

# Stochastic Control of Memory Mean-Field Processes

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*Dedicated to the memory of Salah-Eldin Mohammed*

**Abstract** By a memory mean-field process we mean the solution  $X(\cdot)$  of a stochastic mean-field equation involving not just the current state  $X(t)$  and its law  $\mathcal{L}(X(t))$  at time  $t$ , but also the state values  $X(s)$  and its law  $\mathcal{L}(X(s))$  at some previous times  $s < t$ . Our purpose is to study stochastic control problems of memory mean-field processes.

- We consider the space  $\mathcal{M}$  of measures on  $\mathbb{R}$  with the norm  $\|\cdot\|_{\mathcal{M}}$  introduced by Agram and Øksendal in [?], and prove the existence and uniqueness of solutions of memory mean-field stochastic functional differential equations.
- We prove two stochastic maximum principles, one sufficient (a verification theorem) and one necessary, both under partial information. The corresponding equations for the adjoint variables are a pair of (*time-*) *advanced backward stochastic differential equations*, one of them with values in the space of bounded linear functionals on path segment spaces.
- As an application of our methods, we solve a memory mean-variance problem as well as a linear-quadratic problem of a memory process.

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# 1 Introduction

In this work we are studying a general class of controlled memory mean-field stochastic functional differential equations (mf-sfde) of the form

$$\begin{cases} dX(t) &= b(t, X(t), X_t, M(t), M_t, u(t), u_t)dt + \sigma(t, X(t), X_t, M(t), M_t, u(t), u_t)dB(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t, X(t), X_t, M(t), M_t, u(t), u_t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X(t) &= \xi(t); t \in [-\delta, 0], \\ u(t) &= u_0(t); t \in [-\delta, 0], \end{cases} \quad (1.1) \quad \{\text{mfsfde}\}$$

on a filtered probability space  $(\Omega, \mathbb{F}, \mathbb{P})$  satisfying the usual conditions, i.e. the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  is right-continuous and increasing, and each  $\mathcal{F}_t$ ,  $t \geq 0$ , contains all  $\mathbb{P}$ -null sets in  $\mathbb{F}$ . Here  $M(t) := \mathcal{L}(X(t))$  is the law of  $X(t)$  at time  $t$ ,  $\delta \geq 0$  is a given (constant) memory span and

$$X_t := \{X(t+s)\}_{s \in [-\delta, 0]} \quad (1.2)$$

is the path segment of the state process  $X(\cdot)$ , while

$$M_t := \{M(t+s)\}_{s \in [-\delta, 0]} \quad (1.3)$$

is the path segment of the law process  $M(\cdot) = \mathcal{L}(X(\cdot))$ . The process  $u(t)$  is our control process, and  $u_t := \{u(t+s)\}_{s \in [-\delta, 0]}$  is its memory path segment. The path processes  $X_t, M_t$  and  $u_t$  represent the memory terms of the equation (??). The terms  $B(t)$  and  $\tilde{N}(dt, d\zeta)$  in the mf-sfde (??) denote a one-dimensional Brownian motion and an independent compensated Poisson random measure, respectively, such that

$$\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$$

where  $N(dt, d\zeta)$  is an independent Poisson random measure and  $\nu(d\zeta)$  is the Lévy measure of  $N$ . For the sake of simplicity, we only consider the one-dimensional case, i.e.  $X(t) \in \mathbb{R}, B(t) \in \mathbb{R}$  and  $N(t, \zeta) \in \mathbb{R}$ , for all  $t, \zeta$ .

Following Agram and Øksendal [?], we now introduce the following Hilbert spaces:

## Definition 1.1

- $\mathcal{M}$  is the Hilbert space of random measures  $\mu$  on  $\mathbb{R}$  equipped with the norm

$$\|\mu\|_{\mathcal{M}}^2 := \mathbb{E}[\int_{\mathbb{R}} |\hat{\mu}(y)|^2 e^{-y^2} dy],$$

where  $\hat{\mu}$  is the Fourier transform of the measure  $\mu$ , i.e.

$$\hat{\mu}(y) := \int_{\mathbb{R}} e^{ixy} d\mu(x); \quad y \in \mathbb{R}.$$

- $\mathcal{M}^\delta$  is the Hilbert space of all path segments  $\bar{\mu} = \{\mu(s)\}_{s \in [-\delta, 0]}$  of processes  $\mu(\cdot)$  with  $\mu(s) \in \mathcal{M}$  for each  $s \in [-\delta, 0]$ , equipped with the norm

$$\|\bar{\mu}\|_{\mathcal{M}^\delta} := \int_{-\delta}^0 \|\mu(s)\|_{\mathcal{M}} ds. \quad (1.4)$$

- $\mathcal{M}_0$  and  $\mathcal{M}_0^\delta$  denote the set of deterministic elements of  $\mathcal{M}$  and  $\mathcal{M}^\delta$ , respectively. For simplicity of notation, in some contexts we regard  $\mathcal{M}$  as a subset of  $\mathcal{M}^\delta$  and  $\mathcal{M}_0$  as a subset of  $\mathcal{M}^\delta$ .

The structure of this space  $\mathcal{M}$  equipped with the norm obtained by the Fourier transform, is an alternative to the Wasserstein metric space  $\mathcal{P}_2$  equipped with the Wasserstein distance  $W_2$ . Moreover, the Hilbert space  $\mathcal{M}$  deals with any random measure on  $\mathbb{R}$ , however the Wasserstein space  $\mathcal{P}_2$  deals with Borel probability measures on  $\mathbb{R}$  with finite second moments.

Using the Hilbert space structure for this type of problems has been proposed by P.L. Lions, to simplify the technicalities of the Wasserstein metric space where he considers the Hilbert space of square integrable random variables. Our Hilbert space, however is now.

In the following, we denote by  $\mathcal{C} := \mathcal{C}([-\delta, 0]; \mathbb{R})$  the Banach space of all paths  $\bar{x} := \{x(s)\}_{s \in [-\delta, 0]}$ , equipped with the norm

$$\|\bar{x}\|_{\mathcal{C}} := \mathbb{E} \left[ \sup_{s \in [-\delta, 0]} |x(s)| \right]. \quad (1.5)$$

To simplify the writing, we introduce some notations and the same notations  $E$  and  $E'$  differ but they are clear from the context. The coefficients

$$\begin{aligned} b(t, x, \bar{x}, m, \bar{m}, u, \bar{u}) &= b(t, x, \bar{x}, m, \bar{m}, u, \bar{u}, \omega) & : E &\rightarrow \mathbb{R}, \\ \sigma(t, x, \bar{x}, m, \bar{m}, u, \bar{u}) &= \sigma(t, x, \bar{x}, m, \bar{m}, u, \bar{u}, \omega) & : E &\rightarrow \mathbb{R}, \\ \gamma(t, x, \bar{x}, m, \bar{m}, u, \bar{u}, \zeta) &= \gamma(t, x, \bar{x}, m, \bar{m}, u, \bar{u}, \zeta, \omega) & : E' &\rightarrow \mathbb{R}, \end{aligned}$$

where  $E := [0, T] \times \mathbb{R} \times \mathcal{C} \times \mathcal{M}_0 \times \mathcal{M}_0^\delta \times \mathbb{R} \times \mathcal{C} \times \Omega$  and  $E' := [0, T] \times \mathbb{R} \times \mathcal{C} \times \mathcal{M}_0 \times \mathcal{M}_0^\delta \times \mathbb{R} \times \mathcal{C} \times \mathbb{R}_0 \times \Omega$  and  $\mathbb{R}_0 = \mathbb{R} - \{0\}$ .

We remark that the functionals  $b, \sigma$  and  $\gamma$  on the mf-sfde depend on more than the solution  $X(t)$  and its law  $\mathcal{L}(X(t))$ , both the segment  $X_t$  and the law of this segment  $\mathcal{L}(X_t)$  and this is a new-type of mean-field stochastic functional differential equations with memory.

Let us give some examples: Let  $X(t)$  satisfies the following mean-field delayed sfde

$$\begin{cases} dX(t) &= b(t, \mathbf{X}(t), \mathbb{E}[\mathbf{X}(t)], u(t))dt + \sigma(t, \mathbf{X}(t), \mathbb{E}[\mathbf{X}(t)], u(t))dB(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t, \mathbf{X}(t), \mathbb{E}[\mathbf{X}(t)], u(t), \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X(t) &= \xi(t); t \in [-\delta, 0], \end{cases} \quad (1.6) \quad \{\text{mfd}\}$$

where we denote by the bold  $\mathbf{X}(t) = \int_{-\delta}^0 X(t+s)\mu(ds)$  for some bounded Borel-measure  $\mu$ . As noted in Agram and Røse [?] and Banos et al [?], we have the following:

- If this measure  $\mu$  is a Dirac-measure concentrated at 0 i.e.  $\mathbf{X}(t) = X(t)$  then equation (??) is a *classical mean-field stochastic differential equation*, we refer for example to Anderson and Djehiche in [?] and Hu et al in [?] for stochastic control of such a systems.

- It could also be the Dirac measure concentrated at  $-\delta$  then  $\mathbf{X}(t) = X(t - \delta)$  and in that case the state equation is called a *mean-field sde with discrete delay*, see for instance Meng and Shen [?] and for delayed systems without a mean-field term, we refer to Chen and Wu [?], Dahl et al [?] and Øksendal et al [?].
- If we choose now  $\mu(ds) = g(s)ds$  for any function  $g \in L^1([-\delta, 0])$  thus  $\mathbf{X}(t) = \int_{-\delta}^0 g(s)X(t+s)ds$  and the state is a *mean-field distributed delay*.

It is worth mentioning the papers by Lions [?], Cardaliaguet [?], Carmona and Delarue [?], [?], Buckdahn et al [?] and Agram [?] for more details about systems driven by mean-field equations and stochastic control problems for such a system. These papers, however, use the Wasserstein metric space of probability measures and not our Hilbert space of measures.

The paper is organized as follows: In section 2, we give some mathematical background and define some concepts and spaces which will be used in the paper. In section 3, we prove existence and uniqueness of memory McKean-Vlasov equations. Section 4 contains the main results of this paper, including a sufficient and a necessary maximum principle for the optimal control of stochastic memory mean-field equations. In section 5, we illustrate our results by solving a mean-variance and a linear-quadratic problems of a memory processes.

## 2 Generalities

In this section, we recall some concepts which will be used on the sequel.

a) We first discuss the differentiability of functions defined on a Banach space.

Let  $\mathcal{X}, \mathcal{Y}$  be two Banach spaces with norms  $\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{Y}}$ , respectively, and let  $F : \mathcal{X} \rightarrow \mathcal{Y}$ .

- We say that  $F$  has a directional derivative (or Gâteaux derivative) at  $v \in \mathcal{X}$  in the direction  $w \in \mathcal{X}$  if

$$D_w F(v) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (F(v + \varepsilon w) - F(v))$$

exists.

- We say that  $F$  is Fréchet differentiable at  $v \in \mathcal{X}$  if there exists a continuous linear map  $A : \mathcal{X} \rightarrow \mathcal{Y}$  such that

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathcal{X}}} \frac{1}{\|h\|_{\mathcal{X}}} \|F(v+h) - F(v) - A(h)\|_{\mathcal{Y}} = 0,$$

where  $A(h) = \langle A, h \rangle$  is the action of the linear operator  $A$  on  $h$ . In this case we call  $A$  the *gradient* (or Fréchet derivative) of  $F$  at  $v$  and we write

$$A = \nabla_v F.$$

- If  $F$  is Fréchet differentiable at  $v$  with Fréchet derivative  $\nabla_v F$ , then  $F$  has a directional derivative in all directions  $w \in \mathcal{X}$  and

$$D_w F(v) = \nabla_v F(w) = \langle \nabla_v F, w \rangle.$$

In particular, note that if  $F$  is a linear operator, then  $\nabla_v F = F$  for all  $v$ .

b) Throughout this work, we will use the following spaces:

- $\mathcal{S}^2$  is the set of  $\mathbb{R}$ -valued  $\mathbb{F}$ -adapted càdlàg processes  $(X(t))_{t \in [-\delta, T]}$  such that

$$\|X\|_{\mathcal{S}^2}^2 := \mathbb{E} \left[ \sup_{t \in [-\delta, T]} |X(t)|^2 \right] < \infty,$$

(alternatively  $(X(t))_{t \in [0, T+\delta]}$  with

$$\|X\|_{\mathcal{S}^2}^2 = \mathbb{E} \left[ \sup_{t \in [0, T+\delta]} |X(t)|^2 \right] < \infty,$$

depending on the context.)

- $\mathbb{L}^2$  is the set of  $\mathbb{R}$ -valued  $\mathbb{F}$ -adapted processes  $(Q(t))_{t \in [0, T]}$  such that

$$\|Q\|_{\mathbb{L}^2}^2 := \mathbb{E} \left[ \int_0^T |Q(t)|^2 dt \right] < \infty.$$

- $\mathcal{U}^{ad}$  is a set of all stochastic processes  $u$  required to have values in a convex subset  $\mathcal{U}$  of  $\mathbb{R}$  and adapted to a given subfiltration  $\mathbb{G} = \{\mathcal{G}_t\}_{t \geq 0}$ , where  $\mathcal{G}_t \subseteq \mathcal{F}_t$  for all  $t \geq 0$ . We call  $\mathcal{U}^{ad}$  the set of admissible control processes  $u(\cdot)$ .

- $L^2(\mathcal{F}_t)$  is the set of  $\mathbb{R}$ -valued square integrable  $\mathcal{F}_t$ -measurable random variables.

- $\mathbb{L}_\nu^2$  is the set of  $\mathbb{R}$ -valued  $\mathbb{F}$ -adapted processes  $Z : \mathbb{R}_0 \rightarrow \mathbb{R}$  such that

$$\|Z\|_{\mathbb{L}_\nu^2}^2 := \mathbb{E} \left[ \int_{\mathbb{R}_0} |Z(t, \zeta)|^2 \nu(d\zeta) dt \right] < \infty.$$

- $\mathcal{R}$  is the set of measurable functions  $r : \mathbb{R}_0 \rightarrow \mathbb{R}$ .

- $C_a([0, T], \mathcal{M}_0)$  denotes the set of absolutely continuous functions  $m : [0, T] \rightarrow \mathcal{M}_0$ .

- $\mathbb{K}$  is the set of bounded linear functionals  $K : \mathcal{M}_0 \rightarrow \mathbb{R}$  equipped with the operator norm

$$\|K\|_{\mathbb{K}} := \sup_{m \in \mathcal{M}_0, \|m\|_{\mathcal{M}_0} \leq 1} |K(m)|. \quad (2.1)$$

- $\mathcal{S}_{\mathbb{K}}^2$  is the set of  $\mathbb{F}$ -adapted stochastic processes  $p : [0, T + \delta] \times \Omega \mapsto \mathbb{K}$  such that

$$\|p\|_{\mathcal{S}_{\mathbb{K}}^2}^2 := \mathbb{E} \left[ \sup_{t \in [0, T+\delta]} \|p(t)\|_{\mathbb{K}}^2 \right] < \infty. \quad (2.2)$$

- $\mathbb{L}_{\mathbb{K}}^2$  is the set of  $\mathbb{F}$ -adapted stochastic processes  $q : [0, T + \delta] \times \Omega \mapsto \mathbb{K}$  such that

$$\|q\|_{\mathbb{L}_{\mathbb{K}}^2}^2 := \mathbb{E}[\int_0^{T+\delta} \|q(t)\|_{\mathbb{K}}^2 dt] < \infty. \quad (2.3)$$

- $\mathbb{L}_{\nu, \mathbb{K}}^2$  is the set of  $\mathbb{F}$ -adapted stochastic processes  $r : [0, T + \delta] \times \mathbb{R}_0 \times \Omega \mapsto \mathbb{K}$  such that

$$\|r\|_{\mathbb{L}_{\nu, \mathbb{K}}^2}^2 := \mathbb{E}[\int_0^{T+\delta} \int_{\mathbb{R}_0} \|r(t, \zeta)\|_{\mathbb{K}}^2 \nu(d\zeta) dt] < \infty. \quad (2.4)$$

### 3 Solvability of memory mean-field sfde

For a given constant  $\delta > 0$ , we consider a memory mean-field stochastic functional differential equations (mf-sfde) of the following form:

$$\begin{cases} dX(t) &= b(t, X(t), X_t, M(t), M_t)dt + \sigma(t, X(t), X_t, M(t), M_t)dB(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t, X(t), X_t, M(t), M_t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X(t) &= \xi(t); t \in [-\delta, 0]. \end{cases} \quad (3.1) \quad \{\text{sfde}\}$$

Here  $E := [0, T] \times \mathbb{R} \times \mathcal{C} \times \mathcal{M}_0 \times \mathcal{M}_0^\delta \times \Omega$ ,  $E' := [0, T] \times \mathbb{R} \times \mathcal{C} \times \mathcal{M}_0 \times \mathcal{M}_0^\delta \times \mathbb{R}_0 \times \Omega$  and the coefficients

$$\begin{aligned} b(t, x, \bar{x}, m, \bar{m}) &= b(t, x, \bar{x}, m, \bar{m}, \omega) & : E &\rightarrow \mathbb{R}, \\ \sigma(t, x, \bar{x}, m, \bar{m}) &= \sigma(t, x, \bar{x}, m, \bar{m}, \omega) & : E &\rightarrow \mathbb{R}, \\ \gamma(t, x, \bar{x}, m, \bar{m}, \zeta) &= \gamma(t, x, \bar{x}, m, \bar{m}, \zeta, \omega) & : E' &\rightarrow \mathbb{R}, \end{aligned}$$

are supposed to be  $\mathcal{F}_t$ -measurable and the initial value function  $\xi$  is  $\mathcal{F}_0$ -measurable.

For more information about stochastic functional differential equations, we refer to the seminal work of S.E.A. Mohammed [?] and a recent paper by Banos et al [?].

In order to prove an existence and uniqueness result for the mf-sfde (??), we first need the following lemma:

#### Lemma 3.1

- (i) Let  $X^{(1)}$  and  $X^{(2)}$  be two random variables in  $L^2(\mathbb{P})$ . Then

$$\|\mathcal{L}(X^{(1)}) - \mathcal{L}(X^{(2)})\|_{\mathcal{M}_0}^2 \leq \sqrt{\pi} \mathbb{E}[(X^{(1)} - X^{(2)})^2].$$

- (ii) Let  $\{X^{(1)}(t)\}_{t \geq 0}$ ,  $\{X^{(2)}(t)\}_{t \geq 0}$  be two processes such that

$$\mathbb{E}[\int_0^T X^{(i)2}(s) ds] < \infty \text{ for all } T \text{ with } i = 1, 2.$$

Then

$$\|\mathcal{L}(X_t^{(1)}) - \mathcal{L}(X_t^{(2)})\|_{\mathcal{M}_0^\delta}^2 \leq \sqrt{\pi} \mathbb{E}[\int_{-\delta}^0 (X^{(1)}(s) - X^{(2)}(s))^2 ds].$$

Proof. By definition of the norms and standard properties of the complex exponential function, we have

$$\begin{aligned}
& \|\mathcal{L}(X^{(1)}) - \mathcal{L}(X^{(2)})\|_{\mathcal{M}_0}^2 \\
&= \int_{\mathbb{R}} |\widehat{\mathcal{L}}(X^{(1)})(y) - \widehat{\mathcal{L}}(X^{(2)})(y)|^2 e^{-y^2} dy \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{ixy} d\mathcal{L}(X^{(1)})(x) - \int_{\mathbb{R}} e^{ixy} d\mathcal{L}(X^{(2)})(x) \right|^2 e^{-y^2} dy \\
&= \int_{\mathbb{R}} |\mathbb{E}[e^{iyX^{(1)}} - e^{iyX^{(2)}}]|^2 e^{-y^2} dy \\
&= \int_{\mathbb{R}} |\mathbb{E}[\cos(yX^{(1)}) - \cos(yX^{(2)})] + i\mathbb{E}[\sin(yX^{(1)}) - \sin(yX^{(2)})]|^2 e^{-y^2} dy \\
&= \int_{\mathbb{R}} (\mathbb{E}[\cos(yX^{(1)}) - \cos(yX^{(2)})]^2 + \mathbb{E}[\sin(yX^{(1)}) - \sin(yX^{(2)})]^2) e^{-y^2} dy \\
&\leq \int_{\mathbb{R}} (\mathbb{E}[|\cos(yX^{(1)}) - \cos(yX^{(2)})|^2] + \mathbb{E}[|\sin(yX^{(1)}) - \sin(yX^{(2)})|^2]) e^{-y^2} dy \\
&\leq \int_{\mathbb{R}} (\mathbb{E}[|y(X^{(1)} - X^{(2)})|^2] + \mathbb{E}[|y(X^{(1)} - X^{(2)})|^2]) e^{-y^2} dy \\
&\leq 2 \int_{\mathbb{R}} y^2 e^{-y^2} dy \mathbb{E}[|X^{(1)} - X^{(2)}|^2] \\
&\leq \sqrt{\pi} \mathbb{E}[(X^{(1)} - X^{(2)})^2],
\end{aligned}$$

and similarly, we get that

$$\begin{aligned}
\|\mathcal{L}(X_t^{(1)}) - \mathcal{L}(X_t^{(2)})\|_{\mathcal{M}_0^\delta}^2 &\leq \int_{-\delta}^0 \|\mathcal{L}(X^{(1)}(s) - X^{(2)}(s))\|_{\mathcal{M}_0}^2 ds \\
&\leq \sqrt{\pi} \mathbb{E}[\int_{-\delta}^0 (X^{(1)}(s) - X^{(2)}(s))^2 ds].
\end{aligned}$$

□

We also need the following result, which is Lemma 2.3 in [?]:

**Lemma 3.2** *Suppose that  $X(t)$  is an Itô-Lévy process of the form*

$$\begin{cases} dX(t) = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t, \zeta) \tilde{N}(dt, d\zeta); & t \in [0, T], \\ X(0) = x \in \mathbb{R}, \end{cases} \quad (3.2) \quad \{\text{eq2.1}\}$$

where  $\alpha, \beta$  and  $\gamma$  are predictable processes.

Then the map  $t \mapsto M(t) : [0, T] \rightarrow \mathcal{M}_0$  is absolutely continuous.

It follows that  $t \mapsto M(t)$  is differentiable for a.a.t. We will in the following use the notation

$$M'(t) = \frac{dM(t)}{dt}. \quad (3.3)$$

We are now able to state the theorem of existence and uniqueness of a solution of equation (??). As before we put  $E := [0, T] \times \mathbb{R} \times \mathcal{C} \times \mathcal{M}_0 \times \mathcal{M}_0^\delta \times \Omega$  and  $E' := [0, T] \times \mathbb{R} \times \mathcal{C} \times \mathcal{M}_0 \times \mathcal{M}_0^\delta \times \mathbb{R}_0 \times \Omega$ . Then we have

**Theorem 3.3** *Assume that  $\xi(t) \in \mathcal{C}, b, \sigma : E \rightarrow \mathbb{R}$  and  $\gamma : E' \rightarrow \mathbb{R}$  are progressively measurable and satisfy the following uniform Lipschitz condition  $dtP(d\omega)$ -a.e.:*

*There is some constant  $L \in \mathbb{R}$  such that*

$$\begin{aligned}
& |b(t, x, \bar{x}, m, \bar{m}, \omega) - b(t, x', \bar{x}', m', \bar{m}', \omega)|^2 + |\sigma(t, x, \bar{x}, m, \bar{m}, \omega) - \sigma(t, x', \bar{x}', m', \bar{m}', \omega)|^2 \\
& \quad + \int_{\mathbb{R}_0} |\gamma(t, x, \bar{x}, m, \bar{m}, \zeta, \omega) - \gamma(t, x', \bar{x}', m', \bar{m}', \zeta, \omega)|^2 \nu(d\zeta) \\
& \leq L(|x - x'|^2 + \|\bar{x} - \bar{x}'\|_{\mathcal{C}}^2 + \|m - m'\|_{\mathcal{M}_0}^2 + \|\bar{m} - \bar{m}'\|_{\mathcal{M}_0^\delta}^2), \text{ for a.a. } t, \omega,
\end{aligned} \quad (3.4) \quad \{\text{Lip}\}$$

{existence}

and

$$\begin{aligned} & |b(t, 0, 0, \mu_0, \mu_0, \omega)|^2 + |\sigma(t, 0, 0, \mu_0, \mu_0, \omega)|^2 \\ & + \int_{\mathbb{R}_0} |\gamma(t, 0, 0, \mu_0, \mu_0, \zeta, \omega)|^2 \nu(d\zeta) \leq L \text{ for a.a. } t, \omega, \end{aligned} \quad (3.5) \quad \{\text{Bou}\}$$

where  $\mu_0$  is the Dirac measure with mass at zero. Then there is a unique solution  $X \in \mathcal{S}^2$  of the mf-sfde (??).

Proof. For  $X \in \mathcal{S}^2[-\delta, T]$  and for  $t_0 \in (0, T]$ , we introduce the norm

$$\|X\|_{t_0}^2 := \mathbb{E}[\sup_{t \in [-\delta, t_0]} |X(t)|^2].$$

The space  $\mathbb{H}_{t_0}$  equipped with this norm is a Banach space. Define the mapping  $\Phi : \mathbb{H}_{t_0} \rightarrow \mathbb{H}_{t_0}$  by  $\Phi(x) = X$  where  $X \in \mathcal{S}^2$  is defined by

$$\begin{cases} dX(t) &= b(t, x(t), x_t, m(t), m_t)dt + \sigma(t, x(t), x_t, m(t), m_t)dB(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t, x(t), x_t, m(t), m_t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X(t) &= \xi(t); t \in [-\delta, 0]. \end{cases}$$

We want prove that  $\Phi$  is contracting in  $\mathbb{H}_{t_0}$  under the norm  $\|\cdot\|_{t_0}$  for small enough  $t_0$ . For two arbitrary elements  $(x^1, x^2)$  and  $(X^1, X^2)$ , we denote their difference by  $\tilde{x} = x^1 - x^2$  and  $\tilde{X} = X^1 - X^2$  respectively. In the following  $C < \infty$  will denote a constant which is big enough for all the inequalities to hold.

Applying the Itô formula to  $\tilde{X}^2(t)$ , we get

$$\begin{aligned} \tilde{X}^2(t) &= 2 \int_0^t \tilde{X}(s) (b(s, x^1(s), x_s^1, m^1(s), m_s^1) - b(s, x^2(s), x_s^2, m^2(s), m_s^2)) ds \\ &+ 2 \int_0^t \tilde{X}(s) (\sigma(s, x^1(s), x_s^1, m^1(s), m_s^1) - \sigma(s, x^2(s), x_s^2, m^2(s), m_s^2)) dB(s) \\ &+ 2 \int_0^t \tilde{X}(s) \int_{\mathbb{R}_0} (\gamma(s, x^1(s), x_s^1, m^1(s), m_s^1, \zeta) - \gamma(s, x^2(s), x_s^2, m^2(s), m_s^2, \zeta)) \tilde{N}(ds, d\zeta) \\ &+ \int_0^t (\sigma(s, x^1(s), x_s^1, m^1(s), m_s^1) - \sigma(s, x^2(s), x_s^2, m^2(s), m_s^2))^2 ds \\ &+ \int_0^t \int_{\mathbb{R}_0} (\gamma(s, x^1(s), x_s^1, m^1(s), m_s^1, \zeta) - \gamma(s, x^2(s), x_s^2, m^2(s), m_s^2, \zeta))^2 \nu(d\zeta) ds. \end{aligned}$$

By the Lipschitz assumption (??) combined with standard majorization of the square of a sum (resp. integral) via the sum (resp. integral) of the square (up to a constant), we get

$$\begin{aligned} \tilde{X}^2(t) &\leq C \int_0^t |\tilde{X}(s)| \Delta_{t_0} ds \\ &+ |\int_0^t \tilde{X}(s) \tilde{\sigma}(s) dB(s)| + |\int_0^t \int_{\mathbb{R}_0} \tilde{X}(s) \tilde{\gamma}(s, \zeta) \tilde{N}(ds, d\zeta)| + tC \Delta_{t_0}^{(2)}, \end{aligned}$$

where

$$\begin{aligned} \Delta_{t_0} &:= \|\tilde{x}\|_{\mathcal{S}^2} + \|\tilde{x}\|_{\mathcal{C}} + \|\tilde{m}\|_{\mathcal{M}_0} + \|\tilde{m}\|_{\mathcal{M}_0^\delta} \\ \Delta_{t_0}^{(2)} &:= \|\tilde{x}\|_{\mathcal{S}^2}^2 + \|\tilde{x}\|_{\mathcal{C}}^2 + \|\tilde{m}\|_{\mathcal{M}_0}^2 + \|\tilde{m}\|_{\mathcal{M}_0^\delta}^2. \end{aligned}$$



By the Burkholder-Davis-Gundy inequalities,

$$\mathbb{E}[\sup_{t \leq t_0} |\int_0^t \tilde{X}(s) \tilde{\sigma}(s) dB(s)|] \leq C \mathbb{E}[(\int_0^{t_0} \tilde{X}^2(s) \tilde{\sigma}^2(s) ds)^{\frac{1}{2}}] \leq Ct_0 \|\tilde{X}\|_{t_0} \Delta_{t_0}, \quad (3.6)$$

and

$$\mathbb{E}[\sup_{t \leq t_0} |\int_0^t \tilde{X}(s) \tilde{\gamma}(s) \tilde{N}(ds, d\zeta)|] \leq C \mathbb{E}[(\int_0^{t_0} \tilde{X}^2(s) \tilde{\gamma}^2(s) \nu(d\zeta) ds)^{\frac{1}{2}}] \leq Ct_0 \|\tilde{X}\|_{t_0} \Delta_{t_0}. \quad (3.7)$$

Combining the above and using that

$$\|\tilde{X}\|_{t_0} \Delta_{t_0} \leq C(\|X\|_{t_0}^2 + \Delta_{t_0}^{(2)}),$$

we obtain

$$\|\tilde{X}\|_{t_0}^2 := \mathbb{E}[\sup_{t \leq t_0} \tilde{X}^2(t)] \leq Ct_0(\|\tilde{X}\|_{t_0}^2 + \Delta_{t_0}^{(2)}).$$

By definition of the norms, we have

$$\Delta_{t_0}^{(2)} \leq C\|\tilde{x}\|_{t_0}^2. \quad (3.8)$$

Thus we see that if  $t_0 > 0$  is small enough we obtain

$$\|\tilde{X}(t)\|_{t_0}^2 \leq \frac{1}{2}\|\tilde{x}(s)\|_{t_0}^2, \quad (3.9)$$

and hence  $\Phi$  is a contraction on  $\mathbb{H}_{t_0}$ . Therefore the equation has a solution up to  $t_0$ . By the same argument we see that the solution is unique. Now we repeat the argument above, but starting at  $t_0$  instead of starting at 0. Then we get a unique solution up to  $2t_0$ . Iterating this, we obtain a unique solution up to  $T$  for any  $T < \infty$ .  $\square$

## 4 Optimal control of memory mf-sfde

Consider again the controlled memory mf-sfde (??)

$$\begin{cases} dX(t) &= b(t, X(t), X_t, M(t), M_t, u(t), u_t)dt + \sigma(t, X(t), X_t, M(t), M_t, u(t), u_t)dB(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t, X(t), X_t, M(t), M_t, u(t), u_t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X(t) &= \xi(t); t \in [-\delta, 0]. \end{cases} \quad (4.1) \quad \{\text{exmf sfde}\}$$

The coefficients  $b, \sigma$  and  $\gamma$  are supposed to satisfy the assumptions of Theorem ??, uniformly w.r.t.  $u \in \mathcal{U}^{ad}$ , then we have the existence and the uniqueness of the solution  $X(t) \in \mathcal{S}^2$  of the controlled mf-sfde (??).

Moreover,  $b, \sigma$  and  $\gamma$  have Fréchet derivatives w.r.t.  $\bar{x}, m, \bar{m}$  and are continuously differentiable in the variables  $x$  and  $u$ .

The performance functional is assumed to be of the form

$$J(u) = \mathbb{E}[\int_0^T \ell(t, X(t), X_t, M(t), M_t, u(t), u_t) dt + h(X(T), M(T))]; u \in \mathcal{U}. \quad (4.2) \quad \{\text{perf}\}$$

With  $E := [0, T] \times \mathbb{R} \times \mathcal{C} \times \mathcal{M}_0 \times \mathcal{M}_0^\delta \times \mathcal{U}^{ad} \times \mathcal{C} \times \Omega$ ,  $E' := \mathbb{R} \times \mathcal{M}_0 \times \Omega$  we assume that the functions

$$\begin{aligned} \ell(t, x, \bar{x}, m, \bar{m}, u, \bar{u}) &= \ell(t, x, \bar{x}, m, \bar{m}, u, \bar{u}, \omega) &: E &\rightarrow \mathbb{R}, \\ h(x, m) &= h(x, m, \omega) &: E' &\rightarrow \mathbb{R}, \end{aligned}$$

admit Fréchet derivatives w.r.t.  $\bar{x}$ ,  $m$ ,  $\bar{m}$  and are continuously differentiable w.r.t.  $x$  and  $u$ . We allow the integrand in the performance functional (??) to depend on the path process  $X_t$  and also its law process  $\mathcal{L}(X_t) =: M_t$ , and we allow the terminal value to depend on the state  $X(T)$  and its law  $M(T)$ .

Consider the following optimal control problem. It may be regarded as a partial information control problem (since  $u$  is required to be  $\mathbb{G}$ -adapted) but only in the limited sense, since  $\mathbb{G}$  does not depend on the observation.

**Problem 4.1** Find  $u^* \in \mathcal{U}^{ad}$  such that

$$J(u^*) = \sup_{u \in \mathcal{U}^{ad}} J(u). \quad (4.3) \quad \{\text{eq4.3}\}$$

To study this problem we first introduce its associated Hamiltonian, as follows:

**Definition 4.2** The Hamiltonian

$$H : [0, T + \delta] \times \mathbb{R} \times \mathcal{C} \times \mathcal{M}_0 \times \mathcal{M}_0^\delta \times \mathcal{U}^{ad} \times \mathcal{C} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \mathbb{K} \times \Omega \rightarrow \mathbb{R}$$

associated to this memory mean-field stochastic control problem (??) is defined by

$$\begin{aligned} H(t, x, \bar{x}, m, \bar{m}, u, \bar{u}, p^0, q^0, r^0(\cdot), p^1) &= H(t, x, \bar{x}, m, \bar{m}, u, \bar{u}, p^0, q^0, r^0(\cdot), p^1, \omega) \\ &= \ell(t, x, \bar{x}, m, \bar{m}, u, \bar{u}) + p^0 b(t, x, \bar{x}, m, \bar{m}, u, \bar{u}) \\ &\quad + q^0 \sigma(t, x, \bar{x}, m, \bar{m}, u, \bar{u}) \\ &\quad + \int_{\mathbb{R}_0} r^0(t, \zeta) \gamma(t, \zeta) \nu(d\zeta) + \langle p^1, m' \rangle; t \in [0, T], \end{aligned} \quad (4.4) \quad \{\text{ham1}\}$$

and  $H(t, x, \bar{x}, m, \bar{m}, u, \bar{u}, p^0, q^0, r^0(\cdot), p^1) = 0$ ;  $t > T$ .

The Hamiltonian  $H$  is assumed to be continuously differentiable w.r.t.  $x, u$  and to admit Fréchet derivatives w.r.t.  $\bar{x}, m$  and  $\bar{m}$ .

In the following we let  $L_0^2$  denote the set of measurable stochastic processes  $Y(t)$  on  $\mathbb{R}$  such that  $Y(t) = 0$  for  $t < 0$  and for  $t > T$  and

$$\int_0^T Y^2(t) dt < \infty \quad a.s. \quad (4.5)$$

The map

$$Y \mapsto \int_0^T \langle \nabla_{\bar{x}} H(t), Y_t \rangle dt; \quad Y \in L_0^2$$

is a bounded linear functional on  $L_0^2$ . Therefore, by the Riesz representation theorem there exists a unique process  $\Gamma_{\bar{x}}(t) \in L_0^2$  such that

$$\int_0^T \Gamma_{\bar{x}}(t) Y(t) dt = \int_0^T \langle \nabla_{\bar{x}} H(t), Y_t \rangle dt, \quad (4.6) \quad \{\text{eq4.6}\}$$

for all  $Y \in L_0^2$ . Here  $\langle \nabla_{\bar{x}} H(t), Y_t \rangle$  denotes the action of the operator  $\nabla_{\bar{x}} H(t)$  to the segment  $Y_t = \{Y(t+s)\}_{s \in [-\delta, 0]}$ , where  $H(t)$  is a shorthand notation for

$$H(t, X(t), X_t, M(t), M_t, u(t), u_t, p^0(t), q^0(t), r^0(t, \cdot), p^1(t), \omega).$$

As a suggestive notation (see below) for  $\Gamma_{\bar{x}}$  we will in the following write

$$\nabla_{\bar{x}} H^t := \Gamma_{\bar{x}}(t). \quad (4.7)$$

**Lemma 4.3** *Consider the case when*

$$H(t, x, \bar{x}, p, q) = f(t, x) + F(\bar{x})p + \sigma q,$$

*Then*

$$\Gamma_{\bar{x}}(t) := \langle \nabla_{\bar{x}} F, p^t \rangle \quad (4.8) \quad \{\text{eq4.8}\}$$

*satisfies (??), where  $p^t := \{p(t+r)\}_{r \in [0, \delta]} = \{p(t-s)\}_{s \in [-\delta, 0]}$ .*

*Proof.* We must verify that if we define  $\Gamma_{\bar{x}}(t)$  by (??), then (??) holds. To this end, choose  $Y \in L_0^2$  and consider

$$\begin{aligned} \int_0^T \Gamma_{\bar{x}}(t) Y(t) dt &= \int_0^T \langle \nabla_{\bar{x}} F, p^t \rangle Y(t) dt = \int_0^T \langle \nabla_{\bar{x}} F, \{p(t+r)\}_{r \in [0, \delta]} \rangle Y(t) dt \\ &= \int_0^T \langle \nabla_{\bar{x}} F, \{Y(t)p(t+r)\}_{r \in [0, \delta]} \rangle dt \\ &= \langle \nabla_{\bar{x}} F, \{\int_r^{T+r} Y(u-r)p(u) du\}_{r \in [0, \delta]} \rangle \\ &= \langle \nabla_{\bar{x}} F, \{\int_0^T Y(u-r)p(u) du\}_{r \in [0, \delta]} \rangle \\ &= \int_0^T \langle \nabla_{\bar{x}} F, Y_u \rangle p(u) du \\ &= \int_0^T \langle \nabla_{\bar{x}} H(u), Y_u \rangle du. \end{aligned}$$

□

**Example 4.4** (i) *For example, if*

$$F(\bar{x}) = \int_{-\delta}^0 a(s)x(s) ds \quad (4.9)$$

*when  $\bar{x} = \{x(s)\}_{s \in [-\delta, 0]}$ , then*

$$\langle \nabla_{\bar{x}} F, p^t \rangle = \langle F, p^t \rangle = \int_{-\delta}^0 a(s)p(t-s) ds = \int_0^\delta a(-r)p(t+r) dr. \quad (4.10)$$

(ii) *Similarly, if*

$$G(\bar{x}) = x(-\delta) \text{ when } \bar{x} = \{x(s)\}_{s \in [-\delta, 0]}, \quad (4.11)$$

*then*

$$\langle \nabla_{\bar{x}} G, p^t \rangle = p(t+\delta). \quad (4.12)$$

For  $u \in \mathcal{U}^{ad}$  with corresponding solution  $X = X^u$ , define  $p = (p^0, p^1), q = (q^0, q^1)$  and  $r = (r^0, r^1)$  by the following two adjoint equations:

- The advanced backward stochastic functional differential equation (absfde) in the unknown  $(p^0, q^0, r^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}_\nu^2$  is given by

$$\begin{cases} dp^0(t) &= -[\frac{\partial H}{\partial x}(t) + \mathbb{E}(\nabla_{\bar{x}} H^t | \mathcal{F}_t)]dt + q^0(t)dB(t) + \int_{\mathbb{R}_0} r^0(t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ p^0(t) &= \frac{\partial h}{\partial x}(X(T), M(T)); t \geq T, \\ q^0(t) &= 0; t > T, \\ r^0(t, \cdot) &= 0; t > T. \end{cases} \quad (4.13) \quad \{\text{p0}\}$$

- The operator-valued mean-field advanced backward stochastic functional differential equation (ov-mf-absfde) in the unknown  $(p^1, q^1, r^1) \in \mathcal{S}_{\mathbb{K}}^2 \times \mathbb{L}_{\mathbb{K}}^2 \times \mathbb{L}_{\nu, \mathbb{K}}^2$  is given by

$$\begin{cases} dp^1(t) &= -[\nabla_m H(t) + \mathbb{E}(\nabla_{\bar{m}} H^t | \mathcal{F}_t)]dt + q^1(t)dB(t) + \int_{\mathbb{R}_0} r^1(t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ p^1(t) &= \nabla_m h(X(T), M(T)); t \geq T, \\ q^1(t) &= 0; t > T, \\ r^1(t, \cdot) &= 0; t > T. \end{cases} \quad (4.14) \quad \{\text{p1}\}$$

where  $\nabla_{\bar{m}} H^t$  is defined in the similar way as  $\nabla_{\bar{x}} H^t$  above, i.e. by the property that

$$\int_0^T \Gamma_{\bar{m}}(t) M(t) dt = \int_0^T \langle \nabla_{\bar{m}} H(t), M_t \rangle dt, \quad (4.15) \quad \{\text{eq4.6}\}$$

for all  $M \in L_0^2$ .

Advanced backward stochastic differential equations (absde) have been studied by Peng and Yang [?] in the Brownian setting and for the jump case, we refer to Øksendal et al [?], Øksendal and Sulem [?]. It was also extended to the context of enlargement progressive of filtration by Jeanblanc et al in [?].

When Agram and Røse [?] used the maximum principle to study optimal control of mean-field delayed sfde (??), they obtained a mean-field absfde.

The question of existence and uniqueness of the solutions of the equations above will not be studied here.

## 4.1 A sufficient maximum principle

We are now able to derive the sufficient version of the maximum principle.

**Theorem 4.5 (Sufficient maximum principle)** *Let  $\hat{u} \in \mathcal{U}^{ad}$  with corresponding solutions  $\hat{X} \in \mathcal{S}^2$ ,  $(\hat{p}^0, \hat{q}^0, \hat{r}^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}_\nu^2$  and  $(\hat{p}^1, \hat{q}^1, \hat{r}^1) \in \mathcal{S}_{\mathbb{K}}^2 \times \mathbb{L}_{\mathbb{K}}^2 \times \mathbb{L}_{\nu, \mathbb{K}}^2$  of the forward*

and backward stochastic differential equations (??), (??) and (??) respectively. For arbitrary  $u \in \mathcal{U}$ , put

$$H(t) := H(t, \widehat{X}(t), \widehat{X}_t, \widehat{M}(t), \widehat{M}_t, u(t), u_t, \widehat{p}^0(t), \widehat{q}^0(t), \widehat{r}^0(t, \cdot), \widehat{p}^1(t)), \quad (4.16)$$

$$\widehat{H}(t) := H(t, \widehat{X}(t), \widehat{X}_t, \widehat{M}(t), \widehat{M}_t, \widehat{u}(t), \widehat{u}_t, \widehat{p}^0(t), \widehat{q}^0(t), \widehat{r}^0(t, \cdot), \widehat{p}^1(t)). \quad (4.17)$$

Suppose that

- (Concavity) The functions

$$\begin{aligned} (x, \bar{x}, m, \bar{m}, u, \bar{u}) &\mapsto H(t, x, \bar{x}, m, \bar{m}, u, \bar{u}, \widehat{p}^0, \widehat{q}^0, \widehat{r}^0(\cdot), \widehat{p}^1), \\ (x, m) &\mapsto h(x, m), \end{aligned}$$

are concave  $\mathbb{P}$ -a.s. for each  $t \in [0, T]$ .

- (Maximum condition)

$$\mathbb{E}[\widehat{H}(t)|\mathcal{G}_t] = \sup_{u \in \mathcal{U}^{ad}} \mathbb{E}[H(t)|\mathcal{G}_t], \quad (4.18) \quad \{\max Q\}$$

$\mathbb{P}$ -a.s. for each  $t \in [0, T]$ .

Then  $\widehat{u}$  is an optimal control for the problem (??).

Proof. By considering a sequence of stopping times converging upwards to  $T$ , we see tghat we may assume that all the  $dB$ - and  $\tilde{N}$ - integrals in the following are martingales and hence have expectation 0. We refer to the proof of Lemma 3.1 in [?] for details.

We want to prove that  $J(u) \leq J(\widehat{u})$  for all  $u \in \mathcal{U}^{ad}$ . Application of definition (??) gives for fixed  $u \in \mathcal{U}^{ad}$  that

$$J(u) - J(\widehat{u}) = I_1 + I_2, \quad (4.19) \quad \{J\}$$

where

$$\begin{aligned} I_1 &= \mathbb{E}[\int_0^T \{\ell(t) - \widehat{\ell}(t)\} dt], \\ I_2 &= \mathbb{E}[h(X(T), M(T)) - h(\widehat{X}(T), \widehat{M}(T))], \end{aligned}$$

with

$$\ell(t) := \ell(t, \widehat{X}(t), \widehat{X}_t, \widehat{M}(t), \widehat{M}_t, u(t), u_t), \quad (4.20)$$

$$\widehat{\ell}(t) := \ell(t, \widehat{X}(t), \widehat{X}_t, \widehat{M}(t), \widehat{M}_t, \widehat{u}(t), \widehat{u}_t). \quad (4.21)$$

and similarly with  $b(t), \widehat{b}(t)$  etc. later.

Applying the definition of the Hamiltonian (??), we get

$$\begin{aligned} I_1 &= \mathbb{E}[\int_0^T \{H(t) - \widehat{H}(t) - \widehat{p}^0(t)\tilde{b}(t) - \widehat{q}^0(t)\tilde{\sigma}(t) \\ &\quad - \int_{\mathbb{R}_0} \widehat{r}^0(t, \zeta)\tilde{\gamma}(t, \zeta)\nu(d\zeta) - \langle \widehat{p}^1(t), \widehat{M}'(t) \rangle\} dt], \end{aligned} \quad (4.22) \quad \{I1\}$$

where  $\tilde{b}(t) = b(t) - \widehat{b}(t)$  etc., and

$$\widetilde{M}'(t) = \frac{d\widetilde{M}(t)}{dt} = \frac{d}{dt}(M(t) - \widehat{M}(t)).$$

Using concavity of  $h$  and the definition of the terminal values of the absfde (??) and (??), we get

$$\begin{aligned} I_2 &\leq \mathbb{E}[\frac{\partial \widehat{h}}{\partial x}(T)\widetilde{X}(T) + \nabla_m \widehat{h}(T)\widetilde{M}(T)] \\ &= \mathbb{E}[\widehat{p}^0(T)\widetilde{X}(T) + \langle \widehat{p}^1(T), \widetilde{M}(T) \rangle]. \end{aligned} \quad (4.23) \quad \{\text{I2}\}$$

Applying the Itô formula to  $\widehat{p}^0\widetilde{X}$  and  $\widehat{p}^1\widetilde{M}$ , we have

$$\begin{aligned} \mathbb{E}[\widehat{p}^0(T)\widetilde{X}(T)] &= \mathbb{E}[\int_0^T \widehat{p}^0(t)d\widetilde{X}(t) + \int_0^T \widetilde{X}(t)d\widehat{p}^0(t) + \int_0^T \widehat{q}^0(t)\widetilde{\sigma}(t)dt \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \widehat{r}^0(t, \zeta)\widetilde{\gamma}(t, \zeta)\nu(d\zeta)dt] \\ &= \mathbb{E}[\int_0^T \widehat{p}^0(t)\tilde{b}(t)dt - \int_0^T \frac{\partial \widehat{H}}{\partial x}(t)\widetilde{X}(t)dt - \int_0^T \mathbb{E}(\nabla_{\overline{x}}\widehat{H}^t|\mathcal{F}_t)\widetilde{X}(t)dt \\ &\quad + \int_0^T \widehat{q}^0(t)\widetilde{\sigma}(t)dt + \int_0^T \int_{\mathbb{R}_0} \widehat{r}^0(t, \zeta)\widetilde{\gamma}(t, \zeta)\nu(d\zeta)dt], \end{aligned} \quad (4.24) \quad \{.10\}$$

and

$$\begin{aligned} &\mathbb{E}[\langle \widehat{p}^1(T), \widetilde{M}(T) \rangle] \\ &= \mathbb{E}[\int_0^T \langle \widehat{p}^1(t), d\widetilde{M}(t) \rangle dt + \int_0^T \langle \widetilde{M}(t), d\widehat{p}^1(t) \rangle dt] \end{aligned} \quad (4.25)$$

$$= \mathbb{E}[\int_0^T \langle \widehat{p}^1(t), \widetilde{M}'(t) \rangle dt - \int_0^T \langle \nabla_m \widehat{H}(t), \widetilde{M}(t) \rangle dt - \int_0^T \mathbb{E}(\nabla_{\overline{m}}\widehat{H}^t|\mathcal{F}_t)\widetilde{M}(t)dt], \quad (4.26)$$

where we have used that the  $dB(t)$  and  $\widetilde{N}(dt, d\zeta)$  integrals have mean zero. On substituting (??), (??) and (??) into (??), we obtain \{.11\}

$$\begin{aligned} J(u) - J(\widehat{u}) &\leq \mathbb{E}[\int_0^T \{H(t) - \widehat{H}(t) - \int_0^T \frac{\partial \widehat{H}}{\partial x}(t)\widetilde{X}(t)dt - \int_0^T \nabla_{\overline{x}}\widehat{H}^t\widetilde{X}(t)\}dt \\ &\quad - \int_0^T \langle \nabla_m \widehat{H}(t), \widetilde{M}(t) \rangle dt - \int_0^T \nabla_{\overline{m}}\widehat{H}^t\widetilde{M}(t)dt]. \end{aligned}$$

Since  $\widetilde{X}(t) = 0$  for all  $t \in [-\delta, 0]$  and for all  $t > T$  we see that  $\widetilde{X} \in L_0^2$  and therefore by (??), we have

$$\int_0^T \nabla_{\overline{x}}\widehat{H}^t\widetilde{X}(t)dt = \int_0^T \langle \nabla_{\overline{x}}\widehat{H}(t), \widetilde{X}_t \rangle dt. \quad (4.27) \quad \{\text{estpath}\}$$

Similar considerations give

$$\int_0^T \nabla_{\overline{m}}\widehat{H}^t\widetilde{M}(t)dt = \int_0^T \langle \nabla_{\overline{m}}\widehat{H}(t), \widetilde{M}_t \rangle dt. \quad (4.28) \quad \{\text{stpath2}\}$$

By the assumption that  $H$  is concave and that the process  $u$  is  $\mathcal{G}_t$ -adapted, we therefore get

$$\begin{aligned} J(u) - J(\widehat{u}) &\leq \mathbb{E}[\int_0^T \{\frac{\partial \widehat{H}}{\partial u}(t)\widetilde{u}(t) + \langle \nabla_{\overline{u}}\widehat{H}(t), \widetilde{u}_t \rangle\}dt] \\ &= \mathbb{E}[\int_0^T \mathbb{E}(\frac{\partial \widehat{H}}{\partial u}(t)\widetilde{u}(t) + \langle \nabla_{\overline{u}}\widehat{H}(t), \widetilde{u}_t \rangle | \mathcal{G}_t)dt] \\ &= \mathbb{E}[\int_0^T \{\mathbb{E}[\frac{\partial \widehat{H}}{\partial u}(t)|\mathcal{G}_t]\widetilde{u}(t) + \langle \mathbb{E}[\nabla_{\overline{u}}\widehat{H}(t)|\mathcal{G}_t], \widetilde{u}_t \rangle\}dt] \leq 0. \end{aligned}$$

For the last inequality to hold, we use that  $\mathbb{E}[\widehat{H}(t)|\mathcal{G}_t]$  has a maximum at  $\widehat{u}(t)$ . □

## 4.2 A necessary maximum principle

We now proceed to study the necessary maximum principle. Let us then impose the following set of assumptions.

i) On the coefficient functionals:

- The functions  $b, \sigma$  and  $\gamma$  admit bounded partial derivatives w.r.t.  $x, \bar{x}, m, \bar{m}, u, \bar{u}$ .

ii) On the performance functional:

- The function  $\ell$  and the terminal value  $h$  admit bounded partial derivatives w.r.t.  $x, \bar{x}, m, \bar{m}, u, \bar{u}$  and w.r.t.  $x, m$  respectively.

ii) On the set of admissible processes:

- Whenever  $u \in \mathcal{U}^{ad}$  and  $\pi \in \mathcal{U}^{ad}$  is bounded, there exists  $\epsilon > 0$  such that

$$u + \lambda\pi \in \mathcal{U}^{ad}, \text{ for each } \lambda \in [-\epsilon, \epsilon].$$

- For each  $t_0 \in [0, T]$  and all bounded  $\mathcal{G}_{t_0}$ -measurable random variables  $\alpha$ , the process

$$\pi(t) = \alpha \mathbf{1}_{(t_0, T]}(t),$$

belongs to  $\mathcal{U}^{ad}$ .

In general, if  $K^u(t)$  is a process depending on  $u$ , we define the operator  $D$  on  $K$  by

$$DK^u(t) := D^\pi K^u(t) = \frac{d}{d\lambda} K^{u+\lambda\pi}(t)|_{\lambda=0}, \quad (4.29)$$

whenever the derivative exists.

Define the *derivative process*  $Z(t)$  by

$$Z(t) := DX(t) := \frac{d}{d\lambda} X^{u+\lambda\pi}|_{\lambda=0}.$$

Using matrix notation, note that  $Z(t)$  satisfies the equation

$$\left\{ \begin{array}{l} dZ(t) = (\nabla b(t))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t) dt \\ \quad + (\nabla \sigma(t))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t) B(t) \\ \quad + \int_{\mathbb{R}_0} (\nabla \gamma(t, \zeta))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ Z(t) = 0; t \in [-\delta, 0], \end{array} \right. \quad (4.30) \quad \{\text{dervz}\}$$

where  $(\nabla b)^T = (\frac{\partial b}{\partial x}, \nabla_{\bar{x}} b, \nabla_m b, \nabla_{\bar{m}} b, \frac{\partial b}{\partial u}, \nabla_{\bar{u}} b)^T$ ,  $(\cdot)^T$  denotes matrix transposed and we mean by  $\nabla_{\bar{x}} b(t) Z_t$ , (respectively  $\nabla_{\bar{m}} b(t) DM_t$ ) the action of the operator  $\nabla_{\bar{x}} b(t)$  ( $\nabla_{\bar{m}} b(t)$ ) on the segment  $Z_t = \{Z(t+s)\}_{s \in [-\delta, 0]}$  ( $DM_t = \{DM(t+s)\}_{s \in [-\delta, 0]}$ ) i.e.,  $\langle \nabla_{\bar{x}} b(t), Z_t \rangle$  ( $\langle \nabla_{\bar{m}} b(t), DM_t \rangle$ ) and similar considerations for  $\sigma$  and  $\gamma$ .

**Theorem 4.6 (Necessary maximum principle)** *Let  $\hat{u} \in \mathcal{U}^{ad}$  with corresponding solutions  $\hat{X} \in \mathcal{S}^2$  and  $(\hat{p}^0, \hat{q}^0, \hat{r}^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_\nu$  and  $(\hat{p}^1, \hat{q}^1, \hat{r}^1) \in \mathcal{S}^2_{\mathbb{K}} \times \mathbb{L}^2_{\mathbb{K}} \times \mathbb{L}^2_{\nu, \mathbb{K}}$  of the forward and backward stochastic differential equations (??) and (??) – (??) respectively, with the corresponding derivative process  $\hat{Z} \in \mathcal{S}^2$  given by (??). Then the following, (i) and (ii), are equivalent:*

(i) For all bounded  $\pi \in \mathcal{U}^{ad}$

$$\frac{d}{d\lambda} J(\hat{u} + \lambda\pi)|_{\lambda=0} = 0.$$

(ii)

$$\mathbb{E}[(\frac{\partial H}{\partial u}(t) + \nabla_{\bar{u}} H_t)|\mathcal{G}_t]_{u=\hat{u}} = 0 \text{ for all } t \in [0, T].$$

Proof. Before starting the proof, let us first clarify some notation: Note that

$$\nabla_m \langle p_1^1(t), \frac{d}{dt} m \rangle = \langle p_1^1(t), \frac{d}{dt}(\cdot) \rangle,$$

and hence

$$\langle \nabla_m \langle p_1^1(t), \frac{d}{dt} m \rangle, DM(t) \rangle = \langle p_1^1(t), \frac{d}{dt} DM(t) \rangle = \langle p_1^1(t), DM'(t) \rangle = p_1^1(t) DM'(t).$$

Also, note that

$$dDM(t) = DM'(t)dt. \tag{4.31}$$

Assume that (i) holds. Then

$$\begin{aligned} 0 &= \frac{d}{d\lambda} J(u + \lambda\pi)|_{\lambda=0} \\ &= \mathbb{E}[\int_0^T \{(\nabla \ell(t))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t)\} dt \\ &\quad + \frac{\partial h}{\partial x}(T) Z(T) + \nabla_m h(T) DM(T)]. \end{aligned}$$

Hence, by the definition of  $H$  (??) and the terminal values of the absfde  $p^0(T)$  and  $p^1(T)$ , we have

$$\begin{aligned} 0 &= \frac{d}{d\lambda} J(u + \lambda\pi)|_{\lambda=0} \\ &= \mathbb{E}[\int_0^T \{(\nabla H(t))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t) \\ &\quad - p^0(t)(\nabla b(t))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t) \\ &\quad - q^0(t)(\nabla \sigma(t))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t) \\ &\quad - \int_{\mathbb{R}_0} r^0(t, \zeta)(\nabla \gamma(t, \zeta))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t) \nu(d\zeta)\} dt] \\ &\quad - \int_0^T p^1(t) DM'(t) dt + p^0(T) Z(T) + p^1(T) DM(T)]. \end{aligned}$$

Applying Itô formula to both  $p^0 Z$  and  $p^1 DM$ , we get

$$\begin{aligned} \mathbb{E}[p^0(T) Z(T)] &= \mathbb{E}[\int_0^T p^0(t) dZ(t) + \int_0^T Z(t) dp^0(t) + [p^0, Z]_T] \\ &= \mathbb{E}[\int_0^T p^0(t)(\nabla b(t))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t) dt \\ &\quad - \int_0^T \{ \frac{\partial H}{\partial x}(t) + \nabla_{\bar{x}} H^t \} Z(t) dt \\ &\quad + \int_0^T q^0(t)(\nabla \sigma(t))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t) dt \\ &\quad + \int_0^T \int_{\mathbb{R}_0} r^0(t, \zeta)(\nabla \gamma(t, \zeta))^T (Z(t), Z_t, DM(t), DM_t, \pi(t), \pi_t) \nu(d\zeta) dt], \end{aligned}$$



and

$$\begin{aligned}\mathbb{E}[p^1(T)DM(T)] &= \mathbb{E}[\int_0^T p^1(t)DM'(t)dt + \int_0^T DM(t)dp^1(t)] \\ &= \mathbb{E}[\int_0^T p_1^1(t)DM'(t)dt - \int_0^T \{\nabla_m H(t) + \nabla_{\bar{m}} H^t\}DM(t)dt].\end{aligned}$$

Proceeding as in (??) – (??), we obtain

$$\begin{aligned}\int_0^T \nabla_{\bar{x}} H^t Z(t)dt &= \int_0^T \langle \nabla_{\bar{x}} H(t), Z_t \rangle dt, \\ \int_0^T \nabla_{\bar{m}} H^t DM(t)dt &= \int_0^T \langle \nabla_{\bar{m}} H(t), DM_t \rangle dt.\end{aligned}$$

Combining the above, we get

$$0 = \mathbb{E}[\int_0^T (\frac{\partial H}{\partial u}(t)\pi(t) + \langle \nabla_{\bar{x}} H(t), \pi_t \rangle)dt]. \quad (4.32) \quad \{\mathbf{h\_pi}\}$$

Now choose  $\pi(t) = \alpha \mathbf{1}_{(t_0, T]}(t)$ , where  $\alpha = \alpha(\omega)$  is bounded and  $\mathcal{G}_{t_0}$ -measurable and  $t_0 \in [0, T)$ . Then  $\pi_t = \alpha \{\mathbf{1}_{(t_0, T]}(t+s)\}_{s \in [-\delta, 0]}$  and (??) gives

$$0 = \mathbb{E}[\int_{t_0}^T \frac{\partial H}{\partial u}(t)\alpha dt + \int_{t_0}^T \langle \nabla_{\bar{x}} H(t), \alpha \{\mathbf{1}_{(t_0, T]}(t+s)\}_{s \in [-\delta, 0]} dt].$$

Differentiating with respect to  $t_0$ , we obtain

$$\mathbb{E}[(\frac{\partial H}{\partial u}(t_0) + \nabla_{\bar{x}} H_{t_0})\alpha] = 0,$$

Since this holds for all such  $\alpha$ , we conclude that

$$\mathbb{E}[(\frac{\partial H}{\partial u}(t_0) + \nabla_{\bar{x}} H_{t_0})|\mathcal{G}_{t_0}] = 0, \text{ which is (ii).}$$

This argument can be reversed, to prove that (ii)  $\implies$  (i). We omit the details.  $\square$

## 5 Applications

We illustrate our results by studying some examples.

### 5.1 Mean-variance portfolio with memory

We apply the results obtained in the previous sections to solve the memory mean-variance problem by proceeding as it has been done in Framstad et al [?], Anderson and Djehiche [?] and Røse [?].

Consider the state equation  $X^\pi(t) = X(t)$  on the form

$$\begin{cases} dX(t) &= X(t - \delta)\pi(t)[b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma_0(t, \zeta) \tilde{N}(dt, d\zeta)]; t \in [0, T], \\ X(t) &= \xi(t); t \in [-\delta, 0], \end{cases} \quad (5.1) \quad \{\mathbf{w}\}$$

for some bounded deterministic function  $\xi(t); t \in [-\delta, 0]$ . We assume that the admissible processes are càdlàg processes in  $L^2(\Omega, [0, T])$ , that are adapted to the filtration  $\mathcal{F}_t$  and such

that a unique solution exists. The coefficients  $b_0, \sigma_0$  and  $\gamma_0 > -1$  are supposed to be bounded  $\mathbb{F}$ -adapted processes with

$$|b_0(t)| > 0 \text{ and } \sigma_0^2(t) + \int_{\mathbb{R}_0} \gamma_0^2(t, \zeta) \nu(d\zeta) > 0 \text{ a.s. for all } t.$$

We want to find an admissible portfolio  $\pi(t)$  which maximizes

$$J(\pi) = \mathbb{E}\left[-\frac{1}{2}(X(T) - a)^2\right], \quad (5.2) \quad \{\mathbf{p}\}$$

over the set of admissible processes  $\mathcal{U}^{ad}$  and for a given constant  $a \in \mathbb{R}$ .

The Hamiltonian for this problem is given by

$$H(t, \bar{x}, \pi, p^0, q^0, r^0(\cdot)) = \pi G(\bar{x})(b_0 p^0 + \sigma_0 q^0 + \int_{\mathbb{R}_0} \gamma_0(\zeta) r^0(\zeta) \nu(d\zeta)), \quad (5.3) \quad \{\mathbf{h}\}$$

where

$$G(\bar{x}) = x(-\delta) \text{ when } \bar{x} = \{x(s)\}_{s \in [-\delta, 0]}. \quad (5.4)$$

See Example 4.4 (i). Hence by Lemma 4.3 the triple  $(p^0, q^0, r^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}_\nu^2$  is the adjoint process which satisfies

$$\begin{cases} dp^0(t) &= -\mathbb{E}[\pi(t+\delta)(b_0(t+\delta)p^0(t+\delta) + \sigma_0(t+\delta)q^0(t+\delta) \\ &+ \int_{\mathbb{R}_0} \gamma_0(t+\delta, \zeta) r^0(t+\delta, \zeta) \nu(d\zeta)) | \mathcal{F}_t] dt + q^0(t) dB(t) \\ &+ \int_{\mathbb{R}_0} r^0(t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ p^0(t) &= -(X(T) - a); t \geq T, \\ q^0(t) &= r^0(\cdot) = 0; t > T. \end{cases} \quad (5.5) \quad \{\mathbf{ap}\}$$

Existence and uniqueness of equations of type (??) have been studied by Øksendal et al [?].

Suppose that  $\hat{\pi}$  is an optimal control. Then by the necessary maximum principle, we get for each  $t$  that

$$\begin{aligned} 0 &= \frac{\partial \hat{H}}{\partial \pi}(t, \hat{X}_t, \hat{\pi}(t), \hat{p}^0(t), \hat{q}^0(t), \hat{r}^0(t, \cdot)) \\ &= \hat{X}(t - \delta)(b_0(t)\hat{p}^0(t) + \sigma_0(t)\hat{q}^0(t) + \int_{\mathbb{R}_0} \gamma_0(t, \zeta) \hat{r}^0(t, \zeta) \nu(d\zeta)). \end{aligned} \quad (5.6) \quad \{\mathbf{nc}\}$$

So we search for a candidate  $\hat{\pi}$  satisfying

$$0 = b_0(t)\hat{p}^0(t) + \sigma_0(t)\hat{q}^0(t) + \int_{\mathbb{R}_0} \gamma_0(t, \zeta) \hat{r}^0(t, \zeta) \nu(d\zeta), \text{ for all } t. \quad (5.7) \quad \{\mathbf{pi}\}$$

This gives the following adjoint equation:

$$\begin{cases} d\hat{p}^0(t) &= \hat{q}^0(t) dB(t) + \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ \hat{p}^0(t) &= -(X(T) - a); t \geq T, \\ \hat{q}^0(t) &= \hat{r}^0(\cdot) = 0; t > T. \end{cases} \quad (5.8) \quad \{\mathbf{ap1}\}$$

We start by guessing that  $\hat{p}^0$  has the form

$$\hat{p}^0(t) = \varphi(t)\hat{X}(t) + \psi(t) \quad (5.9) \quad \{\mathbf{p}^\wedge\}$$

for some deterministic functions  $\varphi, \psi \in C^1[0, T]$  with

$$\varphi(T) = -1, \quad \psi(T) = a. \quad (5.10) \quad \{\mathbf{v}\}$$

Using the Itô formula to find the integral representation of  $\widehat{p}^0$  and comparing with the adjoint equation (??), we find that the following three equations need to be satisfied:

$$0 = \varphi'(t)\widehat{X}(t) + \psi'(t) + \varphi(t)\widehat{X}(t - \delta)\widehat{\pi}(t)b_0(t), \quad (5.11) \quad \{\mathbf{d}\}$$

$$\widehat{q}^0(t) = \varphi(t)\widehat{X}(t - \delta)\widehat{\pi}(t)\sigma_0(t), \quad (5.12) \quad \{\mathbf{di}\}$$

$$\widehat{r}^0(t, \zeta) = \varphi(t)\widehat{X}(t - \delta)\widehat{\pi}(t)\gamma_0(t, \zeta). \quad (5.13) \quad \{\mathbf{j}\}$$

Assuming that  $\widehat{X}(t) \neq 0$   $\mathbb{P} \times dt$ -a.e. and  $\varphi(t) \neq 0$  for each  $t$ , we find from equation (??) that  $\widehat{\pi}$  needs to satisfy

$$\widehat{\pi}(t) = -\frac{\varphi'(t)\widehat{X}(t) + \psi'(t)}{\varphi(t)\widehat{X}(t - \delta)b_0(t)}.$$

Now inserting the expressions for the adjoint processes (??), (??) and (??) into (??), the following equation need to be satisfied:

$$0 = b_0(t)[\varphi(t)\widehat{X}(t) + \psi(t)] + \varphi(t)\widehat{X}(t - \delta)\widehat{\pi}(t)(\sigma_0^2(t) + \int_{\mathbb{R}_0} \gamma_0^2(t, \zeta)\nu(d\zeta)).$$

This means that the control  $\widehat{\pi}$  also needs to satisfy

$$\widehat{\pi}(t) = -\frac{b_0(t)[\varphi(t)\widehat{X}(t) + \psi(t)]}{[\sigma_0^2(t) + \int_{\mathbb{R}_0} \gamma_0^2(t, \zeta)\nu(d\zeta)]\varphi(t)\widehat{X}(t - \delta)}. \quad (5.14) \quad \{\mathbf{pih}\}$$

By comparing the two expressions for  $\widehat{\pi}$ , we find that

$$\begin{aligned} b_0^2(t)[\varphi(t)\widehat{X}(t) + \psi(t)] \\ = (\sigma_0^2(t) + \int_{\mathbb{R}_0} \gamma_0^2(t, \zeta)\nu(d\zeta))[\varphi'(t)\widehat{X}(t) + \psi'(t)]. \end{aligned} \quad (5.15) \quad \{\mathbf{co}\}$$

Now define

$$\Lambda(t) := \frac{b_0^2(t)}{\sigma_0^2(t) + \int_{\mathbb{R}_0} \gamma_0^2(t, \zeta)\nu(d\zeta)}. \quad (5.16) \quad \{\mathbf{lam}\}$$

Then from equation (??), we need to have

$$\begin{aligned} \varphi'(t) - \Lambda(t)\varphi(t) &= 0, \\ \psi'(t) - \Lambda(t)\psi(t) &= 0. \end{aligned}$$

Together with the terminal values (??), these equations have the solution

$$\begin{aligned} \varphi(t) &= -\exp(-\int_t^T \Lambda(s)ds), \\ \psi(t) &= a \exp(-\int_t^T \Lambda(s)ds). \end{aligned}$$

Then from equation (??) we can compute

$$\widehat{\pi}(t) = \frac{b_0(t) \left( \widehat{X}(t) - \frac{\psi(t)}{\varphi(t)} \right)}{\sigma_0^2(t) + \int_{\mathbb{R}_0} \gamma_0^2(t, \zeta) \nu(d\zeta) \widehat{X}(t-\delta)} = \Lambda(t) \frac{\left( \widehat{X}(t) - \frac{\psi(t)}{\varphi(t)} \right)}{b_0(t) \widehat{X}(t-\delta)} = \frac{\Lambda(t)}{b_0(t) \widehat{X}(t-\delta)} (\widehat{X}(t) - a).$$

Now, with our choice of  $\widehat{\pi}$ , the corresponding state equation is the solution of

$$\begin{cases} d\widehat{X}(t) &= \frac{\Lambda(t)}{b_0(t)} (\widehat{X}(t) - a) [b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma_0(t, \zeta) \widetilde{N}(dt, d\zeta)]; t \in [0, T], \\ \widehat{X}(t) &= x_0(t); t \in [-\delta, 0]. \end{cases} \quad (5.17) \quad \{\text{wp}\}$$

Put  $Y(t) = \widehat{X}(t) - a$ , then

$$dY(t) = Y(t) [\Lambda(t)b_0(t)dt + \frac{\Lambda(t)}{b_0(t)}\sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \frac{\Lambda(t)}{b_0(t)}\gamma_0(t, \zeta) \widetilde{N}(dt, d\zeta)]. \quad (5.18) \quad \{\text{lw}\}$$

The linear equation (??) has the following explicit solution

$$Y(t) = Y(0) \exp \left[ \int_0^t \Lambda(s)b_0(s)ds + \int_0^t \frac{\Lambda(s)}{b_0(s)}\sigma_0(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} \frac{\Lambda(s)}{b_0(s)}\gamma_0(s, \zeta) \widetilde{N}(ds, d\zeta) \right].$$

So if  $Y(0) > 0$  then  $Y(t) > 0$  for all  $t$ .

We have proved the following:

**Theorem 5.1 (Optimal mean-variance portfolio)** *Suppose that  $\xi(t) > a$  for all  $t \in [-\delta, 0]$ . Then  $\widehat{X}(t - \delta) > 0$  for all  $t \geq 0$  and the solution  $\widehat{\pi} \in \mathcal{U}^{ad}$  of the mean-variance portfolio problem (??) is given in feedback form as*

$$\widehat{\pi}(t) = \frac{\Lambda(t)}{b_0(t) \widehat{X}(t-\delta)} (\widehat{X}(t) - a),$$

where  $\widehat{X}(t)$  and  $\Lambda(t)$  are given by equations (??) and (??) respectively.

## 5.2 A linear-quadratic (LQ) problem with memory

We now consider a linear-quadratic control problem for a controlled system  $X(t) = X^u(t)$  driven by a distributed delay, of the form

$$\begin{cases} dX(t) &= [\int_{-\delta}^0 a(s)X(t+s)ds + u(t)]dt + \alpha_0 dB(t) + \int_{\mathbb{R}_0} \beta_0(\zeta) \widetilde{N}(dt, d\zeta); t \in [0, T], \\ X(t) &= \xi(t); t \in [-\delta, 0], \end{cases} \quad (5.19) \quad \{\text{f}\}$$

where  $\xi(\cdot)$  and  $a(\cdot)$  are given bounded deterministic functions,  $\alpha_0$  is a given constant,  $\beta_0$  is a given function from  $\mathbb{R}_0$  into  $\mathbb{R}$  with

$$\int_{\mathbb{R}_0} \beta_0(\zeta) \nu(d\zeta) < \infty,$$

and  $u \in \mathcal{U}^{ad}$  is our control process. We want to minimize the expected value of  $X^2(T)$  with a minimal average use of energy, measured by the integral  $\mathbb{E}[\int_0^T u^2(t)dt]$ , i.e. the performance functional is of the quadratic type

$$J(u) = -\frac{1}{2} \mathbb{E}[X^2(T) + \int_0^T u^2(t)dt].$$

Our goal is to find  $\hat{u} \in \mathcal{U}^{ad}$ , such that

$$J(\hat{u}) = \sup_{u \in \mathcal{U}^{ad}} J(u). \quad (5.20) \quad \{\text{eq5.20}\}$$

The Hamiltonian in that case takes the form

$$H(t, \bar{x}, u, p^0, q^0) = -\frac{1}{2}u^2 + (F(\bar{x}) + u)p^0 + \beta_0 q^0 + \int_{\mathbb{R}_0} r^0(\zeta) \beta_0(\zeta) \nu(d\zeta),$$

where

$$F(\bar{x}) = \int_{-\delta}^0 a(s)x(s)ds \text{ when } \bar{x} = \{x(s)\}_{s \in [-\delta, 0]}. \quad (5.21)$$

By Lemma 4.3 and Example 4.4 (i) we see that the adjoint absde for  $(p^0, q^0, r^0)$  is the following linear absde

$$\begin{cases} dp^0(t) &= -\mathbb{E}[\int_0^\delta a(-r)p^0(t+r)dr | \mathcal{F}_t]dt + q^0(t)dB(t) + \int_{\mathbb{R}_0} r^0(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ p^0(T) &= -X(T); \quad t \geq T. \end{cases} \quad (5.22) \quad \{\mathbf{b}\}$$

The function  $u \mapsto H(t, \hat{X}(t-\delta), \hat{u}(t), \hat{p}^0(t), \hat{q}^0(t), r^0(t, \zeta))$  is maximal when

$$u(t) = \hat{u}(t) = \hat{p}^0(t). \quad (5.23) \quad \{\text{eq5.23}\}$$

We have proved:

**Theorem 5.2** *The optimal control  $\hat{u}$  of the LQ memory problem (??) is given by (??), where the quadruplet  $(\hat{X}(t) = X^{\hat{u}}(t), \hat{p}^0(t), \hat{q}^0(t), \hat{r}^0(t, \zeta))$  solves the following coupled system of forward-backward stochastic differential equations with distributed delay:*

•

$$\begin{aligned} d\hat{X}(t) &= \left( \int_0^\delta a(-r)\hat{X}(t-r)dr + \hat{p}^0(t) \right) dt + \alpha_0 dB(t) + \int_{\mathbb{R}_0} \beta_0(\zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ \hat{X}(t) &= \xi(t); \quad t \in [-\delta, 0], \end{aligned} \quad (5.24) \quad \{\text{eq5.24}\}$$

•

$$\begin{aligned} d\hat{p}^0(t) &= -\left( \int_0^\delta a(-r)\mathbb{E}[\hat{p}^0(t+r) | \mathcal{F}_t]dr \right) dt + \hat{q}^0(t)dB(t) + \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ \hat{p}^0(T) &= -\hat{X}(T); \quad t \geq T. \end{aligned} \quad (5.25) \quad \{\text{eq5.25}\}$$

**Remark 5.3** *We may regard this coupled system (??)-(??) as the corresponding Riccati equation to our LQ memory problem. See e.g. Hu & Øksendal [?], page 1747.*

Define

$$Y(t) := -\hat{X}(t); \quad t \in [0, T]. \quad (5.26)$$

Note that (??) can be written as a forward SDE as follows:

$$\begin{aligned} dY(t) &= \left( \int_0^\delta a(-r)Y(t-r)dr - \hat{p}^0(t) \right) dt - \alpha_0 dB(t) - \int_{\mathbb{R}_0} \beta_0(\zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ Y(T) &= -\hat{X}(T). \end{aligned} \tag{5.27} \quad \{\text{eq5.26}\}$$

Comparing with (??) we see that if we choose

$$\hat{q}^0(t) = -\alpha_0 \text{ and } \hat{r}^0(t, \zeta) = -\beta_0(\zeta) \tag{5.28}$$

and choose  $\hat{p}^0$  such that

$$\int_0^\delta a(-r)Y(t-r)dr - \hat{p}^0(t) = \int_0^\delta a(-r)\mathbb{E}[\hat{p}^0(t+r)|\mathcal{F}_t]dr, \tag{5.29}$$

then the two equations (??) and (??) become identical, and we can conclude that

$$\hat{p}^0(t) = Y(t) = -X(t); \quad t \in [0, T]. \tag{5.30}$$

We can summarize this as follows:

**Theorem 5.4** *Suppose there exists an adapted solution  $\hat{u}(t)$  of the stochastic Volterra-type equation*

$$\hat{u}(t) + \int_0^\delta a(-r)\mathbb{E}[\hat{u}(t+r)|\mathcal{F}_t]dr = -\int_0^\delta a(-r)\hat{X}(t-r)dr, \quad t \in [\delta, T + \delta]. \tag{5.31}$$

*Then  $\hat{u}(t)$  is an optimal control for the problem (??), expressed in feedback form of the memory of the state process  $\hat{X}(\cdot)$ .*

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