Sheath folds as a strain gauge in simple shear

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Abstract

We investigate initiation and evolution of sheath folds developing in multilayer sequences around slip surfaces in simple shear. The slip surface is initially circular and oriented at 135° to the shearing direction. The flow perturbation around the rotating and deforming slip surface initiates the growth of deflections of the layers, which serves as precursors for the sheath structure. The influence of the perturbed flow on the fold growth decreases with strain as the structure is moved away from the slip surface. For $\gamma > 10$, the sheath fold evolution is dominated by a passive simple shear.

We describe the fold geometry using: 1) interlimb angle (α), 2) hinge angle (β), and 3) aspect ratio of the eye-structures in the section normal to the shearing direction at the fold base (R_{yz}). We show that the fold shapes developing in different interfaces can be characterized by a unique combination of the three parameters depending on strain magnitude. We present three strain gauge diagrams, which can be used to decipher strain from sheath folds based on any combination of two out of three parameters (α, β, R_{yz}).

We approximate the late evolution of the modelled sheath folds by analysing the passive deformation of cone structures in simple shear. We show that R_{yz} is asymptotically proportional to the square root of strain magnitude.

Keywords (6): sheath folds, non-cylindrical folds, eye-structures, simple shear, cone shape, strain gauge

1. Introduction

Non-cylindrical fold structures, which are characterized by a sharp hinge line bend (hinge angle) of more than 90° are termed sheath folds (Ramsay and Huber, 1987). In sections normal to the fold elongation, they give rise to a characteristic eye-shaped pattern. The number of closed contours seen in each section depends on the relation between the position of the section, the size of the fold, and layer thickness (e.g., Reber et al., 2012). Alsop and Holdsworth (2006) used the quotient of the aspect ratios of the outermost and the innermost closed contour to develop sheath fold classification. The authors distinguished three types of folds: analogous-eye, cat's-eye, and bull's-eye, in which the ratio of the outermost closed contour is the same, smaller, and larger from the ratio of the innermost closed contour, respectively. Further, they related the three fold patterns to the corresponding types of bulk strain deformation: plane strain, flattening, and constrictional. Reber et al. (2013a) showed that the sheath fold classification based on the contour aspect ratios may lead to erroneous results and should be carefully used for bulk strain type interpretations. Similar concerns regarding genetic interpretations based on the sheath fold classification were expressed in the work of Marques et al. (2008).

Sheath folds are found in various rock types in a broad spectrum of geological settings across a wide range of scales (Alsop et al., 2007). However, it is generally recognized that they predominantly form during high strain deformation in a simple shear-dominated regime (Cobbold and Quinquis, 1980). Various mechanisms of sheath fold formation in simple shear have been suggested in the literature (Figure 1A). Passive amplification of a pre-existing dome-shaped layer interface perturbation is perhaps the most widespread model used to explain sheath fold development (Quinquis et al., 1978; Minnigh, 1979; Cobbold and Quinquis, 1980; Skjernaa, 1989; Mies, 1993). Approximating the initial layer interface perturbation as an upright cone, Mies (1993) used analytic geometry and graphical simulations to estimate shear strain based on 1) the aspect ratios of the contours, 2) the interlimb angle, and 3) the hinge angle (Figure 1B). Based on analogue experiments, Marques et al. (2008) showed that sheath folds can develop in perturbed layers with dissimilar viscosities of a ratio smaller than 10.

The other group of models explains sheath fold development by various perturbation mechanisms of the simple shear flow field such as 1) above a rigid corrugated basement (Cobbold and Quinquis, 1980), 2) around rigid inclusions (Marques and Cobbold, 1995; Rosas et al., 2002; Marques et al., 2008), and 3) around slip surfaces (Reber et al., 2012;

63   Reber et al., 2013a; Reber et al., 2013b). Cobbold and Quinquis (1980) studied theoretically

64   and experimentally sheath fold evolution above the corrugated surface. The authors derived a

65   two-dimensional analytical solution for the velocity field above a rigid layer with regularly

66   spaced grooves. Rosas et al. (2002) demonstrated how a sheath fold can develop around a

67   rotating rigid inclusion in a shear zone. The impact of various parameters such as the shape of

68   the inclusion and the distance of the marker layer from the inclusion was analysed. The study

69   allowed for indicating configurations that are less likely for the sheath folds to develop. A

70   systematic study of sheath fold formation around a slip surface was performed by Reber et al.

71   (2013a). A range of parameters including slip surface size and orientation, strain magnitude,

72   layer thickness, and cross-section location were investigated in terms of their impact on the

73   aspect ratio of the outermost and innermost contours and the sheath fold length. The three-

74   dimensional analytical flow model allowed for analysis of the structure development in high

75   resolution. The analysis was carried out on densely spaced yz-sections. The slip surface

76   model was also positively tested for the case of a mechanically stratified matrix with

77   viscosity ratio less than 50 between the layers using analogue experiments (Reber et al.,

78   2013b).

79   Alsop and Holdsworth (2012) presented a natural multilayer sheath fold example with

80   rheologically distinct layering. Based on a series of sections, the authors described a three-

81   dimensional shape of the 11 interfaces constituting the structure. They showed that the aspect

82   ratio of the closed contours is larger in folds with larger hinge and interlimb angles. Moreover,

83   the aspect ratio increases towards the fold nose Thus, following Alsop and Holdsworth

84   (2006), the fold was classified as a cats-eye fold. The interlimb angle changes between 12

85   and 40 degrees, whereas the aspect ratio of the closed contour observed in the most distant

86   sections for different interfaces varies between 4 and 5.6. According to the authors, the

87   overall geometrical analysis suggests that the fold was developed during general shear

88   deformation due to the amplification of an initial perturbation.

89   In this paper, we analyse the initiation and evolution of sheath folds that form around

90   slip surfaces in simple shear. The aim of the work is to gain a more detailed insight into

91   sheath fold evolution. Similarly to Reber et al. (2013a), we use the analytical Eshelby

92   solution reduced to the case of an incompressible viscous medium and an inviscid elliptical

93   inclusion (slip surface). However, in contrast to the previous work, we focus on the three-

94   dimensional shape analysis of individual interfaces rather than the eye-shaped structures

95   displaying on the yz section. The approach significantly reduces the complexity of the

96   analysis and allows for a better control on the evolution of fold shape parameters. We use the

97   term sheath structure *sensu lato*, including all non-cylindrical folds exhibiting eye patterns in
98   the yz section, without the hinge angle necessarily below 90°.

99   We show that the deformation of a right cone can be used to approximate the late
100 shape evolution of the sheath fold in the slip surface model. We derive an analytical formula
101 for the change of the aspect ratio of the outermost closed contour, the hinge angle, and the
102 interlimb angle with strain. Finally, we present a strain gauge diagram based on a
103 combination of two out of three parameters: 1) the aspect ratio of the outermost contour, 2)
104 the hinge angle, and 3) the interlimb angle.

105 **2.   Mechanical Model**

106   We study a three-dimensional mechanical model of sheath fold development around a
107 slip surface in simple shear up to shear strain of $\gamma=30$. The slip surface is embedded in a
108 homogeneous, isotropic, and linear viscous matrix. We use a Cartesian coordinate system xyz,
109 with x parallel to the shear direction. The centre of a prescribed circular slip surface is located
110 in the origin of the reference system. The slip surface is initially oriented at $\theta=135°$ to the
111 shearing direction, which corresponds to the mode I fracture orientation (Figure 2A). The
112 spatial coordinates are normalized by the slip surface radius. Thus, the slip surface radius is
113 equal to 1 and its maximum vertical extent is $z_0=0.707$. During deformation, the slip surface
114 can passively deform (rotate and stretch) but it cannot propagate (Means, 1989). Due to the
115 point symmetry, we analyse only the upper part of the model. We use nine planes of passive
116 markers that are equally distributed above the slip surface and located at $z_0=0.8$, 0.95, 1.1,
117 1.25, 1.4, 1.55, 1.7, 1.85, and 2.0 to visualize the fold evolution. Since no mechanical
118 layering is present in the model, the developing folds are passive sheath folds (e.g., Cobbold
119 and Quinquis, 1980). The fold geometry in each interface is described using the interlimb (α)
120 and hinge (β) angles. The interlimb angle is measured as the minimum acute angle between
121 the fold limbs in the area, where the fold forms a sheath structure, whereas the hinge angle is
122 measured as the minimum angle along the hinge line, where hinge line is a curve that joins
123 points of the maximum curvature (Figure 1B). Additionally, we examine the development of
124 flanking structures on the central xz-section and eye-shaped patterns on multiple yz-sections.

125   We track the evolution of interfaces by numerically integrating the velocity field
126 around the slip surface with strain. The velocity field is obtained using a modified Eshelby
127 solution (see Exner and Dabrowski, 2010 for details). In simulations, we use a 4th order
128 Runge-Kutta scheme for time integration (e.g., Keller, 1993).

3. Results

3.1. Sheath fold structure

Figure 3 shows the side and top views of the four selected interfaces after $\gamma=30$, for which $z_0=0.8$, 1.1, 1.4, and 1.7. The thick black lines indicate the positions of the yz section at the sheath structure base and the contour shapes are shown on the right side of the figure. The analysed interfaces form the outermost closed contour of the eye-pattern on the selected yz section.

The folds have notably different geometry and size depending on the interface. In the side view, we can distinguish a long upper limb and a short lower limb. The lower limb is inverted and has a nearly flat shape. The geometry of the upper limb is more complex due to local curvature changes resulting in a second subsidiary hinge zone. The fold appears to have an asymmetric box-shape with a straight segment between the hinges. The straight segment is longer for distant interfaces.

The length and orientation of the inverted limb controls the span along x-axis where the closed contours and thus the sheath structure *sensu lato* can be observed. The length of the inverted limb and, thus, the length of the sheath structure is smaller for distant interfaces. The interlimb angle increases for distant interfaces and is equal to $\alpha=2.3°$ for $z_0=0.8$ and $\alpha=5.0°$ for $z_0=1.7$ (note vertical exaggeration for the side view in Figure 3). The hinge angle (β) is the smallest for the lowermost interface and the largest for the most distant interface. For $z_0=0.8$, β is equal to $13°$ and for $z_0=1.7$ it is $53°$. The closed contour aspect ratio (R_{yz}) is calculated as a ratio between the maximum extents of the closed contour measured in the horizontal and vertical directions. R_{yz} is higher for distant interfaces.

Figure 3 demonstrates the complexity of the sheath structure geometry. The fold outline observed in the top view shows a sine-shaped hinge line variation. The contour shapes in the yz section are asymmetric about the horizontal plane. The asymmetry is more pronounced for proximal interfaces, which exhibit pyramidal shapes with rounded corners. The more distant interfaces develop an almost elliptical geometry.

A small-scale sheath structure develops below the main sheath fold on the other side of the slip surface. In a yz cross-section, we would observe a double-eye fold (see Reber et al., 2012). In the current analysis, we focus on the structures that develop above the plane containing the slip surface.

160   We quantify the shape of the nine studied interfaces after $\gamma=30$ using four parameters:

161   the fold height and length in the central xz section and the aspect ratio and asymmetry of the

162   closed contours appearing on different yz sections (Figure 4). The fold height is measured as

163   the distance between the topmost point on the fold interface and the initial interface plane.

164   The fold length is calculated as a distance between the two most distant yz sections, in which

165   the closed contours can be observed. The aspect ratio of the closed contour is measured for

166   different yz sections and we use r_{yz} to differentiate it from the aspect ratio measured at the

167   sheath structure base (R_{yz}). To quantify the degree of asymmetry, we split each closed

168   contour into the lower and upper part dividing it by the line joining the left- and rightmost

169   hinge points. The degree of asymmetry is defined as the ratio between the heights of the

170   upper and lower parts of the contour. When the asymmetry coefficient is close to 1, the upper

171   and lower part have equal heights and the contour is approximately mirror symmetric around

172   the horizontal plane. For the analysed sheath structure, the asymmetry coefficient is always

173   larger than 1.

174   The height and length of the sheath structure are larger in the proximal interfaces than

175   in the distant ones (Figure 4A and B). The fold height is 5 times larger in the innermost

176   interface $z_0=0.8$ than in the outermost interface $z_0=2.0$, whereas the fold length is more than

177   an order of magnitude greater for $z_0=0.8$ than for $z_0=2.0$. Figure 4C and D shows the variation

178   of the aspect ratio, r_{yz}, and the asymmetry of the closed contour along the x-axis for different

179   interfaces. The aspect ratio r_{yz} is generally higher for distant interfaces and ranges between 6

180   and 11.5. For inner interfaces, r_{yz} variation is characterized by a local minimum along the

181   length of the fold, whereas for the outer interfaces it decreases towards the apex. If we

182   analyse sections at the fold base, we cut through the inner interface with smaller r_{yz}. Thus, we

183   observe bull's-eye-folds. In sections away from the fold base, we can also find analogous-

184   eyes and cat's-eyes.

185   The highest asymmetry is observed close to the fold base and gradually decreases

186   towards the fold apex. The largest variation occurs in the proximal interfaces, e.g., for $z_0=0.8$,

187   where it changes from 7.4 at the fold base to almost 1 at the apex. The eye patterns

188   developing in distant interfaces are more symmetrical e.g., for $z_0=2.0$ the value is between 1.7

189   and nearly 1.

3.2. Interface shape evolution

191   We analyse the shape evolution of interfaces with strain. Figure 5 shows the
192   intersections of interfaces in the central xz section (y=0) for selected strain values γ=1, 5, 10,
193   20, and 30. A rapid rotation of the slip surface is observed in the initial stages of deformation.
194   The orientation measured with respect to the shearing direction changes from the initial value
195   of θ=135° to θ=14° for γ=5 and θ=6° for γ=10. The angle approaches the asymptotic value of
196   θ=0° with further deformation. The length of the slip surface interception in the xz section
197   (slip line) changes during deformation. Initially, it decreases attaining a minimum length at
198   γ=1. Further deformation leads to elongation of the slip line. For γ=10, the aspect ratio of the
199   slip surface is around 6, and for γ=30, it exceeds 20. The rotation of the slip surface promotes
200   interface bending and asymmetric folding. Fold structures develop in the vicinity of both tips
201   of the slip surface. The structures below and above the slip surface are point symmetric with
202   respect to the centre of the slip surface. We focus our analysis on the regions above the slip
203   surface.

204   We distinguish three evolving regions hosting different kinds of structures (Figure 5).
205   The regions are bounded by the extrapolation of the slip line and separated by the vertical
206   lines that intercept the slip line tips. The interfaces in region I are virtually undeformed. Only
207   towards the triangular region III, we observe a narrow zone with slightly deflected interfaces.
208   In region II, which extends above the slip surface, the interfaces form a gentle asymmetrical
209   synform, with a steeper and longer right limb compared to the left limb (note the vertical
210   exaggeration 5:1 in Figure 5). The interface deflections decrease away from the slip surface.
211   Initially, the interface deflections grow and, at the same time, the synform is stretched. The
212   initially rapid evolution of the deflections slows down with strain and the syncline becomes
213   more gentle due to progressive stretching. In the part of the region close to the slip surface,
214   the interfaces that are initially cut by the slip surface develop an antiformal structure. The
215   antiform migrates towards region III, becoming the innermost, core part of the sheath fold
216   structure. For γ=5, the boundary line between region II and III intercepts the fold hinge zone
217   for all the interfaces. Folds developed in the proximal interfaces are tight, strongly
218   asymmetric and their amplitudes are large. Folds developing in the distant interfaces are
219   wider, more symmetrical and have smaller amplitudes. Although further deformation causes
220   only a slight fold growth in the vertical direction, we observe a significant stretch of the hinge
221   zone, rotation and eventually inversion of the right fold limb. The hinge zone located above
222   the tip of the slip surface gets smoother, whereas the hinge zone that defines the apex of the

223   sheath structure sharpens. As a result, an asymmetric box fold forms in region III, which

224   further transforms into a non-cylindrical fold structure and finally a sheath fold. The folds

225   develop earlier in the proximal interfaces than in the distant ones. The position of the yz-

226   section along the x-axis, in which any given interface forms the outermost closed contour is

227   determined by the location of the rearmost (here leftmost) point of the inverted fold limb. The

228   positions of such sections are found closer to the origin of the coordinate system in the

229   proximal interfaces.

230   We show the evolution of the fold height, fold length, the outermost closed contour

231   aspect ratio at the sheath structure base, and the closed contour asymmetry as a function of

232   strain for the nine selected interfaces in Figure 6. The proximal interfaces develop closed

233   contours earlier during deformation compared to the more distant interfaces. Closed contours

234   can be found in the two innermost interfaces $z_0=0.8$ and 0.95 already for $\gamma=6$, whereas it

235   requires $\gamma=24$ to develop closed contours in the $z_0=2.0$ interface.

236   The fold height is always the largest for $z_0=0.8$. For $\gamma=15$, it reaches ca. 0.55 and

237   further deformation leads to only minor changes. The maximum fold height decreases for

238   distant interfaces. The amount of strain required to reach the nearly maximum possible height

239   is larger for outer interfaces. At each stage of deformation, the folds are longer in the

240   proximal than in the distant interfaces. R_{yz} increases with strain for all interfaces and the

241   value is always smaller in the proximal than distant interfaces. For $\gamma=15$, we observe closed

242   contours in the six lowermost interfaces. Their aspect ratios R_{yz} vary between 5 and 6.5. For

243    $\gamma=30$, closed contours are present in all the interfaces and their aspect ratio varies between 6.5

244   and 11.6. The asymmetry of the contours increases with strain. For $z_0=0.8$ it is equal to ca.

245   1.3 at $\gamma=6$ and 7.6 at $\gamma=30$. The asymmetry developed in the distant interfaces is always

246   smaller than in the proximal ones.

247   *3.3. Strain gauge diagrams*

248   Dashed lines in Figure 7A and B show the evolution of R_{yz} for the nine interfaces as a

249   function of the interlimb angle (α) and hinge angle (β), respectively, whereas, in Figure 7C,

250   they illustrate a relation between α and β. Solid lines connect the values obtained for different

251   interfaces for selected strain values $\gamma=10$, 15, 20, 25, and 30. R_{yz} exhibits an approximate

252   linear dependence on both α and β in the double-logarithmic plot. Similarly, a linear trend is

253   observed in the α and β relation. R_{yz} increases and both α and β decrease with strain. For a

254   given strain, R_{yz}, α, and β are always larger for distant interfaces. Hence, the fold shape can

255   be described uniquely using the three parameters for a given strain. However, it is sufficient

256   to use two out of the three geometrical parameters to decipher the shear strain. Thus, the plots

257   presented in Figure 7 can be used as strain gauge diagrams.

258   The grey field in Figure 7B and C delineates the area, where the hinge angle (β) is

259   less than 90° and the non-cylindrical fold satisfy the sheath fold definition. During

260   deformation, the folds that are initially characterized with a large hinge angle evolve into

261   sheath fold *sensu stricto*. However, we can always find distant interfaces that are

262   characterized by a wide hinge angle of more than 90°.

263   *3.4. Analysis of the rate of deformation*

264   In Figure 8, the isocontours show the deviation of the rate of deformation measured in

265   the model (D) from the background rate of deformation ($\bar{D}$), where the rate of deformation is

266   given by the symmetric part of the velocity gradient. The deviation is quantified using the

267   second invariant of the difference between the two rate of deformation tensors $\tilde{D} = D - \bar{D}$.

268   The blue and green contour lines indicate the values of $\tilde{D}$ equal to 0.05 and 0.2, respectively,

269   whereas the thick and thin grey lines show the selected interface shapes.

270   A zone of increased deformation localizes along the slip surface and shows a

271   characteristic butterfly shape at the both slip surface tips. The shape of the zone evolves as

272   the size and orientation of the slip surface changes. At the initial stages, shearing is localized

273   close to the slip surface such that for $\gamma=1$ the $\tilde{D} = 0.05$ contours outline a narrow area and the

274     $\tilde{D} = 0.2$ contours are almost undistinguishable (Figure 8A). Further deformation and the

275   stretching of the slip surface cause that the zone is longer and wider and the butterfly shape is

276   greater (Figure 8B). For $\gamma>10$, the two contours that outline the zone along the slip surface

277   grow horizontally but do not expand vertically, whereas the size of the butterfly shape

278   decreases (Figure 8C, D, and E).

279   At the initial stages of deformation, the perturbation of the interfaces is initiated and

280   grows around the regions of high $\tilde{D}$ (Figure 8A and B). However, already for $\gamma>10$, the

281   perturbed interfaces move to the region ahead of the slip surface tip, where the background

282   deformation dominates, and evolve into sheath folds. The influence of the slip surface on the

283   sheath fold development is negligible for $\gamma>20$.

284   To estimate the role of passive deformation in sheath structure evolution, we compare

285   the fold shape developed in the $z_0=0.8$ interface for $\gamma=9$ (G9) with the fold shapes that are

286   generated in the same interface for larger strains $\gamma=15$ (G15) and $\gamma=21$ (G21). The two latter

287 folds are subject to a passive simple shear backward deformation of γ=-6 and γ=-12
288 respectively, such that the three models illustrate the fold shape for effectively the same strain
289 of γ=9. The contour plots in Figure 9A show the top view of the elevation of the structure for
290 the three folds and the two dashed lines mark the position of the yz and the central xz sections,
291 which are shown in Figure 9B and C, respectively.

292 The three structures exhibit similar shapes in the region past the slip surface tip
293 (Figure 9C). The shape of the inverted fold limb and the fold apex are almost identical in the
294 three models. A small discrepancy is noticeable in the upper fold limb, close to the sheath
295 structure base, where the interface in the actual G9 model plunges slightly below the upper
296 limbs of the retrodeformed folds. As a result, we observe a small misfit between the shapes of
297 the closed contours in Figure 9B. The straight segment of the upper limb is the longest in G21.
298 The significantly different shape of the upper limb is the main reason of the discrepancy
299 between the three fold shapes observable in Figure 9A and Figure 9B.

4. Passive deformation model of the sheath fold evolution

301 To develop a passive deformation model of the sheath structure evolution for large
302 strains, we approximate the fold shape with a right circular cone structure with an initial
303 opening angle α_0. The cone is inclined to be horizontal along the slant height. We analyse the
304 cone shape evolution during progressive simple shear and measure: 1) the aspect ratio of the
305 contour in the yz sections (R_{yz}), 2) the cone angle in the central xz plane that bisects the cone
306 in the centre (α), which is analogous to the interlimb angle and 3) the angle in the plane
307 normal to the xz plane that also bisects the cone in the centre (β), which is analogous to the
308 hinge angle (Figure 10).

309 The cone intercepts with a reference vertical section giving an ellipse (the black line
310 in Figure 10). The lengths of the horizontal and vertical axes of the ellipse are denoted by R_y
311 and R_z, respectively. In the analysis, we set the origin of the coordinate system in the
312 lowermost point of the reference ellipse (the black dot in Figure 10). The aspect ratio of the
313 ellipse is given by (see Appendix A)

$$R_{yz} = \frac{\sqrt{\cos(\alpha_0)}}{\cos\left(\dfrac{\alpha_0}{2}\right)} \qquad (1)$$

315 The aspect ratio of the ellipse increases with strain. The length of the vertical axis
316 remains the same ($R'_z = R_z$), whereas the length of the horizontal axis (R'_y) is inherited from

317   an ellipse embedded in a cone section inclined at an angle φ to the reference vertical section

318   (the white contour in Figure 10). The angle is related to shear strain γ through $\varphi = \tan^{-1} \gamma$.

319   The ellipse aspect ratio R_{yz} changes with strain as (see Appendix B):

320
$$R_{yz} = \frac{\sqrt{\gamma \sin(\alpha_0) + \cos(\alpha_0)}}{\cos\left(\dfrac{\alpha_0}{2}\right)} \qquad (2)$$

321   For the initially elliptical cone shape, (see Appendix C), we obtain

322
$$R_{yz} = f \frac{\sqrt{\gamma \sin(\alpha_0) + \cos(\alpha_0)}}{\cos\left(\dfrac{\alpha_0}{2}\right)} \qquad (3)$$

323   where f denotes the initial cone aspect ratio. Note that Eq. (2) and (3) differ only by the

324   factor f, which indicates that the initial flattening of the cone does not influence the scaling

325   between R_{yz} and α_0.

326   The evolution of the cone angles α and β with strain are given by the following

327   expressions (Appendix D)

328
$$\alpha = \tan^{-1}\left[\frac{\sin(\alpha_0)}{\gamma \sin(\alpha_0) + \cos(\alpha_0)}\right] \qquad (4)$$

329   and

330
$$\beta = 2\tan^{-1}\left[R_{yz} \frac{\sin\left(\dfrac{\alpha}{2}\right)}{\sqrt{\cos(\alpha)}}\right] \qquad (5)$$

331   Note that we have simplified the expression for β by deriving it as a function of α and

332   not α_0.

333   Figure 11 shows the evolution of R_{yz}, α, and β with strain obtained using the

334   analytical expressions (Eq. (2), Eq. (4), and Eq. (5)) for α_0 varying between 10 and 170

335   together with the simulation results for the nine analysed interfaces. The analytical expression

336   for R_{yz} gives an exact linear trend with strain in the double-logarithmic plot only for α_0=90

337
$$R_{yz} = \sqrt{2\gamma} \qquad (6)$$

338 For $\alpha_0 \neq 90$, the relation is visibly non-linear with strain for small strain and almost linear for

339 larger strain. The contribution of the $\cos(\alpha_0)$ term in the nominator in Eq. (2) decreases with

340 strain and for larger strains the expression can be approximated with

341
$$R_{yz} \approx c(\alpha_0)\sqrt{\gamma} \qquad (7)$$

342 where c is a constant dependent on the initial cone angle. In the case of an elliptical cone

343 shape, the slope does not change but it will be shifted upwards or downwards by a factor f

344 (Appendix C). The simulation results for all analysed interfaces (Figure 11B) show a similar

345 trend to the one observed for the right cone.

346   The analytical expression for α is also non-linear for small strains and becomes nearly

347 linear with strain (Figure 11C). It can be approximated with a relation

348
$$\alpha \approx c(\alpha_0)\frac{1}{\gamma} \qquad (8)$$

349 The curves are convex for $\alpha_0 < 90°$ and concave for $\alpha_0 > 90°$. For the case of $\alpha_0 = 90°$, Eq. (4)

350 reduces to

351
$$\alpha = \tan^{-1}\frac{1}{\gamma} \qquad (9)$$

352 The relations shown for the numerical results have concave shape characteristics similar to

353 the analytically derived solution for large $\alpha_0 > 160°$ (cf. Figure 11C and D).

354   The analytical solution for the evolution of β with strain shows approximately linear

355 relation in the double-logarithmic plot for all α_0, whereas the numerical results exhibit

356 slightly curved lines for all interfaces (cf. Figure 11E and F). However, the curves shown in

357 the two plots are characterized by distinctly different slopes in the log-log axis.

358 **5. Discussion**

359 *5.1. Sheath fold geometry*

360   Sheath folds are commonly described as structures with either conical (e.g., Ramsay,

361 1958) or paraboidal (e.g., Carey, 1962) shape. However, we show that sheath folds may

362 exhibit a more complex geometry. In the yz section, the simulated sheath folds at their base

363 show strongly asymmetrical, pyramidal contours with rounded corners, whereas the contours

364 are more elliptical at the apex (Figure 3). In the multilayer sequence, each interface is

365 characterized by a distinct set of geometrical parameters such as interlimb angle (α), hinge

366   angle (β), and aspect ratio of the outermost closed contour (R_{yz}). A single parameter is not

367   sufficient to uniquely describe the interface (Figure 7). Due to a large variation in the shape

368   of individual interfaces, the description of the multilayer sequence is a challenging task.

369   Moreover, the classification of such a structure as sheath fold might be problematic, where

370   e.g. only some layers have a hinge angle below 90° and can be classified as sheath folds

371   *sensu stricto* (see Figure 7B).

372   Based on serial sectioning of a natural example of a multilayer sheath structure, Alsop

373   and Holdsworth (2012) depicted it as a self-similar structure and argued that the bulk three-

374   dimensional fold geometry can be reproduced from the analysis of a single yz section. In this

375   study, sheath folds exhibit higher R_{yz} in the proximal than in the distant interfaces and the

376   increase of R_{yz} is correlated with an increase of α and β. Alsop and Holdsworth (2012)

377   concluded that the fold was formed due to the amplification of the initial layer perturbation in

378   general shear, and the viscous layering played a key role during deformation. Since we

379   describe the sheath structure development due to flow perturbation around a slip surface

380   embedded in a homogeneous rock matrix, the difference in the fold shapes can be attributed

381   to both the different mechanisms of the fold formation and the mechanical role of the layers.

382   Additionally, some discrepancy between observations can be related to the difference in the

383   orientation of the yz-sections. Alsop and Holdsworth (2012) analysed the yz-sections oriented

384   normal to the fold elongation, whereas we orient them normal to the shearing direction. In

385   our opinion, the shearing direction, which in the case of simple shear deformation coincides

386   with orientation of lineation and foliation, is easier to establish accurately in the field than the

387   fold elongation, especially when the analysed fold is not exposed in three-dimensions. For

388   large deformation, the folds are oriented subhorizontally and the difference between different

389   approaches of defining the yz sections diminishes.

390   In our fold shape analysis, we choose to analyse the closed contours in each interface

391   in the yz-section located at the fold base. The analysed interfaces form the outermost closed

392   contour in such sections. The outermost closed contour is one of the most distinct features of

393   the fold and it is generally easy to measure in the field. The same parameter has been used in

394   the sheath fold classification by Alsop and Holdsworth (2006) and the detailed sheath fold

395   analysis by Reber et al. (2012). The results of the R_{yz} analysis presented in the study are

396   consistent with the results presented in the work of Reber et al. (2013a).

5.2. Evolution of the sheath structures

During initial stages of deformation, the flow perturbation around rotating slip surface leads to development of the interface deflections near the slip surface tip, which are a trigger for the sheath structure development. Due to shearing, the sheath structure starts to develop ahead of the slip surface and it is accompanied by the formation of a synformal structure directly above the slip surface. Since the interfaces away from the fold and synformal structures are nearly undeformed, the fold growth must be balanced by the growth of the synform. Formation of the two structures was also described by Adamuszek et al. (2016), for sheath folds developing around layer-parallel slip surfaces.

For $\gamma > 10$, the evolution of the sheath structure is dominated by the passive simple shear deformation and is practically independent from the flow perturbation around the slip surface. Hence, the late evolution of the fold geometry can be easily obtained for an arbitrary strain by subjecting the fold to the simple shear deformation.

5.3. Strain map

The shape of each interface at different deformation stages can be uniquely described using the hinge angle α, the interlimb angle β, and the aspect ratio of the outermost closed contour R_{yz}. This gives a potential tool to decipher strain from the fold geometry. We show that two shape parameters are necessary to close the relation. We provide three strain gauge diagrams that relate strain magnitude to R_{yz} and α, R_{yz} and β, or α and β (Figure 7).

The diagrams illustrate that single geometrical parameter is not sufficient to determine strain and some generalizations can lead to erroneous interpretations e.g., large interlimb angle is not diagnostic for small strain values. For some cases, the diagrams can be used to obtain constraints on the strain magnitude value e.g., $R_{yz}=10$ indicates $\gamma > 25$.

The diagrams are constructed based on the analysis of the folds developed for a specific initial geometry and boundary conditions. However, based on the analysis provided by Reber et al. (2012), we expect that the gauge diagrams would look differently for models with e.g. different shape and orientation of the slip surface. The analysis of sensitivity of an additional factor on the set of three parameters could be a potential tool to gain more information about deformation or initial conditions. Since the exposures of natural sheath structures rarely allow obtaining the three geometrical parameters, we limited the analysis to two parameters and assumed a slip surface geometry and conditions that seem to be most likely to occur in nature in simple shear deformation regime.

14

429   Detailed field measurements of sheath folds such as presented by Alsop and
430   Holdsworth (2012) can be used to estimate strain magnitude. According to our model, the
431   obtained values of α varying between 12 and 40 degrees and R_{yz} varying between 4 and 5.6
432   correspond to shear strains between 8 and 15. However, neglecting effects due to the coaxial
433   component of deformation and viscosity differences between the layers may not be justified
434   in this case.

435   *5.4. Cone shape*

436   Despite the complex geometry of the sheath structure, we used a right cone to
437   approximate the fold shape and analyse how it deforms during simple shear. The approximate
438   model can be only applied to the passive stage of the sheath fold evolution, away from the
439   slip surface, so generally for γ>10. Both the numerical simulations and the analytical solution
440   show that R_{yz} scales with $\sqrt{\gamma}$ for large strain. Since in the analytical solution, we consider
441   the cone with a circular base, and the simulation results and the analytical solution are not
442   exactly the same. If we use an elliptical cone shape, we obtain the scaling that allows
443   matching the two curves. Using the initial cone angle α_0>160°, we could also reproduce the
444   correlation between α and γ. However, we could not obtain the exact scaling between β and γ.
445   This could be attributed to the simplification of the sheath structure while using the cone
446   shape. In the analytical solution, the hinge line variation is approximated with two half-lines
447   originating at the cone apex, whereas the results of the simulation show a sine-shaped hinge
448   line variation (top view of structure in Figure 3).

449   The deformation of the right cone shape to approximate the sheath fold development
450   was also studied by Skjernaa (1989) and Mies (1993). However, the authors examined the
451   fold evolution during passive amplification of the pre-existing perturbation and they used a
452   differently oriented cone. Thus, the results of their analysis are significantly different to those
453   presented in the study.

454   **6. Conclusions**

455   1. The flow perturbation around rotating and deforming slip surfaces in simple shear
456   causes interface deflections, which are precursors of the sheath structure.
457   2. Progressive deformation moves the developing sheath fold away from the slip
458   surface into the regions, where passive simple shear dominates. For shear strain
459   γ>10, passive deformation controls the sheath structure evolution.

15

460   3. Each interface of the sheath fold structure has a unique shape at a given strain. We

461   present three strain gauge diagrams that allow using two out of three parameters 1)

462   the aspect ratio of the outermost closed contour, 2) the hinge angle, and 3) the

463   interlimb angle to estimate the shear strain.

464   4. For large strains, the fold shape evolution can be approximated by the deformation

465   of a horizontally oriented cone shape. We developed an analytical expression that

466   describes the evolution of the aspect ratio of the outermost closed contour, the

467   interlimb angle, and the hinge angle. Due to the simplification of the fold

468   geometry, noticeable differences in the evolution of these parameters between

469   analytical solution and numerical result are recorded only for the hinge angle.

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523

Figure captions

525   Figure 1 A) Models of sheath fold formation in simple shear: passive amplification of
526   a pre-existing perturbation, flow perturbation above a corrugated rigid basement, flow
527   perturbation around a rigid inclusion, and flow perturbation around a slip surface (weak
528   inclusion) (modified after Cobbold and Quinquis, 1980). B) 3-dimensional sketch illustrating
529   interlimb angle (α), hinge angle (β), and aspect ratio of the outermost closed contour (R_{yz}) of
530   a sheath fold.

531

532   Figure 2. Schematic illustration of the model geometry in the xz section showing the
533   slip surface and the analysed interfaces. The model is subject to simple shear deformation
534   that acts in x direction. The slip surface is oriented at θ=135° to the shearing direction.

535

536   Figure 3 Side (xz) and a top (xy) view of the fold shapes for selected interfaces after
537   γ=30 for A) z_0=0.8, B) z_0=1.1, C) z_0=1.4, and D) z_0=1.7. Dotted white lines in the side view
538   show the hinge line of the structure. α and β indicate the interlimb and hinge angle,
539   respectively. Note that vertical exaggeration in the side view is 5:1. The black thick line
540   shows the locations of the outermost contour, which are presented in the yz view (the yz view
541   is in scale). R_{yz} denotes the aspect ratio of the outermost closed contour.

542

543   Figure 4 A) Fold height and B) fold length for different interfaces. Variation of C) the
544   aspect ratio (r_{yz}) and D) asymmetry of the closed contour along the x-axis. The results are
545   presented for γ=30. Star markers in C) and D) refer to the fold geometry presented in Figure 3.

546

547   Figure 5 The central xz section for A) γ=1, B) γ=5, C) γ=10, D) γ=20, and E) γ=30.
548   Selected interfaces are marked with thick lines (z0=±0.8, 1.1, 1.4, 1.7). Dashed lines separate
549   three distinct regions of deformation: I – undeformed or slightly deformed, II – forming basin
550   or dome structures, III – forming the sheath structure. The orientation of the slip surface (θ) is
551   indicated in the right lower corner. Note the vertical exaggeration of 5:1.

552

553   Figure 6 A) Fold height, B) fold length, and C) aspect ratio (R_{yz}) and D) asymmetry
554   of the outermost closed contour presented as a function of strain (γ) for nine different
555   interfaces.

556

557   Figure 7 Aspect ratio of the outermost contour (R_{yz}) as a function of A) interlimb
558   angle (α) and B) hinge angle (β). Solid lines show the simulation results for different
559   interfaces and dashed lines join the points of the same strain. The grey field in B) indicates
560   the area of sheath folds *sensu stricto*.

561

562   Figure 8 Colourmap illustrating the second invariant of the difference between the
563   rate of deformation measured in the model and in the background for γ=1, 5, 10, 20, and 30.
564   The blue and green contour lines mark 0.05 and 0.2 contour values, respectively. Grey lines
565   show the interface shape. For selected interfaces z0=±0.8, 1.1, 1.4, 1.7, and 2.0, we use a
566   thick line. Note that vertical exaggeration is 5:1.

567

568   Figure 9 A) Top view of the z_0=0.8 interface for three cases: at γ=9, at γ=15 subjected
569   to passive backward deformation of γ=-6, and at γ=21 subjected to passive backward
570   deformation of γ=-12. Black dashed lines indicate the yz and xz cross-section positions,
571   which are shown in B) and C), respectively.

572

573   Figure 10 Circular cone shape with initial opening angle $α_0$ used to approximate
574   sheath structure evolution. α and β denote the cone angles measured during deformation in
575   the two orthogonal planes that intersect along the cone axis. The thick black ellipse is a
576   contour of a cone intersection with a vertical section. R_y and R_z denote the horizontal and
577   vertical axis, respectively. The thick white line is an intersection of the cone with an inclined
578   section, which contains the lowermost point (red dot) of the black ellipse. R'_y and R'_z denote
579   the two axes of the white ellipse. The white ellipse corresponds to the location of the black
580   ellipse prior to a certain amount of simple shear. A) shows the cone structure and the
581   contours in a three-dimensional perspective whereas B) and C) are the side xz and yz views,
582   respectively.

583

584   Figure 11 Evolution of A) and B) contour aspect ratio, C) and D) interlimb angle, and
585   E) and F) hinge angle with strain. The plots on the left show the analytical solution for
586   different initial cone angles ($α_0$) varying between 10 and 170 degrees, whereas the plots on
587   the right illustrate simulation results for the outermost contour for the nine analysed
588   interfaces.

589

590   Figure A1 Cone shape illustrating a simplified model of a sheath fold. Dark red line
591   shows the cone axes, whereas the black line shows a contour of a vertical section through the
592   cone. Red line illustrates a contour through the cone normal to the cone axis. See text for
593   details.

594

595   Figure A2 Three-dimensional model of the initial cone shape that is used for the
596   approximation of the initial sheath fold shape. Black line shows the contour of the initial
597   section through the cone normal to the slant height, whereas the white line illustrates the
598   contour at the angle φ. A) shows a 3D perspective and B) and C) show side and top views of
599   the cone, respectively. See text for details.

600

601   Figure A3 A three-dimensional model of the deformed cone shape. White line
602   illustrates the contour that was initially at the angle φ (see Figure A2). Red line is an axillary
603   contour showing the section through the cone normal to the cone axis. A) shows a 3D
604   perspective and B) and C) show side and top view of the cone, respectively. See text for
605   details.

606

607 **Appendix: Evolution of a cone shape in simple shear.**

608 *A. Ellipse ratio for the initial section*

609 We derive the aspect ratio of the ellipse on a vertical section, which cuts the cone with

610 the initial opening angle α_0 at A at the right angle (black contour). The vertical ellipse axis

611 R_z is half of the $\overline{AB}$ segment and it can be easily found in the xz section (Figure A1B),

612 where $\overline{AB} = \overline{AO} \cdot \tan(\alpha_0)$, thus

$$R_z = \frac{\overline{AO}}{2} \tan(\alpha_0) \tag{10}$$

614 In order to determine the horizontal ellipse axis R_y, we construct an axillary section

615 through the cone, which is oriented normal to the cone axis and intersects the vertical section

616 along R_y axis (red contour). The axillary section gives a circle with a radius r. We find R_y

617 using the Pythagoras theorem in the right triangle MCE (see Figure A1C), where the segment

618 $\overline{CE}$ is R_y of an ellipse and $\overline{ME}$ is the circle radius r, thus

$$R_y = \sqrt{r^2 - \overline{MC}^2} \tag{11}$$

620 We define $\overline{MC}$ as the difference between two segments (see Figure A1B)

$$\overline{MC} = \overline{KC} - r \tag{12}$$

622 and use it in Eq. (11)

$$R_y = \sqrt{r^2 - \left(\overline{KC} - r\right)^2} = \sqrt{2 \cdot \overline{KC} \cdot r - \overline{KC}^2} =$$
$$= \sqrt{\overline{KC}^2 \left(\frac{2 \cdot r}{\overline{KC}} - 1\right)} = \overline{KC} \sqrt{\frac{2 \cdot r}{\overline{KC}} - 1} \tag{13}$$

624 Next, we find $\overline{KC}$ as (see Figure A1B):

$$\overline{KC} = \frac{R_z}{\cos\left(\frac{\alpha_0}{2}\right)} = \frac{\overline{AO}}{2} \cdot \frac{\tan(\alpha_0)}{\cos\left(\frac{\alpha_0}{2}\right)} \tag{14}$$

626 and derive r using the relation (see Figure A1B)

$$r = \overline{KM} = \overline{KO} \cdot \sin\left(\frac{\alpha_0}{2}\right) \tag{15}$$

628 Using the fact that

629
$$\overline{KO} = \frac{\overline{AO} + \overline{BO}}{2} \qquad (16)$$

630   and that

631
$$\overline{BO} = \frac{\overline{AO}}{\cos(\alpha_0)} \qquad (17)$$

632   we obtain

633
$$r = \frac{\overline{AO}}{2} \cdot \left[1 + \frac{1}{\cos(\alpha_0)}\right] \cdot \sin\left(\frac{\alpha_0}{2}\right) = \frac{\overline{AO}}{2} \cdot \frac{1 + \cos(\alpha_0)}{\cos(\alpha_0)} \cdot \sin\left(\frac{\alpha_0}{2}\right) \qquad (18)$$

634   We   rearrange   Eq.   (18)   using   the   half   angle   formula,   where

635    $\cos^2(\alpha_0/2) = \left[1 + \cos(\alpha_0)\right]/2$ and   further   the   double   angle   formula,   where

636    $\sin(\alpha_0) = 2\sin(\alpha_0/2)\cos(\alpha_0/2)$ and get

637
$$r = \overline{AO} \cdot \frac{\cos^2\left(\frac{\alpha_0}{2}\right)}{\cos(\alpha_0)} \cdot \sin\left(\frac{\alpha_0}{2}\right) = \frac{\overline{AO}}{2} \cdot \frac{\cos\left(\frac{\alpha_0}{2}\right)}{\cos(\alpha_0)} \cdot \sin(\alpha_0) = \frac{\overline{AO}}{2} \cdot \cos\left(\frac{\alpha_0}{2}\right) \cdot \tan(\alpha_0) \qquad (19)$$

638   We expand the part under the square root in Eq. (13) using Eq. (14) and Eq. (19), and

639   simplify it

640
$$\frac{2 \cdot r}{\overline{KC}} - 1 = \frac{2 \cdot \frac{\overline{AO}}{2} \cdot \cos\left(\frac{\alpha_0}{2}\right) \cdot \tan(\alpha_0)}{\frac{\overline{AO}}{2} \cdot \frac{\tan(\alpha_0)}{\cos\left(\frac{\alpha_0}{2}\right)}} - 1 = 2\cos^2\left(\frac{\alpha_0}{2}\right) - 1 = \cos(\alpha_0) \qquad (20)$$

641   Substituting Eq. (20) and Eq. (14) into Eq. (13), the horizontal ellipse axis is given by

642
$$R_y = \frac{\overline{AO}}{2} \cdot \frac{\tan(\alpha_0)}{\cos\left(\frac{\alpha_0}{2}\right)}\sqrt{\cos(\alpha_0)} \qquad (21)$$

643   Finally, we calculate the elliptical ratio (R_{yz}) using Eq. (21) and Eq. (10), thus

644
$$R_{yz} = \frac{R_y}{R_z} = \frac{\frac{\overline{AO}}{2} \cdot \tan(\alpha_0) \cdot \frac{\sqrt{\cos(\alpha_0)}}{\cos\left(\frac{\alpha_0}{2}\right)}}{\frac{\overline{AO}}{2} \cdot \tan(\alpha_0)} = \frac{\sqrt{\cos(\alpha_0)}}{\cos\left(\frac{\alpha_0}{2}\right)} \qquad (22)$$

645   *B. Ellipse ratio of the deformed cone*

646   Now, we derive the vertical and horizontal ellipse axes of an ellipse R_z' and R_y',

647   accordingly, as a function of φ, which represent different section orientation with respect to

648   the vertical direction. The relation between φ and γ is given by

649
$$\gamma = \tan(\varphi) \tag{23}$$

650   The vertical ellipse axis R_z' is represented by half of the segment PB segment and it

651   can be derived from the relations visible in the xz section (Figure A2B).

652
$$R_z' = \frac{R_z}{\cos(\varphi)} = \frac{\overline{AO}}{2} \cdot \frac{\tan(\alpha_0)}{\cos(\varphi)} \tag{24}$$

653   R_y' is found from the relations visible in Figure A2C using the Pythagoras theorem in

654   the triangle TEM:

655
$$R_y' = \sqrt{r^2 - \overline{ME}^2} \tag{25}$$

656   We find $\overline{ME}$ as the difference between two segments $\overline{ME} = r - \overline{KE}$ (Figure A2B),

657   thus

658
$$R_y' = \sqrt{r^2 - \overline{ME}^2} = \sqrt{r^2 - \left(r - \overline{KE}\right)^2} = \sqrt{2 \cdot \overline{KE} \cdot r - \overline{KE}^2} =$$
$$= \sqrt{\overline{KE}^2 \left(\frac{2 \cdot r}{\overline{KE}} - 1 \right)} = \overline{KE} \sqrt{\frac{2 \cdot r}{\overline{KE}} - 1} \tag{26}$$

659   To get $\overline{KE}$, we use law of sines in the triangle KEP and obtain a relation

660
$$\frac{\overline{KE}}{\sin(90 - \varphi)} = \frac{R_z'}{\sin\left(90 + \dfrac{\alpha_0}{2}\right)} \tag{27}$$

661   which we rearrange and simplify

662
$$\overline{KE} = R_z' \frac{\sin(90 - \varphi)}{\sin\left(90 + \dfrac{\alpha_0}{2}\right)} = R_z' \frac{\cos(\varphi)}{\cos\left(\dfrac{\alpha_0}{2}\right)} \tag{28}$$

663   Substituting R_z' from Eq. (24), we obtain

664
$$\overline{KE} = \frac{\overline{AO}}{2} \cdot \frac{\tan(\alpha_0)}{\cos(\varphi)} \cdot \frac{\cos(\varphi)}{\cos\left(\dfrac{\alpha_0}{2}\right)} = \frac{\overline{AO}}{2} \cdot \frac{\tan(\alpha_0)}{\cos\left(\dfrac{\alpha_0}{2}\right)} \tag{29}$$

665   Next, we derive r from relation (Figure A2B)

$$r = \overline{KM} = \overline{KO} \cdot \sin\left(\frac{\alpha_0}{2}\right) \tag{30}$$

667   where

$$\overline{KO} = \frac{\overline{PO} + \overline{BO}}{2} \tag{31}$$

669   $\overline{PO}$ is defined as

$$\overline{PO} = \overline{PA} + \overline{AO} \tag{32}$$

671   where

$$\overline{PA} = 2R_z \cdot \tan(\varphi) = 2R_z \cdot \gamma \tag{33}$$

673   We get R'_z from Eq. (24) and substitute Eq. (33) into Eq. (32)

$$\overline{PO} = 2\frac{\overline{AO}}{2} \cdot \tan(\alpha_0) \cdot \gamma + \overline{AO} = \overline{AO}\left[\gamma \cdot \tan(\alpha_0) + 1\right] \tag{34}$$

675   We expand Eq. (31) using Eq. (34) and Eq. (17)

$$\overline{KO} = \frac{\overline{AO}\left(\gamma \tan(\alpha_0) + 1\right) + \dfrac{\overline{AO}}{\cos(\alpha_0)}}{2} = \frac{\overline{AO}}{2}\left[\gamma \tan(\alpha_0) + \frac{1 + \cos(\alpha_0)}{\cos(\alpha_0)}\right] \tag{35}$$

677   Using the half angle formula for the cosine function, we get

$$\overline{KO} = \frac{\overline{AO}}{2}\left[\gamma \tan(\alpha_0) + \frac{2\cos^2\left(\dfrac{\alpha_0}{2}\right)}{\cos(\alpha_0)}\right] \tag{36}$$

679   We substitute Eq. (36) into Eq. (30)

$$r = \frac{\overline{AO}}{2}\left[\gamma \tan(\alpha_0) + \frac{2\cos^2\left(\dfrac{\alpha_0}{2}\right)}{\cos(\alpha_0)}\right]\sin\left(\frac{\alpha_0}{2}\right) \tag{37}$$

681   and modify it using the formula for double angles for the sine function

682

$$r = \frac{\overline{AO}}{2}\left[\gamma \tan\left(\alpha_0\right)\sin\left(\frac{\alpha_0}{2}\right) + \frac{\cos\left(\frac{\alpha_0}{2}\right)\sin\left(\alpha_0\right)}{\cos\left(\alpha_0\right)}\right] = \frac{\overline{AO}}{2}\tan\left(\alpha_0\right)\left[\gamma \sin\left(\frac{\alpha_0}{2}\right) + \cos\left(\frac{\alpha_0}{2}\right)\right] \quad (38)$$

683

684   We expand the part under the square root in Eq. (26)

$$\frac{2 \cdot r}{\overline{KE}} - 1 = \frac{2 \cdot \dfrac{\overline{AO}}{2}\tan\left(\alpha_0\right)\left[\gamma \sin\left(\frac{\alpha_0}{2}\right) + \cos\left(\frac{\alpha_0}{2}\right)\right]}{\dfrac{\overline{AO}}{2}\cdot\dfrac{\tan\left(\alpha_0\right)}{\cos\left(\frac{\alpha_0}{2}\right)}} - 1 = $$

685   (39)

$$2\left[\gamma \sin\left(\frac{\alpha_0}{2}\right) + \cos\left(\frac{\alpha_0}{2}\right)\right]\cos\left(\frac{\alpha_0}{2}\right) - 1$$

686   and simplify it

687        $$\frac{2 \cdot r}{\overline{KE}} - 1 = 2\gamma \sin\left(\frac{\alpha_0}{2}\right)\cos\left(\frac{\alpha_0}{2}\right) + 2\cos^2\left(\frac{\alpha_0}{2}\right) - 1 = \gamma \sin\left(\alpha_0\right) + \cos\left(\alpha_0\right) \quad (40)$$

688   Substituting Eq. (40) and Eq. (29) into Eq. (26), we obtain

689        $$R'_y = \frac{\overline{AO}}{2}\cdot\frac{\tan\left(\alpha_0\right)}{\cos\left(\frac{\alpha_0}{2}\right)}\sqrt{\gamma \sin\left(\alpha_0\right) + \cos\left(\alpha_0\right)} \quad (41)$$

690   Therefore, the axial ratio of the tilted ellipse with respect to the initial cone shape is
691   given by

692   $$\frac{R'_y}{R'_z} = \frac{\dfrac{\overline{AO}}{2}\cdot\dfrac{\tan\left(\alpha_0\right)}{\cos\left(\frac{\alpha_0}{2}\right)}\sqrt{\gamma \sin\left(\alpha_0\right) + \cos\left(\alpha_0\right)}}{\dfrac{\overline{AO}}{2}\cdot\dfrac{\tan\left(\alpha_0\right)}{\cos\left(\varphi\right)}} = \cos\left(\varphi\right)\frac{\sqrt{\gamma \sin\left(\alpha_0\right) + \cos\left(\alpha_0\right)}}{\cos\left(\frac{\alpha_0}{2}\right)} \quad (42)$$

693   However, if we take the fact that R'_z after deformation becomes R_z (Eq. (10)), the
694   equation showing an evolution of the cone shape visible on the section has the form

695   $$\frac{R'_y}{R_z} = \frac{\dfrac{\overline{AO}}{2}\cdot\dfrac{\tan\left(\alpha_0\right)}{\cos\left(\frac{\alpha_0}{2}\right)}\sqrt{\gamma \sin\left(\alpha_0\right) + \cos\left(\alpha_0\right)}}{\dfrac{\overline{AO}}{2}\cdot\tan\left(\alpha_0\right)} = \frac{\sqrt{\gamma \sin\left(\alpha_0\right) + \cos\left(\alpha_0\right)}}{\cos\left(\frac{\alpha_0}{2}\right)} \quad (43)$$

696   *C. Ellipse ratio of the elliptic cone during deformation*

697   We also study a modified case of the elliptical cone, where the cone has an elliptical

698   shape in the section normal to the cone axis. The elliptilicity of the section is quantified with

699   a parameter f, which is the ratio of the major to minor ellipse axis $f = r_\beta / r_\alpha$. We

700   distinguish two cone angles α and β, which represent the largest and the smallest angle

701   between the cone's surface in the sections that cut through the cones axis. α is the cone angle

702   that is measured in the xz section. In the derivation, we use $r_\alpha = r$ and $r_\beta = f r_\alpha = f r$.

703   In such a case, R'_z is the same as in the previous analysis. The difference appears in

704   the analysis of R'_y. We employ the ellipse equation, where $\left(\overline{ME}/r_\alpha \right)^2 + \left(R_y / r_\beta \right)^2 = 1$. We

705   rearrange the equation as

706
$$\left(\overline{ME} \right)^2 + \left(\frac{R_y}{f} \right)^2 = r^2 \tag{44}$$

707   thus

708
$$R'_y = f \cdot \sqrt{r^2 - \overline{ME}^2} \tag{45}$$

709   The derivation of the elliptical cone is analogous as in Appendix B, thus, the final

710   formula is

711
$$R_{yz} = \frac{R'_y}{R_z} = f \frac{\sqrt{\gamma \sin \left(\alpha_0 \right) + \cos \left(\alpha_0 \right)}}{\cos \left(\dfrac{\alpha_0}{2} \right)} \tag{46}$$

712   *D. Evolution of the two cone angles*

713   After deformation, the point P moves to the initial position of A (cf. Figure A2 and

714   Figure A3). The length $\overline{PB} = \overline{AB} = 2R_z$ does not change. We can use Figure A3B and C to

715   calculate the evolution of cone angles α and β. The change of angle α can be described

716   through the relation (Figure A3B)

717
$$\alpha = \tan^{-1} \frac{2R_z}{\overline{PO}} \tag{47}$$

718   We substitute Eq. (10) and Eq. (34) and simplify the relation

719
$$\alpha = \tan^{-1} \frac{\overline{AO} \tan \left(\alpha_0 \right)}{\overline{AO} \left[\gamma \cdot \tan \left(\alpha_0 \right) + 1 \right]} = \tan^{-1} \left[\frac{\sin \left(\alpha_0 \right)}{\gamma \cdot \sin \left(\alpha_0 \right) + \cos \left(\alpha_0 \right)} \right] \tag{48}$$

720   The angle β can be described as

$$\beta = 2\tan^{-1}\frac{r_\beta}{\overline{MO}} \tag{49}$$

722   where (see Figure A3B)

$$\overline{MO} = \frac{r_\alpha}{\tan\left(\dfrac{\alpha}{2}\right)} \tag{50}$$

724   We find r_α using the relation

$$r_\alpha = \overline{KO}\cdot\sin\left(\frac{\alpha}{2}\right) \tag{51}$$

726   where $\overline{KO} = \left(\overline{PO}+\overline{BO}\right)/2$ and $\overline{BO} = \overline{PO}/\cos(\alpha)$, thus

$$r_\alpha = \frac{\overline{PO}}{2}\left(1+\frac{1}{\cos(\alpha)}\right)\cdot\sin\left(\frac{\alpha}{2}\right) = \frac{\overline{PO}}{2}\left(\frac{1+\cos(\alpha)}{\cos(\alpha)}\right)\cdot\sin\left(\frac{\alpha}{2}\right) \tag{52}$$

728   We simplify the equation using the half angle formula for the cosine function and the
729   double angle formula for the sine function

$$r_\alpha = \frac{\overline{PO}}{2}\cos\left(\frac{\alpha}{2}\right)\cdot\frac{\sin(\alpha)}{\cos(\alpha)} = \frac{\overline{PO}}{2}\cos\left(\frac{\alpha}{2}\right)\cdot\tan(\alpha) \tag{53}$$

731   We substitute $\overline{PO} = 2R_z/\tan(\alpha)$, so

$$r_\alpha = R_z\cos\left(\frac{\alpha}{2}\right) \tag{54}$$

733   We find r_β using the formula for the ellipse in the point E (Figure A3C)

$$\left(\frac{\overline{MC}}{r_\alpha}\right)^2 + \left(\frac{R_y'}{r_\beta}\right)^2 = 1 \tag{55}$$

735   and rearrange the formula

$$r_\beta = \frac{R_y'}{\sqrt{1-\left(\dfrac{\overline{MC}}{r_\alpha}\right)^2}} = \frac{R_y'}{\sqrt{\dfrac{r_\alpha^2-\overline{MC}^2}{r_\alpha^2}}} = \frac{R_y'\cdot r_\alpha}{\sqrt{r_\alpha^2-\overline{MC}^2}} \tag{56}$$

737   We substitute Eq. (54) and the expression $\overline{MC} = \overline{KC}-r_\alpha$ into Eq. (56) and simplify it

$$r_\beta = \frac{R_y'\cdot R_z\cos\left(\dfrac{\alpha}{2}\right)}{\sqrt{r_\alpha^2-\overline{MC}^2}} = \frac{R_y'\cdot R_z\cos\left(\dfrac{\alpha}{2}\right)}{\sqrt{r_\alpha^2-\left(\overline{KC}-r_\alpha\right)^2}} = \frac{R_y'\cdot R_z\cos\left(\dfrac{\alpha}{2}\right)}{\sqrt{2r_\alpha\overline{KC}-\left(\overline{KC}\right)^2}} \tag{57}$$

739   Expanding the expression under the square root and substituting the relation
740      $\overline{KC} = R_z/\cos(\alpha/2)$ and Eq. (54), we get

27

741
$$2r_\alpha \overline{KC} - \left(\overline{KC}\right)^2 = 2R_z \cos\left(\frac{\alpha}{2}\right)\frac{R_z}{\cos\left(\frac{\alpha}{2}\right)} - \frac{R_z^{\,2}}{\cos^2\left(\frac{\alpha}{2}\right)} \tag{58}$$

742   We simplify the expression and use the double angle formula for the cosine function

743
$$2r_\alpha \overline{KC} - \left(\overline{KC}\right)^2 = 2R_z^{\,2} - \frac{R_z^{\,2}}{\cos^2\left(\frac{\alpha}{2}\right)} = R_z^{\,2}\left(2 - \frac{1}{\cos^2\left(\frac{\alpha}{2}\right)}\right) =$$
$$= \frac{R_z^{\,2}}{\cos^2\left(\frac{\alpha}{2}\right)}\left[2\cos^2\left(\frac{\alpha}{2}\right) - 1\right] = \frac{R_z^{\,2}}{\cos^2\left(\frac{\alpha}{2}\right)}\left[2\cos^2\left(\frac{\alpha}{2}\right) - 1\right] = \frac{R_z^{\,2}}{\cos^2\left(\frac{\alpha}{2}\right)}\cos(\alpha) \tag{59}$$

744   Thus

745
$$r_\beta = \frac{R_y' \cdot R_z \cos\left(\frac{\alpha}{2}\right)}{\sqrt{\dfrac{R_z^{\,2}}{\cos^2\left(\frac{\alpha}{2}\right)}\cos(\alpha)}} = \frac{R_y' \cdot R_z \cos\left(\frac{\alpha}{2}\right)}{\dfrac{R_z}{\cos\left(\frac{\alpha}{2}\right)}\sqrt{\cos(\alpha)}} = \frac{R_y' \cos^2\left(\frac{\alpha}{2}\right)}{\sqrt{\cos(\alpha)}} \tag{60}$$

746   Finally, we substitute Eq. (60) and Eq. (50) into Eq. (49) and simplify it

747
$$\beta = 2\tan^{-1}\left[\frac{\dfrac{R_y' \cos^2\left(\frac{\alpha}{2}\right)}{\sqrt{\cos(\alpha)}}}{\dfrac{R_z \cos\left(\frac{\alpha}{2}\right)}{\tan\left(\frac{\alpha}{2}\right)}}\right] = 2\tan^{-1}\left[\frac{R_y'}{R_z}\frac{\sin\left(\frac{\alpha}{2}\right)}{\sqrt{\cos(\alpha)}}\right] \tag{61}$$

748

Sheath fold shape parameters

$$R_{yz} = \frac{R_y}{R_z}$$

Strain gauge diagrams

A)

B)

$$R_{yz} = \frac{R_y}{R_z}$$

2.0

1.0

0.707

0

θ=135°

Nine interfaces
(passive markers)
with z_0 =0.8 ... 2.0

Circular slip
surface (r=1)

Homogeneous,
isotropic, linear
viscous matrix

z

x

A) $z_0=1.7$ z 5:1   x   subsidiary hinge zone   y 1:1   x $\alpha=5.0$ $\beta=53$ $R_{yz}=10.8$ z 1:1   y

B) $z_0=1.4$ z 5:1   x   y 1:1   x $\alpha=3.3$ $\beta=32$ $R_{yz}=9.5$ z 1:1   y

C) $z_0=1.1$ z 5:1   x   y 1:1   x $\alpha=2.6$ $\beta=20$ $R_{yz}=8.1$ z 1:1   y

D) $z_0=0.8$ z 5:1   x   y 1:1   x $\alpha=2.3$ $\beta=13$ $R_{yz}=6.5$ z 1:1   y

A) γ = 1, θ = 90

B) γ = 5, θ = 14

C) γ = 10, θ = 6

D) γ = 20, θ = 3

E) γ = 30, θ = 2

A) $\gamma=1$

B) $\gamma=5$

C) $\gamma=10$

D) $\gamma=20$

E) $\gamma=30$

z

5:1

A)

G9

G15

G21

B)

| G9: γ = 9 |
| G15: γ = 15 - 6 |
| G21: γ = 21 - 12 |

C)

G21

G15

G9

A)

B)

C)

A) Analytical solutions

B) Simulation results

Legend for A):
- $\alpha_0=10$
- $\alpha_0=30$
- $\alpha_0=50$
- $\alpha_0=70$
- $\alpha_0=90$
- $\alpha_0=110$
- $\alpha_0=130$
- $\alpha_0=150$
- $\alpha_0=170$

Legend for B):
- $z_0=0.8$
- $z_0=0.95$
- $z_0=1.1$
- $z_0=1.25$
- $z_0=1.4$
- $z_0=1.55$
- $z_0=1.7$
- $z_0=1.85$
- $z_0=2.0$

Panel axes: A) R_{yz} vs γ; B) R_{yz} vs γ; C) α vs γ; D) α vs γ; E) β vs γ; F) β vs γ

A)

B)

C)

$90-\alpha_0/2$

α_0

O

R_z

K   A

M

C

L

B

r

E

R_y

K

A

M

C

D

L

B

E

z

y

x

z

x

A)

B)

C)

$90-\varphi$

$90+\alpha_0/2$

P K A O

α_0

R'_z

E

M

$2R_z$

φ

B

L

P

K

A

E

S

R'_y

M

T

r

D

B

L

A)

B)

C)