix with exactly one 0 in every line (row or column). Then  $\nu_h(A) = \nu_v(A) = 2(n - 1)$ , and*

$$\tau(A) = 2(n - 1).$$

*Both the horizontal interval cover and the vertical interval cover are minimum size interval covers.*

**Proof.** Let  $Z = \{(i, j) : a_{ij} = 0\}$  which, by assumption, is the set of positions of the ones of a permutation matrix  $P$  where  $A = (J_n - P)^*$ , and  $|Z| = n$ . For  $n = 1$  the result is trivial, so let  $n \geq 2$ . Each row of  $A$  contains two horizontal intervals, except for the two rows with a 0 in the first or last column where there is one interval. Similar for vertical intervals, so  $\nu_h(A) = \nu_v(A) = 2(n - 1)$ . Associated to each position  $(i, j) \in Z$  there are two, three or four intervals in  $A$  starting next to  $(i, j)$  (up, down, left or right); the number depends on  $i$  and  $j$  (e.g., four if  $1 < i < n$  and  $1 < j < n$ ). Note the crucial property in this case that every interval in  $A$  is associated to *exactly* one  $(i, j) \in Z$  in this way, as  $P$  is a permutation matrix.

Let  $\mathcal{I}$  be an interval cover of  $A$ . We shall prove that  $\mathcal{I}$  has size at least  $2(n - 1)$ . Let  $\mathcal{I}_{ij}$  be the set of intervals of  $\mathcal{I}$  that are associated with  $(i, j) \in Z$ . Then  $\mathcal{I}_{ij}$ ,  $(i, j) \in Z$ , is a partition of  $\mathcal{I}$ .

If  $|\mathcal{I}_{ij}| = 0$  for some  $(i, j) \in Z$ , then  $(i, k)$  for each  $k \neq j$  must be covered by a vertical interval in  $\mathcal{I}$  in column  $k$  and, moreover, each  $(k, j)$  for  $k \neq i$  must be covered by a horizontal interval in  $\mathcal{I}$  in row  $k$ . All these intervals are clearly distinct and hence

$$|\mathcal{I}| \geq (n - 1) + (n - 1) = 2(n - 1).$$

If  $|\mathcal{I}_{ij}| \geq 2$  for all  $(i, j) \in Z$ , then

$$|\mathcal{I}| = \sum_{(i,j) \in Z} |\mathcal{I}_{ij}| \geq \sum_{(i,j) \in Z} 2 = 2n > 2(n - 1).$$

Thus, it remains to consider the case when  $\mathcal{I}$  satisfies  $|\mathcal{I}_{ij}| = 1$  for some  $(i, j) \in Z$ . First we assume that  $(i, j)$  is not on the boundary, so  $1 < i < n$  and  $1 < j < n$ . We may assume that  $\mathcal{I}_{ij}$  contains exactly one vertical interval, but no horizontal interval (the other case of no vertical interval is similar). We remark that the following arguments may be easier to follow by looking at the illustration in Example 4.5. Thus,  $\mathcal{I}$  contains no horizontal interval in row  $i$ , and therefore  $\mathcal{I}$  must contain the unique  $n - 1$  vertical intervals that contain  $(i, k)$  for  $k \neq j$ , respectively. For each  $k \neq j$ , let  $i_k$  be the unique row index such that  $(i_k, k) \in Z$ . Define

$$K_1 = \{k : k \neq j, 1 < i_k < i\}, \text{ and } K_2 = \{k : k \neq j, i < i_k < n\}.$$

Then  $|K_1| + |K_2| = n - 3$  (as  $Z$  contains exactly one position in the first row, and one in the last row, and none of these are in column  $j$ ). Define the set

$$S = \{(i_k - 1, k) : k \in K_1\} \cup \{(i_k + 1, k) : k \in K_2\}. \quad (2)$$

Then  $S$  contains  $n - 3$  positions where no two of these are in the same line. Moreover, no position in  $S$  is covered by the  $n - 1$  vertical intervals we just established in  $\mathcal{I}$ ,

and the same is true for the vertical interval in column  $j$ . Therefore  $\mathcal{I}$  must contain additional  $n - 3$  intervals (vertical or horizontal) to cover the positions in  $S$ . Thus, so far we have shown that  $\mathcal{I}$  contains

$$1 + (n - 1) + (n - 3) = 2n - 3$$

intervals. But  $\mathcal{I}$  must contain at least one more interval. In fact, exactly one of the two positions  $(i - 1, j)$  and  $(i + 1, j)$  is not covered by the vertical interval in  $\mathcal{I}_{ij}$ , and not by any of the just mentioned  $n - 3$  intervals, as they do not intersect rows  $i - 1$  or  $i + 1$ , and not by the first  $n - 1$  vertical intervals (they are in other columns than  $j$ ). So, we conclude that  $\mathcal{I}$  contains at least

$$2n - 3 + 1 = 2(n - 1)$$

intervals.

Next, consider the case when  $(i, j)$  lies on the boundary, where very similar arguments as above may be used. Let  $i = 1$  and  $1 < j < n$ . Assume first that  $\mathcal{I}_{ij}$  consists of a horizontal interval. Then  $\mathcal{I}$  must contain  $n - 1$  horizontal intervals covering positions  $(k, j)$ ,  $k > 1$ . Moreover, similar to the argument in connection with (2) there are  $n - 3$  additional intervals in  $\mathcal{I}$ , to those cover positions next to positions in  $Z$  in other rows than the first. Moreover, in the first row we have one horizontal interval, and the position closest to  $(i, j)$ , but not in this interval, is not covered by the mentioned intervals. Thus  $\mathcal{I}$  contains at least  $2(n - 1)$  intervals, as desired. Similarly we may treat the case when  $\mathcal{I}_{ij}$  consists of a vertical interval. By symmetry this conclusion also holds for all positions  $(i, j)$  on the boundary of the set of positions of  $A$ , except for the four corner positions. Finally, assume  $i = j = 1$ , and that  $\mathcal{I}_{11}$  consists of the horizontal interval in row 1. Then each position  $(k, 1)$   $k > 1$  must be covered, which gives  $n - 1$  horizontal intervals. The the argument in connection with (2) gives  $n - 2$  additional intervals (because in rows other than the first, exactly one row of  $A$  has a 0 in the last column). Finally, we have the interval in row 1, so all together,  $\mathcal{I}$  has at least  $(n - 1) + (n - 2) + 1 = 2(n - 1)$  intervals, as desired.

We have therefore shown that every interval cover contains at least  $2(n - 1)$  intervals. But the horizontal interval cover contains precisely  $2(n - 1)$  intervals, and so does the vertical interval cover. This proves the theorem.  $\square$

**Corollary 3.2.** *Let  $P$  be an  $n \times n$  permutation matrix with  $n \geq 2$ , and let  $A = (J_n - P)^*$ . Then an  $A$ -interval matrix exists.*

**Proof.** This follows by combining Theorem 2.1 and Theorem 3.1.  $\square$

**Example 3.3.** Let  $n = 5$  and let  $A$  be the  $5 \times 5$  matrix

$$\begin{bmatrix} * & * & 0 & * & * \\ 0 & * & * & * & * \\ * & * & * & 0 & * \\ * & 0 & * & * & * \\ * & * & * & * & 0 \end{bmatrix}.$$

So,  $A = (J_5 - P)^*$  where  $P$  is the permutation matrix whose ones correspond to the zeros in  $A$ . An illustration in connection with the proof of Theorem 3.1 is to consider  $(i, j) = (3, 4)$  and where  $\mathcal{I}_{3,4}$  contains the vertical interval  $I = \{(1, 4), (2, 4)\}$ . Then, with letters  $a, b, c, d$  denoting certain vertical intervals,  $\mathcal{I}$  also contains the vertical intervals indicated below:

$$\begin{bmatrix} & b & 0 & & d \\ 0 & b & c & & d \\ a & b & c & 0 & d \\ a & 0 & c & & d \\ a & & c & & 0 \end{bmatrix}.$$

Now, see (2), we get  $S = \{(1, 1), (5, 2)\}$  and to cover these and the position  $(4, 4)$  we need three other intervals, so we get a total of  $8 = 2(n - 1)$ .

In this example there are exactly two  $A$ -interval matrices, namely

$$P_1 = \begin{bmatrix} 1 & & 0 & 1 & \\ 0 & 1 & & & \\ & & 1 & 0 & 1 \\ 1 & 0 & & 1 & \\ & 1 & & & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 1 & & 0 & 1 & \\ 0 & & 1 & & \\ & 1 & & 0 & 1 \\ 1 & 0 & & 1 & \\ & 1 & & & 0 \end{bmatrix}.$$

So, here all ones are forced by the constraints defining an  $A$ -interval matrix, except in the submatrix defined by rows 2, 3 and columns 2, 3.  $\square$

Concerning Corollary 3.2, we remark that if  $A$  has *two* zeros in some line, even with just one more zero than a permutation of zeros, then an  $A$ -interval matrix may not exist, as the next example shows.

**Example 3.4.** Consider

$$A = \begin{bmatrix} 0 & * & * & * & * \\ * & 0 & * & 0 & * \\ * & * & 0 & * & * \\ * & * & * & 0 & * \\ * & * & * & * & 0 \end{bmatrix}.$$

Then  $v_h(A) = v_v(A)$ , so that the number of horizontal and vertical intervals is the same, but an  $A$ -interval matrix does not exist (consider column 5).  $\square$

Next, we show that the case when  $A = (J_n - S)^*$  with  $S$  a subpermutation matrix can be handled directly from the permutation case. Recall that a *subpermutation* matrix is a  $(0, 1)$ -matrix with at most one 1 in every row and column. When  $A$  has this form, the number of horizontal and vertical intervals may not be the same. So, as remarked before, to have the possibility of the existence of an  $A$ -interval matrix, we need to add the condition that  $v_h(A) = v_v(A)$ .

**Corollary 3.5.** *Let  $S$  be an  $n \times n$  subpermutation matrix with  $n \geq 2$ , and let  $A = (J_n - S)^*$ . Then an  $A$ -interval matrix exists if and only if the number of ones in row 1 and  $n$  of  $S$  equals the number of ones in column 1 and  $n$  of  $S$ .*

**Proof.** The condition is necessary by Theorem 2.1. Conversely, let  $S$  be an  $n \times n$  subpermutation matrix. The 1's of  $S$  determine a square  $k \times k$  permutation matrix  $S'$ . Let  $A = (J_n - S)^*$  and  $A' = (J_k - S')^*$ . By Theorem 3.1 we know that  $\tau(A') = 2(k - 1)$  and so there is a choice of  $2(k - 1)$  1's representing all intervals. Now there are 2, 3, or 4 zeros in rows and columns 1 and  $k$  of  $A'$ . Row 1 of  $S'$  (and  $A'$ ) is either row 1 of  $S$  (and  $A$ ) or has an empty row preceding it in  $S$  (so all \*'s in  $A$ ). If row 1 of  $A'$  comes from row 1 of  $A$ , we do nothing. Otherwise, there is a 0 in row 1 of  $A'$  and we choose the 1 in  $A$  above it. We do a similar construction for all "boundary" 1's in our choice. So now we have one 1 in every interval of  $A$  except possibly in those intervals that contained only 1's. In particular, if the number of horizontal intervals equals the number of vertical intervals, then the number of horizontal intervals with no chosen 1's equals the number of vertical intervals with no chosen 1's. These determine a  $p \times p$  submatrix of all 1's and we can then choose any permutation of 1's within this submatrix resulting in every interval of  $A$  having exactly one chosen 1.  $\square$

The construction in the previous proof is illustrated in the following example.

**Example 3.6.** Let

$$A = (J_7 - S)^* = \begin{bmatrix} * & * & * & * & * & * & * \\ * & * & 0 & * & * & * & * \\ * & * & * & * & * & * & * \\ * & 0 & * & * & * & * & * \\ * & * & * & * & 0 & * & * \\ * & * & * & * & * & 0 & * \\ * & * & * & * & * & * & * \end{bmatrix} \rightarrow \begin{bmatrix} * & 0 & * & * \\ 0 & * & * & * \\ * & * & 0 & * \\ * & * & * & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & \\ 0 & 1 & & \\ 1 & & 0 & 1 \\ & & 1 & 0 \end{bmatrix},$$

and then

$$\begin{bmatrix} & & & 1 & & & \\ & 1 & 0 & & 1 & & \\ & & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & & & 0 & 1 & \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & \end{bmatrix} \rightarrow \begin{bmatrix} & & & 1 & & & \\ & 1 & 0 & & 1 & & \\ & & & \mathbf{1} & & & \\ 1 & 0 & 1 & & & & \\ & 1 & & & 0 & 1 & \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & \end{bmatrix}.$$

□

We remark that the case of  $A = (J_{m,n} - S)^*$  for a rectangular subpermutation matrix  $S$  can be treated in the same way.

We now discuss the connection to ASMs, as given in Lemma 1.2. The following result from [4] concerns the *completion* of a matrix into an ASM, in the sense of replacing some zeros in the matrix by ones (and no other changes).

**Theorem 3.7.** ([4]) *Let  $A$  be an  $n \times n$   $(0, -1)$ -matrix such that the first and last rows and columns are zero rows and zero columns, respectively, and the submatrix obtained by deleting the first and last rows and columns has at most one  $-1$  in each row and column. Then  $A$  can be completed to an ASM.*

The ASM  $A$  obtained in Theorem 3.7 is very special, it has at most one  $-1$  in every row and column. It is convenient to refer to a row in a matrix, which is neither the first or the last, as an *inner row*. An *inner column* is defined similarly. If a  $(0, 1)$ -matrix has a 1 in its first or last row or column, that 1 is called a *boundary one*. Now, consider Corollary 3.5, in the special case where the subpermutation matrix  $S$  has no boundary ones, and let  $A = (J_n - S)^*$ . Let  $A'$  be the submatrix of  $A$  corresponding to rows and columns that contain a  $*$ , including the first and last row and column. Then we may apply Corollary 3.3 in [4] (changing the sign of our zeros to  $-1$ ) and obtain an ASM extension which gives a  $A'$ -interval matrix. Then we add ones in a permutation pattern for the deleted rows and columns, and thereby get an  $A$ -interval matrix. Thus, this gives another proof of Corollary 3.5, via ASMs, for this special case. In fact, similar methods can be used to give a proof in the general case, but we omit this, since that proof is quite long, with long combinatorial discussion to handle the ones of  $S$  on the boundary.

Finally, we remark that the symmetric case, when  $A$  is symmetric, can be treated in the same way from ASMs, using Theorem 3 of [4]. As a result one obtains the existence of a symmetric  $A$ -interval matrix when  $A$  is symmetric. We omit these details as the ideas and techniques are similar to what we have discussed.

## 4 Uniqueness

We consider the uniqueness question for  $A$ -interval matrices, and give a characterization of this property.

Let  $A = [a_{ij}]$  be an  $n \times n$   $(0, *)$ -matrix, and let  $\mathcal{G}_A = (\mathcal{I}, \mathcal{J}, E)$  be its corresponding bipartite graph with biadjacency matrix  $B_A$ . Then there is a unique  $A$ -interval matrix if and only if  $\mathcal{G}_A = (\mathcal{I}, \mathcal{J}, E)$  has a unique perfect matching, and this is the case if and only if there is a unique permutation matrix  $Q$  with  $Q \leq B_A$ . It is a basic fact (see e.g. Theorem 1.4.2 in [3]) that given a square  $(0, 1)$ -matrix  $X$ , there is a unique permutation matrix  $Q$  with  $Q \leq X$  if and only if, after row and column permutations,  $X$  takes the form of a triangular matrix with 1's on the main diagonal; in particular,  $X$  has both a row and column with exactly one 1. An algorithm to determine if there is a unique permutation matrix  $Q$  with  $Q \leq X$  and to construct  $Q$  is the following: Choose a row of  $X$  with exactly one 1, delete that row and the column containing its unique 1, and proceed iteratively. If at some iterative step there does not exist a row with exactly one 1, then either there does not exist a permutation matrix  $Q$  with  $Q \leq X$ , or there are at least two. If the algorithm terminates after choosing all rows of  $X$ , there is a unique permutation matrix with  $Q \leq X$ .

This fact can be applied to the biadjacency matrix  $B_A$ . In particular, if every interval has cardinality at least two, either there does not exist an  $A$ -interval matrix or there are at least two.

Let  $A$  be an  $n \times n$   $(0, *)$ -matrix. We adapt the above algorithm for constructing an  $A$ -interval matrix; we call it the *interval elimination algorithm*. Initially, let  $F = \text{supp } A$  be the *support* of  $A$ , i.e., the set of positions where  $A$  contains a  $*$  and thus where an  $A$ -interval matrix may have a 1. Let  $Q = [q_{ij}]$  initially be the zero matrix. Choose, if possible, an interval  $I$  in  $A$  with exactly one position in  $F$ , say  $(i, j)$ . Then, let  $q_{ij} = 1$  and remove from  $F$  all positions in row  $i$  and column  $j$ . We then say that all positions that were removed from  $F$  are *covered*. We repeat this operation, thus, if possible, choose an interval with a unique position in  $F$ , and update as just described. This interval elimination algorithm either terminates with  $F = \emptyset$ , and then the resulting  $Q$  is the unique  $A$ -interval matrix, or,  $F$  is nonempty, and it stops because no interval with a unique position in  $F$  can be found. From the discussion above we have the following result.

**Theorem 4.1.** *Let  $A$  be an  $n \times n$   $(0, *)$ -matrix such that at least one  $A$ -interval matrix exists. Then there is a unique  $A$ -interval matrix if and only if the interval elimination algorithm terminates by covering the support of  $A$ .*

As usual,  $I_m$  denotes the  $m \times m$  identity matrix, and  $L_m$  denotes the  $m \times m$

back-diagonal identity matrix (so  $I_1 = L_1$ ). Consider a permutation matrix

$$P = P_1 \oplus P_2 \oplus \cdots \oplus P_k \quad (3)$$

where each  $P_i$  is either an identity matrix or a back-diagonal identity matrix. Without loss of generality, we assume that every back-diagonal identity matrix (if any) is of order at least 2, and that two consecutive matrices are not both an identity matrix.

**Theorem 4.2.** *Let  $A = (J_n - P)^*$  where  $P$  is a direct sum (3) as specified above. Then there is a unique  $A$ -interval matrix if and only if  $k \leq 3$  and the direct sum (3) contains at most one back-diagonal identity matrix.*

**Proof.** We use the interval elimination (IE) algorithm on  $A$ , and Theorem 4.1. If  $k = 1$ , so  $P$  is an identity or back-diagonal identity matrix, then we see that the IE algorithm covers  $\text{supp } A$ , and the  $A$ -interval matrix is unique. Now assume that  $k \geq 2$ .

Suppose that, for some  $i < j$ , both  $P_i$  and  $P_j$  are back-diagonal identity matrices. Then, it is easy to see that the IE algorithm can only cover intervals that are: (i) in rows and columns before those of  $P_i$ , or (ii) in rows and columns after those of  $P_j$ , or (iii) above or to the left of the ones in  $P_i$ , or (iv) below or to the right of  $P_j$ . In particular, the horizontal interval to the left of the first 1 in  $P_j$  is not covered. Thus, an  $A$ -interval matrix is not unique. This shows the necessity of the condition in the theorem. Thus in (3) there is at most one back-diagonal identity matrix.

The sufficiency of the condition is easy to verify using the IE algorithm for each of the possible cases when  $k$  is 2 or 3. For instance, consider the case  $A = P_1 \oplus P_2$  where  $P_1$  is an identity matrix of order  $t$ , and  $P_2$  is a back-diagonal identity matrix. Then the IE algorithm successively covers each of the first  $t - 1$  vertical intervals below the main diagonal, and similarly the first  $t - 1$  horizontal intervals to the right of the main diagonal. Next, the IE algorithm covers, for each of the positions of the ones in the back-diagonal identity matrix  $P_2$ , the intervals to the left or above that position, and also the remaining intervals in the lower right corner.  $\square$

**Example 4.3.** Consider  $n = 12$  and  $P = I_4 \oplus L_3 \oplus I_5$ . Let  $A = (J_n - P)^*$ . Then,

by Theorem 4.2, the  $A$ -interval matrix is unique, and it is

$$R_1 = \left[ \begin{array}{cccc|cccc|cccc} 0 & 1 & & & & & & & & & & \\ 1 & 0 & 1 & & & & & & & & & \\ \hline & 1 & 0 & 1 & & & & & & & & \\ & & 1 & 0 & & & \mathbf{1} & & & & & \\ \hline & & & & & 1 & 0 & \mathbf{1} & & & & \\ & & & & 1 & 0 & 1 & & & & & \\ & & & \mathbf{1} & 0 & 1 & & & & & & \\ \hline & & & & \mathbf{1} & & & 0 & 1 & & & \\ & & & & & & & 1 & 0 & 1 & & \\ & & & & & & & & 1 & 0 & 1 & \\ & & & & & & & & & 1 & 0 & 1 \\ & & & & & & & & & & 1 & 0 \end{array} \right].$$

Next, let  $n = 9$ ,  $P = L_2 \oplus I_3 \oplus L_4$  and  $A = (J_n - P)^*$ . So, by Theorem 4.2, there are at least two  $A$ -interval matrices. An  $A$ -interval matrix is

$$R_2 = \left[ \begin{array}{cc|cc|cccc} 1 & 0 & 1 & & & & & \\ 0 & 1 & & & & & & \\ \hline 1 & & 0 & 1 & & & & \\ & & 1 & 0 & 1 & & & \\ & & & 1 & 0 & & & 1 \\ \hline & & & & & & 1 & 0 \\ & & & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 & 1 & \\ & & & & 1 & 0 & 1 & & \end{array} \right].$$

We observe that there does not exist an  $A$ -interchange giving a different  $A$ -interval matrix, but another  $A$ -interval matrix is

$$R_3 = \left[ \begin{array}{cc|cc|cccc} 1 & 0 & 1 & & & & & \\ 0 & & & 1 & & & & \\ \hline 1 & & 0 & & & & & 1 \\ & 1 & & 0 & 1 & & & \\ & & 1 & & 0 & & 1 & \\ \hline & & & 1 & & & & 0 \\ & & & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 & 1 & \\ & & & & 1 & 0 & 1 & & \end{array} \right],$$

and this one has an  $A$ -interchange that gives a different  $A$ -interval matrix. Note that, by Theorem 2.4, one can go from  $R_2$  to  $R_3$  by adding suitable  $A$ -cycle matrices, which is a more complicated operation than  $A$ -interchanges.

□

From this example we may conclude: If an  $A$ -interval matrix does not have an  $A$ -interchange, this does not mean that it is unique. Another  $A$ -interval matrix (same  $A$ ) may have an interchange. Therefore, the set of  $A$ -interval matrices is connected with respect the operation of adding  $A$ -cycle matrices as described in Theorem 2.4, but it is not connected with respect to  $A$ -interchanges.

**Theorem 4.4.** *Let  $A = (J_n - P)^*$  where  $P = L_m \oplus P'$  for some permutation matrix  $P'$ , and  $m \geq 2$ . Then an  $A$ -interval matrix is unique if and only if  $P' = I_{n-m}$ .*

**Proof.** The sufficiency of the condition follows from Theorem 4.2. So, assume that an  $A$ -interval matrix is unique. Consider the vertical interval  $I$  in the first column below position  $(m, 1)$ . To (eventually) cover  $I$ ,  $P$  must have its 1 in row  $n - 1$  to the left of its 1 in row  $n$  (as this will force a 0 in position  $(n, 1)$  of an  $A$ -interval matrix). Similarly,  $P$  must have its 1 in row  $n - 2$  to the left of its 1 in row  $n - 1$  (as this will force a 0 in position  $(n - 1, 1)$  of an  $A$ -interval matrix). We can continue like this, and by induction, we conclude that  $P'$  is the identity matrix. □

Let  $A = (J_n - P)^*$  where  $P$  is a permutation matrix equal to  $P_1 \oplus P_2 \oplus \cdots \oplus P_k$  where the  $P_i$ 's are permutation matrices. The following example shows that even if each  $A_i = (J - P_i)^*$  has a unique  $A_i$ -interval matrix, there may not be a unique  $A$ -interval matrix.

**Example 4.5.** Let  $A$  be given by

$$A = \left[ \begin{array}{c|c|c|c|c} * & 0 & * & * & * \\ 0 & * & * & * & * \\ \hline * & * & 0 & * & * \\ * & * & * & * & 0 \\ \hline * & * & * & 0 & * \end{array} \right] = A_1 \oplus A_2$$

where  $A_1$  is the indicated  $2 \times 2$  matrix. Then for  $i = 1$  and  $2$ , there is a unique  $A_i$ -interval matrix. However, there is not a unique  $A$ -interval matrix. This follows from Theorem 4.4, as  $P_1 = L_2$  and  $P_2 \neq I_3$ . In fact, two  $A$ -interval matrices are

$$\left[ \begin{array}{c|c|c|c|c} 1 & 0 & 1 & & \\ 0 & 1 & & & \\ \hline 1 & & 0 & & 1 \\ & & & 1 & 0 \\ \hline & & 1 & 0 & 1 \end{array} \right] \quad \text{and} \quad \left[ \begin{array}{c|c|c|c|c} 1 & 0 & & 1 & \\ 0 & & 1 & & \\ \hline & 1 & 0 & & 1 \\ 1 & & & & 0 \\ \hline & & 1 & 0 & 1 \end{array} \right].$$

□

We conclude this section with the following more general example.

**Example 4.6.** Let  $P$  be a permutation matrix of order  $n$  of the form

$$P = \left[ \begin{array}{c|c|c} Q_1 & P_1 & Q_2 \\ \hline P_2 & O_2 & P_3 \\ \hline Q_3 & P_4 & Q_4 \end{array} \right] \quad (4)$$

where  $O_2$  is the  $2 \times 2$  zero matrix and  $P_i$  contains exactly one 1 ( $1 \leq i \leq 4$ ). Since  $P$  is a permutation matrix, the ones in  $P_2$  and  $P_3$  are in different rows, and the ones in  $P_1$  and  $P_4$  are in different columns. Let  $A = (J_n - P)^*$  where  $P$  is a permutation matrix of the form (4). We claim for this  $A$ , an  $A$ -interval matrix is not unique.

To verify this, assume there exists an  $A$ -interval matrix  $R$ . Let  $O_2$  be in rows  $k, k+1$  and columns  $l, l+1$ . In the biadjacency matrix  $B_A$  corresponding to  $A$ , the two horizontal intervals to the right of the 1 in  $P_2$  and its left border in its other row and to the left of the 1 in  $P_3$  and its right border in its other row, respectively, and the two vertical intervals below the 1 in  $P_1$  and its top border in its other column and above the 1 in  $P_4$  and its bottom border in the other column, respectively, determine a  $2 \times 2$  submatrix  $J_2$  of 1's. It is easy to check that in an  $A$ -permutation matrix, the number of 1's in these two horizontal intervals union with these two vertical intervals is 2. This implies that these two 1's are in the submatrix of  $A^*$  given by  $O_2^*$ . But then we have two choices. For instance, suppose the 1's in  $R$  in the regions determined by  $Q_1, Q_2, Q_3, Q_4$  are not in the first or last rows and columns of  $R$ . Then in the first  $k-1$  rows of  $R$  we have  $2(k-1)$  1's and in the last  $n-(k+1)$  row we have  $2(n-(k+1))$  1's. This is a total of  $2(n-2)$  1's. But  $R$  has  $2(n-1)$  1's. This implies that the remaining two 1's come from the submatrix of  $A^*$  given by  $O_2^*$ . Similar arguments work for the other possibilities. □

## 5 The number of $A$ -interval matrices

Let  $A = [a_{ij}]$  be an  $n \times n$   $(0, *)$ -matrix, and let

$$\#(A) = |\{S : S \text{ is an } A\text{-interval matrix}\}|$$

be the number of  $A$ -interval matrices. We can view  $\#(A)$  as a measure of the complexity of  $A$ , and as a variation of the permanent of  $A$ . In fact,  $\#(A)$  is the permanent of the biadjacency matrix  $B_A$  of the bipartite graph  $\mathcal{G}_A$ .

At first glance one might think that the more 0's in  $A$  the fewer  $A$ -intervals, but our discussion below shows that this not true in general.

First, we consider the case when  $A$  has a single zero. Let  $n \geq 1$  and define the  $n \times n$  matrix  $\Gamma^{(n)} = [\gamma_{ij}^{(n)}]$  as follows: (i)  $\gamma_{ij}^{(n)} = (n-1)^2(n-2)!$  for  $(i, j) \in \{(1, 1), (1, n), (n, 1), (n, n)\}$ , (ii)  $\gamma_{ij}^{(n)} = (i-1)(n-i)(j-1)(n-j)(n-3)!$  for  $1 < i, j < n$ , and (iii)  $\gamma_{ij}^{(n)} = 0$  otherwise. Note that  $\Gamma^{(n)}$  has the symmetries of the square, so  $\gamma_{ij}^{(n)} = \gamma_{i, n-j+1}^{(n)} = \gamma_{n-i+1, j}^{(n)}$  for  $i, j \leq n$ . Moreover, every inner row or column is strictly increasing up to the “center positions” (see Example 5.2 below).

**Proposition 5.1.** *The entry  $\gamma_{ij}^{(n)}$  of the matrix  $\Gamma^{(n)}$  equals  $\#(A^{(i,j)})$  where  $A^{(i,j)}$  is the  $n \times n$  matrix with 0 in position  $(i, j)$  and otherwise all entries are 1. Moreover, for  $n \leq 16$ , we have  $\max_{i,j} \gamma_{ij}^{(n)} = (n-1)^2(n-2)!$  and this maximum is attained in all corner positions. For  $n \geq 17$ , we have*

$$\max_{i,j} \gamma_{ij}^{(n)} = \lfloor (n+1)/2 \rfloor^2 \cdot \lceil (n+1)/2 \rceil^2 \cdot (n-3)!$$

and this maximum is attained in the “center” for  $i, j \in \{\lfloor (n+1)/2 \rfloor, \lceil (n+1)/2 \rceil\}$ .

**Proof.** Any  $A$ -interval matrix is obtained by placing a 1 in each of the intervals incident to position  $(i, j)$ , striking out all rows and columns containing such a 1, and augmenting by ones in a permutation pattern in the remaining submatrix. Let  $1 < i, j < n$ . Then  $A^{(i,j)}$  has four intervals incident to  $(i, j)$ , and

$$(i-1)(n-i)(j-1)(n-j)$$

different choices of the ones covering these intervals. For each such choice the remaining permutation submatrix to be selected is (square and) of order  $n-3$ , so there are  $(n-3)!$  such choices. This gives  $\gamma_{ij}$   $A$ -interval matrices. Next, consider  $A^{(1,1)}$ . Then there are  $(n-1)^2$  different ways of covering the intervals in the first row and the first column, and we are left with a choice of a permutation matrix of order  $n-2$ . So the number of  $A^{(1,1)}$ -interval matrices is  $\gamma_{11}$ . For the remaining three corner positions  $(1, n)$ ,  $(n, 1)$  and  $(n, n)$  we obtain the same expression. For all other positions  $(i, j)$ , i.e., on the boundary, but not in the corner, the number of horizontal and the number of vertical intervals is different, so then no  $A$ -interval matrix exists.

The maximum of the entries follow from properties of  $\Gamma^{(n)}$  mentioned above, and a simple computation comparing  $\gamma_{11}^{(n)}$  and  $\gamma_{ii}^{(n)}$  for  $i = \lfloor (n+1)/2 \rfloor$  shows that the first of these numbers is larger precisely when  $n \leq 16$ .  $\square$

Note that for  $n$  suitably large we have  $\max_{i,j} \gamma_{ij}^{(n)} > n!$ , which shows that adding a zero in  $A$  may increase the number of  $A$ -interval matrices.

**Example 5.2.** The matrices  $\Gamma^{(n)}$  for  $n = 4, 5, 6$  are shown below.

$$\Gamma^{(4)} = \begin{bmatrix} 18 & 0 & 0 & 18 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 4 & 0 \\ 18 & 0 & 0 & 18 \end{bmatrix}, \quad \Gamma^{(5)} = \begin{bmatrix} 96 & 0 & 0 & 0 & 96 \\ 0 & 18 & 24 & 18 & 0 \\ 0 & 24 & 32 & 24 & 0 \\ 0 & 18 & 24 & 18 & 0 \\ 96 & 0 & 0 & 0 & 96 \end{bmatrix},$$

$$\Gamma^{(6)} = \begin{bmatrix} 600 & 0 & 0 & 0 & 0 & 600 \\ 0 & 96 & 144 & 144 & 96 & 0 \\ 0 & 144 & 216 & 216 & 144 & 0 \\ 0 & 144 & 216 & 216 & 144 & 0 \\ 0 & 96 & 144 & 144 & 96 & 0 \\ 600 & 0 & 0 & 0 & 0 & 600 \end{bmatrix}.$$

For  $n = 17$  we compute  $\gamma_{11}^{(n)} = 334,764,638,208,000$  and  $\gamma_{99}^{(17)} = 357,082,280,755,200$  which is the maximum in  $\Gamma^{(17)}$ .  $\square$

Next, we discuss the situation with two zeros in  $A$ . Then there are many cases to consider, and rather than going through all these cases (with repeated arguments), we focus on maximizing  $\#(A)$  when  $A$  is restricted to have two zeros. It simplifies this analysis to consider the case when  $n$  is large. Like in the proof of Proposition 5.1, one can see that  $\#(A)$  is maximized when the two zeros are placed in the “middle” of  $A$ . The reason is that the ones in the four intervals defined by an interior position  $(i, j)$  can be placed approximately in  $(i-1)(n-i) \times (j-1)(n-j)$  different positions, and each of these two quadratic polynomials, in  $i$  and  $j$  respectively, is maximized for the value  $(n+1)/2$ . Also, for  $n$  large enough, placing a zero on the boundary of  $A$  will not maximize  $\#(A)$ . If the two zeros are in the middle, and adjacent in a row or a column,  $\#(A)$  is roughly (we use the symbol  $\sim$  to indicate order notation)

$$\#(A) \sim (n/2)^3 \times (n/2)^3 \times (n-5)! \sim n^6(n-5)! \sim n^7(n-6)!.$$

Alternatively, when the two zeros are in different rows and columns, but diagonally adjacent, we get

$$\#(A) \sim (n/2)^4 \times (n/2)^4 \times (n-6)! \sim n^8(n-6)!.$$

Thus, when  $n$  is suitably large,  $\#(A)$  is maximized by placing the two zeros in the middle and diagonally adjacent positions.

## 6 Interval-intersection bipartite graphs

With an  $m \times n$   $(0, *)$ -matrix  $A$ , we have associated a bipartite graph  $\mathcal{G}_A = (\mathcal{I}, \mathcal{J}, E)$ , whose color classes are the set  $\mathcal{I}$  of horizontal intervals and the set  $\mathcal{J}$  of vertical

intervals, where there is an edge between a horizontal interval  $I \in \mathcal{I}$  and a vertical interval  $J \in \mathcal{J}$  whenever  $I \cap J$  is nonempty. We call such a bipartite graph, and any bipartite graph  $G$  isomorphic to it, an *interval-intersection bipartite graph*. The matrix  $A$  is an *interval-intersection matrix representation* of  $G$ .

We now recall a related concept that was studied in [6]. A *grid intersection graph* is a bipartite graph whose color classes correspond to a set of horizontal and vertical intervals, respectively, in the plane, and where an edge indicates that two such intervals intersect. Here it is assumed that the horizontal intervals are pairwise disjoint, and so are the vertical intervals. These intervals can be regarded as consecutive integer lattice points in a row or column. Thus, every interval-intersection bipartite graph, as defined above, is clearly a grid intersection graph (the  $*$ 's of the matrix  $A$  determine a collection of horizontal and vertical intervals of a grid graph) but, as discussed below, the converse is not true.

All grid intersection graphs  $G$  arise as follows. Let  $N$  be an integer equal to the number of integer lattice points defining  $G$  where these lattice points are listed in some order. Let the number of horizontal intervals be  $h$  and the number of vertical intervals be  $v$ . Let  $A_1$  be the  $h \times N$   $(0, *)$ -matrix whose  $*$ 's in the rows correspond to the horizontal intervals. Similarly, let  $A_2$  be the  $N \times v$   $(0, *)$ -matrix whose columns correspond to the vertical intervals. Replacing in  $A_1$  and  $A_2$ , the  $*$ 's with 1's, the Hadamard-Schur (entrywise) product  $A_1 \circ A_2$  gives a biadjacency matrix of  $G$ . Thus grid intersection graphs are determined by two matrices defining, respectively, the horizontal intervals and the vertical intervals, while the more restrictive interval intersection graphs are determined by only one matrix which simultaneously defines both the horizontal intervals and the vertical intervals.

**Example 6.1.** Let  $m = n = 3$  and consider

$$A_1 = \begin{bmatrix} * & * & 0 \\ 0 & * & * \\ * & * & * \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} * & * & 0 \\ * & * & * \\ * & 0 & * \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then

$$A_1 \circ A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Thus the grid intersection graph determined by  $A_1$  and  $A_2$  is a cycle  $C_6$  of length 6 with biadjacency matrix  $A_1 \circ A_2$ . A geometric picture of the intervals is given in Figure 1. The cycle  $C_6$  is not an interval-intersection bipartite graph. This is straightforward to see: Suppose there is a  $(0, *)$ -matrix  $A$  with  $\mathcal{G}_A$  equal to  $C_6$ . Since each  $*$  determines a horizontal and a vertical interval, each interval (there are three horizontal and three vertical) in  $A$  has exactly two  $*$ 's and these  $*$ 's are consecutive

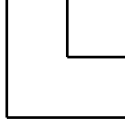


Figure 1: Intervals with grid intersection graph  $C_6$ .

with a horizontal and vertical interval having at most one  $*$  in common. Since there are only three of each of the two types of intervals, a row or column of  $A$  cannot determine two intervals. But this is impossible since the  $*$ 's are consecutive. More generally, each cycle  $C_k$  of even length  $k \geq 6$  is a grid intersection graph but not an interval intersection graph.  $\square$

A  $(0, 1)$ -matrix is  $B$  called *cross-free* [6] if it does not contain a  $3 \times 3$  submatrix of the form

$$\begin{bmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{bmatrix} \quad (5)$$

where the unspecified entries are either 0 or 1. If it is possible to permute rows and columns of a  $(0, 1)$ -matrix  $B$  (change the orders in which the horizontal and vertical intervals are listed) and get a cross-free matrix, then  $B$  is called *cross-freeable* [6]. The following characterization of grid intersection graphs was shown in [6].

**Theorem 6.2.** [6] *A  $(0, 1)$ -matrix  $B$  is the biadjacency matrix of a grid intersection graph if and only if  $B$  is cross-freeable.*

To see that cross-freeable of the biadjacency matrix  $B$  is a necessary condition for a bipartite graph to be a grid intersection graph, we use our matrix representation  $B = A_1 \circ A_2$ . Thus  $A_1$  contains a  $3 \times 3$  submatrix  $A'_1$  and  $A_2$  contains a  $3 \times 3$  submatrix  $A'_2$  such that

$$A'_1 \circ A'_2 = \begin{bmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{bmatrix}$$

Then  $A'_1$  or  $A'_2$  has this form and the ones in  $A_1$  or  $A_2$  are not consecutive, a contradiction.

As a consequence we get a necessary condition for a bipartite graph to be an interval-intersection bipartite graph.

**Corollary 6.3.** *If  $G$  is an interval-intersection bipartite graph, then its biadjacency matrix is cross-freeable.*

The converse of this statement is clearly not true, as interval-intersection bipartite graphs form a strict subclass of the grid intersection graphs.

Let  $B$  be a  $(0, 1)$ -matrix of size  $m \times n$  with at least one 1 in every row and column. We say that  $B$  has the *block-consecutive-ones* property if the following holds: There are integers  $0 = i_0 < i_1 < \dots < i_k = n$  and  $0 = j_0 < j_1 < \dots < j_l = n$ , and corresponding intervals  $I_s = \{i_{s-1} + 1, \dots, i_s\}$  ( $1 \leq s \leq k$ ) and  $J_t = \{j_{t-1} + 1, \dots, j_t\}$  ( $1 \leq t \leq l$ ) such that, for each  $i \leq m$  and  $j \leq n$ ,

- (i) each  $1 \times |J_t|$  submatrix  $B[i, J_t]$  contains at most one 1 ( $t \leq l$ ), and those  $t$  such that  $B[i, J_t]$  contains a 1 are consecutive;
- (ii) each  $|I_s| \times 1$  submatrix  $B[I_s, j]$  contains at most one 1 ( $s \leq k$ ), and those  $s$  such that  $B[I_s, j]$  contains a 1 are consecutive.

**Example 6.4.** The following matrix  $B$  has the block-consecutive-ones property, with blocks as indicated:

$$B = \left[ \begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right].$$

□

**Lemma 6.5.** *Let  $G$  be an interval-intersection bipartite graph. Then there exists an ordering of the vertices in  $G$  (i.e., the underlying horizontal and vertical intervals) so that the corresponding biadjacency matrix of  $G$  has the block-consecutive-ones property.*

**Proof.** Let  $G$  be the interval-intersection graph  $\mathcal{G}_A$  of a  $(0, 1)$ -matrix  $A$  of size  $m \times n$ . Order the horizontal intervals in  $A$  according to row numbers, i.e., intervals in the first row come first (e.g. ordered according to increasing left end-point), then row 2 etc. Similarly, we order the vertical intervals according to column numbers (intervals in first column come first). Let  $B$  be the corresponding biadjacency matrix.

Then the rows of  $B = [b_{ij}]$  may be partitioned into consecutive intervals  $I_1, I_2, \dots, I_k$  where  $I_s$  corresponds to horizontal intervals in  $A$  that are in the  $s$ 'th nonzero row of  $A$  ( $s \leq k$ ). Similarly, the columns of  $B$  may be partitioned into consecutive intervals  $J_1, J_2, \dots, J_l$  where  $J_t$  corresponds to vertical intervals in  $A$  that are in the  $t$ 'th nonzero column of  $A$  ( $t \leq l$ ). So  $k \leq m$  and  $l \leq n$ .

Consider row  $i$  in  $B$ . Then, for  $t \leq l$ , the submatrix  $B[i, J_t]$  has at most one 1, since a horizontal interval in  $A$  cannot intersect more than one vertical interval in a given column of  $A$ . Moreover, the  $i$ 'th horizontal interval in  $A$  intersects certain vertical intervals in consecutive columns of  $A$ . Similar properties hold for a column  $j$  of  $B$ . So, this shows that  $B$  has the block-consecutive-ones property.  $\square$

**Example 6.4, continued.** The matrix  $B$  in Example 6.4 is the biadjacency matrix of the interval-intersection bipartite graph  $\mathcal{G}_A$  with

$$A = \begin{bmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ * & * & 0 & * \\ * & 0 & * & * \end{bmatrix}$$

where we order horizontal interval rowwise and vertical intervals columnwise.  $\square$

The matrix  $B$  in Example 6.4 cannot be permuted to get a matrix with the block-consecutive-ones property, and therefore the corresponding bipartite graph is not an interval-intersection bipartite graph. More generally, the cycle matrix

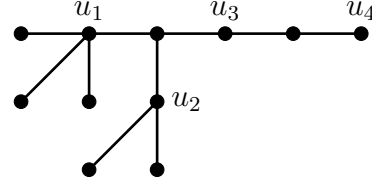
$$\begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 1 & 1 \\ 1 & & & & & 1 \end{bmatrix}$$

can not be permuted to get a matrix with the block-consecutive-ones property. This gives another way to verify this fact already mentioned in Example 6.1.

Many trees are interval-intersection bipartite graphs. An example is shown in Fig. 2, where the vertices  $u_1, u_2, u_3, u_4$  correspond to the four horizontal intervals in the matrix  $A$ , one in each row. However, there are trees which cannot be realized as interval-intersection bipartite graphs; see Theorem 6.7.

Let  $A$  be a  $(0, 1)$ -matrix. Define the associated graph  $G_A$  as the graph with vertex set equal to the support of  $A$ , so the positions of the 1's in  $A$ , and with edges corresponding to adjacent positions in the matrix, i.e., they are next to each other in the same row or column. Thus,  $G_A$  is a subgraph of the full  $m \times n$  grid graph, which is  $G_{J_{m,n}}$ .

**Lemma 6.6.** *Let  $A$  be a  $(0, 1)$ -matrix. Then the interval-intersection bipartite graph  $\mathcal{G}_A$  is acyclic if and only if  $G_A$  is acyclic.*



$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Figure 2: A tree  $T$ , and a matrix  $A$  with  $\mathcal{G}_A$  isomorphic to  $T$ .

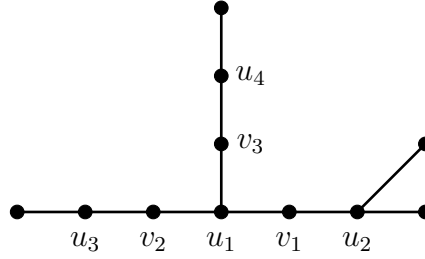


Figure 3: A superstar.

**Proof.** A cycle in  $G_A$  corresponds to some consecutive neighbor vertices in the same row, and therefore a (part of a) horizontal interval in  $A$ , followed by some consecutive neighbor vertices in the same column, and therefore a (part of a) vertical interval in  $A$ , and so on. Thus  $G_A$  has a cycle if and only if  $\mathcal{G}_A$  has a cycle.  $\square$

It follows from Lemma 6.6 that in order to find the interval-intersection bipartite graphs which are trees one must determine trees in a  $m \times n$  grid graph, but these trees satisfy additional constraints.

Let  $N_G(v)$  denote the neighbors of a vertex  $v$  in a graph  $G$ . Let  $G$  be a bipartite graph with a vertex  $u$  such that (i)  $d(u) \geq 3$ , (ii)  $d(v) = 2$  for each  $v \in N_G(u)$ , and (iii)  $d(w) \geq 2$  for each  $w \in N_G(v) \setminus \{u\}$  where  $v \in N_G(u)$ . We then say that  $G$  contains a *superstar* with center  $u$ .

**Theorem 6.7.** *Let  $T$  be a tree, and assume that there exist a  $(0, *)$ -matrix  $A$  such that  $\mathcal{G}_A$  is isomorphic to  $T$ . Then  $T$  has no superstar. In particular, a superstar is not an interval-intersection bipartite graph.*

**Proof.** Let  $I(w)$  denote the interval in  $A$  corresponding to a vertex  $w$  in  $G$ . Assume  $T$  has a superstar with center  $u_1$ , and let the corresponding vertices of the superstar be labeled as in Fig. 3. Assume that the interval  $I(u_1)$  is horizontal and lies in row  $i$  of  $A$ ; the opposite case, when it corresponds to a vertical interval is treated similarly. The interval  $I(u_1)$  covers at least 3 consecutive columns in  $A$ , and intersects corresponding vertical intervals, as  $d(u_3) \geq 3$ . Let  $j_1, j_1 + 1, \dots, j_2$  where  $j_2 \geq j_1 + 2$  be these consecutive columns. One of the vertical intervals intersecting  $I(u_1)$ , say  $I(v_1)$ , must intersect  $I(u_1)$  in a column  $j$  with  $j_1 < j < j_2$  (an “internal” column). By assumption,  $d(v_1) = 2$ , so we may assume  $I(v_1) = \{(i, j), (i - 1, j)\}$  (the case  $I(v_1) = \{(i, j), (i + 1, j)\}$  is similar). Now, the other neighbor of  $v_1$  than  $u_1$  is  $u_2$ , and since  $d(u_2) \geq 2$ , the horizontal interval  $I(u_2)$  contains  $(i - 1, j)$  and either  $(i - 1, j - 1)$  or  $(i - 1, j + 1)$ . In either case  $A$  contains a  $2 \times 2$  matrix of all  $*$ ’s, and with consecutive rows and consecutive columns. Therefore  $G_A$  has a cycle, and, see Lemma 6.6,  $\mathcal{G}_A$  has a cycle, contradiction that it is a tree. Therefore,  $T$  cannot contain a superstar.  $\square$

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