Statistical solutions and Onsager's conjecture

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Abstract

We prove a version of Onsager's conjecture on the conservation of energy for the incompressible Euler equations in the context of statistical solutions, as introduced recently by Fjordholm et al. [12]. As a byproduct, we also obtain an alternative proof for the conservative direction of Onsager's conjecture for weak solutions, under a weaker Besov-type regularity assumption than previously known.

Dedicated to Edriss S. Titi on the occasion of his 60th birthday.

1 Introduction

We consider the d-dimensional incompressible Euler equations: Find a function $v = (v^1, \dots, v^d) : \mathbb{R}_+ \times D \to \mathbb{R}^d$ and a function $p : \mathbb{R}_+ \times D \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t v + \sum_k \partial_{x^k}(v v^k) + \nabla p = 0 \qquad & x \in D,\ t > 0 \\ \nabla \cdot v = 0 \qquad & x \in D,\ t > 0 \\ v(0,x) = v_0(x) \qquad & x \in D. \end{aligned} \tag{1.1}$$

Here and below, the summation limits, when not specified, are always from $k = 1$ to $k = d$. The initial data v_0 is assumed to lie in $L^2(D; \mathbb{R}^d)$. The spatial parameter x takes values in a set D, which we will take as either $\mathbb{R}^d$ or the (d-dimensional) torus $\mathbb{T}^d$ for simplicity[1]. The temporal domain is $[0, T]$ for some $T > 0$.

By a *solution of the Euler equations* we will mean a weak solution of (1.1), i.e. a function $v \in L^2_{loc}\big([0,T] \times D; \mathbb{R}^d\big)$ such that

$$\int_{\mathbb{R}_+} \int_D v \partial_t \varphi + \sum_k v v^k \partial_{x^k} \varphi + p \nabla \varphi \, dx dt + \int_D v_0(x) \varphi(0,x) \, dx = 0 \tag{1.2}$$

for all $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^d)$, as well as satisfying the divergence free condition in the sense of distributions.

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[1] On domains with boundaries, one can show the *local* version of the energy equality with almost no further effort, but in order to deduce from this also the *global* conservation of energy one requires some assumption of continuity at the boundary in addition to one of the usual Besov-type regularity assumptions. For a first result in this direction see [1].

Assume that v is a smooth solution of (1.1). Multiplying the first equation of (1.1) by v gives

$$\partial_t \frac{|v|^2}{2} + \sum_k \partial_{x^k} \left(v_k \left(\frac{|v|^2}{2} + p \right) \right) = 0. \tag{1.3}$$

Integrating this *local energy identity* over $x \in D$ and $t \in [0,T]$, we obtain the *global energy identity*

$$\int_D \frac{|v(T)|^2}{2}\,dx = \int_D \frac{|v_0|^2}{2}\,dx. \tag{1.4}$$

In 1949 Lars Onsager conjectured [21] that if v is Hölder continuous with exponent greater than 1/3 then the above calculations can be made rigorous:

> In fact it is possible to show that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form (...) for any order n greater than 1/3; otherwise the energy is conserved.[2]

Constantin, E and Titi [4] showed that the conjecture is true not just for solutions in $C^{0,\alpha}$, but for any solution in the Besov space $B_3^{\alpha,\infty}$ with $\alpha > \frac{1}{3}$. The proof uses a regularization of (1.1) together with some basic estimates in Besov spaces. Independently, Eyink [9] proved the conjecture in Fourier space under a stronger assumption. Duchon and Robert [8] employed the regularization technique from [4] to quantify the *anomalous energy dissipation* $\mathcal{E}(u)$ of an arbitrary solution u – the amount by which equality in (1.3) fails to hold. The sharp exponent $\alpha = \frac{1}{3}$ was shown to suffice for energy conservation at the cost of a slightly stronger summability assumption for the Besov space in [3], see also [22]. More recently, related results were given for density-dependent Euler models in [20, 11, 7]. On the other hand, there is the question whether energy *can* be dissipated for any Hölder or Besov regularity below the exponent $\frac{1}{3}$. This difficult problem was solved only very recently [18, 2].

When describing turbulent flows it is common not to consider individual realizations of the flow (as in the description above), but rather as an *ensemble* of velocity fields [17]. In his papers [15, 16], C. Foiaş attempted to make this approach rigorous by introducing so-called *statistical solutions* of the incompressible Navier–Stokes equations. A statistical solution is a time-parametrized probability measure μ_t on a Sobolev space $N \subset H^1$, satisfying the Navier–Stokes equations in an averaged sense. The more recent paper [12] defines statistical solutions of hyperbolic conservation laws as probability measures μ_t on some Lebesgue space L^p, satisfying the PDE in a certain sense. The authors show that a statistical solution μ_t (indeed, any probability measure on L^p) can be viewed equivalently as a *correlation measure*, a hierarchy $\boldsymbol{\nu} = (\nu^1, \nu^2, \dots)$ in which each element ν^k gives the joint probability distribution of the unknown at any choice of k spatial points $x_1, \dots, x_k \in D$. The evolution equation for μ_t is most naturally described in terms of its corresponding correlation measure, yielding an infinite hierarchy of evolution equations. In particular, the equation for the one-point distribution ν_x^1 coincides with DiPerna's definition of *measure-valued solutions*, see [5, 6, 23].

The present paper serves several purposes. First, in Section 2 we prove Onsager's conjecture for weak solutions under a more permissive Besov-type criterion than the previously known ones. Second, in Section 3.2 we provide a definition of statistical solutions of the incompressible Euler equations (1.1). Third, in Section 3.3 we prove that statistical solutions conserve energy under a Besov-type regularity condition analogous to the one in Section

[2] As a historical sidenote, Onsager wrote down a formal proof of his own conjecture – close in spirit to the later proof by Duchon and Robert – which he never published; see [10]. Thus, his rather cryptic "it is possible to show" should in fact be interpreted literally rather than hypothetically.

2. The proof for statistical solutions closely follows the one for weak solutions presented in Section 2. The proof is close in spirit to the regularization technique in [8], but is also reminiscent of Kruzhkov's *doubling of variables* technique [19]. We end by comparing the concepts of Besov regularity of functions and of correlation measures in Section 4.

Our main result (Theorem 3.9) says that an uncertain fluid flow – realized as a statistical solution of (1.1) – conserves energy provided it has more than $\frac{1}{3}$ of a derivative. The ease by which statements about regularity can be formulated using correlation measures indicates to us that statistical solutions – and not measure-valued solutions – are the right notion of solutions for uncertain (or *unsteady*) fluid flows. We refer the interested reader to the upcoming paper [14] where we discuss statistical solutions of the Navier–Stokes and Euler equations and the connection to Kolmogorov's theory of turbulence (cf. also Remark 3.11), and we prove that statistical solutions of Navier–Stokes converge to statistical solutions of Euler in the vanishing viscosity limit.

2 Onsager's conjecture

We first write the Euler equation in component form:

$$\partial_t v^i + \sum_k \partial_{x^k}(v^i v^k) + \partial_{x^i} p = 0. \tag{2.1}$$

It is straightforward to see that if (v,p) is any smooth solution of the above equation, then the function $(t,x,y) \mapsto v^i(t,x)v^j(t,y)$ satisfies

$$\begin{aligned}\partial_t \left(v^i(x)v^j(y)\right) + \sum_k \partial_{x^k}\left(v^i(x)v^k(x)v^j(y)\right) + \sum_k \partial_{y^k}\left(v^i(x)v^k(y)v^j(y)\right) \\ + \partial_{x^i}\left(p(x)v^j(y)\right) + \partial_{y^j}\left(v^i(x)p(y)\right) = 0\end{aligned} \tag{2.2}$$

(where we suppress the dependence on t). The proof of this claim consists of evaluating (2.1) at x and at y, multiplying the former by $v^j(y)$ and the latter by $v^i(x)$, and then summing the two. With only a bit more work, one shows that if v is any *weak* solution of the Euler equation then (2.2) is satisfied in the distributional sense.

Lemma 2.1. *If $v : (0,T)\times D \to \mathbb{R}^d$ is a weak solution of the Euler equation then*

$$\begin{aligned}\int_0^T \int_D \int_D v^i(x)v^j(y)\partial_t\varphi + \sum_k v^i(x)v^k(x)v^j(y)\partial_{x^k}\varphi + \sum_k v^i(x)v^k(y)v^j(y)\partial_{y^k}\varphi \\ + p(x)v^j(y)\partial_{x^i}\varphi + v^i(x)p(y)\partial_{y^j}\varphi\,dxdydt + \int_D\int_D v_0^i(x)v_0^j(y)\varphi(0,x,y)\,dxdy = 0\end{aligned} \tag{2.3}$$

for every test function $\varphi = \varphi(t,x,y) \in C_c^\infty([0,T)\times D^2)$ and every $i,j = 1,\dots,d$.

If the local energy inequality (1.3) is to hold for a weak solution, then we need in addition $v \in L^3_{loc}\big([0,T)\times D;\mathbb{R}^d\big)$ and $p \in L^{3/2}_{loc}\big([0,T)\times D;\mathbb{R}^d\big)$ for the equality to make sense distributionally. The Besov regularity of v required to show the equality entails in particular $v \in L^3$, and this in turn implies $p \in L^{3/2}$. Indeed, taking the divergence of the momentum equation in (1.1) we obtain

$$-\Delta p = \sum_{k,l} \partial^2_{x^k,x^l}(v^l v^k), \tag{2.4}$$

and thus $v \in L^3$ implies $p \in L^{3/2}$ by standard elliptic theory.

Theorem 2.2. *Let $v \in L^\infty\big((0,T);L^2\big(D;\mathbb{R}^d\big)\big) \cap L^3((0,T)\times D;\mathbb{R}^d)$ be a weak solution of the Euler equations, and accordingly $p \in L^{3/2}((0,T)\times D;\mathbb{R}^d)$. Assume that*

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^T \int_D \fint_{B_\varepsilon(x)} \big|v(x)-v(y)\big|^3 \,dydxdt = 0. \tag{2.5}$$

Then (v,p) satisfies the local energy identity (1.3).

Remark 2.3. Condition (2.5) is slightly less restrictive than previously known ones.[3] The critical Besov condition in [3], which has been the weakest known, reads (in physical space, cf. also [22])

$$\lim_{|y|\to 0} \int_0^T \int_D \frac{|v(x)-v(x-y)|^3}{|y|} dxdt = 0.$$

Since the average of a function on some set is bounded by its supremum on the same set, our condition (2.5) is even weaker. In fact, the condition can be further weakened, as the average need not be taken over a ball. Indeed, from the following proof it is clear that in (2.5), the average over $B_\varepsilon(x)$ can be replaced by an average over

$$A_\varepsilon := \{x+\varepsilon y : y \in A\},$$

where $A \subset D$ is an arbitrary bounded measurable set of positive measure.

Proof of Theorem 2.2. Set $j = i$ in (2.3) and sum over i. We can then write the resulting identity as

$$\begin{aligned}\int_0^T \int_D \int_D v(x)\cdot v(y)\partial_t\varphi + v(x)\cdot v(y)\left(v(x)\cdot\nabla_x\varphi + v(y)\cdot\nabla_y\varphi\right) + p(x)v(y)\cdot\nabla_x\varphi \\ + p(y)v(x)\cdot\nabla_y\varphi\,dxdydt + \int_D\int_D v_0(x)\cdot v_0(y)\varphi(0,x,y)\,dxdy = 0.\end{aligned} \tag{2.6}$$

Fix a number $\varepsilon > 0$. We choose now the test function $\varphi(t,x,y) = \rho_\varepsilon(x-y)\psi(t,x)$, where $\rho_\varepsilon(z) = \varepsilon^{-d}\rho(\varepsilon^{-1}z)$ for a nonnegative, rotationally symmetric mollifier $\rho \in C_c^\infty(D)$ with unit mass and support in $B_0(1)$ (the unit ball in $\mathbb{R}^d$) and $\psi \in C_c^\infty((0,T)\times D)$. Then

$$\partial_t\varphi = \rho_\varepsilon\partial_t\psi, \qquad \nabla_x\varphi = \psi\nabla\rho_\varepsilon + \rho_\varepsilon\nabla\psi, \qquad \nabla_y\varphi = -\psi\nabla\rho_\varepsilon.$$

Continuing from (2.6), we now have

$$\begin{aligned}\int_0^T \int_D\int_D v(x)\cdot v(y)\rho_\varepsilon\partial_t\psi + v(x)\cdot v(y)\big(v(x)-v(y)\big)\cdot\nabla\rho_\varepsilon\psi \\ + v(x)\cdot v(y)v(x)\cdot\nabla\psi\rho_\varepsilon + \big(p(x)v(y)-v(x)p(y)\big)\cdot\nabla\rho_\varepsilon\psi \\ + p(x)v(y)\cdot\nabla\psi\rho_\varepsilon\,dxdydt = 0.\end{aligned} \tag{2.7}$$

[3] We thank the anonymous referee for this observation.

Making the change of variables $z = x - y$ gives

$$
\begin{aligned}
&\int_0^T \int_D \int_D \partial_t \psi(t,x)\rho_\varepsilon(z) v(x)\cdot v(x-z)\,dxdzdt \\
&+\int_0^T \int_D \int_D \psi(t,x)\nabla\rho_\varepsilon(z)\cdot \big(v(x)-v(x-z)\big)\big(v(x)\cdot v(x-z)\big)\,dxdzdt \\
&+\int_0^T \int_D \int_D \nabla\psi(t,x)\rho_\varepsilon(z)\cdot v(x)\big(v(x)\cdot v(x-z)\big)\,dxdzdt \\
&+\int_0^T \int_D \int_D \psi(t,x)\nabla\rho_\varepsilon(z)\cdot \big(p(x)v(x-z)-v(x)p(x-z)\big)\,dxdzdt \\
&+\int_0^T \int_D \int_D \nabla\psi(t,x)\rho_\varepsilon(z)\cdot v(x-z)p(x)\,dxdzdt = 0.
\end{aligned}
\tag{2.8}
$$

Decompose the above into a sum of five terms $\mathcal{A}_1 + \cdots + \mathcal{A}_5$. By applying the Lebesgue differentiation theorem, it is easy to see that

$$
\begin{aligned}
\mathcal{A}_1 &\to \int_0^T \int_D |v(t,x)|^2 \partial_t \psi(t,x)\,dxdt, \\
\mathcal{A}_3 &\to \int_0^T \int_D |v(t,x)|^2 v(t,x)\cdot\nabla\psi(t,x)\,dxdt
\end{aligned}
$$

as $\varepsilon \to 0$. For $\mathcal{A}_2$ we can write $\mathcal{A}_2 = \mathcal{A}_{2,1} + \mathcal{A}_{2,2}$, where

$$
\begin{aligned}
\mathcal{A}_{2,1} &= -\frac{1}{2}\int_0^T \int_D \int_D \psi(t,x)\nabla\rho_\varepsilon(z)\cdot\big(v(x)-v(x-z)\big)\big|v(x)-v(x-z)\big|^2\,dxdzdt, \\
\mathcal{A}_{2,2} &= \frac{1}{2}\int_0^T \int_D \int_D \psi(t,x)\nabla\rho_\varepsilon(z)\cdot\big(v(x)-v(x-z)\big)\left(|v(x)|^2+|v(x-z)|^2\right)\,dxdzdt.
\end{aligned}
$$

The first term can be bounded by

$$
\begin{aligned}
&\frac{\|\psi\|_\infty}{2}\int_0^T \int_D \int_D |\nabla\rho_\varepsilon(z)|\big|v(x)-v(x-z)\big|^3\,dzdxdt \\
&\qquad \leqslant C\frac{1}{\varepsilon}\int_0^T \int_D \fint_{B_\varepsilon(0)} \big|v(x)-v(x-z)\big|^3\,dzdxdt
\end{aligned}
$$

the inequality following from the fact that $\|\nabla\rho_\varepsilon\|_{L^1(D)} \leqslant C\varepsilon^{-1}$. By the Besov regularity assumption (2.5), the above vanishes along a subsequence $\varepsilon' \to 0$. In the second term $\mathcal{A}_{2,2}$ we make the change of variables $x \mapsto x+z$ in the term $v(x-z)$ and then the change of variables $z \mapsto -z$ to obtain

$$
\mathcal{A}_{2,2} = \frac{1}{2}\int_0^T \int_D \int_D \nabla\rho_\varepsilon(z)\cdot|v(x)|^2\big(v(x)-v(x-z)\big)\big(\psi(t,x)+\psi(t,x-z)\big)\,dxdzdt.
$$

Writing $\psi\nabla\rho_\varepsilon = \nabla(\psi\rho_\varepsilon) - \rho_\varepsilon\nabla\psi$ and using the divergence constraint, we obtain

$$
\mathcal{A}_{2,2} = \frac{1}{2}\int_0^T \int_D \int_D \rho_\varepsilon(z)|v(x)|^2\nabla\psi(t,x-z)\cdot\big(v(x)-v(x-z)\big)\,dxdzdt.
$$

The Lebesgue differentiation theorem now implies that $\mathcal{A}_{2,2} \to 0$ as $\varepsilon \to 0$.

As for $\mathcal{A}_3$, it is easy to see that

$$\mathcal{A}_5 \to \int_0^T \int_D p(t,x)v(t,x)\cdot\nabla\psi(t,x)\,dxdt.$$

We claim that $\mathcal{A}_4$ has the same limit as $\mathcal{A}_5$. The change of variables $x \mapsto x+z$ in the second term gives

$$\mathcal{A}_4 = \int_0^T \int_D \int_D p(x)\nabla\rho_\varepsilon(z)\cdot\big(\psi(t,x)v(x-z) - \psi(t,x+z)v(x+z)\big)\,dxdzdt.$$

The divergence constraint implies that the first term is zero, while the second term gives

$$\mathcal{A}_4 = \int_0^T \int_D \int_D p(x)\rho_\varepsilon(z)\nabla\psi(t,x+z)\cdot v(x+z)\,dxdzdt$$

which converges to $\int_0^T \int_D p(x)\nabla\psi\cdot v\,dxdt$. Summing up all the terms, we conclude that in the limit $\varepsilon \to 0$ we obtain the local energy identity (1.3) in distributional form. □

3 Statistical solutions of the Euler equations

In this section we prove that statistical solutions of Euler's equation are energy conservative under a Besov-type regularity assumption. In Section 3.1 we introduce the necessary technical machinery, in Section 3.2 we define statistical solutions of Euler's equation and in Section 3.3 we carry out the proof of energy conservation. The proof closely follows the proof of energy conservation for weak solutions in Section 2. In particular, there are direct analogues of the weak formulation(s), the divergence constraint, the Besov regularity assumption, and the Lebesgue differentiation theorem.

3.1 Correlation measures

Definition 3.1. Let $d, N \in \mathbb{N}$, $q \in [1,\infty)$ and let $D \subset \mathbb{R}^d$ be an open set (the "space domain") and denote $\mathbf{U} = \mathbb{R}^N$ ("phase space"). A *correlation measure from D to $\mathbf{U}$* is a collection $\boldsymbol{\nu} = (\nu^1, \nu^2, \dots)$ of maps satisfying:

(i) ν^k is a Young measure from D^k to $\mathbf{U}^k$.

(ii) Symmetry: if σ is a permutation of $\{1,\dots,k\}$ and $f \in C_0(\mathbf{U}^k)$ then $\langle \nu^k_{\sigma(x)}, f(\sigma(\xi))\rangle = \langle \nu^k_x, f(\xi)\rangle$ for a.e. $x \in D^k$.

(iii) Consistency: If $f \in C_b(\mathbf{U}^k)$ is of the form $f(\xi_1,\dots,\xi_k) = g(\xi_1,\dots,\xi_{k-1})$ for some $g \in C_0(\mathbf{U}^{k-1})$, then $\langle \nu^k_{x_1,\dots,x_k}, f\rangle = \langle \nu^{k-1}_{x_1,\dots,x_{k-1}}, g\rangle$ for almost every $(x_1,\dots,x_k) \in D^k$.

(iv) L^q integrability:

$$\int_D \langle \nu^1_x, |\xi|^q\rangle\,dx < \infty. \tag{3.1}$$

(v) Diagonal continuity (DC): $\lim_{\varepsilon\to 0} d^q_\varepsilon(\nu^2) = 0$, where

$$d^q_\varepsilon(\nu^2) := \left(\int_D \fint_{B_\varepsilon(x)} \langle \nu^2_{x,y}, |\xi_1-\xi_2|^q\rangle\,dydx\right)^{1/q}. \tag{3.2}$$

We denote the set of all correlation measures by $\mathcal{L}^q(D,\mathbf{U})$.

Remark 3.2. The "modulus of continuity" $d^q_\varepsilon(\nu^2)$ is bounded irrespective of $\varepsilon>0$, due to the L^q bound. Indeed,

$$\begin{aligned}
d^q_\varepsilon(\nu^2) &= \left(\int_D \fint_{B_\varepsilon(x)} \left\langle \nu^2_{x,y},\, |\xi_1-\xi_2|^q\right\rangle dydx\right)^{1/q} \\
&\leqslant \left(\int_D \fint_{B_\varepsilon(x)} \left\langle \nu^2_{x,y},\, |\xi_1|^q\right\rangle dydx\right)^{1/q} + \left(\int_D \fint_{B_\varepsilon(x)} \left\langle \nu^2_{x,y},\, |\xi_2|^q\right\rangle dydx\right)^{1/q} \\
&= 2\left(\int_D \left\langle \nu^1_x,\, |\xi|^q\right\rangle dx\right)^{1/q} < \infty,
\end{aligned}$$

where we have used Minkowski's inequality and then the consistency requirement.

Remark 3.3. An example of a correlation measure is $\nu^k_{x_1,\dots,x_k} = \delta_{u(x_1)}\otimes\cdots\otimes\delta_{u(x_k)}$, where $x_1,\dots,x_k\in D$, $k\in\mathbb{N}$, $u\in L^q(D,\mathbb{R}^N)$ is a measurable function and δ_v is the Dirac measure centered at $v\in\mathbb{R}^N$. The symmetry and consistency conditions *(ii)*, *(iii)* follow immediately, and the L^q integrability condition *(iv)* asserts that $u\in L^q(D,\mathbb{R}^N)$. Moreover, the modulus of continuity d^q_ε in condition *(v)* is

$$d^q_\varepsilon(\nu^2) = \left(\int_D \fint_{B_\varepsilon(x)} |u(x)-u(y)|^q\, dydx\right)^{1/q},$$

which vanishes as $\varepsilon\to0$ due to the Lebesgue differentiation theorem. Thus, diagonal continuity is the assertion that the Lebesgue differentiation theorem – which automatically holds for L^q functions – also holds for $\boldsymbol{\nu}$. Correlation measures which are concentrated on a single function u are called *atomic*.

Definition 3.4. Let $d,N\in\mathbb{N}$ and let $D\subset\mathbb{R}^d$ be an open set (the "space domain"), let $\mathcal{T}\subset\mathbb{R}$ be an interval (the "time domain") and denote $\mathbf{U}=\mathbb{R}^N$ ("phase space"). A *time-dependent correlation measure from $\mathcal{T}\times D$ to $\mathbf{U}$* is a collection $\boldsymbol{\nu}=(\nu^1,\nu^2,\dots)$ of maps satisfying:

(i) ν^k is a Young measure from $\mathcal{T}\times D^k$ to $\mathbf{U}^k$.

(ii) Symmetry: if σ is a permutation of $\{1,\dots,k\}$ and $f\in C_0(\mathbf{U}^k)$ then $\left\langle \nu^k_{t,\sigma(x)},\, f(\sigma(\xi))\right\rangle = \left\langle \nu^k_{t,x},\, f(\xi)\right\rangle$ for a.e. $(t,x)\in\mathcal{T}\times D^k$.

(iii) Consistency: If $f\in C_b(\mathbf{U}^k)$ is of the form $f(\xi_1,\dots,\xi_k)=g(\xi_1,\dots,\xi_{k-1})$ for some $g\in C_0(\mathbf{U}^{k-1})$, then $\left\langle \nu^k_{t,x_1,\dots,x_k},\, f\right\rangle = \left\langle \nu^{k-1}_{t,x_1,\dots,x_{k-1}},\, g\right\rangle$ for almost every $(t,x_1,\dots,x_k)\in\mathcal{T}\times D^k$.

(iv) L^q integrability: There is a $c>0$ such that

$$\int_D \left\langle \nu^1_{t,x},\, |v|^q\right\rangle dx \leqslant c \qquad \text{for a.e. } t\in\mathcal{T}. \tag{3.3}$$

(v) Diagonal continuity (DC): $\lim_{\varepsilon\to0} d^q_\varepsilon(\nu^2)=0$, where

$$d^q_\varepsilon(\nu^2) := \left(\int_0^T \int_D \fint_{B_\varepsilon(x)} \left\langle \nu^2_{t,x,y},\, |\xi_1-\xi_2|^q\right\rangle dydxdt\right)^{1/q}. \tag{3.4}$$

3.2 Statistical solutions

For the following definition, recall that the natural framework to study the local energy (in)equality for the incompressible Euler equations is $v \in L^3_{t,x}$, $p \in L^{3/2}_{t,x}$. Therefore, we need to distinguish the integrability conditions corresponding to the velocity and the pressure, respectively, which leads to the "mixed" integrability and diagonal continuity conditions (3.5) and (3.6). The integration variable of $\nu^k_{t,x}$ will be denoted $\xi = (v^1, \dots, v^d, p) \in \mathbf{U}^k$, with the interpretation of $v^i = (v^i_1, \dots, v^i_k)$ as the i-th component of velocity and of $p = (p_1, \dots, p_k)$ as the scalar pressure at k different spatial points $x = (x_1, \dots, x_k)$. Here, our phase space is $\mathbf{U} = \mathbb{R}^{d+1}$.

Definition 3.5. Let $D \subset \mathbb{R}^d$ be a spatial domain and let $T > 0$. Let $\bar{\boldsymbol{\nu}} \in \mathcal{L}^2(D, \mathbb{R}^d)$ be given initial data. By a *statistical solution* of the incompressible Euler equations, we will mean a time-dependent correlation measure $\boldsymbol{\nu}$ from $[0,T] \times D$ to $\mathbb{R}^{d+1}$, where the integrability condition (3.3) is to be understood as

$$\begin{cases} \int_D \langle \nu^1_{t,x},\, |v|^2 \rangle \, dx \leqslant c & \text{for a.e. } t \in [0,T] \\ \int_0^T \int_D \langle \nu^1_{t,x},\, |v|^3 + |p|^{3/2} \rangle \, dxdt < \infty \end{cases} \tag{3.5}$$

for some $c > 0$, and diagonal continuity is to be understood as

$$\lim_{\varepsilon \to 0} \int_0^T \int_D \fint_{B_\varepsilon(x)} \langle \nu^2_{t,x,y},\, |v_1 - v_2|^2 + |v_1 - v_2|^3 + |p_1 - p_2|^{3/2} \rangle \, dydxdt = 0, \tag{3.6}$$

such that:

(i) For all $k \in \mathbb{N}$, $\nu^k = \nu^k_{t,x_1,\dots,x_k}$ satisfies

$$\begin{aligned} \int_0^T \int_{D^k} \langle \nu^k,\, v_1^{i_1} \cdots v_k^{i_k} \rangle \partial_t \varphi + \sum_{l=1}^k \langle \nu^k,\, v_l v_1^{i_1} \cdots v_k^{i_k} \rangle \cdot \nabla_{x_l} \varphi \\ + \sum_{l=1}^k \langle \nu^k,\, v_1^{i_1} \cdots p_l \cdots v_k^{i_k} \rangle \frac{\partial \varphi}{\partial x_l^{i_l}} \, dxdt = 0 \end{aligned} \tag{3.7}$$

for all $i_1, \dots, i_k = 1, \dots, d$ and for all $\varphi \in C_c^\infty\big((0,T) \times D^k\big)$. (Here we abbreviate $v_1^{i_1} \cdots p_l \cdots v_k^{i_k} = v_1^{i_1} \cdots v_{l-1}^{i_{l-1}} p_l v_{l+1}^{i_{l+1}} \cdots v_k^{i_k}$, the lth component of v being omitted.)

(ii) $\boldsymbol{\nu}$ is divergence-free: For every $\varphi \in C_c^\infty(\mathbb{R}_+ \times D^k)$ and every $\kappa \in C(\mathbf{U}^{k-1})$ for which $\langle \nu^{k-1},\, |\kappa| \rangle < \infty$, we have

$$\int_0^T \int_{D^k} \nabla_{x_k} \varphi(x_k) \cdot \langle \nu^k_{t,x},\, \kappa(v_1, \dots, v_{k-1}) v_k \rangle \, dxdt = 0. \tag{3.8}$$

Remark 3.6. The first two instances of (3.7) are:

$$\int_0^T \int_D \langle \nu^1,\, v^i \rangle \partial_t \varphi + \langle \nu^1,\, v^i v \rangle \cdot \nabla \varphi + \langle \nu^1,\, p \rangle \partial_{x^i} \varphi \, dxdt = 0 \tag{3.9}$$

for all $\varphi \in C_c^\infty\big((0,T) \times D\big)$ and all $i = 1, \dots, k$, and

$$\begin{aligned} \int_0^T \int_D \int_D \langle \nu^2,\, v_1^i v_2^j \rangle \partial_t \varphi + \langle \nu^2,\, v_1^i v_2^j v_1 \rangle \cdot \nabla_x \varphi + \langle \nu^2,\, v_1^i v_2^j v_2 \rangle \cdot \nabla_y \varphi \\ + \langle \nu^2,\, v_2^j p_1 \rangle \partial_{x^i} \varphi + \langle \nu^2,\, v_1^i p_2 \rangle \partial_{y^j} \varphi \, dxdydt = 0 \end{aligned} \tag{3.10}$$

for all $\varphi \in C_c^\infty\big((0,T)\times D^2\big)$ and all $i,j=1,\dots,d$. These are direct analogues of (1.2) and (2.3), respectively, and are the only instances of (3.7) which will be used in the remainder. In particular, the results of this paper still hold if conditions (3.7) are replaced by (3.9) and (3.10).

Remark 3.7. The fact that only the first two marginals ν^1,ν^2 appear in our proof (cf. the previous remark) raises the interesting question of the role of the higher-order marginals $\nu^3,\nu^4,\dots$. As demonstrated in [13, Example 9.1 and Section 9.3], there are simple examples where prescribing only finitely many marginals $\nu^1,\dots\nu^k$ of the initial data yields non-uniqueness of solutions of hyperbolic conservation laws. For this reason we keep all marginals $\nu^1,\nu^2,\dots$ in the definition of statistical solutions, although they are not needed for the results in this paper.

Remark 3.8. By (3.5), all integrals in (3.9) and (3.10) are well-defined.

3.3 Energy conservation for statistical solutions

We are now ready to prove the "energy conservation" part of Onsager's conjecture for statistical solutions. In the same vein as Duchon and Robert [8], we will prove a somewhat stronger result by quantifying the precise energy defect distribution $\mathcal{E}(u)$, and prescribing a sufficient condition that ensures that $\mathcal{E}(u)\equiv 0$.

Theorem 3.9. *Let $\boldsymbol{\nu}$ be a statistical solution of the incompressible Euler equations on $[0,T]\times D$, where either $D=\mathbb{T}^d$ or $D=\mathbb{R}^d$. Let $\rho_\varepsilon(z)=\varepsilon^{-d}\rho(z/\varepsilon)$ be a rotationally symmetric mollifier. Then the distribution $\mathcal{E}(\boldsymbol{\nu})\in\mathcal{D}'(D)$ given by*

$$\mathcal{E}(\boldsymbol{\nu})(\psi) := -\frac{1}{4}\lim_{\varepsilon\to 0}\int_D\int_{B_\varepsilon(x)} \psi(x)\nabla\rho_\varepsilon(z)\cdot\big\langle \nu^2_{x,x-z},\,(v_1-v_2)|v_1-v_2|^2\big\rangle\,dzdx$$

is well-defined and independent of the choice of ρ, and $\boldsymbol{\nu}$ satisfies

$$\partial_t\big\langle \nu^1_{t,x},\,\frac{1}{2}|v|^2\big\rangle + \sum_{k=1}^d \partial_{x^k}\big\langle\nu^1_{t,x},\,\frac{1}{2}|v|^2v^k + v^kp\big\rangle = \mathcal{E}(\boldsymbol{\nu}) \tag{3.11}$$

in the sense of distributions. If $\boldsymbol{\nu}$ satisfies the regularity condition

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T\int_D \fint_{B_\varepsilon(x)} \big\langle \nu^2_{t,x,y},\,|v_1-v_2|^3\big\rangle\,dydxdt = 0 \tag{3.12}$$

then $\mathcal{E}(\boldsymbol{\nu})\equiv 0$.

Remark 3.10. The left-hand side of (3.12) equals $\frac{1}{\varepsilon}d_\varepsilon^3(\nu^2)^3$, cf. (3.4). Whereas the requirement of diagonal continuity merely requires that $d_\varepsilon^3(\nu^2)$ vanishes as $\varepsilon\to 0$, the regularity assumption (3.12) imposes a rate at which it vanishes.

Remark 3.11. In turbulence theory, the Kolmogorov four-fifths law states that in a homogeneous (but not necessarily isotropic) turbulent flow, the left-hand side of (3.11) equals

$$\frac{1}{4}\nabla_z\cdot\big\langle|\delta v(x,z)|^2\delta v(x,z)\big\rangle\Big|_{z=0}, \qquad \delta v(x,z) := v(x)-v(x-z),$$

where the angle brackets denote the expected value over an ensemble of turbulent flows; cf. [17, Section 6.2.5]. It is readily seen that $\mathcal{E}(\boldsymbol{\nu})$ is a distributional version of the above quantity. See also [8, Section 5] and the forthcoming paper [14].

Proof of Theorem 3.9. Set $j = i$ in (3.10) and sum over i:

$$\sum_i \int_0^T \int_D \int_D \langle \nu^2_{x,y},\, v_1^i v_2^i \rangle \partial_t \varphi + \langle \nu^2_{x,y},\, v_1^i v_2^i v_1 \rangle \cdot \nabla_x \varphi + \langle \nu^2_{x,y},\, v_1^i v_2^i v_2 \rangle \cdot \nabla_y \varphi \\ + \langle \nu^2_{x,y},\, v_2^i p_1 \rangle \partial_{x^i} \varphi + \langle \nu^2_{x,y},\, v_1^i p_2 \rangle \partial_{y^i} \varphi \, dxdydt = 0 \tag{3.13}$$

(where we suppress the dependence on t). Fix a number $\varepsilon > 0$. We choose again the test function $\varphi(t,x,y) = \rho_\varepsilon(x-y)\psi(t,x)$, where $\rho_\varepsilon(z) = \varepsilon^{-d}\rho(\varepsilon^{-1}z)$ for a nonnegative, rotationally symmetric mollifier $\rho \in C_c^\infty(D)$ with unit mass and support in $B_0(1)$ and $\psi \in C_c^\infty((0,T)\times D)$. Then as before,

$$\partial_t \varphi = \rho_\varepsilon \partial_t \psi, \qquad \nabla_x \varphi = \psi \nabla \rho_\varepsilon + \rho_\varepsilon \nabla \psi, \qquad \nabla_y \varphi = -\psi \nabla \rho_\varepsilon.$$

Continuing from (3.13), we now have

$$\sum_i \int_0^T \int_D \int_D \langle \nu^2_{x,y},\, v_1^i v_2^i \rangle \rho_\varepsilon \partial_t \psi + \langle \nu^2_{x,y},\, v_1^i v_2^i (v_1 - v_2) \rangle \cdot \nabla \rho_\varepsilon \psi \\ + \langle \nu^2_{x,y},\, v_1^i v_2^i v_1 \rangle \cdot \rho_\varepsilon \nabla \psi \\ + \langle \nu^2_{x,y},\, v_2 p_1 - v_1 p_2 \rangle \cdot \nabla \rho_\varepsilon \psi + \langle \nu^2_{x,y},\, v_2 p_1 \rangle \cdot \rho_\varepsilon \nabla \psi \, dxdydt = 0.$$

Making the change of variables $z = x - y$ yields

$$\int_0^T \int_D \int_D \partial_t \psi(t,x) \rho_\varepsilon(z) \langle \nu^2_{x,x-z},\, v_1 \cdot v_2 \rangle \, dxdzdt \\ + \int_0^T \int_D \int_D \psi(t,x) \nabla \rho_\varepsilon(z) \cdot \langle \nu^2_{x,x-z},\, (v_1 - v_2)(v_1 \cdot v_2) \rangle \, dxdzdt \\ + \int_0^T \int_D \int_D \nabla \psi(t,x) \rho_\varepsilon(z) \cdot \langle \nu^2_{x,x-z},\, v_1 (v_1 \cdot v_2) \rangle \, dxdzdt \\ + \int_0^T \int_D \int_D \psi(t,x) \nabla \rho_\varepsilon(z) \cdot \langle \nu^2_{x,x-z},\, v_2 p_1 - v_1 p_2 \rangle \, dxdzdt \\ + \int_0^T \int_D \int_D \nabla \psi(t,x) \rho_\varepsilon(z) \cdot \langle \nu^2_{x,x-z},\, v_2 p_1 \rangle \, dxdzdt = 0. \tag{3.14}$$

Decompose the above into a sum of five terms $\mathcal{A}_1 + \cdots + \mathcal{A}_5$. We consider each in order.

For $\mathcal{A}_1$ we can write $\mathcal{A}_1 = \mathcal{A}_{1,1} + \mathcal{A}_{1,2}$, where

$$\mathcal{A}_{1,1} = \frac{1}{2} \int_0^T \int_D \int_D \partial_t \psi(t,x) \rho_\varepsilon(z) \langle \nu^2_{x,x-z},\, |v_1|^2 + |v_2|^2 \rangle \, dxdzdt,$$
$$\mathcal{A}_{1,2} = -\frac{1}{2} \int_0^T \int_D \int_D \partial_t \psi(t,x) \rho_\varepsilon(z) \langle \nu^2_{x,x-z},\, |v_1 - v_2|^2 \rangle \, dxdzdt.$$

The first term $\mathcal{A}_{1,1}$ converges as $\varepsilon \to 0$ to $\int_0^T \int_D \partial_t \psi(t,x) \langle \nu^1_{t,x},\, |v|^2 \rangle \, dxdt$. Indeed,

$$
\begin{aligned}
\mathcal{A}_{1,1} &= \frac{1}{2}\int_0^T \int_D \int_D \partial_t\psi(t,x)\rho_\varepsilon(z)\left(\left\langle \nu^2_{x,x-z},\, |v_1|^2\right\rangle + \left\langle \nu^2_{x,x-z},\, |v_2|^2\right\rangle\right) dxdzdt \\
&= \frac{1}{2}\int_0^T \int_D \int_D \partial_t\psi(t,x)\rho_\varepsilon(z)\left\langle \nu^1_{x},\, |v|^2\right\rangle dxdzdt \\
&\quad + \frac{1}{2}\int_0^T \int_D \int_D \partial_t\psi(t,x+z)\rho_\varepsilon(z)\left\langle \nu^2_{x+z,x},\, |v_2|^2\right\rangle dxdzdt \\
&= \int_0^T \int_D \int_D \frac{1}{2}\left(\partial_t\psi(x) + \partial_t\psi(x+z)\right)\rho_\varepsilon(z)\left\langle \nu^1_{x},\, |v|^2\right\rangle dxdzdt,
\end{aligned}
$$

where we have changed variables $x \mapsto x+z$ in the second equality and used the consistency and symmetry properties of ν^2 in the third equality. Letting $\varepsilon \to 0$ we obtain $\mathcal{A}_{1,1} \to \int_0^T \int_D \partial_t\psi(t,x)\left\langle \nu^1_{t,x},\, |v|^2\right\rangle dxdt$.

The second term $\mathcal{A}_{1,2}$ converges to zero as $\varepsilon \to 0$ by diagonal continuity. Thus,

$$
\lim_{\varepsilon\to 0} \mathcal{A}_1 = \int_0^T \int_D \partial_t\psi(t,x)\left\langle \nu^1_{t,x},\, |v|^2\right\rangle dxdt.
$$

For $\mathcal{A}_2$ we can then write $\mathcal{A}_2 = \mathcal{A}_{2,1} + \mathcal{A}_{2,2}$, where

$$
\begin{aligned}
\mathcal{A}_{2,1} &= -\frac{1}{2}\int_0^T \int_D \int_D \psi(t,x)\nabla\rho_\varepsilon(z)\cdot\left\langle \nu^2_{x,x-z},\, (v_1-v_2)|v_1-v_2|^2\right\rangle dxdzdt, \\
\mathcal{A}_{2,2} &= \frac{1}{2}\int_0^T \int_D \int_D \psi(t,x)\nabla\rho_\varepsilon(z)\cdot\left\langle \nu^2_{x,x-z},\, (v_1-v_2)\left(|v_1|^2+|v_2|^2\right)\right\rangle dxdzdt.
\end{aligned}
$$

In the limit $\varepsilon \to 0$, the first term is precisely $2\mathcal{E}(\boldsymbol{\nu})(\psi)$. (This limit is well-defined because all the remaining terms converge as $\varepsilon \to 0$.) Under the regularity assumption (3.12) this term can be bounded as

$$
\begin{aligned}
|\mathcal{A}_{2,1}| &\leqslant \frac{\|\psi\|_\infty}{2}\int_0^T \int_D \int_D |\nabla\rho_\varepsilon(z)|\left\langle \nu^2_{x,x-z},\, |v_1-v_2|^3\right\rangle dxdzdt \\
&\leqslant \frac{C\|\psi\|_\infty}{\varepsilon}\int_0^T \int_D \fint_{B_\varepsilon(0)} \left\langle \nu^2_{x,x-z},\, |v_1-v_2|^3\right\rangle dzdxdt \\
&\to 0,
\end{aligned}
$$

as $\varepsilon \to 0$ (after choosing a suitable subsequence if necessary), where we also used the fact that $\|\nabla\rho_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leqslant C\varepsilon^{-1-d}$.

For the second term $\mathcal{A}_{2,2}$ we use the transformation $x \mapsto x+z$ in the term with $|v_2|^2$ and then the transformation $z \mapsto -z$ and thus get

$$
\mathcal{A}_{2,2} = \frac{1}{2}\int_0^T \int_D \int_D \nabla\rho_\varepsilon(z)\left(\psi(t,x)+\psi(t,x-z)\right)\cdot\left\langle \nu^2_{x,x-z},\, |v_1|^2(v_1-v_2)\right\rangle dxdzdt.
$$

But now the divergence constraint (3.8) implies

$$
\mathcal{A}_{2,2} = \frac{1}{2}\int_0^T \int_D \int_D \rho_\varepsilon(z)\nabla\psi(t,x-z)\cdot\left\langle \nu^2_{x,x-z},\, |v_1|^2(v_1-v_2)\right\rangle dxdzdt,
$$

and then the diagonal continuity condition implies $\lim_{\varepsilon\to 0}\mathcal{A}_{2,2} = 0$.

Using an argument similar to the treatment of $\mathcal{A}_1$, it is not hard to see that diagonal continuity implies

$$\mathcal{A}_3 \to \int_0^T \int_D \langle \nu^1_{t,x}, |v|^2 v\rangle \cdot \nabla\psi(t,x)dxdt$$

as $\varepsilon \to 0$, and similarly

$$\mathcal{A}_5 \to \int_0^T \int_D \langle \nu^1_{t,x}, pv\rangle \cdot \nabla\psi(t,x)dxdt.$$

Finally, for $\mathcal{A}_4$ we translate $x \mapsto x + z$ in the second term and obtain

$$\mathcal{A}_4 = \int_0^T \int_D \int_D \left[\psi(t,x)\nabla\rho_\varepsilon(z) \cdot \langle \nu^2_{x,x-z}, p_1 v_2\rangle - \psi(t,x+z)\nabla\rho_\varepsilon(z) \cdot \langle \nu^2_{x,x+z}, p_1 v_2\rangle\right] dxdzdt.$$

Owing to the divergence condition, the first term is zero, whereas the second term (again invoking the divergence constraint) equals

$$\mathcal{A}_4 = \int_0^T \int_D \int_D \nabla\psi(t,x+z)\rho_\varepsilon(z) \cdot \langle \nu^2_{x,x+z}, p_1 v_2\rangle dxdzdt.$$

Once more invoking diagonal continuity yields

$$\lim_{\varepsilon\to 0} \mathcal{A}_4 = \int_0^T \int_D \nabla\psi(t,x) \cdot \langle \nu^1_x, pv\rangle dxdt.$$

Collecting all terms now gives the desired result.

□

4 Probabilistic versus deterministic regularity

In this section we will compare the regularity of functions and of correlation measures, and we will show that if a correlation measure is concentrated on a family of L^2 functions (soon to be made precise), then this family is at least as regular as the correlation measure (also to be made precise). We will use the notation

$$d^q_\varepsilon(v) := \left(\int_D \fint_{B_\varepsilon(x)} \left|v(x) - v(y)\right|^q dydx\right)^{1/q}$$

for a function $v : D \to \mathbb{R}^d$, and for a correlation measure $\boldsymbol{\nu} = (\nu^1, \nu^2, \dots)$ we write

$$d^q_\varepsilon(\nu^2) := \left(\int_D \fint_{B_\varepsilon(x)} \langle \nu^2_{x,y}, \left|v_1 - v_2\right|^q\rangle dydx\right)^{1/q}.$$

(For notational convenience we only look at space-dependent functions in this section.)

In [12] the authors proved that the set of correlation measures, as defined in Definition 3.4, are equivalent to the set $\mathcal{P}(L^2(D))$ of probability measures on $L^2(D)$. We make this duality more precise in the following definition:

Definition 4.1. A probability measure $\mu \in \mathcal{P}(L^2(D; \mathbf{U}))$ is said to be *dual* to a correlation measure $\boldsymbol{\nu}$ from D to $\mathbf{U}$ provided

$$\int_{D^k} \langle \nu_x^k, g(x, \cdot) \rangle \, dx = \int_{L^2} \int_{D^k} g(x, v(x_1), \dots, v(x_k)) \, dx d\mu(v) \tag{4.1}$$

for every $k \in \mathbb{N}$ and for every Caratheodory function $g : D^k \to C(\mathbf{U}^k)$.

Proposition 4.2. *Let $\mu \in \mathcal{P}(L^2(D; \mathbf{U}))$ be dual to a space-time correlation measure $\boldsymbol{\nu}$, and assume that $\boldsymbol{\nu}$ is α-Besov regular in the sense that*

$$\liminf_{\varepsilon \to 0} \frac{d_\varepsilon^q(\nu^2)}{\varepsilon^\alpha} = 0. \tag{4.2}$$

Then

$$\liminf_{\varepsilon \to 0} \frac{d_\varepsilon^q(v)}{\varepsilon^\alpha} = 0 \tag{4.3}$$

for μ-almost every $v \in L^2(D; \mathbf{U})$. Conversely, if there is a common subsequence $\varepsilon_n \to 0$ such that

$$\lim_{n \to \infty} \frac{d_{\varepsilon_n}^q(v)}{\varepsilon_n^\alpha} = 0 \qquad \textit{boundedly} \tag{4.4}$$

for μ-almost every $v \in L^2(D; \mathbf{U})$, then (4.2) holds.

Proof. Using the duality between μ and ν and Fatou's lemma yields

$$\begin{aligned}
0 &= \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^{\alpha q}} \int_D \fint_{B_\varepsilon(x)} \langle \nu_{x,y}^2, |v_1 - v_2|^q \rangle \, dy dx \\
&= \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^{\alpha q}} \int_{L^2} \int_D \fint_{B_\varepsilon(x)} |v(x) - v(y)|^q \, dy dx d\mu(v) \\
&\geqslant \int_{L^2} \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^{\alpha q}} \int_D \fint_{B_\varepsilon(x)} |v(x) - v(y)|^q \, dy dx d\mu(v) \\
&= \int_{L^2} \liminf_{\varepsilon \to 0} \frac{d_\varepsilon^q(v)^q}{\varepsilon^{\alpha q}} \, d\mu(v).
\end{aligned}$$

The conclusion follows immediately. Conversely, if (4.4) holds then the dominated convergence theorem implies that (4.2) holds along the prescribed subsequence ε_n. □

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References

[1] C. Bardos and E. S. Titi. Onsager's conjecture for the incompressible Euler equations in bounded domains. *Preprint*, 2017. arXiv:1707.03115.

[2] T. Buckmaster, C. De Lellis, L. Székelyhidi Jr., and V. Vicol. Onsager's conjecture for admissible weak solutions. *Preprint*, 2017. arXiv:1701.08678.

[3] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager's conjecture for the Euler equations. *Nonlinearity*, 21(6):1233–1252, 2008.

[4] P. Constantin, W. E, and E. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Communications in Mathematical Physics*, 165(1):207–209, 1994.

[5] R. J. DiPerna. Measure-valued solutions to conservation laws. *Archive for Rational Mechanics and Analysis*, 88:223–270, 1985.

[6] R. J. DiPerna and A. J. Majda. Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Communications in Mathematical Physics*, 108(4):667–689, 1987.

[7] T. Drivas and G. Eyink. An Onsager singularity theorem for turbulent solutions of compressible Euler equations. *Preprint*, 2017. arXiv:1704.03409.

[8] J. Duchon and R. Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity*, 13(1):249–255, 2000.

[9] G. L. Eyink. Energy dissipation without viscosity in ideal hydrodynamics I. Fourier analysis and local energy transfer. *Physica D: Nonlinear Phenomena*, 78(3):222–240, 1994.

[10] G. L. Eyink and K. R. Sreenivasan. Onsager and the theory of hydrodynamic turbulence. *Rev. Mod. Phys.*, 78:87–135, Jan 2006.

[11] Eduard Feireisl, Piotr Gwiazda, Agnieszka Świerczewska-Gwiazda, and Emil Wiedemann. Regularity and energy conservation for the compressible Euler equations. *Arch. Ration. Mech. Anal.*, 223(3):1375–1395, 2017.

[12] U. S. Fjordholm, S. Lanthaler, and S. Mishra. Statistical solutions of hyperbolic conservation laws I: Foundations. *Arch. Ration. Mech. Anal.*, 226(2), pp. 809-849 (2017).

[13] U. S. Fjordholm, S. Mishra and E. Tadmor. On the computation of measure-valued solutions. *Acta Numerica*, 25, 2016, 567-679.

[14] U. S. Fjordholm, S. Mishra, and F. Weber. Statistical solutions of the incompressible Euler equations and Kolmogorov's theory of turbulence. *In preparation*, 2017.

[15] C. Foiaş. Statistical study of Navier–Stokes equations I. *Rend. Sem. Mat. Univ. Padova* 48, 219–348, 1972.

[16] C. Foiaş. Statistical study of Navier–Stokes equations II. *Rend. Sem. Mat. Univ. Padova* 49, 9-123, 1973.

[17] U. Frisch. *Turbulence.* Cambridge University Press, 1995.

[18] P. Isett. A proof of Onsager's conjecture. *Preprint*, 2016. arXiv:1608.08301.

[19] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[20] T. M. Leslie and R. Shvydkoy. The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations. *J. Differential Equations*, 261(6):3719–3733, 2016.

[21] Lars Onsager. Statistical hydrodynamics. In *Nuovo Cimento, Suppl.*, volume 6, pages 279–287, 1949.

[22] J. C. Robinson, J. L. Rodrigo, and J. W. D. Skipper. A simple integral condition for energy conservation in the 3D Euler equations. *Preprint*, 2016.

[23] L. Székelyhidi Jr. and E. Wiedemann. Young Measures Generated by Ideal Incompressible Fluid Flows. *Arch. Ration. Mech. Anal.*, 206(1):333–366, 2012.