\textbf{C}^\infty\textbf{-REGULARIZATION BY NOISE OF SINGULAR ODE'S}

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ABSTRACT. In this paper we construct a new type of noise of fractional nature that has a strong regularizing effect on differential equations. We consider an equation with this noise with a highly irregular coefficient. We employ a new method to prove existence and uniqueness of global strong solutions where classical methods fail because of the "roughness" and non-Markovianity of the driving process. In addition, we prove the rather remarkable property that such solutions are infinitely many times classically differentiable with respect to the initial condition in spite of the vector field being discontinuous. This opens a fundamental question on studying certain classes of interesting partial differential equations perturbed by this noise.

1. Introduction

Consider the ordinary differential equation (ODE)

$$\frac{d}{dt}X_t^x = b(t, X_t^x), \quad X_0 = x, \quad 0 \leq t \leq T$$  \hspace{1cm} (1)

for a vector field $b : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$.

It is well-known that the ODE (1) admits the existence of a unique solution $X_t$, $0 \leq t \leq T$, if $b$ is a Lipschitz function of linear growth, uniformly in time. Further, if in addition $b \in C^k([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$, $k \geq 1$, then the flow associated with the ODE (1) inherits the regularity from the vector field, that is

$$(x \mapsto X_t^x) \in C^k(\mathbb{R}^d; \mathbb{R}^d).$$

However, well-posedness of the ODE (1) in the sense of existence, uniqueness and the regularity of solutions or flow may fail, if the driving vector field $b$ lacks regularity, that is if $b$ e.g. is not Lipschitzian or discontinuous.

In this article we aim at studying the restoration of well-posedness of the ODE (1) in the above sense by perturbing the equation via a specific noise process $\mathbb{B}_t$, $0 \leq t \leq T$, that is we are interested to analyze strong solutions to the following stochastic differential equation (SDE)

$$X_t^x = x + \int_0^t b(t, X_s^x)ds + \mathbb{B}_t, \quad 0 \leq t \leq T,$$  \hspace{1cm} (2)
where the driving process $\mathbb{B}_t$, $0 \leq t \leq T$ is a stationary Gaussian process with non-Hölder continuous paths given by

$$\mathbb{B}_t = \sum_{n \geq 1} \lambda_n B^{H_n}_t.$$  \hfill (3)

Here $B^{H_n}_t$, $n \geq 1$ are independent fractional Brownian motions in $\mathbb{R}^d$ with Hurst parameters $H_n \in (0, \frac{1}{2})$, $n \geq 1$ such that $H_n \searrow 0$ for $n \to \infty$. Further, $\sum_{n \geq 1} |\lambda_n| < \infty$ for $\lambda_n \in \mathbb{R}$, $n \geq 1$. We recall (for $d = 1$) that a fractional Brownian motion $B^H_t$ with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process on some probability space with a covariance structure $R_H(t, s)$ given by

$$R_H(t, s) = E[B^H_t B^H_s] = \frac{1}{2}(s^{2H} + t^{2H} + |t - s|^{2H}), \quad t, s \geq 0.$$  

We mention that $B^H_t$ has a version with Hölder continuous paths with exponent strictly smaller than $H$. The fractional Brownian motion coincides with the Brownian motion for $H = \frac{1}{2}$, but is neither a semimartingale nor a Markov process, if $H \neq \frac{1}{2}$. See e.g. [37] and the references therein for more information about fractional Brownian motion.

Using Malliavin calculus combined with integration-by-parts techniques based on Fourier analysis, we want to show in this paper the existence of a unique global strong solution $X^x_t$ to (2) with a stochastic flow which is smooth, that is

$$(x \mapsto X^x_t) \in C^\infty(\mathbb{R}^d; \mathbb{R}^d) \quad \text{a.e. for all} \quad t,$$  \hfill (4)

when the driving vector field $b$ is singular, that is more precisely, when

$$b \in L^q_{2, p} := L^q([0, T]; L^2(\mathbb{R}^d; \mathbb{R}^d)) \cap L^p([0, T]; L^q(\mathbb{R}^d; \mathbb{R}^d))$$

for $p, q \in (2, \infty]$. We think that the latter result is rather surprising since it seems to contradict the paradigm in the theory of (stochastic) dynamical systems that solutions to ODE’s or SDE’s inherit their regularity from the driving vector fields.

Further, we expect that the regularizing effect of the noise in (2) will also pay off dividends in PDE theory and in the study of dynamical systems with respect to singular SDE’s:

For example, if $X^x_t$ is a solution to the ODE (1) on $[0, \infty)$, then $X : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ may have the interpretation of a flow of a fluid with respect to the velocity field $u = b$ of an incompressible inviscid fluid, which is described by a solution to an incompressible Euler equation

$$u_t + (Du)u + \nabla P = 0, \quad \nabla \cdot u = 0,$$  \hfill (5)

where $P : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is the pressure field.

Since solutions to (5) may be singular, a deeper analysis of the regularity of such solutions also necessitates the study of ODE’s (1) with irregular vector fields. See e.g. Di Perna, Lions [14] or Ambrosio [2] in connection with the construction of (generalized) flows associated with singular ODE’s.
In the context of stochastic regularization of the ODE (1) in the sense of (2), however, the obtained results in this article naturally give rise to the question, whether the constructed smooth stochastic flow in (4) may be used for the study of regular solutions of a stochastic version of the Euler equation (5).

Regarding applications to the theory of stochastic dynamical systems one may study the behaviour of orbits with respect to solutions to SDE’s (2) with singular vector fields at sections on a 2-dimensional sphere (Theorem of Poincaré-Bendixson). Another application may pertain to stability results in the sense of a modified version of the Theorem of Kupka-Smale [43]. We mention that well-posedness in the sense of existence and uniqueness of strong solutions to (1) via regularization of noise was first found by Zvonkin [48] in the early 1970ties in the one-dimensional case for a driving process given by the Brownian motion, when the vector field $\mathbf{b}$ is merely bounded and measurable. Subsequently the latter result, which can be considered a milestone in SDE theory, was extended to the multidimensional case by Veretennikov [44].

Other more recent results on this topic in the case of Brownian motion were e.g. obtained by Krylov, Röckner [23], where the authors established existence and uniqueness of strong solutions under some integrability conditions on $\mathbf{b}$. See also the works of Gyöngy, Krylov [20] and Gyöngy, Martinez [21]. As for a generalization of the result of Zvonkin [48] to the case of stochastic evolution equations on a Hilbert space, we also mention the striking paper of Da Prato, Flandoli, Priola, Röckner [10], who constructed strong solutions for bounded and measurable drift coefficients by employing solutions of infinite-dimensional Kolmogorov equations in connection with a technique known as the "Itô-Tanaka-Zvonkin trick".

The common general approach used by the above mentioned authors for the construction of strong solutions is based on the so-called Yamada-Watanabe principle [46]: The authors prove the existence of a weak solution (by means of e.g. Skorokhod’s or Girsanov’s theorem) and combine it with the property of pathwise uniqueness of solutions, which is shown by using solutions to (parabolic) PDE’s, to eventually obtain strong uniqueness. As for this approach in the case of certain classes of Lévy processes the reader may consult Priola [40] or Zhang [47] and the references therein.

Let us comment on here that the methods of the above authors, which are essentially limited to equations with Markovian noise, cannot be directly used in connection with our SDE (2). The reason for this is that the initial noise in (2) is not a Markov process. Furthermore, it is even not a semimartingale due to the properties of a fractional Brownian motion.

In addition, we point out that our approach is diametrically opposed to the Yamada-Watanabe principle: We first construct a strong solution to (2) by using Malliavin calculus. Then we verify uniqueness in law of solutions, which enables us to establish strong uniqueness, that is we use the following principle:

$$\text{Strong existence} + \text{Uniqueness in law} \Rightarrow \text{Strong uniqueness}.$$
Finally, let us also mention some results in the literature on the existence and uniqueness of strong solutions of singular SDE’s driven by a non-Markovian noise in the case of fractional Brownian motion:

The first results in this direction were obtained by Nualart, Ouknine [35, 36] for one-dimensional SDE’s with additive noise. For example, using the comparison theorem, the authors in [35] are able to derive unique strong solutions to such equations for locally unbounded drift coefficients and Hurst parameters $H < \frac{1}{2}$.

More recently, Catellier, Gubinelli [8] developed a construction method for strong solutions of multi-dimensional singular SDE’s with additive fractional noise and $H \in (0, 1)$ for vector fields $b$ in the Besov-Hölder space $B^{\alpha+1}_{\infty,\infty}, \alpha \in \mathbb{R}$. Here the solutions obtained are even path-by-path in the sense of Davie [9] and the construction technique of the authors rely on the Leray-Schauder-Tychonoff fixed point theorem and a comparison principle based on an average translation operator.

Another recent result which is based on Malliavin techniques very similar to our paper can be found in Baños, Nilssen, Proske [4]. Here the authors proved the existence of unique strong solutions for coefficients $b \in L^1_{\infty,\infty} := L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d))$ for sufficiently small $H \in (0, \frac{1}{2})$.

The approach in [4] is different from the above mentioned ones and the results for vector fields $b \in L^{1,\infty}_{\infty,\infty}$ are not in the scope of the techniques in [8]. See also [5] in the case fractional noise driven SDE’s with distributional drift.

Let us now turn to results in the literature on the well-posedness of singular SDE’s under the aspect of the regularity of stochastic flows:

If we assume that the vector field $b$ in the ODE (1) is not smooth, but merely require that $b \in W^{1,p}$ and $\nabla \cdot b \in L^\infty$, then it was shown in [14] the existence of a unique generalized flow $X$ associated with the ODE (1). See also [2] for a generalization of the latter result to the case of vector fields of bounded variation.

On the other hand, if $b$ in ODE (1) is less regular than required [14, 2], then a flow may even not exist in a generalized sense.

However, the situation changes, if we regularize the ODE (1) by an (additive) noise:

For example, if the driving noise in the SDE (2) is chosen to be a Brownian noise, or more precisely if we consider the SDE

$$dX_t = u(t, X_t)dt + dB_t, \quad s, t \geq 0, \quad X_s = x \in \mathbb{R}^d$$

with the associated stochastic flow $\varphi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$, the authors in [34] could prove for merely bounded and measurable vector fields $b$ a regularizing effect of the Brownian motion on the ODE (1) that is they could show that $\varphi_{s,t}$ is a stochastic flow of Sobolev diffeomorphisms with

$$\varphi_{s,t}, \varphi_{s,t}^{-1} \in L^2(\Omega; W^{1,p}(\mathbb{R}^d; w))$$

for all $s, t$ and $p \in (1, \infty)$, where $W^{1,p}(\mathbb{R}^d; w)$ is a weighted Sobolev space with weight function $w : \mathbb{R}^d \to [0, \infty)$. Further, as an application of the latter result, which rests on techniques similar to those used in this paper, the authors also study solutions of a singular stochastic transport equation with multiplicative noise of Stratonovich type.
Another work in this direction with applications to Navier-Stokes equations, which invokes similar techniques as introduced in [34], deals with globally integrable \( u \in L^{r,q} \) for \( r/d + 2/q < 1 \) (\( r \) stands here for the spatial variable and \( q \) for the temporal variable). In this context, we also mention the paper [16], where the authors present an alternative method to the above mentioned ones based on solutions to backward Kolmogorov equations. See also [15]. We also refer to [40] and [47] in the case of \( \alpha \)-stable processes.

On the other hand if we consider a noise in the SDE (2), which is rougher than Brownian motion with respect to the path properties and given by fractional Brownian motion for small Hurst parameters, one can even observe a stronger regularization by noise effect on the ODE (1): For example, using Malliavin techniques very similar to those in our paper, the authors in [4] are able to show for vector fields \( b \in L^{q,p} \), the existence of higher order Fréchet differentiable stochastic flows

\[
(x \mapsto X^x_t) \in C^k(\mathbb{R}^d) \quad \text{a.e. for all } t,
\]

provided \( H = H(k) \) is sufficient small.

Another work in connection with fractional Brownian motion is that of Catellier, Gubinelli [8], where the authors under certain conditions obtain Lipschitz continuity of the associated stochastic flow for drift coefficients \( b \) in the Besov-Hölder space \( B^{\alpha+1}_{\infty,\infty} \), \( \alpha \in \mathbb{R} \).

We again stress that our approach for the construction of strong solutions of singular SDE’s (2) in connection with smooth stochastic flows is not based on the Yamada-Watanabe principle or techniques from Markov or semimartingale theory as commonly used in the literature. In fact, our construction method has its roots in a series of papers [31], [32], [33], [4]. See also [22] in the case of SDE’s driven by Lévy processes, [17], [34] regarding the study of singular stochastic partial differential equations or [5], [3] in the case of functional SDE’s.

To be more specific, the method we aim at employing in this paper for the construction of strong solutions rests on a compactness criterion for square integrable functionals of a cylindrical Brownian motion from Malliavin calculus, which is a generalization of that in [11], applied to solutions \( X^{s,x,n}_t \)

\[
dX^{s,x,n}_t = b_n(t, X^{s,x,n}_t)dt + dB_t, \quad X^{s,x,n}_s = x, \quad n \geq 1,
\]

where \( b_n, n \geq 0 \) are smooth vector fields converging to \( b \in L^{q,p}_2 \). Then using variational techniques based on Fourier analysis, we prove that \( X_t \) as a solution to (2) is the strong \( L^2 \)-limit of \( X^{n}_t \) for all \( t \).

Based on similar previous arguments we also verify that the flow associated with (2) for \( b \in L^{q,p}_2 \), is smooth by using an estimate of the form

\[
\sup_{s,t} \sup_{x \in U} E \left[ \left\| \frac{\partial^k}{\partial x^k} X^{s,x,n}_t \right\|^\alpha \right] \leq C_{p,q,d,H,k,\alpha,T} \left( \| b_n \|_{L^{q,p}_2} \right), \quad n \geq 1
\]

for arbitrary \( k \geq 1 \), where \( C_{p,q,d,H,k,\alpha,T} : [0, \infty) \to [0, \infty) \) is a continuous function, depending on \( p, q, d, H = \{ H_n \}_{n \geq 1}, k, \alpha, T \) for \( \alpha \geq 1 \) and \( U \subset \mathbb{R}^d \) a fixed bounded domain. See Theorem 5.1.

We also mention that the method used in this article significantly differs from that in [4] and related works, since the underlying noise of \( B \) in (2) is of infinite-dimensional
nature, that is a cylindrical Brownian motion. The latter however, requires in this paper
the application of an infinite-dimensional version of the compactness criterion in \[11\] tailored to the driving noise \(B\).

The organization of our article is as follows: In Section 2 we discuss the mathematical
framework of this paper. Further, in Section 3 we derive important estimates via varia-
tional techniques based on Fourier analysis, which are needed later on for the proofs of
the main results of this paper. Section 4 is devoted to the construction of unique strong
solutions to the SDE (2). Finally, in Section 5 we show \(C^\infty\)–regularization by noise \(B\).

1.1. Notations. Throughout the article, we will usually denote by \(C\) a generic constant.
If \(\pi\) is a collection of parameters then \(C_{\pi}\) will denote a collection of constants depending
only on the collection \(\pi\). Given differential structures \(M\) and \(N\), we denote by \(C_c^\infty(M; N)\)
the space of infinitely many times continuously differentiable function from \(M\) to \(N\) with
compact support. For a complex number \(z \in \mathbb{C}\), \(\bar{z}\) denotes the conjugate of \(z\) and \(i\) the
imaginary unit. Let \(E\) be a vector space, we denote by \(\|x\|\), \(x \in E\) the Euclidean norm.
For a matrix \(A\), we denote \(|A|\) its determinant and \(\|A\|_\infty\) its maximum norm.

2. Framework and setting

In this section we recollect some specifics on Fourier analysis, shuffle products, frac-
tional calculus and fractional Brownian motion which will be extensively used throughout
the article. The reader might consult \[29\], \[28\] or \[13\] for a general theory on Malliavin
calculus for Brownian motion and \[37\, Chapter 5\] for fractional Brownian motion. For
more detailed theory on harmonic analysis and Fourier transform the reader is referred to
\[18\].

2.1. Fourier transform. In the course of the paper we will make use of the Fourier
transform. There are several definitions in the literature. In the present article we have
taken the following: let \(f \in L^1(\mathbb{R}^d)\) then we define its Fourier transform, denoted it by \(\hat{f}\), by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i (x, \xi)} dx, \quad \xi \in \mathbb{R}^d. \tag{6}
\]

The above definition can be actually extended to functions in \(L^2(\mathbb{R}^d)\) and it makes the
operator \(L^2(\mathbb{R}^d) \ni f \mapsto \hat{f} \in L^2(\mathbb{R}^d)\) a linear isometry which, by polarization, implies

\[
\langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R}^d)} = \langle f, g \rangle_{L^2(\mathbb{R}^d)}, \quad f, g \in L^2(\mathbb{R}^d),
\]

where

\[
\langle f, g \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(z) \overline{g(z)} dz, \quad f, g \in L^2(\mathbb{R}^d).
\]
2.2. **Shuffles.** Let $k \in \mathbb{N}$. For given $m_1, \ldots, m_k \in \mathbb{N}$, denote

$$m_{1:j} := \sum_{i=1}^{j} m_i,$$

e.g. $m_{1:k} = m_1 + \cdots + m_k$ and set $m_0 := 0$. Denote by $S_m = \{\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}\}$ the set of permutations of length $m \in \mathbb{N}$. Define the set of shuffle permutations of length $m_{1:k} = m_1 + \cdots m_k$ as

$$S(m_1, \ldots, m_k) := \{\sigma \in S_{m_{1:k}} : \sigma(m_{1:i} + 1) < \cdots < \sigma(m_{1:i+1}), \ i = 0, \ldots, k-1\},$$

and the $m$-dimensional simplex in $[0, T]^m$ as

$$\Delta^m_{t_0,t} := \{(s_1, \ldots, s_m) \in [0, T]^m : t_0 < s_1 < \cdots < s_m < t\}, \quad t_0, t \in [0, T], \quad t_0 < t.$$  

Let $f_i : [0, T] \to [0, \infty)$, $i = 1, \ldots, m_{1:k}$ be integrable functions. Then, we have

$$\prod_{i=0}^{k-1} \int_{\Delta^m_{t_0,t}} f_{m_{1:i}+1}(s_{m_{1:i}+1}) \cdots f_{m_{1:i+1}}(s_{m_{1:i+1}}) ds_{m_{1:i}+1} \cdots ds_{m_{1:i+1}}$$

$$= \sum_{\sigma^{-1} \in S(m_1, \ldots, m_k)} \int_{\Delta^m_{t_0,t}} \prod_{i=1}^{m_{1:k}} f_{\sigma(i)}(w_i) dw_1 \cdots dw_{m_{1:k}}. \quad (7)$$

The above is a trivial generalisation of the case $k = 2$ where

$$\int_{t_0 < s_1 < \cdots < s_{m_1} < t} \prod_{i=1}^{m_1+m_2} f_i(s_i) ds_1 \cdots ds_{m_1+m_2}$$

$$= \sum_{\sigma^{-1} \in S(m_1, m_2)} \int_{t_0 < w_1 < \cdots < w_{m_1+m_2} < t} \prod_{i=1}^{m_1+m_2} f_{\sigma(i)}(w_i) dw_1 \cdots dw_{m_1+m_2}, \quad (8)$$

which can be for instance found in [27].

We will also need the following formula. Given indices $j_0, j_1, \ldots, j_{k-1} \in \mathbb{N}$ such that $1 \leq j_i \leq m_{i+1}$, $i = 1, \ldots, k-1$ and we set $j_0 := m_1 + 1$. Introduce the subset $S_{j_1, \ldots, j_{k-1}}(m_1, \ldots, m_k)$ of $S(m_1, \ldots, m_k)$ defined as

$$S_{j_1, \ldots, j_{k-1}}(m_1, \ldots, m_k) := \{\sigma \in S(m_1, \ldots, m_k) : \sigma(m_{1:i} + 1) < \cdots < \sigma(m_{1:i} + j_i - 1),$$

$$\sigma(l) = l, \ m_{1:i} + j_i \leq l \leq m_{1:i+1}, \ i = 0, \ldots, k-1\}.$$
We have
\[
\int_{\Delta_{t_0,1}\times\Delta_{t_0,2}\times\cdots\times\Delta_{t_0,m_k}} \prod_{i=1}^{m_{1,k}} f_i(s_i) \, ds_1 \cdots ds_{m_{1,k}} \\
= \int_{t_0<s_1<\cdots<s_{m_1+j_1} \leq \cdots \leq t} \prod_{i=1}^{m_{1,k}} f_i(s_i) \, ds_1 \cdots ds_{m_{1,k}} \\
\vdots \\
= \sum_{\sigma^{-1} \in S_{j_1,\ldots,j_{k-1}(m_1,\ldots,m_k)} \cap \Delta_{t_0,1}\times\Delta_{t_0,2}\times\cdots\times\Delta_{t_0,m_k}} \prod_{i=1}^{m_{1,k}} f_{\sigma(i)}(w_i) \, dw_1 \cdots dw_{m_{1,k}}.
\]

where \(\#\) denotes the number of elements in the given set. Then by using Stirling’s approximation, one can show that
\[
\#S(m_1,\ldots,m_k) \leq C_{m_1+\cdots+m_k}
\]
for a large enough constant \(C > 0\). Moreover,
\[
\#S_{j_1,\ldots,j_{k-1}(m_1,\ldots,m_k)} \leq \#S(m_1,\ldots,m_k).
\]

2.3. Fractional calculus. We establish here some basic definitions and properties on fractional calculus. A general theory on this subject may be found in [42] and [26].

Let \(a, b \in \mathbb{R}\) with \(a < b\). Let \(f \in L^p([a,b])\) with \(p \geq 1\) and \(\alpha > 0\). Define the left- and right-sided Riemann-Liouville fractional integrals by

\[
I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) \, dy
\]

and

\[
I_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) \, dy
\]

for almost all \(x \in [a,b]\) where \(\Gamma\) is the Gamma function.

Moreover, for a given integer \(p \geq 1\), let \(I_{a+}^{\alpha}(L^p)\) (resp. \(I_{b-}^{\alpha}(L^p)\)) denote the image of \(L^p([a,b])\) by the operator \(I_{a+}^{\alpha}\) (resp. \(I_{b-}^{\alpha}\)). If \(f \in I_{a+}^{\alpha}(L^p)\) (resp. \(f \in I_{b-}^{\alpha}(L^p)\)) and \(0 < \alpha < 1\) then define the left- and right-sided Riemann-Liouville fractional derivatives by

\[
D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^\alpha} \, dy
\]

and

\[
D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y-x)^\alpha} \, dy.
\]

The left- and right-sided derivatives of \(f\) defined above have the following representations

\[
D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} \, dy \right)
\]

\[
D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(b)}{(b-x)^\alpha} - \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} \, dy \right).
\]
and
\[ D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(b - x)^\alpha} + \alpha \int_x^b \frac{f(y) - f(x)}{(y - x)^{\alpha+1}} dy \right). \]

Finally, observe that by construction, the following formulas hold
\[ I_{a+}^\alpha(D_{a+}^\alpha f) = f \]
for all \( f \in \mathcal{L}^\alpha_1 \) and
\[ D_{a+}^\alpha(I_{a+}^\alpha f) = f \]
for all \( f \in \mathcal{L}^\alpha_2 \) and similarly for \( I_{b-}^\alpha \) and \( D_{b-}^\alpha \).

### 2.4. Fractional Brownian motion.
Let \( B^H = \{ B^H_t, t \in [0, T]\} \) be a \( d \)-dimensional fractional Brownian motion with Hurst parameter \( H \in (0, 1/2) \). In other words, \( B^H \) is a centered Gaussian process with covariance structure
\[
(R_H(t, s))_{i,j} := \mathbb{E}[B^H_t(i)B^H_s(j)] = \delta_{i,j} \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad i, j = 1, \ldots, d,
\]
where \( \delta_{i,j} = 1 \) if \( i = j \) and \( \delta_{i,j} = 0 \) otherwise. Observe that \( \mathbb{E}[|B^H_t - B^H_s|^2] = d|t - s|^{2H} \) and hence \( B^H \) has stationary increments and Hölder continuous trajectories of index \( H - \varepsilon \) for all \( \varepsilon \in (0, H) \). Observe moreover that the increments of \( B^H, H \in (0, 1/2) \) are not independent. This fact makes computations more difficult. Another difficulty one encounters is that \( B^H \) is not a semimartingale, see e.g. [37, Proposition 5.1.1].

Now we give a brief survey on how to construct fractional Brownian motion via an isometry. Since the construction can be done componentwise we present here for simplicity the one-dimensional case. Further details can be found in [37].

Denote by \( \mathcal{E} \) the set of step functions on \([0, T]\) and denote by \( \mathcal{H} \) the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the inner product
\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).
\]
The mapping \( 1_{[0,t]} \mapsto B_t \) can be extended to an isometry between \( \mathcal{H} \) and the Gaussian subspace of \( L^2(\Omega) \) associated with \( B^H \). Denote such isometry by \( \varphi \mapsto B^H(\varphi) \). We recall the following result (see [37, Proposition 5.1.3]) which gives an integral representation of \( R_H(t, s) \) when \( H < 1/2 \):

**Proposition 2.1.** Let \( H < 1/2 \). The kernel
\[
K_H(t, s) = c_H \left[ \left( \frac{t}{s} \right)^{H-\frac{3}{2}}(t - s)^{H-\frac{3}{2}} + \left( \frac{1}{2} - H \right) s^{\frac{3}{2}-H} \int_s^t u^{\frac{1}{2}-H} u^{\frac{3}{2}-H} du \right],
\]
where \( c_H = \sqrt{\frac{2H}{(1-2H)(3H+1/2)}} \) being \( \beta \) the Beta function, satisfies
\[
R_H(t, s) = \int_0^t K_H(t, u)K_H(s, u)du. \tag{10}
\]
The kernel \( K_H \) can also be represented by means of fractional derivatives as follows
\[
K_H(t, s) = c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2}-H} \left( D_t^{\frac{1}{2}-H} u^{\frac{1}{2}-H} \right)(s).
\]
Consider the linear operator \( K^*_H : \mathcal{E} \to L^2([0, T]) \) defined by
\[
(K^*_H \varphi)(s) = K_H(T, s) \varphi(s) + \int_s^T (\varphi(t) - \varphi(s)) \frac{\partial K_H}{\partial t}(t, s) dt
\]
for every \( \varphi \in \mathcal{E} \). Observe that \( (K^*_H1_{[0,t]})(s) = K_H(t, s)1_{[0,t]}(s) \), then from this fact and \( (10) \) we see that \( K^*_H \) is an isometry between \( \mathcal{E} \) and \( L^2([0,T]) \) which can be extended to the Hilbert space \( \mathcal{H} \).

For a given \( \varphi \in \mathcal{H} \) one can show the following two representations for \( K^*_H \) in terms of fractional derivatives
\[
(K^*_H \varphi)(s) = c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2}-H} \left( D_{-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \varphi(u) \right)(s)
\]
and
\[
(K^*_H \varphi)(s) = c_H \Gamma \left( H + \frac{1}{2} \right) \left( D_{-}^{\frac{1}{2}-H} \varphi(s) \right)(s)
+ c_H \left( \frac{1}{2} - H \right) \int_s^T \varphi(t)(t - s)^{H-\frac{3}{2}} \left( 1 - \left( \frac{t-s}{s} \right)^{H-\frac{1}{2}} \right) dt.
\]

One can show that \( \mathcal{H} = L_{\mathbb{R}}^\infty(L^2) \) (see \( [12] \) and \( [1] \) Proposition 6).

Given the fact that \( K^*_H \) is an isometry from \( \mathcal{H} \) into \( L^2([0,T]) \) the \( d \)-dimensional process \( W = \{W_t, t \in [0, T]\} \) defined by
\[
W_t := B^H((K^*_H)^{-1}(1_{[0,t]}))
\]
is a Wiener process and the process \( B^H \) has the following representation
\[
B^H_t = \int_0^t K_H(t, s) dW_s,
\]
see \( [1] \).

We will denote by \( W \) a standard Wiener process on a given probability space equipped with the natural filtration generated by \( W \) augmented by all \( P \)-null sets and \( B := B^H \) the fractional Brownian motion with Hurst parameter \( H \in (0, 1/2) \) given by the representation \( [12] \).

Next, we give a version of Girsanov’s theorem for fractional Brownian motion which is due to \( [12] \) Theorem 4.9). Here we present the version given in \( [35] \) Theorem 3.1 but first we need to define an isomorphism \( K_H \) from \( L^2([0,T]) \) onto \( I_{0+}^{H+\frac{1}{2}}(L^2) \) associated with the kernel \( K_H(t,s) \) in terms of the fractional integrals as follows, see \( [12] \) Theorem 2.1
\[
(K_H \varphi)(s) = I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{3}{2}} \varphi, \quad \varphi \in L^2([0,T]).
\]

From this and the properties of the Riemann-Liouville fractional integrals and derivatives the inverse of \( K_H \) is given by
\[
(K^{-1}_H \varphi)(s) = s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} D_{0+}^{2H} \varphi(s), \quad \varphi \in I_{0+}^{H+\frac{1}{2}}(L^2).
\]

It follows that if \( \varphi \) is absolutely continuous, see \( [35] \), one can show that
\[
(K^{-1}_H \varphi)(s) = s^{H-\frac{1}{2}} D_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} \varphi'(s), \quad a.e.
\]
Theorem 2.2 (Girsanov’s theorem for fBm). Let \( u = \{ u_t, t \in [0, T] \} \) be an \( \mathcal{F} \)-adapted process with integrable trajectories and set \( \tilde{B}_t^H = B_t^H + \int_0^t u_s \, ds, \quad t \in [0, T] \). Assume that

\begin{enumerate}[(i)]
  \item \( \int_0^t u_s \, ds \in L^{H+\frac{3}{2}}(\mathbb{L}^2([0, T])) \), \text{ P.a.s.}
  \item \( E[\xi_T] = 1 \) where
  \[
  \xi_T := \exp \left\{ - \int_0^T K_H^{-1} \left( \int_0^t u_s \, dr \right) (s) \, dW_s - \frac{1}{2} \int_0^T K_H^{-1} \left( \int_0^t u_s \, dr \right)^2 (s) \, ds \right\}.
  \]
\end{enumerate}

Then the shifted process \( \tilde{B}^H \) is an \( \mathcal{F} \)-fractional Brownian motion with Hurst parameter \( H \) under the new probability \( \tilde{P} \) defined by \( \frac{d\tilde{P}}{dP} = \xi_T \).

Remark 2.3. For the multidimensional case, define \( (K_H \varphi)(s) := ((K_H \varphi^{(1)})(s), \ldots, (K_H \varphi^{(d)})(s))^*, \quad \varphi \in L^2([0, T]; \mathbb{R}^d) \), where * denotes transposition. Similarly for \( K^{-1}_H \) and \( K^*_H \).

Finally, we will use a crucial property of the fractional Brownian motion which was proven by [38] for general Gaussian vector fields. This property will essentially help us to overcome the limitations of not having independent increments of the underlying noise.

Let \( m \in \mathbb{N} \) and \( 0 := t_0 < t_1 < \cdots < t_m < T \). Then for every \( \xi_1, \ldots, \xi_m \in \mathbb{R}^d \) there exists a positive finite constant \( C > 0 \) (not depending on \( m \)) such that

\[
\text{Var} \left[ \sum_{j=1}^{m} \langle \xi_j, B_{t_j}^H - B_{t_{j-1}}^H \rangle_{\mathbb{R}^d} \right] \geq C \sum_{j=1}^{m} |\xi_j|^2 \text{Var} \left[ |B_{t_j}^H - B_{t_{j-1}}^H|^2 \right]. \tag{15}
\]

The above property is known as the (strong) local non-determinism property of the fractional Brownian motion. The reader may consult [38] or [45] for more information on this property. See also [19]. A stronger version of the local non-determinism is also satisfied by the fractional Brownian motion. There exists a constant \( K > 0 \), depending only on \( H \) and \( T \), such that for any \( t \in [0, T], 0 < r < t \),

\[
\text{Var} \left[ B_t^H \mid \{ B_s^H : t - s \geq r \} \right] \geq Kr^{2H}. \tag{16}
\]

3. A NEW REGULARIZING PROCESS

Throughout this article we operate on a probability space \( (\Omega, \mathcal{A}, P) \) equipped with a filtration \( \mathcal{F} := \{ \mathcal{F}_t \}_{t \in [0, T]} \) where \( T > 0 \) is fixed, generated by a process \( \mathbb{B}^H = \{ B_t^H, t \in [0, T] \} \) to be defined later and here \( \mathcal{A} := \mathcal{F}_T \).

Let \( H = \{ H_n \}_{n \geq 1} \subset (0, 1/2) \) be a sequence of numbers such that \( \lim_n H_n = 0 \), in particular \( H \) is bounded. Also, consider \( \lambda = \{ \lambda_n \}_{n \geq 1} \subset \mathbb{R} \) a sequence of real numbers such that there exists a bijection

\[
\{ n : \lambda_n \neq 0 \} \to \mathbb{N}, \tag{17}
\]

and

\[
\sum_{n=1}^{\infty} |\lambda_n| \in (0, \infty). \tag{18}
\]
Let \( \{W^n\}_{n \geq 1} \) be a sequence of independent \( d \)-dimensional standard Brownian motions taking values in \( \mathbb{R}^d \) and define for every \( n \geq 1 \),

\[
B^{H_n,n}_t = \int_0^t K_{H_n}(t,s) dW^n_s = \left( \int_0^t K_{H_n}(t,s) dW^{n,1}_s, \ldots, \int_0^t K_{H_n}(t,s) dW^{n,d}_s \right)^*.
\]  

(19)

By construction, \( B^{H_n,n}, n \geq 1 \) are pairwise independent \( d \)-dimensional fractional Brownian motions with Hurst parameters \( H_n \). Observe that \( W^n \) and \( B^{H_n,n} \) generate the same filtrations, see [37, Chapter 5, p. 280]. We will be interested in the following stochastic process

\[
B^H_t = \sum_{n=1}^{\infty} \lambda_n B^{H_n,n}_t, \quad t \in [0, T].
\]  

(20)

Finally, we need two technical conditions on the sequence \( \lambda = \{\lambda_n\}_{n \geq 1} \). The first one is needed in order to apply the compactness criterion in Theorem A.3 whereas the second one is to ensure continuity of the sample paths of \( B^H \). Henceforward, we will assume that the sequence \( \lambda \) satisfies

\[
\sum_{n=1}^{\infty} \frac{|\lambda_n|^2}{1 - 2^{-2(\beta_n - \alpha_n)\delta_n^2}} < \infty,
\]  

(21)

where \( \alpha = \{\alpha_n\}_{n=1}^{\infty}, \beta = \{\beta_n\}_{n=1}^{\infty} \) and \( \delta = \{\delta_n\}_{n=1}^{\infty} \) are the sequences given in Theorem A.3 and

\[
\sum_{n=1}^{\infty} |\lambda_n| E \left[ \sup_{0 \leq s \leq 1} |B^{H_n,n}_s| \right] < \infty,
\]  

(22)

where \( \sup_{0 \leq s \leq 1} |B^{H_n,n}_s| \in L^1(\Omega) \) indeed, see e.g. [6].

The following theorem gives a precise definition of the process \( B^H \) and some of its relevant properties. We remark here that the condition (21) above is only used in Proposition 4.9 and is not needed in the next theorem.

**Theorem 3.1.** Let \( H = \{H_n\}_{n \geq 1} \subset (0, 1/2) \) be a sequence of real numbers such that \( \lim_n H_n = 0 \) and \( \lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{R} \) satisfying (17), (18) and (22). Let \( \{B^{H_n,n}\}_{n=1}^{\infty} \) be a sequence of \( d \)-dimensional independent fractional Brownian motions with Hurst parameters \( H_n, n \geq 1 \), defined as in (19). Define the process

\[
B^H_t := \sum_{n=1}^{\infty} \lambda_n B^{H_n,n}_t, \quad t \in [0, T],
\]

where the convergence is \( P \)-a.s. and \( B^H_t \) is a well defined object in \( L^2(\Omega) \) for every \( t \in [0, T] \).

Moreover, \( B^H_t \) is normally distributed with zero mean and covariance given by

\[
E[B^H_t (B^H_s)^*] = \sum_{n=1}^{\infty} \lambda_n^2 R_{H_n}(t,s) I_d,
\]
where \(*\) denotes transposition, \(I_d\) is the \(d\)-dimensional identity matrix and \(R_{H^n}(t,s) := \frac{1}{2} (s^{2H^n} + t^{2H^n} - |t - s|^{2H^n})\) denotes the covariance function of the components of the fractional Brownian motions \(B^{H^n,n}_{t}\).

The process \(B^H\) has stationary increments. It does not admit any version with Hölder continuous paths of any order. \(B^H\) has no finite \(p\)-variation for any order \(p > 0\), hence \(B^H\) is not a semimartingale. It is not a Markov process and hence it does not possess independent increments.

Finally, under condition (22), \(B^H\) has \(P\)-a.s. continuous sample paths.

**Proof.** One can verify, employing Kolmogorov’s three series theorem, that the series converges \(P\)-a.s. and we easily see that

\[
E[|B^H_t|^2] = d \sum_{n=1}^\infty \lambda_n^2 t^{2H^n} \leq d(1 + t) \sum_{n=1}^\infty \lambda_n^2 < \infty,
\]

where we used that \(x^\alpha \leq 1 + x\) for all \(x \geq 0\) and any \(\alpha \in [0,1]\).

The Gaussianity of \(B^H_t\) follows simply by observing that for every \(\theta \in \mathbb{R}^d\),

\[
E[\exp \{ i \langle \theta, B^H_t \rangle_{\mathbb{R}^d} \}] = e^{-\frac{1}{2} \sum_{n=1}^\infty \sum_{j=1}^d \lambda_n t^{2H^n} \theta^2},
\]

where we used the independence of \(B^{H^n,n}_{t}\) for every \(n \geq 1\). The covariance formula follows easily again by independence of \(B^{H^n,n}_{t}\).

The stationarity follows by the fact that \(B^{H^n,n}_{t}\) are independent and stationary for all \(n \geq 1\).

The process \(B^H\) could a priori be very irregular, since \(B^H\) is a stochastically continuous separable process with stationary increments we know by [30, Theorem 5.3.10] that either \(B^H\) has \(P\)-a.s. continuous sample paths on all open subsets of \([0,T]\) or \(B^H\) is \(P\)-a.s. unbounded on all open subsets on \([0,T]\). Under condition (22) and using the self-similarity of the fractional Brownian motions we see that

\[
E \left[ \sup_{s \in [0,T]} |B^H_s| \right] \leq \sum_{n=1}^\infty |\lambda_n| T^{H^n} E \left[ \sup_{s \in [0,1]} |B^{H^n,n}_s| \right] \leq (1 + T) \sum_{n=1}^\infty |\lambda_n| E \left[ \sup_{s \in [0,1]} |B^{H^n,n}_s| \right] < \infty
\]

and hence by Belyaev’s dichotomy for separable stochastically continuous processes with stationary increments (see e.g. [30, Theorem 5.3.10]) there exists a version of \(B^H\) with continuous sample paths.

Trivially, \(B^H\) is never Hölder continuous since for arbitrary small \(\alpha > 0\) there is always \(n_0 \geq 1\) such that \(H_n < \alpha\) for all \(n \geq n_0\) and since the sequence \(\lambda\) satisfies (17) cancellations are not possible. Further, one also argues that \(B^H\) is neither Markov nor has finite variation of any order \(p > 0\) which then implies that \(B^H\) is not a semimartingale. \(\Box\)

We will refer to (20) as a regularizing cylindrical fractional Brownian motion with associated Hurst sequence \(H\) or simply a regularizing fBm.

Next, we state a version of Girsanov’s theorem which actually shows that equation (26) admits a weak solution. Its proof is mainly based on the classical Girsanov’s theorem for a standard Brownian motion in Theorem 2.2.
Theorem 3.2 (Girsanov). Let \( u : [0, T] \times \Omega \to \mathbb{R}^d \) be a (jointly measurable) \( \mathcal{F} \)-adapted process with integrable trajectories such that \( t \mapsto \int_0^t u_s ds \) belongs to the domain of the operator \( K_{H_{n_0}}^{-1} \) from [13] for some \( n_0 \geq 1 \).

Define the \( \mathbb{R}^d \)-valued process
\[
\widetilde{\mathbb{E}}_t^H := \mathbb{E}_t^H + \int_0^t u_s ds.
\]

Define the probability \( \tilde{P}_{n_0} \) in terms of the Radon-Nikodym derivative
\[
\frac{d\tilde{P}_{n_0}}{dP_{n_0}} := \xi_T,
\]
where
\[
\xi_{T_0}^n := \exp \left\{ \int_0^T K_{H_{n_0}}^{-1} \left( \frac{1}{\lambda_{n_0}} \int_0^T u_s ds \right) (s) dW_{s}^{n_0} - \frac{1}{2} \int_0^T \left| K_{H_{n_0}}^{-1} \left( \frac{1}{\lambda_{n_0}} \int_0^T u_s ds \right) (s) \right|^2 ds \right\}.
\]

If \( E[\xi_{T_0}^n] = 1 \), then \( \widetilde{\mathbb{E}}_t^H \) is a regularizing \( \mathbb{R}^d \)-valued cylindrical fractional Brownian motion with respect to \( \mathcal{F} \) under the new measure \( \tilde{P}_{n_0} \) with Hurst sequence \( H \).

Proof. Indeed, write
\[
\widetilde{\mathbb{E}}_t^H = \int_0^t u_s ds + \lambda_{n_0} B_{t}^{H_{n_0}, n_0} + \sum_{n \neq n_0} \lambda_{n} B_{t}^{H_{n}, n}.
\]

Then it follows from Theorem 2.2 or [36] Theorem 3.1] that
\[
\tilde{B}_{t}^{H_{n_0}, n_0} := \int_0^t K_{H_{n_0}}^{-1} \left( \frac{1}{\lambda_{n_0}} \int_0^r u_r dr \right) (s) ds.
\]

is a fractional Brownian motion with Hurst parameter \( H_{n_0} \) under the measure
\[
\frac{d\tilde{P}_{n_0}}{dP_{n_0}} = \exp \left\{ \int_0^T K_{H_{n_0}}^{-1} \left( \frac{1}{\lambda_{n_0}} \int_0^T u_s ds \right) (s) dW_{s}^{n_0} - \frac{1}{2} \int_0^T \left| K_{H_{n_0}}^{-1} \left( \frac{1}{\lambda_{n_0}} \int_0^T u_s ds \right) (s) \right|^2 ds \right\}.
\]

Hence,
\[
\begin{align*}
\widetilde{\mathbb{E}}_t^H &= \sum_{n=1}^\infty \lambda_{n} \tilde{B}_{t}^{H_{n}, n}.
\end{align*}
\]
where

\[ \tilde{B}^{H,n}_t = \begin{cases} B^{H,n}_t & \text{if } n \neq n_0, \\ B^{H,n_0}_t & \text{if } n = n_0 \end{cases} \]

defines a regularizing \( \mathbb{R}^d \)-valued cylindrical fractional Brownian motion under \( \tilde{P}_{n_0} \).

Remark 3.3. In the above Girsanov theorem we just modify the law of the drift plus one selected fractional Brownian motion with Hurst parameter \( H_{n_0} \). In our proof later, we show that actually \( t \mapsto \int_0^t b(s, \mathbb{B}^H_s)ds \) belongs to the domain of the operators \( K^{-1}_{H_n} \) for any \( n \geq 1 \) but only large \( n \geq 1 \) satisfy Novikov’s condition for arbitrary selected values of \( p, q \in (2, \infty) \).

As we mentioned in the introduction, one of the main limitations of the process \( \mathbb{B}^H \) is that it does not have independent increments. To overcome this restriction we will use the following useful estimate which is weaker than having independent increments, but still very useful for our purposes.

Let \( m \in \mathbb{N} \) and \( 0 < t_0 < \cdots < t_m < T \). Then for every \( \xi_1, \ldots, \xi_m \in \mathbb{R}^d \) there exists a sequence of positive constants \( C = \{ C_n \}_{n=1}^m \subset (0, \infty) \) not depending on \( m \) such that

\[ \text{Var} \left( \sum_{j=1}^m \langle \xi_j, \mathbb{B}^{H,t_j}_j - \mathbb{B}^{H,t_{j-1}}_j \rangle \right) \geq \sum_{n=1}^\infty \lambda_n^2 C_n \sum_{j=1}^m |\xi_j|^2 |t_j - t_{j-1}|^{2H_n}. \]  

(23)

The above is a direct consequence of the (strong) local non-determinism for the fractional Brownian motion and their independence. The next proposition gives an estimate for the determinant of the covariance matrix of \( (\mathbb{B}^{H,t_1}, \ldots, \mathbb{B}^{H,t_m}) \).

Proposition 3.4. Let \( \{ t_i \}_{i=1}^m \subset [0, T] \) an increasing sequence of times. Let \( \mathbb{B}^H \) be as in (20). Denote \( t = (t_1, \ldots, t_m) \) and

\[ \Sigma_t := \left\{ E\left[ \mathbb{B}^{H,t}_{i,j} \right] (\mathbb{B}^{H}_{t_1})^j \right\}_{i,j=1,\ldots,m} = \left\{ \sum_{n=1}^\infty \lambda_n^2 R_{H_n}(t_{i,j}) I_d \right\}_{i,j=1,\ldots,m} \]

the covariance matrix of \( (\mathbb{B}^{H,t_1}, \ldots, \mathbb{B}^{H,t_m}) \). Then the following estimate holds true,

\[ |\Sigma_t| \geq d^n \prod_{i=1}^m \left( \sum_{n=1}^\infty \lambda_n^2 C_n |t_i - t_{i-1}|^{2H_n} \right), \]

(24)

where \( t_0 := 0 \) by convention. Here, the constants \( C_n \) are the ones from (23).

Proof. Write

\[ \mathbb{B}^{H}_t = \sum_{n=1}^\infty \lambda_n \int_0^t K_{H_n}(t,r)dW^n_r + \sum_{n=1}^\infty \lambda_n \int_s^t K_{H_n}(t,r)dW^n_r, \]

which is the sum of an \( \mathcal{F}_s \)-measurable random variable and an \( \mathcal{F}_s \)-independent random variable. Observe that the second term is a Gaussian random variable with zero mean and covariance \( \sigma^2_{s,t}I_d \) where

\[ \sigma^2_{s,t} := \sum_{n=1}^\infty \lambda_n^2 \int_s^t K_{H_n}(t,r)^2 dr. \]
Then we have
\[
\mathbb{B}_t^H - E \left[ \mathbb{B}_t^H | \mathcal{F}_s \right] = \mathbb{B}_t^H - \sum_{n=1}^{\infty} \lambda_n \int_0^s K_{H_n}(t, r) dW^r = \sum_{n=1}^{\infty} \lambda_n \int_s^t K_{H_n}(t, r) dW^r.
\]
This implies
\[
\text{Var} \left[ \mathbb{B}_t^H | \mathcal{F}_s \right] = d \sigma_{s,t}^2.
\]
From the local non-determinism of each \( B^{H_n} \), \( n \geq 1 \), we have
\[
\sigma_{s,t}^2 \geq \sum_{n=1}^{\infty} \lambda_n^2 C_n |t-s|^{2H_n},
\]
where the constants \( C_n \) depend only on \( H_n \).

It is well-known that for zero-mean Gaussian random variables we can express the determinant of the covariance matrix as
\[
|\Sigma| = \text{Var}[\mathbb{B}_1^H] \text{Var}[\mathbb{B}_2^H | \mathbb{B}_1^H] \text{Var}[\mathbb{B}_3^H | \mathbb{B}_2^H, \mathbb{B}_1^H] \cdots \text{Var}[\mathbb{B}_m^H | \mathbb{B}_{m-1}^H, \ldots, \mathbb{B}_1^H].
\]
Then, since \( \sigma(\mathbb{B}_1^H, \ldots, \mathbb{B}_m^H) \subset \mathcal{F}_m \) we have
\[
|\Sigma| \geq \prod_{i=1}^{m} \text{Var}[\mathbb{B}_i^H | \mathcal{F}_{i-1}].
\]
The result follows. \( \square \)

Consider now the following differential equation with the noise \( \mathbb{B}^H_t \) introduced earlier.
\[
X_t = x + \int_0^t b(s, X_s) ds + \mathbb{B}_s^H, \quad t \in [0, T], \tag{25}
\]
where \( x \in \mathbb{R}^d \) and \( b \) is regular. The following result summarises the classical existence and uniqueness theorem and some of the properties of the solution. Existence and uniqueness can be conducted using the classical arguments of \( L^2([0, T] \times \Omega) \)-completeness in connection with a Picard iteration argument.

**Theorem 3.5.** Let \( b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) be continuously differentiable in \( \mathbb{R}^d \) with bounded derivative uniformly in \( t \in [0, T] \) and such that there exists a finite constant \( C > 0 \) independent of \( t \) such that \( |b(t, x)| \leq C(1 + |x|) \) for every \( (t, x) \in [0, T] \times \mathbb{R}^d \). Then equation \( \text{25} \) admits a unique global strong solution which is \( P \)-a.s. continuously differentiable in \( x \) and Malliavin differentiable in each direction \( W^i \), \( i \geq 1 \) of \( \mathbb{B}^H \). Moreover, the space derivative and Malliavin derivatives of \( X \) satisfy the following linear equations
\[
\frac{\partial}{\partial x} X_t = I_d + \int_0^t b'(s, X_s) \frac{\partial}{\partial x} X_s ds, \quad t \in [0, T]
\]
and
\[
D^i_{t_0} X_t = \lambda_i K_{H_i}(t, t_0) I_d + \int_{t_0}^t b'(s, X_s) D^i_{t_0} X_s ds, \quad i \geq 1, \quad t_0, t \in [0, T], \quad t_0 < t,
\]
where \( b' \) denotes the space Jacobian matrix of \( b \), \( I_d \) the \( d \)-dimensional identity matrix and \( D^i_{t_0} \) the Malliavin derivative along \( W^i \), \( i \geq 1 \). Here, the last identity is meant in the \( L^p \)-sense \( [0, T] \).
4. CONSTRUCTION OF THE SOLUTION

We aim at constructing a Malliavin differentiable unique global \( \mathcal{F} \)-strong solutions of the following equation

\[
dX_t = b(t, X_t)dt + dB^H_t,
\]
where the differential is interpreted formally in such a way that if (26) admits a solution \( X \), then

\[
X_t = x + \int_0^t b(s, X_s)ds + \mathbb{B}^H_t
\]
whenever it makes sense. Denote by \( L^{\mathbb{Q}, p} := L^\mathbb{Q}([0, T]; L^p(\mathbb{R}^d; \mathbb{R}^d)) \), \( p, q \in [1, \infty] \) the Banach space of integrable functions such that

\[
\|f\|_{L^{\mathbb{Q}, p}} := \left( \int_0^T \left( \int_{\mathbb{R}^d} |f(t, z)|^p dz \right)^{q/p} dt \right)^{1/q} < \infty,
\]
where we take the essential supremum’s norm in the cases \( p = \infty \) and \( q = \infty \).

In this paper, we want to reach the class of discontinuous coefficients \( b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) in the Banach space

\[
\mathcal{L}_{2,p}^q := L^q([0, T]; L^2(\mathbb{R}^d; \mathbb{R}^d)) \cap L^q([0, T]; L^p(\mathbb{R}^d; \mathbb{R}^d)), \quad p, q \in (2, \infty],
\]
of functions \( f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) with the norm

\[
\|f\|_{\mathcal{L}_{2,p}^q} = \|f\|_{L^2} + \|f\|_{L^p},
\]
for chosen \( p, q \in (2, \infty] \).

Hence, our computations also show the result for uniformly bounded coefficients that are square-integrable.

We will show existence and uniqueness of strong solutions of equation (26) driven by a \( d \)-dimensional regularizing fractional Brownian motion with Hurst sequence \( H \) with coefficients \( b \) belonging to the class \( \mathcal{L}_{2,p}^q \) and whose solution is Malliavin differentiable and infinitely many times differentiable in \( x \) where \( d \geq 1 \), \( p, q \in (2, \infty] \) are arbitrary.

Remark 4.1. We would like to remark that with the method employed in the present article, the existence of weak solutions and the uniqueness in law, hold for drift coefficients in the space \( L^{\mathbb{Q}, p} \). Nevertheless, to obtain strong solutions we need the square integrability of \( b \) in space.

This solution is neither a semimartingale, nor a Markov process, and it has very irregular paths. We show in this paper that the process \( \mathbb{B}^H \) is a right noise to use in order to produce infinitely classically differentiable flows of (26) for highly irregular coefficients.

To construct a solution the main key is to approximate \( b \) by a sequence of smooth functions \( b_n \) a.e. and denoting by \( X^n = \{ X^n_t, t \in [0, T] \} \) the approximating solutions, we aim at using an \textit{ad hoc} compactness argument to conclude that the set \( \{ X^n_t \}_{n \geq 1} \subset L^2(\Omega) \) for fixed \( t \in [0, T] \) is relatively compact.

As for the regularity of the mapping \( x \mapsto X^x_t \), we are interested in proving that it is infinitely many times differentiable. It is known that there irregular drift coefficients for which \( dX_t = b(t, X_t)dt + dB^H_t \), \( X_0 = x \in \mathbb{R}^d \) admits a unique strong solution and the
mapping $x \mapsto X^x_t$ belongs, $P$-a.s., to $C^k$ if $H = H(k) < 1/2$ is small enough. Hence, by adding the noise $\mathbb{B}^H$, we should expect the solution of (26) to have a smooth flow.

Hereunder, we establish the main result of this section.

**Theorem 4.2.** Let $b \in \mathcal{L}^q_{2,\mu}$, $p, q \in (2, \infty)$. Then there exists a unique (global) strong solution $X = \{X_t, t \in [0, T]\}$ of equation (26). Moreover, for every $t \in [0, T]$, $X_t$ is Malliavin differentiable in each direction of the Brownian motions $W^n$, $n \geq 1$ in (19).

The proof of Theorem 4.2 is based on the following steps:

1. First, we construct a weak solution $X$ to (26) by means of Girsanov’s theorem for the process $\mathbb{B}^H$, that is we introduce a probability space $(\Omega, \mathcal{A}, P)$ that carries a regularizing fractional Brownian motion $\mathbb{B}^H$ and a process $X$ such that (26) is fulfilled. However, a priori $X$ is not adapted to the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ generated by $\mathbb{B}^H$.

2. Next, we approximate the drift coefficient $b$ by a sequence of compactly supported and infinitely continuously differentiable functions (which always exists by standard approximation results) $b_n : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $n \geq 0$ such that $b_n(t, x) \to b(t, x)$ for a.e. $(t, x) \in [0, T] \times \mathbb{R}^d$ and such that $\sup_{n \geq 0} \|b_n\|_{\mathcal{L}^q_{2,\mu}} \leq M$ for some finite constant $M > 0$. Then by the previous section we know that for each smooth coefficient $b_n$, $n \geq 0$, there exists unique strong solution $X^n = \{X^n_t, t \in [0, T]\}$ to the SDE

$$dX^n_t = b_n(t, X^n_t)du + dB^H_t, \quad 0 \leq t \leq T, \quad X^n_0 = x \in \mathbb{R}^d. \quad (27)$$

We then show that for each $t \in [0, T]$ the sequence $X^n_t$ converges weakly to the conditional expectation $E[X_t|\mathcal{F}_t]$ in the space $L^2(\Omega)$ of square integrable random variables.

3. By the previous section we have that for each $t \in [0, T]$ the strong solution $X^n_t$, $n \geq 0$, is Malliavin differentiable, and that the Malliavin derivatives $D^i_sX^n_t$, $i \geq 1$, $0 \leq s \leq t$, with respect to $W^i$ in (19) satisfy

$$D^i_sX^n_t = \lambda_i K_{H_i}(t, s)I_d + \int_s^t b'_n(u, X^n_u)D^i_sX^n_udu, \quad (28)$$

for every $i \geq 1$ where $b'_n$ denotes the Jacobian of $b_n$ and $I_d$ the identity matrix in $\mathbb{R}^{d \times d}$. In the next step we then employ a compactness criterion based on Malliavin calculus to show that for every $t \in [0, T]$ the set of random variables $\{X^n_t\}_{n \geq 0}$ is relatively compact in $L^2(\Omega)$, which then admits the conclusion that $X^n_t$ converges strongly in $L^2(\Omega)$ to $E[X_t|\mathcal{F}_t]$. Further we see that $E[X_t|\mathcal{F}_t]$ is Malliavin differentiable as a consequence of the compactness criterion.

4. In the last step we show that $E[X_t|\mathcal{F}_t] = X_t$, which implies that $X_t$ is $\mathcal{F}_t$-measurable and thus a strong solution on our specific probability space.

5. Uniqueness in law is enough to guarantee pathwise uniqueness.

In view of the above scheme, we go ahead with step (1) by first providing some preparatory lemmas in order to verify Novikov’s condition for $\mathbb{B}^H$. Consequently, a weak solution can be constructed via a change of measure.
Lemma 4.3. Let $B^H$ be a $d$-dimensional regularizing fBm and $p, q \in [1, \infty]$. Then for every Borel measurable function $h : [0, T] \times \mathbb{R}^d \rightarrow [0, \infty)$ we have

$$E \left[ \int_0^T h(t, B^H_t) \, dt \right] \leq C \|h\|_{L^p_q}, \quad (29)$$

where $C > 0$ is a constant depending on $p, q, d$ and $H$. Also,

$$E \left[ \exp \left\{ \int_0^T h(t, B^H_t) \, dt \right\} \right] \leq A(\|h\|_{L^p_q}), \quad (30)$$

where $A$ is an analytic function depending on $p, q, d$ and $H$.

Proof. Let $B^H$ be a $d$-dimensional regularizing fBm, then write

$$B^H_t = \sum_{n=1}^\infty \lambda_n \int_0^t K_{H_n}(t, s) \, dW^n_s + \sum_{n=1}^\infty \lambda_n \int_{t_0}^t K_{H_n}(t, s) \, dW^n_s.$$

Note that $\sum_{n=1}^\infty \lambda_n \int_{t_0}^t K_{H_n}(t, s) \, dW^n_s$ is a $d$-dimensional centered Gaussian random variable independent of $\mathcal{F}_{t_0}$ with covariance having diagonal entries

$$\sum_{n=1}^\infty \lambda_n^2 \int_{t_0}^t K_{H_n}(t, s)^2 \, ds \geq \sum_{n=1}^\infty \lambda_n^2 C_n (t - t_0)^{2H_n},$$

where $C_n$ are the constants depending on $H_n$ from the strong local non-determinism of each $B^{H_n}$. On the other hand, $\sum_{n=1}^\infty \lambda_n \int_{t_0}^t K_{H_n}(t, s) \, dW^n_s$ is $\mathcal{F}_{t_0}$-measurable. Hence, by a conditioning argument it is easy to see that for every Borel measurable function $h$ we have

$$E \left[ \int_{t_0}^T h(t_1, B^H_{t_1}) \, dt_1 \big| \mathcal{F}_{t_0} \right] \leq \int_{t_0}^T \int_{\mathbb{R}^d} h(t_1, Y + z)(2\pi)^{-d/2} \sigma_{t_0, t_1}^{-d} \exp \left( -\frac{|z|^2}{2\sigma_{t_0, t_1}^2} \right) \, dz \, dt_1,$$

where

$$\sigma_{t_0, t_1}^2 := \sum_{n=1}^\infty \lambda_n^2 C_n |t_1 - t_0|^{2H_n}.$$

Applying Hölder’s inequality, first w.r.t. $z$ and then w.r.t. $t_1$ we arrive at

$$E \left[ \int_{t_0}^T h(t_1, B^H_{t_1}) \, dt_1 \big| \mathcal{F}_{t_0} \right] \leq C \left( \int_{t_0}^T \left( \int_{\mathbb{R}^d} h(t_1, x_1)^p \, dx_1 \right)^{q/p} \, dt_1 \right)^{1/q} \left( \int_{t_0}^T \left( \sigma_{t_0, t_1}^2 \right)^{-dq'(p'-1)/2p'} \, dt_1 \right)^{1/q'},$$
for some finite constant $C > 0$. The time integral is finite for arbitrary values of $d, q'$ and $p'$. To see this, use the bound $\sum_n a_n \geq a_{n_0}$ for $a_n \geq 0$ and for all $n_0 \geq 1$. Hence,

\[
\int_{t_0}^{T} \left( \sum_{n=1}^{\infty} \lambda_n^2 C_n(t_1 - t_0)^{2H_n} \right)^{-dq'(p'-1)/2p'} \, dt_1 \\
\leq \left( \sum_{n=1}^{\infty} \lambda_n^2 C_n(t_1 - t_0)^{2H_n} \right)^{-dq'(p'-1)/2p'} \int_{t_0}^{T} (t_1 - t_0)^{-H_{n_0}dq'(p'-1)/p'} \, dt_1,
\]

then for fixed $d, q'$ and $p'$ choose $n_0$ so that $H_{n_0}dq'(p'-1)/p' < 1$. Actually, the above estimate already implies that all exponential moments are finite by [39, Lemma 1.1]. Here, though we need to derive the explicit dependence on the norm of $h$.

Altogether,

\[
E \left[ \int_{t_0}^{T} h(t, \mathbb{H}^I_{t_1}) \, dt_1 \bigg| \mathcal{F}_{t_0} \right] \leq C \left( \int_{t_0}^{T} \left( \int_{\mathbb{R}^d} h(t_1, x_1)^p \, dx_1 \right)^{q/p} \, dt_1 \right)^{1/q}, \tag{31}
\]

and setting $t_0 = 0$ this proves (29).

In order to prove (30), Taylor’s expansion yields

\[
E \left[ \exp \left\{ \int_{0}^{T} h(t, \mathbb{H}^I_{t}) \, dt \right\} \right] = 1 + \sum_{m=1}^{\infty} E \left[ \int_{0}^{T} \int_{t_1}^{T} \cdots \int_{t_{m-1}}^{T} \prod_{j=1}^{m} h(t_j, \mathbb{H}^I_{t_j}) \, dt_m \cdots dt_1 \right].
\]

Using (31) iteratively we have

\[
E \left[ \exp \left\{ \int_{0}^{T} h(t, \mathbb{H}^I_{t}) \, dt \right\} \right] \leq \frac{C^m}{(m!)^{1/q}} \left( \int_{0}^{T} \left( \int_{\mathbb{R}^d} h(t, x)^p \, dx \right)^{q/p} \, dt \right)^{m/q} = \frac{C^m \|h\|_{L^p_{\mathcal{F}}}^m}{(m!)^{1/q}},
\]

and the result follows with $A(x) := \sum_{m=1}^{\infty} \frac{C^m}{(m!)^{1/q}} x^m$. 

\[\square\]

**Lemma 4.4.** Let $\mathbb{H}^I$ be a $d$-dimensional regularizing fBm and assume $b \in L^p_{\mathcal{F}}, p, q \in [2, \infty]$. Then for every $n \geq 1$,

\[t \mapsto \int_{0}^{t} b(s, \mathbb{H}^I_{s}) \, ds \in I_{0+}^{H_n + \frac{1}{2}} (L^2([0, T])), \quad P - a.s.,\]

i.e. the process $t \mapsto \int_{0}^{t} b(s, \mathbb{H}^I_{s}) \, ds$ belongs to the domain of the operator $K_{H_n}^{-1}$ for every $n \geq 1, P$-a.s.

**Proof.** Using the property that $D_{0+}^{H_n + \frac{1}{2}} I_{0+}^{H_n + \frac{1}{2}}(f) = f$ for $f \in L^2([0, T])$ we need to show that for every $n \geq 1$,

\[D_{0+}^{H_n + \frac{1}{2}} \int_{0}^{T} |b(s, \mathbb{H}^I_{s})| \, ds \in L^2([0, T]), \quad P - a.s.\]
Indeed,
\[ D_{0+}^{H_n+\frac{1}{2}} \left( \int_0^b(b(s, \mathbb{B}_s^H)|ds) \right)(t) = \frac{1}{\Gamma \left( \frac{1}{2} - H_n \right)} \left( \frac{1}{t^{H_n+\frac{1}{2}}} \int_0^t |b(u, \mathbb{B}_u^H)|du \right) \]
\[ + \left( H + \frac{1}{2} \right) \int_0^t (t-s)^{-H_n-\frac{1}{2}} \int_s^t |b(u, \mathbb{B}_u^H)|duds \]
\[ \leq \frac{1}{\Gamma \left( \frac{1}{2} - H_n \right)} \left( \frac{1}{t^{H_n+\frac{1}{2}}} + \left( H + \frac{1}{2} \right) \int_0^t (t-s)^{-H_n-\frac{3}{2}}ds \right) \int_0^t |b(u, \mathbb{B}_u^H)|du. \]

Hence, for some finite constant \( C_{H,T} > 0 \) we have
\[ \left| D_{0+}^{H_n+\frac{1}{2}} \left( \int_0^b(b(s, \mathbb{B}_s^H)|ds) \right)(t) \right|^2 \leq C_{H,T} \int_0^T |b(u, \mathbb{B}_u^H)|^2 du \]
and taking expectation the result follows by Lemma \ref{lem:4.3} applied to \(|b|^2\). \( \square \)

We are now in a position to show that Novikov’s condition is met if \( n \) is large enough.

**Proposition 4.5.** Let \( \mathbb{B}_t^H \) be a \( d \)-dimensional regularizing fractional Brownian motion with Hurst sequence \( H \). Assume \( b \in L_p^{\infty}, p, q \in (2, \infty) \). Then for every \( \mu \in \mathbb{R} \), there exists \( n_0 \) with \( H_n < \frac{1}{2} - \frac{1}{p} \) for every \( n \geq n_0 \) and such that for every \( n \geq n_0 \) we have
\[ E \left[ \mu \int_0^T K_n^{-1} \left( \frac{1}{\lambda_n} \int_0 b(r, \mathbb{B}_r^H)dr \right)(s) ds \right] \leq C_{\lambda_n, H_n, d, \mu, T}(\|b\|_{L_p^{\infty}}) \]
for some real analytic function \( C_{\lambda_n, H_n, d, \mu, T} \) depending only on \( \lambda_n, H_n, d, T \) and \( \mu \).

In particular, there is also some real analytic function \( \tilde{C}_{\lambda_n, H_n, d, \mu, T} \) depending only on \( \lambda_n, H_n, d, T \) and \( \mu \) such that
\[ E \left[ \mathcal{E} \left( \int_0^T K_n^{-1} \left( \frac{1}{\lambda_n} \int_0 b(r, \mathbb{B}_r^H)dr \right)^*(s)dW_s^n \right)^\mu \right] \leq \tilde{C}_{H_n, d, \mu, T}(\|b\|_{L_p^{\infty}}) \]
for every \( \mu \in \mathbb{R} \).

**Proof.** By Lemma \ref{lem:4.4} both random variables appearing in the statement are well defined. Then, fix \( n \geq n_0 \) and denote \( \theta_n := K_n^{-1} \left( \frac{1}{\lambda_n} \int_0 b(r, \mathbb{B}_r^H)dr \right)(s) \). Then using relation \( \ref{eq:14} \) we have
\[ |\theta_n^\mu| = \frac{1}{\lambda_n} s^{H_n-\frac{1}{2}} \int_0^s \left| b(r, \mathbb{B}_r^H) \right| dr \]
\[ = \frac{1}{\Gamma \left( \frac{1}{2} - H_n \right)} s^{H_n-\frac{1}{2}} \int_0^s (s-r)^{-\frac{1}{2}+H_n-\frac{1}{2}-H_n} |b(r, \mathbb{B}_r^H)|dr. \quad \text{(32)} \]
Observe that since \( H_n < \frac{1}{2} - \frac{1}{p} \), \( p \in (2, \infty) \) we may take \( \varepsilon \in [0, 1) \) such that \( H_n < \frac{1}{1+\varepsilon} - \frac{1}{p} \) and apply Hölder’s inequality with exponents \( 1+\varepsilon \) and \( \frac{1+\varepsilon}{\varepsilon} \), where the case \( \varepsilon = 0 \) corresponds to the case where \( b \) is bounded. Then we get
where
\[ C_{\varepsilon, \lambda_n, H_n} := \frac{\Gamma (1 - (1 + \varepsilon) (H_n + 1/2))}{\lambda_n \Gamma (1/2 - H_n)} \left( \frac{1}{\varepsilon} \right)^{1/2} \]

Squaring both sides and using the fact that \(|b| \geq 0\) we have the following estimate
\[ |\theta^n_s|^2 \leq C^2_{\varepsilon, \lambda_n, H_n} (1 + \varepsilon) |b|^2 \left( \int_0^T |b(r, H_r^n)| \right) \cdot \left( \frac{\varepsilon}{1 + \varepsilon} \right) dP, \quad P \text{-a.s.} \]

Since \(0 < \frac{2\varepsilon}{1+\varepsilon} < 1\) and \(|x|^2 \leq \max\{\alpha, 1 - \alpha\}(1 + |x|)\) for any \(x \in \mathbb{R}\) and \(\alpha \in (0, 1)\) we have
\[ \int_0^T |\theta^n_s|^2 ds \leq C_{\varepsilon, \lambda_n, H_n, T} \left( 1 + \varepsilon \right) |b|^2 \left( \int_0^T |b(r, H_r^n)| \right) \cdot \left( \frac{\varepsilon}{1 + \varepsilon} \right) dP, \quad P \text{-a.s.} \]

for some constant \(C_{\varepsilon, \lambda_n, H_n, T} > 0\). Then estimate (30) from Lemma 4.3 with \(L = C_{\varepsilon, \lambda_n, H_n, T} \mu b^{1+\varepsilon}\) with \(\varepsilon \in [0, 1)\) arbitrarily close to one yields the result for \(p, q \in (2, \infty]\).

Let \((\Omega, \mathcal{F}, P)\) be some given probability space which carries a regularizing fractional Brownian motion \(\tilde{B}^H\) with Hurst sequence \(H = \{H_n\}_{n \geq 1}\) and set \(X_t := x + \tilde{B}^H_t, \ t \in [0, T], \ x \in \mathbb{R}^d\). Set \(\theta_t^n := \left( K_{H_n}^{-1} \left( \frac{1}{\lambda_n} \int_0^T b(r, X_r) dr \right) \right) (t)\) for some fixed \(n_0 \geq 1\) such that Proposition 4.5 can be applied and consider the new measure defined by
\[ \frac{dP_{n_0}}{dP_{n_0}} = Z_{n_0}, \]

where
\[ Z_{n_0} := \prod_{n=1}^\infty \mathcal{E} (\theta_{n_0}) \cdot \exp \left\{ \int_0^T (\theta_{n_0})^2 dW_{s} - \frac{1}{2} \int_0^T |\theta_{n_0}|^2 ds \right\}, \quad t \in [0, T]. \]

In view of Proposition 4.5 the above random variable is a well defined new probability measure and by Girsanov’s theorem, see Theorem 5.2, the process
\[ \mathbb{B}^H_t := X_t - x - \int_0^t b(s, X_s) ds, \quad t \in [0, T] \]

is a regularizing fractional Brownian motion on \((\Omega, \mathcal{F}, P_{n_0})\) with Hurst sequence \(H\). Hence, because of (35), the couple \((X, \mathbb{B}^H)\) is a weak solution of (26) on \((\Omega, \mathcal{F}, P_{n_0})\).

Since \(n_0 \geq 1\) is fixed we will omit the notation \(P_{n_0}\) and simply write \(P\).

Henceforth, we confine ourselves to the filtered probability space \((\Omega, \mathcal{F}, P), \mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}\) which carries the weak solution \((X, \mathbb{B}^H)\) of (26).
Remark 4.6. As outlined in the scheme above, the main challenge to establish existence of a strong solution is now to show that \( X \) is \( \mathcal{F} \)-adapted. Indeed, in that case \( X_t = F_t(\mathbb{B}^H) \) for some family of measurable functionals \( F_t, t \in [0, T] \) on \( C([0, T]; \mathbb{R}^d) \) and for any other stochastic basis \( (\Omega, \mathfrak{A}, \hat{P}, \mathbb{B}) \) one gets that \( X_t := F_t(\mathbb{B}) \), \( t \in [0, T] \), is a \( \mathbb{B} \)-adapted solution to SDE (26). But this means exactly the existence of a strong solution to SDE (26).

We take a weak solution \( X \) of (26) and consider \( E[X_t|\mathcal{F}_t] \). The next result is the content of step (2) in our program.

**Lemma 4.7.** Let \( b_n : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \), \( n \geq 1 \), be a sequence of compactly supported smooth functions converging a.e. to \( b \) such that \( \sup_{n \geq 1} \|b_n\|_{L^p} < \infty \). Let \( t \in [0, T] \) and \( X^n_t \) denote the solution of (26) when we replace \( b \) by \( b_n \). Then for every \( t \in [0, T] \) and continuous function \( \varphi : \mathbb{R}^d \to \mathbb{R} \) of at most linear growth we have that

\[
\varphi(X^n_t) \xrightarrow{n \to \infty} E[\varphi(X_t)|\mathcal{F}_t],
\]

weakly in \( L^2(\Omega) \).

**Proof.** For a moment let us just, without loss of generality, assume that \( x = 0 \). In the course of the proof we always assume that for fixed \( p, q \in (2, \infty] \) then \( n_0 \geq 1 \) is such that \( H_{n_0} < \frac{1}{2} - \frac{1}{p} \). Indeed, from (33) we have a constant \( C_{\varepsilon, \lambda_0, H_{n_0}} > 0 \) such that

\[
E[|K^{-1}_{H_{n_0}}\left(\frac{1}{\lambda_0} \int_0^s b_n(r, \mathbb{B}^H)dr\right)(s) - K^{-1}_{H_{n_0}}\left(\frac{1}{\lambda_0} \int_0^s b(r, \mathbb{B}^H)dr\right)(s)|]
\]

\[
\leq C_{\varepsilon, \lambda_0, H_{n_0}} \varepsilon^{-H_{n_0} - \frac{1}{2}} \left( \int_0^s \|b_n(r, \mathbb{B}^H) - b(r, \mathbb{B}^H)\|_{L^p}^{\frac{1}{\gamma}} dr \right)^{\frac{\gamma}{\gamma - 1}} \to 0
\]

as \( n \to \infty \) by Lemma 4.3.

Moreover, \( \left\{ K^{-1}_{H_{n_0}}\left(\frac{1}{\lambda_0} \int_0^\cdot b_n(r, \mathbb{B}^H)dr\right)(s) \right\}_{n \geq 0} \) is bounded in \( L^2([0, t] \times \Omega; \mathbb{R}^d) \). This is directly seen from (34) in Proposition 4.5.

Consequently

\[
\int_0^t K^{-1}_{H_{n_0}}\left(\frac{1}{\lambda_0} \int_0^s b_n(r, \mathbb{B}^H)dr\right)(s)dW^m_s \to \int_0^t K^{-1}_{H_{n_0}}\left(\frac{1}{\lambda_0} \int_0^s b(r, \mathbb{B}^H)dr\right)(s)dW^m_s
\]

and

\[
\int_0^t \left| K^{-1}_{H_{n_0}}\left(\frac{1}{\lambda_0} \int_0^s b_n(r, \mathbb{B}^H)dr\right)(s) \right|^2 ds \to \int_0^t \left| K^{-1}_{H_{n_0}}\left(\frac{1}{\lambda_0} \int_0^s b(r, \mathbb{B}^H)dr\right)(s) \right|^2 ds
\]
in $L^2(\Omega)$ since the latter is bounded $L^p(\Omega)$ for any $p \geq 1$, see Proposition 4.5.

Using the estimate $|e^x - e^y| \leq e^{x+y}|x - y|$, Hölder’s inequality and the bounds in Proposition 4.5 in connection with Lemma 4.3 it is clear that (36) holds. Similarly, one also shows that

$$\exp\left\{ \left\langle \alpha, \int_s^t b_n (r, \mathbb{B}^H_r) dr \right\rangle \right\} \rightarrow \exp\left\{ \left\langle \alpha, \int_s^t b (r, \mathbb{B}^H_r) dr \right\rangle \right\}$$

in $L^p(\Omega)$ for all $p \geq 1$, $0 \leq s \leq t \leq T$, $\alpha \in \mathbb{R}^d$.

To conclude the proof we note that the set

$$\Sigma_t := \left\{ \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, \mathbb{B}^H_{t_j} - \mathbb{B}^H_{t_{j-1}} \right\rangle \right\} : \{\alpha_j\}_{j=1}^k \subset \mathbb{R}^d, 0 = t_0 < \cdots < t_k = t, k \geq 1 \right\}$$

is a total subspace of $L^2(\Omega, \mathcal{F}_t, P)$ and we may thus restrict ourselves to show the convergence

$$\lim_{n \to \infty} E \left[ (\varphi(X^n_t) - E[\varphi(X_t)|\mathcal{F}_t]) \xi \right] = 0$$

for all $\xi \in \Sigma_t$. To this end, we notice that $\varphi$ is of linear growth and hence $\varphi(\mathbb{B}^H_t)$ has all moments. Consequently we have the following convergence

$$E \left[ \varphi(X^n_t) \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, \mathbb{B}^H_{t_j} - \mathbb{B}^H_{t_{j-1}} \right\rangle \right\} \right]$$

$$= E \left[ \varphi(X^n_t) \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, X^n_{t_j} - X^n_{t_{j-1}} - \int_{t_{j-1}}^{t_j} b_n(s, X^n_s) ds \right\rangle \right\} \right]$$

$$= E[\varphi(\mathbb{B}^H_t) \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, \mathbb{B}^H_{t_j} - \mathbb{B}^H_{t_{j-1}} \right\rangle - \int_{t_{j-1}}^{t_j} b_n(s, \mathbb{B}^H_s) ds \right\}] E \left[ \int_0^t K_{Hn_0}^{-1} \left( \frac{1}{\lambda_{\nn}} \int_0^r b_n(s, \mathbb{B}^H_s) ds \right) \right] (s) dW^n_{s_{\nn}}]$$

$$\rightarrow E[\varphi(\mathbb{B}^H_t) \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, \mathbb{B}^H_{t_j} - \mathbb{B}^H_{t_{j-1}} \right\rangle - \int_{t_{j-1}}^{t_j} b(s, \mathbb{B}^H_s) ds \right\}] E \left[ \int_0^t K_{Hn_0}^{-1} \left( \frac{1}{\lambda_{\nn}} \int_0^r b(s, \mathbb{B}^H_s) ds \right) \right] (s) dW^n_{s_{\nn}}]$$

$$= E[\varphi(X_t) \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, \mathbb{B}^H_{t_j} - \mathbb{B}^H_{t_{j-1}} \right\rangle \right\}]$$

$$= E[E[\varphi(X_t)|\mathcal{F}_t] \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, \mathbb{B}^H_{t_j} - \mathbb{B}^H_{t_{j-1}} \right\rangle \right\}]$$.

We now turn to step (3) of our procedure. For its completion the following estimate is central. It shows how we can get rid of the derivatives of the drift coefficient when approximating the solution.
Proposition 4.8. Let $\mathbb{B}^H$ be a $d$-dimensional regularizing fractional Brownian motion with Hurst sequence $H$. Let $m \geq 1$ be an integer and $b_1, \ldots, b_m$ infinitely continuously differentiable compactly supported functions $b_i : [0, T] \times \mathbb{R}^d \to \mathbb{R}, \ i = 1, \ldots, m$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_i = (\alpha_i^{(1)}, \ldots, \alpha_i^{(d)})$, $i = 1, \ldots, m$ be a multi-index such that $\alpha_i^{(j)} \geq 0$ are natural numbers and $|\alpha_i| = k \geq 1$ for every $i = 1, \ldots, m$ where $|\alpha_i| = \sum_{j=1}^d \alpha_i^{(j)}$ and $|\alpha| = \sum_{i=1}^m |\alpha_i|$. Let $\varepsilon_1, \ldots, \varepsilon_m \in \{0, 1\}$. Then there exists a finite constant $C := C_{p,q,d,H,k,T} > 0$ independent of $\{b_i\}_{i=1}^m$ and $m$ such that

$$
\left| E \left[ \int_{\Delta_{t_0,t}} \left( \prod_{j=1}^m D^{\alpha_j} b_j(t, x + \mathbb{B}^H_{t_j}) K_{H_{t_0,t_0'}}(t_j) \varepsilon_j \right) \ dt_1 \cdots dt_m \right] \right| 
\leq \frac{C}{(m!)^{1/q}} \prod_{j=1}^m \|b_j\|_{L^2} \left| t_0 - t_0' \right|^{\gamma_i \sum_{j=1}^m \varepsilon_j} \left| t_0 \right|^{\gamma_i \sum_{j=1}^m \varepsilon_j} \left( \frac{\left| t - t_0 \right|^{m - \left( \frac{1}{2} - H_i + \gamma_i \right) q' \sum_{j=1}^m \varepsilon_j}}{\Gamma \left( 1 + m - \left( \frac{1}{2} - H_i + \gamma_i \right) q' \sum_{j=1}^m \varepsilon_j \right)} \right)^{1/q'}
$$

for every $i \geq 1$, $x \in \mathbb{R}^d$, $t_0, t_0', t \in [0, T]$, $t_0' < t_0 < t$, $q \in [2, \infty]$, $q'$ is the conjugate exponent of $q$ and $\gamma_i \in (0, H_i)$,

$$
K_{H_{t_0,t_0'}}(t_j) := K_{H_i}(t_j, t_0) - K_{H_i}(t_j, t_0'), \quad j = 1, \ldots, m,
$$

and where $\Gamma$ is the Gamma function. Here, $D^{\alpha_j} = \frac{\partial^{\alpha_j^{(1)}}}{\partial x_j^{(1)}} \cdots \frac{\partial^{\alpha_j^{(d)}}}{\partial x_j^{(d)}}$ denotes the partial derivatives of $b_j$, $j = 1, \ldots, m$.

Similarly, it holds

$$
\left| E \left[ \int_{\Delta_{t_0,t}} \left( \prod_{j=1}^m D^{\alpha_j} b_j(t, x + \mathbb{B}^H_{t_j}) K_{H_i}(t_j, t_0)^{\varepsilon_j} \right) \ dt_1 \cdots dt_m \right] \right| 
\leq \frac{C}{(m!)^{1/q}} \prod_{j=1}^m \|b_j\|_{L^2} \left| t_0 - t_0' \right|^{\gamma_i \sum_{j=1}^m \varepsilon_j} \left| t_0 \right|^{\gamma_i \sum_{j=1}^m \varepsilon_j} \left( \frac{\left| t - t_0 \right|^{m - \left( \frac{1}{2} - H_i \right) q' \sum_{j=1}^m \varepsilon_j}}{\Gamma \left( 1 + m - \left( \frac{1}{2} - H_i \right) q' \sum_{j=1}^m \varepsilon_j \right)} \right)^{1/q'}
$$

Proof. Denote by $z = (z^{(1)}, \ldots, z^{(d)})$ a generic element in $\mathbb{R}^d$ and by $| \cdot |$ the Euclidean norm in $\mathbb{R}^d$.

The vector $(\mathbb{B}^H_{t_1}, \ldots, \mathbb{B}^H_{t_m})$ is normally distributed with zero mean and covariance matrix

$$
\Sigma_t := \Sigma_{t_1, \ldots, t_m} := \left\{ E[\mathbb{B}^H_{t_i}(\mathbb{B}^H_{t_j})^*] \right\}_{i,j=1,\ldots,m} = \left\{ \sum_{n=1}^\infty \chi_n^2 R_{H_i}(t_i, t_j) \right\}_{i,j=1,\ldots,m}.
$$

Here, $I_d$ denotes the $d \times d$ identity matrix and $*$ denotes transposition.

Since the Lebesgue measure is translation invariant on $\mathbb{R}$, we may, without loss of generality, assume that $x = 0$ since the $L^q$-norms will still remain the same.
Denote by $P_{\Sigma}(z_1, \ldots, z_m) = ((2\pi)^{md} |\Sigma|)^{-1/2} \exp\left\{ -\frac{1}{2} z^* \Sigma^{-1} z \right\}$, $z \in (\mathbb{R}^d)^m$, $t > 0$, the normal density in $(\mathbb{R}^d)^m$ with zero mean and covariance matrix $\Sigma$. In the course of the proof we may just write $z = (z_1, \ldots, z_m) \in (\mathbb{R}^d)^m$ and $t = (t_1, \ldots, t_m) \in \Delta_{t_0,t}^m$ when no confusion arises and the corresponding differential forms, $dz$ and $dt$, respectively. Thus, we may express the left-hand side of (37) as

\[
I := \int_{\Delta_{t_0,t}^m} \int_{(\mathbb{R}^d)^m} \left( \prod_{j=1}^m D^{\alpha_j} b_j(t_j, z_j) K_{t_0,t_0}^j (t_j)^{\varepsilon_j} \right) P_{\Sigma}(z_1, \ldots, z_m) dz dt .
\]

We may use integration by parts and the fact that $b_j$, $j = 1, \ldots, m$ are compactly supported to write

\[
I = \left| \int_{\Delta_{t_0,t}^m} \prod_{j=1}^m K_{t_0,t_0}^j (t_j)^{\varepsilon_j} \int_{(\mathbb{R}^d)^m} \left( \prod_{j=1}^m b_j(t_j, z_j) \right) D^{\alpha} P_{\Sigma}(z_1, \ldots, z_m) dz dt \right| .
\] (39)

Observe that the space integral is the usual inner product in $(\mathbb{R}^d)^m$. All factors are in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and hence by Plancherel’s theorem we have

\[
I = \left| \int_{\Delta_{t_0,t}^m} \prod_{j=1}^m K_{t_0,t_0}^j \left( \int_{(\mathbb{R}^d)^m} \left( \prod_{j=1}^m \hat{b}_j(t_j, \xi_j) \right) \overline{D^{\alpha} P_{\Sigma}(\xi_1, \ldots, \xi_m)} d\xi_j \right) d\xi \right| ,
\]

where the Fourier transform used here is given by (6). Since the Fourier transform of the Gaussian kernel is real-valued we may skip conjugation.

Now, apply Cauchy-Schwarz inequality to get

\[
I \leq \left( \int_{\Delta_{t_0,t}^m} \prod_{j=1}^m \| b_j(t_j, \cdot) \|^q_{L^2(\mathbb{R}^d)} \right)^{1/q} \times \left( \int_{\Delta_{t_0,t}^m} \prod_{j=1}^m K_{t_0,t_0}^j (t_j)^{\varepsilon_j} \left( \int_{(\mathbb{R}^d)^m} |D^{\alpha} P_{\Sigma}(\xi_1, \ldots, \xi_m)|^2 d\xi \right)^{q'/2} dt \right)^{1/q'} .
\]

Let us for a moment focus on the first factor in the estimate above. The integrand is a symmetric function over a simplex. So we may write

\[
\left( \int_{\Delta_{t_0,t}^m} \prod_{j=1}^m \| b_j(t_j, \cdot) \|^q_{L^2(\mathbb{R}^d)} dt \right)^{1/q} = \left( \frac{1}{m!} \int_{[t_0,t]^m} \prod_{j=1}^m \| b_j(t_j, \cdot) \|^q_{L^2(\mathbb{R}^d)} dt \right)^{1/q} .
\]

Then, since the integrands are positive and $[t_0, t] \subseteq [0, T]$ we obtain

\[
\left( \int_{\Delta_{t_0,t}^m} \prod_{j=1}^m \| b_j(t_j, \cdot) \|^q_{L^2(\mathbb{R}^d)} dt \right)^{1/q} \leq \frac{1}{(m!)^{1/q}} \prod_{j=1}^m \| b_j \|_{L^2} .
\]
All in a summary we have obtained

\[ J \leq \left( \frac{C^m}{(m!)^{1/q}} \prod_{j=1}^{m} \| b_j \|_{L^q_2} \left( \int_{\Delta_{t_0}^{m,j}} \prod_{j=1}^{m} K_{t_0}^{H_j} (t_j)^{q/2} \left( \int_{(\mathbb{R}^d)^m} \| D^\alpha \widehat{P}_{\Sigma} (\xi_1, \ldots, \xi_m) \| d\xi \right)^{q/2} dt \right)^{1/q} \right)^{1/q}. \] (40)

We turn to the integral in space. Denote

\[ J = \left( \int_{(\mathbb{R}^d)^m} \| D^\alpha \widehat{P}_{\Sigma} (\xi_1, \ldots, \xi_m) \| d\xi \right)^{q/2}. \]

Clearly,

\[ J = \left( \int_{(\mathbb{R}^d)^m} \| \left. 2\pi i \xi_1^{\alpha_1} \cdots 2\pi i \xi_m^{\alpha_m} \right\| \ | \widehat{P}_{\Sigma} (\xi_1, \ldots, \xi_m) \| d\xi \right)^{q/2}, \]

where recall that here \( \xi_1^{\alpha_1} = (\xi_1^{(1)})^{\alpha_1^{(1)}} \cdots (\xi_j^{(d)})^{\alpha_j^{(d)}} \) for every \( j = 1, \ldots, m \).

The Fourier transform of \( \widehat{P}_{\Sigma} \) corresponds to the characteristic function of a normally distributed random vector rescaled according to our definition of Fourier transform in (6). It is given by

\[ \widehat{P}_{\Sigma} (\xi) = e^{-2\pi^2 |\Sigma| \xi}, \quad \xi \in (\mathbb{R}^d)^m. \]

Thus, we have for some constant \( C > 0 \)

\[ J \leq C \left( \int_{(\mathbb{R}^d)^m} \| \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m} \| e^{-4\pi^2 |\Sigma| \xi} d\xi \right)^{q/2}. \]

Observe that

\[ \xi^* \Sigma \xi = \text{Var} \left[ \sum_{j=1}^{m} |\xi_j, B_{t_j}^H \rangle_{\mathbb{R}^d} \right]. \]

Applying the change of variables \( \xi_j = \zeta_j - \zeta_{j+1} \) for every \( j = 1, \ldots, m \) where \( \zeta_{m+1} \) is the \( d \)-dimensional null vector we get

\[ J \leq C \left( \int_{(\mathbb{R}^d)^m} \prod_{j=1}^{m} |(\zeta_j - \zeta_{j+1})^{\alpha_j}|^2 e^{-4\pi^2 \text{Var}[\sum_{j=1}^{m} |\xi_j, B_{t_j}^H - B_{t_{j-1}}^H \rangle_{\mathbb{R}^d}] d\zeta} \right)^{q/2}. \]

By the strong local non-determinism of \( B^H \) from (23) we have

\[ J \leq C \left( \int_{(\mathbb{R}^d)^m} \prod_{j=1}^{m} |(\zeta_j - \zeta_{j+1})^{\alpha_j}|^2 e^{-4\pi^2 \sum_{n=1}^{\infty} \chi_n^2 C_n \sum_{j=1}^{m} |\xi_j|^2 |t_j - t_{j-1}|^{2H_n} d\zeta} \right)^{q/2}, \]

for some finite constants \( C_n > 0 \) not depending on \( m \). Using the inequality \( |\zeta_j - \zeta_{j+1}| \leq |\zeta_j| + |\zeta_{j+1}| \), we can estimate the above integral by sums of integrals with powers of \( |\zeta_j^{\alpha_j}|^2 \). The numbers of summands is at most a constant, still denoted by \( C \), to the power \( m \) and the power of \( |\zeta_j^{\alpha_j}|^2 \) is at most \( |\zeta_j^{\alpha_j}|^4 \). Hence, we need to compute integrals of the form

\[ J \leq C^m \prod_{j=1}^{m} \left( \int_{\mathbb{R}} |\zeta_j|^{4k} e^{-2\pi^2 \zeta_j^2} d\zeta \right)^{q/2}, \]
where \( C > 0 \) is a suitable finite constant independent of \( m \) and here

\[
\sigma_j^2 := \left( 8\pi^2 \sum_{n=1}^{\infty} \lambda_n^2 C_n |t_j - t_{j-1}|^{2H_n} \right)^{-1}.
\]

Therefore,

\[
J \leq C_m \prod_{j=1}^{m} \left( \sum_{n=1}^{\infty} \lambda_n^2 C_n |t_j - t_{j-1}|^{2H_n} \right)^{-\frac{(1+4k)dq'}{4}},
\]

for some finite constant \( C := C_{q,d,k} > 0 \) not depending on \( m \).

Now, we recall that \( K_{t_0,t_0}^H(t_j)^{\varepsilon_j} \) can be estimated by

\[
|K_{t_0,t_0}^H(t_j)^{\varepsilon_j}| \leq C_{H,T} \left| \frac{t_0 - t_0'}{t_0} \right|^{\gamma_{\varepsilon_j}} \left( |t_j - t_{j-1}|^{-(\frac{1}{2} - H_i + \gamma_i)\varepsilon_j} + |t_j - t_{j-1}|^{-(\frac{1}{2} - H_i + \gamma_i)\varepsilon_j} \right),
\]

where \( C_{H,T} > 0 \) is a constant and \( \varepsilon_j \in \{0, 1\} \) for \( j = 1, \ldots, m \), where we used estimate \( (55) \) in the Appendix and the fact that \( |t_j - t_0| \geq |t_j - t_{j-1}| \) for every \( j = 1, \ldots, m \).

The above estimate in connection with \( (40) \) allows us to find a finite constant \( C > 0 \) independent of \( m \) such that

\[
I \leq \frac{C_m}{(m!)^q} \prod_{j=1}^{m} \|b_j\|_{L^q} \sum_{n=1}^{\infty} \lambda_n^2 C_n |t_j - t_{j-1}|^{2H_n} \leq \frac{C_m}{(m!)^q} \prod_{j=1}^{m} \int_{\Delta_0,t} |t_j - t_{j-1}|^{-(\frac{1}{2} - H_i + \gamma_i)\varepsilon_j} \left( \sum_{n=1}^{\infty} \lambda_n^2 C_n |t_j - t_{j-1}|^{2H_n} \right)^{-\frac{(1+4k)dq'}{4}} dt \right)^{1/q'}.
\]

To proceed, we use the estimate \( (\sum_{n=1}^{\infty} \alpha_n)^{-1} \leq \alpha_n^{-1} \) for positive numbers \( \alpha_n > 0 \) and any \( n_0 \geq 1 \). Hence, for any \( n_0 \geq 1 \) we have that the time integral can be bounded by

\[
\left( \lambda_{n_0}^2 C_{n_0} \right)^{-\frac{(1+4k)dq'}{4}} \int_{\Delta_0,t} \prod_{j=1}^{m} \left( |t_j - t_{j-1}|^{-(\frac{1}{2} - H_i + \gamma_i)\varepsilon_j} \right) dt_1 \cdots dt_m.
\]

We have \( 0 < \gamma_i < H_i < 1/2 \) and hence \(-q'/2 < -\left( \frac{1}{2} - H_i + \gamma_i \right) q' < -q'\gamma_i \). Hence, a sufficient condition for \(|t_j - t_{j-1}|^{-(\frac{1}{2} - H_i + \gamma_i)q'}\) to be integrable is that \( q' < 2 \) which is implied by \( q \geq 2 \). Whereas for the other exponent we can always choose \( n_0 \) large enough so that

\[
-\frac{(1+4k)dq'}{2} H_{n_0} > -1.
\]

Integrating iteratively we get

\[
\left( \lambda_{n_0}^2 C_{n_0} \right)^{-\frac{(1+4k)dq'}{4}} \int_{\Delta_0,t} \prod_{j=1}^{m} \left( |t_i - t_{i-1}|^{-(\frac{1}{2} - H_i + \gamma_i)\varepsilon_j} \right) dt_1 \cdots dt_m
\]

\[
\leq C_{n_0} \prod_{j=1}^{m} \Gamma \left( 1 - \frac{(1+4k)dq'}{2} H_{n_0} - \left( \frac{1}{2} - H_i + \gamma_i \right) \varepsilon_j \right) \left| t - t_0 \right|^{m \left( 1 - \frac{(1+4k)dq'}{2} H_{n_0} - \left( \frac{1}{2} - H_i + \gamma_i \right) q' \sum_{j=1}^{m} \varepsilon_j \right)} \frac{1 + m \left( 1 - \frac{(1+4k)dq'}{2} H_{n_0} \right)}{\Gamma \left( 1 + m \left( 1 - \frac{(1+4k)dq'}{2} H_{n_0} \right) - \left( \frac{1}{2} - H_i + \gamma_i \right) q' \sum_{j=1}^{m} \varepsilon_j \right)}.
\]


Taking the limit we have
\[
\int_{\Delta_{t_0,t}^m} \prod_{i=1}^m \left| t_i - t_{i-1} \right| (1 - 4i \delta_i d' t_i H_{n_i} - (\frac{1}{2} - H_{i + \gamma_i}) q' \varepsilon_j) \, dt \leq C_m \frac{|t - t_0| m - (\frac{1}{2} - H_{i + \gamma_i}) q' \sum_{j=1}^m \varepsilon_j}{\Gamma \left( 1 + m - \left( \frac{1}{2} - H_{i + \gamma_i} \right) q' \sum_{j=1}^m \varepsilon_j \right)}
\]
and the result follows.

The second estimate holds trivially by following exactly the same computations and the fact that
\[
|K_{H_i}(t, t_0)| \leq C_H t_0^{H - \frac{1}{2}} (t - t_0)^{H - \frac{1}{2}},
\]
which corresponds to the case \( \gamma_i = 0 \).

The next proposition is a verification of the sufficient condition needed to guarantee relative compactness of the approximating sequence \( \{X^n_t\}_{n \geq 1} \).

**Proposition 4.9.** Let \( b_n : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, n \geq 1, \) be a sequence of compactly supported smooth functions converging a.e. to \( b \) such that \( \sup_{n \geq 1} \| b_n \|_{L^2_{t,p}} < \infty, p, q \in (2, \infty) \). Let \( t \in [0, T] \) and \( X^n_t \) denote the solution of (26) when we replace \( b \) by \( b_n \). Then, for every \( t \in [0, T] \), there exists a sequence \( \beta = \{\beta_i\}_{i=1}^\infty \in (0, 1/2) \) such that
\[
\sup_{n \geq 1} E[\|X^n_t\|^2] < \infty,
\]
\[
\sup_{n \geq 1} \sum_{i=1}^\infty \frac{1}{\delta_i^2} \int_0^t E[\|D_{t_0}X^n_t\|^2] \, dt_0 < \infty,
\]
and
\[
\sup_{n \geq 1} \sum_{i=1}^\infty \frac{1}{\delta_i^2} \int_0^t \int_0^t \frac{E[\|D_{t_0}X^n_t - D_{t_0'}X^n_{t'}\|^2]}{|t_0 - t_0'|^{1 + 2\beta_i}} \, dt_0 \, dt_0 < \infty,
\]
where \( \{\alpha_i\}_{i=1}^\infty, \beta = \{\beta_i\}_{i=1}^\infty \) and \( \delta = \{\delta_i\}_{i=1}^\infty \) are the sequences given in Theorem A.3 where \( D^i \) denotes the Malliavin derivative in the direction of the standard Brownian motion \( W^i, i \geq 1 \). Here, \( \| \cdot \| \) denote any matrix norm.

**Proof.** The most challenging estimate is the third one, the two others can be proven easily. Take \( t_0, t_0' > 0 \) such that \( 0 < t_0' < t_0 < t \). Using the chain rule for the Malliavin derivative, see [37, Proposition 1.2.3], we have
\[
D_{t_0}X^n_t = \lambda_i K_{H_i}(t, t_0) I_d + \int_{t_0}^t b_i'(t, X^n_{t_1}) D_{t_0}X^n_{t_1} \, dt_1
\]
P.a.s. for all \( 0 \leq t_0 \leq t \) where \( b_i'(t, z) = \left( \frac{\partial}{\partial z_j} b_i(z) \right)_{i,j=1,...,d} \) denotes the Jacobian matrix of \( b_i \) at a point \( (t, z) \) and \( I_d \) the identity matrix in \( \mathbb{R}^{d \times d} \). Thus we have

\[
\int_{\Delta_{t_0,t}^m} \prod_{i=1}^m \left| t_i - t_{i-1} \right| (1 - 4i \delta_i d' t_i H_{n_i} - (\frac{1}{2} - H_{i + \gamma_i}) q' \varepsilon_j) \, dt \leq C_m \frac{|t - t_0| m - (\frac{1}{2} - H_{i + \gamma_i}) q' \sum_{j=1}^m \varepsilon_j}{\Gamma \left( 1 + m - \left( \frac{1}{2} - H_{i + \gamma_i} \right) q' \sum_{j=1}^m \varepsilon_j \right)}
\]
On the other hand, observe that one may again write

\[ D_{t_0}^i X^n_t - D_{t_0}^i X^n_{t_0} = \lambda_i (K_{H_i}(t, t_0) I_d - K_{H_i}(t, t_0') I_d) \]

\[ + \int_{t_0}^t b'_n(t, X^n_{t_1}) D_{t_0}^i X^n_{t_1} dt_1 - \int_{t_0'}^t b'_n(t, X^n_{t_1}) D_{t_0}^i X^n_{t_1} dt_1 \]

\[ = \lambda_i (K_{H_i}(t, t_0) I_d - K_{H_i}(t, t_0') I_d) \]

\[ - \int_{t_0}^t b'_n(t, X^n_{t_1}) D_{t_0}^i X^n_{t_1} dt_1 + \int_{t_0}^t b'_n(t, X^n_{t_1})(D_{t_0}^i X^n_{t_1} - D_{t_0}^i X^n_{t_1}) dt_1 \]

\[ = \lambda_i K_{H_i}(t, t_0) I_d - (D_{t_0}^i X^n_{t_0} - \lambda_i K_{H_i}(t_0, t_0') I_d) \]

\[ + \int_{t_0}^t b'_n(t, X^n_{t_1})(D_{t_0}^i X^n_{t_1} - D_{t_0}^i X^n_{t_1}) dt_1, \]

where as in Proposition 4.8 we define

\[ \mathcal{K}_{t_0, t_0}^H(t) = K_{H_i}(t, t_0) - K_{H_i}(t, t_0'). \]

Iterating the above equation we arrive at

\[ D_{t_0}^i X^n_t - D_{t_0}^i X^n_{t_0} = \lambda_i K_{H_i}(t, t_0) I_d \]

\[ + \lambda_i \sum_{m=1}^\infty \int_{\Delta_{t_0, t_0}^m} \prod_{j=1}^m b'_n(t_j, X^n_{t_j}) K_{H_i}^m(t_m) I_d dt_m \cdots dt_1 \]

\[ - \left( I_d + \sum_{m=1}^\infty \int_{\Delta_{t_0, t_0}^m} \prod_{j=1}^m b'_n(t_j, X^n_{t_j}) dt_m \cdots dt_1 \right) \left( D_{t_0}^i X^n_{t_0} - \lambda_i K_{H_i}(t_0, t_0') I_d \right). \]

On the other hand, observe that one may again write

\[ D_{t_0}^i X^n_{t_0} - \lambda_i K_{H_i}(t_0, t_0') I_d = \lambda_i \sum_{m=1}^\infty \int_{\Delta_{t_0, t_0}^m} \prod_{j=1}^m b'_n(t_j, X^n_{t_j})(K_{H_i}(t_m, t_0') I_d) dt_m \cdots dt_1. \]

In a summary,

\[ D_{t_0}^i X^n_t - D_{t_0}^i X^n_{t_0} = \lambda_i I_1(t_0', t_0) + \lambda_i I_2^n(t_0', t_0) + \lambda_i I_3^n(t_0', t_0), \]

where

\[ I_1(t_0', t_0) := K_{H_i}^H(t_0, t_0') I_d = K_{H_i}(t, t_0) I_d - K_{H_i}(t, t_0') I_d \]

\[ I_2^n(t_0', t_0) := \sum_{m=1}^\infty \int_{\Delta_{t_0, t_0}^m} \prod_{j=1}^m b'_n(t_j, X^n_{t_j}) K_{H_i}^m(t_m) I_d dt_m \cdots dt_1 \]

\[ I_3^n(t_0', t_0) := - \left( I_d + \sum_{m=1}^\infty \int_{\Delta_{t_0, t_0}^m} \prod_{j=1}^m b'_n(t_j, X^n_{t_j}) dt_m \cdots dt_1 \right) \times \left( \sum_{m=1}^\infty \int_{\Delta_{t_0, t_0}^m} \prod_{j=1}^m b'_n(t_j, X^n_{t_j})(K_{H_i}(t_m, t_0') I_d) dt_m \cdots dt_1. \right) \]
Hence,
\[ E[\|D_t^n X_t^n - D_{t_0}^n X_t^n\|^2] \leq C \lambda^2 \left( E[\|I_1(t'_0, t_0)\|^2] + E[\|I_2^n(t'_0, t_0)\|^2] + E[\|I_3^n(t'_0, t_0)\|^2] \right). \]
It follows from Lemma B.1 and condition (21) that
\[ \sum_{i=1}^{\infty} \frac{\lambda^2}{1 - 2(3, -\alpha_i) \delta^2_i} \int_0^t \frac{\|I_1(t'_0, t_0)\|^2_{L^2(\Omega)}}{|t_0 - t'_0|^{1+2\beta}} dt_0 dt'_0 \]
\[ \leq \sum_{i=1}^{\infty} \frac{\lambda^2}{1 - 2(3, -\alpha_i) \delta^2_i} \int_0^t \int_0^t \frac{\|I_1(t'_0, t_0)\|^2_{L^2(\Omega)}}{|t_0 - t'_0|^{1+2\beta}} dt_0 dt'_0. \]
for a suitable choice of sequence \( \{\beta_i\}_{i \geq 1} \subset (0, 1/2). \)

Let us continue with the term \( I_2^n(t'_0, t_0) \). Then Theorem 3.2 Cauchy-Schwarz inequality and Lemma 4.5 imply
\[ E[\|I_2^n(t'_0, t_0)\|^2] \]
\[ \leq C(\|b_n\|_{L^p}) E \left[ \left\| \sum_{m=1}^{\infty} \int_{\Delta_{t_0,t}} \prod_{j=1}^m b_n(t_j, x + H_{t_j}) K_{t_0,t_0} H_{t_0} (t_m) I_d dt_m \cdots dt_1 \right\|^2 \right]^{1/2}, \]
where \( C : [0, \infty) \to [0, \infty) \) is the function from Lemma 4.5. Taking the supremum over \( n \) we have
\[ \sup_{n \geq 0} C(\|b_n\|_{L^p}) =: C_1 < \infty. \]

Let \( \| \cdot \| \) from now on denote the matrix norm in \( \mathbb{R}^{d \times d} \) such that \( \|A\| = \sum_{i,j=1}^d |a_{ij}| \) for a matrix \( A = \{a_{ij}\}_{i,j=1,...,d} \), then we have
\[ E[\|I_2^n(t'_0, t_0)\|^2] \leq C_1 \left( \sum_{m=1}^{\infty} \sum_{j,k=1}^d \sum_{l_m=1}^d \left\| \int_{\Delta_{t_0,t}} \frac{\partial}{\partial x_1} b_n(j)(t_1, x + H_{t_1}) \right. \right. \]
\[ \times \left. \left. \frac{\partial}{\partial x_2} b_n(l_1)(t_2, x + H_{t_2}) \cdots \frac{\partial}{\partial x_k} b_n(l_{m-1})(t_{m-1}, x + H_{t_{m-1}}) K_{t_0,t_0} H_{t_0} (t_m) I_d dt_m \cdots dt_1 \right\|_{L^1(\Omega, \mathbb{R})}\right)^2. \]

Now, the aim is to shuffle the four integrals above. Denote
\[ J_2^n(t'_0, t_0) := \int_{\Delta_{t'_0,t}} \frac{\partial}{\partial x_1} b_n(j)(t_1, x + H_{t_1}) \cdots \frac{\partial}{\partial x_k} b_n(l_{m-1})(t_{m-1}, x + H_{t_{m-1}}) K_{t_0,t_0} H_{t_0} (t_m) dt. \quad (41) \]
Then, shuffling \( J_2^n(t'_0, t_0) \) as shown in (3), one can write \( (J_2^n(t'_0, t_0))^2 \) as a sum of at most \( 2^{2m} \) summands of length \( 2m \) of the form
\[ \int_{\Delta_{t'_0,t}} g_1^n(t_1, x + H_{t_1}) \cdots g_2^n(t_{2m}, x + H_{t_{2m}}) dt_{2m} \cdots dt_1, \quad (42) \]
where for each \( l = 1, \ldots, 2m, \)
\[ g_l^n(\cdot, x + H_{t_1}) \in \left\{ \frac{\partial}{\partial x_k} b_n(j)(\cdot, x + H_{t_1}), \frac{\partial}{\partial x_k} b_n(j)(\cdot, x + H_{t_1}) K_{t_0,t_0} (\cdot), j, k = 1, \ldots, d \right\}. \]
Repeating this argument once again, we find that \( J^n_\gamma (t'_0, t_0) \) can be expressed as a sum of, at most, \( 2^{4m} \) summands of length \( 4m \) of the form

\[
\int_{\Delta_{t_0,t}^{4m}} g^n_{1}(t_1, x + \mathbb{B}^H) \cdots g^n_{4m}(t_{4m}, x + \mathbb{B}^H_{t_{4m}}) dt_{4m} \cdots dt_1,
\]

(43)

where for each \( l = 1, \ldots, 4m \),

\[
g^n_l(\cdot, x + \mathbb{B}^H) \in \left\{ \frac{\partial}{\partial x_k} b^{(j)}_n(\cdot, x + \mathbb{B}^H) \frac{\partial}{\partial x_k} b^{(j)}_n(\cdot, x + \mathbb{B}^H) K^{H}(\cdot, t), \ j, k = 1, \ldots, d \right\}.
\]

It is important to note that the function \( K^{H}(\cdot, t) \) appears only once in term (41) and hence only four times in term (43). So there are indices \( j_1, \ldots, j_4 \in \{1, \ldots, 4m\} \) such that we can write (43) as

\[
\int_{\Delta_{t_0,t}^{4m}} \left( \prod_{j=1}^{4m} b^{(j)}_n(t_j, x + \mathbb{B}^H_{t_j}) \right) \prod_{l=1}^{4m} K^{H}(t_l, t_{l0}) \ dt_{4m} \cdots dt_1,
\]

where

\[
b^{(j)}_n(\cdot, x + \mathbb{B}^H) \in \left\{ \frac{\partial}{\partial x_k} b^{(j)}_n(\cdot, x + \mathbb{B}^H), \ j, k = 1, \ldots, d \right\}, \ l = 1, \ldots, 4m.
\]

The latter enables us to use the estimate from Proposition 4.8 with \( \sum_{i=1}^{4m} \delta_i(H) = 4 \) and thus we obtain that

\[
\left( E(J^n_\gamma (t'_0, t_0))^4 \right)^{1/4} \leq C^m m^{\gamma_0 - \frac{1}{2}} H^{(\gamma_0 + \gamma_0)} \left( \frac{|t - t_0|^{m-4(\frac{1}{2} - H_i + \gamma_i)}q'}{(1 + 4m - 4(\frac{1}{2} - H_i + \gamma_i) q')} \right)^{1/4},
\]

for some constant \( C \) not depending on \( m \), where \( q' \) is the conjugate exponent of \( q \geq 2 \).

The remaining series is summable over \( j, k, l_1, \ldots, l_{m-1} \) and \( m \) so we just need to verify that the double integral is finite for suitable \( \gamma_i \)'s and \( \beta_i \)'s. Indeed,

\[
\int_0^t \int_0^t \frac{|t_0 - t_0'|^{2\gamma_0 - 2 - 2\beta_i}}{|t_0 - t_0'|^{2\gamma_0}} \left( t - t_0 \right)^{2/q - 2(\frac{1}{2} - H_i + \gamma_i)} dt_0 dt_0' < \infty,
\]

whenever \( 2/q - 2(\frac{1}{2} - H_i + \gamma_i) < -1, 2\gamma_0 - 2 - 2\beta_i < -1 \) and \( -2 \left( \frac{1}{2} - H_i + \gamma_i \right) - 2\gamma_0 > -1 \) which is fulfilled if for instance \( \gamma_i < H_i/2 \) and \( 0 < \beta_i < \gamma_i \), for all \( i = 1, \ldots, m \). Finally the sums over \( i \geq 1 \) also converge since we have \( \lambda_i \) satisfying (21).

For the term \( I^n_\gamma \) we may use Theorem 3.2 Cauchy-Schwarz inequality twice and observe that the first factor of \( I^n_\gamma \) is bounded uniformly in \( t_0, t \in [0, T] \) by a simple application of Proposition 4.8 with \( \varepsilon_j = 0 \) for all \( j = 1, \ldots, m \). Then, the remaining estimate is fairly similar to the case of \( I^n_\gamma \) by using the second estimate from Proposition 4.8. As for the estimate for the Malliavin derivative the reader may agree that the arguments are analogous. \( \square \)

The following is a consequence of combining Lemma 4.7 and Proposition 4.9.
Corollary 4.10. For every \( t \in [0,T] \) and continuous function \( \varphi : \mathbb{R}^d \to \mathbb{R} \) with at most linear growth we have
\[
\varphi(X_t^n) \xrightarrow{n \to \infty} \varphi(E[X_t|\mathcal{F}_t])
\]
strongly in \( L^2(\Omega) \). In addition, \( E[X_t|\mathcal{F}_t] \) is Malliavin differentiable along any direction \( W^i, i \geq 1 \) of \( \mathbb{B}^H \). Moreover, the solution \( X \) is \( \mathcal{F} \)-adapted, thus being a strong solution.

Proof. This is an immediate consequence of the relative compactness from Theorem A.3 in connection with Proposition 4.9 and by Lemma 4.7 we can identify the limit as being \( E[X_t|\mathcal{F}_t] \) then the convergence holds for any continuous functions as well. The Malliavin differentiability of \( E[X_t|\mathcal{F}_t] \) is shown by taking \( \varphi = I_d \) and the second estimate in Proposition 4.9 together with [37, Proposition 1.2.3]. \( \square \)

Finally, we can verify item (4) of our scheme.

Corollary 4.11. The constructed solution \( X \) of (26) is strong.

Proof. It remains to prove that \( X_t \) is \( \mathcal{F}_t \)-measurable for every \( t \in [0,T] \) and by Remark 4.6 it then follows that there exists a strong solution in the usual sense that is Malliavin differentiable. Indeed, let \( \varphi \) be a globally Lipschitz continuous function, then by Corollary 4.10 we have, for a subsequence \( n_k, k \geq 0 \), that
\[
\varphi(X_t^{n_k}) \to \varphi(E[X_t|\mathcal{F}_t]), \quad P - a.s.
\]
as \( k \to \infty \).

On the other hand, by Lemma 4.7 we also have
\[
\varphi(X_t^{n_k}) \to E[\varphi(X_t)|\mathcal{F}_t]
\]
weakly in \( L^2(\Omega) \). By the uniqueness of the limit we immediately have
\[
\varphi(E[X_t|\mathcal{F}_t]) = E[\varphi(X_t)|\mathcal{F}_t], \quad P - a.s.
\]
which implies that \( X_t \) is \( \mathcal{F}_t \)-measurable for every \( t \in [0,T] \). \( \square \)

Finally, we turn to step (5) and complete this section by showing pathwise uniqueness. Following the same argument as in [41, Chapter IX, Exercise (1.20)] we see that strong existence and uniqueness in law implies pathwise uniqueness. The argument does not rely on the process being a semimartingale. Hence, uniqueness in law is enough. The following lemma actually implies the desired uniqueness by estimate (33) in connection with [25, Theorem 7.7].

Lemma 4.12. Let \( X \) be a strong solution of (26) where \( b \in L^p_q, p, q \in (2, \infty] \). Then the estimates (29) and (30) hold for \( X \) in place of \( \mathbb{B}^H \). As a consequence, uniqueness in law holds for equation (26) and since \( X \) strong, pathwise uniqueness follows.

Proof. Assume first that \( b \) is bounded. Fix any \( n \geq 1 \) and set
\[
\eta^n_s = K_{H_n}^{-1} \left( \frac{1}{\lambda_n} \int_0^s b(r, X_r)dr \right) (s).
\]
Since $b$ is bounded it is easy to see from (32) by changing $B^H$ with $X$ and bounding $b$ that for every $\kappa \in \mathbb{R}$,

$$E_{\tilde{P}} \left[ \exp \left\{ -\kappa \int_0^T (\eta^n_s)^* dW^n_s - \kappa^2 \int_0^T |\eta^n_s|^2 ds \right\} \right] = 1,$$

(44)

where

$$\frac{d\tilde{P}}{dP} = \exp \left\{ -\int_0^T (\eta^n_s)^* dW^n_s - \frac{1}{2} \int_0^T |\eta^n_s|^2 ds \right\}.$$

Hence, $X_t - x$ is a regularizing fractional Brownian motion with Hurst sequence $H$ under $\tilde{P}$. Define

$$\xi^\kappa_T := \exp \left\{ -\kappa \int_0^T (\eta^n_s)^* dW^n_s - \kappa^2 \int_0^T |\eta^n_s|^2 ds \right\}.$$

Then,

$$E_{\tilde{P}}[\xi^\kappa_T] = E_{\tilde{P}} \left[ \exp \left\{ -\kappa \int_0^T (\eta^n_s)^* dW^n_s - \frac{\kappa}{2} \int_0^T |\eta^n_s|^2 ds \right\} \right]$$

$$= E_{\tilde{P}} \left[ \exp \left\{ -\kappa \int_0^T (\eta^n_s)^* dW^n_s - \kappa^2 \int_0^T |\eta^n_s|^2 ds \right\} \exp \left\{ \left( \kappa^2 + \frac{\kappa}{2} \right) \int_0^T |\eta^n_s|^2 ds \right\} \right]$$

$$\leq \left( E_{\tilde{P}} \left[ \exp \left\{ 2 \kappa^2 + \frac{\kappa}{2} \int_0^T |\eta^n_s|^2 ds \right\} \right] \right)^{1/2}$$

in view of (44).

On the other hand, using (34) with $X$ in place of $B^H$ we have

$$\int_0^T |\eta_s|^2 ds \leq C_{\varepsilon, \lambda_n, H_n, T} \left( 1 + \int_0^T |b(r, X_r)|^{\frac{1+\varepsilon}{2}} dr \right), \quad P - a.s.$$

for any $\varepsilon \in (0, 1)$. Hence, applying Lemma 4.3 we get

$$E_{\tilde{P}}[\xi^\kappa_T] \leq e^{\kappa^2 + \frac{\kappa}{2} \left( A \left( C_{\varepsilon, \lambda_n, H_n, T} \left| \kappa^2 + \frac{\kappa}{2} \right| \right) \right)^{1/2}},$$

where $A$ is the analytic function from Lemma 4.3.

Furthermore, observe that for every $\kappa \in \mathbb{R}$ we have

$$E_P[\xi^\kappa_T] = E_{\tilde{P}}[\xi^{\kappa-1}_T].$$

(45)

In fact, (45) holds for any $b \in L_p^2$ by considering $b_n := b1_{\{|b| \leq n\}}, \ n \geq 1$ and then letting $n \to \infty$.

Finally, let $\delta \in (0, 1)$ and apply Hölder’s inequality in order to get

$$E_P \left[ \int_0^T h(t, X_t) dt \right] \leq T^\delta \left( E_P[(\xi^1_T)^{\frac{1+\delta}{2}}] \right)^{\frac{1}{1+\delta}} \left( E_P \left[ \left( \int_0^T h(t, X_t)^{1+\delta} dt \right)^{\frac{1}{1+\delta}} \right] \right)^{\frac{1}{1+\delta}},$$

and

$$E_P \left[ \exp \left\{ \int_0^T h(t, X_t) dt \right\} \right] \leq T^\delta \left( E_P[(\xi^1_T)^{\frac{1+\delta}{2}}] \right)^{\frac{1}{1+\delta}} \left( E_P \left[ \exp \left\{ (1 + \delta) \int_0^T h(t, X_t) dt \right\} \right] \right)^{\frac{1}{1+\delta}},$$
for every Borel measurable function. Since we know that \(X_t - x\) is a regularizing fractional Brownian motion with Hurst sequence \(H\) under \(\tilde{P}\), the result follows by Lemma 4.3 by choosing \(\delta\) close enough to 0.

\[\square\]

5. INFINITELY DIFFERENTIABLE FLOWS

From now on, we denote by \(X_t^{s,x}\) the solution to the following SDE driven by a regularizing fractional Brownian motion \(B^H\) with Hurst sequence \(H\) satisfying the assumptions from previous sections,

\[dX_t^{s,x} = b(t, X_t^{s,x})dt + dB^H_t, \quad s, t \in [0, T], \quad s \leq t, \quad X_s^{s,x} = x \in \mathbb{R}^d.\tag{46}\]

We will then assume the hypotheses from Theorem 4.2 on \(b\) and \(H\). The next result tells us that the stochastic mapping \(x \mapsto X_t^{s,x}\) is \(P\)-a.s. infinitely many times continuously differentiable. In particular, it shows that the strong solution constructed in the former section, in addition to being Malliavin differentiable, is also smooth in \(x\) and, although we will not prove it explicitly here, it is also smooth in the Malliavin sense, and since Hörmander’s condition is met then implies that the densities of the marginals are also smooth.

**Theorem 5.1.** Let \(b \in C^\infty_c([0, T] \times \mathbb{R}^d; \mathbb{R}^d)\) and \(U \subset \mathbb{R}^d\) a bounded, open subset. Fix \(\alpha \geq 1\). Then, we have

\[\sup_{s,t \in [0,T]} \sup_{x \in U} E \left[ \left\| \frac{\partial^k}{\partial x^k} X_t^{s,x} \right\|^q \right] \leq C_{p,q,d,H,k,\alpha,T}(\|b\|_{L_p^2}),\]

for arbitrary \(k \geq 1\), where \(C_{p,q,d,H,k,\alpha,T} : [0, \infty) \to [0, \infty)\) is an analytic function, depending on \(k, d, H, \alpha\) and \(T\).

**Proof.** Given a \(k\)-times continuously differentiable vector field \(f : \mathbb{R}^d \to \mathbb{R}^d\), \(f(x) = (f^{(1)}(x), \ldots, f^{(d)}(x))\), we have that \(Df : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)\) where \(L(\mathbb{R}^d, \mathbb{R}^d)\) denotes the space of linear forms from \(\mathbb{R}^d\) to \(\mathbb{R}^d\) and hence \(Df(x) \in L(\mathbb{R}^d, \mathbb{R}^d)\). The second derivative is then \(D^2f : \mathbb{R}^d \to L(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d))\), \(D^2f(x) : L(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d)) \cong L^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)\) which is the space of bilinear forms from \(\mathbb{R}^d \times \mathbb{R}^d\) to \(\mathbb{R}^d\). Further, the \(k\)-th derivative of \(f\) can be seen as an operator \(D^k f : \mathbb{R}^d \to L(\mathbb{R}^d \times \cdots \times \mathbb{R}^d, \mathbb{R}^d)\) and hence \(D^k f(x) \in L(\mathbb{R}^d \times \cdots \times \mathbb{R}^d, \mathbb{R}^d)\) being a \(k\)-multilinear form from \(\mathbb{R}^d \times \cdots \times \mathbb{R}^d\) to \(\mathbb{R}^d\). We will further identify the space of \(k\)-multilinear forms with the \(k + 1\)-times tensor product of \(\mathbb{R}^d\), i.e.

\[L(\mathbb{R}^d \times \cdots \times \mathbb{R}^d, \mathbb{R}^d) \cong \otimes_{i=1}^{k+1} \mathbb{R}^d.\]

We endow \(\mathbb{R}^{\otimes_{i=1}^{k+1} \otimes d}\) with the Kronecker product and denote

\[D^k f(x) = \left\{ \frac{\partial^k}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_k}} f^{(i)}(x) \right\}_{i_1, \ldots, i_k, j = 1, \ldots, d}.\]

Since \(b \in C^\infty_c([0, T] \times \mathbb{R}^d)\), we know (compare [24]) that the solution of (46), \(X_t^{s,x}\) is smooth in the initial value \(x\) and that

\[\frac{\partial}{\partial x} X_t^{s,x} = I_d + \int_s^t Db(u, X_u^{s,x}) \frac{\partial}{\partial x} X_u^{s,x} du.\]
Using Picard’s iteration we get
\[
\frac{\partial}{\partial x} X^{s,x}_t = I_d + \sum_{m_1 \geq 1} \int_{\Delta_{s,t}^{m_1}} Db(u_1, X^{s,x}_{u_1}) \cdots Db(u_m, X^{s,x}_{u_m}) du_m \cdots du_1. \quad (47)
\]

Now apply \( \frac{\partial}{\partial x} \) again, then by dominated convergence we have
\[
\frac{\partial^2}{\partial x^2} X^{s,x}_t = \sum_{m \geq 1} \int_{\Delta_{s,t}^{m}} \frac{\partial}{\partial x} \left[ Db(u_1, X^{s,x}_{u_1}) \cdots Db(u_m, X^{s,x}_{u_m}) \right] du_m \cdots du_1. \quad (48)
\]

We can expand (48) using Leibniz’s rule as follows
\[
\frac{\partial}{\partial x} \left[ Db(u_1, X^{s,x}_{u_1}) \cdots Db(u_m, X^{s,x}_{u_m}) \right] = \sum_{r=1}^{m} Db(u_1, X^{s,x}_{u_1}) \cdots D^2 b(u_r, X^{s,x}_{u_r}) \frac{\partial}{\partial x} X^{s,x}_t \cdots Db(u_m, X^{s,x}_{u_m}).
\]

Inserting the representation (47) for \( \frac{\partial}{\partial x} X^{s,x}_t \) in this case we have that
\[
\frac{\partial^2}{\partial x^2} X^{s,x}_t = \sum_{m_1 \geq 1} \int_{\Delta_{s,t}^{m_1}} \sum_{r=1}^{m_1} Db(u_1, X^{s,x}_{u_1}) \cdots D^2 b(u_r, X^{s,x}_{u_r})
\]
\[
\times \left( I_d + \sum_{m_2 \geq 1} \int_{\Delta_{s,t}^{m_2}} Db(v_1, X^{s,x}_{v_1}) \cdots Db(v_{m_2}, X^{s,x}_{v_{m_2}}) dv_{m_2} \cdots dv_1 \right)
\]
\[
\times Db(u_{r+1}, X^{s,x}_{u_{r+1}}) \cdots Db(u_{m_1}, X^{s,x}_{u_{m_1}}) du_{m_1} \cdots du_1.
\]

We reallocate terms by dominated convergence and respecting the order of matrices
\[
\frac{\partial^2}{\partial x^2} X^{s,x}_t = \sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \int_{\Delta_{s,t}^{m_1}} Db(u_1, X^{s,x}_{u_1}) \cdots D^2 b(u_r, X^{s,x}_{u_r}) \cdots Db(u_{m_1}, X^{s,x}_{u_{m_1}}) du_{m_1} \cdots du_1
\]
\[
+ \sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \sum_{m_2 \geq 1} \int_{\Delta_{s,t}^{m_1}} \int_{\Delta_{s,t}^{m_2}} Db(u_1, X^{s,x}_{u_1}) \cdots D^2 b(u_r, X^{s,x}_{u_r})
\]
\[
\times Db(v_1, X^{s,x}_{v_1}) \cdots Db(v_{m_2}, X^{s,x}_{v_{m_2}}) Db(u_{r+1}, X^{s,x}_{u_{r+1}}) \cdots Db(u_{m_1}, X^{s,x}_{u_{m_1}}) dv_{m_2} \cdots dv_1 du_{m_1} \cdots du_1.
\]
\[
= : I_1 + I_2.
\]
Now, we iterate this scheme, up to step $k \geq 2$. We will obtain that \( \frac{\partial^k}{\partial x^k} X_i^{s,x} \) is a sum of $2^{k-1}$ terms. That is

\[
\frac{\partial^k}{\partial x^k} X_i^{s,x} = I_1 + \cdots + I_{2^{k-1}},
\]

where each $I_i$, $i = 1, \ldots, 2^{k-1}$ is a sum of iterated integrals over sets of the form $\Delta_{s,u}^{m_j}$, $s < u < t$, $j = 1, \ldots, k$ with integrands having at most one product factor $D^k b$ and the other factors are of the form $D^j b$, $j \leq k - 1$.

In order to simplify the reading we introduce some notation. For given indices $m := (m_1, \ldots, m_k)$ and $r := (r_1, \ldots, r_{k-1})$ denote

\[
m_{1:j} := \sum_{i=1}^j m_i \quad \text{and} \quad m_{j:k} := \sum_{i=j}^k m_i
\]

and

\[
\sum_{m \geq 1 \atop r_1 \leq m_1, \ldots, r_{k-1} \leq m_{k-1}} := \sum_{m_1 \geq 1} \sum_{r_1 = 1}^{m_1} \sum_{m_2 \geq 1} \sum_{r_2 = 1}^{m_2} \cdots \sum_{m_{k-1} \geq 1} \sum_{r_{k-1} = 1}^{m_{k-1}}.
\]

Moreover, denote

\[
\int_{\Delta^m} \cdot du := \int_{\Delta_{s,u}^{m_1}} \int_{\Delta_{s,u}^{m_2}} \cdots \int_{\Delta_{s,u}^{m_{k-1}}} \cdot du
\]

where

\[
u = (u_1^{k}, \ldots, u_1^{k}, \ldots, u_1^{1}, \ldots, u_1^{1}) \in [0, T]^{m_{1:k}}.
\]

Then, more generally, \( \frac{\partial^k}{\partial x^k} X_i^{s,x} = \sum_{j=0}^{2^{k-1}} I_{2^j}, k \geq 2 \). We will carry out the computations for $I_{2^{k-1}}$, as it can be seen, all terms are treated analogously by choosing $j = 1, \ldots, 2^{k-1}$. Then $I_{2^{k-1}}$ will take the following form

\[
I_{2^{k-1}} = \sum_{m \geq 1 \atop r_1 \leq m_1, \ldots, r_{k-1} \leq m_{k-1}} \int_{\Delta^m} G_k^X (u) du_1^k \cdots du_{m_k}^k \cdots du_1^1 \cdots du_{m_1}^1,
\]

where the integrand $G_k^X (u) = \{ g_{i_1, \ldots, i_{k-1}, j} (u) \}_{i_1, \ldots, i_{k-1}, j = 1, \ldots, d}$ is an element in $\mathbb{R}^{d \otimes \cdots \otimes d}$ with entries given by sums of at most $C(d, k)^{m_1 + \cdots + m_k}$ terms, which are products of length $m_1 + \cdots + m_k$ with respect to functions belonging to the set

\[
g_{i_1, \ldots, i_{k-1}, j} (u) \in \left\{ \frac{\partial^k}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_{k-1}}} b^{(s)} (u, X_s^{u,x}), i, l_1, \ldots, l_{k-1}; j = 1, \ldots, d \right\}.
\]

Let $\alpha \in [1, \infty)$ choose $r, s \in [1, \infty)$ such that $s \alpha = 2^{\alpha'}$ for some integer $\alpha'$ and $\frac{1}{r} + \frac{1}{s} = 1$. Then using Girsanov’s theorem in connection with Proposition 4.5 and
Hölder’s inequality we have for a constant $C := C(||b||_{L^p}) > 0$

$$E [||I_{2k-1}||^\alpha] \leq CE \left[ \sum_{m \geq 1} \sum_{\substack{r_l \leq m_1, i \leq \cdots \leq m, l = 1, \ldots, k-1}} \left\| \int_{\Delta^m} G_k^B (u) du_1^k \cdots du_{m_1}^1 \right\|^{2\alpha/2\alpha'} \right].$$

Now using the maximum norm on $\mathbb{R}^{d \otimes (k+1)}$ we get

$$E [||I_{2k-1}||^\alpha] \leq C \left( \sum_{m \geq 1} \sum_{\substack{r_l \leq m_1, i \leq \cdots \leq m, l = 1, \ldots, k-1}} \left\| \int_{\Delta^m} \mathcal{H}_{k,i}^B (u) du_1^k \cdots du_{m_1}^1 \right\|^{\alpha} \right),$$

where $\# I \leq C(d,k)^{m_1+\cdots+m_k}$ and

$$\mathcal{H}_{k,i}^B (u) := \prod_{l=1}^{m_1} h_l(u_l), \quad h_l \in \Lambda, \quad l = 1, \ldots, m_1:k,$$

being $h_l$ elements in the set of functions

$$\Lambda := \left\{ \frac{\partial^k}{\partial x_i \partial x_{i_1} \cdots \partial x_{i_{k-1}}} b^{(i)}(u_l, x + \mathbb{B}^H u_l), \ i, i_1, \ldots, i_{k-1}, j = 1, \ldots, d \right\}.$$

Using (9), one actually shows that

$$\int_{\Delta^m} \mathcal{H}_{k,i}^B (u) du = \sum_{\sigma \in A_m} \int_{\Delta^m} \prod_{l=1}^{m_1} f_{\sigma(l)}(w_l) dw, \quad f_l \in \Lambda, \quad l = 1, \ldots, m_1:k,$$

for a set of indices $A_m$ such that $\# A_m \leq C^{m_1:k}$ for a sufficiently large constant $C > 0$. As a consequence

$$E [||I_{2k-1}||^\alpha] \leq C \left( \sum_{m \geq 1} \sum_{\substack{r_l \leq m_1, i \leq \cdots \leq m, l = 1, \ldots, k-1}} \left\| \int_{\Delta^m} \prod_{l=1}^{m_1} f_{\sigma(l)}(w_l) dw \right\|^{\alpha} \right),$$

Define

$$J := \int_{\Delta^m} \prod_{l=1}^{m_1} f_{\sigma(l)}(w) dw, \quad f_l \in \Lambda, \quad l = 1, \ldots, m_1:k.$$

Then using the same argument as in (42) by exploiting the identity in (9) repeatedly, we find that $J$ can be written to the power 2 as a sum of, at most $2^{2m_1:k}$ of length $2m_1:k$ of the form

$$\int_{\Delta^m} \prod_{l=1}^{2m_1:k} f_{\sigma(l)}(w) dw, \quad f_l \in \Lambda, \quad l = 1, \ldots, 2m_1:k.$$
Repeating this argument, we find that we can write $J^{2^{\alpha'}}$ as a sum of at most $2^{\alpha'}m_{1,k}$ of the form

$$\int_{\Delta^{2^{\alpha'}m_{1,k}}} \prod_{l=1}^{2^{\alpha'}m_{1,k}} f_{\sigma(l)}(w)dw, \quad f_l \in \Lambda, \quad l = 1, \ldots, 2^{\alpha'}m_{1,k}.$$

Finally, taking expectation we can apply the estimate from Proposition 4.8 with $\varepsilon_j = 0$ for all $j$. Then we can find a constant $C_{p,q,d,H,k,T} > 0$ such that

$$(E\|I_{2k-1}\|^\alpha)^{1/\alpha} \leq C^{1/\alpha} \sum_{m_1,\ldots,m_k \geq 1} \frac{C(d,k)^{m_{1,k}}C_{p,q,d,H,k,T}^{2^{\alpha'}m_{1,k}}\|b\|_{L^q_{2^k}}}{[(2^{\alpha'}m_{1,k})!]^{1/2^{\alpha'}}} \left( \frac{|t-s|^{m_{1,k}}}{\Gamma(2^{\alpha'}m_{1,k}+1)2^{-\alpha'}} \right)^{1/\alpha},$$

$$\leq \sum_{m_1,\ldots,m_k \geq 1} \frac{C_{p,q,d,H,k,T}^{m_{1,k}}}{[(2^{\alpha'}m_{1,k})!]^{1/2^{\alpha'}}} \frac{\Gamma(2^{\alpha'}m_{1,k}+1)^{1/2^{\alpha'}}}{\Gamma \left( (2^{\alpha'}m_{1,k}) \right)}\frac{\|b\|_{L^q_{2^k}}^{m_{1,k}}}{\|b\|_{L^q_{2^k}}},$$

where $C_{p,q,d,H,k,\alpha,T} > 0$ denotes the collection of all constants obtained so far.

Now,

$$(E\|I_{2k-1}\|^\alpha)^{1/\alpha} \leq \sum_{m_1,\ldots,m_k \geq 1} C_{p,q,d,H,k,\alpha,T}^{m_{1,k}} \frac{\|b\|_{L^q_{2^k}}^{m_{1,k}}}{\Gamma(2^{\alpha'}m_{1,k}+1)^{1/2^{\alpha'}}},$$

where we used the inequality $\Gamma(x+y) \geq \Gamma(x)\Gamma(y)$, $x, y \geq 1$ in the second inequality.

As a result,

$$\sup_{s,t \in [0,T]} \sup_{x \in U} E \left[ \left\| \frac{\partial^k}{\partial x^k} X_t^{s,x} \right\| \right]^{\alpha} \leq C_{p,q,d,H,k,\alpha,T} \left( \|b\|_{L^q_{2^k}} \right)^k$$

for an analytic function $C_{p,q,d,H,k,\alpha,T} : [0, \infty) \to [0, \infty)$, depending on $p, q, d, H, \alpha$, and $T$.

The following is the main result of this section and shows that the regularizing fractional Brownian motion $B^H$ generates an infinitely continuously differentiable random field $x \mapsto X_t^b$ when $b$ belongs to $C^{2\alpha}_{\alpha}$ for any $p, q \in (2, \infty)$. □
**Theorem 5.2.** Assume \( b \in L^4_{2,p} \), \( p, q \in (2, \infty] \). Let \( U \subset \mathbb{R}^d \) and open and bounded subset and \( X = \{X_t, t \in [0, T]\} \) the solution of (26). Then it follows that

\[
X_t \in \bigcap_{k \geq 1} \bigcap_{\alpha > 1} L^2(\Omega, W^{k, \alpha}(U)).
\]

**Proof.** First of all, approximate the irregular drift vector field \( b \) by a sequence of functions \( b_n : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, n \geq 0 \) in \( C^\infty_c([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \) in the sense of (27). Denote by \( X^{n,x} = \{X^{n,x}_t, t \in [0, T]\} \), the corresponding solution to (26) starting from \( x \in \mathbb{R}^d \) when \( b \) is replaced by \( b_n \).

Observe that for any test function \( \varphi \in C^\infty_0(U, \mathbb{R}^d) \) and fixed \( t \in [0, T] \) the set of random variables

\[
\langle X^{n,x}_t, \varphi \rangle := \int_U \{X^{n,x}_t, \varphi(x)\} dx, \quad n \geq 0
\]

is relatively compact in \( L^2(\Omega) \). To show this, we use the compactness criterion from Appendix, in Theorem \( 1.3 \) in terms of the Malliavin derivative. Since the Malliavin derivative is a closed linear operator we have

\[
E[\int_0^T |D_s^{i,j}(X^{n,x}_t, \varphi)|^2 ds] = \sum_{l=1}^d \left( \int_U E[D_s^{i,j}(X^{n,x}_t, \varphi)](x) dx \right)^2 \leq d\|\varphi\|_{L^2(\mathbb{R}^d)}^2 \lambda(\text{supp}(\varphi)) \sup_{x \in U} \sum_{i=1}^\infty \int_0^T \|D_s^{i,j}X^{n,x}_t\|^2 ds,
\]

where \( D_s^{i,j} \) denotes the Malliavin derivative in the direction of \( W^{i,j} \) where \( W^i \) is the \( d \)-dimensional standard Brownian motion defining \( B^{H,i} \) and \( W^{i,j} \) its \( j \)-th component, \( \lambda \) the Lebesgue measure on \( \mathbb{R}^d \), \( \text{supp}(\varphi) \) the support of \( \varphi \) and \( \| \cdot \| \) a matrix norm. Then taking the sum over all \( j = 1, \ldots, d \) and using Proposition \( 4.9 \) we obtain

\[
\sup_{n \geq 0} \sum_{i=1}^\infty \|D_s^{i}(X^{n,x}_t, \varphi)\|^2_{L^2(\Omega \times [0, T])} \leq C\|\varphi\|^2_{L^2(\mathbb{R}^d)} \lambda(\text{supp}(\varphi))
\]

In a similar manner we have

\[
\sup_{n \geq 0} \sum_{i=1}^\infty \int_0^T \int_0^T E[|D_{s'}^{i}(X^{n,x}_t, \varphi) - D_{s'}^{i}(X^{n,x}_s, \varphi)|^2] \frac{ds'}{|s' - s|^{1 + 2\beta}} < \infty
\]

for some sequence \( \beta = \{\beta_i\}_{i=1}^\infty \in (0, 1/2). \) Hence \( \langle X^{n,x}_t, \varphi \rangle, n \geq 0 \) is relatively compact in \( L^2(\Omega) \). Let us denote by \( Y_t(\varphi) \) its limit after taking (if necessary) a subsequence.

Following exactly the same reasoning as in Lemma \( 4.7 \) one can show that

\[
\langle X^{n,x}_t, \varphi \rangle \xrightarrow{n \to \infty} \langle X_t, \varphi \rangle
\]

weakly in \( L^2(\Omega) \). Then by uniqueness of the limit we can establish that

\[
Y_t(\varphi) = \langle X_t, \varphi \rangle
\]

in \( L^2(\Omega) \).

Note that there exists a subsequence \( n(j) \) such that \( \langle X^{n(j),x}_t, \varphi \rangle \) converges for every \( \varphi \), that is, \( n(j) \) is independent of \( \varphi \).
We have that $X^n_{t}$ is bounded in the Sobolev norm $L^2(\Omega, W^{k,\alpha}(U))$ for each $n \geq 0$ and $k \geq 1$. Indeed, by Proposition 5.1 we have

$$\sup_{n \geq 0} \|X^n_{t}\|_{L^2(\Omega, W^{k,\alpha}(U))}^2 = \sup_{n \geq 0} \sum_{i=0}^k E \left[ \left\| \frac{\partial^i}{\partial x^i} X^n_{t} \right\|_{L^2(U)}^2 \right]$$

$$\leq \sum_{i=0}^k \int_U \sup_{n \geq 0} E \left[ \left\| \frac{\partial^i}{\partial x^i} X^n_{t} \right\|_{L^2(U)}^\alpha \right] \, dx$$

$$< \infty.$$  

Since $L^2(\Omega, W^{k,\alpha}(U))$, $\alpha \in (1, \infty)$ is reflexive we get that the set $\{X^n_{t}\}_{n \geq 0}$ is weakly compact in $L^2(\Omega, W^{k,\alpha}(U))$ for every $k \geq 1$. Thus, there exists a subsequence $n(j)$, $j \geq 0$ such that

$$X^n_{t} \rightarrow X^*_t \quad \text{weakly} \quad \text{in} \quad L^2(\Omega, W^{k,\alpha}(U)).$$

On the other hand, we have proven that $X^n_{t} \rightarrow X^*_t$ strongly in $L^2(\Omega)$, so by uniqueness of the limit we can conclude that

$$X^n_{t} = Y \in L^2(\Omega, W^{k,\alpha}(U)), \quad P - a.s.$$  

Moreover, for all $A \in \mathcal{F}$ and $\varphi \in C^\infty_c(\mathbb{R}^d, \mathbb{R}^d)$ we have

$$E[1_A \langle X^n_t, \varphi' \rangle] = \lim_{j \rightarrow \infty} E[1_A \langle X^n_{t(j)}, \varphi' \rangle] = \lim_{j \rightarrow \infty} -E[1_A \langle \frac{\partial}{\partial x} X^n_{t(j)}, \varphi \rangle] = -E[1_A \langle Y^*, \varphi \rangle]$$

and thus

$$\langle X^n_t, \varphi \rangle = -\langle Y^*, \varphi \rangle, \quad P - a.s.$$  

Since $k \geq 1$ is arbitrary, the proof follows.  

**APPENDIX A. A COMPACTNESS CRITERION FOR SUBSETS OF $L^2(\Omega)$**

The following result which is originally due to [11] in the finite dimensional case and which can be e.g. found in [7], provides a compactness criterion of square integrable functionals of cylindrical Wiener processes on a Hilbert space:

**Theorem A.1.** Let $B_t$, $0 \leq t \leq T$ be a cylindrical Wiener process on a separable Hilbert space $H$ with respect to a complete probability space $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is generated by $B_t$, $0 \leq t \leq T$. Further, let $L_{HS}(H, \mathbb{R})$ be the space of Hilbert-Schmidt operators from $H$ to $\mathbb{R}$ and let $D : \mathbb{D}^{1,2} \rightarrow L^2(\Omega; L^2([0, T]) \otimes L_{HS}(H, \mathbb{R}))$ be the Malliavin derivative in the direction of $B_t$, $0 \leq t \leq T$, where $\mathbb{D}^{1,2}$ is the space of Malliavin differentiable random variables in $L^2(\Omega)$.

Suppose that $C$ is a self-adjoint compact operator on $L^2([0, T]) \otimes L_{HS}(H, \mathbb{R})$ with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbb{D}^{1,2} : \|G\|_{L^2(\Omega)} + \|C^{-1}DG\|_{L^2(\Omega; L^2([0, T]) \otimes L_{HS}(H, \mathbb{R}))} \leq c \right\}$$

is relatively compact in $L^2(\Omega)$.  


In this paper we aim at using a special case of the previous theorem, which is more suitable for explicit estimations. To this end we need the following auxiliary result from [11].

**Lemma A.2.** Denote by $v_s, s \geq 0$ with $v_0 = 1$ the Haar basis of $L^2([0, 1])$. Define for any $0 < \alpha < \frac{1}{2}$ the operator $A_\alpha$ on $L^2([0, 1])$ by

$$A_\alpha v_s = 2^{k_\alpha} v_s, \text{ if } s = 2^k + j, \ k \geq 0, \ 0 \leq j \leq 2^k$$

and

$$A_\alpha 1 = 1.$$  

Then for $\alpha < \beta < \frac{1}{2}$ we have that

$$\|A_\alpha f\|_{L^2([0, 1])}^2 \leq 2 \left( \|f\|_{L^2([0, 1])}^2 + \frac{1}{1 - 2^{2(\beta - \alpha)}} \int_0^1 \int_0^1 \frac{|f(t) - f(u)|^2}{|t - u|^{1 + 2\beta}} dt du \right).$$

**Theorem A.3.** Let $D^i$ be the Malliavin derivative in the direction of the $i$-th component of $B_t$, $0 \leq t \leq 1$, $i \geq 1$. In addition, let $0 < \alpha_i < \beta_i < \frac{1}{2}$ and $\delta_i > 0$ for all $i \geq 1$. Define the sequence $\lambda_{s, i} = 2^{-k\alpha_i} \delta_i$, if $s = 2^k + j$, $k \geq 0, 0 \leq j \leq 2^k, i \geq 1$. Assume that $\lambda_{s, i} \to 0$ for $s, i \to \infty$. Let $c > 0$ and $G$ the collection of all $G \in D^{1/2}$ such that

$$\|G\|_{L^2(\Omega)} \leq c,$$

$$\sum_{i \geq 1} \delta_i^{-2} \left\| D^i G \right\|_{L^2(\Omega; L^2([0, 1])))}^2 \leq c$$

and

$$\sum_{i \geq 1} \frac{1}{(1 - 2^{2(\beta_i - \alpha_i)}) \delta_i^2} \int_0^1 \int_0^1 \frac{\left| D^i G - D^j G \right|_{L^2(\Omega)}^2}{|t - u|^{1 + 2\beta_i}} dtdu \leq c.$$

Then $G$ is relatively compact in $L^2(\Omega)$.

**Proof.** As before denote by $v_s, s \geq 0$ with $v_0 = 1$ the Haar basis of $L^2([0, 1])$ and by $e_i^* = (e_i, \cdot)_H, i \geq 1$ an orthonormal basis of $L_{HS}(H, \mathbb{R})$ where $e_i, i \geq 1$ is an orthonormal basis of $H$. Define a self-adjoint compact operator $C$ on $L^2([0, 1]) \otimes L_{HS}(H, \mathbb{R})$ with dense image by

$$C(v_s \otimes e_i^*) = \lambda_{s, i} v_s \otimes e_i^*, \ s \geq 0, \ i \geq 1.$$
Then it follows for $G \in D^{1,2}$ from Lemma A.2 that
\[
\|C^{-1}DG\|_{L^2(\Omega;L^2([0,1]) \otimes L_{HS}(H;\mathbb{R}))}^2 = \sum_{i \geq 1} \sum_{s \geq 0} \lambda_{s,i}^{-2} E[(DG,v_s \otimes \epsilon_s^\ast)]_{L^2([0,1]) \otimes L_{HS}(H;\mathbb{R})}^2 \\
= \sum_{i \geq 1} \delta_i^{-2} \|A_{\alpha_i}D_i^rG\|_{L^2(\Omega;L^2([0,1]))}^2 \\
\leq 2 \sum_{i \geq 1} \delta_i^{-2} \|D_i^rG\|_{L^2(\Omega;L^2([0,1]))}^2 \\
+ 2 \sum_{i \geq 1} \frac{1}{(1 - 2^{-2(\beta - \alpha_i)})} \delta_i^2 \int_0^1 \int_0^1 \frac{\|D^r_i G - D^r_i G\|_{L^2(\Omega)}^2}{|t-u|^{1+2\beta}} dt du \\
\leq M
\]
for a constant $M < \infty$. So using Theorem A.1 we obtain the result. \hfill \square

APPENDIX B. TECHNICAL ESTIMATE

The following technical estimate is used in the course of the paper.

**Lemma B.1.** Let $H \in (0, 1/2)$ and $t \in [0, T]$ be fixed. Then, there exists a $\beta \in (0, 1/2)$ such that
\[
\int_0^t \int_0^t \frac{|K_H(t', t_0) - K_H(t, t_0)|^2}{|t_0 - t_0'|^{1+2\beta}} dt_0 dt' < \infty. \tag{54}
\]

**Proof.** Let $t_0, t'_0 \in [0, t]$, $t'_0 < t_0$ be fixed. Write
\[
K_H(t, t_0) - K_H(t, t_0') = c_H \left[ f_t(t_0) - f_t(t'_0) + \left( \frac{1}{2} - H \right) (g_t(t_0) - g_t(t'_0)) \right],
\]
where $f_t(t_0) := \left( \frac{t}{t_0} \right)^{H-\frac{1}{2}} (t - t_0)^{H-\frac{1}{2}}$ and $g_t(t_0) := \int_t^t \frac{f_s(t_0)}{s} ds, t_0 \in [0, t]$.

We will proceed to estimating $K_H(t, t_0) - K_H(t, t'_0)$. First, observe the following fact,
\[
\frac{y^{-\alpha} - x^{-\alpha}}{(x-y)^\gamma} \leq C y^{-\alpha-\gamma}
\]
for every $0 < y < x < \infty$ and $\alpha := (\frac{1}{2} - H) \in (0, 1/2)$ and $\gamma < \frac{1}{2} - \alpha$. This implies
\[
f_t(t_0) - f_t(t'_0) = \left( \frac{t}{t_0} (t - t_0) \right)^{H-\frac{1}{2}} - \left( \frac{t}{t'_0} (t - t'_0) \right)^{H-\frac{1}{2}} \\
\leq C \left( \frac{t}{t_0} (t - t_0) \right)^{H-\frac{1}{2}-\gamma} \frac{t^2 (t_0 - t'_0)^\gamma}{(t_0 t'_0)^\gamma} \\
\leq C \frac{(t_0 - t'_0)^\gamma}{(t_0 t'_0)^\gamma} \frac{(t - t_0)^{H-\frac{1}{2}-\gamma}}{(t_0 t'_0)^\gamma} \\
\leq C \frac{(t_0 - t'_0)^\gamma}{(t_0 t'_0)^\gamma} t_0^{H-\frac{1}{2}-\gamma} (t - t_0)^{H-\frac{1}{2}-\gamma}.
\]
Further,

\[ g_t(t_0) - g_t(t'_0) = \int_{t_0}^{t} \frac{f_u(t_0) - f_u(t'_0)}{u} \, du - \int_{t'_0}^{t_0} \frac{f_u(t'_0)}{u} \, du \]

\[
\leq \int_{t_0}^{t} \frac{f_u(t_0) - f_u(t'_0)}{u} \, du
\]

\[
\leq C \left( t_0 - t'_0 \right)^\gamma (t_0 t'_0)^\gamma \int_{t_0}^{t} \frac{(u - t_0)^{H-\frac{1}{2} - \gamma}}{u} \, du
\]

\[
\leq C \left( t_0 - t'_0 \right)^\gamma t_0^{H-\frac{1}{2} - \gamma} \int_{1}^{\infty} \frac{(u - 1)^{H-\frac{1}{2} - \gamma}}{u} \, du
\]

\[
\leq C \left( t_0 - t'_0 \right)^\gamma t_0^{H-\frac{1}{2} - \gamma} t_0 - t_0)^{H-\frac{1}{2} - \gamma},
\]

As a result, we have for every \( \gamma \in (0, H) \), \( 0 < t'_0 < t_0 < t < T \),

\[ K_H(t, t_0) - K_H(t, t'_0) \leq C_{H,T} \frac{(t_0 - t'_0)^\gamma}{(t_0 t'_0)^\gamma} t_0^{H-\frac{1}{2} - \gamma} (t - t_0)^{H-\frac{1}{2} - \gamma}, \quad (55) \]

for some constant \( C_{H,T} > 0 \) depending only on \( H \) and \( T \).

Thus

\[
\int_{0}^{t} \int_{0}^{t_0} \frac{(K_H(t, t_0) - K_H(t, t'_0))^2}{|t_0 - t'_0|^{1 + 2\beta}} \, dt'_0 \, dt_0
\]

\[
\leq C \int_{0}^{t} \int_{0}^{t_0} \frac{|t_0 - t'_0|^{1 - 2\beta + 2\gamma}}{(t_0 t'_0)^{2\gamma}} t_0^{2H-1-2\gamma} (t - t_0)^{2H-1-2\gamma} dt'_0 \, dt_0
\]

\[
= C \int_{0}^{t} t_0^{2H-1-4\gamma} (t - t_0)^{2H-1-2\gamma} \int_{0}^{t_0} \frac{|t_0 - t'_0|^{1 - 2\beta + 2\gamma}}{(t'_0)^{2\gamma}} dt'_0 \, dt_0
\]

\[
= C \int_{0}^{t} t_0^{2H-1-4\gamma} (t - t_0)^{2H-1-2\gamma} \frac{\Gamma(-2\beta + 2\gamma) \Gamma(-2\gamma + 1)}{\Gamma(-2\beta + 1)} t_0^{2\beta} \, dt_0
\]

\[
\leq C \int_{0}^{t} t_0^{2H-1-4\gamma - 2\beta} (t - t_0)^{2H-1-2\gamma} \, dt_0
\]

\[
= C \frac{\Gamma(2H - 2\gamma) \Gamma(2H - 4\gamma - 2\beta)}{\Gamma(4H - 6\gamma - 2\beta)} t^{4H-6\gamma-2\beta-1} < \infty,
\]

for appropriately chosen small \( \gamma \) and \( \beta \).
On the other hand, we have that

\[
\begin{align*}
&\int_0^t \int_{t_0}^t \frac{(K_H(t, t_0) - K_H(t, t_0'))^2}{|t_0 - t_0'|^{1+2\beta}} dt_0' dt_0 \\
&\leq C \int_0^t \int_{t_0}^t (t - t_0)^{2H-1-2\gamma} dt_0' \int_{t_0}^t |t_0 - t_0'|^{-1-2\beta+2\gamma} dt_0' dt_0 \\
&\leq C \int_0^t \int_{t_0}^t (t - t_0)^{2H-1-6\gamma} dt_0' \int_{t_0}^t |t_0 - t_0'|^{-1-2\beta+2\gamma} dt_0' dt_0 \\
&= C \int_0^t \int_{t_0}^t (t - t_0)^{2H-1-6\gamma} dt_0 \\
&\leq Ct^{1H-6\gamma-2\beta-1}.
\end{align*}
\]

Hence

\[
\begin{align*}
&\int_0^t \int_{t_0}^t \frac{(K_H(t, t_0) - K_H(t, t_0'))^2}{|t_0 - t_0'|^{1+2\beta}} dt_0' dt_0 < \infty.
\end{align*}
\]

□

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