3-D sensitivity kernels of the Rayleigh wave ellipticity

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SUMMARY
The ellipticity of the Rayleigh wave at the surface depends on the seismic structure beneath and in the vicinity of the seismological station where it is measured. We derive here the expression and compute the 3-D kernels that describe this dependence with respect to S-wave velocity, P-wave velocity and density. Near-field terms as well as coupling to Love waves are included in the expressions. We show that the ellipticity kernels are the difference between the amplitude kernels of the radial and vertical components of motion. They show maximum values close to the station, but with a complex pattern, even when smoothing in a finite-frequency range is used to remove the oscillatory pattern present in mono-frequency kernels. In order to follow the usual data processing flow, we also compute and analyse the kernels of the ellipticity averaged over incoming wave backazimuth. The kernel with respect to P-wave velocity has the simplest lateral variation and is in good agreement with commonly used 1-D kernels. The kernels with respect to S-wave velocity and density are more complex and we have not been able to find a good correlation between the 3-D and 1-D kernels. Although it is clear that the ellipticity is mostly sensitive to the structure within half-a-wavelength of the station, the complexity of the kernels within this zone prevents simple approximations like a depth dependence times a lateral variation to be useful in the inversion of the ellipticity.

Key words: Seismic tomography; Theoretical seismology.

1 INTRODUCTION
It is well-known that Rayleigh waves are elliptically polarized and that the ratio between their surface radial and vertical displacements depends upon the structure in which they have propagated. This is partly the basis for the classical HV ratio method which is extensively used in near-surface geophysics (Nakamura 1989). The usage of the Rayleigh wave ellipticity as a source of information in larger-scale seismology is however more recent. Measurements of the Rayleigh wave ellipticity on single recordings have proven unstable (Ferreira & Woodhouse 2007; Tanimoto & Rivera 2008; Yano et al. 2009), but averaged values measured either on noise recordings (Lin et al. 2012; Workman et al. 2017) or using azimuthally well-distributed sources (Tanimoto & Rivera 2008; Yano et al. 2009; Berbellini et al. 2016) have shown that stable measurements can be obtained.

The Rayleigh wave ellipticity may be inverted in terms of structure either alone (Yano et al. 2009; Berbellini et al. 2017) or in combination with phase or group velocity data (Lin et al. 2012), as their more shallow sensitivity makes them a good complement to the more traditional velocity data. In addition, as it is assumed that the ellipticity depends on the local structure beneath the station, inversion of ellipticity can be done independently for several stations and does not require a tomographic inversion as velocity data do.

The 1-D sensitivity kernels used in recent inversions were first calculated numerically (Tanimoto & Rivera 2008) until older results (Tsuboi & Saito 1983) based on the variational principle were rediscovered, enabling also the analytical computation of the kernels (Tanimoto & Tsuboi 2009). First-order perturbation theory provides an additional way to calculate analytically the sensitivity kernels of the eigenfunctions (Maupin 1987, 1989), but has been used principally to analyse the effect of anisotropy on the modes eigenfunctions.

Irrespective of the method chosen to calculate them, these 1-D sensitivity kernels express how the Rayleigh wave surface ellipticity is affected by a perturbation of the laterally homogeneous structure in which the modes are computed, implying implicitly that the perturbation is localized in depth but infinite in the horizontal directions. This is in contrast with the assumption made during inversion that the ellipticity at a given station depends on the structure right beneath the station and not on the structure along the propagation path.

Ray theory solves this apparent contradiction. Ray theory for surface waves tells us that the eigenfunctions of the modes can be computed at a given location using the local structure (Woodhouse & Wong 1986). The consequence of this is in particular that the ellipticity of any Rayleigh wave mode at a seismological station depends, as long as ray theory is valid, on the structure beneath the station, and not on the structure along the wave path. This has the advantage to provide local information on the structure that can be exploited directly, without a lateral tomographic inversion. Ray theory is valid if the lateral variations occur over a large scale compared to the...
wavelength, leading to the implicit assumption that the ellipticity is affected by the average structure over a significant area around the station. Inversions of ellipticity are on the other hand done individually beneath stations, without considering, also because this is unknown, what is the region around the station that is actually affecting the measurement at that station (Lin et al. 2012; Berbellini et al. 2017).

Surface wave velocity inversion also traditionally relied on ray theory, assuming that the phase depends on the integral of the phase slowness along the source-to-station ray, and depth inversion was, and is still usually done locally using the 1-D sensitivity kernels for phase velocity. These surface wave 1-D kernels have been extended in recent years to 3-D kernels (Zhou et al. 2004; Dahlen & Zhou 2006), the surface wave cousins of the banana-doughnut kernels for body waves (Dahlen et al. 2000). These 3-D kernels free us from the ray theory assumption and enable tomographic inversions that take into account the sensitivity of the observables in 3-D, in particular how far away from the great circle any structural perturbation affects an observable.

In the present work, we extend the 3-D surface wave sensitivity kernels calculated previously for phase, group time, amplitude and path deviation (Zhou et al. 2004; Dahlen & Zhou 2006) to Rayleigh wave surface ellipticity. The goal is to be able to quantify the dimension of the region around a receiver that affects the ellipticity at a given frequency. After deriving the necessary equations, we present the kernels with respect to S-wave velocity, P-wave velocity and density for a surface wave at 20 s period, both for a single ellipticity measurement and for azimuthally averaged measurements. We compare them with previous results and discuss the possible challenges related to ellipticity inversions.

2 Theory - Methodology for the Computation of the Kernels

2.1 The ellipticity of the Rayleigh wave fundamental mode

The ellipticity of the Rayleigh wave can simply be defined as the ratio $\eta$ of the radial to vertical amplitudes, hereafter called HV ratio:

$$\eta(\omega) = \frac{R(\omega)}{V(\omega)}$$

(1)

where $\omega$ is frequency, $R$ and $V$ are the amplitudes of the Fourier transforms of the radial and vertical components respectively, taken in an appropriately chosen time window. This notation is in accordance to the commonly used HV ratio, and also corresponds to the conventions used by Berbellini et al. (2016). This is slightly different from the convention used by Tanimoto & Rivera (2008), where the inverse ratio is used to avoid values going to infinity. Although the expressions derived further down are also valid for overtones, we will restrict applications to the fundamental Rayleigh mode, which is usually well-isolated enough to allow ellipticity measurements.

The definition of the ellipticity above (eq. 1) overlooks the possibility of polarization distortion by anisotropy. As shown by Yano et al. (2009), the precision of the measurements does not usually enable to determine anisotropy from this kind of polarization analysis and we will limit ourselves to isotropic structures.

Different approaches can be used to measure the Rayleigh wave ellipticity. Tanimoto & Rivera (2008) propose two methods, based on well-known characteristics of the Rayleigh wave fundamental mode, namely a polarization in the radial-vertical plane with a $\pi/2$ phase shift between the two components. The first method is based on measuring the envelope of the vertical and radial components and taking the ratio of the maxima; the second is based on cross-correlating the two components after a $90^\circ$ phase advance of the vertical component. Both methods give directly the value of $\eta$ in eq. (1). An alternative is to compute the attributes of the wave polarization in a more general way using the covariance matrix between the three components (Park et al. 1987; Workman et al. 2017). We have verified that this more general definition of the ellipticity leads, under the assumption that the ellipticity is measured in the direction of the unperturbed radial direction, to eq. (1).

2.2 First-order variations of the ellipticity

First-order variations of the HV ratio are given simply by derivation of eq. (1):

$$\frac{\delta \eta(\omega)}{\eta(\omega)} = \frac{\delta R(\omega)}{R(\omega)} - \frac{\delta V(\omega)}{V(\omega)}.$$  

(2)

This is equivalent to

$$\delta \ln \eta(\omega) = \delta \ln R(\omega) - \delta \ln V(\omega).$$  

(3)

Zhou et al. (2004) show that the perturbation of the amplitude $A$ on a given component is expressed in the Born approximation as

$$\delta \ln A(\omega) = \Re \left( \frac{\delta s}{s} \right),$$  

(4)

where $s$ is the waveform on the particular component in the unperturbed structure and $\delta s$ is its perturbation by a scatterer.

This implies that the perturbation of the ellipticity can be expressed as

$$\delta \ln \eta(\omega) = \Re \left( \frac{\delta s_r}{s_r} - \frac{\delta s_v}{s_v} \right).$$  

(5)

where $s_r$ and $s_v$ are the waveforms on the radial and vertical components respectively and $\delta$ indicates their perturbations.

2.3 The scattered wavefield

The wavefield recorded at position $\mathbf{x}$ scattered by an isotropic heterogeneity located at position $\mathbf{x}'$ characterized by a perturbation in elastic Lamé elastic parameters $\delta \lambda$ and $\delta \mu$ and density $\delta \rho$ has the general expression (rewritten after Dahlen et al. 2000):

$$\delta s_r(\mathbf{x}) = \int \delta \rho(\mathbf{x}') \omega^2 G_{\omega}(\mathbf{x}, \mathbf{x}') s_t(\mathbf{x}')$$

$$- \delta \lambda(\mathbf{x}') (\partial_i G_{\omega}(\mathbf{x}, \mathbf{x}')) \partial_i s_t(\mathbf{x}')$$

$$- \delta \mu(\mathbf{x}') (\partial_i G_{\omega}(\mathbf{x}, \mathbf{x}')) \partial_i s_t(\mathbf{x}')$$

$$- \delta \rho(\mathbf{x}') \partial_i G_{\omega}(\mathbf{x}, \mathbf{x}') \partial_i s_t(\mathbf{x}') d\mathbf{x}'$$  

(6)

where $G_{\omega}(\mathbf{x}, \mathbf{x}')$ is the Green’s function of the medium and where all derivatives are to be taken with respect to the position $\mathbf{x}'$ of the scatterer as indicated by the ‘$i$’ on the $\partial$ operators. For the purpose of calculating the ellipticity perturbation, we need the expression for $\delta s_r(\mathbf{x}')$ for the vertical and radial components.

2.4 The Green’s function

For surface waves in laterally homogeneous structures, the Green’s function can be written as a sum of modes which propagate independently from each other. Zhou et al. (2004) use a far-field expression for the Green’s function valid on a spherical Earth. As we expect that the scatterers close to the receiver have a dominant influence
on the ellipticity of the Rayleigh wave, we need to use an expression for the Green’s function that also includes the near-field terms. The Green’s function with near-field terms in spherical geometry is used in Liu & Zhou (2016), where they study the influence of the near-field terms on the Rayleigh wave sensitivity kernels in phase and amplitude, measured on vertical components. In the present work, we use a Green’s function that includes the near-field terms but is expressed in a flat Cartesian/cylindrical coordinate system. This provides expressions that are slightly simpler and give a better physical insight, and is justified by the fact that we expect a kernel localized close to the receiver, with little dependence on wave path.

The Green’s function for surface waves can be written as a sum of modes:

\[
G_q(\vec{x}, \vec{x}') = \sum_m Z_q^m(\vec{z})g_m^q(\vec{x}, \vec{y}, \vec{z}),
\]

which we express, like Friederich et al. (1993) and Maupin (2001), as an operator:

\[
Z_q^m(\vec{z}) = \left( \frac{1}{k_m} - W^m(\vec{z}) \frac{\partial}{\partial x} + W^m(\vec{z}) \frac{\partial}{\partial y} \right)
\]

acting on a potential:

\[
g_m^q(\vec{x}, \vec{y}, \vec{z}) = -\frac{i}{8\epsilon_m \mu_m J_n^m} Z_q^m(\vec{z}) H_0^{(2)}(k_m r),
\]

where \( U^m \) and \( \ell^m \) are respectively the vertical and horizontal components of the eigenfunction of Rayleigh wave mode \( m \), and \( W^m \) is the eigenfunction of Love wave mode \( m \). \( k_m, \epsilon_m, \mu_m, \) and \( J_n^m \) are the wavenumber, phase velocity, group velocity and energy integral of the modes, respectively. \( x, y \) and \( z \) denote the three components of \( \vec{z} \) in a right-handed coordinate system where \( x \) and \( y \) are the two horizontal directions and \( z \) points downwards. \( r \) is the distance in the horizontal plane between \( \vec{x} \) and \( \vec{x}' \), and \( H_0^{(2)} \) is the Hankel function of second kind at order 0. In the far-field approximation, the operator \( Z_q^m(\vec{z}) \) simply reduces to containing the eigenfunction of the mode and some \( \frac{1}{2} \pi \) phase factors. Let us note that the operator \( Z \) is also present in the expression of the potential, but then acting on the source side. In the far-field approximation, the potential can be viewed as simply representing the amplitude and the phase of mode \( m \) in the horizontal plane.

Eq. (7) can also be rewritten:

\[
G_q(\vec{x}, \vec{x}') = -\frac{i}{8\epsilon_m \mu_m J_n^m} \sum_m Z_q^m(\vec{z}) Z_\ell^m(\vec{z}') H_0^{(2)}(k_m r)
\]

provided the eigenfunctions are normalized so that \( 8\epsilon_m \mu_m J_n^m = 1 \). The derivatives in \( Z_q^m(\vec{z}) \) are to be taken with respect to \( \vec{x} \) whereas those in \( Z_\ell^m(\vec{z}') \) are to be taken with respect to \( \vec{x}' \).

2.5 The modal decomposition

Decomposing the reference field as a sum of modes:

\[
s_q(\vec{x}') = \sum_n Z_q^n(\vec{z}') \Phi_n(\vec{x}', y')
\]

where \( \Phi_n(\vec{x}', y') \) is the potential of mode \( n \) at scatterer location \((x', y')\), and inserting eq. (10) for the Green’s function in eq. (6), we can write the scattered wavefield as a sum of modes:

\[
\delta s_q(\vec{x}) = \sum_m Z_q^m(\vec{z}) \Phi_m(x, y).
\]

where \( \delta \Phi_m(x, y) \), the perturbation of the potential of mode \( m \) at receiver location \((x, y)\), is given by

\[
\delta \Phi_m(x, y) = -i \int \delta \rho(\vec{x}) \omega^2 (\Phi_m(x') H_0^{(2)}(k_m r)) \sum_n Z^n_q(\vec{z}') \Phi_n(x', y') - \delta \lambda(\vec{x}) (\Phi_m(\vec{z}') H_0^{(2)}(k_m r)) (\frac{\partial}{\partial x} \sum_n Z^n_q(\vec{z}') \Phi_n(x', y'))
\]

\[
- \delta \mu(\vec{x}) (\Phi_m(\vec{z}') + \delta \rho(\vec{z}') H_0^{(2)}(k_m r)) \sum_n Z^n_q(\vec{z}') \Phi_n(x, y') d\vec{x}'
\]

In the present case, we want to compute the perturbation of the ellipticity of the Rayleigh wave fundamental mode. The reference field is therefore composed of the Rayleigh wave fundamental mode only \((n = 0) \) only in eqs 11 and 13. The scattered field on the other hand (modes \( m \)) is composed of both the Rayleigh wave fundamental mode, Rayleigh overtones, and Love modes that are excited by the scatterers and introduce a perturbation of the HV ratio at the receiver by constructive interference with the incident field. The \( \sum_m \) in eq. (12) should therefore be kept.

2.6 The perturbation of the vertical and radial displacements

The perturbation of the vertical component is obtained by inserting the last component of the operator \( Z \) (eq. 8) into eq. (12), and derives from the simple expression:

\[
\delta s_x(\vec{x}) = \sum_m U^m(\vec{z}) \delta \Phi_m(x, y).
\]

The expressions for the horizontal components are slightly more complicated as they involve horizontal derivatives as well as Love and Rayleigh modes:

\[
\delta s_x(\vec{x}) = \sum_m (1/k_m (V^m(\vec{z}) \partial/\partial x + W^m(\vec{z}) \partial/\partial y)) \delta \Phi_m(x, y)
\]

\[
\delta s_x(\vec{x}) = \sum_m (1/k_m (V^m(\vec{z}) \partial/\partial x - W^m(\vec{z}) \partial/\partial y)) \delta \Phi_m(x, y)
\]

Let us note here that the difference between the scattered fields on the vertical and horizontal components, as expressed in the equations above, is at the core of the ellipticity sensitivity. The vertical field depends simply on the perturbed potential of the Rayleigh waves only. The horizontal components show additional horizontal derivatives which enhance the singularity of the potential close to the station, and contain in addition Love wave contributions.

Eq. (5) requires that we calculate the scattered field in the vertical and radial directions. The scattered field on the vertical component derives directly from eq. (14), but the scattered field on the radial component requires a rotation of the scattered field from the horizontal directions to the radial direction, which we simply define as the direction that maximizes the energy of the reference field at the receiver.

2.7 Sensitivity kernels

By definition, the 3-D sensitivity kernels enable us to express the variation in an observable as an integral over the Earth volume at first order in structural variations. Combining eq. (5) with
\[ \delta \ln \eta = \int \left( \delta \lambda(x) \hat{K}_\lambda(x) + \delta \mu(x) \hat{K}_\mu(x) + \delta \rho(x) \hat{K}_\rho(x) \right) \, dx, \]  
\text{where the kernels } \hat{K}_\lambda(x), \hat{K}_\mu(x) \text{ and } \hat{K}_\rho(x) \text{ are for perturbations in the Lamé elastic parameters and density. The sensitivity kernels are more commonly expressed in terms of relative variations of compressional wave velocity } \alpha, \text{ shear wave velocity } \beta, \text{ and density, leading to the expression:} 
\delta \ln \eta = \int \left( \delta \ln \alpha(x) K_\alpha(x) + \delta \ln \beta(x) K_\beta(x) + \delta \ln \rho(x) K_\rho(x) \right) \, dx, \]  
\text{where the relative kernels can be expressed in terms of the absolute ones:} 
\begin{align*}
K_\alpha &= 2(\lambda + 2\mu) \hat{K}_\lambda, \\
K_\beta &= 2\mu(\hat{K}_\mu - 2\hat{K}_\lambda), \\
K_\rho &= \lambda \hat{K}_\lambda + \mu \hat{K}_\mu + \rho \hat{K}_\rho.
\end{align*} 

3 NUMERICAL ASPECTS

3.1 Singularities of the Hankel functions

Due to the presence of horizontal derivatives in the operators \(Z\) in eq. (13), the potential of the scattered field, and consequently its vertical displacement, varies with the distance between scatterer and receiver as Hankel functions of degrees 0 to 2 (see detailed expressions in Maupin 2001). The horizontal components involve an additional horizontal derivative and therefore Hankel functions of degree 3 in addition. As Hankel functions are singular at the origin, the computation of the kernels requires careful attention to numerical aspects. In the examples shown in the next sections, the kernels are calculated on a regular grid positioned such that the receiver is located at a centre of a cell and not on a grid point. This ensures that we do not need to calculate Hankel functions at the receiver location itself. The horizontal derivatives in eqs (15) and (16) are performed analytically and lead to terms in Hankel functions of up to degree 3 and azimuthal variations in up to 36.

As our grid excludes the very close vicinity of the receiver, and as singularities occur in this region, we have to look closer at their contributions to the scattered field. The contribution \(A\) to the scattered potential by scatterers located in a vertical cylinder of radius \(a\) around the receiver can be computed by integrating \(\delta \Phi_\alpha(r, \theta)\):
\[ A \approx \int_0^a \int_0^{2\pi} \delta \Phi_\alpha(r, \theta) \, d\theta \, dr, \]  

where \((r, \theta)\) are horizontal coordinates in cylindrical symmetry.

Examining the expression for \(\delta \Phi_\alpha\) in eq. (13), we see that the dominant horizontal dependence results from the product of a Hankel function with the reference potential or its horizontal derivatives. Assuming that the source is far away from the receiver, the reference potential and its derivatives can be expressed as slowly varying functions times an exponential, thereby singling out the horizontal variation in eq. (20):
\[ A = A_0 \int_0^a \int_0^{2\pi} H_n^{(2)}(kr) \exp(-ikr \cos \theta) \, d\theta \, dr, \]  
where \(A_0\) depends weakly on \((r, \theta)\), \(n\) varies from 0 to 2 and the propagation is in direction \(\theta = 0\) without loss of generality. It is shown in appendix B of Maupin (2001) that these integrals are not singular and that their values go to zero when the radius \(a\) goes to zero. This ensures that the vicinity of the receiver does not produce a dominant factor that would be omitted in our approach and that the final results are not dependent on grid size. In a tomographic analysis, it would be possible to use integrated values over small volumes close to the source instead of values at point locations in order to avoid problems related to the singularities.

We have also checked numerically that our results are not dependent on grid size. Due to the singularity of the Hankel functions close to the receiver, the maximum values computed for a given kernel are however of course very dependent on grid size. This requires careful attention in the choice of colour scale to ensure that one single point does not blur the rest of the figure, but also that the high values in the vicinity of the receiver are properly represented.

Detailed information on choice of colour scale is given in each figure.

3.2 Frequency-band kernels

As seismological observables are usually the result of a filtering process with a finite-frequency band on a windowed signal, sensitivity kernels computed for one single frequency do not represent the real sensitivity of measurements, and kernels are commonly computed in a frequency band (Zhou et al. 2004). If filtering is done using multitapers, the amplitude kernels for a frequency band around \(\omega\) is defined in Zhou et al. (2004) as:
\[ \delta \ln A(\omega) = \text{Re} \left( \sum \frac{\delta s_i s_j^*}{\sum \delta s_i s_j^*} \right), \]  
where \(s_i(\omega) = s(\omega) \otimes h_i(\omega)\), the displacement convolved by taper \(j\), and similarly for \(\delta s_i\).

Using the observation window as unique taper, assuming that the central time of the observation window corresponds to the group time of the reference field, where phase is stationary, and that the amplitude of the reference field is constant in the appropriate frequency band, eq. (22) can be rewritten as:
\[ \delta \ln A(\omega) = \text{Re} \left( \sum \frac{\delta s(\omega - v_i) s^*(\omega - v_j) h(v_i)}{\sum s(\omega - v_j) s^*(\omega - v_i) h(v_j)} \right), \]  
where \(h(v_i)\) is the spectrum of the observation window at frequency \(v_i\), \(h(v_j)\) can also be seen as a factor used to weight contributions to the kernel at frequency \(\omega\) from the scattering field at neighbouring frequencies.

In the following section, we will present an example of mono-frequency kernel but will thereafter focus the discussion on the more useful frequency-band kernels.

4 KERNELS IN \(\beta\)

We first present the sensitivity kernels of the HV ratio with respect to shear wave velocity. We have chosen to present all our results at a frequency of 0.05 Hz (20 s period, corresponding to 75 km wavelength) which is appropriate for crustal studies (Berbellini et al. 2016), but
these results can easily be scaled to other frequencies. The surface wave modes needed to calculate the kernels are computed in the continental version of the PREM model (Dziewonski & Anderson 1981) with a software package from Saito (1988).

4.1 Mono-frequency kernels

Horizontal sections of the shear wave sensitivity kernels of the radial amplitude, vertical amplitude and HV ratio are shown at 25 km depth in Fig. 1. The reference wavefield is a plane wave coming from 60° North. The plots on the left-hand side show the kernels computed at a single frequency of 0.05 Hz.

The kernels for the radial and vertical amplitudes (upper and middle plots) have quite similar features, with a broad negative zone to the Northeast, in the direction of the wave arrival, corresponding to forward scattered waves, surrounded by an oscillatory pattern off-azimuth and a strongly oscillatory pattern to the Southwest of the receiver, corresponding to the contribution from backscattered waves.

The lower plot shows the sensitivity kernel for the HV ratio, which is just the difference between the two plots right over. As forward-scattered waves perturb the radial and vertical amplitudes in similar ways, they do not perturb the HV ratio, resulting in a low HV kernel in the northeastern corner of the plot. On the opposite, backscattered waves act with opposite signs on the amplitudes of the two components and yield a large, albeit very oscillatory, kernel in the southwestern corner of the plot.

4.2 Frequency-band kernels

The plots on the right-hand side of Fig. 1 show the kernels for a frequency range centred on 0.05 Hz, with a total bandwidth of 0.024 Hz including a linear ramp over 0.01 Hz on each side.

As expected, the frequency-band kernels have strongly reduced oscillations compared to the mono-frequency ones. The sensitivities of the radial and vertical amplitudes are still very similar away from the receiver. The sensitivity kernel for the vertical component compares well with those presented by Zhou et al. (2004) and Liu & Zhou (2016), with a negative lobe towards the source, surrounded by two smaller positive ones. The width of the main negative lobe is not as broad as the corresponding lobe for phase variations (not shown here), also in accordance with the results of Zhou et al. (2004) and Liu & Zhou (2016). The main difference between the two kernels is in the vicinity of the receiver, resulting in the HV kernel (right lower plot) being now clearly concentrated close to the receiver. The amplitude of the HV ratio kernel increases sharply close to the receiver, in particular the amplitude of two strongly localized negative lobes. It has a complex oscillatory pattern dominated by a \( \cos 2\theta \) azimuthal variation.

4.3 Influence of coupling with Love waves and Rayleigh higher modes

As discussed in Section 2.5, the field scattered by an incident Rayleigh wave fundamental mode incident on an inhomogeneity is composed of the Rayleigh fundamental mode, overtones and Love modes. The Rayleigh wave kernels, for ellipticity or for any other observable, depend on this mode coupling at the heterogeneity.

The scattered field used to compute the kernels presented in the previous section is composed of the Rayleigh fundamental mode and the Love wave fundamental mode. Coupling to both Rayleigh and Love first and second higher modes proved not to contribute significantly to the kernels, their amplitudes being about 30 times smaller than the fundamental modes contributions to the HV kernels. We show in Fig. 2 the kernels computed with the Rayleigh fundamental mode only and with the Love fundamental mode only. The kernel with the Love wave only is of the same order of magnitude as the kernel with the Rayleigh wave only and completes it nicely with energy coming at larger angles from the propagation direction of the reference field. Both modes contribute significantly to the total kernels.

4.4 Kernel for azimuthally averaged ratio

Zooming closer to the station, in an area corresponding to 4/3 of a wavelength, we will now examine the HV kernel more in details. The left plot of Fig. 3 shows the HV kernel with respect to shear wave velocity at 25 km depth in a smaller area around the receiver, calculated with a 2 km grid spacing. The \( \cos 2\theta \) azimuthal variation appears more clearly than in Fig. 1. We see that the maximum sensitivity is clearly concentrated close to the receiver but with a complex pattern. This pattern may explain why measurements of HV ratio on single recordings yield unstable measurements (Ferreira & Woodhouse 2007; Tanimoto & Rivera 2008), and that averaging over many azimuths is necessary to obtain reliable data (Tanimoto & Rivera 2008; Berbellini et al. 2016).

The kernels for azimuthally averaged HV ratios are the azimuthally averages of the kernels for single propagation directions. We have therefore computed the kernels for the average ratio by averaging kernels for plane waves arriving from azimuths varying in 30° increments. This azimuthal sampling proved good enough to remove any azimuthal variation, in agreement with the maximum azimuthal variation of the kernels in 30°. The plot on the right-hand side of Fig. 3 shows the horizontal section at 25 km depth of the azimuthally averaged kernel with respect to S-wave velocity. It shows a circular pattern with a maximum value in the middle, surrounded by a negative one, and then a decreasing oscillatory pattern with distance.

Vertical sections of this kernel are shown in Fig. 4, together with the averaged kernels for the radial and vertical amplitudes. The averaged kernels for the amplitudes are not very meaningful by themselves, as one would not invert averaged amplitudes at a given station, but their comparison is meaningful as their difference yields the HV ratio kernel. We see that the vertical component has a sensitivity that is larger and more concentrated beneath the receiver than the radial component. A major difference between these kernels arises from Love wave contributing only to radial amplitude variations and not to vertical ones. The HV ratio kernel has a lower amplitude than the kernels of both vertical and radial components, a positive maximum at 30 km depth beneath the station, surrounded by negative values.

The lower plot of Fig. 4 shows depth profiles of the HV kernel at increasing distances from the receiver. This representation is closer to the representation of the sensitivity kernels used previously (Tanimoto & Rivera 2008; Berbellini et al. 2016) and allows better comparison. We see that the kernel varies significantly with distance both in amplitude and in shape. There is no simple decrease with distance that would allow us to define a clear and simple sensitivity zone. Except in the distance range 8–18 km where there is some similarity, there is no good correlation with the depth sensitivity curves shown in Berbellini et al. (2016) and in Tanimoto & Rivera.
Figure 1. Horizontal sections at 25 km depth of the sensitivity kernels for the Rayleigh wave radial amplitude (upper plots), vertical amplitude (middle plots) and HV ratio (lower plots) with respect to shear wave velocity. Left plots: mono-frequency kernels at 0.05 Hz; right plot: kernels in a frequency band 0.037–0.063 Hz. The reference field is a plane wave coming from 60° north and the receiver is located at (0, 0) in the middle of the plot. The grid spacing is 5 km, the units are km$^{-3}$ and the colour scale saturates at the HV ratio kernel maximum value divided by 25.
Surface waves and free oscillations

Figure 2. The same as the HV ratio sensitivity kernel in the lower right plot of Fig. 1 but computed with the Rayleigh wave fundamental mode only (left plot) and the Love wave fundamental mode only (right plot).

Figure 3. Left plot: zoom on the HV ratio sensitivity kernel with respect to shear wave velocity shown in the lower right plot of Fig. 1. Right plot: azimuthal average of the sensitivity kernel shown on the left-hand side. For both plots, the frequency band 0.037–0.063 Hz and the units are km$^{-3}$. Note that the colour scales differ and that the colour scale on the left-hand side saturates at the kernel maximum value divided by 36. The grid spacing is 2 km.

(2008), taking into account that the inverse ratio is used in this last paper.

Figs 5 and 6 show the decomposition of the kernels shown in Fig. 4 into contributions respectively from Rayleigh wave and Love wave fundamental modes. The dominant negative values at 30 km depth beneath the receiver (at 0 km horizontal distance) in the kernels of the vertical and radial kernels calculated with the Rayleigh wave fundamental mode only (Fig. 5) cancel each other in the HV kernel, yielding much smaller maximum values in the HV kernels. This gives the Love wave, which has in the first place smaller kernels, the opportunity to contribute significantly to the total HV kernels. The oscillatory pattern of the HV kernel beneath the station seen in the lower panel of Fig. 4 is in particular a result of this complex interaction of different contributions.

As the curves from Tanimoto & Rivera (2008) and Berbellini et al. (2016) represent the sensitivity of the observable to a variation with depth in a laterally homogeneous structure, we have integrated our kernels laterally in order to make a meaningful comparison. Integrating our sensitivity kernels in vertical cylindrical areas around the receiver, we can analyse how the
Figure 4. Vertical sections of the azimuthally averaged sensitivity kernels of the Rayleigh wave radial amplitude (upper plot), vertical amplitude (upper middle plot) and HV ratio (lower middle plot) with respect to shear wave velocity. The sections pass at 1 km distance from the receiver and the frequency band is 0.037–0.063 Hz. The lower plot shows the depth dependence of the HV ratio kernel as a function of distance from the receiver.

integrated sensitivity curve evolves with the radius of integration. This is shown in the first five plots of Fig. 7 with blue lines for $S$-wave sensitivity kernels. The integration out to 18 km radius yields depth curves which are rather similar to those presented in Tanimoto & Rivera (2008) and Berbellini et al. (2016), but our curves continue to evolve significantly for larger radius and stabilize with a sign that is opposite to expected and with a deeper sensitivity.
For comparison, the sensitivity of the phase measured on the vertical and radial components is shown in the rightmost plot of Fig. 7 for a radius of 50 km. They have the same shape but an opposite sign to the classical phase velocity sensitivity curves, as expected. Their shape is much more stable with radius than the curves for the HV ratio, and very similar for the radial and vertical component. Although this does not prove that our HV ratio curves are right, the correctness of these phase sensitivity curves rules out a global sign error in our computation of the kernels. Likewise, the agreement of our kernels for the vertical component (Fig. 1) with those in Dahlen & Zhou (2006) argue for the correctness of our code.

5 Kernels in $\alpha$ and $\rho$

Fig. 7 also shows the laterally integrated sensitivity curves with respect to the $P$-wave velocity and density. For the $P$-wave velocity, the sensitivity is always negative, concentrated close to the surface.
and with a simple linear decrease with depth. This corresponds well with the curves presented by Tanimoto & Rivera (2008) and Berbellini et al. (2016). Looking at the kernel itself, before lateral integration (two upper plots of Fig. 8), we see that the kernel has strong positive values just beneath the receiver and that the main negative contribution arises from surface layers at about 15 km distance from the receiver.

We have a rather similar pattern for the sensitivity with respect to density, with maximum values at very shallow depth, alternating from positive beneath the station to negative at 15 km distance. The radius integrated values (Fig. 7) also show a significant variation with radius of integration and, as for the $S$-wave kernels, do not correspond well with those presented by Tanimoto & Rivera (2008) and Berbellini et al. (2016).

Figure 6. The same as Fig. 4 but with the contribution of the Love wave fundamental mode only.
Figure 7. Kernels with respect to S-wave velocity (blue), P-wave velocity (grey) and density (brown) integrated with respect to distance from the receiver over growing areas. The limit of radius integration is indicated at the top of each panel. The plots are at the same scale, with tick marks every 0.01. The plot to the right is the sensitivity kernel of the phase variation measured on the vertical component (full line) and radial component (dashed line) integrated over 50 km radius.

6 DISCUSSION AND CONCLUSION

Rayleigh wave ellipticity has previously been inverted using depth sensitivity curves, assuming the ellipticity measured at a station reflects the structure right beneath, but the lateral extent of the sensitivity zone around the station had not been investigated. In order to quantify this lateral extent, we have derived the equations and computed the 3-D sensitivity kernels for the ellipticity of the Rayleigh wave fundamental mode with respect to S-wave velocity, P-wave velocity and density. We show that these kernels are just the difference between the kernels of the amplitudes of the radial and vertical components. As these two last kernels differ mostly close to the receiver, the sensitivity of the HV ratio is concentrated around the station, as expected. Due to this localization, near-field terms cannot be neglected. Coupling to the Love wave fundamental mode must be taken into account for a proper computation of the kernels, but coupling to overtones proved to be negligible. The 1-D kernels for ellipticity are more shallow than other kernels, for example phase velocity kernels. We expect therefore, but have not verified, that coupling to non-trapped body waves, which would increase energy at depth, is not a major issue.

A possible extension of the present work would be to allow for a laterally heterogeneous background instead of a laterally homogeneous one, and develop kernels based on the adjoint operator. Since far-range contributions are small in the HV kernels compared to the more classical arrival time or amplitude kernels which have significant amplitude along the whole source-station path, it is likely that one would have less to gain in the present case, except for taking into account a possible deviation from great-circle path. This deviation can also be taken into account in a simpler way by using the appropriate arrival angle instead of the great circle direction at the station.

The kernels have complex shapes, even when smoothing in a finite-frequency range is used to remove the oscillatory pattern present in mono-frequency kernels. The kernels for a single event have an integrable singularity at the receiver, an azimuthal dependence of up to order $\cos 3\theta$, and show a dominant $\cos 2\theta$ pattern. This complex and strong dependence on the structure close to the receiver may explain why single measurements of the Rayleigh wave ellipticity often show a large spread at a single station (Ferreira & Woodhouse 2007; Yano et al. 2009).

As averaging the measurements with azimuth has proved to stabilize the results (Tanimoto & Rivera 2008; Yano et al. 2009; Berbellini et al. 2016), we have calculated and focused on azimuthally averaged kernels. Our expressions show that the sensitivity of the HV kernels has a maximum azimuthal variation in $\cos 3\theta$, and we have therefore used and found appropriate to use a $30^\circ$ sampling for computing the azimuthal average. In real data, the sampling will likely be less regular, but our expressions give some indications on the sampling which is necessary to achieve an appropriate azimuthal average at a given station.

Although simpler than the kernels for one individual direction, the azimuthally averaged kernels are not simple either. The kernel with respect to S-wave velocity is particularly complex, with an alternation of positive and negative values both in depth and in distance from receiver. As the kernels derived in Tanimoto & Rivera (2008) are computed in the 1-D assumption of a model perturbation extending infinitely in the horizontal plane, we would expect that integration of our kernels in growing cylinders around the receiver would converge to the kernels derived in Tanimoto & Rivera (2008). This is not the case for the S-wave velocity kernel, and we have not found any explanation for this. The sensitivity kernels with respect to P-wave velocity are simpler, more concentrated close to the surface and in much better agreement with those derived in Tanimoto & Rivera (2008) and Berbellini et al. (2016), but, as they are smaller than the S-wave sensitivity kernels, they will have less influence on the data. The sensitivity with respect to density is also complex.

The structure within half-a-wavelength of the receiver contributes most to the Rayleigh wave ellipticity. There is however considerable variation of the kernels within this area, and it would be misleading to use this one number to quantify the lateral extent of the structure around the station that controls the variation of the ellipticity. Although it would have been very convenient, it would not be right, event for the simplest P-wave dependence, to replace the 3-D kernels by simplified kernels consisting of the classical 1-D kernels in vertical cylindrical boxes with a given lateral dimension around the station.
Figure 8. The same as Fig. 4 for the HV ratio sensitivity kernels with respect to $P$-wave velocity (two upper plots) and density (two lower plots). The colour scale is the same as in Fig. 4 and saturates by a factor of 2 for the $P$-wave velocity kernel.

Yano et al. (2009) observe that the inversion of the Rayleigh wave ellipticity using 1-D kernels is easily unstable and may lead to spurious low-velocity zones. This might be related to the difference between the kernels used in the inversion and the real sensitivity of the data. The Rayleigh wave ellipticity shows a complex dependence on structural parameters and its inversion using 3-D kernels is probably not a simple issue, although it should of course be attempted.

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