On ergodic invariant states and irreducible representations

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

In this thesis, based heavily on the work of Huang and Wu [HW17], we set out to study a specific C*-algebra defined therein, as the crossed-product $C^*(\mathbb{Z}(\frac{1}{pq})) \rtimes \mathbb{Z}^2$. We will also look at the irreducible representations of the C*-algebra satisfying some additional conditions. These representations are shown to be in a 1-1 correspondence with the ergodic states on $C^*(\mathbb{Z}(\frac{1}{pq})) \rtimes \mathbb{Z}^2$ and turn out to be of particular interest to the *Furstenberg* conjecture.

The connection between certain pure states on the crossed product and the ergodic states on the underlying C*-algebra was proven in the article for the case when the C*-algebra is unital and the group is discrete. We will give an alternative proof to this result.

For readability we have also added the needed prerequisites from ergodic theory and crossed product C*-algebra, either in the preliminary part, or in the appendices.

The thesis is divided into three main parts. In the Preliminary part we present the general theory of crossed product C*-algebras and their representations. In the next part, entitled "a review of the article of Huang and Wu", we retrace some of the results in [HW17]. In the final part (supplementary results) we construct the explicit representation associated with the Lebesgue measure, and show it satisfies the conditions stipulated in Corollary 2.12. Additionally, we formulate a characterisation of the representations associated with invariant measures on the torus, by showing they must have an irreducible decomposition of a certain type.

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Contents

CHAPTER 1

Preliminaries

1.1 A review of C*-crossed products

In this section we introduce the definitions and notation used in the article of Huang and Wu [HW17] and try to motivate the construction of the crossed product C*-algebra. Unless stated otherwise, $\mathcal{A}$ will denote a unital C*-algebra, G a discrete group and H will always denote a complex Hilbert space.

Preliminary definitions

Definition 1.1 (C*-dynamical system)**.** A C*-dynamical system is a triple $(\mathcal{A}, G, \alpha)$, where $\alpha : G \to Aut(\mathcal{A})$ is a group homomorphism into the group of $\star$-automorphisms of $\mathcal{A}$.[1]

Definition 1.2 (Covariant representation)**.** A covariant representation of a dynamical system $(\mathcal{A}, G, \alpha)$ is defined as a pair (π, U) where $\pi : \mathcal{A} \to B(H)$ is a *non-degenerate* representation of $\mathcal{A}$ on some Hilbert space H, $U : G \to B(H)$ is a unitary representation of G on H, satisfying the "covariance" relation:[2]

$$\pi(\alpha(g)(a)) = U(g)\pi(a)U(g)^* \qquad \forall\, g \in G, a \in \mathcal{A} \tag{1.1}$$

The next proposition and proof, in addition to its intrinsic value, will come in handy later, when we define two norms on a dense subset of the crossed-product C*-algebra.

Proposition 1.3 (Existence of covariant representations)**.** *For any dynamical system $(\mathcal{A}, G, \alpha)$ there exists at least one covariant representation associated*

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[1] Generally the homomorphism α is required to be "strongly continnous", that is, the map $g \mapsto \alpha_g(a)$ is continuous for all $a \in \mathcal{A}$, but we dropped this requirement, since G is assumed to be discrete.

[2] Henceforth we will often denote $U(g)$ and $\alpha(g)$ simply by U_g and α_g respectively.

with any non-degenerate representation $\pi : \mathcal{A} \to B(H)$ of $\mathcal{A}$, namely the **regular covariant representation**, *which we will denote by $(\tilde{\pi}, \lambda^H)$.*[3]

Proof. Letting $\pi : \mathcal{A} \to B(H)$ be any non-degenerate representation of $\mathcal{A}$, we define $(\tilde{\pi}, \lambda^H)$ to be the covariant representation on the Hilbert space direct sum

$$\bigoplus_{g \in G} H = \{f : G \to H \mid \sum_{g \in G} ||f(g)||^2 < \infty\} =: l^2(G, H),$$

given by

$$(\tilde{\pi}(a)\xi)(s) = \pi(\alpha_{s^{-1}}(a))\xi(s) \qquad \text{(for all } s \in G, \ a \in \mathcal{A} \text{ and } \xi \in l^2(G, H),)$$
$$(\lambda_g^H \xi)(s) = \xi(g^{-1}s). \qquad \text{(for all } s, g \in G, \text{ and } \xi \in l^2(G, H))$$

One can check these are indeed well defined representations, and that λ_g^H is unitary, since it acts on $\xi \in l^2(G, H)$ by permuting the indices G, hence $||\lambda_g^T \xi||^2 = \sum_{h \in G} ||\xi(g^{-1}h)||^2 = \sum_{h \in G} ||\xi(h)||^2 = ||\xi||^2$. Additionally

$$((\lambda_g^H \tilde{\pi}(a)(\lambda_g^H)^*)\xi)(s) = (\tilde{\pi}(a)(\lambda_g^H)^*)\xi)(g^{-1}s) = (\tilde{\pi}(a)(\lambda_{g^{-1}}^H))\xi)(g^{-1}s)$$
$$= [\pi(\alpha_{(g^{-1}s)^{-1}}(a))](\lambda_{g^{-1}}^H)(\xi)(g^{-1}s) = \pi(\alpha_{s^{-1}}(\alpha_g(a)))\xi(s)$$
$$= (\tilde{\pi}(\alpha_g(a))\xi)(s)$$

for all $\xi \in l^2(G, H)$ and $s \in G$, where we used that $(\lambda_g^H)^* = \lambda_{g^{-1}}^H$. So the covariance relation (1.1) also holds. ∎

Universal property of the crossed-product C*-algebra

Before stating the definition of the crossed product, note that if (π, U) is a covariant representation of $(\mathcal{A}, G, \alpha)$, by the covariance relation, we have the equalities

$$(\pi(a)U_g)^* = \pi(\alpha_{g^{-1}}(a^*))U_{g^{-1}} \quad \text{and} \quad (\pi(a)U_g)(\pi(b)U_h) = \pi(a\alpha_g(b))U_{gh}.$$

Consequently to any covariant representation (π, U) of a dynamical system $(\mathcal{A}, G, \alpha)$ we can associate the C*-algebra

$$C^*(\pi, U) = \overline{\text{span}\{\pi(a)U_g \ : \ a \in \mathcal{A}, \ g \in G\}}.$$

A natural question would then be if the above defined C^*-algebra could be a working definition for the crossed product of $\mathcal{A}$ and G. One issue is that we

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[3] Some authors denote the regular representation by $ind_e\pi$, since it is equivalent to the "induced" representation of π with respect to the trivial subgroup $\{e\} \subset G$ (see [Wil07] remark 5.7 on page 156), but a treatment of induced representations is outside the scope of this thesis.

have not address the arbitrariness in the choice of covariant representation.

To remedy this, let's instead define the crossed product $G \rtimes_\alpha \mathcal{A}$ by the following universal property which was first introduced in [Rae88] in a slightly more general form. It relies on the concept of a *covariant homomorphism* of a dynamical system $(\mathcal{A}, G, \alpha)$ into a C^*-algebra $\mathcal{B}$ which is defined as a pair $(j_\mathcal{A}, j_G)$, where $j_\mathcal{A} : \mathcal{A} \to \mathcal{B}$ is a $\star$-homomorphism and $j_G : G \to U\mathcal{B}$ (the unitaries) is a group homomorphisms satisfying the relation $j_\mathcal{A}(\alpha_g(a)) = j_G(g)j_\mathcal{A}(a)j_G(g)^*$.

Definition 1.4 (Raeburn)**.** The C*-crossed product associated with $(\mathcal{A}, G, \alpha)$, denoted $\mathcal{A} \rtimes_\alpha G$, is the C*-algebra satisfying the following universal properties

- There exists a covariant homomorphism $(j_\mathcal{A}, j_G)$ of $(\mathcal{A}, G, \alpha)$ into $\mathcal{A} \rtimes_\alpha G$

- For any covariant representation (π, U) of $(\mathcal{A}, G, \alpha)$, there is a non-degenerate representation $L = L_{(\pi,U)}$ of $\mathcal{A} \rtimes_\alpha G$ such that

$$L \circ j_\mathcal{A} = \pi$$
$$L \circ j_G = U$$

- $\mathcal{A} \rtimes_\alpha G = \overline{\text{span}\{j_\mathcal{A}(a)j_G(g) \mid a \in \mathcal{A}, \ g \in G\}}$

and is unique in the following sense; If $(\mathcal{B}, j'_\mathcal{A}, j'_G)$ also satisfy the above conditions for some unital C*-algebra $\mathcal{B}$, then there exists an isomorphism $\phi : \mathcal{A} \rtimes_\alpha G \to B$ such that $j_\mathcal{A} = \phi \circ j'_\mathcal{A}$ and $j_G = \phi \circ j_G$.

Below we show the existence of such a crossed product C*-algebra. As one would expect, historically that definition preceded the the one given in Definition 1.4, but working with the universal definition above has its benefits, as evidences in appendix B.

The Skew-Algebra

To highlight the similarities with the semidirect product of groups, we try to give the algebraic intuition behind the the construction of the concrete realisation of the crossed product $\mathcal{A} \rtimes_\alpha G$.

Let's start with the vector space $W = \{\sum_{(a,g) \in I}(a,g) \mid I \subset \mathcal{A} \times G, I \text{ is finite}\}$ over $\mathbb{C}$ consisting of all finite formal sums of elements in $\mathcal{A} \times G$ with scalar multiplication distribuiting over sums and defined elementwise by $\lambda(a,g) = (\lambda a, g)$. Addition is treated formally, that is $\sum_{(a,g) \in I}(a,g) + \sum_{(a,g) \in J}(a,g) = \sum_{(a,g) \in I \sqcup J}(a,g)$ where $\sqcup$ is the disjoint union of the index sets.

Just as with the semidirect product, we may define a product operation on W, by

$$(a, g_1) \cdot (b, g_2) = (a\alpha_{g_1}(b), g_1 g_2).$$

making W an algebra over $\mathbb{C}$.

The subspace $V \subset W$ defined as the span

$$V = \text{span}\{(a_1, g) + (a_2, g) - (a_1 - a_2, g) \mid (a_1, g), (a_2, g) \in \mathcal{A} \times G\}$$

is a two sided ideal of W since

$$
\begin{aligned}
(b, h) \cdot [(a_1, g) + (a_2, g) - (a_1 - a_2, g)] &= (b\alpha_h(a_1), hg) + (b\alpha_h(a_2), hg) \\
&\quad - (b\alpha_h(a_1) + b\alpha_h(a_2), hg) \in V \\
[(a_1, g) + (a_2, g) - (a_1 - a_2, g)] \cdot (b, h) &= (a_1\alpha_g(b), gh) + (a_2\alpha_g(b), gh) \\
&\quad - (a_1\alpha_g(b) + a_2\alpha_g(b), gh) \in V
\end{aligned}
$$

The quotient algebra W/V, consisting of finite sums on the form $\sum_{g \in J \subset G}(a_g, g)$, contains a copy of $\mathcal{A}$, (by the inclusion $a \mapsto (a, e)$) and a copy of G (by the inclusion $g \mapsto (1, g)$), and is called the *skew algebra*.

We can identify this algebra with $C_c(G, \mathcal{A})$, the algebra of finitely supported functions from G to $\mathcal{A}$, in a natural way by the map

$$\sum_{g \in G}(a_g, g) \mapsto f \qquad \text{where} \qquad f(g) = a_g.$$

Under this identification, the above defined product induce a product on $C_c(G, \mathcal{A})$ given by:

$$(f_1 \star f_2)(h) = \sum_{g \in G} f_1(g)\alpha_g(f_2(g^{-1}h)).$$

We define the involution $f^\star(g) = \alpha_g(f(g^{-1})^*)$, making $C_c(G, \mathcal{A})$ a unital *-algebra. In addition, with $\delta_{a,g} \in C_c(G, \mathcal{A})$ given by

$$\delta_{a,g}(h) = \begin{cases} a & g = h \\ 0 & g \neq h \end{cases} \tag{1.2}$$

the previously defined inclusion maps of $\mathcal{A}$, and G, lift to the maps

$$\iota_{\mathcal{A}}(a) = \delta_{a,e} \qquad \iota_G(g) = \delta_{1,g} \tag{1.3}$$

Some authors write a and u_g for the inclusions $\delta_{a,e}$ and $\delta_{1,g}$ respectively, but we have chosen to stick with this δ notation.

The concrete crossed-product C*-algebra

Now let (π, U) be a covariant representation of $(\mathcal{A}, G, \alpha)$ on a Hilbert space H. Define $\pi \times U : C_c(G, \mathcal{A}) \to B(H)$ by [4]

$$\pi \times U(f) = \sum_{g \in G} \pi(f(g))U_g \qquad \text{for all } f \in C_c(G, \mathcal{A})$$

then $\pi \times U$ is a representation of $C_c(G, \mathcal{A})$ on H, which is non-degenerate, since $\pi \times U(\delta_{1,e}) = \pi(1)U(e) = I_H$ for the identity element $\delta_{1,e}$ of $C_c(G, \mathcal{A})$.

Next we wish to complete this *-algebra with respect to a C*-norm, and here there are two common choices. For the first, we need the following lemma

Lemma 1.5. *If $\pi : \mathcal{A} \to B(H)$ is a faithful non-degenerate representation, then $\tilde{\pi} \times \lambda^H$ is a non-degenerate faithful representation of $C_c(G, \mathcal{A})$, where $(\tilde{\pi}, \lambda^H)$ is the regular covariant representation of Proposition 1.3.*

Proof. Let π be a faithful non-degenerate representation of $\mathcal{A}$, let $f \in C_c(G, \mathcal{A})$ be non-zero and let $g_0 \in supp(f)$. Since π is faithful and $\alpha_{g_0^{-1}}$ is an automorphism, we may pick an $x \in H$ such that $\pi(\alpha_{g_0^{-1}}(f(g_0)))x \neq 0$. Define, $\Delta_{x,g} \in C_c(G, H) \subset l^2(G, H)$ by

$$\Delta_{x,g}(s) = \begin{cases} x & \text{if } s = g \\ 0 & \text{else} \end{cases}$$

Note that $\lambda_g^H \Delta_{x,e} = \Delta_{x,g}$ for all $g \in G$. Hence we get

$$\begin{aligned}
\left[\tilde{\pi} \times \lambda^H(f)(\Delta_{x,e})\right](g_0) &= \sum_{g \in G} \left[(\tilde{\pi}(f(g))\lambda_g^H)(\Delta_{x,e})\right](g_0) \\
&= \sum_{g \in G} \left[\pi(\alpha_{g_0^{-1}}(f(g)))(\lambda_g^H(\Delta_{x,e}))\right](g_0) \\
&= \sum_{g \in G} \left[\pi(\alpha_{g_0^{-1}}(f(g)))(\Delta_{x,g})\right](g_0) \\
&= \pi(\alpha_{g_0^{-1}}(f(g_0)))x \neq 0
\end{aligned}$$

So $\tilde{\pi} \times \lambda^H(f) \neq 0$ and $\tilde{\pi} \times \lambda^H$ is injective. ∎

The definition now reads

Definition 1.6 (Reduced Crossed Product C*-algebra)**.** The **reduced crossed product C*-algebra** associated with the dynamical system (A, G, α), denoted

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[4]Note that since G is discrete any function $f \in C_c(G, \mathcal{A})$ has finite support, and the sum is well defined.

$\mathcal{A} \rtimes_{r,\alpha} G$, or $\mathcal{A} \rtimes_r G$, is defined as the C*-completion of $C_c(G, \mathcal{A})$ with respect to the following C*-norm

$$||f||_r = ||\tilde{\pi} \times \lambda^H(f)||$$

Where $\tilde{\pi} \times \lambda^H$ is the regular representation associated with any faithful non-degenerate representation π of $\mathcal{A}$.

Remark 1.7. It is straightforward to check that $|| \cdot ||_r$ is indeed a C*-seminorm, and employing Lemma (1.5) we see that $||f||_r = 0 \Rightarrow f = 0$, hence it is a C*-norm. The above definition can also be shown to be independent of choice of faithful representation π (see prop. 4.1.5 in [BO08])

Now we introduce a second norm on $C_c(G, \mathcal{A})$, called the universal norm, defined by

$$||f||_u = \sup ||\pi \times U(f)|| \tag{1.4}$$

where the supremum is taken over all covariant representations (π, U) of $(\mathcal{A}, G, \alpha)$. In [HW17] the supremum is taken over all representations of $C_c(G, \mathcal{A})$, which gives the same norm, but the above definition seems clearer, since it is immediate that there is an upper bound.

Proposition 1.8. $|| \cdot ||_u$ *is a well defined C*-norm on* $C_c(G, \mathcal{A})$*, satisfying the C*-equality.*

Proof. Submultiplicativity, and the C*-equality of $|| \cdot ||_u$ are straightforward to check. The supremum is finite, since we have a uniform bound given by

$$||\pi \times U(f)|| = ||\sum_{g \in G} \pi(f(g))U_g|| \leq \sum_{g \in G} ||f(g)|| := ||f||_1 < \infty$$

since f has finite support. Lastly since $||f||_u \geq ||f||_r$ we have that $||f||_u = 0 \Rightarrow ||f||_r = 0 \Rightarrow f = 0$, so it is indeed a C*-norm. ∎

Definition 1.9 (Crossed Product C*-algebra)**.** The **(full) crossed product C*-algebra** associated with a dynamical system (A, G, α), denoted $\mathcal{A} \rtimes_\alpha G$, is defined as the C*-completion of $C_c(G, \mathcal{A})$ with respect to the universal norm (1.4).

We now prove that the above constructed C*-algebra together with the inclusion maps $\iota_\mathcal{A}$ and ι_G defined by equation (1.3), is the universal object of Definition 1.4.

Proof of universality. By inspection we can see that

$$\delta_{\alpha_g(a),e} = \delta_{1,g} \star \delta_{a,e} \star \delta_{1,g}^*$$

making the previously defined inclusion maps $\iota_\mathcal{A}, \iota_G$ a covariant homomorphism of the system $(\mathcal{A}, G, \alpha)$.

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If (π, U) is a covariant representation of $(\mathcal{A}, G, \alpha)$, and $a \in \mathcal{A}$, $g \in G$ are arbitrary, we have

$$\pi \rtimes U \circ \iota_{\mathcal{A}}(a) = \pi(a)$$
$$\pi \rtimes U \circ \iota_G(g) = U(g)$$

so with $L_{(\pi, U)} = \pi \rtimes U$ the two first conditions of Proposition 1.4 are satisfied. Lastly we have that $\delta_{a,e} \star \delta_{1,g} = \delta_{a,g}$, hence

$$\operatorname{span}\{\iota_{\mathcal{A}}(a)\iota_G(g) \mid a \in \mathcal{A}, \ g \in G\} = C_c(G, \mathcal{A})$$

which is dense in $\mathcal{A} \rtimes_\alpha G$. ∎

Representations of Crossed products

For a C*-algebra $\mathcal{B}$ we may define the category $\operatorname{Rep}(\mathcal{B})$ of all non-degenerate representations of $\mathcal{B}$[5]. If π and π' are two non-degenerate representations of $\mathcal{B}$ on Hilbert spaces H and H' respectively, then we define the morphisms from π to π' to be the bounded equivariant linear maps $\phi : H \to H'$, where "equivariant" means

$$\pi'(a)(\phi v) = \phi(\pi(a)v) \qquad \text{for all } a \in \mathcal{B} \text{ and } v \in H.$$

We denote the set of such morphisms $\operatorname{Hom}_{\mathcal{B}}(\pi, \pi')$. As usual we define composition of morphisms as the composition of maps, and the identity morphisms are the identity maps.

Similarly for a discrete group G, we let $\operatorname{Rep}(G)$ denote the category of all unitary representations of a group G whose morphisms are again the bounded equivariant maps. We denote the set of all morphism from U to U' by $\operatorname{Hom}_G(U, U')$.

For a dynamical system $(\mathcal{A}, G, \alpha)$ we let $\operatorname{Rep}(A, G, \alpha)$ be the category whose objects are the covariant representations of $(\mathcal{A}, G, \alpha)$. As for the morphisms, if (π, U) and (π', U') are covariant representation on Hilbert spaces H and H' respectively, then we simply define

$$\operatorname{Hom}_{(A,G,\alpha)}((\pi, U), (\pi', U')) = \operatorname{Hom}_G(U, U') \cap \operatorname{Hom}_{\mathcal{A}}(\pi, \pi')$$

That is, the continuous linear maps from H to H' that commute with both actions.

Next, we define a map

$$L : \operatorname{Rep}(A, G, \alpha) \to \operatorname{Rep}(\mathcal{A} \rtimes_\alpha G)$$

by sending the objects $(\pi, U) \mapsto \pi \times U$ and acting as the identity on the morphisms.

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[5] This is clearly not a well defined set, as we have not fixed a Hilbert space for our representations. Such categories are called large, and the ways to overcome the set theoretic paradoxes is outside the scope of this thesis, and the competence of the author.

Proposition 1.10. *The map L is a functor and determines an isomorphism of categories*

$$Rep(\mathcal{A} \rtimes_\alpha G) \simeq Rep(A, G, \alpha).$$

Proof. First let's show $(\pi, U) \mapsto \pi \times U$ is indeed a bijection.
If ρ is a non-degenerate representation of $\mathcal{A} \rtimes_\alpha G$ on a H, let

$$\pi_\rho(a) = \rho(\iota_\mathcal{A}(a))$$
$$U_\rho(g) = \rho(\iota_G(g))$$

where $\iota_\mathcal{A}$ and ι_G are the inclusion maps defined earlier. Note that $\pi(1_\mathcal{A}) = I_H$ so π is also non-degenerate. One readily checks that (π_ρ, U_ρ) is a covariant representation of $(\mathcal{A}, G, \alpha)$ with $\pi_\rho \times U_\rho = \rho$. So L is surjective.

If $\pi \times U = \pi' \times U'$ then we have that

$$\pi(a) = (\pi \times U)(\iota_\mathcal{A}(a)) = (\pi' \times U')(\iota_\mathcal{A}(a)) = \pi'(a)$$
$$U(g) = (\pi \times U)(\iota_G(g)) = (\pi' \times U')(\iota_G(g)) = U'(g).$$

Thus L is also injective.

Lastly, we need to check that

$$\operatorname{Hom}_{(A,G,\alpha)}((\pi, U), (\pi', U')) = \operatorname{Hom}_{\mathcal{A} \rtimes_\alpha G}(\pi \times U, \pi' \times U').$$

For the first inclusion, let $\phi \in \operatorname{Hom}_{(A,G,\alpha)}((\pi, U), (\pi', U'))$, then

$$\begin{aligned}
\phi((\pi \times U)(f)(v)) &= \phi(\sum_{g \in G} \pi(f(g))U_g(v)) \\
&= \sum_{g \in G} \phi((\pi(f(g))U_g)(v))) \\
&= \sum_{g \in G} (\pi'(f(g))U'_g)(\phi(v)) \\
&= (\pi' \times U')(f)(\phi(v)).
\end{aligned}$$

We can conclude that $\phi \in \operatorname{Hom}_{\mathcal{A} \rtimes_\alpha G}(\pi \times U, \pi' \times U')$ and

$$\operatorname{Hom}_{(A,G,\alpha)}((\pi, U), (\pi', U')) \subset \operatorname{Hom}_{\mathcal{A} \rtimes_\alpha G}(\pi \times U, \pi' \times U').$$

Conversely, if $\phi \in \operatorname{Hom}_{\mathcal{A} \rtimes_\alpha G}(\pi \times U, \pi' \times U')$, we have that

$$\begin{aligned}
\phi(\pi(a)(v)) &= \phi((\pi \times U)(\iota_\mathcal{A}(a))(v)) \\
&= \pi' \times U'(\iota_\mathcal{A}(a))(\phi(v)) \qquad \text{(by equivariance)} \\
&= \pi'(a)(\phi(v))
\end{aligned}$$

so $\phi \in \mathrm{Hom}_{\mathcal{A}}(\pi, \pi')$. A similar argument shows that $\phi \in \mathrm{Hom}_G(U, U')$, which means $\phi \in \mathrm{Hom}_{(\mathcal{A}, G, \alpha)}((\pi, U), (\pi', U'))$ and

$$\mathrm{Hom}_{\mathcal{A} \rtimes_\alpha G}(\pi \times U, \pi' \times U') \subset \mathrm{Hom}_{(\mathcal{A}, G, \alpha)}((\pi, U), (\pi', U')).$$

Clearly for the identity maps (morphisms) we have $L(id) = id$ and $L(f \circ g) = L(f) \circ L(g)$ for any two morphisms (when compositions make sense). Hence L is a functor with inverse L^{-1} sending $\pi \times U \mapsto (\pi, U)$, and acting as the identity on the morphisms. ∎

Remark 1.11. We will not do much about this isomorphism, which is a restatement of Proposition 2.40 of [Wil07][p.59], but it should be mentioned that the functor L preserve direct sums and "irreducibility"[6]. For a proof of these facts, see the cited proposition.

If a representation π of $\mathcal{A}$ is not unitary equivalent to π', then $\mathrm{Hom}_{\mathcal{A}}(\pi, \pi')$ contains no unitary operator, and so neither will $\mathrm{Hom}_G(U, U') \cap \mathrm{Hom}_{\mathcal{A}}(\pi, \pi') = \mathrm{Hom}_{\mathcal{A} \rtimes_\alpha G}(\pi \times U, \pi' \times U')$, so we get the immediate corollary,

Corollary 1.12. *If either $\pi \not\sim \pi'$ or $U \not\sim U'$ then $\pi \rtimes U \not\sim \pi' \rtimes U'$, where $\sim$ denotes unitary equivalence.*

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[6]We say, as in [Wil07], that a covariant representation is irreducible if it has no proper closed subspace, invariant under both actions of π and U.

CHAPTER 2

A review of the article of Huang and Wu

From here on out, by a measure on a topological space, we will always mean a regular *Borel measure*. For a continuous map $T : \mathbb{T} \to \mathbb{T}$ we will denote the entropy of T with respect to a T-invariant measure μ by $h_\mu(T)$. For two integers p, q, a measure μ on $\mathbb{T}$ is said to be $\times p, \times q$-invariant if it is invariant with respect to the maps $T_p(z) = z^p$ and $T_q(z) = z^q$ on $\mathbb{T}$, that is $\mu = \mu \circ T_p^{-1}$ and $\mu = \mu \circ T_q^{-1}$. It is said to be ergodic if for every measurable A with $T_p^{-1}(A) = T_q^{-1}(A) = A$ we have that $\mu(A) \in \{0, 1\}$.

2.1   Statement of the Furstenberg Conjecture

For two integers $p, q > 1$, the group $\mathbb{Z}[\frac{1}{pq}]$ (the pq-adic rationals) and its dual group S_{pq} (pq-solenoid) will play an important role in the following sections, so we have collecting most of the needed background material on these groups in Appendix A.

We say that two positive integers $p, q > 1$ are *multiplicatively independent* independent if $\frac{log(p)}{log(q)} \notin \mathbb{Q}$.

One of the main results of the article [HW17] is a reformulation of the *Furstenberg* $\times p, \times q$ *conjecture*, which reads

Theorem 2.1 (Furstenberg $\times p, \times q$ Conjecture)**.** *For two multiplicatively independent integers $p, q > 1$, an ergodic $\times p, \times q$-invariant probability measure on $\mathbb{T}$ is either the Lebesgue measure, or has finite support.*

In the original paper of Furstenberg [Fur67][part IV] the theory was developed using the notion of *non-lacunary semigroups*, which where defined as multiplicative semi-subgroup E of $\mathbb{Z}$ with the property that there is no positive integer a such that $E \cap \mathbb{N} \subset \{a^0, a^1, ...\}$.

Note that $\frac{log(p)}{log(q)} \notin \mathbb{Q}$ if and only if for all non-zero positive integers n, m we have $p^n \neq q^m$. This implies that the multiplicative semigroup

$$E = \{p^n q^m \mid n, m = 1, 2, ...\}$$

is non-lacunary if p, q are multiplicatively independent. Indeed, if there are integers $n, m > 0$ such that $pq = a^n$, and $pq^2 = a^m$, it follows that $q = a^{k_1}$, where $k_1 = m - n$. Similarly we get that $p = a^{k_2}$ for some positive integer k_2. But then $p^{k_2} = q^{k_1}$, which is a contradiction.

Proposition IV.2 in [Fur67] says any closed subset of $\mathbb{T}$ invariant under the (multiplicative) action of a non-lacunary semigroup is either finite or $\mathbb{T}$ itself. In Appendix C we have shown how finite $\times p, \times q$-invariant subsets can be used to construct ergodic measures of finite support. One can also show that the Lebesgue measure is $\times p, \times q$-invariant and ergodic, and it is conjectured that these are all the possible $\times p, \times q$-invariant ergodic measures on $\mathbb{T}$.

From here on out we assume the integers $p, q > 1$ are multiplicatively independent. A well known result supporting the conjecture is the following (Theorem A of [Joh92])

Theorem 2.2 (Rudolph-Johnson). *If p, q are two positive integers greater than 1, $\frac{log(p)}{log(q)}$ is irrational, and μ is an ergodic $\times p, \times q$-invariant probability measure on $\mathbb{T}$, then either $h_\mu(T_p) = h_\mu(T_q) = 0$, or μ is the Lebesgue measure.*

Unfortunately having zero entropy does not generally entail the measure is finitely supported, so the Furstenberg conjecture remains unsolved, but it does give us a new formulation of the conjecture:

> The Furstenberg conjecture is true if and only if the only ergodic $\times p, \times q$-invariant probability measures μ for which $h_\mu(T_p) = h_\mu(T_q) = 0$ are finitely supported.

The starting point of the article is an idea presented to the authors in a private communication by J. Cuntz, which we will try to retrace now.

When μ is a $\times p, \times q$-invariant on $\mathbb{T}$, T_p, T_q induce two commuting isometries on $L^2(\mathbb{T}, \mu)$ given by the maps maps σ_p, σ_q determined by $\sigma_i(f)(x) = f(T_i(x))$. This follows from the change of variable formula,

$$||\sigma_p(f)||_2^2 = \int_\mathbb{T} |\sigma_p(f)|^2(x)d\mu(x) = \int_\mathbb{T} |f|^2(T_p x)d\mu(x) = \int_\mathbb{T} |f|^2(x)d\mu \circ T_p^{-1}(x) = ||f||_2^2$$

and similarly for σ_q.

If in addition $h_\mu(T_p) = h_\mu(T_q) = 0$, it was noted that σ_p, σ_q are invertible (Corollary 4.14.3 [Wal00][p.93]), hence unitaries. Together with the unitary operator $M \in B(L^2(\mathbb{T}, \mu))$ given by $M(f)(z) = zf(z)$, the three operators satisfy the relations

$$\sigma_p \sigma_q = \sigma_q \sigma_p \qquad \sigma_p M = M^p \sigma_p \qquad \sigma_q M = M^q \sigma_q.$$

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Universal C*-algebras generated by unitaries (a slight digression)

Now let s, t, u be three (indeterminate) unitaries that satisfy the relations;

$$st = ts \qquad su = u^p s \qquad tu = u^q t \qquad (2.1)$$

By the C*-algebra generated by unitaries $v_1, ..., v_n$ satisfying relations $R_1, ..., R_m$, we mean a unital C*-algebra $\mathcal{U}$ which is generated as a C*-algebra by $v_1, ..., v_n$, (that is, the closure of the non-commuting polynomials in variables $v_1, ..., v_n, v_1^*, ..., v_n^*$), and satisfies the following universal property:

> For every unital C*-algebra $\mathcal{B}$ generated by unitaries $V_1, ..., V_n$ satisfying the relations $R_1, ..., R_m$, there exists a surjective *-homomorphism $\phi : \mathcal{U} \to \mathcal{B}$ such that $\phi(v_i) = V_i$ for $i = 1, ..., n$.

We denote $\mathcal{U}$ by $C^*(u_1, ..., u_n \mid R_1, ..., R_m)$ and note that it is clearly unique up to (unique) isomorphism. One can show the existence of $C^*(u_1, ..., u_n \mid R_1, ..., R_m)$ as follows:

First form the group G with presentation $G = \langle g_1, ..., g_n \mid R_1, ..., R_n \rangle$. The group G has the property that for any other group K generated by elements $k_1, ..., k_n$ satisfying the relations $R_1, ..., R_m$ there exists a surjective group homomorphism $\phi : G \to K$ sending $g_i \mapsto k_i$ for $i = 1, ..., n$.

Hence if $U_1, ..., U_n$ are unitary operators on a Hilbert space H satisfying the relations $R_1, ..., R_m$ we may form the group (under composition) $K = \langle U_1, ..., U_n \rangle$. The surjective group homomorphism $\phi : G \to K$ is now a unitary representation of G on H, and thus lifts to a surjective *-homomorphism from $C^*(G)$ to the C*-algebra generated by $U_1, ..., U_n$. The crux of the mater is that there is a 1-1 correspondence between the unitary representations of G and the (non-degenerate) representations of $C^*(G)$.

In our concrete example, the group presentation associated with the three generators (s, t, u) and the relations (2.1), has a concrete realisation given by the isomorphism

$$\langle s, t, u \mid st = ts, \ su = u^p s, \ tu = u^q p \rangle \simeq \mathbb{Z}(\frac{1}{pq}) \rtimes \mathbb{Z}^2$$

which sends $s \mapsto a = (0, (1, 0))$, $t \mapsto b = (0, (0, 1))$ and $u \mapsto c = (1, (0, 0))$, and where $\mathbb{Z}^2$ acts on $\mathbb{Z}(\frac{1}{pq})$ by $(l, k) \cdot \frac{m}{(pq)^n} = p^l q^k \frac{m}{(pq)^n}$. The elements a, b, c generate $\mathbb{Z}(\frac{1}{pq}) \rtimes \mathbb{Z}^2$, since it can be checked that

$$(\frac{m}{(pq)^n}, (k, l)) = a^n b^n c^m a^{-n} b^{-n} a^k b^l.$$

We need to check that there are no further relation between the generators (a, b, c). Note that the relations (2.1) imply that

$$t^{-1}s = st^{-1} \qquad su^{-1} = u^{-p}s \qquad tu^{-1} = u^{-q}t$$

since

$$su^{-1} = (su^{-1})(s^{-1}s) = s(su)^{-1}s = s(u^p s)^{-1}s = u^{-p}s$$
$$tu^{-1} = (tu^{-1})(t^{-1}t) = t(tu)^{-1}t = t(u^q t)^{-1}t = u^{-q}t$$
$$st^{-1} = st^{-1}(s^{-1}s) = s(st)^{-1}s = s(ts)^{-1}s = t^{-1}s.$$

Using these relations we may write any element in a group generated by unitaries s, t and u satisfying the relations (2.1) as $u^k s^l t^j$ by grouping together terms of s, t and u.

To show that the group $\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2$ has no more relations between its generators is thus equivalent to showing that $c^k a^l b^j = 1$ implies that $k = l = j = 0$, where $1 = (0, (0,0))$ is the unit in $\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2$. But,

$$c^k a^l b^j = (k, (0,0)) \cdot (0, (l,0)) \cdot (0, (0,j)) = (k, (l,j)) = (0, (0,0)) \Leftrightarrow k = l = j = 0$$

and thus

$$C^*(s,t,u) = C^*\Big(\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2\Big) \simeq C^*\Big(\mathbb{Z}[\frac{1}{pq}]\Big) \rtimes \mathbb{Z}^2 \simeq C(S_{pq}) \rtimes \mathbb{Z}^2.$$

These identifications are explained in Appendix B.1. The group $\mathbb{Z}^2$ act on the dual group S_{pq} of $\mathbb{Z}[\frac{1}{pq}]$ by $(n,m) \cdot (z_i) = (z_i^{p^n q^m})$. This action lifts to an action on $C(S_{pq})$ given by $[(n,m)f]((z_i)) = f((z_i^{p^n q^m}))$.

Notations and an overview of the article

We will call a state on $C(S_{pq})$ $\times p, \times q$-invariant if it is invariant under the action of $\mathbb{Z}^2$ on $C(S_{pq})$, the set of all such states will be denoted $S_{\mathbb{Z}^2}(C(S_{pq}))$, the corresponding measures are denoted $M_{p,q}(S_{pq})$, and the extremal points of $S_{\mathbb{Z}^2}(C(S_{pq}))$ which we refere to as the **ergodic states** on S_{pq}, will be denoted by $E_{\mathbb{Z}^2}(C(S_{pq}))$, with $EM_{p,q}(S_{pq})$ their corresponding measures.

Similarly, we let $M_{p,q}(\mathbb{T})$ denote the space of $\times p, \times q$-invariant probability measures on $\mathbb{T}$ with associated states which we denote (by abuse of notation) $S_{\mathbb{Z}^2}(C(\mathbb{T}))$. We will refere these states as $\times p, \times q$-**invariant states** on $C(\mathbb{T})$.

The extremal points of $M_{p,q}(\mathbb{T})$, that is, the ergodic measures, will be denote by $EM_{p,q}(\mathbb{T})$ and the corresponding states (called the ergodic states on $C(\mathbb{T})$) we denote by $E_{\mathbb{Z}^2}(C(\mathbb{T}))$. We will see later that the action on $C(S_{pq})$ defined here is an extension of the action on $C(\mathbb{T})$ when treated as a subalgebra of $C(S_{pq})$ (see the diagram prior to Proposition 2.6 in section 2.2).

Each finitely supported ergodic $\times p, \times q$-invariant measure μ on $\mathbb{T}$ induces a representation $\pi_\mu : C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2 \to B(L^2(\mathbb{T}, \mu))$, determined by the equations

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$$\pi_\mu(\delta_{1,(0,0)})) = M \qquad \pi_\mu(\delta_{0,(1,0)}) = V_p \qquad \pi_\mu(\delta_{0,(0,1)}) = V_q. \qquad (2.2)$$

In [HW17] the authors sets out to characterise the representations of $C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2$ which are induced by $\times p, \times q$-ergodic measures, and those induced by ergodic $\times p, \times q$-invariant measures of finite support.

The first step is to construct a homeomorphism between the space of ergodic states on $C(\mathbb{T})$, and the space of ergodic states on $C(S_{pq})$.

Next a bijective correspondence is established between the space of ergodic states on $C(S_{pq})$ and the irreducible representations of $C(S_{pq}) \rtimes \mathbb{Z}^2$ whose restriction to $\mathbb{Z}^2$ contains the trivial representation. Among these the ones associated with measures of finite support are also characterised by the additional condition given by Theorem 4.2 of [HW17], which reads

Theorem 2.3. *A representation $\pi : C(S_{pq}) \rtimes \mathbb{Z}^2 \to B(H)$ is induced by an ergodic $\times p, \times q$-invariant measure on the circle $\mathbb{T}$ with finite support, if and only if*

- *π is irreducible*

- *$H_{\mathbb{Z}^2} = \{x \in H \mid \pi(g)x = x \ \forall \ g \in \mathbb{Z}^2\} \neq \{0\}$*

- *There exists a non-zero $N \in \mathbb{N}$ such that $\pi(\delta_{E_N,(0,0)})x = x$ for every $x \in H_{\mathbb{Z}^2}$, where $E_N \in C(S_{pq})$ is given by $E_N((z_i)) = z_0^N$.*

The first two conditions in the above proposition ensures that the representation is induced by an ergodic $\times p, \times q$-invariant measure on $\mathbb{T}$, while the last ensures it has finite support. Hence the authors were led to the following reformulation of the Furstenberg conjecture:
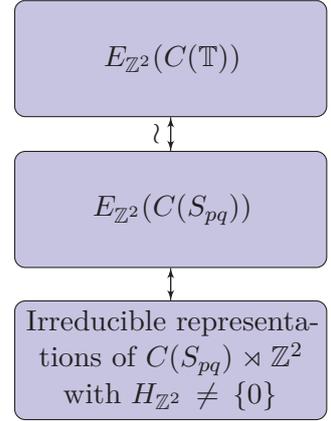
Corollary 2.4. *The Furstenberg conjecture is true if and only if there is precisely one irreducible representation $\pi : C(S_{pq}) \rtimes \mathbb{Z}^2 \to B(H)$, with $H_{\mathbb{Z}^2} \neq 0$, that does not satisfy the third point in Theorem 2.3 above, ie. such that*

$$\pi(\delta_{E_N,(0,0)})x \neq x \qquad \text{for all } N \in \mathbb{Z} \text{ and for all nonzero } x \in H_{\mathbb{Z}^2}.$$

The above representation is then the one induced by the Lebesgue measure on the unit circle. Later on we will find an explicit formula for this representation, but first lets assert that, alas, the regular representation does not meet the criterion of corollary 2.4,

Proposition 2.5 (Non-examples)**.** *The regular representation $\tilde{\pi} \times \lambda^H$, for any representation π of $C(S_{pq})$, does not satisfy the criterion in corollary 2.4*

Proof. Since $(\tilde{\pi} \rtimes \lambda^H)|_{\mathbb{Z}^2} = \lambda^H : \mathbb{Z}^2 \to U(l^2(G, H))$, and λ_g^H permuting the "indices" in G, it is not difficult to verify that $l^2(G, H)_{\mathbb{Z}^2} = \{0\}$, so it is not induced by an ergodic measure. ∎

$E_{\mathbb{Z}^2}(C(\mathbb{T}))$

$\updownarrow$

$E_{\mathbb{Z}^2}(C(S_{pq}))$

$\updownarrow$

Irreducible representations of $C(S_{pq}) \rtimes \mathbb{Z}^2$ with $H_{\mathbb{Z}^2} \neq \{0\}$

2.2 Main Results of the article

Now we restate the theorems and try to add some details to the proofs when needed. A crucial observation for the whole process is that we can identify the ergodic $\times p, \times q$ - invariant Borel probability measures on $\mathbb{T}$ or S_{pq} with the $\times p, \times q$-invariant states on $C(\mathbb{T})$ and $C(S_{pq})$ respectively. For this reason the two notions will be used interchangeably.

Identifying ergodic states on $C(S_{pq})$ with ergodic states on $C(\mathbb{T})$.

As mentioned in the previous section, the first step in the process is to show that the space of ergodic probability measures on S_{pq} and $\mathbb{T}$ are actually homeomorphic, this is Proposition 4.1 in [HW17] and it was mentioned that it is probably a previously known result.

In appendix A we showed that the algebra $C(\mathbb{T})$ embeds into $C(S_{pq})$, so it makes sense to talk about a "restriction" map $R : M_{p,q}(S_{pq}) \to M_{p,q}(\mathbb{T})$ when the measures are viewed as states on these algebras.

As we have seen, the action of $\mathbb{Z}^2$ on $C(S_{pq})$ is determined by the maps $\tau_p(f((z_i))) = f((z_i^p))$ and $\tau_q(f((z_i))) = f((z_i^q))$. Similarly, let $\sigma_p, \sigma_q : C(\mathbb{T}) \to C(\mathbb{T})$ be defined by $\sigma_p(f)(z) = f(z^p)$ and $\sigma_q(f)(z) = f(z^p)$, respectively.

Recall that a probability measure μ on S_{pq} (reps. on $\mathbb{T}$) is $\times p, \times q$-invariant if and only if the corresponding state ϕ_μ on $C(S_{pq})$ (resp. $C(\mathbb{T})$) is invariant under τ_p and τ_q (resp. σ_p and σ_q).

Now let $\iota : C(\mathbb{T}) \hookrightarrow C(S_{pq})$ be the natural inclusion defined by $(\iota \circ f)((z_i)) = f(z_0)$ (Appendix A). Since $(\iota \circ \sigma_p)(f)((z_j)) = f(z_0^p)$ and $(\tau_p \circ \iota)(f)((z_j)) = \iota(f)((z_j^p)) = f(z_0^p)$, we see that the following diagrams commute.

$$
\begin{array}{ccc}
C(\mathbb{T}) \xrightarrow{\ \iota\ } C(S_{pq}) & & C(\mathbb{T}) \xrightarrow{\ \iota\ } C(S_{pq}) \\
\downarrow{\sigma_p} \qquad\qquad \downarrow{\tau_p} & & \downarrow{\sigma_q} \qquad\qquad \downarrow{\tau_q} \\
C(\mathbb{T}) \xrightarrow{\ \iota\ } C(S_{pq}) & & C(\mathbb{T}) \xrightarrow{\ \iota\ } C(S_{pq})
\end{array}
$$

As in the article, we will refer to the map $R : M(S_{pq}) \to M(\mathbb{T})$, sending $\phi \mapsto \phi^\circ := \phi \circ \iota$, as the restriction map, where $M(S_{pq})$ and $M(\mathbb{T})$ denotes the space of probability measures on S_{pq} and $\mathbb{T}$ respectively. It has the following property

Proposition 2.6. *When equipped with the weak*-topologies, the restriction map $R : M(S_{pq}) \to M(\mathbb{T})$ is an (affine) homeomorphism between $M_{p,q}(S_{pq})$ and $M_{p,q}(\mathbb{T})$, hence also determines a homeomorphism between $EM_{p,q}(S_{pq})$ and $EM_{p,q}(\mathbb{T})$*

The proof is rather long, and uses the correspondence between (normalised) positive definite functions on $\mathbb{Z}[\frac{1}{pq}]$ and states on $C(S_{pq})$, but is essential for the subsequent parts of the thesis, so we add it here.

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Proof. To ease notation, we will denote $R(\psi) = \psi^\circ$ and $R^{-1}(\phi) = \tilde{\phi}$ for $\psi \in S(C(S_{pq}))$ and $\phi \in S(C(\mathbb{T}))$. First we check that R restricts to a well defined map from $M_{p,q}(S_{pq})$ to $M_{p,q}(\mathbb{T})$. By the above diagrams we know that if a state ψ on $C(S_{pq})$ is invariant under τ_p and τ_q, then for $f \in C(\mathbb{T})$, we have

$$\psi^\circ(\sigma_p(f)) = \psi(\iota(\sigma_p(f))) = \psi(\tau_p(\iota(f))) = \psi(\iota(f)) = \psi^\circ(f).$$

so ψ° is σ_p invariant. A similar argument shows it is σ_q invariant.

It is not hard to check that R sends positive functionals to positive functionals. The constant function $1_\mathbb{T} \in C(\mathbb{T})$ is sent to $\iota(1_\mathbb{T}) = 1_{S_{pq}}$, hence if $\psi \in S(C(S_{pq}))$, then $\psi^\circ(1_\mathbb{T}) = \psi(1_{S_{pq}}) = 1$, so R sends states to states.

Conversely, let ϕ be a σ_p, σ_q-invariant state on $C(\mathbb{T})$. The associated positive definite function P_ϕ on $\mathbb{Z}$ is given by

$$P_\phi(n) = \phi(e_n)$$

where $e_n \in C(\mathbb{T})$ is given by $e_n(z) = z^n$.

Now for any $l \in \mathbb{N}$ we note that $\sigma_p^l(f)(z) = f(z^{p^l})$ for all $f \in C(\mathbb{T})$, so we get $e_{p^l n}(z) = z^{p^l n} = (z^{p^l})^n = \sigma_p^l(e_n)(z)$. Hence

$$P_\phi(p^l n) = \phi(e_{p^l n}) = \phi(\sigma_p^l(e_n)) = \phi(e_n) = P_\phi(n) \qquad \text{for all } n \in \mathbb{Z}.$$

So we can conclude that P_ϕ is invariant under multiplication by p. A similar argument shows P_ϕ is also invariant under multiplication by q. We extend P_ϕ to a function $\tilde{P}_\phi$ defined on the whole of $\mathbb{Z}\left[\frac{1}{pq}\right]$ by

$$\tilde{P}_\phi\left(\frac{k}{(pq)^l}\right) = P_\phi(k).$$

This function is well defined, since if $\frac{k}{(pq)^l} = \frac{k'}{(pq)^{l'}}$, with $l' \leq l$ then $k = k'(pq)^{l-l'}$ and

$$P_\phi(k) = P_\phi(k'(pq)^{l-l'}) = P_\phi(k').$$

$\tilde{P}_\phi$ is also invariant under the group action of $\mathbb{Z}^2$ on $\mathbb{Z}[\frac{1}{pq}]$ (which is given by $(n,m) \cdot \frac{k}{(pq)^l} = p^n q^m \frac{k}{(pq)^l}$), since

$$\tilde{P}_\phi\left(\frac{pk}{(pq)^l}\right) = P_\phi(pk) = P_\phi(k) = \tilde{P}_\phi\left(\frac{k}{(pq)^l}\right),$$

$$\tilde{P}_\phi\left(\frac{1}{p}\frac{k}{(pq)^l}\right) = \tilde{P}_\phi\left(\frac{qk}{(pq)^{l+1}}\right) = P_\phi(qk) = P_\phi(k) = \tilde{P}_\phi\left(\frac{k}{(pq)^l}\right),$$

and similarly for q and $\frac{1}{q}$. Now we check that $\tilde{P}_\phi$ is indeed positive definite on $\mathbb{Z}[\frac{1}{pq}]$. Let $\lambda_1, ..., \lambda_n \in \mathbb{C}$ and $g_1, ..., g_n \in \mathbb{Z}[\frac{1}{pq}]$ be arbitrary. Then we may choose a $k \in \mathbb{N}$ such that $a_j := (pq)^k g_j \in \mathbb{Z}$ for $j = 1, ..., n$. We get

$$\sum_{i,j=1}^{n} \tilde{P}_\phi(g_i - g_j)\lambda_i\overline{\lambda}_j = \sum_{i,j=1}^{n} \tilde{P}_\phi((pq)^{-k}(a_i - a_j))\lambda_i\overline{\lambda}_j$$

$$= \sum_{i,j=1}^{n} \tilde{P}_\phi(a_i - a_j)\lambda_i\overline{\lambda}_j$$

$$= \sum_{i,j=1}^{n} P_\phi(a_i - a_j)\lambda_i\overline{\lambda}_j \geq 0$$

where the last inequality follows by the positive definiteness of P_ϕ.

Define $E_{\frac{k}{(pq)^l}} \in C(S_{pq})$ by $E_{\frac{k}{(pq)^l}}((z_j)) = z_l^k$. It is shown in Appendix that the span of all such functions is a dense subalgebra of $C(S_{pq})$.

For any E_g, with $g = \frac{m}{(pq)^n} \in \mathbb{Z}[\frac{1}{pq}]$, the state $\tilde{\phi}$ on $C(S_{pq})$ associated with $\tilde{P}_\phi$ is given by

$$\tilde{\phi}(E_g) = \tilde{P}_\phi(g).$$

With $(\tilde{\phi})^\circ = \tilde{\phi} \circ \iota$, we get that

$$(\tilde{\phi})^\circ(e_k) = \tilde{\phi}(\iota \circ e_k) = \tilde{\phi}(E_{\frac{k}{pq}}) = \tilde{P}_\phi(\frac{k}{pq}) = P_\phi(k) = \phi(e_k)$$

for all $k \in \mathbb{Z}$. It follows that $(\tilde{\phi})^\circ = \phi$ and that R is surjective.

Lastly, for a $\times p$, $\times q$-invariant state ψ on $C(S_{pq})$ and for all $g = \frac{m}{(pq)^n} \in \mathbb{Z}[\frac{1}{pq}]$, we have

$$(\widetilde{\psi^\circ})(E_g) = \tilde{P}_{\psi\circ\iota}(g) = P_{\psi\circ\iota}(m) = \psi(\iota(e_m))$$

$$= \psi(E_{\frac{m}{pq}}) = \psi(E_{\frac{(pq)^{n-1}m}{(pq)^n}})$$

$$= \psi(\tau_p^{n-1}\tau_q^{n-1}E_g) = \psi(E_g),$$

so $(\widetilde{\psi^\circ}) = \psi$, which shows that R is injective, hence a bijection. Being a restriction map R is clearly continuous and can easily be seen to be affine. It follows that R sends $EM_{p,q}(S_{pq})$ to $EM_{p,q}(\mathbb{T})$. Finally since all the spaces are compact, R is a homeomorphism. ∎

Associating a representation to ergodic states on $C(S_{pq})$

The next step in the process is to associate to each ergodic probability measure on S_{pq} a representation of $C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2$.

If $(\mathcal{A}, G, \alpha)$ is a C*-dynamical system, where $\mathcal{A}$ is unital and the group G is discrete, we may, for any $\phi \in S_G(\mathcal{A})$ associate a representation

$$\rho_\phi = \pi_\phi \rtimes U_\phi : \mathcal{A} \rtimes_\alpha G \to B(H_\phi). \qquad (2.3)$$

$\pi_\phi : \mathcal{A} \to B(H_\phi)$ is the usual GNS-representation on H_ϕ, and U_ϕ is the *Koopmann* representation, given by

$$U_\phi(g)(\widehat{a}) = \widehat{\alpha_g(a)}, \qquad \text{where } \widehat{a} = a + I_\phi \in H_\phi,$$

and $I_\phi = \{a \in \mathcal{A} \mid \phi(a^*a) = 0\}$.

Note that G-invariance of ϕ makes $U_\phi(g)$ a well defined unitary operator for each $g \in G$. The pair (π_ϕ, U_ϕ) is indeed a covariant representation of $(\mathcal{A}, G, \alpha)$, since they satisfy the covariance relation:

$$U_\phi(g)\pi_\phi(a)U_\phi(g)^\star \widehat{x} = \widehat{\alpha_g(a\alpha_{g^{-1}}(x))} = \widehat{\alpha_g(a)x} = \pi_\phi(\alpha_g(a))\widehat{x}.$$

We will henceforth refer to the representation ρ_ϕ as the one associated with, or induced by, the G-invariant state ϕ on $\mathcal{A}$.

In the case where $\mathcal{A}$ is commutative, that is, $\mathcal{A} = C(X)$ for some compact Hausdorff space X, any state ϕ on $\mathcal{A}$ is associated with a probability measure ν on X. If ϕ is invariant under the action of G, we may represent $C(X) \rtimes_\alpha G$ on $L^2(X, \nu)$ by a representation

$$\rho_\nu = \pi_\nu \times U_\nu : C(X) \rtimes_\alpha G \to B(L^2(X, \nu)) \qquad (2.4)$$

determined by the covariant representation (π_ν, U_ν), where $\pi_\nu : C(X) \to L^2(X, \nu)$ is given by $\pi_\nu(f)(h) = fh$ and $U_\nu : G \to U(L^2(X, \nu))$ is determined by $U_\nu(g)(h) = \alpha_g(h)$ for all $h \in C(X)$ and extended by continuity. As the notation suggest we have a unitary equivalence $\rho_\phi \sim \rho_\nu$, established by the map $L : H_\phi \to L^2(X, \nu)$ sending $\widehat{h} \mapsto h$. It is not hard to check that L is indeed a unitary operator. We also have that

$$L\pi_\phi(f)L^\star(h) = L(\widehat{fh}) = fh = \pi_\nu(f)(h)$$
$$LU_\phi(g)L^\star(h) = L(\widehat{\alpha_g(h)}) = \alpha_g(h) = U_\nu(g)(h)$$

for all $h, f \in C(X)$, and $g \in G$, hence $L \in \mathrm{Hom}_\mathcal{A}(\pi_\phi, \pi_\nu) \cap \mathrm{Hom}_G(U_\phi, U_\nu)$. It follows by Proposition 1.10 that L determines a unitary equivalence of representations $\pi_\phi \times U_\phi$ and $\pi_\nu \times U_\nu$.

For any representation $\rho : \mathcal{A} \rtimes_\alpha G \to B(H)$, we denote

$$H_G = \{x \in H \mid \rho(\delta_{1,g})x = x \text{ for all } g \in G\}.$$

The overarching goal in this section will be to prove the following.

Theorem 2.7. *If $\mathcal{A}$ is a commutative unital C*-algebra and G is a discrete abelian group then every irreducible representation $\pi : \mathcal{A} \rtimes_\alpha G \to B(H)$ satisfies that $dim(H_G) \leq 1$. Additionally, π is unitarily equivalent to $\rho_\phi = \pi_\phi \times U_\phi$ for some uniquely determined ergodic G-invariant state ϕ on $\mathcal{A}$ if and only if π is irreducible and $dim(H_G) = 1$.*

We start by introducing the following two subsets of the state space of $\mathcal{A} \rtimes_\alpha G$, where $\mathcal{A}$ is a unital C*-algebra, and G a discrete group. Adhering to the notation in [HW17], we define

$$S^1(\mathcal{A} \rtimes_\alpha G) = \{\psi \in S(\mathcal{A} \rtimes_\alpha G) \ : \ \psi(\delta_{1_\mathcal{A}, g}) = 1\}$$
$$P^1(\mathcal{A} \rtimes_\alpha G) = \{\psi \in P(\mathcal{A} \rtimes_\alpha G) \ : \ \psi(\delta_{1_\mathcal{A}, g}) = 1\}$$

that is, the states and pure states such that $\psi(\iota_G(g)) = 1$ for all $g \in G$. These spaces are easily seen to be weak*-closed.

Theorem 2.8. *When equipped with the weak*-topology, the restriction maps $R : S^1(\mathcal{A} \rtimes_\alpha G) \to S_G(\mathcal{A})$ and $R : P^1(\mathcal{A} \rtimes_\alpha G) \to E_G(\mathcal{A})$, sending $\psi \mapsto \psi \circ \iota_\mathcal{A}$ are well defined affine homeomorphisms.*

To spice things up, we opt for an alternative proof, based on Proposition 7.6.10 of [Ped79]. There the author extends the notion of positive definite functions on a group G (see Appendix B) to functions with values in the dual of a C*-algebra $\mathcal{A}$.

A function $F : G \to \mathcal{A}^\star$, is called **positive definite** (with respect to an action $\alpha : G \to Aut(\mathcal{A})$) if there exists a positive linear functional ψ on $\mathcal{A} \rtimes_\alpha G$ such that $F(g)(a) = \psi(\delta_{a,g})$ for all $g \in G$ and $a \in \mathcal{A}$. It is said to be normalised if $||F(e)|| = 1$. The set of all normalised positive definite functions will be denoted $B^1_+(\mathcal{A} \rtimes_\alpha G)$. Analogous to the group case, there is also a 1-1 correspondence between normalised positive definite functions, and states on $\mathcal{A} \rtimes_\alpha G$. The equivalence is established by relations very similar to those in Appendix B, sending a normalised positive definite function F to the state ψ on $\mathcal{A} \rtimes_\alpha G$, determined by

$$\psi(f) = \sum_{g \in G} F(g)(f(g)) \qquad \text{for all } f \in C_c(G, \mathcal{A}). \qquad (2.5)$$

with inverse correspondence determined by the equation

$$F(g)(a) = \psi(\delta_{a,g}).$$

Proposition 7.6.10 in [Ped79] then says that when $B^1_+(\mathcal{A} \rtimes_\alpha G)$ is equipped with the topology of pointwise weak*-convergence, and the state space $S(\mathcal{A} \rtimes_\alpha G)$ with the weak*-topology, the correspondence is actually an affine homeomorphism. A proof of these results will be omitted here, but can be found in the cited reference.

We wish to show the that the above correspondence restricts to a (still affine) homeomorphism

$$L : S^1(\mathcal{A} \rtimes_\alpha G) \to B_+^{1,G}(\mathcal{A} \rtimes_\alpha G),$$

where

$$B_+^{1,G}(\mathcal{A} \rtimes_\alpha G) = \{F \in B_+^1(\mathcal{A} \rtimes_\alpha G) \mid F = F_\phi, \text{ where } F_\phi(g) := \phi \in S_G(\mathcal{A}) \text{ for all } g \in G\}$$

showing that $S^1(\mathcal{A} \rtimes_\alpha G)$ (and $S_G(\mathcal{A})$) can be viewed as a subspace of the space of positive definite functions $B_+^1(\mathcal{A} \rtimes_\alpha G)$.

Proof. First let's assert that the map L is *well defined*. Let $\psi \in S^1(\mathcal{A} \rtimes_\alpha G)$, then $F = L(\psi)$ is given by $F(g)(a) = \psi(\delta_{a,g})$.

Since $\delta_{1_\mathcal{A},g}$ is in the multiplicative domain of ψ, it follows that $F(g)(a) = \psi(\delta_{a,g}) = \psi(\delta_{a,e}\delta_{1_\mathcal{A},g}) = \psi(\delta_{a,e})\psi(\delta_{1_\mathcal{A},g}) = \psi(\delta_{a,e})$, showing that F is indeed independent of g. Next if we set $\phi = \psi \circ \iota_\mathcal{A}$, so $F(g) = \phi$ for all $g \in G$, then

$$\phi(\alpha_g(a)) = \psi(\delta_{\alpha_g(a),e}) = \psi(\delta_{1_\mathcal{A},g}\delta_{a,e}\delta_{1_\mathcal{A},g^{-1}}) = \psi(\delta_{a,e}) = \phi(a)$$

again due to the fact that $\delta_{1_\mathcal{A},g}$ is in the multiplicative domain of ψ. Hence $F(g) \in S_G(\mathcal{A})$ so $F \in B_{G+}^1(\mathcal{A} \rtimes_\alpha G)$ and L is well defined.

Now we check that L *is surjective*. Pick any $F \in B_+^{1,G}(\mathcal{A} \rtimes_\alpha G)$ let $\phi := F(g) \in S_G(\mathcal{A})$. Using equation (2.5) above, we find the corresponding element $\psi \in S(\mathcal{A} \rtimes_\alpha G)$ by

$$\psi(f) = \sum_{g \in G} F(g)(f(g)) = \sum_{g \in G} \phi(f(g)).$$

Now $\psi(\delta_{1_\mathcal{A},g}) = \phi(1_\mathcal{A}) = 1$ for all $g \in G$, hence $\psi \in S^1(\mathcal{A} \rtimes_\alpha G)$. From this we deduce that the map L also surjective. Being the restriction of an affine homeomorphism, we know L must also be injective and affine, so L is an affine homeomorphism.

The final part of the proof is to show that there exists an (affine) homeomorphsm between $B_+^{1,G}(\mathcal{A} \rtimes_\alpha G)$ and $S_G(\mathcal{A})$. This is established in the obvious way, by the map sending $F \in B_+^{1,G}(\mathcal{A} \rtimes_\alpha G)$ to $F(e)$. Denoting $F_\phi : G \to S_G(\mathcal{A})$ the constant map $g \mapsto \phi$, we readily check that the above map is a homeomorphism since for any converent net F_{ϕ_β} in $\in B_+^{1,G}(\mathcal{A} \rtimes_\alpha G)$, we get

$$
\begin{aligned}
F_{\phi_\beta} \to F_\phi &\Leftrightarrow F_{\phi_\beta}(g) \to F_\phi(g) && \text{(in the weak*-topology for all } g \in G) \\
&\Leftrightarrow \phi_\beta(a) \to \phi(a) && \text{(for all } a \in \mathcal{A}) \\
&\Leftrightarrow \phi_\beta \to \phi && \text{(in the weak*-topology)}
\end{aligned}
$$

So by composition, we have established an affine homeomorphism,

$$S^1(\mathcal{A} \rtimes_\alpha G) \xleftrightarrow{\ \sim\ } B_+^{1,G}(\mathcal{A} \rtimes_\alpha G) \xleftrightarrow{\ \sim\ } S_G(\mathcal{A})$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{R}$$

Next, we show that the set $P^1(\mathcal{A} \rtimes_\alpha G)$ are the extremal points of $S^1(\mathcal{A} \rtimes_\alpha G)$. Here we follow the proof given in [HW17]. Since $P^1(\mathcal{A} \rtimes_\alpha G)$ consists of extremal points of $S(\mathcal{A} \rtimes_\alpha G)$, and since $S^1(\mathcal{A} \rtimes_\alpha G) \subset S(\mathcal{A} \rtimes_\alpha G)$ it follows that $P^1(\mathcal{A} \rtimes_\alpha G)$ are extremal points of $S^1(\mathcal{A} \rtimes_\alpha G)$. It remains to check that the extremal points of $S^1(\mathcal{A} \rtimes_\alpha G)$ are contained in $P^1(\mathcal{A} \rtimes_\alpha G)$. This amount to showing that any extremal point of $S^1(\mathcal{A} \rtimes_\alpha G)$ must be a pure state. To see this, let $\phi = t\phi_1 + (1-t)\phi_2$, where $\phi_1, \phi_2 \in S(\mathcal{A} \rtimes_\alpha G)$, then

$$\phi(\delta_{1,g}) = 1 \qquad \Rightarrow \qquad \phi_1(\delta_{1,g}) = \phi_2(\delta_{1,g}) = 1 \qquad \text{for all } g \in G$$

so $\phi_1, \phi_2 \in S^1(\mathcal{A} \rtimes_\alpha G)$. But since ϕ was an extremal point of $S^1(\mathcal{A} \rtimes_\alpha G)$ we get that $\phi_1 = \phi_2 = \phi$, hence ϕ is pure, and $\phi \in P^1(\mathcal{A} \rtimes_\alpha G)$. Since L is affine homeomorphism, we get that $E_G(\mathcal{A}) \simeq P^1(\mathcal{A} \rtimes_\alpha G)$. ■

As the following application of the above theorem is used throughout the article, it seems sensible to write out a short proof:

Lemma 2.9. *Let $\phi \in S_G(\mathcal{A})$ and $R^{-1}(\phi) = \psi \in S^1(\mathcal{A} \rtimes_\alpha G)$, then*

$$\rho_\phi \sim \pi_\psi.$$

That is, the representation ρ_ϕ (defined in equation (2.3)), is unitarily equivalent to the GNS representation of $\mathcal{A} \rtimes_\alpha G$ with respect to ψ. In particular it follows by Theorem 2.8 that ρ_ϕ is irreducible if an only if the state is ergodic.

Proof. This follows by noting that

$$\langle \pi_\psi(\delta_{a,g})\widehat{\xi}, \widehat{\xi} \rangle = \psi(\delta_{a,g}) = \psi(\delta_{a,e}) = \phi(a)$$

where $\widehat{\xi} = \widehat{1_{\mathcal{A} \rtimes_\alpha G}} \in H_\psi$. With $\widehat{\zeta} = \widehat{1_\mathcal{A}} \in H_\phi$, since $\pi_\phi \rtimes U_\phi(\delta_{a,g})\widehat{\zeta} = \pi_\phi(a)U_\phi(g)\widehat{\zeta} = \pi_\phi(a)\widehat{\alpha_g(1_\mathcal{A})} = \pi_\phi(a)\widehat{\zeta}$, we also have that

$$\langle \pi_\phi \rtimes U_\phi(\delta_{a,g})\widehat{\zeta}, \widehat{\zeta} \rangle = \langle \pi_\phi(a)\widehat{\zeta}, \widehat{\zeta} \rangle = \phi(a).$$

We thus have an equality $\langle \pi_\phi \rtimes U_\phi(\delta_{a,g})\widehat{\zeta}, \widehat{\zeta} \rangle = \langle \pi_\psi(\delta_{a,g})\widehat{\xi}, \widehat{\xi} \rangle$ and so, by linearity,

$$\langle \pi_\phi \rtimes U_\phi(f)\widehat{\zeta}, \widehat{\zeta} \rangle = \langle \pi_\psi(f)\widehat{\xi}, \widehat{\xi} \rangle$$

for all $f \in C_c(G, \mathcal{A})$. The claim now follows by uniqueness of the GNS-construction. ■

We are now ready to retrace the proof of Theorem 2.7:

Proof. We will first show that a representation π of a $\mathcal{A} \rtimes_\alpha G$ on H is unitarily equivalent to a representation ρ_ϕ associated with some ergodic state $\phi \in E_G(\mathcal{A})$ if and only if π is irreducible and $\dim(H_G) \neq 0$. [1]

Assume π is an irreducible representation of $\mathcal{A} \rtimes_\alpha G$ on H, with $dim(H_G) \neq 0$, then any unit vector $x \in H_G$ produces a vector state

$$\psi(b) = \langle \pi(b)x, x \rangle.$$

By Theorem 5.1.7 of [Mur04] ψ is a pure state on $\mathcal{A} \rtimes_\alpha G$, and π is unitarily equivalent to the GNS representation π_ψ. Clearly $\psi \in P^1(\mathcal{A} \rtimes_\alpha G)$, since $\psi(\delta_{1_\mathcal{A}, g}) = \langle \pi(\delta_{1_\mathcal{A}, g})x, x \rangle = \langle x, x \rangle = 1$, so by Theorem 2.8, we have $\phi = \psi \circ \iota_\mathcal{A} \in E_G(\mathcal{A})$, and finally by Lemma 2.9, we know that $\pi \sim \pi_\psi \sim \rho_\phi$

Conversely, assume $\pi : \mathcal{A} \rtimes_\alpha G \to B(H)$ is any representation such that $\pi \sim \rho_\phi$ for some $\phi \in E_G(\mathcal{A})$. Since ρ_ϕ is known to be irreducible by Lemma 2.9, π must be irreducible. It's easy to check that the span of $\widehat{1_\mathcal{A}} \in H_\phi$ is a G-invariant subspace of ρ_ϕ, and since G-invariance is preserved under unitary equivalence we have $dim(H_G) = dim((H_\phi)_G) = \neq 0$.

If $\mathcal{A}$ is commutative, say $\mathcal{A} = C(X)$ for some compact Hausdorff space X, the claim that $dim(H_G) \leq 1$ follows by using the equivalent representation ρ_μ on $L^2(X, \mu)$ defined in equation (2.4). For an ergodic measure μ it is known that the only invariant functions in $L^2(X, \mu)$ are constant functions ([Gla03][3.10, p. 67]), hence $L^2(X, \mu)_G = \mathbb{C}1_X$, which is 1-dimensional. [2]

Now it remains to show that the state $\phi \in E_G(C(X))$ is uniquely determined. Assume $\rho_{\phi_1} \sim \rho_{\phi_2}$ for two $\phi_1, \phi_2 \in E_G(C(X))$, denote the Hilbert spaces of ρ_{ϕ_1} and ρ_{ϕ_2} by H^{ϕ_1} and H^{ϕ_2} respectively. Let $U : H^{\phi_1} \to H^{\phi_2}$ be the map that establishes the unitary equivalence, then U maps $H_G^{\phi_1}$ to $H_G^{\phi_2}$, and thus $U(\widehat{1_{\phi_1}}) = \lambda \widehat{1_{\phi_2}}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. It then follows that

$$\phi_1(f) = \langle \rho_{\phi_1}(f)\widehat{1_{\phi_1}}, \widehat{1_{\phi_1}} \rangle = \langle U^\star \rho_{\phi_2}(f)U\widehat{1_{\phi_1}}, \widehat{1_{\phi_1}} \rangle = \langle \rho_{\phi_2}(f)\lambda\widehat{1_{\phi_2}}, \lambda\widehat{1_{\phi_2}} \rangle = \phi_2(f).$$

∎

Characterising the finitely supported ergodic measures

Finally we are ready to characterise those representations in Theorem 2.7 that are induced by measures of finite support, but first a quick recap of what has been done so far.

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[1] If $\mathcal{A}$ is not commutative this is the best we can do.

[2] Reverting back to the GNS-representation H_ϕ by the unitary operator $L : H_\phi \to L^2(X, \mu)$ of equation 2.4, we actually get that $(H_\phi)_G = \text{span}(\widehat{1})$, where $\widehat{1} \in H_\phi$ is the element corresponding to the unit in $\mathcal{A}$.

To a $\times p, \times q$-invariant ergodic measure μ on $\mathbb{T}$ with associated ergodic state ψ_μ on $C(\mathbb{T})$, we associate a $\times p, \times q$-invariant ergodic measure $\nu = R^{-1}(\mu)$ on S_{pq} by the homeomorphism R of Proposition 2.6, and let ψ_ν denote the corresponding ergodic state on $C(S_{pq})$. We then defined the representation ρ_{ψ_ν} of $C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2$ on the Hilbert space H_{ψ_ν} (the Hilbert space of the GNS-representation associated with ψ_ν) by $\rho_{\psi_\nu} = \pi_{\psi_\nu} \times U_{\psi_\nu}$, where π_{ψ_ν} is the GNS-representation of $C(S_{pq})$ on H_{ψ_ν} and U_{ψ_ν} is the Koopmann representation of $\mathbb{Z}^2$ on H_{ψ_ν} (see equation (2.3)). We called this the representation induced by the $\times p, \times q$-invariant ergodic measure μ on $\mathbb{T}$. Any representation π of $C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2$ is said to be induced by the $\times p, \times q$-invariant ergodic measure μ on $\mathbb{T}$ if it is unitarily equivalent to ρ_{ψ_ν}. We also showed in equation (2.4) that ρ_{ϕ_ν} is unitarily equivalent to a representation on $L^2(S_{pq}, \nu)$ which we denoted ρ_ν.

At the beginning of the article, in equation (2.2), another representations was associated with the ergodic $\times p, \times q$ invariant measure μ on $\mathbb{T}$ which we denoted π_μ (not to be confused with the GNS-representation of the state ϕ_μ associated with μ). The next proposition (formulated in [HW17] for finitely supported measures) shows that these constructions are actually equivalent. We will expand on this in the next section.

Proposition 2.10. *If the $\times p, \times q$-invariant ergodic measure μ on $\mathbb{T}$ has $h_\mu(T_p) = h_\mu(T_q) = 0$, then the representation π_μ defined in equation (2.2) is (unitarily equivalent to) the representation induced by μ, that is, it is unitarily equivalent to the representation ρ_{ψ_ν}, defined in equation (2.3), or equivalently ρ_ν of equation (2.4).*

Proof. As in equation (2.4), we denote the representation of $C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2$ on $L^2(S_{pq}, \nu)$ induced by the finitely supported $\times p, \times q$-invariant ergodic measure μ on $\mathbb{T}$ by ρ_ν.

Since ψ_ν restricts to ψ_μ on $C(\mathbb{T}) \subset C(S_{pq})$, which is dense in $L^2(\mathbb{T}, \mu)$, we immediately get that $L^2(\mathbb{T}, \mu) \subset L^2(S_{pq}, \nu)$. So we may treat π_μ as a (possibly degenerate) representation on $L^2(S_{pq}, \nu)$, determined on the inclusion $\iota(C(\mathbb{T})) \subset C(S_{pq}) \subset L^2(S_{pq}, \nu)$ by [3]

$$\pi_\mu(\delta_{1,(0,0)})(\iota(f))((z_i)) = z_0 f(z_0)$$
$$\pi_\mu(\delta_{0,(1,0)})(\iota(f))((z_i)) = f(z_0^p)$$
$$\pi_\mu(\delta_{0,(0,1)})(\iota(f))((z_i)) = f(z_0^q)$$

and acting trivially outside of $L^2(\mathbb{T}, \mu)$. We wish to show that π_μ and ρ_ν agree on $L^2(\mathbb{T}, \mu)$. First recall that under the identification $C^*(\mathbb{Z}(\frac{1}{pq})) \simeq C(S_{pq})$ one sends $1 \mapsto E_1$, where $E_1((z_i)) = z_0$. Now for any $f \in C(\mathbb{T})$, we have

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[3]Recall that $\iota(f)((z_i)) = f(z_0)$.

$$\rho_\nu(\delta_{1,(0,0)})(\iota(f))((z_i)) = E_1((z_i))\iota(f)((z_i)) = z_0 f(z_0) = \pi_\mu(\delta_{1,(0,0)})(\iota(f))((z_i)).$$

And similarly

$$\rho_\nu(\delta_{0,(1,0)})(\iota(f))((z_i)) = \tau_p\iota(f)((z_i)) = \pi_\mu(\delta_{0,(1,0)})(\iota(f))((z_i))$$

$$\rho_\nu(\delta_{0,(0,1)})(\iota(f))((z_i)) = \tau_q\iota(f)((z_i)) = \pi_\mu(\delta_{0,(0,1)})(\iota(f))((z_i)).$$

But from this we can deduce that $L^2(\mathbb{T}, \mu)$ is a closed invariant subspace of $L^2(S_{pq}, \nu)$, and since ρ_ν is irreducible we must have $L^2(S_{pq}, \nu) = L^2(\mathbb{T}, \mu)$, which concludes the proof. ∎

The representations $\pi : C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2 \to B(H)$ induced by $\times p, \times q$-invariant ergodic measures on $\mathbb{T}$ were characterised in Theorem 2.7 as those that are irreducible with $dim(H_{\mathbb{Z}^2}) = 1$, where

$$H_{\mathbb{Z}^2} = \{x \in H \mid \pi(\delta_{1,g})x = x \text{ for all } g \in \mathbb{Z}^2\}.$$

Now we wish to determine which of these are induced by finitely supported $\times p, \times q$-invariant ergodic measures on $\mathbb{T}$. As mentioned earlier, the theorem that addresses this, is Theorem 4.2 of [HW17], which reads

Theorem 2.11. *A representation $\pi : C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2 \to B(H)$ is induced by a finitely supported ergodic $\times p, \times q$-invariant measure on $\mathbb{T}$, if and only if*

- *π is irreducible,*

- *$H_{\mathbb{Z}^2} = \{x \in H \mid \pi(\delta_{1,g})x = x \ \ for \ all \ \ g \in \mathbb{Z}^2\} \neq \{0\}$,*

- *There exists a non-zero $N \in \mathbb{N}$ such that $\pi(\delta_{E_N,(0,0)})x = x$ for every $x \in H_{\mathbb{Z}^2}$, where $E_N \in C(S_{pq})$ is the function $(z_i) \mapsto z_0^N$.*

Proof. Assume $\mu \in EM_{p,q}(T)$, with $supp(\mu) = \{z_1, ..., z_n\} \subset \mathbb{T}$ finite, and let ν be the measure on S_{pq} which "restricts" to μ (Proposition 2.6). Since we know that $\pi_\mu \sim \rho_\nu$ (by Proposition 2.10), we know by previous results (Theorem 2.7) that $H_{\mathbb{Z}^2} \neq 0$, and π_μ is irreducible.

Though not explicitly mentioned, under the identification $\mathbb{T} \sim I = [0, 1)$, we must have $supp(\mu) \subset \mathbb{Q} \cap I$. To see this, note that if some atom z of μ is irrational, then by Furstenberg's theorem (Theorem 4.1 [Fur67]) $A = \{p^i q^j z \mid i, j \in \mathbb{N}\}$ is dense in $\mathbb{T}$, and hence infinite. But since $\mu(\{p^i q^j z\}) = \mu(\{z\})$ for all $i, j \in \mathbb{N}$, we see that $\mu(A)$ cannot be finite.

Hence there exists an integer N such that $supp(\mu) \subset \{e^{\frac{2\pi i k}{N}}\}_{k=0}^{N-1}$. Now the function $e_N(z) = z^N = 1$ (μ-a.e.) on $\mathbb{T}$ since $(e^{\frac{2\pi i k}{N}})^N = e^{2\pi i k} = 1$, and so $\pi_\mu(\delta_{E_N,(0,0)})x = \pi_\mu(\delta_{E_1,(0,0)})^N x = z^N x = x$ (ν-a.e.) for every $x \in H_{\mathbb{Z}^2}$.

Conversely, assume π is a representation of $C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2$ on H, that satisfies the conditions in the theorem. By Theorem 2.7, we know that π is induced by some $\times p, \times q$-invariant measure ν on S_{pq} (ref. equation (2.4)). Let μ be the $\times p, \times q$-invariant ergodic probability measure given by the "restriction" of ν to $\mathbb{T}$ (Proposition 2.6). Our goal is to show that μ is finitely supported. Recall that the induced representation ρ_ν is a representation on the Hilbert space $H = L^2(S_{pq}, \nu)$, and we have seen that $L^2(\mathbb{T}, \mu) \subset L^2(S_{pq}, \nu)$. Hence, to show that μ is finitely supported, it suffices to show that $L^2(S_{pq}, \nu)$ is finite dimensional. By assumption there exists a non-zero $N \in \mathbb{N}$ such that $\pi(\delta_{E_N,(0,0)})y = y$ for all $y \in H_{\mathbb{Z}^2}$. We will show $dim(L^2(S_{pq}, \nu)) \leq N$.

The group action of $\mathbb{Z}^2$ on the dense subalgebra of $C(S_{pq})$ spanned by the functions E_k, is given by $\alpha_{(n,m)}(E_k) = E_{p^n q^m k}$. Combined with the covariance relation (1.1), we have that for any $y \in H_{Z^2}$;

$$\pi(\delta_{E_{p^n q^m k},(0,0)})y = \pi(\delta_{1,(n,m)})\pi(\delta_{E_k,(0,0)})\pi(\delta_{1,(n,m)})^\star y = \pi(\delta_{1,(n,m)})\pi(\delta_{E_k,(0,0)})y$$

For any positive integer K write $KN = Mp^i q^j$, where $i, j \geq 0$ are integers, and M is coprime to p and q. We get that

$$\pi(\delta_{E_{p^i q^j M},(0,0)})y = \pi(\delta_{E_{KN},(0,0)})y = \pi(\delta_{E_N^K,(0,0)}) = \pi(\delta_{E_N,(0,0)})^K y = y.$$

Combined, these two equations yield

$$\pi(\delta_{1,(i,j)})\pi(\delta_{E_M,(0,0)})y = \pi(\delta_{E_{p^i q^j M},(0,0)})y = y$$

Since $\pi(\delta_{1,(i,j)})$ is injective (being unitary), and fixes y, we must have that

$$\pi(\delta_{E_M,(0,0)})y = y \qquad\qquad (*)$$

By assumption M is coprime to p and q, so we know there exists integers l, r such that
$$lp^i q^j + rM = \gcd(M, p^i q^j) = 1.$$

Thus we may factorise an element $g = \frac{k}{p^i q^j} \in \mathbb{Z}[\frac{1}{pq}]$ as $\frac{k}{p^i q^j} = \frac{k(lp^i q^j + rM)}{p^i q^j} = kl + \frac{krM}{p^i q^j}$. Now we have that

$$\pi(\delta_{1,(i,j)})\pi(\delta_{E_{\frac{krM}{p^i q^j}},(0,0)})y = \pi(\delta_{E_{krM},(0,0)})y = \pi(\delta_{E_M,(0,0)})^{kr}y = y,$$

where the last equality follows from (*). Again since $\pi(\delta_{1,(i,j)})$ is unitary and fixes y, we get that
$$\pi(\delta_{E_{\frac{krM}{p^i q^j}},(0,0)})y = y.$$

By inspection we have that for any $g_1, g_2 \in \mathbb{Z}[\frac{1}{pq}]$, $E_{g_1+g_2}((z_i)) = E_{g_1}((z_i))E_{g_2}((z_i))$ (see Appendix B), hence

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$$\pi(\delta_{E_{\frac{k}{p^i q^j}},(0,0)})y = \pi(\delta_{E_{kl+\frac{krM}{p^i q^j}},(0,0)})y$$
$$= \pi(\delta_{E_{kl+\frac{krM}{p^i q^j}},(0,0)})y = \pi(\delta_{E_{kl},(0,0)})\pi(\delta_{E_{\frac{krM}{p^i q^j}},(0,0)})y$$
$$= \pi(\delta_{E_{kl},(0,0)})y$$

which shows

$$\text{span}\{\pi(\delta_{E_g,(0,0)})y \mid g \in \mathbb{Z}[\frac{1}{pq}]\} = \text{span}\{\pi(\delta_{E_n,(0,0)})y \mid n \in \mathbb{Z}\}.$$

For an arbitrary $n \in \mathbb{Z}$, let $n = q \pmod{N}$, say $n = q + sN$ for some integer s. Now we get that

$$\pi(\delta_{E_n,(0,0)})y = \pi(\delta_{E_{q+sN},(0,0)})y = \pi(\delta_{E_q,(0,0)})\pi(\delta_{E_{sN},(0,0)})y = \pi(\delta_{E_q,(0,0)})y$$

so $\{\pi(\delta_{E_n,(0,0)})y \mid n \in \mathbb{Z}\} \subset \text{span}\{\pi(\delta_{E_n,(0,0)})y\}_{n=0}^{N-1}$, and since y is cyclic (as π is irreducible), this shows $L^2(S_{pq}, \nu)$ is finite dimensional and concludes the proof. $\blacksquare$

Finally the authors state the following corollary,

Corollary 2.12. *The Furstenberg conjecture is true if and only if there is precisely one irreducible representation $\pi : C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2) \to B(H)$ satisfying the conditions*

- $H_{\mathbb{Z}^2} \neq \{0\}$

- $\pi(\delta_{E_M,(0,0)})y \neq y$ *for every non-zero $M \in \mathbb{N}$ and non-zero $y \in H_{\mathbb{Z}^2}$.*

Proof. To each $\times p, \times q$-invariant ergodic measure on $\mathbb{T}$ Theorem 2.7 states that the induced representation ρ is irreducible and has a 1-dimensional $\mathbb{Z}^2$ invariant subspace, so these conditions hold if and only if the representation is induced by an ergodic measure.

If $\rho(\delta_{E_N,(0,0)})y = y$ for some non-zero positive integer N, and $y \in H_{\mathbb{Z}^2}$ then Theorem 2.11 guaranties that ρ is induced by a measure with finite support on $\mathbb{T}$.

The Furstenberg conjecture asks if the only measure on $\mathbb{T}$ which is not finitely supported, is the Lebesgue measure. Since the correspondence between ergodic measures and induced representations in Theorem 2.7 is 1-1, this is equivalent to asking if there is only one irreducible representation, which fails to satisfy the third criterion in Theorem 2.11. $\blacksquare$

<div align="center">

CHAPTER 3

Supplementary Results

</div>

Here we set out to expand on the article showing how we can use the above machinery to find a decomposition of the representations associated with arbitrary $\times p, \times q$-invariant probability measures on $\mathbb{T}$. Unfortunately, due to time limitations, we didn't get as far as we had hoped.

3.1 The "Lebesgue representation"

As a concrete example, we construct an explicit representation induced by the Lebesgue measure and show it satisfies the conditions in Corollary 2.12. Let λ be the normalised Lebesgue measure on $\mathbb{T}$ and ρ_ν the induced representation given by equation (2.4), where ν is the measure on S_{pq} that restricts to λ (Proposition 2.6).

Example 3.1 (The Lebesgue Representation)**.** The representation ρ_ν associated with the Lebesgue measure on $\mathbb{T}$ satisfies the conditions in corollary 2.12.

Lets start with the following lemma

Lemma 3.2. *The representation* $\rho_\nu : C(S_{pq}) \rtimes \mathbb{Z}^2 \to B(H_\nu)$*, acts by sending* $f \in C_c(\mathbb{Z}^2, C(S_{pq}))$ *to the operator* $\rho_\nu(f)$*, determined by*

$$\rho_\nu(f)(h) = \sum_{g \in \mathbb{Z}^2} f(g)\alpha_g(h) \qquad \textit{for all } h \in H_\nu,$$

where

$$H_\nu = L^2(S_{pq}, \nu), \qquad and \qquad \int_{S_{pq}} (E_{\frac{m}{(pq)^n}}) d\nu = \int_{\mathbb{T}} z^m d\lambda = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

and α_g *is the continuous extension of* α_g *to* $L^2(S_{pq}, \nu)$ *from* $C(S_{pq})$*.*

Proof. The representation associated with the (normalised) Lebesgue measure is $\rho_\nu = \pi_\nu \times U_\nu : C(S_{pq} \rtimes \mathbb{Z}^2) \to B(H_\nu)$ (see equation (2.4)). A quick

computation shows that for any $f \in C_c(\mathbb{Z}^2, C(S_{pq}))$ and $h \in L^2(S_{pq}, \nu)$ we have,

$$\rho_\nu(f)(h) = \sum_{g \in \mathbb{Z}^2} \pi_\nu(f(g))U_\nu(g)(h) = \sum_{g \in \mathbb{Z}^2} \pi_\nu(f(g))(\alpha_g(h)) = \sum_{g \in \mathbb{Z}^2} f(g)\alpha_g(h).$$

∎

of example. Irreducibility of ρ_ν follows directly from Theorem 2.7 since the Lebesgue measure is ergodic. As we have noted before, since ν is ergodic, $L^2(S_{pq}, \nu)_{\mathbb{Z}^2}$ are precisely the constant functions ([Gla03][3.10, p. 67]). Since ν restricts to λ, for any $f \in C(\mathbb{T})$, we must have that

$$\int_{S_{pq}} \iota(f)d\nu = \int_{\mathbb{T}} fd\lambda.$$

Now for the constant function $1_{S_{pq}}((z_i)) = 1$, and positive integer $N \neq 0$,

$$\begin{aligned}
||\rho_\nu(\delta_{E_N,(0,0)})(1_{S_{pq}}) - 1_{S_{pq}}||^2 &= \int_{S_{pq}} (E_N - 1_{S_{pq}})(E_N - 1_{S_{pq}})^* d\nu \\
&= \int_{S_{pq}} \iota\left[(e_N - 1_{\mathbb{T}})(e_N - 1_{\mathbb{T}})^*\right] d\nu \\
&= \int_{\mathbb{T}} (e_N - 1_{\mathbb{T}})(e_N - 1_{\mathbb{T}})^* d\lambda \\
&= ||e_N - 1_{\mathbb{T}}||^2 \neq 0.
\end{aligned}$$

The result easily extends to any non-zero constant function. We may conclude that $\rho_\nu(\delta_{E_N,(0,0)})x \neq x$ for all non-zero $x \in L^2(S_{pq}, \nu)_{\mathbb{Z}^2}$, so ρ_ν does indeed satisfy the criterion of Corollary 2.12. ∎

3.2 Decompositions of induced representations

As mentioned in Appendix C, the ergodic measures are the building blocks of ergodic theory, as any invariant probability measure can be decomposed into its ergodic components. This is formalised by what is referred to as the *ergodic decomposition theorem* (see [Gla03] Theorem 3.22), which states that any invariant probability measure on $\mathbb{T}$ can be written as a certain integral over the ergodic measures. [1]

In representation theory the fundamental building blocks are the irreducible representations, and we should note that the correspondence of $\times p$, $\times q$-invariant

---

[1] Since the ergodic measures are the extremal points of the ergodic states, which is a weak*-compact closed and convex set, this is an extension of the usual Krein-Millman theorem, called Choquet's Theorem

probability measures on $\mathbb{T}$ and induced representations of $C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2$ in a sense preserves this decomposition, though not completely.

First we will verify the claim for finite convex combinations, and in the process we will strengthen Proposition 2.10. The general case, that is the case of decompositions of invariant measures into arbitrary convex combinations of ergodic measure, is significantly more subtle, and the correspondence unfortunately does not always hold. But we will come back to this shortly.

Recall that for a C*-algebra $\mathcal{B}$ the direct sum of two representations $\pi_1 : \mathcal{B} \to B(H_1)$ and $\pi_2 : \mathcal{B} \to B(H_2)$ is defined as the representation $\pi_1 \oplus \pi_2 : \mathcal{B} \to B(H_1 \oplus H_2)$ given by $\pi_1 \oplus \pi_2(b)(x,y) = (\pi_1(b)x, \pi_2(b)y)$.

Proposition 3.3. *Let μ be a measure on $\mathbb{T}$ given by a convex combination,*

$$\mu = \sum_{i=1,\ldots,n} t_i \mu_i, \qquad \text{where } t_i > 0, \quad \sum_{i=1,\ldots n} t_i = 1,$$

and the μ_i's are $\times p, \times q$-invariant ergodic probability measures on $\mathbb{T}$. Let ν and ν_i denote the lift of μ and μ_i respectively to S_{pq} (Proposition 2.6).

Then the induced representation of μ (denoted ρ_ν) is given as the direct sum

$$\rho_\nu = \bigoplus_{i=1,\ldots,n} \rho_{t_i \nu_i} : C(S_{pq}) \rtimes \mathbb{Z}^2 \to B(\bigoplus_{i=1}^n L^2(S_{pq}, t_i \nu_i)),$$

where $\rho_{t_i \nu_i}$ are the usual induced representations(equation (2.4)).

If $h_{\mu_i}(T_p) = h_{\mu_i}(T_q) = 0$ for all $i = 1, \ldots, n$, we have that

$$\rho_\nu = \bigoplus_{i=1,\ldots,n} t_i \pi_{\mu_i} : C(S_{pq}) \rtimes \mathbb{Z}^2 \to B(\bigoplus_{i=1}^n L^2(\mathbb{T}, t_i \mu_i)),$$

where π_{μ_i} are the representations given in equation (2.2).[2]

Proof. Let ν and ν_i be the lifts of μ and μ_i to S_{pq} (Proposition 2.6). Since the lift is affine we know that

$$\nu = \sum_{i=1}^n t_i \nu_i$$

where ν_i are ergodic measures on S_{pq}. Let ρ_ν and $\rho_{t_i \nu_i}$ denote the usual induced representations on $L^2(S_{pq}, \nu)$ and $L^2(S_{pq}, t_i \nu_i)$ respectively, defined in equation (2.4).

Let's verify that the map $U : L^2(S_{pq}, \nu) \to \bigoplus_{i=1}^n L^2(S_{pq}, t_i \nu_i)$ sending $f \mapsto (f, f, \ldots, f)$ is a unitary operator.

---

[2]The representations $\pi_{t_i \mu_i}$ (and $\rho_{t_i \nu_i}$) are easily seen to be unitary equivalent to the representation π_{μ_i} (and $\rho_{t_i \nu_i}$) by the unitary operator $U : L^2(S_{pq}, t_i \mu_i) \to L^2(S_{pq}, \mu_i)$ given by $U(f) = t_i f$, so they are irreducible

First off, U is **well defined** since for all measurable $E \subset \mathbb{T}$ we have $t_i \mu_i(E) \leq \mu(E)$, from which we get that

$$f = g \ (\mu\text{- a.e.}) \ \Rightarrow \ f = g \ (\mu_i\text{- a.e.})$$

for all $i = 1, ..., n$, and so U is well defined.

To show U is an **isometry**, we need the following fact: If two ergodic measures on $\mathbb{T}$ are distinct then they are mutually singular (Theorem 4.2 [Gla03][p. 97]). It follows that

$$||f||_2^2 = \int_{S_{pq}} |f|^2 d\nu = \int_{S_{pq}} |f|^2 d(t_1\nu_1 \oplus ... \oplus t_n\nu_n)$$
$$= \sum_{i=1}^{n} \int_{S_{pq}} |f|^2 d(t_i\nu_i) = ||(f, ..., f)||_2^2 = ||U(f)||_2^2$$

so U is an isometry.

Now we show U is **surjective**. Since the measures are mutually singular, we can construct measurable sets $\mathcal{A}_i \subset S_{pq}$ such that

$$\nu_i(A_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Let χ_i be the characteristic function associated with the set A_i. For any $(f_1, ..., f_n) \in \bigoplus_{i=1}^{n} L^2(S_{pq}, t_i\nu_i)$ define $f = \sum_{i=1}^{n} \chi_i f_i \subset L^2(S_{pq}, \nu)$. It is now not hard to check that

$$U(f) = (f_1, f_2, ..., f_n)$$

which shows that U is surjective.

We will now show U is **equivariant**. The elements of the form $\delta_{E_N, g}$, where $N \in \mathbb{Z} \subset \mathbb{Z}[\frac{1}{pq}]$ and $g = (n, m) \in \mathbb{Z}^2$, span $C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2$, and

$$U\rho_\nu(\delta_{E_N, g})U^\star(f_1, ..., f_n) = U(E_N\alpha_g(f))$$
$$= (E_N\alpha_g(f), ..., E_N\alpha_g(f)),$$

but $\alpha_g(f) = \sum_{i=1}^{n} \alpha_g(\chi_i f_i) = \alpha_g(f_i) \ (\nu_i\text{-a.e.})$, since $\chi_j = \begin{cases} 1_{S_{pq}} \ \nu_j \ a.e. & i = j \\ 0 \ \nu_j \ a.e. & i \neq j \end{cases}$, and α_g is a unitary operator on each $L^2(S_{pq}, \nu_i)$. Hence

$$U\rho_\nu(\delta_{E_N,(n,m)})U^\star(f_1, ..., f_n) = \bigoplus_{i=1}^{n} t_i\rho_{\nu_i}(\delta_{a,(n,m)})(f_1, ..., f_n).$$

The last assertion of the proposition follows from Proposition 2.10 where we showed that $\rho_{\nu_i} \sim \pi_{\mu_i}$. ∎

Remark 3.4. It should also be noted that the entropy map, that is, the map sending $\mu \mapsto h_\mu(T)$ is affine by Theorem 8.1 [Wal00], so if $h_{\mu_i}(T_p) = h_{\mu_i}(T_q) = 0$ for all μ_i in the above proposition, we get that $h_\mu(T_p) = h_\mu(T_q) = 0$. Thus there is nothing in the way of creating a representation π_μ on $L^2(\mathbb{T}, \mu)$ as in (2.2). A completely analogous argument shows that

$$\pi_\mu \simeq \bigoplus_{i=1}^n \pi_{t_i \mu_i}$$

but then $\pi_\mu \sim \bigoplus_{i=1}^n \pi_{t_i \mu_i} \sim \bigoplus_{i=1}^n \rho_{t_i \nu_i} \sim \rho_\nu$, which shows that π_μ is (equivalent to) the induced representation associated with the measure μ.

Remark 3.5. Note that the above proposition holds also for finite sums of $\times p, \times q$-invariant measures as long as they are mutually singular.

To get an idea of the situation for a general decomposition into ergodic components of a $\times p, \times q$-invariant probability measure ν on S_{pq}, we quickly state the criterion known to give sufficient and necessary condition for the above correspondence to persist, but will skimp on the definitions, which can all be found in many introductory books to operator theory, like [Tak79].

For an arbitrary $\times p, \times q$-invariant measure $\nu \in M_{p,q}(S_{pq})$ the ergodic decomposition theorem guaranties the existence of a probability measure η on $EM_{p,q}(S_{pq})$ such that

$$\nu = \int_{EM_{p,q}(S_{pq})} \mu d\eta(\mu)$$

that is, for every function $f \in C(S_{pq})$ we have $\nu(f) = \int_{EM_{p,q}(S_{pq})} \mu(f) d\eta(\mu)$.

In this setting one could hope that the induced representation is isomorphic to the representations on the *Hilbert direct integral* $\int_{EM_{p,q}(S_{pq})}^\oplus H_\mu d\eta(\mu)$, given by the (measurable field of) induced representations $\mu \mapsto \rho_\mu$, and $H_\mu = L^2(S_{pq}, \mu)$. That is the representation sending $a \in C(S_{pq})$ to the (decomposable) operator $\rho_\mu(a)$, which acts by sending an element $f \in \int_{M_{p,q}(S_{pq})}^\oplus L^2(S_{pq}, \mu) d\eta(\mu)$ to $\widehat{f}$ given by $\widehat{f}(\mu) = \rho_\mu(f(\mu))$.

Unfortunately, this might not hold in general. A sufficient and necessary criterion for this to hold is that the measure η is **orthogonal**, which here means that for each measurable $E \subset EM_{p,q}(S_{pq})$, we have

$$\int_E \nu d\eta(\nu) \perp \int_{E^c} \nu d\eta(\nu)$$

where E^c denotes the complement of E in $EM_{p,q}(S_{pq})$ (and $\perp$ of course means the measures are singular) (this is Theorem 8.31 in [Tak79]).

In particular, if η is finitely supported, that is if ν is a finite convex combination of ergodic measures, it is straightforward to verify that it is orthogonal in the above sense, since the ergodic measures are mutually singular. The Hilbert direct integral when η is finitely supported reduces to $\int_{EM_{p,q}(S_{pq})}^\oplus H_\mu d\eta(\mu) \simeq \bigoplus_{i=1}^n H_{t_i \mu}$, where $t_i = \eta(\mu_i)$, and the isomorphism is

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established by the obvious map, sending $f \mapsto (f(\mu_1), f(\mu_2), ..., f(\mu_n))$. From this one can also retrieve Proposition 3.3 as a special case.

If we can write our measure ν as a countable convex combination of finitely supported ergodic measures, things seem to work out much better, as the following proposition will show. First let's recall some definitions. A sequence of measures $\{\mu_i\}$ on $\mathbb{T}$ converges in the vague topology (or weak*-topology) to μ if

$$\int_{\mathbb{T}} f \, d\mu = \lim_{i \to \infty} \int_{\mathbb{T}} f \, d\mu_i \qquad \text{for all } f \in C(\mathbb{T}).$$

Also, recall that the *Hilbert direct sum* of a countable family of Hilbert spaces $\{H_i\}_{i=1}^{\infty}$ is defined as the (Hilbert space) completion of the space

$$\bigoplus_{i=1}^{\infty} H_i = \{f \in \prod_{i=1}^{\infty} H_i \mid \sum_{i=1}^{\infty} {}_i\|f(i)\|_2^2 < \infty\}$$

with respect to the norm induced by the inner product

$$\langle f, g \rangle := \sum_{i=1}^{\infty} {}_i\langle f(i), g(i) \rangle$$

where ${}_i\| \cdot \|$ is the norm and ${}_i\langle \cdot, \cdot \rangle$ the inner product of the Hilbert space H_i. When convenient, we will denote elements in the Hilbert direct sum as sequences $(f_1, f_2, ...)$.

Proposition 3.6 (Countable convex combinations)**.** *Let $\{\mu_i\}_{i=1}^{\infty}$ be a series of finitely supported ergodic measures on $\mathbb{T}$, such that $\mu = \sum_{i=1}^{\infty} t_i \mu_i$ where $t_i > 0$, $\sum_{i=1}^{\infty} t_i = 1$, then*

$$\rho_\nu = \bigoplus_{i=1}^{\infty} \pi_{t_i \mu}$$

where ν is the measure on S_{pq} that restricts to ν, ρ_ν the representation on $L^2(S_{pq}, \nu)$ given in equation (2.4), $\bigoplus_{i=1}^{\infty} \pi_{t_i \mu_i}$ is the representation on $\bigoplus_{i=1}^{\infty} L^2(\mathbb{T}, t_i \mu_i)$ and $\pi_{t_i \mu_i}$ are the induced representations of equation (2.2) with respect to the scaled measures $t_i \mu_i$.

Proof. We start by lifting the measures μ_i and μ to measures ν_i and ν on S_{pq} respectively, by the affine homeomorphism of Proposition 2.6. We have

$$\nu = \sum_{i=1}^{\infty} t_i \nu_i$$

Now we will show that $L^2(S_{pq}, \nu) \simeq \bigoplus_{i=1}^{\infty} L^2(S_{pq}, t_i \nu_i)$. Let χ_i be the characteristic function of the set $A_i := supp(\nu_i) \subset S_{pq}$. All A_i's are finite sets, since we have seen that $L^2(\mathbb{T}, \mu_i) \simeq L^2(S_{pq}, \nu_i)$, hence $L^2(S_{pq}, \nu_i)$ is finite dimensional, and so ν_i is finitely supported.

We denote the L^2-norm in $L^2(S_{pq}, \nu)$ by $||\cdot||_2$, and the L^2-norm in $L^2(S_{pq}, \nu_i)$ (that is $\int_{S_{pq}} |\cdot|^2 d\nu_i$) by $_{\nu_i}||\cdot||_2$.

As in Proposition 3.3 above, we define the linear map

$$U : L^2(S_{pq}, \nu) \to \bigoplus_{i=1}^{\infty} L^2(S_{pq}, t_i\nu) \qquad \text{by} \qquad f \mapsto (f, f, ...).$$

A collection of results known as Portmanteau's theorem states (among other things) that a bounded series of measures ν_i on a metric space converges to ν in the vague topology if an only if

$$||h||_2^2 \geq \limsup_{m \to \infty} \sum_{i=1}^{m} t_i {}_{\nu_i}||f||_2^2 \qquad \text{for all upper semicontinuous functions } h.$$

To show that U is **well defined** we must verify that if $f = 0$ ν-a.e. then $f = 0$ ν_i-a.e. Convince yourself that for each i the function $\chi_i|f| = 0$ (ν-a.e.) and that $\chi_i|f|$ is indeed upper-semicontinuous since A_i is a finite set. From this we get that

$$0 = ||\chi_i f||_2^2 \geq \limsup_{m \to \infty} \sum_{i=1}^{m} t_i {}_{\nu_i}||\chi_i f||_2^2 = {}_{\nu_i}||f||_2^2$$

which shows $f = 0$ ν_i-a.e. and U is well defined.

U is an **isometry** since for any $f \in C(S_{pq}) \subset L^2(S_{pq}, \nu)$ we have

$$||f||_2^2 = \int_{S_{pq}} |f|^2 \nu := \sum_{i=1}^{\infty} t_i \int_{S_{pq}} |f|^2 d\nu_i = ||U(f)||^2$$

and since $C(S_{pq})$ is dense in $L^2(S_{pq}, \nu)$.

Surjectivity of U is slightly more involved. Let

$$S = \{(f_1, f_2, ...) \in \bigoplus_{i=1}^{\infty} L^2(S_{pq}, t_i\nu) \mid \sup_i\{||f_i|_{A_i}||_\infty\} < \infty\}$$

where $||f_i|_{A_i}||_\infty$ is the sup-norm of f_i on A_i. S is dense in $\bigoplus_{i=1}^{\infty} L^2(S_{pq}, t_i\nu)$ since for any

$$(g_1, g_2, ...) \in \bigoplus_{i=1}^{\infty} L^2(S_{pq}, t_i\nu)$$

one can verify that the sequence

$$(g_1, 0, 0, ...), (g_1, g_2, 0, 0, ...), (g_1, g_2, g_3, 0, 0, ...), ...$$

converges in $\bigoplus_{i=1}^{\infty} L^2(S_{pq}, t_i\nu)$ to $(g_1, g_2, ...)$, and clearly each of these elements are in S, since the A_i's are finite.

We will show that the series $\sum_{i=1}^{\infty} \chi_i f_i$ is convergent in $L^2(S_{pq}, \nu)$ for any $(f_1, f_2, ...) \in S$ by showing the sequence of partial sums are Cauchy. Let n, m be positive integers, with $1 < n < m$, employing Urysohn's lemma, we can find a continuous real valued function $h \in C(S_{pq})$ with the following properties

- $0 < h((z_i)) \leq M = \sup\{||f_i|_{A_i}||_\infty\} < \infty$ for all $(z_i) \in S_{pq}$.

- $h((z_i)) = \begin{cases} f_i((z_i)) & \text{if } (z_i) \in A_i \text{ for } i \in \{n, ..., m\} \\ 0 & \text{if } (z_i) \in A_i \text{ for } i \in \{1, ..., n-1, m+1, ..., N_\epsilon\}. \end{cases}$

- Choose N_ϵ so large that $\sum_{i=N_\epsilon}^{\infty} t_i < \frac{\epsilon}{M^2}$

Now $h((z_i)) \geq |\sum_{i=n}^{m} \chi_i((z_i)) f_i((z_i))|$ for all $(z_i) \in S_{pq}$, so $||h||_2 \geq ||\sum_{i=n}^{m} \chi_i f_i||_2$, and

$$\begin{aligned} ||h||_2^2 &= \sum_{i=1}^{\infty} t_i \nu_i ||h||_2^2 && \text{(since } h \text{ is continuous.)} \\ &= \sum_{i=n}^{m} t_i \nu_i ||f_i||_2^2 + \sum_{i=N}^{\infty} t_i \nu_i ||h||_2^2 \\ &\leq \sum_{i=1}^{m} t_i \nu_i ||f_i||_2^2 + \epsilon. \end{aligned}$$

Since ϵ was arbitrary we get

$$||\sum_{i=n}^{m} \chi_i f_i||_2^2 \leq \sum_{i=1}^{m} t_i \nu_i ||f_i||_2^2 \leq M \sum_{i=n}^{\infty} t_i \xrightarrow{n \to \infty} 0$$

which shows that $\sum_{i=1}^{\infty} \chi_i f_i$ converges for all elements $(f_1, f_2, ...) \in S$. But then it is immediate that

$$U(\sum_{i=1}^{\infty} \chi_i f_i) = (\sum_{i=1}^{\infty} \chi_i f_i, \sum_{i=1}^{\infty} \chi_i f_i, ...) = (f_1, f_2, ...)$$

which shows that the image of U is dense in $\bigoplus_{i=1}^{\infty} L^2(S_{pq}, t_i \nu)$, and so U is surjective.

By the above, we see that $\sum_{i=1}^{\infty} \chi_i$ converges in $L^2(S_{pq}, \nu)$, and it's not too hard to check that $\sum_{i=1}^{\infty} \chi_i = 1$. This means $\rho_\nu(\delta_{E_k,(n,m)})(f) = \rho_\nu(\delta_{E_k,(n,m)})(\sum_{i=1}^{\infty} \chi_i f) = (\sum_{i=1}^{\infty} \rho_\nu(\delta_{E_k,(n,m)})(\chi_i f) = \rho_{\nu_i}(\delta_{E_k,(n,m)})(f)$ ν_i-a.e. for all $f \in L^2(S_{pq}, \nu)$, where $\delta_{E_K,(n,m)} \in C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2$. From this we see that for any $(f_1, f_2, ...) \in S$, and any $a \in C(S_{pq}) \rtimes_\alpha \mathbb{Z}^2$, we have

$$\begin{aligned} U\rho_\nu(a)U^*(f_1, f_2, ...) &= U\rho_\nu(\sum_{i=1}^{\infty} \chi_i f_i) = (\sum_{i=1}^{\infty} \chi_i f_i, \sum_{i=1}^{\infty} \chi_i f_i, ...) \\ &= (\rho_{t_1 \nu_1}(a) f_1, \rho_{t_2 \nu_2}(a) f_2, ...) = \bigoplus_{i=1}^{\infty} \rho_{t_i \nu_i}(a)(f_1, f_2, ...) \end{aligned}$$

which shows that U is equivariant.

Lastly, we have

$$\bigoplus_{i=1}^{\infty} \pi_{t_i\mu_i} \sim \bigoplus_{i=1}^{\infty} \rho_{t\nu_i}$$

again by Proposition 2.10. $\blacksquare$

If we try to extend the above result to series $\sum_{i=1}^{\infty} t_i\mu_i$ of $\times p, \times q$-invariant ergodic probability measures whose support is not finite, we immediately get into trouble, since, even though we can construct disjoint sets $A_i \in S_{pq}$ such that

$$\nu_i(A_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else,} \end{cases}$$

there need not be any continuous function which separates these sets.

Since the finitely supported functions have disjoint support contained in the rationals, we know they are countably many, hence we have the following corollary,

Corollary 3.7. *The Furstenberg conjecture is true if and only if, every representation induced by a $\times p, \times q$-invariant probability measure on $\mathbb{T}$ has the following irreducible decomposition*

$$(\bigoplus_{i=1}^{\infty} \pi_{t_i\mu}) \oplus \rho_{t\nu} : C(S_{pq}) \rtimes \mathbb{Z}^2 \to B\left(\left(\bigoplus_{i=1}^{\infty} L^2(\mathbb{T}, t_i\mu_i)\right) \oplus L^2(S_{pq}, t\nu)\right)$$

for some choice of $0 \leq t_i, t$ such that $\sum_{i=1}^{\infty} t_i + t = 1$.

Here ν is the measure on S_{pq} which restricts to the Lebesgue measure λ on $\mathbb{T}$, the μ_i's are finitely supported $\times p, \times q$-invariant ergodic measures on $\mathbb{T}$ and the representations $\pi_{t_i\mu_i}$ and $\rho_{t\nu}$ are given by the equations (2.2) and (2.4) respectively.

Appendix A: p-adic Solenoid

Since the p-adic solenoid, where p is a positive integer, play such a central role in work of [HW17] it might be helpful to list and prove some of the relevant properties of these spaces and their relation to the p-adic rationals. First of all, let's start with the definitions we need, most of which can be formulated in several equivalent ways.

Definition A.1 (p-adic Solenoid)**.** The p-adic solenoid S_p is defined as the projective limit of the the system

$$\mathbb{T} \xleftarrow{p} \mathbb{T} \xleftarrow{p} \mathbb{T} \xleftarrow{p} \mathbb{T}...$$

where $p(z) = z^p$. In other words, it is the group[1]

$$S_p = \{(z_i)_{i \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}} \mid z_{i+1}^p = z_i\}.$$

With group operation $(z_j)(w_j) = (z_j w_j)$ and the weak topology induced by the projection maps, this becomes a topological group. The topology is the same as the subspace topology inherited from $\mathbb{T}^{\mathbb{N}}$, and it is not hard to check it is closed, hence by Tychonoff's theorem it is compact.

Note that since each map p in the above diagram is surjective, the projection map $\pi_i((z_j)) = z_i \in \mathbb{T}$ must also be surjective, hence lift to an imbedding $C(\mathbb{T}) \hookrightarrow C(S_p)$, by sending $f \mapsto f \circ \pi_i$. We will use the first projection map π_0 for our canonical imbedding and denote it by ι.

Definition A.2. We define $\mathbb{Z}[\frac{1}{p}]$ is as the group

$$\mathbb{Z}[\frac{1}{p}] = \{\frac{m}{p^n} \ : \ m \in \mathbb{Z}, \ n \in \mathbb{N}\}$$

with the discrete topology, and the additive group operation inherited from $\mathbb{Q}$.

The next proposition shows the groups $\mathbb{Z}[\frac{1}{p}]$ and S_p are dual groups. In Appendix B we define the dual group of a locally compact abelian group G,

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[1]We use the convention $\mathbb{N} = \{0, 1, ...\}$ throughout this thesis.

denoted $\widehat{G}$, and give it the compact open topology. We note that if the group G is discrete, the compact open topology is just the topology of pointwise convergence.

Proposition A.3. *The dual group of* $\mathbb{Z}[\frac{1}{p}]$ *is isomorphic to* S_p, *i.e.*

$$\widehat{\mathbb{Z}[\frac{1}{p}]} \simeq S_p$$

Proof. We define the map $L : S_p \to \widehat{\mathbb{Z}[\frac{1}{q}]}$ by sending $s = (z_i) \in S_p$ to the map γ_s defined by $\gamma_s(\frac{m}{p^n}) = z_n^m$. We will show L is group isomorphism, but first observe that γ_s is indeed well defined, since $\gamma_s(\frac{p^k m}{p^{k+n}}) = z_{k+n}^{mp^k} = z_n^m$. The last equality follows from the fact that $z_{i+1}^p = z_i$, so γ_s is independent of choice of representative $\frac{m}{p^n}$.

γ_s is also a group homomorphism, since

$$\begin{aligned}
\gamma_s(\frac{m_1}{p^{n_1}} + \frac{m_2}{p^{n_2}}) &= \gamma_s(\frac{m_1 p^{n_2} + m_2 p^{n_1}}{p^{n_2} p^{n_1}}) \\
&= z_{n_1+n_2}^{m_1 p^{n_2} + m_2 p^{n_1}} \\
&= z_{n_1+n_2}^{m_1 p^{n_2}} z_{n_1+n_2}^{m_2 p^{n_1}} \\
&= z_{n_1}^{m_1} z_{n_2}^{m_2} = \gamma_s(\frac{m_1}{p^{n_1}})\gamma_s(\frac{m_2}{p^{n_2}}).
\end{aligned}$$

This shows that the map L is well defined.

L is a group homomorphism, since with $s_1 = (w_i), s_2 = (z_i) \in S_p$, recalling that $s_1 s_2 = (w_i z_i)$, we have

$$\gamma_{s_1 s_2}(\frac{m}{p^n}) = (w_n z_n)^m = w_n^m z_n^m = \gamma_{s_1}(\frac{m}{p^n})\gamma_{s_2}(\frac{m}{p^n})$$

If $s_1 = (w_i) \neq s_2 = (z_i)$ then $z_j \neq w_j$ for some index j, hence

$$\gamma_{s_1}(\frac{1}{p^j}) = w_j \neq z_j = \gamma_{s_2}(\frac{1}{p^j})$$

so L is injective.

For each $\gamma \in \widehat{\mathbb{Z}[\frac{1}{p}]}$ define $s = (\gamma(\frac{1}{p^i}))$. Since γ is a group homomorphism we get that $\gamma(\frac{1}{p^{i+1}})^p = \gamma(\frac{p}{p^{i+1}}) = \gamma(\frac{1}{p^i})$, hence $s \in S_p$, and it is straightforward to check that $\gamma_s = \gamma$, which shows that L is surjective.

Lastly, if $s_\alpha = (z_i^\alpha) \to s = (z_i)$ is a converging net in S_p, (ie. $z_i^\alpha \to z_i$ for all $i \in \mathbb{N}$), then $\gamma_{s_\alpha}(\frac{m}{p^n}) = (z_n^\alpha)^m \to z_n^m = \gamma_s(\frac{m}{p^n})$. Since L is a continuous bijection between compact Hausdorff spaces it is a homeomorphism. ∎

APPENDIX B

Appendix B: Basics from harmonic analysis and Gelfand Theory

B.1 Dual group and Gelfand Transform

Definition B.1 (Pontryagin Duality)**.** For a locally compact abelian group G the (Pontryagin) dual $\widehat{G}$ is defined as the collection of all continuous group homomorphisms $\gamma : G \to \mathbb{T}$, which, with the compact-open topology and group operation given by pointwise multiplication, becomes a locally compact abelian group in its own right.

If the group G is discrete and abelian, its dual $\widehat{G}$ will thus be the collection of all group homomorphisms $\gamma : G \to \mathbb{T}$, which is a compact group, with respect to the topology of *pointwise convergence*. This can be deduced from Tychonoff's theorem, since $\widehat{G} \subset \prod_{g \in G} \mathbb{T}$ is closed.

As a concrete example, the dual group of $\mathbb{Z}$ is given by $\widehat{\mathbb{Z}} \simeq \mathbb{T}$ where $z \in \mathbb{T}$ acts on $n \in \mathbb{Z}$, by $z \cdot n = z^n$.

The group C*-algebra is defined as the completion of $C_c(G, \mathbb{C})$ with respect to the universal norm defined by equation (1.4) on page 6. When G is a discrete abelian group, we have the following useful relation between the dual group and the group C*-algebra:

$$C^*(G) \simeq C(\widehat{G}).$$

By Gelfand theory, it is enough to establish a homeomorphism between $\widehat{G}$ and the character space of $C^*(G)$, denoted $\widehat{C^*(G)}$. This can be shown by employing the restriction map $R : \widehat{C^*(G)} \to \widehat{G}$, sending a character γ on $C^*(G)$ to the group homomorphism $\tilde{\gamma} \in \widehat{G}$ defined by $\tilde{\gamma}(g) = \gamma(\delta_g)$, where (as in 1.2)

on page 4)

$$\delta_g := \delta_{1,g} = \begin{cases} 1, & g = e \\ 0 & \text{else} \end{cases}$$

Writing this out, the isomorphism $C^*(G) \simeq C(\widehat{G})$ is determined by the map sending $f \in C_c(G)$ to $\hat{f} \in C(\widehat{G})$, where $\hat{f}(\hat{\gamma}) = \sum_{g \in G} \hat{\gamma}(g) f(g)$ and $\hat{\gamma} \in \widehat{G}$, or equivalently, determined on the spanning set of functions $\delta_g \subset C^*(G)$ by

$$\delta_g \mapsto e_g \qquad \text{where } e_g \text{ is the "evaluation at g" homomorphism}$$

that is $e_g(\hat{\gamma}) = \widehat{\gamma(g)}$.

For any semidirect product $N \rtimes_\alpha H$ between two discrete groups N and H, determined by $\alpha : H \to Aut(N)$, we have another useful identity;

$$C^*(N) \rtimes_\alpha H \simeq C^*(N \rtimes_\alpha H)$$

where the action α of H on $C^*(N)$ is determined, with a slight abuse of notation, by

$$\alpha_h(\delta_g) = \delta_{\alpha_h(g)} \qquad \text{with } g \in N, \text{ and } h \in H.$$

We prove this by employing Definition 1.4, that is, we show that both algebras share the same universal property. Adhering to the notation there, with $\mathcal{A} = C^*(N)$, $f \in C_c(N)$, we define the maps $j_{\mathcal{A}} : \mathcal{A} \to C^*(N \rtimes_\alpha H)$ and $j_H : H \to C^*(N \rtimes_\alpha H)$, by

$$j_{\mathcal{A}}(f) = \hat{f}, \qquad \text{where } \hat{f}(g,h) := \sum_{n \in N} f(n) \delta_{(n,e_H)}(g,h) = \begin{cases} 0 & h \neq e_H \\ f(g) & h = e_H \end{cases}$$

$$j_H(h) = \delta_{(e_N,h)}.$$

Proof. The pair $(j_{\mathcal{A}}, j_H)$ is a covariant homomorphism (defined in the paragraph preceding Definition 1.4) of the system $(\mathcal{A}, H, \alpha)$, since for any $f \in C_c(N) \subset \mathcal{A}$ and $h \in H$ we have

$$\begin{aligned}
j_{\mathcal{A}}(\alpha_h(f)) &= \sum_{g \in N} f(g) \delta_{(\alpha_h((g)),e_H)} \\
&= \sum_{g \in N} f(g) \delta_{(e_N,h)(g,e_H)(e_N,h)^*} \\
&= \sum_{g \in N} f(g) \delta_{(e_N,h)} \delta_{(g,e_H)} \delta_{(e_N,h)^*} \\
&= \delta_{(e_N,h)} \left(\sum_{g \in N} f(g) \delta_{(g,e_H)} \right) \delta_{(e_N,h)^*} \\
&= j_H(h) j_{\mathcal{A}}(f) j_H(h)^*.
\end{aligned}$$

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Next, noting that for $\delta_n \in C^*(N)$ we have

$$j_{\mathcal{A}}(\delta_n) j_H(h) = \delta_{(n,h)}$$

The elements $\delta_{(n,h)}$ are clearly a spanning set for $C^*(H \rtimes_\alpha H)$ hence we get that

$$C^*(N \rtimes_\alpha H) = \overline{\text{span}\{j_{\mathcal{A}}(f) j_H(h) \mid f \in C_c(N) \text{ and } h \in H\}}$$

Lastly, for any covariant representation (π, U) of $(\mathcal{A}, H, \alpha)$ on a Hilbert space V, let $L = L_{(\pi,U)} : C^*(N \rtimes_\alpha H) \to B(V)$ be the representation determined by[1]:

$$L(n, h) = \pi(n) U(h)$$

Now it is easy to check that $L \circ j_{\mathcal{A}} = \pi$ and $L \circ j_H = U$. All the criteria of 1.4 are met, thus we have a unique isomorphism

$$C^*(N) \rtimes_\alpha H \simeq C^*(N \rtimes_\alpha H).$$

$\blacksquare$

In the article [HW17], the authors addresses the existence of extreme points in Proposition 3.4, since, by Krein-Milman, this amounts to $S_G(\mathcal{A})$ being non-empty. In particular, when the group $G = \mathbb{Z}^2$ is abelian, hence amenable, and $\mathcal{A} = C(S_{pq})$ is unital, it is known that $S_G(\mathcal{A}) \neq \emptyset$.

To see this, note that G being amenable is equivalent to the existence of a translation invariant state m on $l^\infty(G)$, that is, there exists a state m such that $m(l_g(f)) = m(f)$ for all $f \in l^\infty(G)$, for all $g \in G$, where $l_g(f)(h) = f(gh)$. Now let ψ be any state on $\mathcal{A}$ and define a map $\phi : \mathcal{A} \to \mathbb{C}$ by $\phi(a) = m(f_a)$, where $f_a(g) = \alpha_{g^{-1}}(a)$ is in $l^\infty(G)$. ϕ can be shown to be a state on $\mathcal{A}$, and

$$\phi(\alpha_g(a)) = m(f_{\alpha_g(a)}) = m(g \cdot (f_a)) = m(f_a) = \phi(a) \qquad \text{so } \phi \in S_G(\mathcal{A}).$$

B.2 Positive Functionals and Positive Definite functions

One important correspondence, which we will use in the proof of proposition 2.6 is that between the positive definite functions on the group and the positive linear functionals on the group C*-algebra. First let's state the necessary definitions, bearing in mind that there are several equivalent definitions (see for instance [Ped79] prop. 7.1.9).

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[1]Tacitly we exploit that there is a 1-1 correspondence between the non-degenerate representations of the group C*-algebra and unitary representations of the group. So $\pi(h)$ is unitary, and so is L, hence determine a representation is non-degenerate representation of $C^*(H \rtimes_\alpha H)$.

Definition B.2 (Positive Definite function)**.** Let G be a discrete group. A function $f : G \to \mathbb{C}$ is called positive definite when

- $\sum_{i,j=1}^{n} f(s_i^{-1} s_j) \lambda_i \overline{\lambda_j} \geq 0$

for any combinations of $s_1, ..., s_n \in G$, $\lambda_1, ..., \lambda_n \in \mathbb{C}$, and $n \in \mathbb{N}$.

The following consequence of Bochner's theorem [Rud62][p. 19] will be important for the article.

Theorem B.3. *There is a bijective correspondence between the positive definite functions on G and positive linear functionals on $C^*(G)$, given by sending a positive definite function F to the functional ϕ_F, determined by*

$$\phi_F(f) = \sum_{s \in G} F(s) f(s) \qquad \text{for all } f \in C_c(G).$$

Conversely, for a positive linear functional ϕ on $C^*(G)$, we may retrieve the positive function associated with it by the equation

$$\phi \mapsto F \qquad F(g) = \phi(\delta_g).$$

As we have seen, for discrete abelian group G we can identify $C^*(G)$ with $C(\widehat{G})$ (see section B.1). This identification is determined by sending the basis element $\delta_g \mapsto e_g$, where $e_g \in C(\widehat{G})$ is the "evaluation at g" map. Under this identifucation, the positive definite function F in the above definition induces a positive linear functional $\tilde{\phi}_F$ on $C(\widehat{G})$ given by

$$\tilde{\phi}_F(e_g) = F(g).$$

In this article we will primarily be dealing with the groups $\mathbb{Z}$ or $\mathbb{Z}[\frac{1}{pq}]$, which have dual groups $\mathbb{T}$ and S_{pq} respectively. Positive functionals ϕ on $C(\mathbb{T})$ and ψ on $C(S_{pq})$ induce positive definite functions on P_ϕ on $\mathbb{Z}$ and P_ψ on $\mathbb{Z}[\frac{1}{pq}]$ respectively, given by

$$P_\phi(n) = \phi(e_n) \qquad\qquad \text{where } e_n \in C(\mathbb{T}), \text{ and } e_n(z) = z^n$$

$$P_\psi(\frac{m}{(pq)^n}) = \psi(E_{\frac{m}{(pq)^n}}) \quad \text{with } E_{\frac{m}{(pq)^n}} \in C(S_{pq}), \text{ and } E_{\frac{m}{(pq)^n}}((z_i)) = z_n^m.$$

We will also use the fact that for any $g_1, g_2 \in \mathbb{Z}[\frac{1}{pq}]$, $E_{g_1+g_2}((z_i)) = E_{g_1}((z_i)) E_{g_2}((z_i))$ since with $g_1 = \frac{m_1}{(pq)^{n_1}}$ and $g_2 = \frac{m_2}{(pq)^{n_2}}$, we may write $g_1 + g_2 = \frac{m_1(pq)^{n_2} + m_2(pq)^{n_2}}{(pq)^{n_1+n_2}}$, hence

$$E_{g_1+g_2}((z_i)) = z_{n_1+n_2}^{m_1(pq)^{n_2}+m_2(pq)^{n_2}} = z_{n_1+n_2}^{m_1(pq)^{n_2}} z_{n_1+n_2}^{m_2(pq)^{n_1}} = z_{n_1}^{m_1} z_{n_2}^{m_2} = E_{g_1}((z_i)) E_{g_2}((z_i))$$

Note that E_g (with $g \in \mathbb{Z}[\frac{1}{pq}]$) are the "evaluation at g" homomorphisms, sending $g \mapsto \widehat{g} = E_g \in C(S_{pq})$, where $\widehat{g}(\gamma) = \gamma(g)$. So this is actually the image of the Gelfand transform of $C^*(\mathbb{Z}[\frac{1}{pq}])$ onto $C(\widehat{\mathbb{Z}[\frac{1}{pq}]})$ after the identification of $\widehat{\mathbb{Z}[\frac{1}{pq}]} \simeq S_{pq}$, which is known to be dense and well defined. Hence span$\{E_k \mid k \in \mathbb{Z}[\frac{1}{pq}]\}$ is dense in $C(S_{pq})$

APPENDIX C

Ergodic Theory Basics

Here are some of the definitions regarding ergodic theory, which will be used later.

Given a measurable space $(X, \mathcal{B})$ and a measurable map $T : X \to X$, a probability measure $\mu : \mathcal{B} \to [0, 1]$ is said to be **T-invariant** if

$$\mu(T^{-1}A) = A \qquad \text{for all } A \in \mathcal{B}.$$

It is said to be **ergodic** if for all $E \in \mathcal{B}$ with $T^{-1}E = E$, we have $\mu(E) \in \{0, 1\}$

The ergodic measures are considered to be the building blocks of ergodic theory. By this we mean that, if $(X, \mathcal{B}, T, \mu)$ is a probability space, with $\mathcal{B}$ the Borel σ-algebra, $T : X \to X$ continuous, and μ is T-invariant, and assume there is a subset $B \subset X$ such that $T^{-1}B = B$, we may restrict the system to two (possibly) simpler systems defined on B and B^c by restrict the map T, the σ-algebra $\mathcal{B}$ and the measure μ in the obvious way to B and B^c. The ergodic measures (or rather systems) are those which have no (non-trivial) decomposition, since if $T^{-1}B = B$, then by definition $\mu(B) \in \{0, 1\}$, which means either B or B^c has measure zero, and is thus measure theoretically negligible.

In the present context we will also need the following definitions.

Definition C.1 (*G*-invariant States)**.** Given a C*-dynamical system $(\mathcal{A}, G, \alpha)$, a G-invariant state of $\mathcal{A}$ is a state ϕ of $\mathcal{A}$ such that

$$\phi(\alpha_g(a)) = \phi(a)$$

for all $g \in G$ and all $a \in \mathcal{A}$. We will denote the set of G-invariant states of $\mathcal{A}$ by $S_G(\mathcal{A})$.

Definition C.2 (Ergodic States)**.** Given a C*-dynamical system $(\mathcal{A}, G, \alpha)$, with $\mathcal{A}$ unital and G discrete, the ergodic G-invariant states of $\mathcal{A}$ are defined as the extremal points of the set of G-invariant states of $\mathcal{A}$. We will denote the set of ergodic G-invariant states of $\mathcal{A}$ by $E_G(\mathcal{A})$.

One can check that $S_G(\mathcal{A})$ is a convex and weak*-compact set, hence extremal points exists by the Krein-Milman theorem, provided $S_G(\mathcal{A}) \neq \emptyset$.

To motivate the above definition, let X be a compact metric space, and let $T : X \to X$ be a continuous map. Then μ is a $T-$invariant ergodic probability measure if and only if it is an extreme point of the convex set of T-invariant probability measures on X (see prop 6.10 [Wal00]), which is non-empty by Krylov–Bogolyubov theorem.

By a $\times p, \times q$-invariant probability measure on $\mathbb{T}$ we mean a (regular Borel) probability measure μ on $\mathbb{T}$ which is invariant under both maps $T_p, T_q : \mathbb{T} \to \mathbb{T}$ defined respectively by $z \mapsto z^p$ and $z \mapsto z^q$. It is called **ergodic** if for each $E \in \mathbb{B}$ with $T_p^{-1}E = T^{-1}E = E$ we have $\mu(E) \in \{0, 1\}$. Note that these are not the intersection of the T_q and T_q ergodic measure, but rather the extremal points of the $\times p, \times q$-invariant measures (see [Gla03] theorem 4.2).

The simplest example of a finitely supported $\times p, \times q$-invariant measure on $\mathbb{T}$ is the point measure δ_1 with support on $\{1\}$. Almost as simple is the measure with values $\frac{1}{2}$ on 1 and -1. For a less trivial example, identify $\mathbb{T}$ with the interval $[0, 1)$, and pick a point $\frac{m}{p^j q^k}$, where m, j, k are arbitrary integers, then

$$ p^r q^l \frac{m}{p^j q^k} = p^{r-j} q^{l-k} m $$

By letting $r, l \in \mathbb{N}$ vary, we may produce at most $(r - j)(l - k)$ numbers which are distinct modulo 1. On this "orbit", we may construct a probability measure μ with constant constant values on each point of this collection.

All of the next definitions can be found in any introductory book to ergodic theory, like [Wal00]. By a measurable partition ξ of $\mathbb{T}$ we mean a countable collection of subsets of $\mathbb{T}$ whose pairwise intersection have measure zero, that cover $\mathbb{T}$ almost everywhere. The entropy of a measurable partition is defined as[1]

$$ H_\mu(\xi) = - \sum_{C \in \xi} \mu(C) \log(\mu(C)). $$

The entropy of a measurable μ-invariant map T is given by

$$ h_\mu(T, \xi) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\bigvee_{k=0}^{n-1} T^{-k}\xi). $$

here $\vee$ denotes the "join" of the partitions, defined for two partitions ξ, η to be

$$ \xi \vee \eta = \{C \cap D \mid C \in \xi, \ D \in \eta, \ \mu(C \cap D) \neq 0\}. $$

The limit does exist and is always finite, as the sequence can be shown to be monotone non-increasing.

---

[1]We could admit uncountable partitions, by defining $H_\mu(\xi) = \infty$ if the union of all sets of measure zero in ξ have positive measure.

Definition C.3 (Entropy)**.** The (measure theoretic) entropy of a μ-invariant measurable map $T : \mathbb{T} \to \mathbb{T}$ is defined as the quantity

$$h_\mu(T) = \sup\{h_\mu(T, \xi) \mid \xi \text{ is a measurable partition of } \mathbb{T}, \text{ with } H_\mu(\xi) < \infty\}.$$

Remark C.4. Some authors call this the metric entropy, but this seems highly misleading, as the definition of Adler et. al. in [AKM65] explicitly makes use of a metric. The two definitions are related by the Variational Principle (see [Wal00]).

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