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INFLUENCE OF SHAPE AND STRUCTURE ON LIGHT SCATTERING BY MARINE PARTICLES

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Abstract

New formulae for the light scattering and absorption by thin disks, long cylinders and hollow spheres are presented and discussed. An appropriate measure of size for irregular particles may be the ratio between volume and mean geometrical cross section. Existing works disagree as to whether the scattering efficiency of irregular particles has a maximum.

The internal structure may have a considerable effect on the scattering, but less on the absorption. There is no general relation between the spectral dispersion of absorption and scattering by marine particles. The relation will depend on size, internal structure and the ratio between refractive and absorption indices.

CONTENTS

PART I

DISCUSSION

1. THEORIES OF LIGHT SCATTERING

1.1. The problem of scattering theory

The scattering of light by marine particles is a physical phenomenon which is very difficult to describe theoretically. Although the physical equations are known, analytic solutions have so far only been obtained for a few very idealized types of particles.

The introduction of electronic computers in science started a new era in scattering theory, and especially during the last years great progress has been made. The scattering by non-spherical inhomogeneous particles can now be calculated by different methods. A brief, and not complete, outline of the actual branches in scattering theory is given in the next sections.

Unfortunately most of the presented metods require rather complicated computer programs, and these are perhaps too complicated to attract the non-optical scientist. Analytical solutions, on the other hand, can usually be applied more easily, and often much of the behaviour of the solution can be understood at once just by looking at the expression. It then seems worthwhile to search for such solutions. In this work a few analytical solutions of some special kinds of particles are presented.

1.2. Diffraction and geometrical optics

Fraunhofer diffraction and geometrical optics (ray tracing) are perhaps the most classical theories which are still used. The scattering of light by particles much larger than the wavelength can be approximated by the sum of three different processes:

Scattering = Diffraction + reflection + refraction

In principle the method can be applied to particles of any shape and structure, in practice the computer programs will be the limiting factor.

HODKINSON and GREENLEAVES (1963) compared results obtained by this method with results from Mie theory for spherical particles, and found satisfactory agreement. MULLANEY (1970) applied the method to biological particles (spheres), while WENDLING et al. (1979) calculated the scattering by hexagonal ice crystals.

POLLACK and CUZZI (1980) used a modified version of this method on cubes, octahedra, convex and concave particles and flat plates. Still another version is applied by LIOU et al. (1983) on cubes and parallelepipeds.

1.3. The oscillating dipole

The original works of RAYLEIGH considered particles much smaller than the wavelength which was assumed to behave like an oscillating dipole. The theory could be extended to larger particles provided the phase shift of all rays passing through the particle was small. Later contributions came from GANS and DEBYE, and the theory is sometimes called Rayleigh-Gans scattering, other times Rayleigh-Debye scattering, or as a compromise the RGD method. References to the numerous original works of RGD are given by VAN DE HULST (1957) and KERKER (1969).

CROSS and LATIMER (1972) apply the RGD method to obtain an analytical expression for spheroids, while BARBER and WANG (1978) compare similar results with results derived from other methods. LATIMER and WAMBLE (1982) investigate the scattering by large colloidal aggregates.

An interesting work by LATIMER (1984) compares the scattering by spheres with and without spines.

The RGD method regards each volume element of the particle as an oscillator driven solely by the incident light field. If interaction between the oscillators and the fields of the scattered light should be taken into account, the mathematics would become far more complicated. PURCELL and PENNYPACKER (1973), however, solve this problem by an iterative procedure. They apply their method to particles of spherical and rectangular shapes. The method has also been used by DRUGER et al. (1979).

1.4. The Maxwell equations

The classical Mie theory (MIE, 1908, first derived by LORENZ, 1890) is a solution of the Maxwell equations with the appropriate boundary conditions. The solution for a homogeneous sphere has the form of rather complicated series. The larger the sphere, the more slowly will the series converge. This made it rather difficult during the pre-computer era to see how the solution really varied with refractive index, size and wavelength. Even as late as 1947 British chemical engineers obviously had no knowledge of the results that the solution predicts (ICE & SCI, 1947).

The introduction of electronic computers eliminated the work problem of the Mie theory. However, it was discovered that for absorbing spheres, the calculation became numerically unstable. KATTAWAR and PLASS (1967) describe a procedure which solves this problem.

In 1951 ADEN and KERKER extended the Mie solutions to a two-layered sphere, that is a sphere with a concentric spherical shell. The series became far more complicated, as was to be expected. BRUNSTING and MULLANEY (1972), MEYER and BRUNSTING (1975) and MEYER (1979) have applied the Aden and Kerker solution to biological particles.

The classical electromagnetic equations and solutions were fitted to the shape of spheroids by ASANO and YAMAMOTO (1975), ASANO (1979) and ASANO and SATO (1980).

A new, and non-Mie type of approach to the solutions of the electromagnetic equations were introduced by WATERMAN (1971). The method, called the extended boundary condition method, was carried further by BARBER and YEH (1975), BARBER and WANG (1978), WANG and BARBER (1979) and WANG et al. (1979). The method can be applied to multilayered, dielectric objects of any shape.

Another method, called the field average method, was tested by KERKER et al. (1978) by comparison with the ADEN-KERKER solution for concentric spheres. A comparison with other methods were made by DRUGER et al (1979).

Finally the interesting attempt by CHYLEK et al. (1976), to modify the Mie theory in order to describe the scattering by irregular randomly oriented particles, should be mentioned.

1.5. Anomalous diffraction

VAN DE HULST (1957) used the term anomalous diffraction for the scattering process when the refractive index of the particle was close to 1 and the dimensions of the particle were much larger than the wavelength. The approximated theory he put forward gave very simple and useful equations, especially for the integrated scattering. His results for non-absorbing spheres are often used and agree fairly well with the more rigorous Mie theory.

Due to the simplicity of the expressions, it is rather surprising that no one seems to have used the theory on non-spherical particles, particularly since the theory is so well suited to marine particles. VAN DE HULST only gave expressions for long cylinders with optical axes normal to the incident light, and for thin disks with axes parallel with the light. In the present work analytic solutions for the total scattering and attenuation coefficients for randomly oriented cylinders and disks are deduced. An analytic solution for the hollow sphere is also presented.

The van de Hulst method can be applied to large particles of any shape and internal structure, provided the refractive index is close to 1, and will always give a solution in the form of an integral. In the present work the integral form is applied to a two-layered sphere.

2. THE BASIC CONCEPTS

2.1. The attenuation, absorption, scattering and geometrical cross sections

Assume that a parallel beam of light has a density of light flux or irradiance E_0 at the point $r=0$ and E at the point r. The relation between E and E_0 is given by Bouger-Lamberts law:

$$E = E_0\ e^{-cr} \qquad 2.1$$

where $\underline{c}$ is the attenuation coefficient. The relation holds provided $\underline{c}$ is not too great. The attenuation coefficient $\underline{c}$ can be regarded as composed of two coefficients, an absorption coefficient $\underline{a}$ for processes where light energy is transformed to heat or chemical energy, and a scattering coefficient $\underline{b}$ for processes where the light energy is conserved as light, but where the direction is changed. The relation between the coefficients is

$$a + b = c \qquad 2.2$$

If we consider a medium which consists of water with suspended particles, we may make a further subdivision

$$c = c_w + c_p = a_w + a_p + b_w + b_p \qquad 2.3$$

where the subscript w denotes contributions from the pure water, and p denotes contributions from the particles.

Up to certain limits c_p will be proportional with the particle concentration. This is the so-called Beer's law, which for a mono-dispersion can be written

$$c_p = a_p + b_p = N\ C_c = N\ C_a + N\ C_b \qquad 2.4$$

N is the concentration expressed as number of particles per volume unit, C_c is the attenuation cross section of the particle, C_a is the absorption cross section and C_b is the scattering cross section. Each particle within a parallel beam of light will remove an amount of light flux from the beam corresponding to the flux within a cross section of area C_c. An amount of flux corresponding to the flux within an area C_a is converted to heat or chemical energy (included fluorescence), another amount corresponding to the area C_b is scattered away from the beam.

The optical cross sections can be smaller, equal or even greater than the geometrical cross section or shadow area C_g of the particle. The cross sections will depend on the orientation of the particle, if the particle is non-spherical. For randomly oriented particles we will denote the mean values of C_a, C_b, C_c and C_g by A, B, C and G respectively.

Eq.2.4 can for a polydispersion be given the form

$$c_p = \int_0^\infty C(D) \frac{dN}{dD} dD \qquad 2.5$$

where D is the "diameter" of the particle, and dN/dD is the size distribution.

Unfortunately the concept of a "diameter" is only unique with regard to spheres. It is possible to define a volume diameter D_v which means the diameter of a sphere of equivalent volume. One may also define a surface diameter D_s which is the diameter of a sphere of equivalent surfacee. But very often in the literature the term diameter only means a subjective estimate of a distance across the center of the particle.

An important theorem with regard to the surface of a particle, is that the mean geometrical cross section of a convex particle is one fourth of its total surface (VAN DE HULST, p.110). Thus for all convex particles

$$G = \pi D_s^2/4 \qquad 2.6$$

2.2. The volume scattering function

The angular distribution of scattered light is given by the volume scattering function $\beta(\theta,\varphi)$, defined by

$$\beta(\theta,\varphi) = \frac{db(\theta,\varphi)}{d\Omega} \qquad 2.7$$

where θ is the angle between incident and scattered light, the "zenith" angle, and φ is the azimuth angle. db is the part of the scattering coefficient which is due to light scattered in the direction (θ,φ) within the solid angle $d\Omega$. For an assembly of natural particles oriented at random db will depend only on θ, and not on φ. We then se that

$$b = \int_{4\pi} \beta(\theta)d\Omega = 2\pi \int_0^\pi \beta(\theta) \sin\theta \, d\theta \qquad 2.8$$

$\beta(\theta)$ will be related to the concentration N by

$$\beta(\theta) = \bar{C}_\beta(\theta)N \qquad 2.9$$

where $\bar{C}_\beta(\theta)$ is the mean cross section for light scattered in the direction θ. Eqs.2.9 and 2.4 give that

$$B = \int_{4\pi} \bar{C}_\beta(\theta)\, d\Omega = 2\pi \int_0^\pi \bar{C}_\beta(\theta) \sin\theta \, d\theta \qquad 2.10$$

which is another form of eq.2.8.

2.3. The attenuation, absorption and scattering efficiencies

The ratio between the mean attenuation cross section C and the mean geometrical cross section or shadow area G of the particle, may be termed the attenuation efficiency

$$Q_c = C/G \qquad 2.11$$

(VAN DE HULST, p.14, denotes C_c/C_g as the attenuation efficiency, which will then depend on the particle's orientation).

Similarly the absorption efficiency may be defined as

$$Q_a = A/G \tag{2.12}$$

the scattering efficiency as

$$Q_b = B/G \tag{2.13}$$

and the scattering function efficiency as

$$Q_\beta(\theta) = \bar{C}_\beta(\theta)/G \tag{2.14}$$

We see that

$$Q_c = Q_a + Q_b \tag{2.15}$$

and that

$$Q_b = \int_{2\pi} Q_\beta(\theta)\, d\Omega = 2\pi \int_0^\pi Q_\beta(\theta) \sin\theta\, d\theta \tag{2.16}$$

2.4. The refractive index

So far we have defined the optical properties of a particle suspension. On a smaller scale the particles will possess internal absorption and scattering coefficients. The internal scattering, due to the molecular structure of the particle, may be regarded as included in the integrated scattering properties of the particle. The internal absorption, however, which in the case of phytoplankton is due to the pigment content, can be expressed by the internal absorption coefficient $\underline{a}_i$. Often it is more practical to introduce a dimensionless absorption index κ defined by

$$\kappa = \frac{a_i \lambda}{4\pi} \tag{2.17}$$

where λ is the wavelength of light in the medium surrounding the particle.

The refractive index $\underline{n}$ of the particle will in this text mean the ratio between the velocity of light outside the particle and the velocity on the inside. It is usually more practical to work with a complex index of refraction <u>m</u> defined by

$$m = n - i\kappa \qquad 2.18$$

where

$$i = (-1)^{1/2} \qquad 2.19$$

Marine particles, and especially the phytoplankton, have the property that

$$|m-1| \ll 1 \qquad 2.20$$

which simplifies considerably the theoretical work, as will be shown later.

3. THE RAYLEIGH REGION

If a particle is much smaller than the wavelength of light, it is not possible to see the shape of the particle, and its optical properties will be independent of the shape. We shall call this size region the Rayleigh region. Protein and starch molecules, the smallest viruses and the smallest clay particles (see Fig.1) belong to this category of particles, for which the Rayleigh scattering formula is valid (e.g. VAN DE HULST, p.68).

$$C_\beta = \frac{\pi^2 V^2}{2\lambda^4} |m^2 - 1|^2 (1 + \cos^2\theta) \qquad 3.1$$

The formula requires that the particles are isotropic. θ is the angle between incident and scattered light. V is the volume of the particle. Since m is close to 1 for marine particles, we may approximate

$$|m^2 - 1|^2 = |(m + 1)(m - 1)|^2 \approx 4|m - 1|^2 \qquad 3.2$$

and eq.3.1 may be written

$$C_\beta = 2 \frac{\pi^2 V^2}{\lambda^4} |m - 1|^2 (1 + \cos^2\theta) \qquad 3.3$$

C_β is independent of the particle's orientation, and the scattering cross section becomes

$$C_b = B = 2\pi \int_0^\pi C_\beta \sin\theta \, d\theta$$

$$= \frac{32\pi^3 V^2}{3\lambda^4} |m-1|^2 = \frac{32\pi^3 V^2}{3\lambda^4} [(n-1)^2 + \kappa^2] \qquad 3.4$$

The absorption cross section is given by

$$C_a = A = \frac{4\pi}{\lambda} \kappa V = a_i V \qquad 3.5$$

The expressions above show that the optical properties are independent of particle shape and orientation, but depend on volume, wavelength, absorption and refraction. If m is constant throughout the spectrum, then B will be proportional to λ^{-4} while A will be proportional to λ^{-1}. Usually, however, if κ has any significant value, it will have a strong dependence on wavelength, so that A will be dominated by the variation in κ, and not by λ^{-1}.

The volume of the particle may be expressed by the equivalent volume diameter D_V

$$V = \frac{\pi}{6} D_V^3 \qquad 3.6$$

and the mean geometrical cross section, provided the particle is convex, by the equivalent surface diameter D_S

$$G = \frac{\pi}{4} D_S^2 \qquad 3.7$$

The scattering and absorption efficiencies can then be written

$$Q_\beta = \frac{2}{9} \pi^3 \left(\frac{D_V}{\lambda}\right)^4 \left(\frac{D_V}{D_S}\right)^2 |m-1|^2 (1+\cos^2\theta) \qquad 3.8$$

$$Q_b = \frac{32}{27} \pi^4 \left(\frac{D_V}{\lambda}\right)^4 \left(\frac{D_V}{D_S}\right)^2 |m-1|^2 \qquad 3.9$$

$$Q_a = \frac{8}{3} \pi\kappa \left(\frac{D_V}{\lambda}\right)\left(\frac{D_V}{D_S}\right)^2 = \frac{2}{3} a \left(\frac{D_V}{\lambda}\right)\left(\frac{D_V}{D_S}\right)^2 \qquad 3.10$$

The attenuation efficiency becomes

$$Q_c = \frac{8}{3} \pi \left(\frac{D_V}{\lambda}\right)\left(\frac{D_V}{D_S}\right)^2 \left[\kappa + \frac{4}{9} \pi^3 \left(\frac{D_V}{\lambda}\right)^3 |m-1|^2\right] \qquad 3.11$$

For all particle shapes, except the sphere, $D_V/D_S<1$. Since $D_V/\lambda<<1$, the efficiencies are very small numbers. Eq.3.11 also illustrates that if κ is of the same order as $|m-1|^2$, then $Q_c \approx Q_a$. The attenuation by these very small particles is then dominated by absorption.

4. THE RAYLEIGH-DEBYE REGION

Particles in the Rayleigh-Debye region satisfy the two requirements:

$$|m - 1| \ll 1 \qquad 4.1$$

and

$$Dk|m - 1| \ll 1 \qquad 4.2$$

where the "diameter" D should be the maximum distance across the particle, and k is the wavenumber

$$k = \frac{2\pi}{\lambda} \qquad 4.3$$

Clay particles, bacteria and phytoplankton less than 1 μm belong to this region.

The first condition above ensures that reflection and refraction effects can be neglected. The second condition means that every point within the particle can be regarded as having the same phase as the undisturbed light field at that point. The Rayleigh-Debye theory now assumes that all volume elements within the particle are independent Rayleigh scatterers with the phase of the undisturbed light field, and that the resulting distribution of scattered light is due to interference between the light from all the individual scatterers. The reader is referred to the text by VAN DE HULST for a further discussion as well as for references. It suffices for our purposes here to render only the final results.

The absorption cross section is the same as in the case of pure Rayleigh scattering

$$C_a = A = 2k\kappa V = a_i V \qquad 4.4$$

The mean cross section $\overline{C}_\beta$, however, now is the cross section for Rayleigh scattering multiplied with a shape factor $f(\theta)$

$$\overline{C}_\beta = 2\pi^2 \frac{V^2}{\lambda^4} |m - 1|^2 (1 + \cos^2\theta) f(\theta) \qquad 4.5$$

The mean scattering cross section B is the expression for Rayleigh scattering multiplied with another shape factor F.

$$B = \frac{32 \pi^3 V^2}{3 \lambda^4} |m - 1|^2 F \qquad 4.6$$

Since integration of $\overline{C}_\beta$ should give B, the relation between F and $f(\theta)$ must be

$$F = \frac{3}{8} \int_0^\pi (1 + \cos^2\theta)\sin\theta f(\theta)d\theta \qquad 4.7$$

Expressions for $f(\theta)$ and F for spheres and for thin disks and long cylinders oriented at random are given by VAN DE HULST (p.88-98). $f(\theta)$ is always less than one, except for $\theta=0$, which means that the distribution of scattered light will equal that of pure Rayleigh scattering in the forward direction. The angular distribution of scattered light is no longer symmetrical around $\theta=\pi/2$, as in the case of Rayleigh scattering, but is now elongated in the forward direction. The greater the value of kD is, the more pronounced will be the forward elongation.

F will decrease with increasing particle size, but since B is proportional to V^2, the combined effect is still that the cross section increases.

The scattering efficiency of a sphere will be, according to eq.3.9,

$$Q_b = \frac{2 k^4 D^4}{27} |m - 1|^2 F$$

$$= \frac{2}{27} [k^2 D^2 |m - 1|^2] (k^2 D^2 F) \qquad 4.8$$

In the Rayleigh region kD is a small number while F=1, so that

Q_b becomes a very small number. In the Rayleigh-Debye region the term in the brackets of eq.4.8 is required to be a small number (eq.4.2), while kD may be a great number. Whether the term in the parenthesis of eq.4.8 will be a small or great number, will then depend on the value of F. The solution by Rayleigh gives that

$$f(\theta) = \frac{9}{u^6} (\sin u - u \cos u)^2 \qquad 4.9$$

where

$$u = k\, D \sin \frac{\theta}{2} \qquad 4.10$$

According to eq.4.7 the shape factor F becomes

$$F = \frac{27}{X^6} [- 56 + 56 \cos X - 8 X \sin X + 20 X^2 + X^4$$

$$+ (64 - 16 X^2)(\gamma_0 + \ln X - \mathrm{Ci}\ (X))] \qquad 4.11$$

where

$$X = 2\, k\, D \qquad 4.12$$

and γ_0 and the cosine integral Ci are defined by eq.8.1. When $k\, D \leqq 1$, the shape factors can be expanded in series:

$$f(\theta) \approx 1 - \frac{k^2 D^2}{5} \sin^2 \frac{\theta}{2} + \frac{3\, k^4 D^4}{175} \sin^4 \frac{\theta}{2} - \ldots \qquad 4.13$$

$$F \approx 1 - \frac{k^2 D^2}{10} + \frac{3\, k^4 D^4}{500} \qquad 4.14$$

When kD>>1, that is the sphere is much greater than the wavelength, eq.4.11 gives that

$$F \approx \frac{27}{4k^2\, D^2} \qquad 4.15$$

The scattering efficiency then becomes

$$Q_b = \frac{1}{2} k^2 D^2 |m - 1|^2 \qquad 4.16$$

which is a small number. The absorption efficiency of a sphere can be written (eq.3.10)

$$Q_a = \frac{4}{3} k D \kappa \qquad 4.17$$

which also, according to eq.4.2, must be a small number. Large spheres in the Rayleigh-Debye region then have the attenuation efficiency

$$Q_c = \frac{4}{3} k D \kappa + \frac{1}{2} k^2 D^2 [(n - 1)^2 + \kappa^2] \qquad 4.18$$

The same efficiency is obtained for small spheres in the lower van de Hulst region (Chapter 6.1).

Cylinders or disks oriented at random have a mean shape factor $f(\theta)$ which can be written

$$f(\theta) = \int_0^{\pi/2} f_1 f_2 \sin \beta \, d \beta \qquad 4.19$$

where

$$f_1 = \frac{\sin^2 (v \cos \beta)}{v^2 \cos^2 \beta} \qquad 4.20$$

$$v = k L \sin \frac{\theta}{2} \qquad 4.21$$

$$f_2 = [\frac{2 J_1 (w \sin \beta)}{w \sin \beta}]^2 \qquad 4.22$$

$$w = k D \sin \frac{\theta}{2} \qquad 4.23$$

D is the diameter and L the length or thickness. J_1 is the Bessel function of the first kind and first order, defined by eq.8.14. Unfortunately no general analytic solution to the integral has been found. An approximated solution, however, can easily be found in the case when both L and D are of wavelength dimension or less, so that $kD \lesssim 1$ and $kL \lesssim 1$. The functions f_1 and f_2 can then be approximated by

$$f_1 \approx 1 - \frac{v^2}{3} \cos^2\beta + \frac{2\, v^4}{45} \cos^4\beta - \dots \qquad 4.24$$

$$f_2 \approx 1 - \frac{w^2}{4} \sin^2\beta + \frac{5\, w^4}{192} \sin^4\beta - \dots \qquad 4.25$$

and the integral of eq.4.19 becomes

$$f(\theta) \approx 1 - \frac{v^2}{9} - \frac{w^2}{6} + \frac{2\, v^4}{225} + \frac{v^2 w^2}{90} + \frac{w^4}{72} - \dots$$

The integrated shape factor F is found to be

$$F \approx 1 - \frac{k^2}{6} \left(\frac{L^2}{3} + \frac{D^2}{2}\right) + \frac{7\, k^4}{180} \left(\frac{2\, L^4}{25} + \frac{L^2 D^2}{10} + \frac{D^4}{8}\right) - \dots \qquad 4.27$$

The ratio between the scattering cross sections of equal-volume spheres and cylinders, will be the ratio between their shape factors:

$$\frac{f(\theta)_{cyl}}{f(\theta)_{sph}} \approx \frac{1 - k^2 \sin^2 \frac{\theta}{2} \left(\frac{L^2}{9} + \frac{D^2}{6}\right)}{1 - k^2 \sin^2 \frac{\theta}{2} \frac{D_v^2}{5}} \qquad 4.28$$

The diameter D of the cylinder is related to the equal-volume diameter D_v by

$$\frac{\pi}{4} D^2 L = \frac{\pi}{6} D_v^3 \qquad 4.29$$

or

$$D_V = \left(\frac{3}{2} D^2 L\right)^{1/3} \qquad 4.30$$

The ratio above can then be written

$$\frac{f(\theta)_{cyl}}{f(\theta)_{sph}} = \frac{1 - k^2L^2\sin^2\frac{\theta}{2}\left(\frac{1}{9} + \frac{D^2}{6\,L^2}\right)}{1 - k^2L^2\sin^2\frac{\theta}{2}\,\frac{1}{5}\left(\frac{3\,D^2}{2\,L^2}\right)^{2/3}} \qquad 4.31$$

The greatest deviation of this ratio from 1 will occur when $\theta=\pi$, that is for the backward scattered light. For long cylinders with $L/D=10$, $kL=1$ and $\theta=\pi$, the ratio above becomes 0.90. Equi-dimensional cylinders with $k^2(L^2+D^2) = 1$ and $L/D=1$ give similarly the ratio 0.99. Thin disks with $kD=1$ and $D/L=10$ give the ratio 0.88. Thus it seems as if particles of wavelength size may scatter up to 10-15% less light than equal-volume spheres in the backward direction. This deviation, however, will be less than half as great in the forward hemisphere, where most of the light is scattered. The observation by POLLACK and CUZZI (1980), that non-spherical particles can be substituted by equal-volume spheres, when $\pi D_v/\lambda \leqq 1-15$, depending on particle shape, thus seems quite reasonable from a theoretical point of view.

5. THE VAN DE HULST REGION

5.1. The importance of large particles in the sea

The term "Mie scattering" is often applied to scattering by particles greater or much greater than the wavelength of light, but the term is somewhat misleading, since the theory is valid for spheres of any size. It is convenient, however, to have a name for the size region of particles greater than the wavelength, and it may be called the Mie region.

The importance of light scattering by particles in the Mie region is due to the fact that the scattering efficiencies in this region become of order 1, while they are less than 10^{-2} in the Rayleigh-Debye region and of order 10^{-4} to 10^{-10} in the Rayleigh region. A marine particle in the Rayleigh region (diameter of order 1-10 nm) will then have a scattering cross section of order 10^{-15} to 10^{-6} μm^2, in the Rayleigh-Debye region (diameter less than 10 μm) of order less than 10^{-2} μm^2, while in the Mie region (diameter 1-100 μm) it may amount to an order of 10^5 μm^2. Thus one large particle from the Mie region may scatter just as much light as 10^{20} small particles from the Rayleigh region.

Often the major part of the contribution to the total geometrical cross section comes from particles in the range 10-100 μm (e.g. JERLOV, 1976, table VI), but at other times particles smaller than 2.5 μm also seem to give a significant contribution (OCHAKOVSKY, 1966; BEARDSLEY et al., 1970; SUGIHARA et al., 1981). The variation of the size distribution of marine particles shall not be discussed here. It suffices for our purposes to note that most of the optically important particles obviously belong to the Mie region and the upper part of the Rayleigh-Debye region. These particles are then well fitted for the van de Hulst method, since they also have a refractive index close to 1.

5.2. The concept of the van de Hulst region

It will be convenient to have a special term for particles larger than the wavelength and with a relative refractive index close to 1, and we shall call it the van de Hulst region. It is a subregion of the Mie region.

Since a particle in the van de Hulst region has a refractive index close to that of the surrounding medium, reflection and refraction at the boundaries of the particle can be neglected. On the other hand the particle will have dimensions greater than the wavelength of light, and the phase difference between a ray passing through the particle and a ray of the undisturbed light field must be taken into account. This is the fundamental difference from the optics of the Rayleigh-Debye region, where phase shifts between neighbouring points inside and outside the particle could be neglected.

For large particles one may consider scattering as consisting of three separate processes, as already mentioned: reflection, refraction and diffraction. In the van de Hulst region scattering is then almost only a diffraction phenomenon, but not quite, as shown by VAN DE HULST (p.183-187).

The distribution of scattered light will not be discussed here, we may once again refer to the text by VAN DE HULST. We shall instead limit ourselves to look at the expressions for the integrated effects of scattered and absorbed light, which become particularly simple by the van de Hulst method.

The attenuation cross section is found by integrating the phase shift of each ray which passes through the particle, and the absorption cross section is similarly found by integrating the absorptance of each ray. The general expressions are given in eqs.9.1-2. We shall now look at some of the results.

6. THE INFLUENCE OF SHAPE

The three geometrical shapes to be studied here are the sphere, the thin disk and the long cylinder. They have been chosen due to their mathematical simplicity, but they still cover some of the natural occurring variation of particle shape. Several phytoplankton species are spherical, and especially diatoms may attain disk-like and cylindrical shapes. Mineral particles are usually rather irregular, but minerals like quartz and feldspar may be rounded off by chemical dissolution or mechanical abrasion. Mica will appear as thin flakes, while amphiboles may have needle-like shapes.

In order to simplify the calculations, the particles will be assumed to be homogeneous.

6.1. Homogeneous spheres

The attenuation and absorption efficiencies of the homogeneous sphere are (eqs.10.8 and 13)

$$Q_c = 2 - 4 \text{ Re } \{\frac{ie^{-i\zeta}}{\zeta} + \frac{e^{-i\zeta}}{\zeta^2} - \frac{1}{\zeta^2}\} \qquad 6.1$$

$$Q_a = 1 + \frac{e^{-2\gamma}}{\gamma} + \frac{e^{-2\gamma}}{2\gamma^2} - \frac{1}{2\gamma^2} \qquad 6.2$$

where

$$\zeta = (m - 1)kD = \varrho - i\gamma \qquad 6.3$$

$$\varrho = (n - 1)kD \qquad 6.4$$

$$\gamma = \kappa kD \qquad 6.5$$

$$m = n - i\kappa \qquad 6.6$$

$$k = 2\pi/\lambda \qquad 6.7$$

m: complex refractive index (relative to that of the surrounding medium) n: real part of m

κ: absorption index

λ: wavelength of light in the surrounding medium

D: diameter of the sphere

When $|\zeta| \ll 1$, the efficiencies become (eqs.10.17-19)

$$Q_c = \frac{4}{3}\gamma + \frac{1}{2}(\varrho^2 - \gamma^2) \qquad 6.8$$

$$Q_a = \frac{4}{3}\gamma - \gamma^2 \qquad 6.9$$

$$Q_b = Q_c - Q_a \approx \frac{1}{2}(\varrho^2 + \gamma^2) = \frac{1}{2}k^2D^2|m - 1|^2 \qquad 6.10$$

We see that absorption is a first order and scattering a second order effect, and that the attenuation will be dominated by absorption, provided that ϱ and γ are of the same order. The expression for the scattering efficiency of small spheres in the van de Hulst region agrees with that given earlier (eq.4.10) for large spheres in the Rayleigh-Debye region. This confirms the soundness of the van de Hulst method.

If $|\zeta|$ is a great number, then Q_c will be close to 2, and when γ is great, Q_a will be close to 1. All the rays which hit the sphere are then absorbed, and an equally great amount of light is scattered.

The behaviour of Q_c as a function of the particle size is perhaps best seen when the sphere is non-absorbing ($\kappa=\gamma=0$). Q_c is then

$$Q_c = Q_b = 2 - \frac{4}{\varrho}\sin\varrho + \frac{4}{\varrho^2} - \frac{4}{\varrho^2}\cos\varrho \qquad 6.11$$

The function $Q_b(\varrho)$ rises first steeply, proportional with ϱ^2, according to eq.6.10, reaches the maximum value 3.2 when $\varrho=4.1$ and continues to oscillate around the value 2 with smaller and smaller amplitude as ϱ increases. The period of the oscillation is given by $\varrho=2\pi$, or $D=\lambda/(n-1)$. The maxima or minima of Q_b will then roughly appear for values of D with a distance $\lambda/(n-1)$, or if D is fixed, for values of λ with a distance $D(n-1)$.

Laboratory measurements on spheres by LEWIS and LOTHIAN (1954) confirm partly the oscillating form of $Q_b(\varrho)$.

6.2. Thin disks

The mean attenuation and absorption efficiencies of thin disks oriented at random are (eqs.12.14 and 22)

$$Q_c = 2 - 2\,\mathrm{Re}\,\{e^{-i\zeta} - i\zeta e^{-i\zeta} - \zeta^2 E_1(i\zeta)\} \qquad 6.12$$

$$Q_a = 1 - e^{-2\gamma} + 2\gamma e^{-2\gamma} - 4\gamma^2 E_1(2\gamma) \qquad 6.13$$

where

$$\zeta = (m-1)kL = \varrho - i\gamma \qquad 6.14$$

$$\varrho = (n-1)kL \qquad 6.15$$

$$\gamma = \kappa kL \qquad 6.16$$

L is the thickness of the disk. It is assumed that the diameter D of the disk is much greater than L.

Equation 6.12 is valid when $|\zeta| \geqslant 1$. E_1 is the exponential integral, defined by eq.8.7.

It may seem possible from the last term of eq.6.12, that Q_c goes to infinity when ζ goes to infinity. This is not the case. As shown in Chapter 12.4, the asymptotic forms of Q_c and Q_a are (eqs.12.29-30)

$$Q_c \approx 2 + 4\,\mathrm{Re}\,\{\frac{i\,e^{-i\zeta}}{\zeta} - \frac{3\,e^{-i\zeta}}{\zeta^2}\} \qquad 6.17$$

$$|\zeta| \gg 1$$

$$Q_a \approx 1 - \frac{e^{-2\gamma}}{\gamma} + \frac{3\,e^{-2\gamma}}{2\,\gamma^2} \qquad 6.18$$

$$\gamma \gg 1$$

Q_c oscillates around the value 2, and Q_a increases towards 1 with increasing argument. When the disks are non-absorbing, eq.6.17 becomes

$$Q_c = Q_b = 2 + \frac{4\sin\varrho}{\varrho} - \frac{12\cos\varrho}{\varrho^2} \qquad 6.19$$

$$\varrho \gg 1$$

which strongly resembles the earlier formula for non-absorbing spheres (eq.6.11).

The formulae 6.12-19 have been obtained by taking into account the light which passes through the flat top and bottom of the disk, and neglecting the light which passes through the circular cylinder walls. This can be done because the last area constitutes only a minor part of the total surface of the thin disk, and when a number of disks are oriented at random only a minor part of the light will pass through the cylinder walls. When the disk decreases in size, however, the importance of this minor part changes. Especially the situation when the axis of the disk is normal to the incident, light becomes important. The light passes then a much longer distance through the disk than when the axis is parallel with the incident light, and the greater phase shift produced by the disk in this situation compensates partly for the much smaller geometrical cross section. The result is that the contribution from the cylinder walls can no longer be neglected in the mean attenuation and absorption cross sections. Unfortunately, exact analytical solutions to the problem is very difficult to obtain in this case. A crude estimate is given in Chapter 12.3, and a perhaps somewhat better estimate is given in Chapter 9.3. The results coincide, however, for terms up to the second order. They are (eqs.9.50-52, 12.26-28)

$$Q_c \approx 4\gamma + 2(\varrho^2 - \gamma^2)\ \ln D/L \qquad 6.20$$

$$Q_a \approx 4\gamma - 4\gamma^2\ \ln D/L \qquad 6.21$$

$$Q_b = Q_c - Q_a \approx 2(\varrho^2 + \gamma^2)\ \ln\ (D/L) \qquad 6.22$$

While the efficiencies for medium and large values of ζ and γ only depend on the thickness L of the disk, the efficiencies for small values are influenced by the diameter D.

6.3. Long cylinders

For randomly oriented long cylinders of length L and diameter D, $L \gg D$, the attenuation and absorption efficiencies can be written (eqs.13.14 and 26)

$$Q_c \approx 4 \int_0^{\pi/2} \sin^2\theta\ \mathrm{Re}\ \{i\ J_1(u) + H_1(u)\}\ d\theta \qquad 6.23$$

$$Q_a \approx 2 \int_0^{\pi/2} \sin^2\theta\ [-iJ_1(w) + H_1(w)]\ d\theta \qquad 6.24$$

where

$$u = \frac{\zeta}{\sin\theta} \qquad 6.25$$

$$w = iv = i\ \frac{2\gamma}{\sin\theta} \qquad 6.26$$

$$\zeta = (m - 1)kD$$
$$= (n - 1)kD - i\kappa kD = \varrho - i\gamma \qquad 6.27$$

$$\gamma = \kappa kD \qquad 6.28$$

Eq.6.23 is valid when $|\zeta| \gg 1$ and eq.6.24 when $\gamma \gg 1$.

When $|\zeta|$ and γ are great numbers, the efficiencies become (eqs.13.49 and 13.54)

$$Q_c \approx 2 + \mathrm{Re}\ \{\frac{3}{2\ \zeta^2} + (\frac{i}{\zeta} + \frac{3}{8\ \zeta^2})\ e^{i\frac{\pi}{2} - i\zeta}\} \qquad 6.29$$

$$Q_a \approx 1 - \frac{3}{16\ \gamma^2} \qquad 6.30$$

Non-absorbing large cylinders have the efficiency

$$Q_c = Q_b \approx 2 - \frac{\cos \varrho}{\varrho} + \frac{3}{2\,\varrho^2} + \frac{3 \sin \varrho}{8\,\varrho^2} \qquad 6.31$$

which is a form quite similar to the expression for spheres, eq.6.11.

In the former section we saw that when the size of the disks decreased, "cylinder" effects had to be taken into account. Similarly we might expect here that when the size of the cylinders decrease, "disk" effects become important. By "disk"effects we mean the effect of light passing through the flat top and bottom of the cylinder. These areas constitute a minor part of the total surface of the long cylinder, and for values of $|\zeta|$ and γ larger than 1 the "disk" effects can be neglected, as shown in Chapter 13.

Perhaps somewhat surprising we find in Chapter 13 that the "disk" effects can be neglected also when $|\zeta|$ is small and $D/L \ll 1$, and we obtain (eqs.13.30, 32 and 33)

$$Q_c \approx 2\,\gamma + \frac{4}{3}\,(\varrho^2 - \gamma^2) \qquad 6.32$$

$$Q_a \approx 2\,\gamma - \frac{8}{3}\,\gamma^2 \qquad 6.33$$

$$Q_b = Q_c - Q_a \approx \frac{4}{3}\,(\varrho^2 + \gamma^2) \qquad 6.34$$

The deduction in Capter 9 gives a slightly different result (eqs.9.56-58)

$$Q_c \approx 2\,\gamma + \frac{\pi^2}{8}\,(\varrho^2 - \gamma^2) \qquad 6.35$$

$$Q_a \approx 2\,\gamma - \frac{\pi^2}{4}\,\gamma^2 \qquad 6.36$$

$$Q_b \approx \frac{\pi^2}{8}\,(\varrho^2 + \gamma^2) \qquad 6.37$$

Since $4/3 \approx 1.33$ while $\pi^2/8 \approx 1.23$ the difference between the methods is not great. The eqs.6.32-34, however, were only assumed to give the correct order of the efficiencies.

But why can "disk" effects be neglected for small cylinders, while "cylinder" effects have to be taken into account for small disks? The answer is the random orientation of disks and cylinders. A thin disk will yield "cylinder" effects when its axis is tilted an angle about $\pi/2$ from the incident light. The relative number of disks with this orientation will be proportional to the solid angle $2\pi \sin(\pi/2)\Delta\theta = 2\pi\ \Delta\theta$. A long cylinder, however, will only yield "disk" effects when its axis is nearly or quite parallel with the incident light, that is inside the solid angle $2\pi \sin(\Delta\theta)\Delta\theta = 2\pi\ (\Delta\theta)^2$. This solid angle is much smaller than $2\pi\ \Delta\theta$, and the relative number of cylinders with this orientation will then also be smaller than the number of disks with axes inside $2\pi\ \Delta\theta$.

6.4. The appropriate size parameter for natural particles

Fig.2 illustrates the similarity between the attenuation or scattering efficiencies of non-absorbing spheres, long cylinders and thin disks. The efficiencies are given as functions of ϱ, and it can be imagined that by a change of scale in the abscissa, so that the first maxima of the curves will coincide, even better similarity can be obtained.

A problem arises, however, if we want to apply these results to natural particles. The ϱ value of a sphere is a function of the diameter, that is the maximum measure across the particle, while the ϱ values of cylinders and disks are both functions of minimum measures across the particle. What is then the appropriate size parameter for a natural particle? The problem is an old one in industrial engineering dealing with powder analysis. Several methods and constants have been proposed in order to relate volume and surface to some size parameter which is easy to observe (e.g. DALLAVALLE, 1948). There are, however, two dimension properties which are independent of the particle orientation, and those are volume and surface. In order to see if D_s or D_v, that is the equivalent surface and volume diameter, could be a convenient size parameter, the efficiencies are presented as functions of D_s and D_v in Figs.3-4. The results of Fig.4 agree qualitatively with those obtained by ASANO and SATO (1980, fig.2) for prolate and oblate spheroids.

The curves of Figs.2-4 consist of a rising part, from zero to the maximum value, followed by an oscillating part, around the asymptotic value of the efficiencies. For a natural size distribution the oscillations will be meaned out, and the efficiency in this size region could just as well be replaced by its asymptotic value. This means that if we are working with the scattering of very large non-absorbing particles, D_s will be the ideal size measure, since B then is given by $2\ G = \pi\ D_s^2/2$. For a size distribution where a significant part of the scattering comes from the rising part of the efficiency curves, however, D_s is not a good parameter, as Fig.3 illustrates.

On the other hand we know from Chapter 3 that the scattering cross section for very small particles, that is particles from the Rayleigh region, will be proportional to D_v^6. In the size region between very small and very large particles, B will then depend both on D_v and D_s.

It is now clear that neither the surface nor the volume alone of an arbitrary particle is able to provide a satisfactory measure through all the size regions. If, however, we choose the parameter V/G, we find that for spheres the ratio is 2 D/3, for thin disks 2 L and for long cylinders D, which are parameters very close to those applied in Fig.1. Further support for this choice can be found from eqs.6.8-9, 6.20-21 and 6.32-33 where we find that the first order term of the attenuation and absorption efficiencies can be written $2\kappa kV/G = a_i V/G$ for all kinds of particle shapes. This is also shown in Chapter 9.

Q_b is given as a function of $x=(n-1)kV/G$ in Fig.5. The rising parts of the curves are seen to coincide fairly well.

Still we have not got rid of the shape factor, since the functional relations between Q_c, Q_a and ϱ or V/G depend on particle shape. If we draw Q_c and Q_a as functions of V/G, we can at least say that the true values of arbitrary natural particles probably lie between the functions of the three extreme shapes, as given by the spheres, disks and cylinders.

But if we want to predict the optical properties of a suspension of homogeneous particles, of which we know the refractive index, volume, surface and shape, this knowledge of the limits is not satisfactory. The present paper is not able to give a general solution to the problem. But the results of Chapter 9 suggest that it may be possible to construct power series of V/G, including constants which only depend on shape and not on size. We shall now take a closer look at those ideas.

6.5. Irregular particles

In Chapter 9 it is shown that the attenuation and absorption efficiencies can be written (eqs.9.24-25)

$$Q_c = 2 \operatorname{Re} \left\{ \sum_{n=1}^{N} -(-i)^n (m-1)^n k^n \frac{f_n}{n!} \left(\frac{V}{G}\right)^n \right\} \qquad 6.38$$

$$Q_a = \sum_{n=1}^{N} -(-1)^n (2\kappa)^n k^n \frac{f_n}{n!} \left(\frac{V}{G}\right)^n \qquad 6.39$$

where f_n is a dimensionless number depending on the particle shape. If f_n does not vary too much from one kind of natural particles to another kind of similar shapes, then perhaps only a few f_n series are needed in order to describe the natural occuring variation. The remaining problem is to find the f_n values. These may be obtained if the necessary geometrical data of a sample of irregular particles are known, but so far such data are lacking.

The values for particles of known geometrical shapes, however,

TABLE 1

	f_1	f_2	f_3	f_4
Spheres	1	1.13	1.35	3.38
Cylinders L/D=10	1	1.12	1.61	3.57
Disks D/L=10	1	1.33	2.56	6.66

may help us somewhat. From the results of Chapter 9 and 10 we have the f_n values given by Table 1.

If we are working with particles which are so small that the 4th term of the series expansion of eqs.6.38-39 may be omitted, the error introduced by setting, for instance, $f_1=1$, $f_2=1.1$ and $f_3=1.5$, is probably very small. We then only need to know the refractive index of the particles, and the ratio V/G.

In order to estimate V/G, Table 2 may give some support. The mean geometrical cross section G can be related to the volume V by a dimensionless shape factor s, so that

$$G = s\,V^{2/3} \qquad 6.40$$

If the particle is a sphere, s becomes $(9\pi/16)^{1/3}=1.21$.

The table consists of three parts. The first part gives values of s for different geometrical shapes. The second part gives s for phytoplankton. The third part gives s for mineral particles. It is calculated from values of the volume-shape factor ν and the surface-shape factor σ. The volume of the particle is then related to a diameter D by

$$V = \nu\,D^3 \qquad 6.41$$

TABLE 2

Particle	L [μm]	$\frac{L}{D}$	$\frac{s}{(9\pi/16)^{1/3}}$	From data by
SPHERE			1	
ICOSAHEDRON			1.06	
DODECAHEDRON			1.10	
CYLINDER		1	1.14	
OCTAHEDRON			1.18	
CUBE			1.24	
TETRAHEDRON			1.49	
CYLINDER		10	1.73	
DISK		0.1	2.13	
DINOFLAGELLATES				
Gonyaulux polyedra	47	1	1	a
COCCOLITHOPHORIDS				
Coccolithus huxleyi	13	1	1	"
Cricosphaera elongata	14	1.1	1.1	"
MICRO-FLAGELLATES				
Monochrysis lutherii	6	1	1	"
Dunaliella tertiolecta	6	1.6	1.2	"
DIATOMS				
Thalassiosira rotula	12	0.4	1.3	"
Coscinodiscus wailesii	100	0.7	1.2	"
Rhizosolenia stolterfothii	126	7.3	1.6	"
Rhizosolenia hebeta f. semispina	440	61	3.0	b

a: EPPLEY et al., 1967.   b: PAASCHE, 1960.

TABLE 2 (cont.)

Particle	D [μm]	$\frac{\sigma}{\pi}$	$\frac{v}{(\pi/6)}$	$\frac{\sigma}{v}$	$\frac{s}{(9\pi/16)^{1/3}}$	From data by
Quartz, crushed	220-830	0.78-0.79	0.52-0.59	8.1-9.2	1.13-1.22	c
Quartz	2-64		0.27			d
Granite	2-72		0.27			"
Calcite	1-69		0.26			"
Sand, rounded					1.02	e
" worn					1.07	"
" sharp					1.17	"
" angular					1.29	"
Sand, rounded	200	0.93	0.65	8.6	1.24	f
Coal, natural dust	200	0.81	0.38	13	1.53	"
Glass, angular	200	1.0	0.53	11	1.54	"
Mica, D/L≈26	200	0.53	0.057	56	3.58	"
" D/L≈260	3000	0.51	0.0057	530	15.8	g
Sillimanite	2400	0.81	0.48	10	1.33	h
Coal, crushed	2400	0.81	0.48	10	1.33	"
Quartz	1900		0.32			i
"	890		0.29			"
Feldspar	40		0.50			"
Dolomite	40		0.36			"
Hornblende	50		0.042			"
White sand, smooth	2900	0.67				"
"	800	0.86				"
"	400	0.83				"
Filter sand, smooth	600	0.86				"
"	500	0.92				"
Quartz	1-49			14		j
Diamond	1-4			10		k

c: MARTIN, 1927.

d: HATCH and CHOATE, 1929.

e: FAIR and HATCH, 1933.

f: HEYWOOD, 1933.

g: HEYWOOD, 1938.

h: SKINNER et al., quoted by HAWKSLEY, 1951.

i: DALLAVALLE, 1948, p.331.

j: PROCTOR and HARRIS, 1974.

k: PROCTOR and BARKER, 1974.

and the surface S is related to D by

$$S = \sigma D^2 \qquad 6.42$$

D was measured in microscopic counts by MARTIN (1927) as the projection of the particle on a fixed line. HEYWOOD (1933) defined D as the area-equivalent diameter of the particle when it rested on a horizontal plane in its most stable position. If the geometrical projection of the particle on a horizontal plane was C_g, then

$$D = 2\ (C_g/\pi)^{1/2} . \qquad 6.43$$

These two different definitions of D influence v and σ, but not s. From eqs.6.40-42 we find that for convex particles

$$s = \frac{\sigma}{4\ v^{2/3}} \qquad 6.44$$

HEYWOOD claimed that for a large number of randomly oriented particles the mean diameter of equivalent area will become very close to the mean "statistical" diameter of MARTIN.

For a sphere $v=\pi/6=0.524$ and $\sigma=\pi=3.14$. In Table 2 σ has been divided with π, v with $\pi/6$ and s with $(9\pi/16)^{1/3}$ in order to make a comparison with the spherical particle easier. It is seen that the mineral particles have smaller surfaces and volumes than a sphere of the same diameter. The particles have, however, a greater surface than a sphere of the same volume. This means that if the total particle surface of a water sample is estimated from the volume measurements by a Coulter counter, the special assumption is likely to give a surface which is 20-50% too small. In extreme cases, when the dominating particles are disk-shaped, like mica; the error may even amount to several hundred per cent.

We shall now look at the scattering efficiency of non-absorbing crushed quartz particles. The scattering coefficient b due to the particles of a suspension, and their mass concentration M may be written

$$b = N\,G\,Q_b = N\,\frac{\sigma\,D^2}{4}\,Q_b \qquad 6.45$$

$$M = N\,V\,d = N\,v\,D^3\,d \qquad 6.46$$

where N is the number of particles per volume unit, and d is the density of the particles. By eliminating N we find that

$$b = \frac{Q_b}{D}\,\frac{M\,\sigma}{4\,v\,d} \qquad 6.47$$

which shows that when the mass concentration is constant, b will be inversely proportional to D as long as Q_b remains constant.

TOLMAN et al. (1919) were the first to observe such a proportionality for silica particles down to a diameter of about 2 μm. The mean relative values of Q_b obtained from curve 2 and 3 in their plate 4, are presented in Fig.6. The values have been normalized so that the mean of the three greatest values equals 2. (One of their points lies outside our figure). TOLMAN et al. observed visually with "white" light, and the representative wavelength of the light has here been set equal to the wavelength of maximum eye sensitivity, that is 555 nm in air or 417 nm in water. Since quartz has the refractive index $n=1.16$ relative to water, $\varrho=(n-1)2\pi/\lambda$ is obtained by multiplying D with $2.41\ \mu m^{-1}$.

Measurements by JERLOV and B. KULLENBERG (1953) confirm the proportionality observed by TOLMAN et.al. Their table 1 gives values of b, M and D. The scattering coefficient was measured with blue light (470 nm in air or 353 nm in water), and the particles were a mixture of quartz and feldspar. Their density can be estimated to $2.65\ g\ cm^{-3}$. If eq.6.47 is written

$$Q_b = \frac{4\,v\,d}{\sigma}\,\frac{b\,D}{M} \qquad 6.48$$

and if the value 10 is chosen from Table 2 for the ratio σ/v, we see that Q_b can be computed. ϱ can be found by multiplying D with

2.84. The values of Q_b so obtained, however, are about one third of the values presented by JERLOV in 1976 (table VII) as derived from the original 1953 measurements. Instead of using the 1976 values, the originally obtained values have been normalized by the requirement that the mean value of the three largest size fractions should equal 2, and the result is presented in Fig.6. (Two of the five points lie outside the figure).

HODKINSON (1963) observed b, N and D, and found that for a certain size fraction b was independent of the wavelength. By assuming that the corresponding values of Q_b should be 2, he was able to determine the value of σ in eq.6.45, and Q_b was then calculated for the other size fractions as a function of $\pi D/\lambda$. The last term gives ϱ when multiplied with 0.32. The values are presented in Fig.6.

The figure illustrates that Q_b is constant down to a ϱ value of 5 or 6. We can use $x=(n-1)kV/G$ as an independent variable instead of $\varrho=(n-1)kD$. The relation between x and ϱ is

$$\frac{x}{\varrho} = \frac{V}{G\,D} = \frac{4\,v}{\sigma} = 0.4 \qquad 6.49$$

with the earlier chosen ratio $\sigma/v=10$.

Eq.6.38 gives with $f_2=1.1$, $f_4=3.5$ and $\gamma=0$ at the actual wavelengths that

$$Q_b \approx 1.1\,x^2 - 0.29\,x^4 \qquad 6.50$$

This function has been drawn in Fig.6 for x values between 0 and 1. The function seems to make a natural continuation of the observations into the region of small particles.

By comparing the observations presented in Fig.6 with the theoretical results in Fig.5, the most striking difference is the lack of oscillating character in Q_b which the irregular particles seem to demonstrate. Partly this may be brought about by the differences in shape and size inside each size fraction. If we take the mean value of the three curves for each x value in Fig.5, some of the oscillations will be meaned out, and if we further construct a gliding mean value for a certain size region Δx, even less oscillations will occur. Still remnants of the oscillations, especially the first maximum, are likely to be present. For regular particles like spheres the oscillations are real phenomena, as the observations by LEWIS and LOTHIAN (1954) show. Bur for irregular particles like crushed quartz, they are obviously absent, according to the figure.

CHYLEK et al. (1976) assume that resonance phenomena are not present in scattering by irregular randomly oriented particles, and they modify the Mie theory in accordance with this. They then find very good agreement between theoretical and observed angular scattering patterns. Transferred to our integrated scattering efficiencies, their assumption may mean that Q_b should have a a constant value in the oscillating region, as the observations in Fig.6 suggest. This value should be the asymptotic value 2.

Measurements by PROCTOR and HARRIS (1974), however, are in contradiction with the results of Fig.6. Rather than a constant value of Q_b, they found that Q_b was decreasing from a maximum value about 2.4 at $\varrho \approx 11$ to the asymptotic value 2 about $\varrho \approx 60$. An even more pronounced maximum was observed for irregular diamond particles by PROCTOR and BARKER (1974). Some older measurements by ROSE (1952) also give values of Q_b significantly greater than 2 for diameters less than 5 μm (ϱ probably less than 12).

The discrepancy between the three first mentioned works and the three last ones may be due to several factors. The most natural to think of is that one of the groups has made systematic errors of measurements. Another factor is different particle roundness. ROSE used desert sand which he assumed to be wind-blown and therefore well rounded, and silica sand which possibly had been abraded during the milling process. Such fairly regular particles are more likely to obtain resonance effects than more irregular ones. A third factor which may have some influence is the difference in definition of size. For instance HODKINSON defines D as the root-mean-square of the diameters of a certain size fraction, while PROCTOR and HARRIS and PROCTOR and BARKER define D as $\sum_i D_i^3/\sum_i D_i^2$. Difference in methods may also be of importance. We shall here only conclude that so far the final answer to the scattering by irregular particles does not seem to have been obtained.

6.6. Conclusions about the influence of shape

For all kind of shapes we have that for very small particles (the Rayleigh region)

$$B \sim V^2 \sim D_V^6 \qquad 6.51$$

For very large non-absorbing particles

$$B = 2\,G \sim D_S^2 \qquad 6.52$$

In between these extremes the scattering cross section will be a function both of the volume-equivalent diameter D_V and the surface-equivalent diameter D_S. The function will depend on the shape of the particles as Chapters 6.1-3 show. For instance, in the upper part of the Rayleigh-Debye region (:kD>>1)a non-absorbing sphere will have the scattering cross section

$$B = \frac{\pi}{8} k^2 (n-1)^2 D_V^4 \qquad 6.53$$

according to eq.6.10, while randomly oriented cylinders and disks, according to eqs.9.24 and 36, will have

$$B = \frac{\pi}{4} \frac{k^2 (n-1)^2 D^2 L^2}{1+(\frac{4}{\pi}\frac{L}{D})^2} (\ln \frac{\pi}{4}\frac{D}{L} + 2\frac{L}{D})$$

$$= \frac{\pi}{8} k^2 (n-1)^2 D_V^4 \left[\frac{2(\frac{2}{3}\frac{L}{D})^{2/3}}{1+(\frac{4}{\pi}\frac{L}{D})^2} (\ln \frac{\pi}{4}\frac{D}{L} + 2\frac{L}{D})\right] \qquad 6.54$$

Here the volume-equivalent diameter D_V, defined by eq.4.30, has been introduced. When the term inside the brackets above is close to 1, then the scattering cross section equals that of an equal-volume sphere, as comparison with eq.6.53 shows. If L/D=0.1, 1 or 10, the term becomes 0.73, 1.02 and 0.76 respectively. This suggests that fairly equi-dimensional particles may be replaced by equal-volume spheres in this size region, but not particles of more extreme shapes, like thin disks and long cylinders.

Thus the situation has changed from the lower part of the Rayleigh-Debye region, described in Chapter 4, where even thin disks and long cylinders of wavelength size could be substituted by equal-volume spheres.

The appropriate size parameter for theoretical treatment of irregular particles may be the ratio V/G. So far there does not seem to be a definite answer as to whether the scattering efficiency of irregular particles have a primary maximum or not.

7. THE INFLUENCE OF INTERNAL STRUCTURE

Mineral particles may be regarded as homogeneous bodies. Phytoplankton on the other hand, except the blue-green alga, show definite internal structures. Most important for the optical properties may be the shape, number and distribution of the light-absorbing chloroplasts. These will vary between the species, but in this chapter we shall do a very great simplification. We will assume that phytoplankton can be regarded as consisting of two main constituents: dry matter and water, and restrict the discussion to spherical structures.

7.1. The layered sphere

The constituents may be organized in three extreme ways:

- with all the dry matter as an outer hard shell filled with water,
- with the dry matter as a dense core surrounded by waterish substance,
- and with dry matter and water perfectly mixed into a homogeneous substance. For the last case we shall assume that the resulting refractive index n follows the Gladstone-Dale relation

$$n = n_1 v_1 + n_2 v_2 \qquad 7.1$$

where v_1 and v_2 are the partial volumes of the two constituents and n_1 and n_2 their refractive indices.

As an example we may choose $n_1=1.15$ and $v_1=0.2$ for the dry mass and $n_2=1$, $v_2=0.8$ for the water content. The homogeneous particle then obtains the refractive index $n=1.03$, which is a reasonable value (AAS, 1981).

Q_c, Q_b and Q_a for the homogeneous sphere, the hollow sphere and the sphere with a dense core are given in Chapters 10 and 11. The values of Q_b for the non-absorbing, homogeneous sphere, as given by eq.10.12, are presented in Fig.7. The refractive index is 1.03 relative to water. If all the dry mass, with a partial volume 0.2 is concentrated into an outer shell, the ratio f of the inner radius to the outer radius R can be found from the relation

$$\frac{4}{3}\pi (fR)^3 = 0.8 \frac{4}{3}\pi R^3 \qquad 7.2$$

which gives that

$$f = 0.8^{1/3} \qquad 7.3$$

Eq.11.20 was then used to produce the function Q_b for a hollow sphere in Fig.7.

The last case occurs when the non-water constituents of the sphere are concentrated into a dry core. The ratio f between the radius of the core and the radius of the total sphere will be

$$f = 0.2^{1/3} \qquad 7.4$$

The scattering efficiency Q_b can then be calculated, as explained in Chapter 10, and the result is presented in Fig.7.

The last case gives clearly a much smaller scattering efficiency than the homogeneous and hollow spheres, during the oscillating part of the curve. During the rising part, however, that is for small particles, the situation is different. The scattering efficiency will then be

$$Q_b = \frac{\rho^2}{2} = \frac{k^2}{2} D^2 (n-1)^2 = \frac{k^2}{2} D^2 (1.03-1)^2$$

$$= \frac{k^2}{2} D^2 \, 0.0009 \qquad 7.5$$

for homogeneous spheres according to eq.10.20,

$$Q_b = \frac{k^2}{2} D^2 (n-1)^2 \left(1 - f - f^3 + f^4 + \frac{(1-f^2)^2}{2} \ln \frac{1+f}{1-f}\right)$$

$$= \frac{k^2}{2} D^2 (1.15-1)^2 \, . \, 0.0458 = \frac{k^2}{2} D^2 \, 0.00103 \qquad 7.6$$

for hollow spheres according to eq.11.26, and

$$Q_b = \frac{k^2}{2} D^2 \ (n-1)^2 \ f^4 = \frac{k^2}{2} D^2 \ (1.15-1)^2 \ . \ 0.117$$

$$= \frac{k^2}{2} D^2 \ 0.00263 \qquad 7.7$$

for dry cores according to Chapter 10.6.

Homogeneous small spheres (D<3 μm) have a scattering efficiency which is approximately equal to that of the related hollow spheres, only about 10% smaller. The dry cores, however, scatter more than twice as much light as the other two spheres.

If the particles are absorbing, so that $m=1.15-0.05\ i$ for the dry matter, the homogeneous sphere will have an index $m=1.03-0.01\ i$. In the expressions given above, 0.0009, 0.00103 and 0.00263 will then be substituted by 0.001, 0.00114 and 0.00292, respectively. The absorption effect will consequently increase the scattering efficiency of the smaller spheres with about 10%. The larger spheres, as seen from Fig.7, will have their efficiency greatly reduced, to about half of their earlier values.

The absorption efficiencies are also presented in Fig.7. It is interesting that the hollow and homogeneous spheres have practically identical curves, while the dry core have a much smaller asymptotic value. To a first approximation the absorption cross section for small particles can be written (eq.9.25)

$$A = a_i V \qquad 7.8$$

where a_i is the mean internal absorption coefficient of the volume V. If the amount of absorbing material within the particle remains constant, a_i will not be influenced by the internal distribution of the material, neither will A nor Q_a.

7.2. The scattering dispersion of an alga

We shall now see how a great variation in the absorption index may influence the scattering of a particle. CHARNEY and

BRACKETT (1961) observed that the chloroplast of the spherical green alga <u>Chlorella pyrenoidosa</u> had a refractive index 1.05-1.06 relative to water, while the rest of the cytoplasm had the index 1.015. The chloroplast would occupy about 1/3 to 1/2 of the cell.

We may now construct an idealized spherical alga by assuming that its chloroplast of partial volume 1/3 has the refractive index 1.06, while the 2/3 rest of the cell has the refractive index 1.015. If all the cytoplasm were mixed together, the mean refractive index should be 1.03 according to eq.7.1. A pigment concentration of 7 mg cm^{-3} may be a reasonable value (AAS, 1981, p.36). To simplify matter the pigment has been assumed to consist only of chlorophyll <u>a</u> and <u>b</u> in the ratio 3.6:1 (RABINOWITCH, 1945, p.410). The absorption index κ of the pigments relative to the refractive index of water is shown in Figs.8 and 9. κ has been calculated from values of the absorption coefficient given by RABINOWITCH (1951, p.605-10). Since the chloroplast constitutes 1/3 of the total volume, the values of κ have been multiplied by 3 to give the actual values of the chloroplast. Outside the chloroplast κ is assumed zero.

Q_b and Q_a have been calculated numerically by means of the general equations 11.9 and 11.10, and are presented as functions of the wavelength in air. Fig.8 shows the efficiencies when the algal diameter is 10 μm, and Fig.9 when the diameter is 50 μm. Three different internal structures are considered, the homogeneous case (n=1.03) and the cases when the chloroplast (n=1.06) acts either as an outer shell or as an inner core, with the rest of the particle consisting of cytoplasm (n=1.015).

The figures illustrate that the internal structure has little influence on Q_a when Q_a is of order 0.1 or less. This can also be seen in Fig.7.

Absorption peaks can influence the scattering of a particle in entirely different ways, depending on the size of the particle. For instance, the absorption and scattering efficiencies of a small homogeneous sphere are approximately (eqs.10.18 and 19)

$$Q_a \approx \frac{4}{3}\gamma \qquad 7.9$$

$$Q_b \approx \frac{1}{2}(\varrho^2 + \gamma^2) \approx \frac{\varrho^2}{2} + \frac{9}{16}Q_a^{\ 2} \qquad 7.10$$

If γ is of the same order as ϱ, that is κ is of the same order as $(n-1)$, then a positive sharp peak in κ will give positive sharp peaks in Q_a and Q_b.

On the other hand, for very large particles we usually have that

$$Q_b = Q_c - Q_a \approx 2 - Q_a \qquad 7.11$$

A positive peak in Q_a will now correspond to a negative peak in Q_b.

A theoretical possibility for large spheres is that the sphere is very waterish, almost non-refracting, but strongly absorbing at all wavelengths, so that $\gamma \gg \varrho$. According to eq.10.27 the scattering efficiency can then be written

$$Q_b \approx 7\,Q_a - 6 \qquad 7.12$$

which shows that positive peaks will appear simultaneously in Q_b and Q_a.

In between these extremes, as given by eqs.7.10-12, the ratio between ϱ and γ as well as the size and the internal structure will influence the resultant scattering dispersion.

Fig.8, for a diameter of 10 µm, shows that the influence of the internal structure may be important. When the chloroplast has the form of an inner core, the scattering will decrease from the UV region towards the green region. When the chloroplast has the form of an outer shell, or is homogeneously distributed, however, the scattering will increase from UV towards green. In the green part of the spectrum the outer shell structure scatters more than twice as much light as the inner core.

When the diameter is 50 µm, the course of the scattering efficiencies with increasing wavelength is again quite different, as illustrated by Fig.9. Especially the outer shell structure may seem to be influenced by the absorption effect, and has a marked maximum in the green region, which is 50% higher than those of the other structures. The high scattering efficiency of 3, however, is a resonance phenomenon, and it is doubtful whether it will be observed in a natural population of varying sizes.

7.3. Conclusions about the influence of internal structure

We can now draw the qualitative conclusions:

- Very small particles, which only differ in internal structure, will absorb the same amount of light, but will scatter more light the more the refractive and absorbing material is concentrated towards the centre.

- Very large particles will absorb and scatter less light the more the refractive and absorbing material is concentrated towards the centre of the particle. The concentration of material towards the surface of the sphere has a much smaller effect on the scattering, and almost none on the absorption.

- There is no general relationship between the wavelength dispersion of the absorption and scattering efficiencies. The correlation may be positive, negative or absent, depending on the size and internal structure of the particle, and on the ratio between the refractive and absorption indices.

PART II

DERIVATION OF THE EQUATIONS

8. SOME USEFUL MATHEMATICAL FORMULAE

In the following chapters some less common mathematical functions will frequently appear. It will be practical to define them and gather them in one chapter. The notations are taken from ABRAMOWITZ and STEGUN (1972).

COSINE INTEGRAL Ci(z)

Integral form

$$Ci(z) = \gamma_0 + \ln z + \int_0^z \frac{\cos t - 1}{t}\, dt = - \int_z^\infty \frac{\cos t}{t}\, dt \qquad 8.1$$

γ_0 = Euler's constant = 0.5772

Series expansion

$$Ci(z) = \gamma_0 + \ln z - \frac{z^2}{2!2} + \frac{z^4}{4!4} - \ldots.. \qquad 8.2$$

Asymptotic expansion

$$Ci(z) = \frac{\sin z}{z}\left(1 - \frac{2!}{z^2} + \ldots\right) - \frac{\cos z}{z^2}\left(1 - \frac{3!}{z^2} + \ldots\right) \qquad 8.3$$

SINE INTEGRAL Si(z)

Integral form

$$Si(z) = \int_0^z \frac{\sin t}{t}\, dt = \frac{\pi}{2} - \int_z^\infty \frac{\sin t}{t}\, dt \qquad 8.4$$

Series expansion

$$Si(z) = \frac{z}{1!1} - \frac{z^3}{3!3} + \frac{z^5}{5!5} - \ldots.. \qquad 8.5$$

Asymptotic expansion

$$Si(z) = \frac{\pi}{2} - \frac{\cos z}{z}\left(1 - \frac{2!}{z^2} + \ldots\right) - \frac{\sin z}{z^2}\left(1 - \frac{3!}{z^2} + \ldots\right) \qquad 8.6$$

EXPONENTIAL INTEGRAL $E_1(z)$

Integral form

$$E_1(z) = -\gamma_0 - \ln z + \int_0^z \frac{1-e^{-t}}{t}\,dt = \int_z^\infty \frac{e^{-t}}{t}\,dt \qquad 8.7$$

Series expansion

$$E_1(z) = -\gamma_0 - \ln z + \frac{z}{1!1} - \frac{z^2}{2!2} + \ldots \qquad 8.8$$

Asymptotic expansion

$$E_1(z) = \frac{e^{-z}}{z}\left(1 - \frac{1!}{z} + \frac{2!}{z^2} - \frac{3!}{z^3} + \ldots\right) \qquad 8.9$$

Relation between E_1, Ci and Si

$$E_1(iz) = -\mathrm{Ci}(z) + i\left(\mathrm{Si}(z) - \frac{\pi}{2}\right) \qquad 8.10$$

ERROR FUNCTION erf z

Integral form

$$\mathrm{erf}\, z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\,dt \qquad 8.11$$

Series expansion

$$\mathrm{erf}\, z = \frac{2}{\sqrt{\pi}}\left(\frac{z}{1} - \frac{z^3}{1!3} + \frac{z^5}{2!5} - \ldots\right) \qquad 8.12$$

Asymptotic expansion

$$\mathrm{erf}\, z = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \ldots\right) \qquad 8.13$$

BESSEL FUNCTION $J_1(z)$

Integral form

$$J_1(z) = \frac{2\,z}{\pi} \int_0^{\pi/2} \cos(z \cos t) \sin^2 t \, dt \qquad 8.14$$

Series expansion

$$J_1(z) = \frac{z}{2} \left[1 - \frac{z^2}{1!2!4} + \frac{z^4}{2!3!4^2} - \ldots\right] \qquad 8.15$$

Asymptotic expansion

$$J_1(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left[\cos(z - \frac{3}{4}\pi)\left(1 + \frac{15}{128z^2}\right) - \sin(z - \frac{3}{4}\pi)\left(\frac{3}{8z} - \ldots\right)\right] \qquad 8.16$$

STRUVE FUNCTION $H_1(z)$

$$H_1(z) = \frac{2z}{\pi} \int_0^{\pi/2} \sin(z \cos t) \sin^2 t \, dt \qquad 8.17$$

Series expansion

$$H_1(z) = \frac{2z}{\pi} \left[\frac{z}{3} - \frac{z^3}{3^2.5} + \frac{z^5}{3^2.5^2.7} - \ldots\right] \qquad 8.18$$

Asymptotic expansion

$$H_1(z) = \frac{2}{\pi}\left[1 + \frac{1}{z^2} - \ldots\right] + \left(\frac{2}{\pi z}\right)^{1/2} \left[\sin(z - \frac{3}{4}\pi)\left(1 + \frac{15}{128z^2} - \ldots\right)\right.$$

$$\left. + \cos(z - \frac{3}{4}\pi)\left(\frac{3}{8z} - \ldots\right)\right] \qquad 8.19$$

9. THE IRREGULAR PARTICLE

9.1. General expressions

Provided the refractive index of the particle relative to the fluid around it is close to 1, and provided the dimension of the particle is so much greater than the wavelength that the concept of a ray passing through the particle has any meaning, VAN DE HULST (1957, p.174-183) deduced two very simple relations which can be written

$$C_c = 2 \text{ Re } \{\int_{C_g} (1 - e^{-i\varphi}) \, dg\} \tag{9.1}$$

$$C_a = \int_{C_g} (1 - e^{-\alpha}) \, dg \tag{9.2}$$

Here C_g is the geometrical cross section or shadow area of the particle in a plane normal to the incident light field. φ is the phase difference between the ray passing through the small shadow area dg and the undisturbed ray. If we let the z axis be parallel with the incident light, then φ is given by

$$\varphi = \int_Z (m - 1) \, k \, dz \tag{9.3}$$

where m is the complex refraction index of the particle relative to the surrounding medium. k is the wavenumber, defined as

$$k = \frac{2\pi}{\lambda} \tag{9.4}$$

where λ is the wavelength of light in the surrounding medium. Z is the thickness of the particle normal to dg.

α is the absorption path of the ray through dg. α can be written as

$$\alpha = \int_Z a_i \, dz = 2 \, k \int_Z \kappa \, dz \tag{9.5}$$

Here a_i means the internal absorption coefficient of the particle material, not to be confused with the resultant absorption coefficient of the particle suspension (eq.2.4). κ is the absorption index, defined as

$$\kappa = \frac{a_i}{2\ k} \tag{9.6}$$

The ratios between the optical cross sections and the geometrical cross section are

$$\frac{C_c}{C_g} = 2 - \frac{2}{C_g} \mathrm{Re} \{ \int_{C_g} e^{-i\varphi} dg \} \tag{9.7}$$

$$\frac{C_a}{C_g} = 1 - \frac{1}{C_g} \int_{C_g} e^{-\alpha} dg \tag{9.8}$$

$$\frac{C_b}{C_g} = \frac{C_c}{C_g} - \frac{C_a}{C_g} \tag{9.9}$$

We see that as α increases from 0 to ∞, C_a/C_g will increase from 0 to 1. In the last case all incident light at the particle is absorbed.

The refractive index m can be written (Chapter 2.4)

$$m = n - i\kappa = n - \frac{i\ a_i}{2\ k} \tag{9.10}$$

n is the real part of the refractive index and $-\kappa$ is the imaginary part. The exponent in the integral of eq.9.7 will then be

$$\begin{aligned} -i\ \varphi &= - i \int_z (n - i\kappa - 1)\ k\ dz \\ &= - i \int_z (n - 1)\ k\ dz - \int_z \kappa\ k\ dz \\ &= - i \int_z (n - 1)\ k\ dz - \frac{1}{2} \int_z a_i\ dz \end{aligned} \tag{9.11}$$

The integrand of eq.9.7 consists of an oscillating part, due to the imaginary part of eq.9.11, and a "damping" part, due to the real part of 9.11. The damping part can only reduce the value of the integral, so that if the internal absorption coefficient a_i goes towards infinity, C_c/C_g will obtain 2 as the asymptotic value.

If φ is real, then the real part of the integral in eq.9.7 may vary between -1 and +1, and so C_c/C_g may vary between 0 and 4. This does not mean that particles cannot have a ratio C_c/C_g higher than 4, only that particles with m close to 1 will have 4 as an upper limit. Higher values can be obtained from the more general Mie theory for particles with higher refractive indices.

The ratio C_b/C_g may vary between 0 and 4 if κ or a_i is zero. If κ is different from zero, the upper limit will be smaller, but not smaller than 1.

What has been said here about the variation of the ratios C_c/C_g, C_a/C_g and C_b/C_g, also applies to the efficiencies Q_c, Q_a and Q_b.

The expressions 9.1, 9.2, 9.7, 9.8 and 9.9 can be applied to particles of any shape and orientation, but convenient analytic solutions of the integrals can only be obtained for particles with a certain symmetry. Examples of such shapes will be given in later chapters. Here we shall now look at irregular particles of such dimensions that the exponential functions in the integrals can be expanded into power series of finite length.

9.2. Irregular particles oriented at random

Eqs.9.1-2 can be expanded into

$$C_c = 2 \text{ Re } \{ \int_{C_g} (1 - 1 + (i\varphi) - \frac{(i\varphi)^2}{2} + \frac{(i\varphi)^3}{6} - \ldots) dg \} \qquad 9.12$$

$$C_a = \int_{C_g} (1 - 1 + \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{6} - \ldots) dg \qquad 9.13$$

If the particle is homogeneous, φ and α are given by

$$\varphi = (n - 1 - i\kappa)kZ \qquad 9.14$$

$$\alpha = 2\,k\,\kappa\,Z = a_i Z \tag{9.15}$$

and eqs.9.12 and 13 become

$$C_c = 2\,k\,\kappa \int_{C_g} Z\,dg + ((n-1)^2 - \kappa^2)k^2 \int_{C_g} Z^2 dg$$
$$- ((n-1)^2\,\kappa - \frac{\kappa^3}{3})\,k^3 \int_{C_g} Z^3 dg + \ldots.. \tag{9.16}$$

$$C_a = 2\,k\,\kappa \int_{C_g} Z\,dg - 2\,k^2\kappa^2 \int_{C_g} Z^2\,dg$$
$$+ \frac{4}{3}\,k^3\kappa^3 \int_{C_g} Z^3\,dg + \ldots.. \tag{9.17}$$

The scattering cross section of the particle becomes

$$C_b = C_c - C_a = ((n-1)^2 + \kappa^2)k^2 \int_{C_g} Z^2\,dg$$
$$- ((n-1)^2\kappa + \kappa^3)k^3 \int_{C_g} Z^3\,dg + \ldots$$

The mean attenuation and absorption cross sections for a particle oriented at random are

$$C = 2\,\mathrm{Re}\,\{i\,\frac{(m-1)k}{1!}\,\overline{\int_{C_g} Z\,dg} + \frac{(m-1)^2k^2}{2!}\,\overline{\int_{C_g} Z^2\,dg}$$
$$- i\,\frac{(m-1)^3k^3}{3!}\,\overline{\int_{C_g} Z^3\,dg} + \ldots.\} \tag{9.19}$$

$$A = \frac{2k\kappa}{1!}\,\overline{\int_{C_g} Z\,dg} - \frac{(2k\kappa)^2}{2!}\,\overline{\int_{C_g} Z^2\,dg} + \frac{(2k\kappa)^3}{3!}\,\overline{\int_{C_g} Z^3\,dg} + \ldots \tag{9.20}$$

The first integral defines the volume V of the particle, which is independent of the particle's orientation. The other integrals , however, depend on the orientation. We may write their mean values as

$$\overline{\int_{C_g} z^2 \, dg} = f_2 \left(\frac{V}{G}\right)^2 G \qquad 9.21$$

$$\overline{\int_{C_g} z^3 \, dg} = f_2 \left(\frac{V}{G}\right)^3 G \qquad 9.22$$

and so on. $\underline{f}$ is a dimensionless shape factor and G is the mean geometrical cross section of the particle. The f's are defined by

$$f_n = \frac{\frac{1}{G} \overline{\int_{C_g} z^n \, dg}}{(V/G)^n} \qquad 9.23$$

The value of f_1 is then by definition 1.

The attenuation and absorption efficiencies can now be written

$$Q_c = \frac{C}{G} = 2 \, \mathrm{Re} \left\{ \sum_{n=1}^{N} - (-i)^n (m-1)^n k^n \frac{f_n}{n!} \left(\frac{V}{G}\right)^n \right\} \qquad 9.24$$

$$Q_a = \frac{A}{G} = \sum_{n=1}^{N} - (-1)^n (2\kappa)^n k^n \frac{f_n}{n!} \left(\frac{V}{G}\right)^n \qquad 9.25$$

So far, according to the well-known principle of conservation of difficulty, we have of course only reformulated the problem. Instead of one integral, given by eq.9.1 or.9.2, we now have a series of integrals, as given by eq.9.23. The advantage, however, is that these integrals only depend on shape, and not on the optical properties, and that their solution may be less difficult to find than the original ones.

Still the integration by analytical methods can be very difficult, even for fairly simple geometrical shapes. The relation below may then be useful:

$$\left(\frac{V}{C_g}\right)^n C_g \leqq \int_{C_g} z^n dg \leqq z^n_{max} \, C_g \qquad 9.26$$

Here Z_{max} is the greatest possible distance through the particle. V/C_g becomes the arithmetic mean value of Z in the area C_g. The relation is valid for all orientations of the particle, and so also for the mean value of the terms. We then find that

$$G^{n-1}\ \overline{C_g^{\,1-n}} \leqq f_n \leqq \frac{Z^n_{max}}{(V/G)^n} \tag{9.27}$$

The left equal sign is always valid when $n=1$. The greater n becomes, the closer f_n will become to the term on the right side.

9.3. Application to cylinders and disks

As a crude estimate of f_n, we may assume it equal to the term on the left side of eq.9.27. Spheres have $C_g=G$, and so all $f_n \approx 1$. By comparing with f_2, f_3 and f_4 for spheres as given in Table 1, Chapter 6.5, we see that the error by this approximation increases with increasing n, and that f_n becomes underestimated. This will be the case for all kinds of particle shapes.

Disks and cylinders which have axes tilted an angle θ from the incident light, have the projected geometrical cross section

$$C_g = \frac{\pi\ D^2}{4} \cos\theta + DL \sin\theta \tag{9.28}$$

where D is the diameter and L is the length or thickness. In order to find f_2, we have to calculate the value of

$$\overline{C_g^{\,-1}} = \int_0^{\pi/2} \frac{\sin\theta\ d\theta}{\frac{\pi}{4} D^2 \cos\theta + DL \sin\theta} \tag{9.29}$$

By introducing the new variable

$$t = \tan\frac{\theta}{2} \tag{9.30}$$

the expression can be written

$$\overline{C_g^{-1}} = \frac{4}{\pi D^2(1+p^2)} \int_0^1 \left(\frac{2t+2p}{1+t^2} - \frac{1}{q+t} + \frac{1}{r-t} \right) dt \qquad 9.31$$

where

$$p = \frac{4}{\pi}\frac{L}{D} \qquad 9.32$$

$$q = (1 + p^2)^{1/2} - p \qquad 9.33$$

$$r = (1 + p^2)^{1/2} + p \qquad 9.34$$

The integration gives

$$\overline{C_g^{-1}} = \frac{4\pi}{\pi^2 D^2 + 16L^2} \left[\ln \frac{1+t^2}{1+rt-qt-t^2} + 2\,p \arctan t \right]_0^1$$

$$= \frac{4\pi}{\pi^2 D^2 + 16L^2} \left(\ln \frac{\pi}{4}\frac{D}{L} + 2\,\frac{L}{D}\right) \qquad 9.35$$

and finally we find that

$$f_2 = G\,\overline{C_g^{-1}} = \frac{1/2+L/D}{1+(4L/\pi D)^2} \left(\ln \frac{\pi}{4}\frac{D}{L} + 2\,\frac{L}{D}\right) \qquad 9.36$$

For a very thin disk $D/L \gg 1$, and f_2 becomes

$$f_2 \approx \frac{1}{2} \ln \frac{\pi}{4}\frac{D}{L} \approx \frac{1}{2} \ln \frac{D}{L} \qquad 9.37$$

For a very long cylinder $D/L \ll 1$, and f_2 becomes

$$f_2 \approx \frac{\pi^2}{16} \left(\frac{D}{L} \ln \left(\frac{\pi}{4}\frac{D}{L}\right) + 2\right) \approx \frac{\pi^2}{8} \qquad 9.38$$

The shape factor f_3 is, with the approximation of eq.9.27, given as

$$f_3 = G^2\,\overline{C_g^{-2}} \qquad 9.39$$

By means of eq.9.28 we see that

$$\overline{C_g^{-2}} = \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{(\frac{\pi}{4} D^2 \cos\theta + DL \sin\theta)^2} \qquad 9.40$$

This integral can be written

$$\overline{C_g^{-2}} = (\frac{4}{\pi D^2})^2 \int_0^1 \frac{4t\, dt}{(1-t^2+2pt)^2}$$

$$= (\frac{4}{\pi D^2})^2 \frac{1}{2s^3} \int_0^1 [- \frac{1+q^2}{(q+t)^2} + \frac{r-q}{q+t} + \frac{1+r^2}{(r-t)^2} + \frac{r-q}{r-t}]\, dt$$

$$= (\frac{4}{\pi D^2})^2 \Big[\frac{2(1+pt)}{s^2(1-t^2+2pt)} - \frac{p}{s^3} \ln \frac{r-t}{q+t} \Big]_0^1$$

$$=(\frac{4}{\pi D^2})^2 [\frac{1-p}{s^2 p} + \frac{p}{s^3} \ln \frac{1+r}{1-q}] \qquad 9.41$$

where t, p, q and r are given by eqs.9.30 and 32-34, respectively, and s by

$$s = (1 + p^2)^{1/2} \qquad 9.42$$

Since

$$G^2 = (\frac{\pi D^2}{8} + \frac{\pi DL}{4})^2 = (\frac{\pi D}{4})^4 (\frac{2}{\pi} + p)^2 \qquad 9.43$$

the shape factor f_3 becomes

$$f_3 = G^2 \overline{C_g^{-2}} = \frac{\pi^2}{16} (\frac{2}{\pi} + p)^2 [\frac{1-p}{s^2 p} + \frac{p}{s^3} \ln \frac{1+p+s}{1+p-s}] \qquad 9.44$$

If the particles are very thin disks, that is $D \gg L$, then $p \ll 1$ and

$$s \approx 1 + \frac{1}{2} p^2 \approx 1 \qquad 9.45$$

f_3 is approximately

$$f_3 \approx \frac{1}{4} [\frac{1}{p} + p \ln \frac{2}{p}] \approx \frac{1}{4} \frac{1}{p} = \frac{\pi D}{16L} \qquad 9.46$$

For long cylinders L>>D, which leads to p>>1 and

$$s \approx p \qquad 9.47$$

f_3 is now approximately

$$f_3 \approx \frac{\pi^2 p^2}{16}\left[-\frac{1}{p^2} + \frac{1}{p^2}\ln 2p\right] \approx \frac{\pi^2}{16}\ln 2p \approx \frac{\pi^2}{16}\ln\frac{L}{D} \qquad 9.48$$

The shape factor f_4 is approximated by

$$f_4 = G^3\,\overline{C_g^{-3}} = G^3\int_0^{\pi/2}\frac{\sin\theta\, d\theta}{\left(\frac{\pi}{4}D^2\cos\theta + DL\sin\theta\right)^3} \qquad 9.49$$

By means of the transformation given by eq.9.30, and by the definitions of eqs.9.32-34 and 9.42, the integral above can be written

$$\overline{C_g^{-3}} = \frac{4^4}{\pi^3 D^6}\int_0^1 \frac{t(1+t^2)dt}{(1-t^2+2pt)^3} = \frac{4^4}{\pi^3 D^6}\int_0^1\frac{t^3+t}{(q+t)^3(r-t)^3}\,dt$$

$$= \frac{4^4}{\pi^3 D^6 8s^3}\int_0^1\left[-\frac{q^3+q}{(q+t)^3} + \frac{1}{(q+t)^3} + \frac{r^3+r}{(r-t)^3} - \frac{1}{(r-t)^2}\right]dt$$

$$= \frac{4^4}{\pi^3 D^6}\left[\frac{t^2}{2(1-t^2+2pt)^2}\right]_0^1 = \frac{4^4}{\pi^3 D^6}\,\frac{1}{2(2p)^2} = \frac{32}{\pi^3 D^6 p^2} \qquad 9.50$$

The factor f_4 becomes

$$f_4 \approx \left(\frac{\pi D^2}{4}\right)^3\left(\frac{1}{2}+\frac{L}{D}\right)^3 \cdot \frac{32}{\pi^3 D^6 p^2}$$

$$= \frac{1}{2p^2}\left(\frac{1}{2}+\frac{L}{D}\right)^3 = \frac{1}{2}\left(\frac{\pi}{4}\frac{D}{L}\right)^2\left(\frac{1}{2}+\frac{L}{D}\right)^3 \qquad 9.51$$

For a very thin disk

$$f_4 \approx \frac{\pi^2}{4^4} \frac{D^2}{L^2} \tag{9.52}$$

while a very long cylinder has

$$f_4 \approx \frac{\pi^2}{32} \frac{L}{D} \tag{9.53}$$

For thin disks

$$\frac{V}{G} \approx 2\ L \tag{9.54}$$

Eqs.9.24 and 25 now finally give us the efficiencies of thin disks as

$$Q_c = 4\ \kappa\ k\ L + 2[(n-1)^2 - \kappa^2]\ k^2 L^2 \ln D/L$$

$$- \frac{\pi}{2} [(n-1)^2 \kappa - \frac{\kappa^3}{3}]\ k^3 L^3\ \frac{D}{L}$$

$$- \frac{\pi^2}{192} [(n-1)^4 - 6(n-1)^2 \kappa^2 + \kappa^4]\ k^4 L^4\ \frac{D^2}{L^2} - \ldots\ldots$$

$$= 4\ \gamma + 2(\varrho^2 - \gamma^2) \ln D/L - \frac{\pi}{2} (\varrho^2 \gamma - \frac{\gamma^3}{3}) \frac{D}{L}$$

$$- \frac{\pi^2}{192} (\varrho^4 - 6\ \varrho^2 \gamma^2 + \gamma^4) \frac{D^2}{L^2} - \ldots\ldots \tag{9.55}$$

$$Q_a = 4\ \gamma - 4\ \gamma^2 \ln D/L + \frac{2\ \pi}{3} \gamma^3 \frac{D}{L} - \frac{\pi^2}{24} \gamma^4 \frac{D^2}{L^2} + \ldots. \tag{9.56}$$

$$Q_b = Q_c - Q_a = 2(\varrho^2 + \gamma^2) \ln D/L - \frac{\pi}{2} (\varrho^2 \gamma \pm \gamma^3) \frac{D}{L}$$

$$- \frac{\pi^2}{192} (\varrho^4 - 6\ \varrho^2 \gamma^2 - 7\ \gamma^4) \frac{D^2}{L^2} + \ldots\ldots \tag{9.57}$$

where

$$\varrho = (n - 1)\ k\ L \qquad 9.58$$

$$\gamma = \kappa\ k\ L \qquad 9.59$$

For long cylinders

$$\frac{V}{G} \approx D \qquad 9.60$$

and the efficiencies become

$$Q_c = 2\ \gamma + \frac{\pi^2}{8}\ (\varrho^2 - \gamma^2) - \frac{\pi^2}{16}\ (\varrho^2 \gamma - \frac{\gamma^3}{3})\ \ln L/D$$

$$- \frac{\pi^2}{384}\ (\varrho^4 - 6\ \varrho^2 \gamma^2 + \gamma^4)\ \frac{L}{D} + \ldots\ldots \qquad 9.61$$

$$Q_a = 2\ \gamma - \frac{\pi^2}{4}\ \gamma^2 + \frac{\pi^2}{12}\ \gamma^3\ \ln L/D - \frac{\pi^2}{48}\ \gamma^4\ \frac{L}{D} + \ldots\ldots \qquad 9.62$$

$$Q_b = \frac{\pi^2}{8}\ (\varrho^2 + \gamma^2) - \frac{\pi^2}{16}\ (\varrho^2\ \gamma + \gamma^3)$$

$$- \frac{\pi^2}{384}\ (\varrho^4 - 6\ \varrho^2 \gamma^2 - 7\ \gamma^4)\ \frac{L}{D} + \ldots\ldots \qquad 9.63$$

where

$$\varrho = (n - 1)\ k\ D \qquad 9.64$$

$$\gamma = \kappa\ k\ D \qquad 9.65$$

10. THE HOMOGENEOUS SPHERE

10.1. Attenuation by homogeneous spheres

We assume that the incident light is parallel with the z-axis so that the geometrical cross section G of the sphere lies in the x-y plane. Origo is placed at the center of the sphere (Fig.10). An arbitrary ray (AB at Fig.10), which passes at a distance x from the center, will have a phase difference φ from the undisturbed light field given by

$$\varphi = (m - 1)\, k\, Z = (m - 1)\, k\, 2\, R \sin \tau \qquad 10.1$$

according to Fig.10. R is the radius of the sphere. However, all rays passing through the sphere with the same distance x from the center will experience the same phase difference. Together they form a ring with projected area dg in the x-y plane equal to

$$dg = 2\pi\, x\, dx = 2\pi (R \cos \tau)(-d(R \cos \tau))$$

$$= 2\pi\, R^2 \sin \tau \cos \tau\, d\tau \qquad 10.2$$

Eq.9.1 gives

$$C_c = 4\pi\, R^2\, \mathrm{Re}\, \{ \int_0^{\pi/2} (1 - e^{-i\zeta \sin \tau}) \sin \tau \cos \tau\, d\tau \} \qquad 10.3$$

where

$$\zeta = (m - 1)\, k\, D = \rho - i\, \gamma \qquad 10.4$$

$$\rho = (n - 1)\, k\, D \qquad 10.5$$

$$\gamma = \kappa\, k\, D \qquad 10.6$$

D is the diameter of the sphere. The integral of 10.3 is found by partial integration

$$C_c = \pi D^2 \, \mathrm{Re} \left\{ \frac{1}{2} - \frac{i}{\zeta} e^{-i\zeta} - \frac{e^{-i\zeta}}{\zeta^2} + \frac{1}{\zeta^2} \right\} \qquad 10.7$$

Since the particle is a sphere, the cross sections are independent of the particle's orientation. The attenuation efficiency is then obtained by dividing 10.7 with the geometrical cross section $\pi D^2/4$

$$Q_c = 2 - 4 \, \mathrm{Re} \left\{ \frac{ie^{-i\zeta}}{\zeta} + \frac{e^{-i\zeta}}{\zeta^2} - \frac{1}{\zeta^2} \right\} \qquad 10.8$$

By means of eq.10.4 we obtain the real part of eq.10.8. We find that

$$Q_c = 2 - \frac{4\,\varrho\, e^{-\gamma}}{\varrho^2+\gamma^2} \left(1 + \frac{2\,\gamma}{\varrho^2+\gamma^2}\right) \sin \varrho$$

$$+ \frac{4\, e^{-\gamma}}{\varrho^2+\gamma^2} \left(\gamma - \frac{\varrho^2-\gamma^2}{\varrho^2+\gamma^2}\right) \cos \varrho + 4 \, \frac{\varrho^2-\gamma^2}{(\varrho^2+\gamma^2)^2} \qquad 10.9$$

When $\gamma=0$, the expression reduces to

$$Q_c = Q_b = 2 - \frac{4}{\varrho} \sin \varrho - \frac{4}{\varrho^2} \cos \varrho + \frac{4}{\varrho^2} \qquad 10.10$$

This is the often-quoted formula by VAN DE HULST for a non-absorbing sphere. It shows clearly how Q_c will oscillate as a function of ϱ with 2 as its asymptotic value.

If the particle is very waterish, so that $\varrho \approx 0$, while $\gamma \gg \varrho$, eq.10.9 gives that

$$Q_c \approx 2 + \frac{4\, e^{-\gamma}}{\gamma^2} (\gamma + 1) - \frac{4}{\gamma^2} \qquad 10.11$$

10.2. Absorption by homogeneous spheres

By the same procedure as we used to obtain C, we find that eq.9.2 gives for spheres

$$A = \frac{\pi D^2}{2} \left(\frac{1}{2} + \frac{e^{-2\gamma}}{2\gamma} + \frac{e^{-2\gamma}}{4\gamma^2} - \frac{1}{4\gamma^2}\right) \qquad 10.12$$

The absorption efficiency becomes

$$Q_a = 1 + \frac{e^{-2\gamma}}{\gamma} + \frac{e^{-2\gamma}}{2\gamma^2} - \frac{1}{2\gamma^2} \qquad 10.13$$

It is seen that the asymptotic value of Q_a is 1.

10.3. Scattering by homogeneous spheres

The scattering cross section is

$$B = C - A \qquad 10.14$$

where C and A are given by eq.10.7 and 10.11 respectively. Similarly the scattering efficiency becomes

$$Q_b = Q_c - Q_a \qquad 10.15$$

where Q_c is given by eq.10.8 or 10.9, and Q_a by eq.10.13. If $\kappa \neq 0$, the asymptotic value of Q_b is 1, if $\kappa = 0$ the value is 2.

Q_b is given by eq.10.10 when $\kappa = 0$, as already noted. Another interesting case is when $\rho \approx 0$ and $\gamma \gg \rho$. Eqs.10.9 and 10.13 then give that

$$Q_b \approx 1 + \frac{1}{\gamma}\left(4\, e^{-\gamma} - e^{-2\gamma}\right) + \frac{1}{2\gamma^2}\left(8\, e^{-\gamma} - e^{-2\gamma} - 7\right) \qquad 10.16$$

10.4. Very small homogeneous spheres

When ϱ and γ are small numbers, e.g. less than 1, the harmonic and exponential functions of eq. .10.9 can be expanded in series, and Q_c can be approximated by

$$Q_c \approx \frac{4}{3}\gamma + \frac{1}{2}(\varrho^2 - \gamma^2) - \frac{2}{15}(3\varrho^2\gamma - \gamma^3) \qquad 10.17$$

(The expression is obtained more easily directly from eq.10.8.) Eq.10.13 gives similarly that

$$Q_a \approx \frac{4}{3}\gamma - \gamma^2 + \frac{8}{15}\gamma^3 \qquad 10.18$$

The scattering efficiency will then be

$$Q_b \approx \frac{1}{2}(\varrho^2 + \gamma^2) - \frac{2}{5}(\varrho^2\gamma + \gamma^3) \qquad 10.19$$

For non-absorbing spheres (γ=0)

$$Q_c = Q_b \approx \frac{1}{2}\varrho^2 \qquad 10.20$$

For non-refracting spheres (ϱ=0)

$$Q_c \approx \frac{4}{3}\gamma - \frac{1}{2}\gamma^2 + \frac{2}{15}\gamma^3 \qquad 10.21$$

$$Q_b \approx \frac{1}{2}\gamma^2 - \frac{2}{5}\gamma^3 \qquad 10.22$$

10.5. Very large homogeneous spheres

When $|\zeta| \gg 1$, eq.10.8 gives that

$$Q_c \approx 2 - 4 \operatorname{Re} \left\{ \frac{i\, e^{-i\zeta}}{\zeta} - \frac{1}{\zeta^2} \right\}$$

$$= 2 - \frac{4\, e^{-\gamma}}{\varrho^2 + \gamma^2} [\varrho \sin \varrho - \gamma \cos \varrho] + 4 \frac{\varrho^2 - \gamma^2}{(\varrho^2 + \gamma^2)^2} \qquad 10.23$$

Similarly, when $\gamma \gg 1$, eq.10.13 gives that

$$Q_a \approx 1 - \frac{1}{2\, \gamma^2} \qquad 10.24$$

If the sphere is non-absorbing

$$Q_c = Q_b \approx 2 - \frac{4}{\varrho} \sin\varrho \qquad 10.25$$

while for a non-refracting large sphere

$$Q_c \approx 2 - \frac{4}{\gamma^2} \qquad 10.26$$

$$Q_b \approx 1 - \frac{7}{2\, \gamma^2} = 7\, Q_a - 6 \qquad 10.27$$

In the last case the spectral variation of scattering will be proportional to the variation of absorption.

10.6. The sphere with a dense core

If the sphere, rather than being homogeneous, has most of its non-water material concentrated into a core, the equations will be very similar to that of the homogeneous sphere. We will assume that the material outside the core has a refractive index so close to 1 that it can be neglected, and that only the core will contribute to the attenuation. The attenuation and absorption cross sections become similar to eqs.10.7 and 10.12, except that D is replaced by fD, where f is the core diameter's fraction of the total diameter. ζ and γ are replaced by

$$\zeta_c = (m_c - 1)\ k\ f\ D \qquad 10.28$$

$$\gamma_c = \kappa_c\ k\ f\ D \qquad 10.29$$

where m_c and κ_c are the indices of the concentrated core. If the indices are assumed to follow the simple Gladstone-Dale equation, they are related to the indices m and κ of the homogeneous sphere by

$$(m_c - 1)\ f^3 = (m - 1) \qquad 10.30$$

$$\kappa_c\ f^3 = \kappa \qquad 10.31$$

We then see that ζ_c and γ_c can be written

$$\zeta_c = \zeta/f^2 \qquad 10.32$$

$$\gamma_c = \gamma/f^2 \qquad 10.33$$

where ζ and γ are the values for the homogeneous sphere.

Q_c and Q_a are obtained by dividing C and A with $G=\pi D^2/4$.

11. THE HOLLOW SPHERE

11.1. The general expressions for a two-layered sphere

We now assume that the sphere consists of a shell of refractive index m_1, which surrounds an inner sphere of refractive index m_2. The outer radius of the shell is R, and the inner radius fR, where f is a number between 0 and 1.

A ray which passes through the sphere at a distance x from the center, through the points AB of Fig.11, will obtain the phase difference

$$\varphi_1 = (m_1 - 1)\ k\ d \sin\tau = \zeta_1 \sin\tau \qquad 11.1$$

$$\text{for } x \geqq f\,R$$

$$\varphi_2 = (m_2-1)\ k\ f\ D \sin\mu + (m_1-1)\ k\ D\ (\sin\tau - f\sin\mu)$$

$$= \zeta_2\ f \sin\mu + \zeta_1\ (\sin\tau - f\sin\mu) \qquad 11.2$$

$$\text{for } x \leqq f\,R$$

The absorption path becomes similarly

$$\alpha_1 = \kappa_1\ k\ D \sin\tau = \gamma_1 \sin\tau \qquad 11.3$$

$$\text{for } x \geqq f\,R$$

$$\alpha_2 = \kappa_2\ k\ f\ D \sin\mu + \kappa_1\ k\ D\ (\sin\tau - f\sin\mu)$$

$$= \gamma_2\ f\sin\mu + \gamma_1\ (\sin\tau - f\sin\mu) \qquad 11.4$$

$$\text{for } x \leqq f\,R$$

where κ_1 and κ_2 are the absorption indices of m_1 and m_2. The angles τ and μ are linked together (see Fig.11) by the relation

$$x = R\cos\tau = f\,R\cos\mu \qquad 11.5$$

which gives that

$$\sin \mu = (f^2 - 1 + \sin^2 \tau)^{1/2}/f \tag{11.6}$$

These equations give that $x=fR$ and $\mu=0$ when $f^2-1+\sin^2\tau_0=0$, that is when

$$\sin \tau_0 = (1 - f^2)^{1/2} \tag{11.7}$$

Instead of eq.10.3 we now find that the attenuation cross section is

$$C = \pi D^2 \text{ Re } \{ \int_0^{\tau_0} (1 - e^{-i\varphi_1}) \sin \tau \cos \tau \, d\tau$$

$$+ \int_{\tau_0}^{\pi/2} (1 - e^{-i\varphi_2}) \sin \tau \cos \tau \, d\tau\}$$

$$= \pi D^2 \text{ Re } \{ \int_0^{\pi/2} \sin \tau \cos \tau \, d\tau - \int_0^{\tau_0} e^{-i\varphi_1} \sin \tau \cos \tau \, d\tau$$

$$- \int_{\tau_0}^{\pi/2} e^{-i\varphi_2} \sin \tau \cos \tau \, d\tau\} \tag{11.8}$$

and the attenuation efficiency becomes

$$Q_c = 2 -4 \text{ Re } \{ \int_0^{\tau_0} e^{-i\varphi_1} \sin \tau \cos \tau \, d\tau$$

$$+ \int_{\tau_0}^{\pi/2} e^{-i\varphi_2} \sin \tau \cos \tau \, d\tau\} \tag{11.9}$$

In the same way we deduce that

$$Q_a = 1 - 2 [\int_0^{\tau_0} e^{-\alpha_1} \sin \tau \cos \tau \, d\tau$$

$$+ \int_{\tau_0}^{\pi/2} e^{-\alpha_2} \sin \tau \cos \tau \, d\tau] \tag{11.10}$$

Eqs.11.9 and 19 are the general expressions for a two-layered sphere. The integrals can easily be calculated by numerical methods, but analytical solutions are more complicated to obtain. The special case that the inner sphere has the refractive index 1, however, gives a fairly simple solution as we now shall see.

11.2. Attenuation by hollow spheres

If the inner sphere consists of water, that is the sphere is hollow, then $m_2=1$, and we may simplify the notation so that $m=m_1$, and $\zeta=\zeta_1$.

In the first integral of eq.11.9 it is convenient to introduce the new variable

$$u = \varphi_1 = \zeta \sin \tau \qquad 11.11$$

so that

$$\sin \tau \cos \tau \, d\tau = u \, du/\zeta^2 \qquad 11.12$$

The value of u when $\tau=\tau_0$ will be called s:

$$s = \zeta \sin \tau_0 = \zeta \, (1 - f^2)^{1/2} \qquad 11.13$$

In the second integral of eq.11.9 we introduce, by means of eq.11.6,

$$v = \varphi_2 = \zeta \, (\sin \tau - f \sin \mu)$$

$$= \zeta \, (\sin \tau - (f^2 - 1 + \sin^2 \tau)^{1/2}) \qquad 11.14$$

The value of v when $\tau=\pi/2$ will be called S:

$$S = \zeta(1 - f) \qquad 11.15$$

When $\tau=\tau_0$, $v=s$. Eq.11.14 gives $\sin\tau$ as a unique function of v:

$$\sin\tau = \frac{v}{2\,\zeta} + \frac{\zeta(1-f^2)}{2\,v} \qquad 11.16$$

from which we find that

$$\sin\tau\cos\tau\,d\tau = \left(\frac{v}{2\,\zeta} + \frac{\zeta(1-f^2)}{2\,v}\right)\left(\frac{1}{2\,\zeta} - \frac{\zeta(1-f^2)}{2\,v^2}\right) dv$$

$$= \frac{1}{4\,\zeta^2}\left(v - \frac{s^4}{v^3}\right) dv \qquad 11.17$$

Eq.11.9 can now be written

$$Q_c = 2 - \mathrm{Re}\,\{\frac{4}{\zeta^2}\int_0^s e^{-iu}\,u\,du + \frac{1}{\zeta^2}\,e\int_s^S {}^{-iv}\,v\,dv$$

$$- \frac{s^4}{\zeta^2}\int_s^S \frac{e^{-iv}}{v^3}\,dv\} \qquad 11.18$$

The first and second integral above have already been solved in the preceding chapter. The third integral gives

$$\int \frac{e^{-iv}}{v^3}\,dv = -\frac{e^{-iv}}{2\,v^2} + \frac{i\,e^{-iv}}{2\,v} - \frac{1}{2}\int \frac{e^{-iv}}{v}\,dv \qquad 11.19$$

Eq.11.18 becomes

$$Q_c = 2\ \mathrm{Re}\,\{\frac{4}{\zeta^2}(i\,s\,e^{-is} + e^{-is} - 1$$

$$+ \frac{1}{\zeta^2}(i\,S\,e^{-iS} + e^{-iS} - i\,s\,e^{-is} - e^{-is})$$

$$+ \frac{s^4}{2\,\zeta^2}\left(\frac{e^{-iS}}{S^2} - \frac{i\,e^{-iS}}{S} - \frac{e^{-is}}{s^2} + \frac{i\,e^{-is}}{s}\right)$$

$$+ \frac{s^4}{2\,\zeta^2}(E_1(is) - E_1(iS))\} \qquad 11.20$$

Here we have introduced the exponential integral E_1, defined by eq.8.7.

If the shell fills out the whole sphere, that is $f \to 0$, then $S \to s \to \zeta$, and eq.11.20 will be reduced to eq.10.8 as it should.

11.3. Absorption by hollow spheres

Since $m_2=1$ and $\kappa_2=0$, we will write $\gamma=\gamma_1$ and $\kappa=\kappa_1$. In order to find the integrals of eq.11.10, we may use the same procedure as in Chapter 11.2. By writing

$$t = 2\gamma(1 - f^2)^{1/2} \qquad 11.21$$

$$T = 2\gamma(1 - f) \qquad 11.22$$

we find that

$$\begin{aligned} Q_a = 1 &+ \frac{1}{2\gamma^2}(t\,e^{-t} + e^{-t} - 1) \\ &+ \frac{1}{8\gamma^2}(T\,e^{-T} + e^{-T} - t\,e^{-t} - e^{-t}) \\ &+ \frac{t^4}{16\gamma^2}\left(\frac{e^{-T}}{T} - \frac{e^{-T}}{T^2} - \frac{e^{-t}}{t} + \frac{e^{-t}}{t^2}\right) \\ &+ \frac{t^4}{16\gamma^2}(E_1(t) - E_1(T)) \end{aligned} \qquad 11.23$$

If $f \to 0$, then $T \to t \to 2\gamma$, and eq.11.23 will reduce to eq.10.13, which gives the absorption efficiency of the homogeneous sphere.

11.4. Very small hollow spheres

When ϱ and γ have values much less than 1, then s, S, t and T will also be much less than 1. By means of eq.8.8 the exponential integral E_1 of eq.11.20 can be expanded in series. Further expansion of eq.11.20 gives

$$Q_c = \mathrm{Re}\ \{\frac{4}{3}\ i\ \zeta\ (1 - f^3)$$

$$+ \frac{\zeta^2}{2}\ (1 - f - f^3 + f^4 + \frac{(1-f^2)^2}{2}\ \ln \frac{1+f}{1-f})$$

$$+ \frac{i\ \zeta^3}{15}\ (2 + 6f + 2f^2 - 4\ (1-f^2)^{5/2})\}$$

$$= \frac{4}{3}\ \gamma\ (1 - f^3)$$

$$+ \frac{\varrho^2-\gamma^2}{2}\ (1 - f - f^3 + f^4 + \frac{(1-f^2)^2}{2}\ \ln \frac{1+f}{1-f})$$

$$+ \frac{3\varrho^2\gamma-\gamma^3}{15}\ ((2+6f+2f^2)(1-f)^3 - 4(1-f^2)^{5/2}) \qquad 11.24$$

With the same conditions eq.11.23 can be approximated by

$$Q_a = \frac{4}{3}\ \gamma\ (1 - f^3)$$

$$- \gamma^2\ (1 - f - f^3 + f^4 + \frac{(1-f^2)^2}{2}\ \ln \frac{1+f}{1-f})$$

$$- \frac{\gamma^3}{15}\ ((8 + 24f + 8f^2)(1 - f)^3 - 16\ (1-f^2)^{5/2}) \qquad 11.25$$

The scattering efficiency becomes

$$Q_b = Q_c - Q_a$$

$$= \frac{\varrho^2+\gamma^2}{2} \left(1 - f - f^3 + f^4 + \frac{(1-f^2)^2}{2} \ln \frac{1+f}{1-f}\right)$$

$$+ \frac{2}{5} (\varrho^2 \gamma + \gamma^3)((1 + 3f + f^2)(1-f)^3 - 2(1-f^2)^{5/2}) \qquad 11.26$$

11.5. Very large hollow spheres

When the argument is great, the exponential integral can be approximated by means of eq.8.9. Eq.11.20 then reduces to

$$Q_c \approx 2 - \mathrm{Re}\left\{\frac{4}{\zeta^2} (i\, s\, e^{-is} + e^{-is} - 1)\right.$$

$$\left. + \frac{1}{\zeta^2} (i\, S\, e^{-iS} + e^{-iS} - i\, s\, e^{is} - e^{-is})\right\} \qquad 11.27$$

which is the attenuation efficiency for very large hollow spheres. We see that the last two parentheses of eq.11.20 have cancelled each other. The absorption efficiency becomes, by means of similar approximations,

$$Q_a = 1 + \frac{1}{2\gamma^2} (t\, e^{-t} + e^{-t} - 1)$$

$$+ \frac{1}{8\gamma^2} (T\, e^{-T} + e^{-T} - t\, e^{-t} - e^{-t}) \qquad 11.28$$

12. THIN DISKS

12.1. Attenuation by disks

We consider a thin circular disk with its axis of symmetry in the x-z plane (Fig.12). The angle between this axis and the z axis is θ. The geometrical cross section of the disk becomes

$$C_g(\theta) \approx \frac{\pi}{4} D^2 \cos\theta \qquad 12.1$$

where D is the diameter. The contribution from the side walls of the disk has been assumed so small that it can be neglected.

Each ray passing through the disk travels the same distance $Z=L/\cos\theta$, where L is the thickness of the disk. The phase difference then becomes

$$\varphi = (m - 1)\, k\, L/\cos\theta = \zeta/\cos\theta \qquad 12.2$$

for all rays, and the attenuation cross section (eq.9.1) can at once be written

$$C_c(\theta) = 2\,\mathrm{Re}\,\{(1 - e^{-i\varphi})\, \frac{\pi D^2}{4} \cos\theta\} \qquad 12.3$$

This expression can be regarded as valid except when θ is close to $\pi/2$. According to eq.12.3 $C_c(\pi/2)=0$, while in reality and according to eq.13.18 it should be

$$C_c(\pi/2) = \pi\, L\, D\, \mathrm{Re}\,\{i\, J_1(U) + H_1(U)\} \qquad 12.4$$

which is the attenuation cross section of a cylinder. U is defined as

$$U = (m - 1)\, k\, D \qquad 12.5$$

J_1 is the Bessel function of the first kind and first order, and H_1 is the Struve function of the first order.

We shall now assume that eq.12.4 gives the cross section for the small interval

$$\frac{\pi}{2} - \varepsilon = \theta_0 \leqq \theta \leqq \frac{\pi}{2} \qquad 12.6$$

where ε is of order L/D, and that eq.12.3 is valid when $\theta < \theta_0$.

The mean attenuation cross section C for randomly oriented disks is found by integrating $C_c(\theta)$ over all possible directions of the symmetry axis.

$$C = \frac{\pi D^2}{2} \text{Re} \{ \int_0^{\theta_0} (1 - e^{-i\varphi}) \cos\theta \sin\theta \, d\theta \}$$

$$+ \pi L D \, \text{Re} \{ iJ_1(U) + H_1(U) \} \int_{\theta_0}^{\pi/2} \sin\theta \, d\theta \} \qquad 12.7$$

Eq.12.2 gives that

$$\sin\theta \cos\theta \, d\theta = \frac{\zeta^2}{\varphi^3} d\varphi \qquad 12.8$$

We may then use φ as the variable in the first integral above. We may further introduce, by means of eq.12.6,

$$\varphi_0 = \frac{\zeta}{\cos\theta_0} = \frac{\zeta}{\sin\varepsilon} \approx \frac{\zeta}{\varepsilon} \qquad 12.9$$

After some calculations we then find that

$$C = \frac{\pi D^2}{4} \text{Re} \{ [1 - e^{-i\zeta} + i\,\zeta\, e^{-i\zeta} + \zeta^2 E_1(i\zeta)]$$

$$- [\varepsilon^2 - \varepsilon^2 e^{-i\varphi_0} + \varepsilon\, i\, \zeta\, e^{-i\varphi_0} + \zeta^2 E_1(i\varphi_0)] \}$$

$$+ \varepsilon \pi L D \, \text{Re} \{ i\, J_1(U) + H_1(U) \} \qquad 12.10$$

where $E_1(z)$ is the exponential integral defined by eq.8.7.

If $|\zeta|$ is of order 1 or greater, then $|\varphi_0| >> 1$ and $|U| >> 1$.

$|J_1(U)|$ and $|H_1(U)|$ become of order $U^{-1/2}$ and 1, respectively. $|E_1(i\varphi_0)|$ will be much smaller than $|E_1(i\zeta)|$ and can be neglected. We may omit all the terms which contains ε. The mean attenuation cross section then becomes

$$C \approx \frac{\pi D^2}{4} \text{Re } \{1 - e^{-i\zeta} + i \zeta e^{-i\zeta} + \zeta^2 E_1(i\zeta)\} \qquad 12.11$$

The mean geometrical cross section can be found either be integrating eq.12.1 for all possible orientations, or more simply by dividing the total surface of the thin disk by 4. We then find the mean value

$$G \approx \frac{\pi D^2}{8} \qquad 12.12$$

The mean attenuation efficiency for thin disks of random orientation becomes, when $|\zeta| \gtrsim 1$,

$$Q_c = \frac{C}{G} = 2 \text{ Re } \{1 - e^{-i\zeta} + i \zeta e^{-i\zeta} + \zeta^2 E_1(i\zeta)\} \qquad 12.13$$

When $|\zeta| \ll 1$, however, the terms which were neglected above become important, as we will see in Chapter 12.3.

12.2. Absorption by disks

The absorption cross section will be defined in a way similar to that of the attenuation cross section. We assume that

$$C_a(\theta) = \frac{\pi D^2}{4} (1 - e^{-\alpha}) \cos \theta \qquad 12.14$$

$$\text{for } 0 \leqq \theta \leqq \theta_0,$$

$$C_a(\theta) = \frac{\pi L D}{2} [-i J_1(W) + H_1(W)] \qquad 12.15$$

$$\text{for } \theta_0 \leqq \theta \leqq \pi/2,$$

where

$$\alpha = \frac{2\,\kappa\,k\,L}{\cos\theta} = \frac{2\,\gamma}{\cos\theta} \qquad 12.16$$

and

$$W = i\,2\,\kappa\,k\,D \qquad 12.17$$

By the same procedure as in the preceding chapter we find that

$$A = \frac{\pi\,D^2}{4} \int_0^{\theta_0} (1 - e^{-\alpha}) \cos\theta \sin\theta \, d\theta$$

$$+ \frac{\pi}{2} L\,D\,[-i\,J_1(W) + H_1(w)] \int_{\theta_0}^{\pi/2} \sin\theta \, d\theta$$

$$= \frac{\pi\,D^2}{8} [(1 - e^{-2\gamma} + 2\,\gamma\,e^{-2\gamma} - 4\,\gamma^2 E_1(2\gamma))$$

$$- (\varepsilon^2 - \varepsilon^2 e^{-\alpha_0} + 2\,\varepsilon\,\gamma\,e^{-\alpha_0} - 4\,\gamma^2 E_1(\alpha_0))]$$

$$+ e\,\frac{\pi}{2} L\,D\,[-i\,J_1(W) + H_1(W)] \qquad 12.18$$

where

$$\alpha_0 = \frac{2\,\gamma}{\cos\theta_0} = \frac{2\,\gamma}{\varepsilon} \qquad 12.19$$

If γ is of order 1 or greater, $\alpha_0 >> 1$ and $|W| >> 1$, and A can be approximated by

$$A \approx \frac{\pi\,D^2}{8} [1 - e^{-2\gamma} + 2\,\gamma\,e^{-2\gamma} - 4\,\gamma^2 E_1(2\gamma)] \qquad 12.20$$

For values of $\gamma \gtrsim 1$, Q_a can be written

$$Q_a = \frac{A}{G} = 1 - e^{-2\gamma} + 2\,\gamma\,e^{-2\gamma} - 4\,\gamma^2 E_1(2\gamma) \qquad 12.21$$

12.3. Very small disks

When $|\zeta|$ and $|\varphi_0|$ are much smaller than 1, the exponential integrals of eq.12.10 can be approximated by eq.8.8. Since U and φ_0 are of the same order, the Bessel and Struve functions can be approximated by eqs.8.15 and 18. If the exponential functions are expanded in series too, we find that

$$C = \frac{\pi D^2}{4} \text{Re} \{2 i \zeta(1 - \varepsilon) - \frac{i \zeta^3}{3} (\varphi_0 - \zeta) + \zeta^2 \ln (1/\varepsilon)\}$$

$$+ \varepsilon \pi L D \text{Re} \{i \frac{U}{2} - \frac{i U^3}{8} + \frac{2}{\pi} \frac{U^2}{3}\} \qquad 12.22$$

In order to proceed, we have to estimate the value of ε. Earlier we have said that ε was of order L/D, and we will now make the assumption that

$$\varepsilon = L/D \qquad 12.23$$

Since

$$\zeta = (m - 1) k = (n - 1) k - i \kappa k = \varrho - i \gamma \qquad 12.24$$

we may then, by means of eqs.12.12 and 22, write Q_c as

$$Q_c \approx 2 \{2 \kappa k L (1 - \frac{L}{D}) + ((n-1)^2 - \kappa^2) k^2 L^2 \ln(D/L)$$

$$+ ((n-1)^2 \kappa - \frac{\kappa^3}{3}) k^3 L^3 (1 - \frac{D}{L})\}$$

$$+ 4 \frac{L^2}{D^2} \{\kappa k D + \frac{4}{3\pi} ((n-1)^2 - \kappa^2) k^2 D^2 - \frac{1}{4} (3(n-1)^2 \kappa - \kappa^3) k^3 D^3\}$$

$$\approx 4 \gamma + 2 (\varrho^2 - \gamma^2) \ln D/L - 5 (\varrho^2 \gamma - \frac{\gamma^3}{3}) \frac{D}{L} \qquad 12.25$$

Similarly we may expand the functions of eq.12.17 in series, and we find that Q_a is

$$Q_a \approx 4 \gamma - 4 \gamma^2 \ln D/L + \frac{20}{3} \gamma^3 \frac{D}{L} \qquad 12.26$$

The scattering efficiency becomes

$$Q_b = Q_c - Q_a \approx 2(\varrho^2+\gamma^2)\ \ln D/L - 5(\varrho^2\gamma+\gamma^3)\ \frac{D}{L} \qquad 12.27$$

It should be noted once more that the expressions of eqs.12.25-27 are based on several rather crude assumptions. The first is that eq.12.3 is valid for angles less than $\pi/2-\varepsilon$, the second that eq.12.4 is valid for all other angles. The third assumption is that $\varepsilon \approx L/D$. A comparison with eqs.9.55-57, however, indicates that our guessings may not be so bad after all. The first and second order terms of this chapter coincide with those of Chapter 9. Only the third order terms differ.

12.4. Very large disks

For great values of ζ eq.12.13 becomes, by means of eq.8.9,

$$Q_c \approx 2\ \mathrm{Re}\ \{1 + \frac{2i\ e^{-i\zeta}}{\zeta} - \frac{6\ e^{-i\zeta}}{\zeta^2}\} \qquad 12.28$$

and eq.12.22 gives similarly for great values of γ

$$Q_a \approx 1 - \frac{e^{-2\gamma}}{\gamma} + \frac{3\ e^{-2\gamma}}{2\ \gamma^2} \qquad 12.29$$

For non-absorbing disks eq.12.28 reduces to

$$Q_c = Q_b \approx 2 + \frac{4\ \sin\ \varrho}{\varrho} - \frac{12\ \cos\ \varrho}{\varrho^2} \qquad 12.30$$

which has the same oscillatory character as the formula for non-absorbing spheres (eq.10.10).

13. LONG CYLINDERS

13.1. Attenuation by cylinders

The axis of symmetry of a long cylinder and the z direction of the incident light defines the x-z plane. The angle between the z axis and the cylinder axis is θ (Fig.13). The length of the cylinder is L and the diameter is D=2R.

A cross section through the cylinder parallel with the y-z plane, will have the shape of an ellipsis with minor half-axis R and major half-axis R/sin θ. The phase difference φ of a ray passing through the cylinder will then be a function of its y coordinate:

$$\varphi = (m-1)\ k\ 2\ R\ (1 - \frac{y^2}{R^2})^{1/2} / \sin\theta \qquad 13.1$$

The attenuation cross section can be written

$$C_c(\theta) = 2\ \mathrm{Re}\ \{ \int_{C_g} (1 - e^{-i\varphi})\ dg\} = 2\ \mathrm{Re}\ \{\int_{C_g} \int (1 - e^{-i\varphi})\ dx\ dy\}$$

$$= 2\ L \sin\theta\ \mathrm{Re}\ \{2 \int_0^R (1 - e^{-i\varphi})\ dy\} \qquad 13.2$$

If we introduce a new variable t, defined by

$$\frac{y}{R} = \sin t \qquad 13.3$$

then φ can be written

$$\varphi = (m - 1)\ k\ D \cos t/\sin\theta = u \cos t \qquad 13.4$$

where

$$u = (m - 1)\ k\ D/\sin\theta = \zeta/\sin\theta \qquad 13.5$$

Eq.13.2 becomes

$$C_c(\theta) = 2\ L\ D \sin\theta \ \mathrm{Re}\ \{\int_0^{\pi/2} (1 - e^{-i\ u \cos t}) \cos t\ dt\}$$

$$= 2\ L\ D \sin\theta [1 - \mathrm{Re}\ \{\int_0^{\pi/2} e^{-i\ u \cos t} \cos t\ dt\}] \qquad 13.6$$

This expression is approximately valid except when θ is close to 0. The cross section for $\theta=0$ is the cross section of a disk, as given in the previous chapter (eq.12.3):

$$C_c(0) = \frac{\pi\ D^2}{2} \mathrm{Re}\ \{1 - e^{-i\psi}\} \qquad 13.7$$

where

$$\psi = (m - 1)\ k\ L \qquad 13.8$$

We shall assume that eq.13.7 is valid in the range $0 \leq \theta \leq \theta_0$, where θ_0 is a small angle of order D/L, and that eq.13.6 is valid in the range $\theta_0 \leq \theta \leq \pi/2$. The mean attenuation cross section for cylinders of random orientation then becomes

$$C = \int_0^{\pi/2} C_c(\theta) \sin\theta\ d\theta$$

$$= \frac{\pi\ D^2}{2} \mathrm{Re}\ \{1 - e^{-i\psi}\} \int_0^{\theta_0} \sin\theta\ d\theta + \int_{\theta_0}^{\pi/2} \sin\theta\ C_c(\theta)\ d\theta$$

$$= \frac{\pi\ D^2}{4} \theta_0^{\ 2}\ \mathrm{Re}\ \{1 - e^{-i\psi}\} + \int_{\theta_0}^{\pi/2} \sin\theta\ C_c(\theta)\ d\theta$$

$C_c(\theta)$ in the last integral is given by eq.13.6.

The total surface of a long cylinder is approximately πDL, so the mean geometrical cross section becomes

$$G = \pi\ D\ L/4 \qquad 13.10$$

The mean attenuation efficiency is then

$$Q_c = \frac{C}{G} = \frac{D}{L}\,\theta_0^{\,2}\ \mathrm{Re}\ \{1 - e^{-i\psi}\} + \frac{4}{\pi DL}\int_{\theta_0}^{\pi/2} \sin\theta\ C_c(\theta)d\theta \qquad 13.11$$

$C_c(\theta)$ can be given an alternative form. By partial integration eq.13.6 becomes

$$C_c(\theta) = 2\ L\ D\ \sin\theta\ \mathrm{Re}\ \int_0^{\pi/2} i\ u\ e^{-i\ u\ \cos\ t}\sin^2 t\ dt\}$$

$$= 2\ L\ D\ \sin\theta\ \mathrm{Re}\ \{i\ u\int_0^{\pi/2} \cos(u\ \cos\ t)\ \sin^2 t\ dt$$

$$+ u\int_0^{\pi/2} \sin(u\ \cos\ t)\ \sin^2 t\ dt\} \qquad 13.12$$

If we compare this expression with the Bessel function J_1 of the first kind and first order, eq.8.14, and the Struve function H_1 of the first order, eq.8.17, we see that eq.13.12 can be written

$$C_c(\theta) = \pi\ L\ D\ \sin\theta\ \mathrm{Re}\ \{i\ J_1(u) + H_1(u)\} \qquad 13.13$$

If $|\zeta| \gtrsim 1$, then the integral of $\mathrm{Re}\ \{i\ J_1(u) + H_1(u)\}$ in eq.13.11 will be of order 1. The first term on the right side of eq.13.11 will then be very small compared with the second, and can be neglected. The lower limit of the integral can also be changed to 0 without any significant error. The attenuation efficiency in this case becomes

$$Q_c \approx 4\int_0^{\pi/2} \sin^2\theta\ \mathrm{Re}\ \{i\ J_1(u) + H_1(u)\}\ d\theta \qquad 13.14$$

where $u=\zeta/\sin\theta$. The asymptotic values of $J_1(u)$ and $H_1(u)$ as u goes to infinity, are 0 and $2/\pi$, respectively, so the integrand remains well defined for all values of θ.

13.2. Absorption by cylinders

The absorption cross section is obtained from

$$C_a(\theta) = \int_{C_g} (1 - e^{-\alpha})\, dg = \iint_{C_g} (1 - e^{-\alpha})\, dx\, dy$$

$$= 2\ L \sin\theta \int_0^R (1 - e^{-\alpha})\, dy \qquad 13.15$$

where α is the absorption path.

$$\alpha = 2\ \gamma\ (1 - \frac{y^2}{R^2})^{1/2} / \sin\theta = 2\ \gamma \cos t / \sin\theta = v \cos t \qquad 13.16$$

Here

$$v = 2\ \gamma / \sin\theta = 2\ \kappa\ k\ D / \sin\theta \qquad 13.17$$

t is defined by eq.13.3. This cross section, which neglects effects from the ends of the cylinder, is valid except for very small values of θ. The cross section for $\theta=0$ is the cross section of a disk, and is according to eq.12.14

$$C_a(0) = \frac{\pi\ D^2}{4} (1 - e^{-\chi}) \qquad 13.18$$

where

$$\chi = 2\ \kappa\ k\ L \qquad 13.19$$

By assuming that eq.13.18 is valid in the small range $0-\theta_0$, and that 13.15 is valid in the range $\theta_0-\pi/2$, the mean absorption cross section becomes

$$A = \overline{C}_a = \frac{\pi\ D^2}{4} (1 - e^{-\chi}) \int_0^{\theta_0} \sin\theta\ d\theta + \int_{\theta_0}^{\pi/2} C_a(\theta) \sin\theta\ d\theta$$

$$= \frac{\pi\ D^2}{8} (1 - e^{-\chi})\ \theta_0^{\ 2} + \int_{\theta_0}^{\pi/2} \sin\theta\ d\theta \qquad 13.20$$

$C_a(\theta)$ in the last integral is given by eq.13.15 as

$$C_a(\theta) = L\ D \sin\theta \int_0^{\pi/2} (1 - e^{-v \cos t}) \cos t\ dt \qquad 13.21$$

By partial integration it becomes

$$C_a(\theta) = L\ D \sin\theta\ v \int_0^{\pi/2} e^{-v \cos t} \sin^2 t\ dt \qquad 13.22$$

If we introduce the imaginary variable w defined by

$$w = i\ v \qquad 13.23$$

eq.13.22 can be written

$$C_a(\theta) = \frac{\pi}{2} L\ D \sin\theta\ [-i\ J_1(w) + H_1(w)] \qquad 13.24$$

When $|w| \geq 1$, the first term on the right side of eq.13.20 will be much smaller than the second, and we can make the approximation

$$A \approx \frac{\pi}{2} L\ D \int_0^{\pi/2} \sin^2\theta\ [-i\ J_1(w) + H_1(w)]\ d\theta \qquad 13.25$$

The absorption efficiency becomes

$$Q_a \approx 2 \int_0^{\pi/2} \sin^2\theta\ [-i\ J_1(w) + H_1(w)]\ d\theta \qquad 13.26$$

13.3. Very small cylinders

When the argument is small, J_1 and H_1 can be approximated by eqs.8.15 and 18, and we can expand the exponential function of eq.13.11. By using eq.13.13, eq.13.11 gives

$$Q_C \approx \frac{D}{L}\,\theta_0^{\,2}\ \mathrm{Re}\ \{i\ \psi + \frac{\psi^2}{2} - \frac{i\ \psi^3}{6}\}$$

$$+ 4\ \mathrm{Re}\ \{\int_{\theta_0}^{\pi/2} (\frac{i\ \zeta}{2}\sin\theta + \frac{2\ \zeta^2}{3\ \pi} - \frac{i\ \zeta^3}{8\ \sin\theta})\ d\theta\}$$

$$= \theta_0^{\,2}\ [\kappa\ k\ D + ((n-1)^2 - \kappa^2)\ \frac{k^2 D\ L}{2}$$

$$- (3(n-1)^2\ \kappa - \kappa^3)\ \frac{k^3 L^2 D}{6}]$$

$$+ [2\ \kappa\ k\ D\cos\theta_0 + \frac{8}{3\ \pi}\ ((n-1)^2 - \kappa^2)k^2D^2(\frac{\pi}{2} - \theta_0)$$

$$+ (3(n-1)^2\ \kappa - \kappa^3)\ \frac{k^3D^3}{2}\ \ln\frac{\theta_0}{2}] \qquad 13.27$$

We do not know the value of θ_0, only that it is likely to be of order D/L. As a crude estimate we may set

$$\theta_0 = D/L \qquad 13.28$$

and

$$\cos\theta_0 \approx 1 - \frac{\theta_0^{\,2}}{2} = 1 - \frac{D^2}{2\ L^2} \qquad 13.29$$

We then find that Q_C becomes

$$Q_C \approx 2\ \kappa\ k\ D + ((n-1)^2 - \kappa^2)\ k^2D^2(\frac{4}{3} + \frac{D}{2L} - \frac{8\ D}{3\pi L})$$

$$- (3(n-1)^2\ \kappa - \kappa^3)\ k^3D^3\ (\frac{1}{6} + \frac{1}{2}\ln\frac{2\ L}{D})$$

$$\approx 2\ \gamma + \frac{4}{3}\ (\varrho^2 - \gamma^2) - \frac{3}{2}\ (\varrho^2\gamma - \frac{\gamma^3}{3})\ \ln L/D \qquad 13.30$$

Similarly the absorption efficiency for small cylinders can be obtained from eqs. 13.20 and 24. We find that

$$Q_a = \frac{D\,\theta_0^{\;2}}{2\,L}\,(\chi - \frac{\chi^2}{2} + \frac{\chi^3}{6}$$

$$+ 2\int_{\theta_0}^{\pi/2} (\gamma \sin\theta - \frac{8}{3}\frac{\gamma^2}{\pi} + \frac{\gamma^3}{\sin\theta})\,d\theta$$

$$= \theta_0^{\;2}(\kappa\,k\,D - \kappa^2 k^2\,L\,D + \frac{2}{3}\,\kappa^3 k^3 L^2\,D)$$

$$+ 2\,\kappa\,k\,D\cos\theta_0 - \frac{16}{3\pi}\,\kappa^2 k^2 D^2\,(\frac{\pi}{2} - \theta_0)$$

$$- 2\,\kappa^3 k^3 D^3\,\ln\frac{\theta_0}{2} \qquad 13.31$$

By means of eqs.13.28-29 we find

$$Q_a \approx 2\,\kappa\,k\,D - \kappa^2 k^2 D^2\,(\frac{8}{3} + \frac{D}{L} - \frac{16\,D}{3\,\pi\,L})$$

$$+ \kappa^3 k^3 D^3\,(\frac{2}{3} + 2\ln\frac{2\,L}{D})$$

$$\approx 2\,\gamma - \frac{8}{3}\,\gamma^2 + 2\,\gamma^3\,\ln L/D \qquad 13.32$$

The scattering efficiency for small cylinders becomes approximately

$$Q_b = Q_c - Q_a \approx \frac{4}{3}\,(\varrho^2+\gamma^2) - \frac{3}{2}\,(\varrho^2\gamma+\gamma^3)\,\ln L/D \qquad 13.33$$

When we compare the expressions 13.30, 32 and 33 with 9.61-63, which were obtained in a quite different way, we see that the first order terms coincide exactly, the second order terms coincide fairly well since $4/3=1.33$ and $\pi^2/8=1.23$, while the third order terms differ significantly.

Since the method of Chapter 9 will underestimate the terms, and the method here is based on crude assumptions, this result is as satisfactory as we could expect.

13.4. Very large cylinders

When the argument is very large, J_1 and H_1 can be approximated by eqs.8.16 and 19. We then find that

$$[i\,J_1(u) + H_1(u)] \approx \frac{2}{\pi} + \frac{2}{\pi u^2} + \left(\frac{2}{\pi\,u}\right)^{1/2}\left(i + \frac{3}{8\,u}\right) e^{-i(u-3\pi/4)} \qquad 13.34$$

According to eq.13.14 the attenuation efficiency becomes

$$Q_c = 2 + \mathrm{Re}\,\left\{\frac{3}{2\,\zeta^2} + \int_0^{\pi/2} \sin^2\theta\,\left(\frac{2}{\pi\,u}\right)^{1/2}\left(i + \frac{3}{8\,u}\right) e^{-i(u-3\pi/4)} d\theta\right\} \qquad 13.35$$

The integral above can be written

$$I = \left(\frac{2}{\pi\,\zeta}\right)^{1/2} e^{i3\pi/4} \int_0^{\pi/2} \left[i(\sin\theta)^{5/2} + \frac{3(\sin\theta)^{7/2}}{8\,\zeta}\right] e^{-i\frac{\zeta}{\sin\theta}}\, d\theta \qquad 13.36$$

The term in the brackets is a monotonously increasing function of θ, while the real part ϱ of ζ will make the exponential function oscillate between positive and negative values. The larger the value of $\varrho/\sin\theta$, the more oscillations per unit of θ will occur, and the less net contribution to the integral per unit of θ will be obtained. The oscillations will be much more rapid when θ is small than when θ is close to $\pi/2$, since in the last case $\sin\theta$ is almost constant in value and close to 1. The most significant contribution to the integral will come from values of θ close to $\pi/2$.

It is now convenient to introduce the new variable

$$x = \frac{\pi}{2} - \theta \qquad 13.37$$

For the exponent of eq.13.36 we make the approximation

$$\frac{1}{\sin\theta} = \frac{1}{\cos x} \approx \frac{1}{1-x^2/2} \approx 1 + \frac{x^2}{2} \qquad 13.38$$

When $x \ll 1$, eq.13.38 will give a fairly correct exponent. But as x increases, the exponent will become smaller than its true value. This error is assumed to influence the integral very little, in the first place because of the rapid oscillations of the exponential function as explained above, and then because the exponential function is multiplied with $\cos^p x$ which decreases for increasing x.

The function $\cos^p x$, where p is 5/2 or 7/2, can generally be expanded in a series of the form

$$\cos^p x = 1 - a_2 x^2 + a_4 x^4 - a_6 x^6 + \dots \qquad 13.39$$

Each of the two terms in the integral of eq.13.36 can be written as

$$\int_0^{\pi/2} (1 - a_2 x^2 + a_4 x^4 - \dots)\, e^{-i\zeta(1+x^2/2)}\, dx$$

$$= e^{-i\zeta} \int_0^{\pi/2} (1 - a_2 x^2 + a_4 x^4 - \dots)\, e^{-i\zeta x^2/2}\, dx \qquad 13.40$$

By introducing

$$t^2 = i\,\zeta\, x^2/2 = q^2 x^2 \qquad 13.41$$

where

$$q^2 = i\,\zeta/2 \qquad 13.42$$

the expression above becomes

$$e^{-i\zeta}\, \frac{1}{q} \int_0^{T} (1 - b_2 t^2 + b_4 t^4 - \dots)\, e^{-t^2} dt \qquad 13.43$$

where

$$\begin{aligned} b_2 &= a_2/q^2 \\ b_4 &= a_4/q^4 \qquad 13.44 \\ &\dots\dots\dots\dots \end{aligned}$$

T is given by

$$T = q\ \pi/2 \qquad 13.45$$

Since ζ is a large number, q and T will also be large. The integral of eq.13.43 is easily found by partial integration:

$$e^{-i\zeta}\frac{1}{2q}\ [\pi^{1/2}\text{erf } T + b_2\ (T\ e^{-T^2} - \frac{\pi}{2}^{1/2}\text{ erf } T)$$

$$- b_4\ ((T^3 + \frac{3}{2}T)\ e^{-T^2} - \frac{3}{2}\pi^{1/2}\text{ erf } T)$$

$$+ b_6\ ((T^5 + \frac{5}{2}T^3 + \frac{5}{2}\frac{3}{2}T)\ e^{-T^2} - \frac{5}{2}\frac{3}{2}\frac{1}{2}\pi^{1/2}\text{erf } T) - ......] \qquad 13.46$$

Here we have introduced the error function of T, erf T, defined by eq.8.11. For large values of T it will reach the asymptotic form of eq.8.13.

In eq.13.46 we will now replace T with $q\pi/2$, except in the exponential functions. By means of eqs.13.44 and 8.13, and by omitting all terms within the brackets which contains the factors $1/q^2$, $1/q^3$ and so on, the expression becomes

$$e^{-i\zeta}\frac{1}{2\ q}\ [\pi^{1/2} - \frac{2\ e^{-T^2}}{\pi\ q}\ (1 - a_2(\frac{\pi}{2})^2 + a_4(\frac{\pi}{2})^4 - ....)]$$

$$= e^{-i\zeta}\frac{1}{2\ q}\ [\pi^{1/2} - \frac{2\ e^{-T^2}}{\pi\ q}\cos^p\frac{\pi}{2}]$$

$$\approx e^{-i\zeta}\frac{\pi^{1/2}}{2\ q} = e^{-i\zeta}\ (\frac{\pi}{2\ i\ \zeta})^{1/2} \qquad 13.47$$

This approximation is valid both for $p=5/2$ and for $p=7/2$. The equation 13.36 can now be written

$$I \approx (\frac{i}{\zeta} + \frac{3}{8\ \zeta^2})\ e^{i\pi/2\ -\ i\zeta} \qquad 13.48$$

and the attenuation efficiency (eq.13.35) becomes

$$Q_c = 2 + \text{Re}\ [\frac{3}{2\ \zeta^2} + (\frac{i}{\zeta} + \frac{3}{8\ \zeta^2})\ e^{i\pi/2\ -\ i\zeta}\} \qquad 13.49$$

When ζ is real, that is the cylinders are non-absorbing, the attenuation efficiency is

$$Q_c = 2 - \frac{\cos \varrho}{\varrho} + \frac{3}{2\, \varrho^2} + \frac{3 \sin \varrho}{8\, \varrho^2} \qquad 13.50$$

which resembles eq.10.10 for spheres.

We shall now estimate the absorption efficiency. When γ and w are very large, we may make the approximation

$$-i\, J_1(w) + H_1(w) \approx \frac{2}{\pi} + \frac{2}{\pi w^2} - \left(\frac{2}{\pi w}\right)^{1/2} \left(i - \frac{3}{8\, w}\right) e^{i(w-3/4\pi)} \qquad 13.51$$

Since $w=iv=i2\gamma/\sin\theta$, this expression can also be written

$$\frac{2}{\pi} - \frac{2}{\pi v^2} + i \left(\frac{2}{\pi v}\right)^{1/2} \left(1 + \frac{3}{8\, v}\right) e^{-v} \qquad 13.52$$

Since the absorption efficiency only contains real values, it may seem to be an error that we have obtained an imaginary term in eq.13.52. However, this asymptotic expression was obtained by the assumption that v should be a great number. But then the absolute value of the last term, which contains an exponential function of $-v$, will be much smaller than the second term which is proportional to v^{-2}. Consequently we can write

$$-\, i\, J_1(w) + H_1(w) \approx \frac{2}{\pi} - \frac{\sin^2\theta}{2\, \pi\, \gamma^2} \qquad 13.53$$

and by means of eq.13.26, the absorption efficiency becomes

$$Q_a \approx 1 - \frac{3}{16\, \gamma^2} \qquad 13.54$$

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LIST OF SYMBOLS

A mean absorption cross section

B mean scattering cross section

C mean attenuation cross section

C_a absorption cross section

C_b scattering cross section

C_c attenuation cross section

C_g geometrical cross section

$\bar{C}_\beta$ scattering function's mean cross section

D diameter

D_s surface equivalent diameter

D_v volume equivalent diameter

E irradiance

F shape factor

G mean geometrical cross section

L thickness, length

M mass concentration of particles

N number of particles per volum unit

Q_a absorption efficiency

Q_b scattering efficiency

Q_c attenuation efficiency

Q_β scattering function efficiency

R radius

S see eq.11.5

T eqs.11.22, 13.45

U eq.12.5

V volume

W eq.12.17

Z thickness

a absorption coefficient of suspension

a_i internal absorption coefficient of a particle

a_p absorption coefficient of particles

a_w absorption coefficient of water

b scattering coefficient of suspension

b_p scattering coefficient of particles
b_w scattering coefficient of water
c attenuation coefficient of suspension
c_p attenuation coefficient of particles
c_w attenuation coefficient of water
d mass density of a particle
f shape factor
f fraction
f function (eqs.4.16-17)
g area
i imaginary unit
k wave number
m complex index of refraction
n real part of refractive index
p see eq.9.32
q eqs.9.33, 13.42
r eq.9.34
s shape factor (eq.6.40)
s eq.9.42
t eqs.9.30, 11.21, 13.3, 13.41
u eqs.4.18, 6.25, 11.11, 13.5
v volume fraction
v eqs.4.19, 6.26, 11.14, 13.17
w eqs.6.26, 13.23
x coordinate
x eqs.6.49, 13.37
y coordinate
z coordinate

α absorption path
α_0 see eq.12.19
β volume scattering function
β angle (eq.4.15)
γ eqs.6.5, 6.16, 6.28
γ_c eq.10.29
ε eqs.12.6, 12.23
ζ eqs.6.3, 6.14, 6.27

ζ_c eq.10.28

θ angle

κ absorption index

κ_c eq.10.31

λ wavelength

μ angle

ν shape factor (eq.6.41)

ϱ eqs.6.4, 6.15, 6,27

σ shape factor (eq.6.42)

τ angle

τ_0 eq.11.7

φ angle

φ phase difference

φ_0 eqs.12.9, 13.28

χ eq.13.19

ψ eq.13.8

Ω solid angle

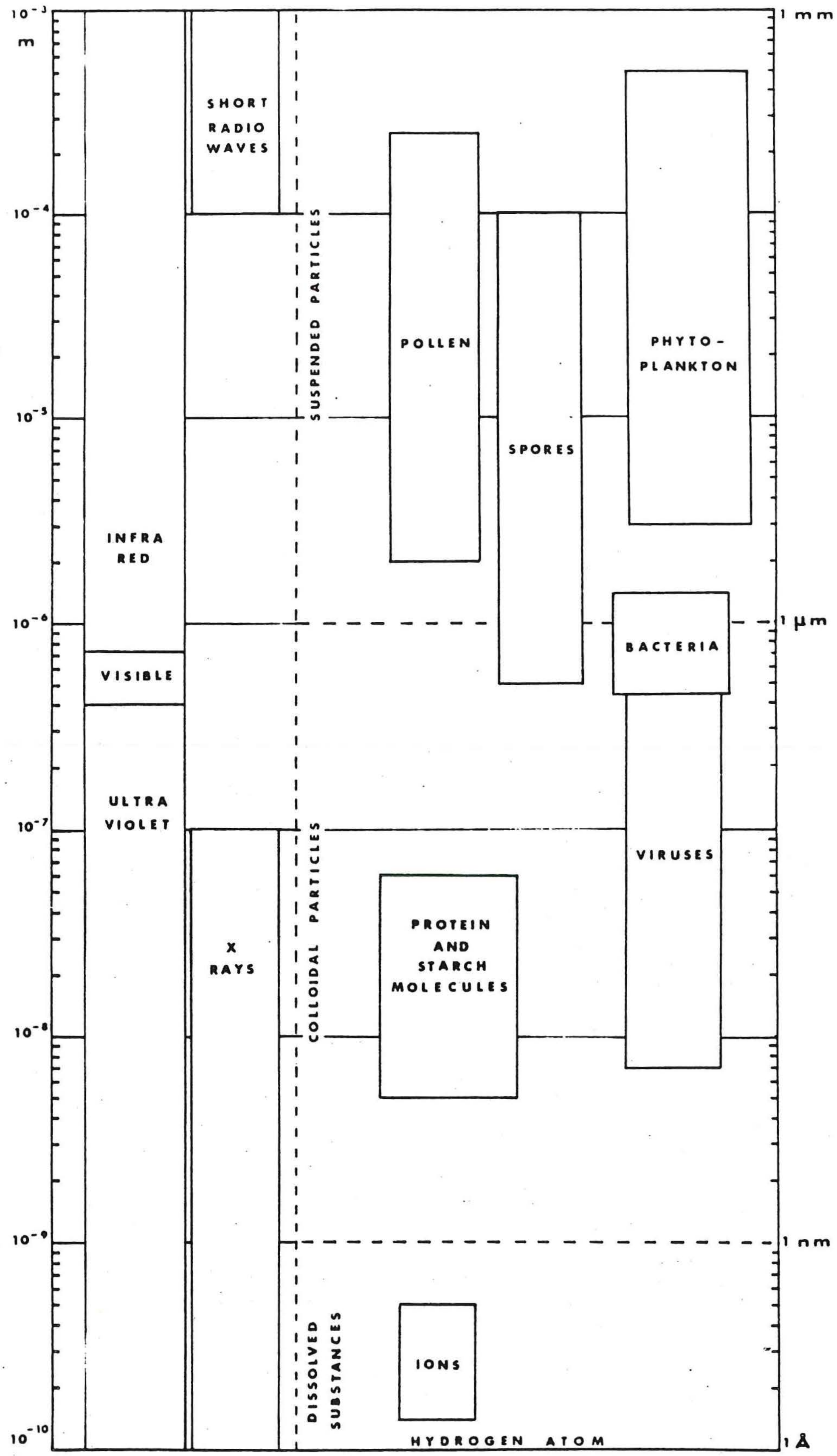

Fig.1. Dimensions of different kinds of particles in relation to the wavelength of electromagnetic waves. The size regions of suspended and colloidal particles and dissolved substances are denoted by hatched lines.

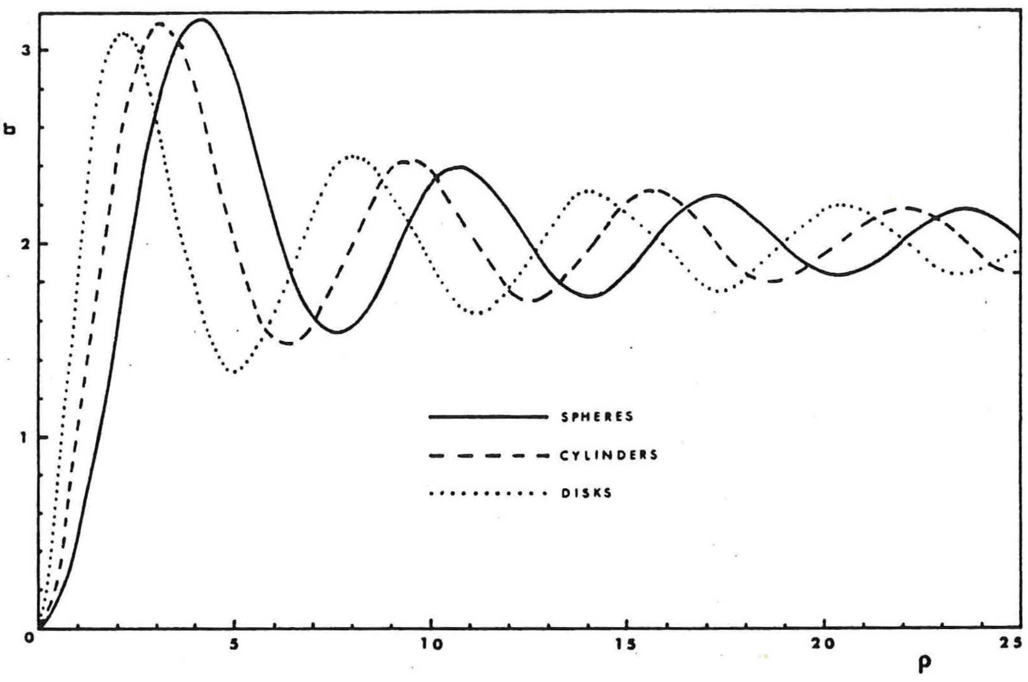

Fig.2. The scattering efficiency of non-absorbing particles as a function of the size parameter ρ.

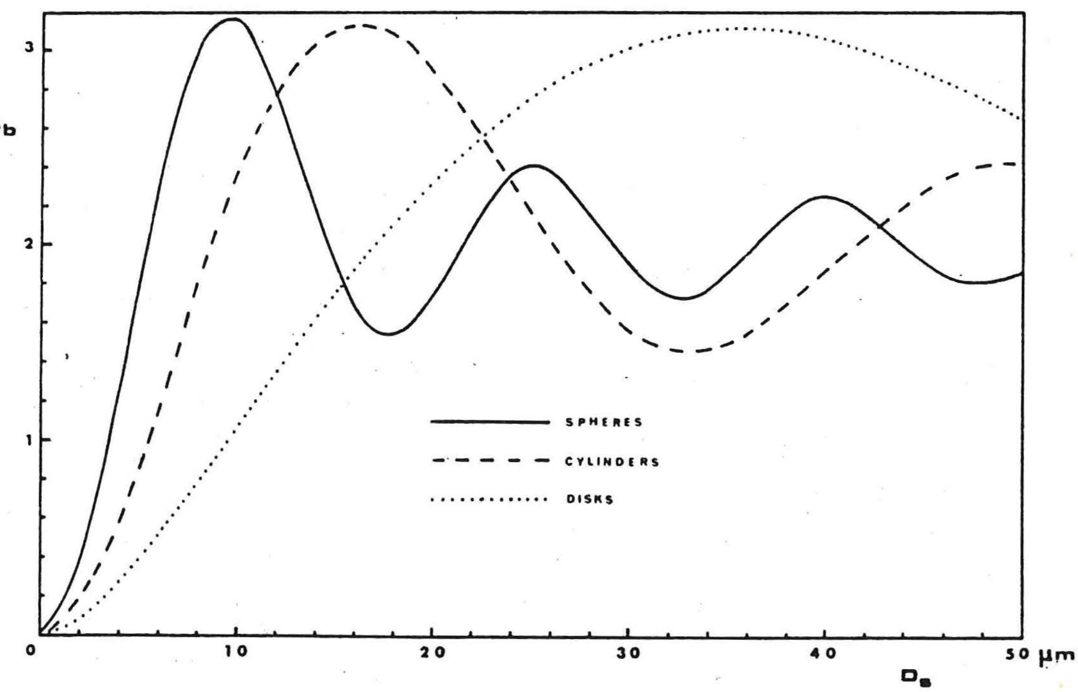

Fig.3. The scattering efficiency of non-absorbing particles as a function of the equivalent surface diameter D_s.

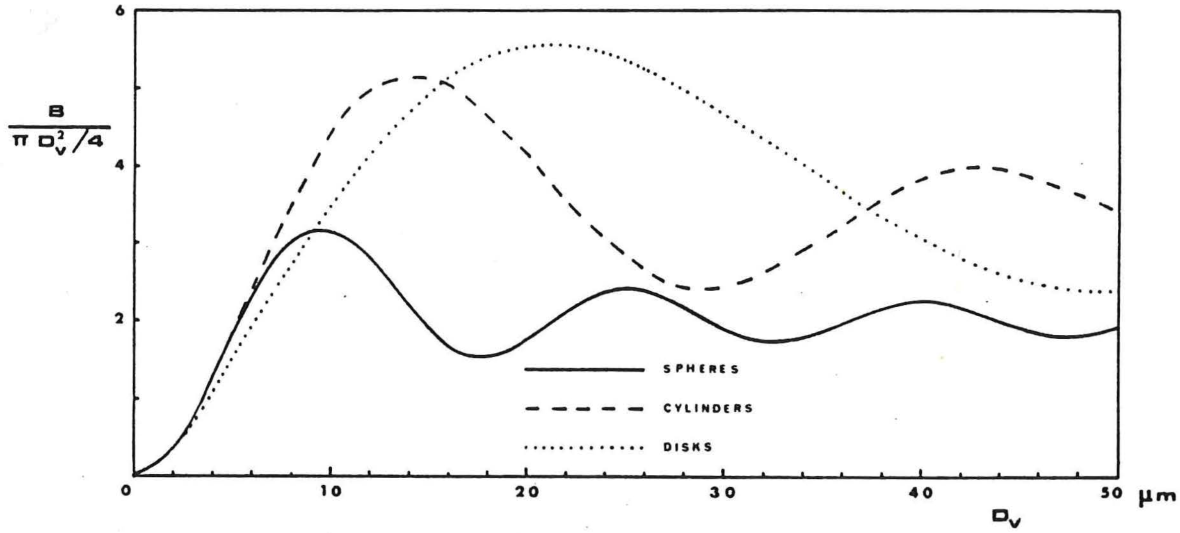

Fig.4. The ratio between the mean scattering cross section and the equivalent volume-spherical cross section of non-absorbing particles as a function of the volume equivalent diameter D_v.

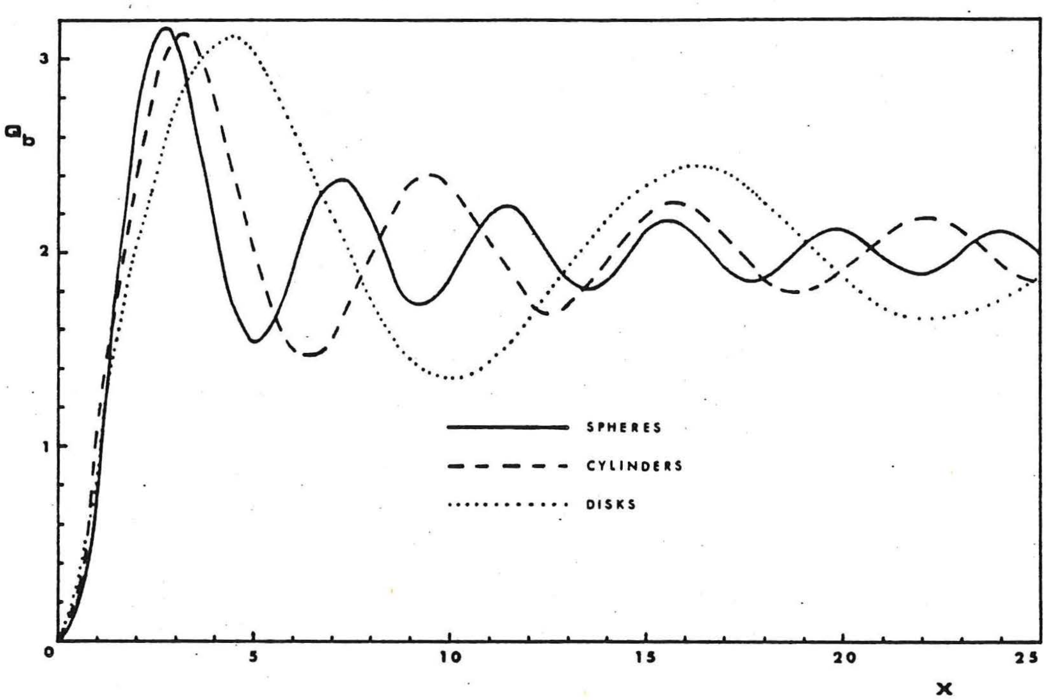

Fig.5. The scattering efficiency of non-absorbing particles as a function of the size parameter x.

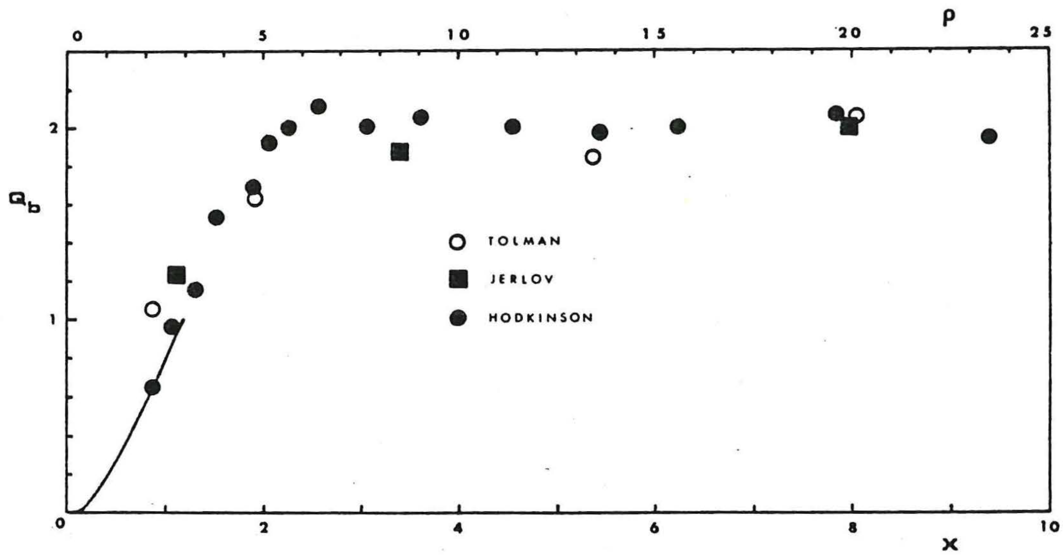

Fig.6. The scattering efficiency of irregular quartz particles as a function of the size parameters ρ and x, according to data by TOLMAN et al. (1919), JERLOV and KULLENBERG (1953) and HODKINSON (1963).

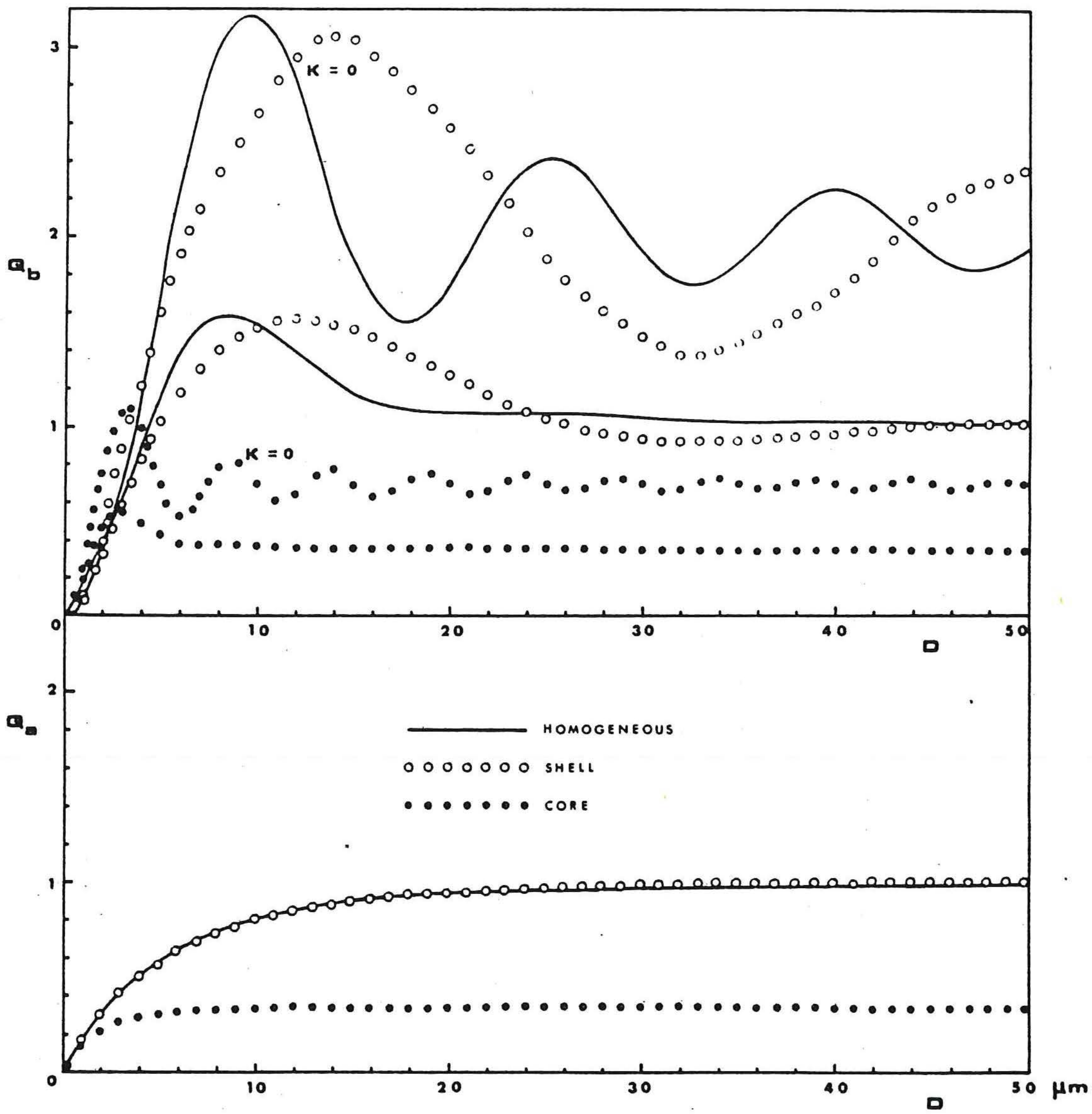

Fig.7. The scattering and absorption efficiencies of spheres, when the non-water constituents are concentrated into an outer shell or an inner core, or are homogeneously distributed, in the non-absorbing (κ=0) and absorbing cases. In the last case the absorption index κ of the homogeneous sphere has been given the value 0.01, while the real part of the refractive index is 1.03.

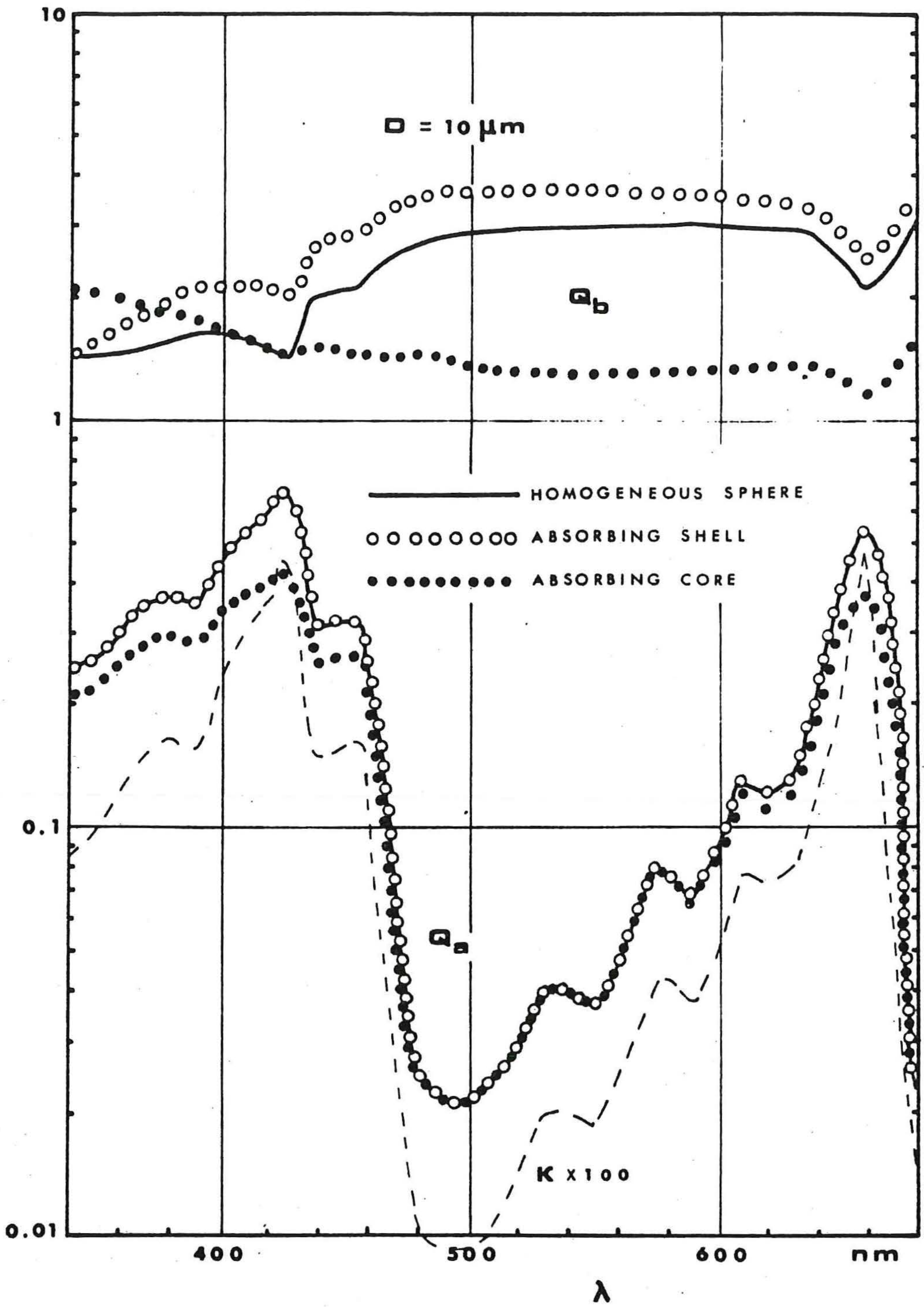

Fig.8. Scattering and absorption efficiencies of an idealized spherical alga as a function of wavelength. The diameter is 10 μm. The chloroplasts are assumed to be distributed as an outer shell, an inner core or homogeneously.

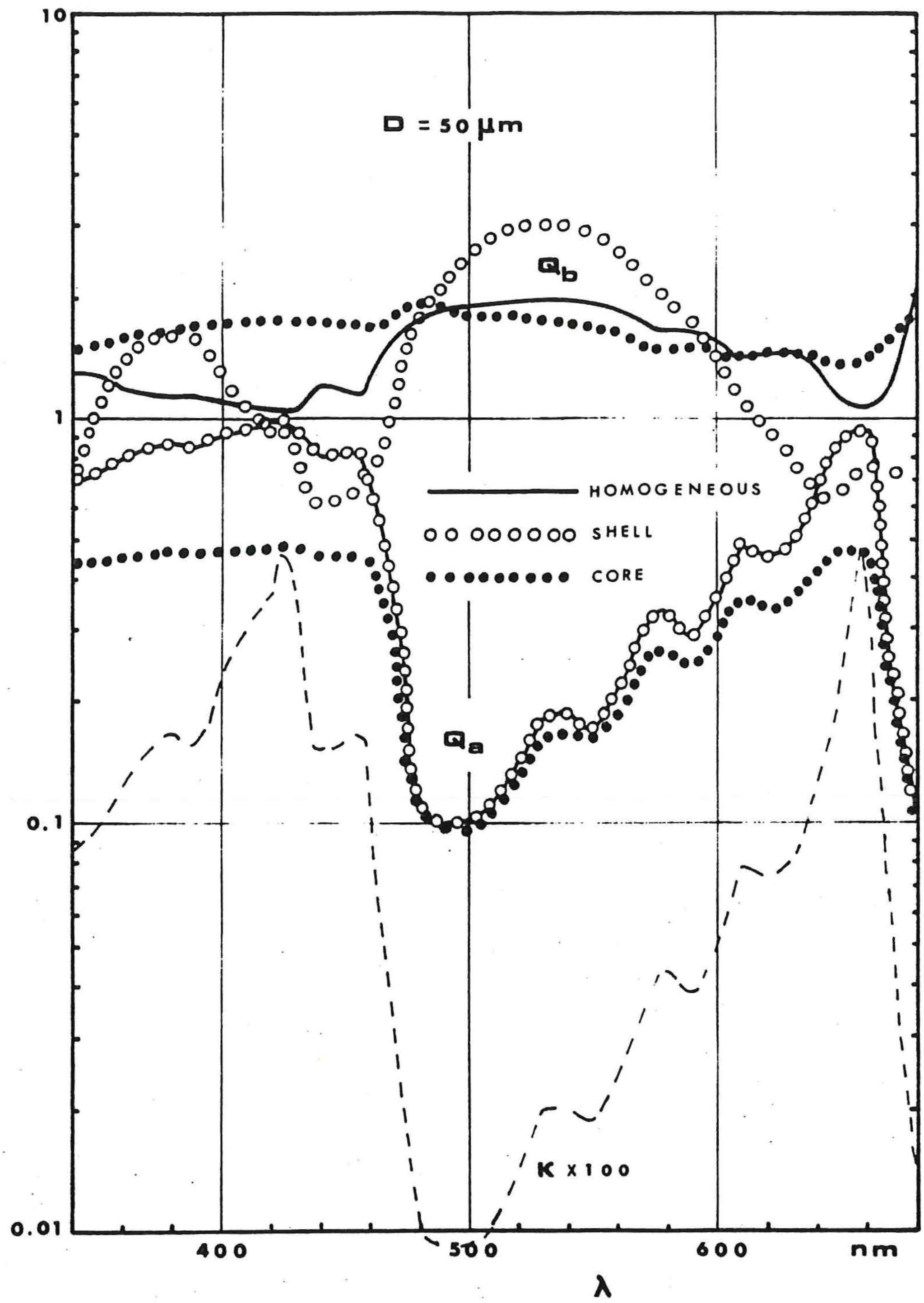

Fig.9. Scattering and absorption efficiencies of an idealized spherical alga as a function of the wavelength. The diameter is 50 µm. The chloroplasts are assumed to be distributed as an outer shell, an inner core or homogeneously.

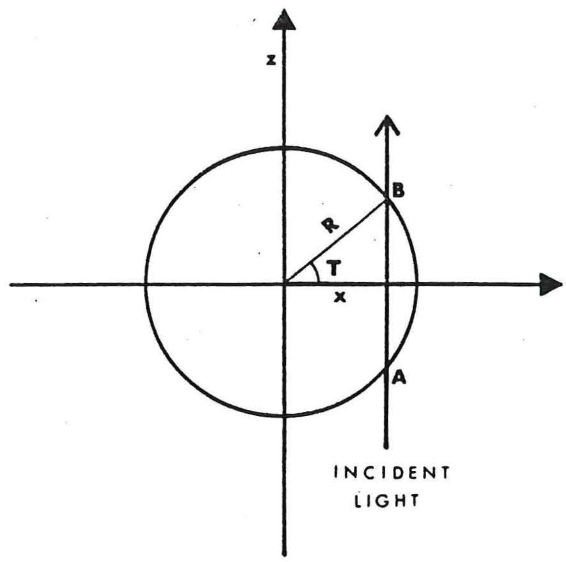

Fig.10. Cross section through the center of a homogeneous sphere.

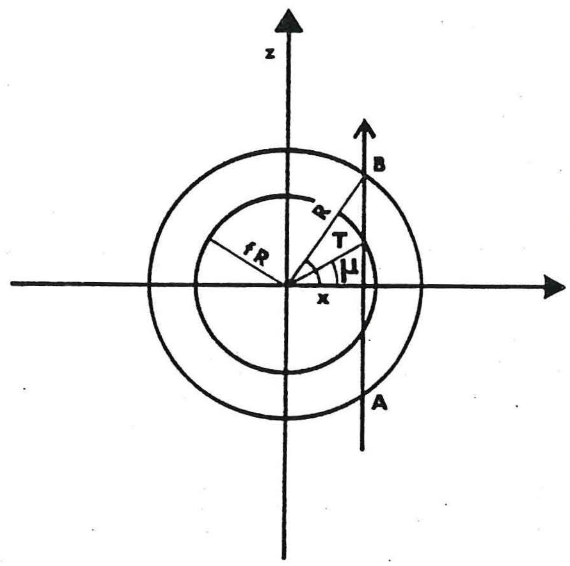

Fig.11. Cross section through the center of a two-layered sphere

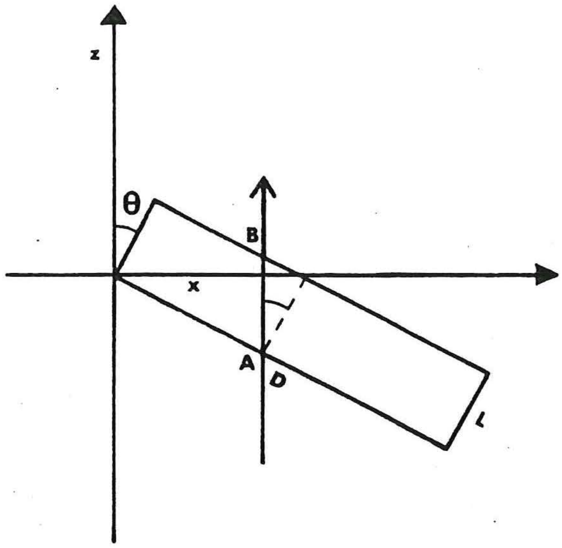

Fig.12. Cross section containing the symmetry axis of a thin disk.

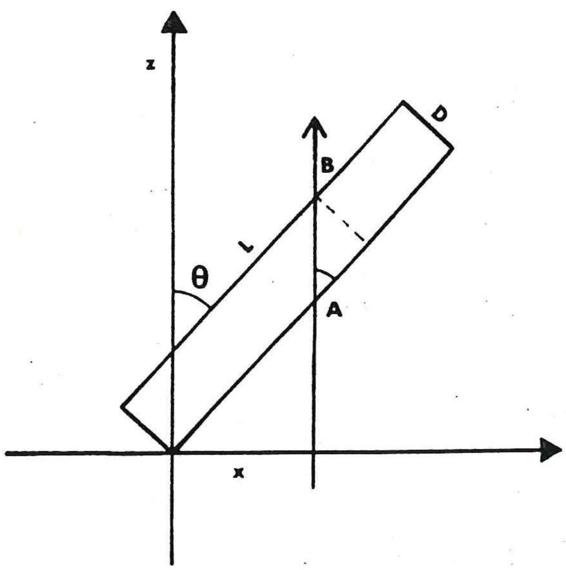

Fig.13. Cross section containing the symmetry axis of a long cylinder.