Kinetic screening of fields and stability of NEC violating configurations

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Abstract:

Following the discovery of the late-time acceleration, a multitude of extensions to General Relativity have been developed, seeking to shed light on this new phenomenon. Some of these models involve couplings to ordinary matter, suggesting an impact on gravitational effects. However, local measurements enforce severe constraints on these models, rendering them inadequate as cosmological theories. Thus there is purpose in finding mechanisms to hide the signature of these modifications in local environments, and as such restore the models viability. Screening mechanisms have been explored in the past, and successfully implemented in scalar field theories. In this thesis we will consider a screening mechanism inspired by k-mouflage on general vector field theories of the form $f(F^2)$ and $f(F^2, F\tilde{F})$. Further, we will analyse the stability about the screened solutions and look for healthy violations to the null energy condition.

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Contents

1 Introduction

Vacuum by definition means emptiness of space. According to quantum mechanics however, there is no such place. Vacuum is the state of minimal energy and the laws of quantum mechanics dictate that such a state must have fluctuations. Therefore it is not completely empty and this lack of emptiness is reflected in the fact that its energy is non-zero. As it is the greater part of the cosmos it is an important subject to research in order to unravel the mystery of the Universe. The full picture of how the vacuum energy contribute to the cosmological evolution is yet to be known. However, since the discovery that the Universe is expanding with increasing velocity it is widely believed that vacuum energy is the cause of it. In truth it is not known exactly what it is that generate this accelerated expansion, it could even be some other exotic component we have yet to discover. For now, we label it *dark energy*.

Dark energy is characterized by its property of having negative pressure. In terms of its dynamics, this means that the equation of state of dark energy, $w = P/\rho$ is negative, where P is the pressure and ρ is the energy density. In fact, to realize the rate of the accelerated expansion we observe today, we know that dark energy must have an equation of state less than $w = -1/3$. The most common model which applies this feature is the *cosmological constant* Λ. First introduced by Albert Einstein as a means to realize a static universe when he discovered that his equations of motion, derived in the framework of General Relativity, did in fact produce a dynamic universe. Later observations proved that the Universe was not static after all, and by the end of the century discoveries were made that the Universe expands with increasing velocity. The already known cosmological constant became the natural explanation for this late time acceleration, representing dark energy as a constant parameter through time and space with the equation of state $w = -1$.

Although the cosmological constant model is consistent with observations, the unknown nature of dark energy has led to the propositions of alternative models which offer different perspective on the subject. Opportunities such as an equation of state depending on space and time, realized for instance through some scalar field, could lead to interesting dynamical features. However certain aspects of relating dark energy to vacuum energy leaves unsolved theoretical problems. Moreover, such a relation suggests it necessarily should be a coupling between any dark energy field and the fields from the standard model. In the close vicinity of the solar system we would expect to detect such couplings through measurements of gravity. Yet local gravity experiments fit General Relativity to a precise degree, so any modified theory would be heavy constrained. Motivated by the questions these problems pose, a new development in the form of *screening mechanisms* have been proposed as a method to hide the effects of dark energy in dense environments.

The screening models proposed to date are mostly developed for scalar fields Lagrangian, where the screening mechanisms usually kicks in either in regions of high Newtonian potential or when higher derivatives of the scalar field dominates. However, theories involving vector field modifications to General Relativity have been less exhaustively studied. In this thesis screening mechanisms for a model involving a vector field Lagrangian will be attempted. We will consider a model similar to the Euler-Heisenberg Lagrangian, which describes the non-linear dynamics of electromagnetic fields in vacuum. In detail the model in question consists of a Maxwell-term supplemented by a quartic interaction in the vector field strength tensor in addition to a coupling to a conserved current. Further we will establish the stability of the model by investigating the presence of pathologies and examine healthy violations to the null energy condition.

Following this introduction a short review of cosmology, General Relativity and the cosmological constant will be given. Thereafter we introduce several alternative dark energy theories succeeded by examining various existing screening mechanisms in order to ensure that the reader is up to date. After that the model in question will be presented together with attempts to screen its effects in dense environments. Finally we will review shortly the various energy conditions, analyse the stability of the theory and look for healthy violations to the null energy condition.

1.1 Cosmology

Cosmology is the scientific study of the structure and dynamics of the Universe as a whole. While the birth of astronomy dates back thousands of years, the modern science of cosmology is fairly new. In fact, it was not until the theory of General Relativity that we started to think of the Universe as a physical system governed by a set of equations which we can study mathematically. Later, the discovery of the cosmic microwave background (CMB) pointed to the Hot Big Bang model, which is the prevailing cosmological model for the Universe today.

The foundation of the physical system of the Universe is based on some principles deduced from some observations. The first observation is that the Universe is approximately *isotropic*. It means that no matter where we look, beyond our immediate surroundings such as our galaxy, the Universe looks the same in every direction. If it looks the same in each direction, it must also mean that it looks the same regardless of where we stand. There is one exception of this rule: if the scenario is such that the matter of the Universe were distributed evenly in shells from some center. If that is the case then the only place the Universe would look isotropic is from the center. So either we are placed exactly in the center of the Universe, otherwise the Universe must be *homogeneous*. Thus as far we can see, space can be approximated to be uniformly filled with matter. These principles were first merely suggestions for the basis of cosmology, but has later been confirmed by measurements of the CMB and large scale structures to be true on an average level.

In the visible universe there are about 10^{11} galaxies, each galaxy containing about 10^{11} stars. All together, that is 10^{22} stars in our visible universe. Galaxies are neutral in charge, but they interact with each other through gravity. So the seemingly only contributing force for the system is the gravitational force. But if the Universe is isotropic and homogeneous, there would be an equal amount of mass on every side of every galaxy, hence the galaxy would not move and the Universe would be static! This was the belief for a very long time, and even though results from evaluating dynamics told otherwise, physicists went through great lengths to realize theories with a static Universe. Not before galaxies were observed to move away from us in the 1920s were scientists actually convinced that the Universe was not static at all. But how can the expansion be possible if every galaxy is pulled equally much from every direction?

First let us introduce a set of coordinates suitably for an expanding universe. Consider a grid made out of uniformly distributed galaxies, such that every point in the coordinate system, e.g (x_1, y_1, z_1) or (x_3, y_5, z_2), corresponds to the position of a galaxy. If every galaxy was moving in entirely random directions, this system would not work very well, but that is not the case. We observe that galaxies move very coherently like they are frozen in the grid, and that the grid itself is expanding. At one point in time, lets say the distance r between two galaxies in the grid is Δx. At any point in time, we can write the distance between the same two galaxies as

$$r = a(t)\Delta x \tag{1}$$

where $a(t)$ is the *scale factor* of the grid. This factor is what governs the size of the grid. If it is constant the Universe is static, if it changes in time the Universe either expands or contracts. If $a(t) \to 0$ the galaxies get crammed on top of each other and the density goes to infinity. If $a(t) \to \infty$ the density goes to zero. The velocity v with which the galaxies move away from each other is $v = dr/dt = \dot{a}\Delta x$. The ratio of the velocity to the distance of the galaxies is

$$\frac{v}{r} = \frac{\dot{a}\Delta x}{a(t)\Delta x} = \frac{\dot{a}}{a} = H \tag{2}$$

where H is the *Hubble constant*. Notice that the distance Δx cancel, hence it does not matter which galaxies we are considering. The relation can be rewritten into the *Hubble law*: the velocity between any galaxy in the Universe is given by the Hubble constant times the distance between the two galaxies,

$$\Rightarrow v = Hr. \tag{3}$$

Newton found that the gravitational force from a source acting on a object is

$$F = -\frac{mMG}{r^2} \tag{4}$$

where m is the mass of the object, M is the mass of the source and r is the distance between them. Newtons second law states that $F = mA$, where A is the acceleration of the object. We can rewrite the law of gravity to find the acceleration of the object,

$$A = -\frac{MG}{r^2}. \tag{5}$$

The acceleration A is given by taking the derivative of the velocity with time, $A = \ddot{a}\Delta x$. However instead of restricting ourselves to the x-axis, lets generalize the position to any coordinate in the grid, $\Delta x \rightarrow R = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. Exchanging this into the equation together with the distance $r = a(t)R$ we found above gives

$$\frac{\ddot{a}}{a} = -\frac{MG}{a^3 R^3}. \tag{6}$$

Newton postulated a theorem which says that in a system of multiple bodies the gravitational force acting on one object is due to the combined mass of all other bodies within a sphere drawn by the object around the original source, and all the masses of all these bodies can be estimated to lie at the very center of this sphere.

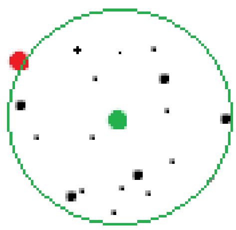

Figure 1: Newton's theorem: The gravitational force acting on the red galaxy can be estimated to be due to the green source which contains the mass of all other galaxies within the circle. The condition is that the distribution of matter must be isotropic from the center of the circle.

Hence equation (4) is still valid, where M now is the combined mass of all the bodies within the sphere. The volume of the sphere is $V = 4\pi a^3 R^3/3$. Plugging this into equation (6) gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi MG}{3V} = -\frac{4\pi G}{3}\rho \tag{7}$$

where $\rho = M/V$ is the density of the source. Note that ρ doesn't really depend on R any more. It doesn't matter where we put our origin. If we expand the sphere to entail the entire Universe, this means that the density of the Universe depend only on the scale factor a. Moreover if the Universe is static, a must be constant, and hence the time derivative of a must be zero. By equation (7) it can only be zero if ρ is zero, which since we exist we know is not true. Thus the Universe is not static.

A cell in the grid with volume $V = a^3\Delta x\Delta y\Delta z$ has the mass $M = \nu\Delta x\Delta y\Delta z$ where ν is a constant since the cell is constant. Hence the density of an arbitrary cell in the grid is $\rho = M/V = \nu/a^3$. Inserting this into equation (7) we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\nu}{3a^3}. \tag{8}$$

This is an equation of motion for the scale factor. Although we have derived it with Newton mechanics, it was actually first discovered in the context of General Relativity by the russian physicist Alexander Friedmann. The fact that the double derivative of the scale factor is negative, where as the Hubble constant is positive means that the Universe is still expanding, but with decreasing velocity. However we now know this is not true by the discovery of the late time acceleration about 20 years ago.

Friedmann uncovered another useful equation of motion for the scale factor, which is more frequently referred to. It is derived from the key concept of conservation of energy. Every physical system has potential energy and kinetic energy. The total energy of the system is the sum of these, and by energy conservation this sum is constant. This means that no energy can suddenly and dramatically pop up or disappear out of the blue. Consider a scenario where we have a particle of mass m, ejected away from a larger spherically symmetric, static object of mass M as shown in the figure (2).

The kinetic energy of the ejected particle is $(1/2)mv^2$, where v is the velocity of the particle. The potential energy it has due to the gravitational field of the larger object is $-mMG/r$, where G is Newton's gravitational constant and r is the distance between the two objects. The total energy is thus

$$E = \frac{1}{2}mv^2 - \frac{mMG}{r}. \tag{9}$$

Energy conservation states that this is time independent. Since it is constant the only real difference that matter is if it is positive, negative or zero. If the energy

Figure 2: The two body system. The particle with mass m has initial velocity pointing in the direction of the arrow away from the more heavy object of mass M. The only active force in the system is the gravitational force between the two bodies.

is positive the kinetic energy is larger than the potential energy. Hence if the velocity of the particle points outward as shown in the figure, it will continue to travel outwards, first with decreasing velocity but eventually, as it escapes the gravitational pull, it will fly away with constant speed. If the energy is negative the particle will eventually turn around, as its kinetic energy is not great enough to escape the potential well of the heavy object. The boundary between these two cases is when the velocity is such that the kinetic energy is exactly the same as the potential energy. This is the *escape velocity* and the particle will move away from the object slower and slower but the potential energy is never great enough to turn it around.

In light of Newton's theorem the equations of motion for the Universe has the same property. If the initial starting point has all of the galaxies moving outward relative to each other, according to the Hubble law but with a big enough Hubble constant, hence big enough velocity relative to each other, then the galaxies would fly off and away without any tendency of falling back. If on the other hand the initial velocity of the galaxies are too slow, the gravitational pull of the combined galaxies would force them together, and the Universe would collapse. If the velocity is equal to the escape velocity of the system, then the galaxies would move away slower and slower, but never turning back.

The energy of the Universe can thus also be written as

$$v^2 - 2\frac{MG}{r} = k \tag{10}$$

where k is the constant value of energy in the system. The large mass M now contains all the mass within the circle in figure 1. It is the same equation as (9) only multiplied by 2 and divided by m, where on the right hand side both the factors are absorbed into the constant k. Recall that the distance between two galaxies is $r = a(t)\Delta x$ and the relative velocity is $v = \dot{r} = \dot{a}\Delta x$,

$$\Rightarrow \dot{a}^2\Delta x^2 - \frac{2MG}{a\Delta x} = \dot{a}^2\Delta x^2 - \frac{8\pi G}{3}a^2\Delta x^2\rho = k \tag{11}$$

where we have used that $M = \rho V = \rho(4/3)\pi a^3\Delta x^3$. Notice that in the equation above all the terms are equal everywhere in the Universe except for Δx. Both terms on the left hand side are proportional to Δx^2, thus for the equation to make sense the right hand side also must be proportional to Δx^2. Remember that even though we said the right hand side is constant it is constant in time only. In other words for each Δx it is constant in time. Physically it means that the energy of galaxies further apart from each other is larger than those closer together. Intuitively it makes sense because they are moving faster. The equation then reads

$$\dot{a}^2\Delta x^2 - \frac{8\pi G}{3}a^2\Delta x^2\rho = k\Delta x^2. \tag{12}$$

Now we can divide away the Δx^2 terms, not only making the right hand side a constant with respect to space and time, but also fabricating the whole equation to be independent of which galaxies we look at. Then we can rewrite the equation as

$$\Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{13}$$

Here, aside from dividing by a^2, we choose a negative sign in front of the constant k simply to achieve the historically correct form of the equation. It is called the *Friedmann equation*. The fact that it is a negative sign doesn't mean anything, it is merely the authentic terminology. The constant k itself can be positive, negative or zero. Either of these cases apply to the scenarios we discussed above for the total energy of the system. If it is positive the universe would expand first decreasingly, when the density is sufficiently large, then with more uniform velocity as the density decrease. If it is negative the universe will eventually collapse. If it is zero the universe will continue to grow slower and slower, but never recede. The whole term (k/a^2) can actually be interpreted as the *spatial curvature* in any time slice of the universe. It decides its geometrical shape. Typically the values of k are normalized to either $+1$, where the universe is a closed sphere, -1 where the universe takes the shape of an open hyperbolic, and 0 where the universe is flat.

As implied above the different configurations of the equation leads to different solutions. As an example, let us solve the equation when we set $k = 0$. If we replace ρ with the density we found earlier, $\rho = \nu/a^3$, we find that

$$\frac{\dot{a}^2}{a^2} = H^2 \propto \frac{1}{a^3}. \tag{14}$$

An expanding universe with zero energy will keep expanding, but slower and slower, the expansion velocity going to zero towards infinity. This is a separable differential equation and by integration we obtain a solution on the form $a \sim \lambda t^p$, where λ is a constant. If a was linear in t it would grow in proportion with time, which by the equation above is not true. Therefore we would expect t to be of some power p. Plugging in our solution into the equation above we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\lambda p t^{p-1}}{\lambda t^p}\right)^2 = \left(\frac{p}{t}\right)^2 \tag{15}$$

$$\frac{1}{a^3} = \frac{1}{\lambda^3 t^{3p}} \tag{16}$$

$$\Rightarrow \left(\frac{p}{t}\right)^2 = \frac{1}{\lambda^3 t^{3p}}. \tag{17}$$

In order to match the two sides we would like to have $t^2 = t^{3p}$, which gives $p = 2/3$ and $p^2 = 1/\lambda^3$. This tells us that the Universe expands as

$$a \propto t^{2/3}. \tag{18}$$

Newton never did these calculations even though he was right on the threshold to predicting an expanding universe. Although the Universe we know today is expanding increasingly fast, it doesn't mean that these equations are wrong.

They are exactly valid in a *flat* space. The difference between the calculations of Newton and Einstein is that while Newton explained forces, Einstein treated gravity as curvature of spacetime. Space may indeed be curved on an average level, but in the close proximity of our galaxy it is approximately flat. Hence the equations we have derived is legitimate and do agree with the Einstein equations in small fractions of the Universe below a *non-relativistic* limit. A "small" fraction in this sense may be billions of light years. The point is that to explain the whole Universe as a system, we would need the theory of General Relativity.

1.2 General Relativity

1.2.1 Basics

The motion of free particles can in general be described by the shortest path between two points. In other words we say that free particles follow geodesics. The shortest path between two points is analogous to the stationary distance S between them. To calculate the geodesic, we minimize differential elements dS along the curve and add them all up,

$$S = \int dS \tag{19}$$

We can think of this as the action of a particle moving from one point to the other. A more formal definition of the action is the integral of the Lagrangian $\mathcal{L}$,

$$S = \int d^4x \mathcal{L} \tag{20}$$

The equation contains all dynamics of the system at all times. The principle of least action states that the path taken is the one where the action is stationary to the first order. Thus a small variation of the action yields

$$\delta S = 0. \tag{21}$$

From this principle the equation of motion can be derived, which is what we are ultimately interested in. This is the basic cornerstone of mechanics and usually a starting point for most theories of gravity. Indeed, General Relativity treats the effects of gravity as a manifestation of the curvature of spacetime itself. Particles moving through space takes the shortest path from point to point. Thus geodesics are an important tool to describe the environment of the universe.

Another important key concept in General Relativity is the *metric tensor* $g_{\mu\nu}$ which captures all geometry and casual structure of spacetime. It is constructed from the distribution of matter, and in return creates the landscape in which geodesics are drawn upon. Naturally it comes from the *metric*. Say we have, in a homogeneous and isotropic universe, the four dimensional metric

$$ds^2 = -dt^2 + a(t)[dx^2 + dy^2 + dz^2] \tag{22}$$

then the metric tensor takes the form

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t) & 0 \\ 0 & 0 & 0 & a(t) \end{bmatrix} \tag{23}$$

The metric tensor is also a very handy mathematical tool, as it has the ability to raise and lower indices. For example consider the Faraday tensor $F_{\mu\nu}$. From this we can construct different tensors,

$$F_{\mu\nu} = g_{\mu\alpha} F^{\alpha}_{\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}. \tag{24}$$

The only rule is that the sum of indices has to be the same on both sides of the quality sign, and they have to be in the right place (upper or lower). For example we can write $F^2 = F_{\mu\nu} F^{\mu\nu}$, where the equal upper and lower indices cancel. Thus with this in mind we see that the remaining indices on both sides of the equation is lower μ and ν. Actually an equal upper and lower index means that the indices are summed over,

$$T^{ab}_{bc} = \sum_b T^{ab}_{bc} = T^{a1}_{1c} + T^{a2}_{2c} + \cdots + T^{an}_{nc} = U^a_c. \tag{25}$$

The tensor T^{ab}_{bc} is *contracted* to form a new tensor U^a_c. Even easier, consider an ordinary three dimensional vector $\vec{V}$, which is a sum over its vector components,

$$\vec{V} = V^1 \mathbf{e}_1 + V^2 \mathbf{e}_2 + V^3 \mathbf{e}_3 = V^m \mathbf{e}_m \tag{26}$$

This is all part of the notation of calculations in General Relativity. If we stay within these rules, we can trust that our calculations hold. An upper and a lower index has not the same meaning however. They are not necessarily the same because space is not necessarily flat. We say that an upper index is *contravariant* while a lower index is *covariant*. Physically they have different interpretations. Contravariant components can be thought of as the same as ordinary indices as in the example (26). It is the indices we use to construct a vector out of the basis vectors. Covariant components are the dot product of the vector with the basis vector, $V_n = \vec{V} \cdot \mathbf{e}_n$. They are related through the metric tensor,

$$V_n = \vec{V} \cdot \mathbf{e}_n = V^m \mathbf{e}_m \cdot \mathbf{e}_n = V^m g_{mn}. \tag{27}$$

In Euclidean coordinates both components are numerically equal. The key difference between them is how they change under coordinate transformation.

Already we have mentioned the concept of tensors, but has yet to explain what it actually means. A tensor is a class of objects with well defined and specific transformation properties under the change of coordinates. It can be of several orders, depending on what kind of object it is. The zeroth-order tensor is a scalar, the first-order a vector, the second-order a matrix and so on. What makes the tensor special is its property that it has the same meaning in every frame of reference. It is independent of a particular choice of coordinate system. Therefore tensor equations are of crucial importance in the formulation of General Relativity.

1.2.2 The Einstein field equations

In 1915 Albert Einstein published the robust theory of General Relativity which describes the fundamental interaction of gravitation as a result of spacetime being curved by mass and energy. Even today, a hundred years later, the theory still holds as the framework of theoretical cosmology and gravitation in modern physics. To quote one of Einstein's late collaborators, John Archibald Wheeler, "Spacetime tells matter how to move; matter tells spacetime how to curve." There is a two way street. In the context of Newtonian mechanics we would say: masses affect the gravitational field and the gravitational field affect how the masses move,

$$F = -m\nabla\Phi(x) = ma. \tag{28}$$

In order to intuitively understand the Einstein equations, let us use logic and find the analogue case of Newtonian mechanics. Everywhere in space, due to whatever reason, there is a gravitational field $\Phi(x)$. It varies from place to place. The gradient of the gravitational field gives the force acting on a particle with mass m, or the acceleration of the particle, $a = -\nabla\Phi(x)$. On the other hand the distribution of masses are the things that are causing the field to be there. The equation that gives this relation is Poisson's equation,

$$\nabla^2\Phi = 4\pi G\rho \tag{29}$$

where G is Newtons constant and ρ is the density of mass. We can solve the equation for the following special case: Consider a spherically symmetric object of mass M. On the exterior of the object, in other words on the outside of where there is any mass, the solution of the equation is

$$\Phi = -\frac{MG}{r}. \tag{30}$$

This solution is valid everywhere except inside the object. If we plug this into (28) we get Newtons gravitational law.

From a relativistic point of view we have that $E = mc^2$, which if we set $c = 1$ says that mass and energy are the same. Thus the density ρ in equation (29) turns into the energy density of the system. In turn, this energy density is part of a more complex tensor which other components has other meanings. The tensor in question is the *energy momentum tensor* $T_{\mu\nu}$. It is similar to the four-momentum vector $P^\mu = (E, P^m)$, only as a second order tensor it contains more information. It describes the density and flow of energy and momentum in spacetime. Along its diagonal we find the energy density ρ and the pressure p_i. The remaining temporal components, T^{0n} and T^{m0}, contains the flow of energy and density of momentum, while the remaining spatial components T^{mn} contain the flow of momentum in each direction. This tensor is the source of the gravitational field in Einstein's field equations and is the analogue to the mass density of Newtonian physics.

The energy momentum tensor has the property that it is symmetric, thus the indices can change place without consequence. For example $T^{m0} = T^{0m}$, meaning that the density of momentum is equal to the flow of energy. Also, since energy and momentum are conserved quantities, the energy-momentum tensor obey the continuity equation:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{31}$$

Now if we look at equation (29) we see that it contains the energy density ρ, or as we now know T^{00}. However, in order to achieve the fulfilled theory we must generalize to $T^{\mu\nu}$. Thus since the right hand side of the equation is a tensor of second order, the left hand side must be as well. It must be symmetric, since $T^{\mu\nu}$ is symmetric, otherwise the equation doesn't make sense. Since General Relativity treats gravity as curvature of spacetime, the left hand side should also involve derivatives of the metric. Actually, to best fit the equation (29) we would like to involve two derivatives of the metric. For now, lets call the tensor $G^{\mu\nu}$. Thus

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}. \tag{32}$$

Lets explore the possibilities of what $G^{\mu\nu}$ could be. In curved space there are some features which makes calculations slightly more complicated compared to flat space. To demonstrate this, consider a scenario where we want to differentiate a vector $\vec{V}$ in Gaussian coordinates and transform it to other coordinates. In flat space the vector will stay the same. In curved space however it is not necessarily the same. One would have to take into account that the coordinates might have changed from one point to the next. In order to do that we must use the *covariant derivative*,

$$\nabla_j V_i = \frac{\partial V_i}{\partial x_j} - \Gamma^t_{ij} V_t. \tag{33}$$

A way to find out if a space is flat or not is to search for diagnostic quantities which are built out of the metric and its derivatives. If they are zero everywhere throughout the space, the space is flat. If they are non zero any place, space is curved. The diagnostic quantity we are looking for is the curvature tensor. It is built from derivatives of the metric tensor through a connection $\Gamma^\mu_{\nu\lambda}$ called the *Christoffel symbol*, defined as

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\alpha} (\partial_\lambda g_{\alpha\nu} + \partial_\nu g_{\alpha\lambda} - \partial_\alpha g_{\nu\lambda}) \tag{34}$$

where the subscript of the derivatives has the meaning $\partial_\lambda = \partial/\partial\lambda$. The Riemann curvature tensor $R^\rho_{\sigma\mu\nu}$ was introduced by the mathematician Bernhard Riemann in order to express the curvature of Riemann manifolds. It is the quantity that tells us if space is curved. Constructed by the Christoffel symbols, it can be written as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}. \tag{35}$$

Notice that the tensor contains the derivatives of the Christoffel symbols, which in turn contains first derivatives of the metric. Hence the Riemann tensor contains derivatives of the metric tensor to second order, which is what we are looking for. The third and fourth terms are squared Christoffel symbols, thus contains squared first derivatives. Because of symmetries of the Christoffel symbols, by contraction there is only two possible tensors we can build from two derivatives acting on the metric. The first is called the Ricci tensor $R_{\mu\nu}$. Since it is contracted it has less components than the Riemann tensor and hence less information, meaning for example it can be zero while the Riemann tensor is not necessarily zero. However it has the nice property that it is symmetric in its indices, such that $R_{\mu\nu} = R_{\nu\mu}$. Of course, we are allowed to raise and lower it's indices, thus there are several Ricci tensors, R^{μ}_{ν}, $R^{\mu\nu}$ and so on. The only other possible tensor is found by contracting again, which gives the Ricci scalar $R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}$. Both these tensors seems like they would be good candidates for the tensor $G^{\mu\nu}$, since they have two indices and contains two derivatives of the metric (we can multiply the Ricci scalar with the metric $g^{\mu\nu}$ so that we obtain a tensor with two indices on the left hand side). There is, however, one more thing we know. Since the energy-momentum tensor obey the continuity equation, the left hand side tensor $G^{\mu\nu}$ must obey it as well,

$$\Rightarrow \nabla_\mu G^{\mu\nu} = 0. \tag{36}$$

For the Ricci scalar we have,

$$\nabla_\mu (g^{\mu\nu}R) = \nabla_\mu g^{\mu\nu} R + g^{\mu\nu}\nabla_\mu R \tag{37}$$

The covariant derivative of the metric is zero. The reason is that covariant derivatives are by definition tensors which in the special good frame of reference are equal to ordinary derivatives, where the special good frame of reference being the frame of reference in which the ordinary derivatives of the metric is zero. The Christoffel symbols are precisely constructed as the only symmetric connection that gives a vanishing derivative of the metric. In other words when $g^{\mu\nu}$ has constant components, e.g. in a Minkowski background, the covariant derivative is just the ordinary derivative. The covariant derivative of a scalar is also just an ordinary derivative since scalars do not have any indices, hence no direction. Therefore,

$$\nabla_\mu (g^{\mu\nu} R) = g^{\mu\nu} \partial_\mu R \tag{38}$$

But we know that in general the derivative of R is not zero, that would mean there is no curvature anywhere. So this equation does not satisfy the continuity equation alone. For the Ricci tensor we get that

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} g^{\mu\nu} \partial_\mu R. \tag{39}$$

Neither is this zero, but it happen to be exactly one half of that of the Ricci scalar. Combined they obey the continuity equation,

$$\nabla_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0. \tag{40}$$

Thus finally we can write that the equations of motion, named after Albert Einstein, have to be

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu} \tag{41}$$

where $G^{\mu\nu}$ is in fact called the Einstein tensor. We deduced this equation by following the same principles behind equation (29), but there is an important difference between the two. The source of the curvature does not solely depend on the energy density ρ, but all the components which are incorporated into $T^{\mu\nu}$. However the momentum density, flow of momentum and even the flow of energy are typically much smaller quantities than the energy density. The reason is if we insert the speed of light into the equations, we see that the energy is multiplied by c^2, while momentum is multiplied only by the velocity v, which is usually much smaller. So in the non-relativistic limit, energy density is by far the largest contribution to curvature. Outside the non-relativistic limit, the other components do contribute to gravity. Newtons equations works remarkably well in the solar system, so as a test of the Einstein equations we would like the matching case to give the same result, to some appropriate non-relativistic limit. The matching case would be when $\mu = \nu = 0$ such that $T^{00} = \rho$. It turns out by numerical approximations that they are very well matched.

The more elegant way to derive the Einstein equations is by mathematically varying an action S to obtain the equations of motion. To construct a theory of gravity within the framework of General Relativity it is essential to involve a curvature factor in some way. David Hilbert knew that the simplest contribution of curvature would be the Ricci scalar, and so he proposed this as the Lagrangian. In 1917, however, Einstein included the cosmological constant as an additional term in order to realize a static Universe. The Einstein-Hilbert action takes the form,

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}(R - 2\Lambda) + S_m \tag{42}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$. This is the metric volume element, and it is present because the integral is over curved space rather than flat space. The factor $\kappa^2 = 8\pi G$ where G is the gravitational constant. Taking the variation of the action with respect to $g^{\mu\nu}$ yields

$$\delta S = \frac{1}{2\kappa} \int d^4x \delta[\sqrt{-g}((R - 2\Lambda)] + \delta S_m \tag{43}$$

$$= \frac{1}{2\kappa} \int d^4x [\delta\sqrt{-g}(g^{\mu\nu} R_{\mu\nu} - 2\Lambda) + \sqrt{-g}\delta g^{\mu\nu} R_{\mu\nu} \tag{44}$$

$$+ \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}] + \delta S_m. \tag{45}$$

We can write the determinant g as $g_{\mu\nu}\mathcal{M}^{(\mu\nu)}$, where $\mathcal{M}^{(\mu\nu)}$ is the determinant of the cofactor matrix which itself is not dependent on $g_{\mu\nu}$. Replacing $\mathcal{M}^{(\mu\nu)} = gg^{\mu\nu}$ we get the relation

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \tag{46}$$

For the variation of the Ricci tensor, we have that

$$\delta R_{\mu\nu} = \nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\alpha_{\mu\alpha}) \tag{47}$$

$$\Rightarrow g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu_{\mu\nu}). \tag{48}$$

Inside the integral, this term vanishes by Gauss's theorem,

$$\int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{-g} \nabla_\alpha (g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu_{\mu\nu}) = 0. \tag{49}$$

Thus we can write the variation of the action as

$$\delta S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu} + \delta S_m. \tag{50}$$

From the variation of the matter action S_m we get the energy momentum tensor $T_{\mu\nu}$,

$$\delta S_m = -\frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \tag{51}$$

and finally we end up with

$$\delta S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi G T_{\mu\nu}\right)\delta g^{\mu\nu}. \tag{52}$$

The principle of least action states that $\delta S = 0$. An integral which always vanishes can only be achieved by the vanishing of the integrand itself. Hence we obtain the equations of motion, also known as the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{53}$$

When Einstein first introduced the theory of General Relativity it was common belief that the Universe neither expanded nor contracted. To his despair, the equations of his theory would not recreate such a static Universe. In order to make the equations comply with the current observations, Einstein reluctantly introduced the cosmological constant Λ, an additional term which counteracts the effects of gravity. He was not fond of the idea, as he felt it soiled the beauty of the theory, however it was necessary to achieve a static Universe. About ten years later the combined research of Slipher and Hubble proved that other galaxies actually moved away from us, and hence that the Universe was in fact not static after all. In the aftermath of this observation, Einstein abandoned the cosmological constant. Nevertheless, at the end of the century, observations pointed to a Universe which not only expands, but with increasing acceleration. After many years trying to figure out why Λ should be zero, this was a totally surprising discovery. The cosmological constant was given new perspective, it's counteracting gravity effects accounting for the late-time acceleration. Turns out that the introduction of Λ was not such a blunder after all as it became the natural candidate for what is called dark energy. It leaves the question however, of what the cosmological constant actually is.

1.3 The cosmological constant

In a conversation sometime in the 30's, Pauli and Dirac were talking about the infinities of quantum field theory. Dirac said "The zero point vacuum energy comes out infinite, and therefore it cannot mean anything." Pauli replied "Just because something is infinite does not mean it is zero".

The history of the problem with the cosmological constant is long and rich and this discussion between Pauli and Dirac illustrates its essence. Recall the Einstein-Hilbert action (42). Here the cosmological constant is merely another parameter to the Lagrangian. In this form, what it means physically is not clear yet, there is no fundamental way to decide its value and the only thing that can be done is to use observations to constrain it. But we know it has the dimension of inverse length squared, and some nice properties such as compatibility with general covariance and with a conserved energy momentum tensor. Yet since we are working within the framework of low energy field theory, which will be explained in a moment, we are compelled to include it into the equations of our gravitational theory. In fact, since the cosmological constant is a constant parameter throughout the whole Universe, it is natural to think of the cosmological constant as the energy from vacuum. The fact that the vacuum energy gravitates comes from assuming that the equivalence principle applies to the zero point fluctuations. In General Relativity all forms of energy induce some sort of gravitational attraction, and after all, the zero point fluctuations are just another kind of energy. In this section we will pursue a deeper understanding of the nature of the cosmological constant and the luggage that comes with it.

From (42) we derived the Einstein equations of motion,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{54}$$

The fact that the cosmological constant is present means that the energy momentum tensor $T_{\mu\nu}$ no longer only represent ordinary matter, but also a contribution from vacuum energy as well. The source of this energy is two folded. In the classical regime we can think of the vacuum energy as the ground state of the vacuum energy field, meaning that the field sits at the minimum of its potential. Secondly, quantum mechanics claims that there will be fluctuations about this ground state. Thus the energy momentum tensor is supplemented by a contribution from the vacuum state. The expected value of this contribution, $\langle T_{\mu\nu}\rangle$, can be found by the following arguments.

Since vacuum is not empty it must have an energy density ρ_{vac}. In flat Minkowski space-time the only invariant tensor is the metric $\eta_{\mu\nu}$, hence by tensor principles it should be proportional to the vacuum energy momentum tensor, $\langle T_{\mu\nu}\rangle \propto \eta_{\mu\nu}$. Also we claim that the energy momentum tensor must be conserved, $\nabla^\mu T_{\mu\nu} = 0$, such that ρ_{vac} must be constant. Thus in curved coordinates, exchanging $\eta_{\mu\nu}$ for $g_{\mu\nu}$, the expected value is

$$\langle T_{\mu\nu}\rangle = -\rho_{vac} g_{\mu\nu}. \tag{55}$$

Inserting this new energy momentum tensor, the Einstein equations now reads

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T^{matter}_{\mu\nu} + 8\pi G\langle T_{\mu\nu}\rangle. \tag{56}$$

Here the first term on the right side of the equation represent the energy and momentum from ordinary matter while the latter term is that of the vacuum energy. We can rewrite this again by defining an effective cosmological constant as a sum of the bare cosmological constant parameter and the vacuum density, $\Lambda_{eff} g_{\mu\nu} = \Lambda g_{\mu\nu} + 8\pi G\langle T_{\mu\nu}\rangle = \Lambda g_{\mu\nu} + 8\pi G \rho_{vac} g_{\mu\nu}$. Hence we arrive at

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda_{eff} g_{\mu\nu} = 8\pi G T^{matter}_{\mu\nu}. \tag{57}$$

This effective cosmological constant is what we can measure from observations. However, here is where the problems arise. The supplemented vacuum energy is made out of various contributions which are all huge compared to the observed Λ_{eff}.

1.3.1 The cosmological constant problem

The classical contribution to the vacuum energy comes from the value of the potential minimum which will affect the cosmological constant. However it is possible to construct a situation where this minimum is zero. Consider a scalar field ϕ which at its potential minimum gives the vacuum contribution

$$\langle T_{\mu\nu}\rangle = -V(\phi_{min}) g_{\mu\nu}. \tag{58}$$

with $V(\phi_{min}) = \rho_{vac}$. The potential of the field is given by

$$V(\phi) = V_0 + \frac{\lambda}{4}(\phi^2 - v^2)^2. \tag{59}$$

Here V_0 is the classical offset, λ is the self-interacting coupling constant and v is the value of the field at its minimum. Consider a situation where the scalar field ϕ is interacting with another scalar field ψ such that

$$V(\phi, \psi) = V(\phi) + \frac{\hat{g}}{2}\phi^2\psi^2 \tag{60}$$

where $\hat{g}$ is a dimensionless coupling constant between the two fields. Let us examine the dynamics of a phase transition.

At finite temperatures, quantum field theory allows us to apply usual statistical measures [20]. Actually a given process is made out of contributions from the many particles involved, but we can simplify by taking thermodynamic averages. Then we are left with only a few remaining parameters, like the temperature. So assuming ψ is in thermal equilibrium, we can write ψ^2 as the averaged value $\langle\psi^2\rangle_T$ in a thermal state T. On dimensional grounds we expect that $\langle\psi^2\rangle_T \propto T^2$. Redefining the coupling constant $\hat{g}$ to account for the proportionality constant between the thermal average and the temperature allows us to exchange ψ^2 with T^2 in equation (60). This gives the effective potential

$$V_{eff}(\phi) = V_0 + \frac{\lambda}{4}(\phi^2 - v^2)^2 + \frac{\hat{g}}{2}\phi^2 T^2. \tag{61}$$

Defining $T_{crit} = v\sqrt{\lambda/\hat{g}}$ as the temperature where the phase transition takes place, we can rewrite the potential above as

$$V_{eff}(\phi) = V_0 + \frac{\lambda v^4}{4} + \frac{\lambda v^2}{2}\left(\frac{T}{T_{crit}} - 1\right)\phi^2 + \frac{\lambda}{4}\phi^4. \tag{62}$$

We obtain the effective mass at the origin of the field by evaluating $m^2_{eff} = V_{,\phi\phi}$ for $\phi = 0$,

$$\Rightarrow m^2_{eff} = \lambda v^2\left(\frac{T}{T_{crit}} - 1\right) \tag{63}$$

Before the phase transition, when $T > T_{crit}$, the effective mass is positive. In this situation the minimum of the potential is located at $\phi = 0$, and from equation (59) we find that the vacuum energy is $V(\phi_{min}) = V_0 + \lambda v^2/4$. In order to neglect the contribution to the cosmological constant we must require $V_0 = -\lambda v^2/4$. After the phase transition, when $T < T_{crit}$, the effective mass is negative. Here the minimum is located at $\phi = v$ and the vacuum energy takes the value $V(\phi_{min}) = V_0$.

Thus for ρ_{vac} to be zero, V_0 must be zero. This is the root of the problem: we can choose ρ_{vac} to vanish before or after the phase transition, but we can not have both, see figure (3). Any phase transition of ϕ is likely to have happened in the early Universe, hence the latter option would be preferred since observational measurements are limited at that time. However the solution is not very satisfactory. For instance, estimating the situation for the electroweak transition shows that the vacuum energy was huge prior to the phase transition [15].

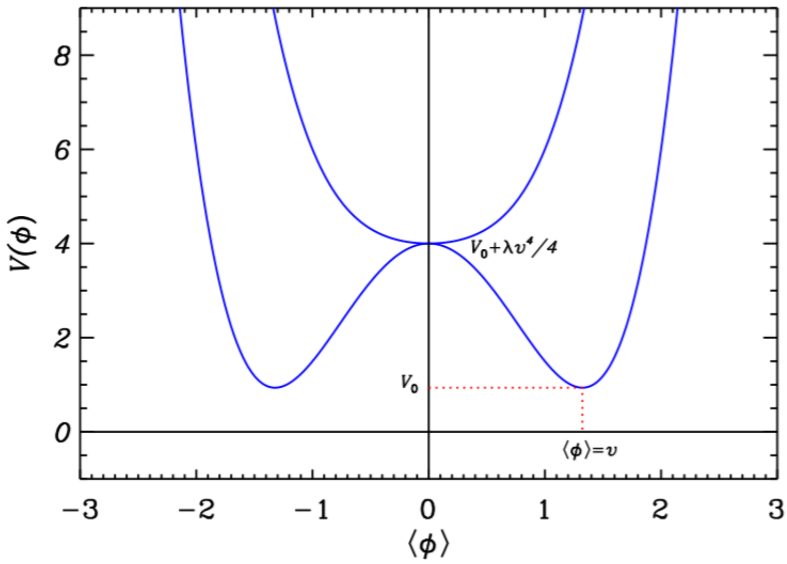

Figure 3: The classical cosmological constant problem. The figure shows the potential $V(\phi)$ before and after the phase transition. Before, the minimum is located at $\phi = 0$ while after the transition it is located at $\phi = v$. The vacuum energy cannot be zero in both situations. Reproduced from [15].

The second aspect of the cosmological constant problem comes from quantum mechanics. Indeed, even if we tune the potential minimum to be zero, ρ_{vac} receives contributions from zero point oscillations from any free quantum field present in the Universe. However in quantum physics when we integrate over all possible contributions we find that the integrals diverge. Thus the additional energy that comes from these self-interactions are infinite. Usually when infinities appear in quantum mechanics they are accepted since the true value of the quantity can never be observed, only how it changes. Hence we can work around the infinities by renormalization, which by altering the property of the quantity takes into account the possible effects of its self-interaction, fabricating finite expressions.

High precision experiments have proven that by renormalization, theory and observations coincide remarkably well. One exception however, is the cosmological constant. The zero point oscillations are indeed renormalizable, yet the value we derive from finite expressions do not fit measurements by a long shot. It seems that when gravity comes into play the renormalization scheme is inadequate.

The cosmological constant problem lies at the boundary between different branches of physics. Observational evidence of the late time acceleration could indicate that vacuum energy is able to curve space-time. As the effects of the cosmological constant is visible only at cosmological scales, this would be the natural place to explore in order to gain further insight.

1.4 Effective field theory

In quantum field theory, all kinds of virtual processes at arbitrarily high energies can contribute to what we observe at low energies [3]. However, there are ways in which we can avoid these complications. Scalar fields in theoretical cosmology are often interpreted as classical approximations to some quantum field. These approximations are possible due to an *effective field theory.* In this robust approach the result of these quantum mechanical processes fades away by averaging over the behaviour of energy processes at lower scales. Thus it allows us to create simplified models that can be trusted up to some *UV-cutoff scale.* At higher energies new physics will kick in and effective field theory ceases to be predictive. Beyond this scale the theory must be completed by some fundamental theory called a *UV-completion* which we implicitly assume exists. We can use this to suppress possible high order terms in the Lagrangian, thus leave them out and dramatically simplify our theory to a given accuracy. Lately it has become apparent that extensions to the ΛCDM model is best explored in the robust scene of effective field theory. However a number of challenges such as ghosts, gradient instabilities or superluminal propagation around certain backgrounds may appear. Therefore it is of importance to take care in its usage.

Although some of the models we will discuss here will need further fine tuning of their solutions since they are beyond their regime of validity, we underline that this is an important issue that should be taken into consideration when studying cosmological theories.

2 Dark energy models

Since the discovery of the late-time acceleration various theories designed to explain this new phenomenon have been developed. In this section we will explore some of the alternative models for dynamical dark energy, with the purpose of attaining an insight on how they would affect the evolution of the Universe. Aside from the cosmic expansion, most theories also seek to find explanations of why the acceleration of the Universe happened exactly when it did. The present value of the energy density of dark energy and ordinary matter are of the same order of magnitude to each other, and the fact that they decrease at different rates as the Universe expands suggests that there is just an extremely short time in the evolution of the Universe that the two are comparable to each other. This is in fact a remarkable coincidence. If, for instance the acceleration began earlier, structures such as galaxies would never have time to form, and life as we know it would never exist. Any later and it is likely that gravity would counteract the expansion. Theoretically the main problem is that either the effective equation of state is always so that it satisfy cosmic acceleration, or the equation of state changed exactly when needed, triggered by something.

Apart from the cosmological constant, there are mainly two different branches of dark energy models. Either one modifies the content of the Universe, by introducing a new element which would represent dark energy, or one modifies the gravitational sector itself. Both branches seek to modify the cosmological dynamics in order to realize the accelerated expansion. First we will review *quintessence*, a scalar field theory with a slowly varying equation of state. After quintessence we present a related theory involving non-canonical kinetic terms called *k-essence*. Finally we examine the $f(R)$ theory, as an example on modifying gravity.

2.1 Quintessence

One of the simplest alternatives to the cosmological constant is the quintessence model. It introduces a canonical scalar field ϕ with a potential $V(\phi)$ which interacts with all other components only through standard gravity. The cosmological constant is constrained by the background fluid density ρ_M in the early Universe in order to realize the development of the present Universe. However, the equation of state of quintessence changes in time, allowing the existence of tracker fields in which the field energy density ρ_ϕ tracks the background ρ_M. Therefore, unlike with the cosmological constant, the energy density of quintessence does not need to be very small compared to that of radiation and matter in the early

Universe.

The quintessence model is described by the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + \mathcal{L}_\phi\right] + S_M \tag{64}$$

were $\kappa^2 = 8\pi G$, R is the Ricci scalar and S_M is the matter action. The Lagrangian for the scalar field is given by $\mathcal{L}_\phi$,

$$\mathcal{L}_\phi = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{65}$$

For the background we consider a perfect fluid with the energy density ρ_M, the pressure P_M and the equation of state $w_M = P_M/\rho_M$. The subscript M is used for a general perfect fluid without specifying non-relativistic matter or radiation. The fluid satisfies the continuity equation

$$\dot{\rho_M} + 3H(\rho_M + P_M) = 0. \tag{66}$$

The energy-momentum tensor of quintessence is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right]. \tag{67}$$

In flat FLRW background we can write the energy density ρ_ϕ and the pressure P_ϕ of the field as

$$\rho_\phi = -T_0^0 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{68}$$

$$P_\phi = \frac{1}{3}T_i^i = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{69}$$

which gives the equation of state

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}. \tag{70}$$

With the above statements the Friedmann equations can be written as

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M\right] \tag{71}$$

$$\dot{H} = -\frac{\kappa^2}{2}(\dot{\phi}^2 + \rho_M + P_M) \tag{72}$$

where $H = \dot{a}/a$ is the Hubble parameter. while the variation of the action (64) with respect to ϕ gives

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{73}$$

In the early eras of the Universe the energy density of the fluid dominates over the energy density of quintessence, $\rho_M >> \rho_\phi$. As mentioned above we need ρ_ϕ to track ρ_M so that dark energy density emerges at late times. As we will see later, this sort of tracking behaviour can be achieved by suitably choosing the form of the potential $V(\phi)$. In order to realize late time cosmic acceleration we require $\ddot{a} > 0$, which in turn gives the condition $w_\phi < -1/3$. In our case, this can be translated into $\dot{\phi}^2 < V(\phi)$. Hence the potential needs to be sufficiently shallow such that the field evolve slowly along the potential. Several potentials for quintessence have been proposed. They can be classified into two types: freezing models and thawing models. In the freezing models the field was rolling along the potential in the past, but the movement gradually slows down after the system enters the phase of cosmic acceleration. In thawing models the field (with mass m_ϕ) has been frozen in time by Hubble friction ($H\dot{\phi}$) until recently and then it begins to evolve once H drops below m_ϕ.

Any cosmological model can be thought of as a dynamical system consisting of a space and a set of mathematical rules which describe the evolution at any point in that space. The state of the system is described by a set of quantities or variables which are important about the system. In the presence of a scalar field ϕ and a background fluid, such as the quintessence model, it is convenient to introduce the following dimensionless variables:

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H} \tag{74}$$

$$y = \frac{\kappa V}{\sqrt{3}H} \tag{75}$$

Then the equation of motion (71) can be written as

$$\Omega_M = \frac{\kappa^2 \rho_M}{3H^2} = 1 - x^2 - y^2 \tag{76}$$

The energy fraction of dark energy is

$$\Omega_\phi = \frac{\kappa^2 \rho_\phi}{3H^2} = x^2 + y^2 \tag{77}$$

which satisfy the relation $\Omega_M + \Omega_\phi = 1$. Together with equation (72) we obtain

$$\frac{\dot{H}}{H^2} = -3x - \frac{3}{2}(1 + w_M)(1 - x^2 - y^2) \tag{78}$$

The effective equation of state is a useful parameter and defined as the total pressure divided by the total energy density, $w_{eff} = P_t/\rho_t$. In this case we have

$$w_{eff} = w_M + (1 - w_M)x^2 - (1 + w_M)y^2 \tag{79}$$

while the equation of state of dark energy (70) can be expressed as

$$w_\phi = \frac{x^2 - y^2}{x^2 + y^2}. \tag{80}$$

An important element in understanding the cosmological dynamics is to study fixed points of the system and examine them in phase space. Phase space is a space in which all possible states of a dynamical system are represented. Each state corresponds to a unique point in the phase space. We can derive the fixed points by differentiating the variables x and y with respect to the number of e-foldings $N = lna$ and setting them equal to zero, $dx/dN = dy/dN = 0$. In phase space the trajectories with respect to $x(N)$ and $y(N)$ run from unstable fixed points to stable points, coasting along saddle points. Simply put, a fixed point is stable if all the solutions of, for instance $x(N)$, starting near a point x^c stay close to it. The point is unstable if the solutions drift away. If some solutions move away while others stay close we have a saddle point [2]. It should be noted though that even without fixed points we could have flow. However fixed points are important to understanding the asymptotic behaviour of the system.

To find stability at the fixed points (x^c, y^c), we consider linear perturbations $(\delta x, \delta y)$ such that

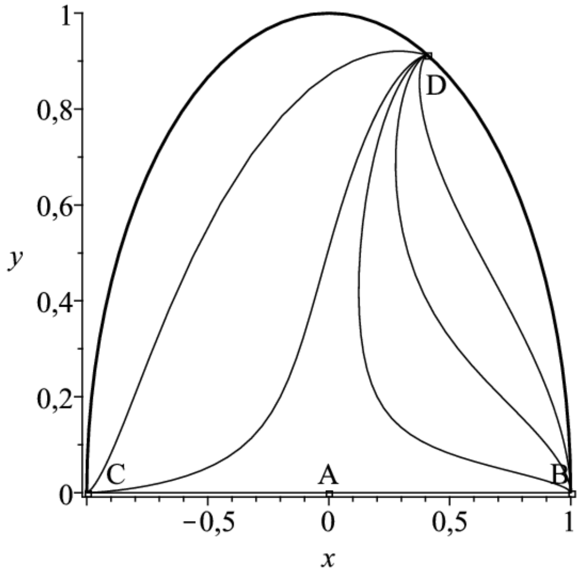

Figure 4: The figure shows the trajectories for the phase space solutions for the exponential potential (87) in the x and y plane. Here A is a saddle, while B and C are unstable points. The stable point D is the tracker solution. Figure from [12].

$$x = x^c + \delta x \tag{81}$$

$$y = y^c + \delta y. \tag{82}$$

Using these perturbed variables we linearise the equations dx/dN and dy/dN and find the first order differential equation

$$\frac{d}{dN}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \tag{83}$$

where $\mathcal{M}$ is a 2 x 2 matrix. The components of the matrix, a_{ij}, depend on x^c and y^c and its eigenvalues are given by

$$u_{1,2} = \frac{1}{2}\left[a_{11} + a_{22} \pm \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}\right]. \tag{84}$$

The eigenvalues of $\mathcal{M}$ decide if the point is stable or not. A fixed point is stable if all the real parts of the eigenvalues is negative, unstable if they are positive and a saddle point if there are both positive and negative parts.

Phase spaces can also be characterized by special trajectories that attracts other trajectories, such that eventually all trajectories converge along these special lines. These cases are called tracker solutions. In the context of cosmology and quintessence, a stable fixed point could give rise to such an attractor. To realize this tracking behaviour we would require that the slope of the potential $V(\phi)$ gradually decreases (for example as in the freezing model $V(\phi) = M^{4+n}\phi^{-n}$). In such a case, the stable point would result in late time acceleration, after radiation and matter dominated epochs which can be realized by saddle points or unstable points. The tracking solution for the potential $V(\phi) = M^{4+n}\phi^{-n}$ is characterized by

$$\Omega_\phi \simeq \frac{3(1+w_\phi)}{\lambda^2} \tag{85}$$

with the equation of state

$$w_\phi \simeq \frac{nw_M - 2}{n+2}. \tag{86}$$

The quantity $\lambda \equiv -\frac{V_{,\phi}}{\kappa V}$ describes the slope of the potential. If λ is constant, integration would result in an exponential potential,

$$V(\phi) = V_0 e^{-\kappa\lambda\phi}. \tag{87}$$

The exponential potential would correspond to the border that separates solutions where tracking behaviour occurs from where it does not [1]. Figure (4) illustrates the projected phase space evolution for the potential (87) with $\lambda = 1$ and $w_M = 0$. Here the point D is stable, which attracts solutions at the saddle point A and the unstable points B and C. The matter and radiation dominated eras are typically realized by A, B and C. The solutions eventually comes together at the point D, which would represent the latest epoch, namely the late time acceleration.

There may also exist fixed points which has the properties that the ratio Ω_ϕ/Ω_M is constant and $w_\phi = w_M$. This is the so called *scaling solutions*. Because of the simplicity that these particular points offer, scaling solutions are useful when analysing the cosmological evolution. For the exponential potential mentioned above, the existence of a scaling solution demands the condition $\lambda^2 > 3(1+w_M)$.

This is a stable fixed point. Hence, in radiation and matter dominated epochs, the solutions will approach this point rather than saddle points. This is convenient because it attracts the solutions regardless of the initial conditions of the field. Unlike the ΛCDM model, the energy density of quintessence can contribute to the total energy density in the early epochs. It is possible to constrain the density parameter Ω_ϕ from the Big Bang Nucleosynthesis and the CMB background because it directly affects the expansion rate of the universe in early epochs. Thus it can alter the ratio of neutrons and protons at freeze-out and lead to a deviation in the position of peaks in the CMB power spectrum.

Since Ω_ϕ/Ω_M is constant, the energy density of the field decreases in proportion to the background fluid. For radiation and matter eras, $\Omega_\phi = 4/\lambda^2$ and $\Omega_\phi = 3/\lambda^2$ respectively. Hence for scaling solutions to exist during these eras, we must have $\lambda^2 > 4$. However, the exponential potential $V(\phi) = V_0 e^{-\kappa\lambda\phi}$ with $\lambda^2 > 4$ is too steep to give rise to late-time acceleration. The solutions therefore has to move from the scaling regime to the dark energy dominated epoch. In order to realize this we could consider a doubled exponential potential,

$$V(\phi) = V_1 e^{-\kappa\lambda\phi} + V_2 e^{-\kappa\mu\phi}. \tag{88}$$

If for example $\lambda^2 > 4$ and $\mu < 2$, the solutions will follow the steep $V_1 e^{-\kappa\lambda\phi}$ term during the eras dominated by radiation and matter. Later the solutions will move towards the latter term when the slope μ becomes important, which would cause the late time acceleration. In fact this method can be generalized to a model where N scalar fields work together to drive cosmic acceleration. The potential is the sum of the exponential potentials of multiple fields $(\phi_1, \phi_2, ..., \phi_N)$,

$$V = \sum_{i=1}^{N} V_i e^{-\kappa\lambda_i\phi_i}. \tag{89}$$

All these scalar fields evolve to give dynamics matching a single field model with an effective slope λ_{eff} defined by

$$\frac{1}{\lambda_{eff}^2} = \sum_{i=1}^{N} \frac{1}{\lambda_i^2} \tag{90}$$

If $\lambda_{eff} < 2$ the solutions will move towards the stable accelerated attractor.

There are alterations to quintessence which involve a coupling between the scalar field and ordinary matter. Such interactions would influence the scalar field in dense regions, thus adapting different behaviour depending on the surrounding environment. On cosmological scales the interesting effect resulting from such couplings is that the matter-dominated era is replaced by a ϕ-matter-dominated epoch which gives changes to the background expansion history of the Universe. Further analysis of a coupled quintessence scenario will show that the equation for matter perturbations is different than that of the uncoupled case, and will in fact lead to a larger growth of matter structures. Constraints can thus be placed upon the coupling from observational data. As we will se later, such couplings between the scalar field ϕ and matter could lead to intriguing effects on smaller scales as well. Decoupling of baryons, for instance, ensure that constraints from local gravity experiments can be avoided.

We have seen that a canonical scalar field can drive the cosmic acceleration. From an observational point of view the main contrast between the cosmological constant and quintessence is the variation of the equation of state with time. This does pose some interesting phenomenon such as, by suitably choosing the potential, one can achieve a position where the energy density of the field tracks the matter and radiation densities at early times, and later grows to dominate, resulting in late time acceleration. This is ideal in terms of solving the coincidence issue of dark energy. In short, the essence of quintessence can be captured by the idea that the potential does not have a minimum, it just slowly converges down towards some value, and the scalar field ϕ is on a very slow roll down the potential. Eventually in the future w_ϕ approaches arbitrarily close to -1, and the Universe will expand as a non-zero cosmological constant.

2.2 k-essence

While quintessence is based on a canonical scalarfield with a slowly varying potential, other candidates with non-canonical kinetic structure have also been much studied for cosmic acceleration. These kind of models go by the name *k-essence*. The action for such models generally takes the form

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + P(\phi, X)\right] + S_m \tag{91}$$

where S_m is the matter action and $P(\phi, X)$ is a funtion dependent on the scalar field ϕ and the kinetic term

$$X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \equiv -\frac{1}{2}(\nabla\phi)^2. \tag{92}$$

The idea is that, in the context of cosmology, the cosmic acceleration is driven by the kinetic term X of the scalar field ϕ. The energy-momentum tensor derived from the scalar field term $P(\phi, X)$ in the action (91) is

$$T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}P}{\delta g^{\mu\nu}} = P_{,x}\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}P. \tag{93}$$

For the k-essence model, $T_{\mu\nu}$ takes the form of a perfect fluid,

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + g_{\mu\nu}P \tag{94}$$

where $u_\mu = \partial_\mu\phi/\sqrt{2X}$ is the velocity, $P = P_\phi$ is the pressure and $\rho_\phi = 2XP_{,x} - P$ is the energy density. Thus the equation of state of k-essence is

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{P}{2XP_X - P}. \tag{95}$$

In the presence of a matter fluid with pressure P_M and energy density ρ_M we have the following equations of motion in a flat FLRW background,

$$3H^2 = \kappa^2(\rho_\phi + \rho_M) \tag{96}$$

$$2\dot{H} = -\kappa(2XP_{,X} + \rho_M + P_M) \tag{97}$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0. \tag{98}$$

Scaling solutions play a crucial part in understanding the dynamics of k-essence. As in the quintessence case late-time cosmic acceleration is reached by the tracking of the k-essence field. It is necessary to derive conditions for the existence of the scaling solutions where the energy density of dark energy is constant (but non-zero) relative to the energy density of the background fluid. Scaling solutions restricts the Lagrangian to be

$$P(\phi, X) = Xg(Xe^{\kappa\lambda,\phi}) \tag{99}$$

where λ is a constant and g is a function in terms of $Y = Xe^{\kappa\lambda\phi}$. We can for example recover quintessence with an exponential potential by choosing $g(Y) = 1 - c/Y$, which will result in $p = X - Xe^{-\kappa\lambda\phi}$.

There are several variations of k-essence. The present observational data allows phantom fields with an equation of state w_{DE} to be smaller than -1. In order to explain such an equation of state we can consider a field with negative kinetic energy $-X$ and field potential $V(\phi)$. Such fields with negative kinetic energy is called a ghost and are in general associated with unstable solutions. These fields roll up the potential because of their negative kinetic energy, and the field energy density will grow toward infinity if the potential is unbounded from above. Usually these kind of models leads to several severe problems and complications, however we will look at one of these variations of k-essence, titled the ghost condensate model, which give rise to a viable cosmic evolution. The Lagrangian of the ghost condensate model is given by

$$P = -X + \frac{X^2}{M^4} \tag{100}$$

where M is a constant with dimension mass. The negative kinetic energy may cause serious problems because the energy density is not bounded from below, and hence may generate ultra-violet quantum instabilities. The model takes into account higher orders of X to stabilize the vacuum and avoid this issue. The equation of state for a ghost condensate model with the above Lagrangian is given by equation (95),

$$w_\phi = \frac{1 - X/M^4}{1 - 3X/M^4} \tag{101}$$

which for $1/2 < X/M^4 < 2/3$ gives $-1 < w_\phi < -1/3$. Thus the de Sitter solution $(w_\phi = -1)$ can be established at $X/M^4 = 1/2$. This gives an field energy density $\rho_\phi = M^4/4$ which can explain the cosmic acceleration for $M \sim 10^{-3}$.

To derive the stability conditions of k-essence let us consider small fluctuations about the FLRW solution,

$$\phi(t, x) = \phi_0(t) + \delta\phi(t, x) \tag{102}$$

Because we are dealing with ultra-violet instabilities however, we cannot limit our analysis to a Minkowski background. Actually the model is called the ghost condensate model because it gives a ghost around a Minkowski background. The Lagrangian and the following Hamiltonian is found by expanding $P(\phi, X)$ to second order in $\delta\phi$,

$$\delta\mathcal{H} = (P_{,X} + 2XP_{,XX})\frac{(\delta\dot{\phi})^2}{2} + P_{,X}\frac{(\nabla\delta\phi)^2}{2} - P_{,\phi\phi}\frac{(\delta\phi)^2}{2} \tag{103}$$

which gives a positive Hamiltonian for the conditions

$$\varepsilon_1 \equiv P_{,X} + 2XP_{,XX} \geq 0, \tag{104}$$

$$\varepsilon_2 \equiv P_{,X} \geq 0, \tag{105}$$

$$\varepsilon_3 \equiv -P_{,\phi\phi} \geq 0. \tag{106}$$

In cosmological perturbation theory the speed of sound c_s is frequently used and emerge as a coefficient of the term k^2/a^2, where k is the comoving wavenumber and a is the scale factor. The speed of sound is defined by

$$c_s^2 \equiv \frac{P_{\phi,X}}{\rho_{\phi,X}} = \frac{\varepsilon_2}{\varepsilon_1} \tag{107}$$

In classical perturbation theory, the fluctuation is regarded as stable as long as c_s is real and c_s^2 is positive. However since we are handling quantum fluctuations the stability of the fluctuations demands both ε_1 and ε_2 to be positive. If these two conditions are met we can avoid any unpleasant issues concerning negative energy ghost states. These requirements can thus be used as a criteria for the consistence of the theory. It is also worth noting that in order to avoid superluminal speed of sound ($c_s > 1$), which could cause causality to be violated, ε_1 and ε_2 gives the condition $P_{,XX} \geq 0$.

To illustrate the cosmological dynamics of k-essence let us introduce a modification of the ghost condensate model, named the dilatonic ghost condensate model, with the Lagrangian

$$P = -X + e^{\kappa\lambda\phi}X^2/M^4 \tag{108}$$

Like we did with the quintessence model we can analyse k-essence as a dynamical system. We designate convenient variables,

$$x_1 \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H} \tag{109}$$

$$x_2 \equiv \frac{\dot{\phi}^2 e^{\kappa\lambda\phi}}{2M^4} \tag{110}$$

$$x_3 \equiv \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H} \tag{111}$$

With these new definitions we rewrite the equations of state and the density parameters,

$$w_{eff} = -1 - \frac{2\dot{H}}{3H^2} = -x_1^2 + x_1^2 x_2 + \frac{1}{3}x_3^2 \tag{112}$$

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{1 - x_2}{1 - 3x_2} \tag{113}$$

$$\Omega_\phi = -x_1^2 + 3x_1^2 x_2 \tag{114}$$

$$\Omega_r = x_3^2 \tag{115}$$

$$\Omega_m = 1 + x_1^2 - 3x_1^2 x_2 - x_3^2 \tag{116}$$

In order to analyse fixed points, we take the derivatives with respect to N = ln a and set them equal to zero ($dx_i/dN = 0$). We require the condition $x_2 \geq 1/2$ to guarantee quantum stability. The resulting fixed points gives a radiation point at $(x_1, x_2, x_3) = (0, 1/2, 1)$, a matter point at $(x_1, x_2, x_3) = (0, 1/2, 0)$ and an accelerated point at $(x_1, x_2, x_3) = (\sqrt{6}\lambda f_-(\lambda)/4, 1/2 + \lambda^2 f_+(\lambda)/16, 0)$, where

$$f_\pm(\lambda) = 1 \pm \sqrt{1 + 16/(3\lambda^2)}. \tag{117}$$

The accelerated point is stable for $0 \leq \lambda \leq \sqrt{3}$, which give the restriction $1/2 \leq x_2 \leq 2/3$. Acceleration is realized for $-1 \leq w_{eff} \leq -1/3$. The energy density of the field will be negligible relative to the background fluid in radiation and matter dominated eras. After x_1 grows to order unity, the energy density of the field begins to dominate and solutions begins to approach the accelerated point. The speed of sound can be rewritten in terms of our designated variables as

$$c_s^2 = \frac{2x_2 - 1}{6x_2 - 1}. \tag{118}$$

With the restriction on x_2 it can be approximated to be $0 \leq c_s \leq 1/3$ and we avoid superluminal propagation of the field. Thus with the dilatonic ghost condensate model a successful cosmological evolution can be achieved without violating causality and ensuring quantum stability. Nevertheless, it is worth noting that k-essence in general struggles to alleviate the coincidence problem. An adequate cosmological model should give rise to a cosmic evolution where the solutions eventually approach the accelerated attractor even if the initial conditions for the field energy density Ω_ϕ is relatively high. The k-essence models can achieve this, however in doing so, they will eventually force superluminal propagation of the sound speed [1].

Although they are similar as they both involve the dynamics of light scalar fields, quintessence and k-essence differ in the fact that quintessence relies on the functional forms of its potential, while the behaviour of k-essence comes from the presence of non-canonical kinetic terms. By comparison the simplest scalar field model is quintessence as it avoid theoretical problems with negative energy, such as ghosts and other instabilities. However it requires the derivatives of the field to be extremely small. The present observational data has yet to identify which of the two scalar field models is the best fit for our Universe. Future experiments may establish in more detail the dynamical properties of dark energy which will probably allow us to distinguish between the two.

2.3 Dark energy as a modification of gravity

The dark energy models we have discussed so far has been models where the origin of dark energy is realized through a scalar field ϕ with potential $V(\phi)$, which give rise to cosmic acceleration by modifying the ways matter react to the evolution of the Universe. Another alternative to model dark energy is by modifying the gravitational sector. In such models late-time acceleration is realized without recourse to an explicit dark energy matter component. Models in this category includes scalar-tensor theories, braneworld models and Gauss-Bonnet gravity. Scalar-tensor theories in particular includes both a scalar field and a tensor to represent a certain interaction, hence the name. Their general action is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(\varphi, R) - \frac{1}{2}\zeta(\varphi)(\nabla\varphi)^2\right] + S_m(g_{\mu\nu}, \Psi) \tag{119}$$

where we have set $\kappa^2 = 1$. Here f is a function of the scalar field φ and the Ricci scalar R, ζ is a function of φ and S_m is the matter action. A multitude of different theories can be derived from this action. The simplest is perhaps $f(R)$ gravity, where $f(\varphi, R) = f(R)$ and $\zeta = 0$. The action is given by ,

$$S = \frac{1}{2} \int d^4x \sqrt{-g} f(R) + S_m(g_{\mu\nu}, \Psi_m) \tag{120}$$

where S_m is the matter action and the matter fields Ψ_m obey standard conservation equations. There are two approaches to derive the field equations of this theory. First we have the metric formalism, where the Christoffel symbols are as usual defined in terms of the metric. The variation of the action (120) with respect to $g_{\mu\nu}$ leads to the equations of motion

$$F(R)R_{\mu\nu}(g) - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = \kappa^2 T_{\mu\nu} \tag{121}$$

where $F(R) = \partial f(R)/\partial R$ and $T_{\mu\nu}$ is the energy-momentum tensor. The trace of (121) is given by

$$3\Box F(R) + F(R)R - 2f(R) = \kappa^2 T. \tag{122}$$

The alternative way to derive the field equations is the Palatini formalism. In this approach the connections $\Gamma^\lambda_{\alpha\beta}$ and the metric $g_{\mu\nu}$ are treated as independent variables. Hence we must vary the action with respect to both $g_{\mu\nu}$ and $\Gamma^\lambda_{\alpha\beta}$. Varying with respect to $g_{\mu\nu}$ gives

$$F(R)R_{\mu\nu}(\Gamma) - \frac{1}{2} f(R) g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{123}$$

and the trace

$$F(R)R - 2f(R) = \kappa^2 T. \tag{124}$$

The term $R(\Gamma)$ is the Ricci tensor in terms of $\Gamma^\lambda_{\alpha\beta}$ and $R(T) = g^{\mu\nu} R_{\mu\nu}(\Gamma)$ is directly related to T. Varying the action with respect to $\Gamma^\lambda_{\alpha\beta}$ and using equation (123) gives

$$
\begin{aligned}
R_{\mu\nu}(g) - \frac{1}{2}R(g) = \frac{\kappa^2 T_{\mu\nu}}{F} - \frac{FR(T) - f}{2F}g_{\mu\nu} \\
+\frac{1}{F}(\nabla_\mu \nabla_\nu F - g_{\mu\nu}\Box F) \\
-\frac{3}{2F^2}\left[\partial_\mu F \partial_\nu - \frac{1}{2}g_{\mu\nu}(\nabla F)^2\right].
\end{aligned}
\tag{125}
$$

We can retrieve General Relativity by choosing $f(R) = R - 2\Lambda$ and $F(R) = 1$. Consequently, the term $\Box F(R)$ in (124) vanish and both formalism give $R = -\kappa^2 T = \kappa^2(\rho - 3P)$. Hence the Ricci scalar is determined by the trace T. Notice the difference in the trace of the equations of motion: in (122) the term $F(R)$ give rise to a propagating degree of freedom $\psi \equiv F(R)$ whose dynamics is regulated by the trace. In (124) however, term $\Box F(R)$ is not present. The scalar degree of freedom thus does not propagate freely for the Palatini case. This trait makes a critical difference in the dynamics of the two formalisms. As such they should be analysed separately. In our presentation of $f(R)$ gravity we will mainly focus on the metric formalism, but for completion we will briefly comment on differences with the Palatini case as well.

The de Sitter point resemble a vacuum solution in a Universe which is considered spatially flat. Hence the Ricci scalar is constant and $\Box F(R) = 0$. In such a case both the metric and Palatini formalisms gives

$$
F(R)R - 2f(R) = 0. \tag{126}
$$

This condition is satisfied by the model $f(R) = \alpha R^2$ which gives an asymptotically exact de Sitter solution. Because of this, the squared term has been a popular supplement to describe the inflation epoch. Early in the 1980s the model $f(R) = R + \alpha R^2$ was proposed, where the squared term is responsible for the expansion of inflation. Once R^2 is smaller than the linear term, inflation ends. However, the difference between R^2 and R at the present epoch is insignificant, and so the model is not sufficient to realize the current cosmic acceleration. As a possible solution a model on the form $f(R) = R - \alpha/R^n$, where $(\alpha > 0, n > 0)$, was suggested. This model accounts for modification for small R and late-time acceleration is in fact reached. However, it provokes instabilities related to negative values of $f_{,RR}$ and thus they do not satisfy local gravity constrains. Another problem is a large coupling between the Ricci scalar and non-relativistic matter, causing the standard matter epoch to vanish. This is not the case for the Palatini formalism, because the kinetic term $\Box F(R)$ is not present. The $f(R) = R - \alpha/R^n$

model can in fact lead to a viable cosmic evolution in the Palatini case, under certain conditions.

To construct an applicable $f(R)$ dark energy model with the metric formalism, we must require the following

(i) $f_{,R} > 0$ for $R \geq R_0 (> 0)$, where R_0 is the Ricci scalar at the present epoch. This is required to avoid anti-gravity [1].

(ii) $f_{,RR} > 0$ for $R \geq R_0$. This is required for consistency with local gravity tests, for the presence of the matter-dominated epoch and for the stability of cosmological perturbations.

(iii) $f(R) \to R - 2\Lambda$ for $R \gg R_0$. This as well is required for consistency with local gravity tests and for the presence of the matter-dominated epoch.

(iv) $0 < \frac{Rf_{,RR}}{f_{,R}}(r = -2) < 1$ at $r = \frac{-Rf_{,R}}{f} = -2$. This is required for stability of the late time de Sitter point.

Although there are several proposed models that satisfy the above conditions, we can perform a general analysis of the dynamics without specifying $f(R)$. In a flat FLRW background we can write the Ricci scalar as

$$R = 6(2H^2 + \dot{H}). \tag{127}$$

We consider non-relativistic matter and radiation which satisfy the conservation equations $\dot{\rho}_m + 3H\rho_m = 0$ and $\dot{\rho}_r + 4H\rho_r = 0$. From (121) and (122) we obtain

$$3FH^2 = \kappa^2(\rho_m + \rho_r) + \frac{1}{2}(FR - f) - 3H\dot{F} \tag{128}$$

and

$$-2F\dot{H} = \kappa^2\left(\rho_m + \frac{4}{3}\rho_r\right) + \ddot{F} - H\dot{F} \tag{129}$$

We define the following variables

$$x_1 \equiv -\frac{\dot{F}}{HF} \tag{130}$$

$$x_2 \equiv -\frac{f}{6FH^2} \tag{131}$$

$$x_3 \equiv \frac{R}{6H^2} \tag{132}$$

$$x_4 \equiv \frac{\kappa^2 \rho_r}{3FH^2} \tag{133}$$

Together with the quantities

$$\Omega_m \equiv \frac{\kappa^2 \rho_m}{3FH^2} = 1 - x_1 - x_2 - x_3 - x_4 \tag{134}$$

$$\Omega_r \equiv x_4 \tag{135}$$

$$\Omega_{DE} \equiv x_1 + x_2 + x_3 \tag{136}$$

Differentiating x_i with the number of e-foldings $N = \ln\, a$ gives the differential equations

$$\frac{dx_1}{dN} = -1 - x_3 - 3x_2 + x_1^2 - x_1 x_3 + x_4 \tag{137}$$

$$\frac{dx_2}{dN} = \frac{x_1 x_3}{m} - x_2(2x_3 - 4 - x_1) \tag{138}$$

$$\frac{dx_3}{dN} = -\frac{x_1 x_3}{m} - 2x_3(x_3 - 2) \tag{139}$$

$$\frac{dx_4}{dN} = -2x_3 x_4 + x_1 x_4. \tag{140}$$

with the following definitions

$$m \equiv \frac{dlnF}{dlnR} = \frac{Rf_{,RR}}{f_{,R}} \tag{141}$$

$$r \equiv -\frac{dlnf}{dlnR} = -\frac{Rf_{,R}}{f} = \frac{x_3}{x_2} \tag{142}$$

The effective equation of state is

$$w_{eff} = -\frac{1}{3}(2x_3 - 1) \tag{143}$$

A closer look at m reveals its interesting nature. From the equation of r we can write R as a function of x_3/x_2. Because m is a function of R, we can also write m as a function of r, $m = m(r)$. We can recover the ΛCDM model $f(R) = R - 2\Lambda$ by choosing $m = 0$. Hence m act as a measurement of the deviation from the ΛCDM model. At the present epoch m can be allowed to be of the order 0.1. However at more dense regions ($R \gg R_0$), like the radiation dominated epoch, it is constrained to be of the order $m \ll 10^{-9}$ to be consistent with local gravity tests.

The $f(R)$ models

$$f(R) = R - \mu R_c \frac{(R/R_c)^{2n}}{(R/R_c)^{2n} + 1} \tag{144}$$

$$f(R) = R - \mu R_c \left[1 - (1 + R^2/R_c^2)^{-n}\right] \tag{145}$$

$$f(R) = R - \mu R_c tanh(R/R_c) \tag{146}$$

with n, μ, $R_c > 0$, are constructed to sustain a suppressed m at the early eras of the cosmic evolution while rapidly increasing towards an acceptable value ($m \lesssim \mathcal{O}(0.1)$) at the present epoch.

The main difference between the action for scalar-tensor theories (119) and the previous discussed models is the couplings of the scalar field to matter. In a way, $f(R)$ generalizes Einstein equations, where the $f(R)$ is equal to the Ricci scalar. As a consequence of introducing an arbitrary function of R there is freedom to explain the accelerated expansion without introducing unknown forms of dark energy or dark matter. Yet as one might expect, modifications to gravity are severely restricted by local gravity constraints.

3 Screening mechanisms

Although there exist plenty of viable dark energy models, many which offer interesting opportunities in terms of dynamical cosmic evolution, they are constrained by local gravity measurements within our solar system. However there are ways in which these constraints can be avoided. By hiding the effects of dark energy through a screening mechanism we can recover General Relativity in high density areas. Such screening mechanisms can be achieved through different ways, usually depending on what kind of model one is working with. Usually the mechanism is implemented in some term in the Lagrangian and is provoked when certain conditions are met. To illustrate the effect at work, consider the coupled quintessence field we briefly mentioned earlier, whose effective mass changes depending on the surrounding environment. If we are in a high density region the field acquires a sufficiently high mass about the potential minimum such that it cannot propagate freely, thus making its impact short range and negligible. On the other hand when we are in a low density region such as deep space, the field has a lighter mass and we can observe its effect. This type of screening mechanism, where the scalar mass blend in with the high density of the surrounding environment, has the appropriate name chameleon.

There are several screening mechanisms out there with the shared purpose to recover General Relativity and make sure our theory is compatible with observational measurements. A suited way to classify them is to recognize the nature of their screening criterion: (i) *Screening based on* ϕ and (ii) *Screening with higher-derivative interactions*: $\partial\phi$; $\partial^2\phi$

3.1 Screening by deep potentials

In this category screening occur as a result of field self-interactions, regulated by the field's potential $V(\phi)$. Whether or not the screening process is triggered depends on the local value of ϕ. The chameleon mechanism mentioned earlier fall under this category, as well as the symmetron and dilaton screening mechanisms. The criterion for these mechanisms is met when the gravitational potential Φ exceeds some critical value $\Phi \gtrsim \Lambda$, where Λ is set by the parameters of the theory. This kind of screening is usually implemented in scalar-tensor theories, with an action of the kind

$$S = \int d^4x\sqrt{-g}\left(\frac{M_{Pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right) + S_m(\tilde{g}_{\mu\nu}, \psi). \tag{147}$$

Here $\tilde{g}_{\mu\nu} = A^2(\phi)g_{\mu\nu}$ is the Jordan-frame metric and M_{Pl} is the Planck mass. The matter fields ψ, described by the matter action S_m, couple to the field ϕ through the conformal factor $A(\phi)$.

To achieve screening we want to write the equation of motion for the action (147) in the form of an effective potential for the system, which is written as a sum of the bare potential and a density-dependent term. To do this we first find the original equation of motion by taking the variation of the action above

$$\Box\phi = V_{,\phi} - A^3(\phi)A_{,\phi}\tilde{T} \tag{148}$$

where $\tilde{T}$ is the trace of the Jordan-frame energy-momentum tensor, $\tilde{T} = \tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu}$. This energy-momentum tensor is covariantly conserved, $\nabla^\mu\tilde{T}_{\mu\nu} = 0$, and we have the relation $\tilde{T} = A^{-4}T_{matter}$.

The gravitational part of the action is governed by the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = G_{\mu\nu} = \frac{1}{M_{Pl}^2}\left(T_{\mu\nu}^{matter} + T_{\mu\nu}^{\phi}\right). \tag{149}$$

By the Bianchi identity, $\nabla_{\mu\nu}G^{\mu\nu} = 0$, we find that the matter energy-momentum tensor in the Einstein frame is not conserved,

$$\nabla T_{matter}^{\mu\nu} = \frac{A_{,\phi}}{A}T_{matter}\partial^\nu\phi \tag{150}$$

However, if we consider an FLRW background we find that the 0-component of (150) is in fact conserved,

$$\dot{\rho} + 3H\rho = 0 \tag{151}$$

where we have defined $\rho \equiv A^{-1}T_{matter}$. Hence we can rewrite the equation of motion (148) to

$$\Box\phi = V_{eff}(\phi) \tag{152}$$

where $V_{eff}(\phi) = V(\phi) + A(\phi)\rho$ is the effective potential. The idea is to choose the potential $V(\phi)$ and it's coupling to matter $A(\phi)$ such that the scalar field propagates freely in low density areas where the Newtonian potential Φ is small, but in high density regions the effects of the field vanish. This idea can be realized by different methods, such as the chameleon mechanism.

3.1.1 Chameleon

As explained, the chameleon is based on the concept that scalar field obtain sufficiently large mass to limit it's effective range to non-observable values in high density regions. Hence the scalar field need to develop a density-dependent mass,

$$m_{eff}^2(\phi) = V_{,\phi\phi}^{eff}(\phi) = V_{,\phi\phi}(\phi) + A_{,\phi\phi}(\phi)\rho. \tag{153}$$

In order to ensure the effects we are looking for we need some requirements to constrain the potential $V(\phi)$ and the coupling function $A(\phi)$ over the regions of interest.

(i) We want to balance the contributions of $V(\phi)$ and $A(\phi)$ to the effective potential. Without loss of generality we can assume that $A(\phi)$ is monotonically increasing and that $V(\phi)$ is monotonically decreasing, over the relative field values.

(ii) The potential $V(\phi)$ is usually the dominant contribution to the effective mass. We therefore require $V_{,\phi\phi} > 0$ to secure stability.

(iii) Also we require $V_{,\phi\phi\phi} < 0$ to ensure that the effective mass increase with the density of the surrounding environment.

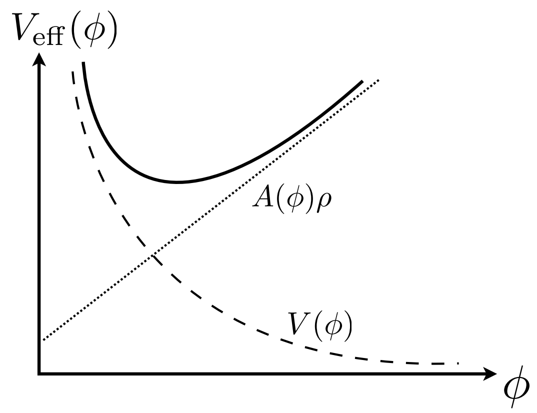

Figure 5: The solid line illustrate the effective potential felt from the chameleon field. It is the sum of the bare potential $V(\phi)$ (dashed line) and a density dependent term (dotted line). Reproduced from [7].

By analysis of small scale phenomena, it can be shown that the coupling function can be well approximated by the linear form

$$A(\phi) \simeq 1 + \varepsilon \frac{\phi}{M_{pl}} \tag{154}$$

where the constant ε must be positive and of order $\mathcal{O}(1)$ for a gravitational-strength scalar force [7]. As for the potential, let us consider the quintessence tracker model

$$V(\phi) = M^{4+n}\phi^{-n} \tag{155}$$

where $n > 0$ is a constant. With the coupling function (154) we can approximate the minimum of the effective potential as

$$\phi(\rho) \approx \left(\frac{nM^{4+n}M_{Pl}}{\varepsilon\rho}\right)^{\frac{1}{n+1}} . \tag{156}$$

The effective mass of the ϕ fluctuations is given by

$$m_{eff}^2(\rho) \approx n(n+1)M^{-\frac{4+n}{1+n}} \left(\frac{\varepsilon\rho}{nM_{Pl}}\right)^{\frac{n+2}{n+1}} . \tag{157}$$

Hence the effective mass of the scalar field increases with the background density.

To illustrate how exactly the chameleon mechanism would work in practice, consider a scenario for the field profile in the presence of a massive compact object. We assume the object is spherically symmetric, with constant radius R, density ρ_{obj}, and total mass M. The object is also assumed to exist in a static homogeneous background with density ρ_{amb}. For this case it is convenient to rewrite the equation of motion in spherical coordinates. Since we are dealing with a static, spherical symmetric source, we only concern ourself with the radial coordinates. Thus the Laplacian is translated to

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \tag{158}$$

Substituting this into equation (148), together with the coupling function (154), we get the equation of motion

$$\phi'' + \frac{2}{r}\phi' = V_{,\phi} + \varepsilon\frac{\rho}{M_{Pl}} \tag{159}$$

As boundaries we have $d\phi/dr = 0$ at $r = 0$ and that the field approach it's ambient-density minimum as $r \to \infty$. The density is divided into two regions, $\rho(r) = \rho_{obj}$ when $r < R$ and $\rho(r) = \rho_{amb}$ when $r > R$. We would like to know how ϕ looks like at the exterior of the object, but close enough such that Φ is sufficiently large. In the 1930s, Hideki Yukawa showed that the exchange from a massive scalar field mediates a force with a range inversely proportional to the mass of the mediator particle. So inside the object, at $r < R$, we have $\phi \simeq \phi_{obj}$, naturally. Outside the object however, within the range $R < r < m_{amb}^{-1}$, we can approximate the field profile as

$$\phi \simeq \frac{A}{r} + B \tag{160}$$

where A and B are constants. As $r \to \infty$ and $\phi \to \phi_{amb}$ we find that $B = \phi_{amb}$. Inside the object, where $\phi(R) = \phi_{obj}$, we find $A = -R(\phi_{amb} - \phi_{obj})$. Thus within the distance $R < r < m_{amb}^{-1}$ equation (160) can be rewritten as

$$\phi \simeq -\frac{R}{r}(\phi_{amb} - \phi_{obj}) + \phi_{amb}. \tag{161}$$

To see the actual screening at work we can imagine that the object acts as a conducting sphere, where any chameleon "charge" is limited to a thin shell of thickness ΔR near the surface. Gauss law yields

$$\frac{d\phi}{dr}|_{r=R_+} = \frac{\varepsilon\rho}{M_{Pl}}\Delta R \tag{162}$$

where $\varepsilon\rho\Delta R/M_{Pl}$ is the "charge density" on this shell. Together with equation (161) we can solve for the shell thickness,

$$\frac{\Delta R}{R} = \frac{\phi_{amb} - \phi_{obj}}{6\varepsilon M_{Pl}\Phi} \tag{163}$$

where $\Phi = M/8\pi M_{Pl}^2 R$ is the surface gravitational potential. This *thin-shell factor* is the mechanism that will trigger screening. Solving equation (163) for $\phi_{amb} - \phi_{obj}$, we can rewrite equation (161) as

$$\phi(r > R) \simeq -\frac{3\varepsilon}{4\pi M_{Pl}}\frac{\Delta R}{R}\frac{Me^{-m_{amb}(r-R)}}{r} + \phi_{amb}. \quad (164)$$

The exponential factor comes from the Yukawa potential, since the field is massive. The source is screened if the thin-shell factor $\Delta R/R \ll 1$, hence reducing the coupling and the field profile can be approximated to ϕ_{amb}. This can be realized by a sufficiently large gravitational potential Φ. If Φ is weak however, and $\Delta R/R \gtrsim 1$ the object is unscreened,

$$\phi(r > R) \simeq -\frac{3\varepsilon}{4\pi M_{Pl}}\frac{Me^{-m_{amb}(r-R)}}{r}. \quad (165)$$

Thus the chameleon effectively couple only to a thin shell beneath the surface, while gravity couple to the entire mass of the object, suppressing the effects of the dark energy field.

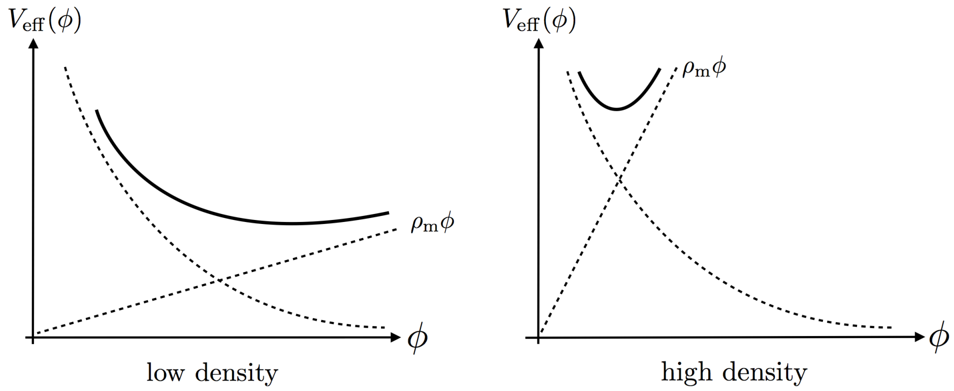

Figure 6: The figure shows the chameleon effective potential in low density regions and high density regions. In low density regions the field exhibits a long range force while in regions of higher density the term $\rho_m\phi$ is much larger, thus the force is suppresed. Figure from [7].

3.1.2 Symmetron

The symmetron mechanism is another way to suppress the coupling of the scalar field to matter in high density areas. It does so by choosing an effective potential such that the coupling of the scalar field to matter is proportional to the vacuum expectation value (VEV). In high density regions the field has zero VEV and as such does not couple to matter. In low density regions however the field spontaneously breaks some symmetry and acquire a nonzero VEV, allowing a coupling to matter and thus mediates a force. To illustrate the symmetron mechanism, consider the action (147) with a potential

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4. \tag{166}$$

This is a $\mathbb{Z}_2$-symmetric action of the form

$$S = \int d^4x\sqrt{-g}\left(\frac{M_{Pl}}{2}R - \frac{1}{2}(\partial\phi)^2 + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4\right) + S_m\left[\left(1 + \frac{2}{M^2}\phi^2\right)^2 g_{\mu\nu}, \psi\right] \tag{167}$$

with the coupling to matter

$$A(\phi) = 1 + \frac{1}{2M^2}\phi^2 + \mathcal{O}\left(\frac{\phi^4}{M^4}\right). \tag{168}$$

The higher order terms are negligible since $\phi \ll M$, where M is some high mass scale of the system. Recall that the effective potential can be written as $V_{eff}(\phi) = V(\phi) + A(\phi)\rho$. Thus in this case the effective potential in the presence of a non-relativistic source is given by

$$V_{eff}(\phi) = \frac{1}{2}\left(\frac{\rho}{M^2} - \mu^2\right)\phi^2 + \frac{\lambda}{4}\phi^4 \tag{169}$$

When the effective mass ρ is low, $\rho \ll M^2\mu^2$, the scalar field ϕ acquires a vacuum expectation value which spontaneously breaks the reflection symmetry of the Lagrangian $\phi \to -\phi$,

$$\bar{\phi} = \sqrt{\frac{\mu^2}{\lambda}}. \tag{170}$$

In high density regions, $\rho \gg M^2\mu^2$, the scalar is trapped around $\phi = 0$ and the potential does not break symmetry. Fluctuations of the scalar field $\delta\phi$ couple to matter as $(\bar{\phi}/M^2)\delta\phi\rho$. This is proportional to the vacuum expectation value $\bar{\phi}$.

Hence, when the ambient density is high and ϕ has no VEV, the fluctuations $\delta\phi$ does not couple to matter. The force mediated by ϕ in these regions is therefore non existent. See figure (7).

Lets consider the static, spherically-symmetric scenario where we assume the source is in a vacuum with homogeneous density $\rho > \mu^2 M^2$. Deep inside the object the symmetron couples weakly to matter, while far away from the object the symmetry breaks as $r \to \infty$. Hence at the exterior of the object the field must increase from $\phi = 0$. Like the chameleon case, a thin-shell screening effect occurs. The effect depend on the parameter

$$\alpha \equiv \frac{\rho R^2}{M^2} = 6\frac{M_{Pl}^2}{M^2}\Phi \tag{171}$$

where we have thin-shell screening if $\alpha \gg 1$, and the force due to the scalar field is suppressed compared to the strong gravitational force. In the case $\alpha \ll 1$, analogous to a weak gravitational potential Φ, the shell does not exist and the scalar field is not suppressed.

3.2 Screening with higher-derivative interactions

If the effective mass of the field is zero then the chameleon and symmetron mechanism would fail. However there are other mechanisms which do not rely on the mass of the field, but rather on kinetic suppression through higher derivative terms.We can distinguish to cases, kinetic screening and the Vainshtein effect. The kinetic type arise when the first derivative of the field becomes important. This class includes theories like k-mouflage. The criterion for screening is met when the local gravitational acceleration exceeds $|\nabla\Phi| \gtrsim \Lambda^2$. Related to kinetic screening, the Vainshtein mechanism relies on the second derivative of the field becoming important, while higher order derivatives remain small. The screening mechanism occur when the local curvature or density, $R \backsim \nabla^2\Phi$, surpass the critical value $|\nabla^2\Phi| \gtrsim \Lambda^3$. In such theories with non-trivial derivative interactions there are a number of challenges we would want to evade, such as contributions from a multitude of Lorentz-invariant terms involving derivatives. Within a carefully defined effective field theory it is possible to construct models that avoid these problems and implement screening mechanisms.

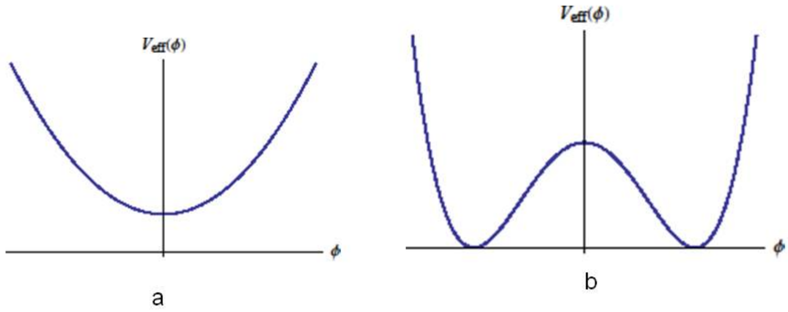

Figure 7: The figure displays the symmetron potential. In high density regions (a) the scalar is trapped around $\phi = 0$. On the other hand in low density regions (b) when the symmetric state of (a) is no longer a true ground state, the scalar will make a transition to one of the ground states in (b) giving rise to a spontaneous breaking of symmetry. Figure from [33].

3.2.1 Kinetic screening

To illustrate the mechanisms of this class of screening we will investigate the kinetic type, also known as k-mouflage. Such screening mechanisms often occur in models that involve a Lagrangian with first derivative kinetic terms referred to as $P(\phi, X)$, like the *k-essence* model we discussed earlier.

Consider a model where the scalar field ϕ accept a shift symmetry,

$$\phi(x) \to \phi(x) + c \tag{172}$$

where c is a constant. Further, we assume that the theory is invariant under the discrete symmetry $\phi \to -\phi$. The lowest-order Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{\alpha}{4\Lambda^4}(\partial\phi)^4 + \frac{g}{M_{Pl}}\phi T \tag{173}$$

where Λ has units of mass and α and g are dimensionless numbers. Here the shift symmetry is broken by the coupling to matter ϕT, however by requiring $M_{Pl} \gg \Lambda$ the breaking is accepted. Also we can absorb the magnitude of α into Λ, and thus we only need to consider $\alpha = \pm 1$. From the Lagrangian (173) we obtain the equation of motion

$$\Box\phi - \frac{\alpha}{\Lambda^4}\partial_\mu((\partial\phi)^2\partial^\mu\phi) + \frac{g}{M_{Pl}} = 0 \tag{174}$$

As we did previously with the chameleon mechanism, let us examine our model around a static, spherically-symmetric background. Consider a point source, $T = -M\delta^{(3)}(\vec{x})$. With these prerequisites, the equation of motion can be rewritten to

$$\vec{\nabla}\cdot\left(\vec{\nabla}\phi - \frac{\alpha}{\Lambda^4}(\vec{\nabla}\phi)^2\vec{\nabla}\phi\right) = \frac{gM}{M_{Pl}}\delta^{(3)}(\vec{x}) \tag{175}$$

Integrating both sides yields

$$\phi' - \frac{\alpha}{\Lambda^4}\phi'^3 = \frac{1}{4\pi r^2}\frac{gM}{M_{Pl}} \tag{176}$$

Far from the source the linear term of ϕ' is dominant, while close to the source the cubic term ϕ'^3 is more influential. The transition between these two regimes happens at $\phi' \sim \Lambda^2$. Inserting this into the equation gives

$$r_* = \frac{1}{\Lambda}\left(\frac{gM}{M_{Pl}}\right)^{1/2}. \tag{177}$$

Thus far from the source, $r \gg r_*$, we have

$$\phi'(r) = \frac{\Lambda^2}{4\pi}\left(\frac{r_*}{r}\right)^2 \tag{178}$$

while close to the source, $r \ll r_*$,

$$\phi'(r) = (-\alpha)^{1/3}\Lambda^2\left(\frac{r_*}{r}\right)^{2/3}. \tag{179}$$

From these forms it is apparent that in order to have a consistent continuous solution we must choose $\alpha = -1$. The force acting on the source due to the scalar field is

$$\vec{F}_\phi(x) = \frac{g}{M_{Pl}} \vec{\nabla}\phi = \hat{r}\frac{g}{M_{Pl}}\phi'(r) \tag{180}$$

while the force due to gravity around the source is

$$F_{grav} = \frac{M}{8\pi M_{Pl}^2}\frac{1}{r^2} = \frac{\Lambda^2}{8\pi M_{Pl}}\left(\frac{r_*}{r}\right)^2 \tag{181}$$

From the asymptotic regimes of ϕ', (178) and (227), we see that far from a source the scalar field convey a gravitational-strength force. Close to the source ϕ is undermined by the force of gravity, where the ratio between the two will go to zero as $\sim r^{4/3}$. This is the essence of kinetic screening.

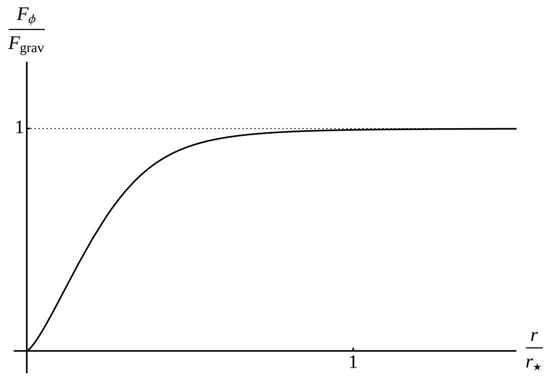

Figure 8: The ratio of the force mediated by the scalar field and the gravitational force. Figure from [7].

3.2.2 The Vainshtein mechanism

There are special classes of scalar field theories whose Lagrangian contain higher order derivatives while still have second order equations of motion. Galileons are an example of such theories, where apart from standard kinetic terms the Lagrangian also includes specific non-linear derivative terms. Galileons have been used to express cosmic acceleration and the origin of density perturbations due to inflation. In this section we will use such a theory to illustrate the effects of the Vainshtein screening mechanism. The strong non-linear terms become active around a massive body below the Vainshtein radius, where they effectively suppress the force mediated from the scalar field. The cubic galileon has the Lagrangian

$$\mathcal{L} = -3(\partial\phi)^2 - \frac{1}{\Lambda^3}\Box\phi(\partial\phi)^2 + \frac{g}{M_{Pl}}\phi T^\mu_\mu \tag{182}$$

where Λ is the strong-coupling scale and $g \sim \mathcal{O}(1)$ for gravitational strength coupling. Galileons have the property of being invariant under the galileon shift symmetry $\phi(x) \to \phi(x) + x + b_\mu x^\mu$. In fact, the coupling to matter breaks these symmetries, but the breaking is soft and we may ignore it. The second-order equation of motion derived by the Lagrangian above reads

$$6\Box\phi + \frac{2}{\Lambda^3}\left((\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2\right) = -\frac{g}{M_{Pl}}T^\mu_\mu. \tag{183}$$

Theories like the galileon are sensitive since other possible operators consistent with the symmetries, generated quantum mechanically, may appear in the Lagrangian as well. The Vainshtein mechanism relies on the term $\Box\phi(\partial\phi)^2/\Lambda^3$ dominating over the kinetic term $(\partial\phi)^2$ near massive objects. In this regime $\partial^2\phi \gg \Lambda^3$. Here, possible higher order operators becomes important and one would expect the effective field theory to break down. That said, the galileon theory has two parameters that measure the importance of classical non-linearities and quantum mechanical contributions which can allow scenarios where the galileon term is large while the other operators remain negligible:

The classical parameter,

$$\alpha_{cl} \equiv \frac{\partial^2\phi}{\Lambda^3} \tag{184}$$

and the quantum parameter,

$$\alpha_q \equiv \frac{\partial^2}{\Lambda^2}. \tag{185}$$

Possible desirable situations would have that $\alpha_{cl} \gg 1$ while $\alpha_q \ll 1$. Around a static, spherically-symmetric point source M the equation of motion can be written as

$$\vec{\nabla}\cdot\left(6\vec{\nabla}\phi+\hat{r}\frac{4}{\Lambda^3}\frac{(\vec{\nabla}\phi)^2}{r}\right)=\frac{gM}{M_{Pl}}\delta^{(3)}(\vec{x}) \tag{186}$$

where $T^\mu_\mu = -M\delta^{(3)}(\vec{x})$. Integrating this equation gives

$$6\phi'+\frac{4}{\Lambda^3}\frac{\phi'^2}{r}=\frac{gM}{4\pi r^2 M_{Pl}} \tag{187}$$

Solving this for $\phi'(r)$ as $\phi\to 0$ we get

$$\phi'(r)=\frac{3\Lambda^3 r}{4}\left(-1+\sqrt{1+\frac{1}{9\pi}\left(\frac{r_V}{r}\right)^3}\right) \tag{188}$$

where r_V is the Vainshtein radius,

$$r_V\equiv\frac{1}{\Lambda}\left(\frac{gM}{M_{Pl}}\right)^{1/3}. \tag{189}$$

In the asymptotic regimes we have that far from the source, $r\gg r_V$, the profile goes as $1/r^2$,

$$\phi'(r\gg r_V)\simeq\frac{g}{3}\cdot\frac{M}{8\pi M_{Pl}r^2} \tag{190}$$

and the ratio of the galileon force and gravity is

$$\frac{F_\phi}{F_G}\simeq\frac{g^2}{3}. \tag{191}$$

Here both the classical and the quantum parameters are small,

$$\alpha_{cl}\sim\left(\frac{r_V}{r}\right)^3\ll 1 \tag{192}$$

$$\alpha_q\sim\frac{1}{(r\Lambda)^2}\ll 1 \tag{193}$$

Close to the source, $r\ll r_V$, we have

$$\phi'(r \ll r_V) \simeq \frac{\Lambda^3 r_V}{2}\sqrt{\frac{r_V}{r}} \sim \frac{1}{\sqrt{r}}. \tag{194}$$

In this regime, the galileon force is strongly suppressed by the gravitational force as the ratio between the two is

$$\frac{F_\phi}{F_G} \sim \left(\frac{r}{r_V}\right)^{3/2} \ll 1. \tag{195}$$

For this screening to happen, the classical parameter must be very large,

$$\alpha_{cl} \sim \left(\frac{r_V}{r}\right)^{3/2} \gg 1. \tag{196}$$

However, the quantum parameter has the same form as before,

$$\alpha_q \sim \frac{1}{(r\Lambda)^2}. \tag{197}$$

This can be large or small depending on the distance r compared to Λ^{-1}. If the distance is sufficiently small, $r \ll \Lambda^{-1}$, the parameter becomes significantly large, thus quantum effects become important and the effective field theory breaks down. But at distances $r \gg \Lambda^{-1}$ the parameter remains small and quantum corrections are under control. Hence there is a regime, $\Lambda^{-1} \ll r \ll r_V$, where both parameters are small and the classical solution is viable.

4 Screening in vector field theories

Scalar field theories have been intensively studied as a way to realize the different epochs in the evolution of the Universe. However less extensively studied, in the same way can vector fields lead to interesting cosmological phenomenology. We know for a fact that vector fields are already present in the Universe through electromagnetic radiation. Indeed, they are abundant in several physical theories and have been explored in the context of the Universe as well.

In the following, we will consider vector fields described through some function $f(F)$ where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the Faraday tensor, i.e. the electromagnetic field tensor, and A_μ is the electromagnetic 4-potential. The Faraday tensor hold the electric and magnetic field components in any direction,

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}. \tag{198}$$

By multiplication with the corresponding covariant tensor we find the standard Maxwell Lagrangian $(1/4)F_{\mu\nu}F^{\mu\nu} = (1/2)(\mathbf{E}^2 - \mathbf{B}^2)$. In the absence of sources, the Maxwell equations are symmetric under a duality transformation, interchanging the electric and magnetic field components. Hence the dual Faraday tensor is defined as $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and has the explicit matrix form

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix} \tag{199}$$

which together with the standard electromagnetic field tensor give the identity $(1/4)F_{\mu\nu}\tilde{F}^{\mu\nu} = \mathbf{E}\cdot\mathbf{B}$. Further, the Maxwell equations on tensor form emerge as $\partial_\mu F^{\mu\nu} = -J^\nu$ and $\partial_\mu \tilde{F}^{\mu\nu} = 0$, with and without sources respectively.

Several models where the cosmological dynamics are driven by vector fields have been debated. Examples of such theories involve non-linear functions such as $f(F^2)$, $f(F^2, F\tilde{F})$ and $f(F^2)R$ with non-minimal coupling to the metric. Naturally, such theories must also abide by local gravity experiments. The present theories involving screening mechanisms are generally developed for scalar fields. While vector fields are different objects than scalar fields, the same principles can be adopted. In this section we will study a screening process inspired by the k-mouflage mechanism we examined above for scalar fields. We will use the metric signature (-+++). The exact model consists of a standard Maxwell term supplemented by a function f of the vector field strength tensor and a coupling to a conserved current $A_\mu J^\mu$,

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + f(X,Y) + A_\mu J^\mu\right] \tag{200}$$

The function $f(X,Y)$, where $X = F_{\mu\nu}F^{\mu\nu}$, $Y = F_{\mu\nu}\tilde{F}^{\mu\nu}$, is given by

$$f(X,Y) = \frac{1}{\Lambda_\nu^4}\left(c_1(F_{\mu\nu}F^{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right) \tag{201}$$

where c_1 and c_2 are constants, and Λ_ν^4 is some energy scale at which the non-linear contributions arise. The model is similar to the Euler-Heisenberg Lagrangian, which describes the non-linear dynamics of electromagnetic fields in vacuum. It was originally used in analysing light-by-light scattering, and is obtained from QED when fermions are integrated out.

In the following steps we show in detail how the equations of motion is acquired from (200). Varying the action with respect to A_ν yields

$$\delta S = \int d^4x\delta\left(\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + f(X,Y) + A_\mu J^\mu\right]\right). \tag{202}$$

$$= \int d^4x\sqrt{-g}\left[-\frac{1}{4}\delta(F_{\mu\nu}F^{\mu\nu}) + f_{,X}\delta X + f_{,Y}\delta Y + J^\mu\delta A_\mu\right] \tag{203}$$

where $f_{,X} = \partial f/\partial X$ and $f_{,Y} = \partial f/\partial Y$.

The first term of the Lagrangian can be written as

$$\delta(F_{\mu\nu}F^{\mu\nu}) = (\delta F_{\mu\nu})F^{\mu\nu} + F_{\mu\nu}(\delta F^{\mu\nu}). \tag{204}$$

To illustrate the approach we will treat these terms separately. Expanding the former gives

$$(\delta F_{\mu\nu})F^{\mu\nu} = (\partial_\mu\delta A_\nu - \partial_\nu\delta A_\nu)F^{\mu\nu} = F^{\mu\nu}\partial_\mu\delta A_\nu - F^{\mu\nu}\partial_\nu\delta A_\nu. \tag{205}$$

Integration by parts result in

$$-(\partial_\mu F^{\mu\nu}\delta A_\nu - \partial_\nu F^{\mu\nu}\delta A_\nu) = -(\partial_\mu F^{\mu\nu}\delta A_\nu + \partial_\nu F^{\nu\mu}\delta A_\nu) = -2\partial_\mu F^{\mu\nu}\delta A_\nu \quad (206)$$

In the second equality the sign changes because of the fact that the electromagnetic field tensor is totally antisymmetric, i.e. $F^{\mu\nu} = -F^{\nu\mu}$. The latter term of (204) can be derived in a similar way,

$$F_{\mu\nu}(\delta F^{\mu\nu}) = g_{\mu\alpha}g^{\nu\beta}F^{\alpha\beta}(\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu) \quad (207)$$

Partial integration gives

$$-g_{\mu\alpha}g^{\nu\beta}(\partial^\mu F^{\alpha\beta}\delta A^\nu - \partial^\nu F^{\alpha\beta}\delta A^\mu) = -(\partial_\alpha F^{\alpha\beta}\delta A_\beta - \partial_\beta F^{\alpha\beta}\delta A_\alpha) \quad (208)$$

$$= -(\partial_\alpha F^{\alpha\beta}\delta A_\beta + \partial_\beta F^{\beta\alpha}\delta A_\alpha) = -2\partial_\alpha F^{\alpha\beta}\delta A_\beta \quad (209)$$

Equation (204) becomes

$$\delta(F_{\mu\nu}F^{\mu\nu}) = -4\partial_\mu F^{\mu\nu}\delta A_\nu \quad (210)$$

In the second term of equation (203), $\delta X = \delta(F_{\mu\nu}F^{\mu\nu})$ results in the same as we derived above, giving

$$f_{,X}\delta X = -4\partial_\mu(f_{,X}F^{\mu\nu}\delta A_\nu) \quad (211)$$

Expanding the third term $f_{,Y}\delta Y$ yields

$$f_{,Y}\delta Y = f_{,Y}\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\delta(F_{\mu\nu}F_{\alpha\beta}) = 4f_{,Y}\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}\partial_\alpha\delta A_\beta \quad (212)$$

$$= 4f_{,Y}\tilde{F}^{\alpha\beta}\partial_\alpha\delta A_\beta = -4\partial_\alpha(f_{,Y}\tilde{F}^{\alpha\beta}\delta A_\beta) \quad (213)$$

Finally by pulling the pieces together, equation (203) can be rewritten as

$$\delta S = \int d^4x\sqrt{-g}\left[\partial_\mu F^{\mu\nu} - 4\partial_\mu(f_{,X}F^{\mu\nu}) - 4\partial_\mu(f_{,Y}\tilde{F}^{\mu\nu}) + J^\mu\right]\delta A_\nu \quad (214)$$

The principle of least action states that the path taken is the one where the action is stationary to the first order, $\Rightarrow \delta S = 0$. The only possible way the integral always yield zero is if the integrand itself is zero, imposing

$$\nabla_\mu \left[F^{\mu\nu} - 4f_{,X} F^{\mu\nu} - 4f_{,Y} \tilde{F}^{\mu\nu} \right] + J^\mu = 0 \tag{215}$$

From (201) we find that the partial derivatives $f_{,X}$ and $f_{,Y}$ are given by

$$f_{,X} = 2\frac{c_1}{\Lambda_\nu^4}(F_{\mu\nu}F^{\mu\nu}) \tag{216}$$

$$f_{,Y} = 2\frac{c_2}{\Lambda_\nu^4}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \tag{217}$$

Thus the equation of motion takes the form

$$\Rightarrow \nabla_\mu \left[F^{\mu\nu} - \frac{8}{\Lambda_\nu^4} c_1 (F_{\mu\nu}F^{\mu\nu}) F^{\mu\nu} - \frac{8}{\Lambda_\nu^4} c_2 (F_{\mu\nu}\tilde{F}^{\mu\nu}) \tilde{F}^{\mu\nu} \right] + J^\mu = 0. \tag{218}$$

Adopting the methods shown with kinetic screening and a scalar field, we will consider a static and spherical symmetric source with $J^\mu = (\rho(r), 0)$ and use a static and spherical symmetric ansatz for the vector field. Note that we now have a gauge symmetry $A_\mu \to A_\mu + \partial_\mu \theta$.

For $\nu = 0$ equation (218) gives :

$$\nabla_i \left[(1 - \frac{8}{\Lambda_\nu^4} c_1 F^2) F^{i0} - \frac{8}{\Lambda_\nu^4} c_2 (F\tilde{F}) \tilde{F}^{i0} \right] = -\rho. \tag{219}$$

For $\nu = i$ equation (218) gives:

$$\nabla_j \left[(1 - \frac{8}{\Lambda_\nu^4} c_1 F^2) F^{ji} - \frac{8}{\Lambda_\nu^4} c_2 (F\tilde{F}) \tilde{F}^{ji} \right] = 0. \tag{220}$$

Around a static point charge there are no magnetic fields, so we can assume that $\vec{B} = 0$. Since $B = 0$ the spatial components $F^{ij} = \varepsilon^{ijk} B_k$ vanish. Similarly, in the dual tensor where the electric and magnetic field components are interchanged, the time components vanishes, giving $\tilde{F}^{0i} = 0$. In fact we require that this assumption holds for $r = 0$ to infinity. No matter where we place the charge, since it is static it will never induce any magnetic field. Hence we need only to consider an electric background. In terms of the electromagnetic field tensor we can write the identities $F^2 = F_{\mu\nu}F^{\mu\nu} = 2(E^2 - B^2) = 2E^2$ and $F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B} = 0$. If we insert this into (219) and (220) we see that the assumption is consistent with the equations. From (219) we are left with

$$\nabla\left[(1 - \frac{16c_1}{\Lambda_\nu^4}E^2)\vec{E}\right] = -\rho. \tag{221}$$

Integrating over space and using Gauss law gives the cubic equation

$$\Rightarrow (1 - \frac{16c_1}{\Lambda_\nu^4}E^2)\vec{E} = \frac{Q}{4\pi r^3}\vec{r} \tag{222}$$

By taking the absolute value the equation can be written as

$$\Rightarrow E - \frac{16c_1}{\Lambda_\nu^4}E^3 = \frac{Q}{4\pi r^2}. \tag{223}$$

The only real solution resulting from this cubic equation is

$$\begin{aligned} E(r) = \frac{1}{4}&\left(\frac{r^2\Lambda}{3^{1/3}(18c_1^2r^4r_c^2\Lambda^6 + \sqrt{3}\sqrt{-c_1^3r^{12}\Lambda^{12} + 108c_1^4r^8r_c^4\Lambda^{12}})^{1/3}}\right) \\ -\frac{1}{4}&\left(\frac{3^{1/3}(18c_1^2r^4r_c^2\Lambda^6 + \sqrt{3}\sqrt{-c_1^3r^{12}\Lambda^{12} + 108c_1^4r^8r_c^4\Lambda^{12}})^{1/3}}{3^{2/3}c_1r^2}\right). \end{aligned} \tag{224}$$

It doesn't look too appealing. However, we can deduce its meaning by analysing the more lenient cubic equation (223). In the asymptotic regimes the linear term E dominates far from the source, while close to the source the cubic term E^3 dominates. The scale at which the transition between the two regimes happens is when the two terms are equal. Absorbing $16c_1$ into Λ, we find the transition when $E = \Lambda_\nu^2$. Far from the source this gives

$$\Lambda_\nu^2 = \frac{Q}{4\pi r^2}. \tag{225}$$

Hence the transition happens at a distance $r_c = \frac{1}{\Lambda_\nu}\left(\frac{Q}{4\pi}\right)^{1/2}$ from the source.

For $r \gg r_c$,

$$E(r) \simeq \Lambda_\nu^2 \left(\frac{r_c}{r}\right)^2 \tag{226}$$

while for $r \ll r_c$,

$$E(r) \simeq \Lambda_\nu^2 \left(\frac{r_c}{r}\right)^{2/3} \tag{227}$$

At this stage let us underline that, although operating with electrodynamic quantities, the vector field is simply an analogue to this case. However, for the sake of illustration, we will imagine that the force due to the vector field is like the electric field, i.e. $F_E = q\vec{E}$. Hence, from (226) we find that far away from the source the force behaves as $F_E \propto r^{-2}$, similar to that of a gravitational field. Close to the source however, the electric field is proportional to $r^{-2/3}$. In this regime, the gravitational force diverges much quicker, effectively suppressing any effect from the force mediated by the vector field, see figure (9). The ratio between the two goes as

$$\frac{F_E}{F_G} \propto \frac{r^{-2/3}}{r^{-2}} = r^{-4/3}. \tag{228}$$

As r decrease the ratio goes to zero, and we can conclude that inside the boundary radius r_c, the vector field is efficiently screened as shown in figure (10). Thus far we have successfully adopted the kinetic screening mechanism originally developed for kinetic scalar field theories.

4.0.1 Generalization

The screening mechanism illustrated above can be applied to a more generalized Lagrangian. For instance, consider a Lagrangian $\mathcal{L} = f(X)/\Lambda_\nu^4$ where $X = (1/4)F_{\mu\nu}F^{\mu\nu}$ and Λ_ν is some energy scale. Variation with respect to X leads to

$$\delta\mathcal{L} = f_X \delta X + \delta\mathcal{L}_m = \nabla(f_X E) + \delta\mathcal{L}_m \tag{229}$$

where $\mathcal{L}_m$ is the matter Lagrangian. Around a static, spherically symmetric source we use the same anzats as before, where $\mathcal{L}_m$ consists of a current $J^\mu = (\rho, \vec{0})$ coupled to matter. Integrating over space and using Gauss law gives

$$|f_X E| = \frac{Q}{4\pi r^2} \tag{230}$$

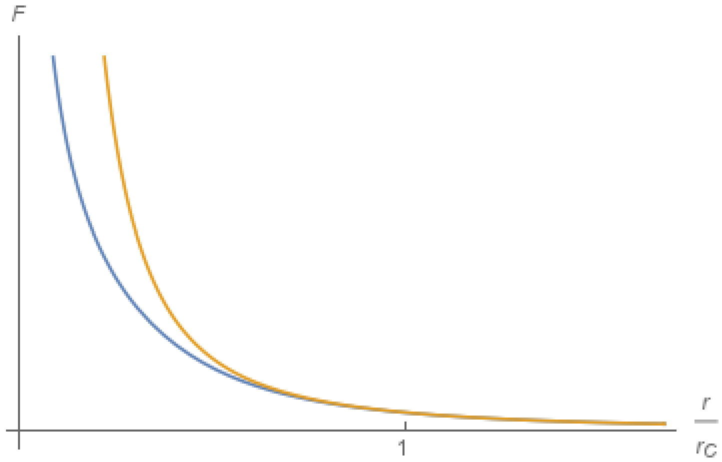

Figure 9: In the image, force is plotted versus r/r_c where r_c is the boundary between the two regimes. The blue line represent the force from the vector field and the orange represent the gravitational force. At low r, the force due to gravity diverges much quicker and hence suppress the force from the vector field.

As we have seen earlier, since $E \propto 1/r^n$ we have $X << 1$ far away from the source. In this region we assume that $f(X) \simeq X$ so that we recover Maxwell equations. Generally, as a rule of thumb we should always recover Maxwell in flat spacetime to ensure that the theory is consistent. Also, in certain classes of vector theories the only ghost free Lagrangian in flat spacetime is the standard Maxwell Lagrangian. For $r >> r_c$ we have that

$$E \propto \frac{1}{r^2} \tag{231}$$

Closer to the source, when $r << r_c$ we, expect higher order terms to dominate, $f(X) \simeq X^\alpha$. Then, from equation (230), we have

$$|f_X E| = |\alpha X^{\alpha-1} E| = |\alpha E^{3(\alpha-1)}| \Rightarrow E \propto \left(\frac{1}{r}\right)^{\frac{2}{3(\alpha-1)}} \tag{232}$$

The crossover between the two regimes happen at $X \sim 1$. Hence we find that for vector theories involving non-linear terms, or rather for $\alpha > 1$, the kinetic screening process is effective.

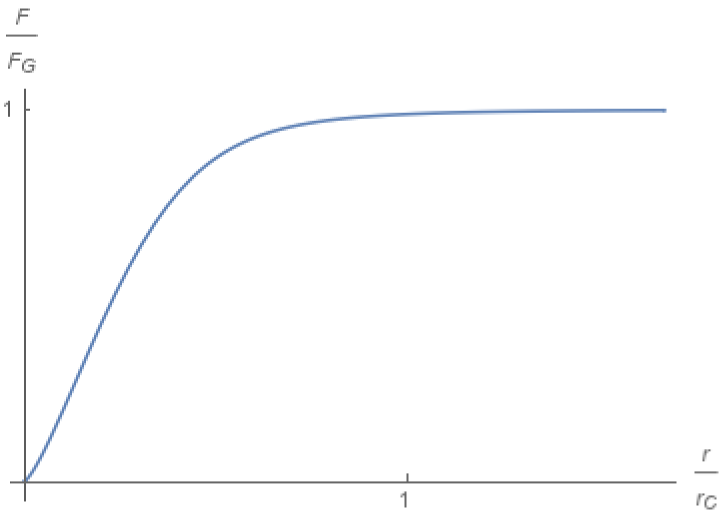

Figure 10: Here the ratio of the vector field force to that of the gravitational force is plotted. Once r reach below the cross over limit r_c, the ratio goes to zero fast. Far from the source the force from the vector field follow $1/r^2$, like gravity. Hence the ratio remains constant as $r \to \infty$.

5 Stability analysis

In order to ensure that a theory is reasonably applicable in cosmology, it should undergo stability tests of some form. While its evolution might produce the right epochs at the right time, it is imperative that it happens without violating basic physical principles. We have already without further explanation discussed models which produce instabilities such as ghost degrees of freedom or superluminal propagation of the field. Such pathologies should in general be avoided, hence in a theory it is important to identify which scenarios allow these kind of unstable solutions.

There are certain energy conditions we can impose on a theory which ensure that the solutions stay within reasonable boundaries, and shuts out solutions that is regarded as physically unrealistic and weird. Such energy conditions impose certain criteria that can be applied to the matter content in a cosmological model, which is usually bestowed by the energy-momentum tensor. There are several conditions, however they do force severe restrictions. As such, there are many matter configurations which violate some of them in order to achieve the desired outcome. For instance, the strong energy condition stipulates that for any future pointing time-like vector field $\vec{X}$, the trace of the tidal tensor measured by corresponding observers is always non-negative,

$$\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) X^\mu X^\nu \geq 0 \tag{233}$$

where the tidal tensor on the left hand side describes the tidal forces, or in the sense of General Relativity static gravity, and is part of the Riemann curvature tensor. Several theories of cosmic acceleration as well as any inflationary theory in cosmology violate this condition. The weak energy condition stipulates that for any time-like vectorfield $\vec{X}$, the matter density observed the corresponding observers is always non-negative,

$$\rho = T_{\mu\nu} X^\mu X^\nu \geq 0. \tag{234}$$

In addition, the dominant energy condition states that for any casual vector field (either time-like or null), the vector field $-T^\mu_\nu Y^\nu$ must be a future pointing casual vector. Hence mass-energy can never be observed to flow faster than light. As mentioned above, in some scenarios violating a condition is physically accepted and even necessary, however in doing so it is imperative to undergo proper stability analysis.

5.1 Null energy condition

The conditions mentioned above implicate the weakest of the energy conditions, widely believed that any physical system should respect, namely the null energy condition (NEC). It states that the energy momentum tensor satisfy

$$T_{\mu\nu} n^\mu n^\nu \geq 0 \tag{235}$$

for any light-like vector n^μ which accept $g_{\mu\nu} n^\mu n^\nu = 0$, in other words a null vector. In a homogeneous, isotropic universe with a FLRW background,

$$ds^2 = dt^2 - a(t)^2 \gamma_{ij} dx^i dx^j \tag{236}$$

the non-vanishing components of the energy momentum tensor are,

$$T_{00} = \rho \tag{237}$$

$$T_{ij} = a^2 \gamma_{ij} p \tag{238}$$

where ρ is the energy density and p is the pressure. The Einstein equations can thus be written as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \tag{239}$$

$$\dot{H} = -4\pi G(\rho + p) + \frac{\kappa}{a^2}. \tag{240}$$

The condition (235) implies we need to choose a set of null-vectors. Choosing $n^\mu = (1, a^{-1}\nu^i)$, where $\gamma_{ij}\nu^i\nu^j = 1$, one finds that in the cosmological setting NEC can be translated to the condition

$$\rho + p \geq 0. \tag{241}$$

We see that from the conservation of the energy momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, gives

$$\frac{d\rho}{dt} = -3H(\rho + p) \tag{242}$$

Thus because of condition (241), NEC indicate that in an expanding universe the energy density always decrease.

The null energy condition impose some interesting consequences to the Universe. First of it is a crucial component of the singularity theorem, a set of solutions in General Relativity which addresses singularities produced by gravity. It states that a universe described by General Relativity and with matter respecting NEC, a contracting universe always ends up as a singularity. Similarly an expanding universe has a singularity in the past, which in our Universe is the Big Bang.

As an intriguing alternative to such a singularity, a bouncing universe scenario has been suggested. A bouncing universe would mean that the universe first contracted (negative k) to a point in time and then resumed to expand. In other words $\dot{H}$ would go from negative to positive. However if the spatial curvature term k was negative and NEC holds, we see from equation (240) that $\dot{H}$ would remain negative. Hence in order to switch the sign of $\dot{H}$, NEC must be violated. Otherwise a bouncing universe scenario is forbidden.

In the case of dark energy, it is possible to realize an accelerated expansion within the boundaries of NEC. With negative pressure, we find that the condition (241) holds if $\rho \geq p$. Hence an equation of state $w \geq -1$ is allowed. The cosmological constant model for instance, where $w = -1$, respects NEC.

Situations may rise in which one could be motivated to finding healthy theories in which NEC could be violated. However, violating NEC often induce pathologies like ghost fields, tachyonic instabilities or gradient instabilities. As an example, consider a scalar field ϕ in flat Minkowski space. If there are any pathologies, they will show up in the conduct of small perturbations about the background, $\phi = \bar{\phi} + \chi$. Typically, if the linearised field equations for χ is of second order in derivatives, the quadratic Lagrangian takes the form

$$L_{\chi}^{(2)} = \frac{1}{2}U\dot{\chi}^2 - \frac{1}{2}V(\partial_i\chi)^2 - \frac{1}{2}W\chi^2 \tag{243}$$

where U, V and W are time dependent quantities. The dispersion relation for conventional excitations is

$$U\omega^2 = V\mathbf{p}^2 + W. \tag{244}$$

and the energy density for perturbations is positive,

$$T_{00}^{(2)} = \frac{1}{2}U\dot{\chi}^2 + \frac{1}{2}V(\partial_i\chi)^2 + \frac{1}{2}W\chi^2. \tag{245}$$

If we consider the high-momentum regime, meaning that χ varies much more frequent in time and space than the variations of the background $\bar{\phi}$, we can neglect the time dependence of those quantities. This simplifies much and we can directly read of the stability conditions from the equations above. In the following sections we will present various configurations of U, V and W which lead to pathologies at the level of the effective field theory.

5.1.1 Gradient instability

$U > 0$, $V < 0$, or $U < 0$, $V > 0$

At high momenta, the dispersion relation (244) cause imaginary ω. This causes instability in the background since the perturbations grow arbitrary fast. Such an instability is called gradient instability, or Laplacian instability in terms of vector field theories. This is related to the wrong sign of the spatial gradients. As an example take the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - c_s^2|\vec{\nabla}\phi|^2 - m^2\phi^2). \tag{246}$$

with the equation of motion $\ddot{\phi} - c_s^2\nabla^2\phi + m^2\phi^2 = 0$. In Fourier space, we find solutions on the form

$$\phi_\omega \sim e^{i\omega t} \tag{247}$$

which gives the corresponding equation of motion $\omega^2 = c_s^2k^2 + m^2$, which for high momenta, $\omega^2 \simeq c_s^2k^2$. Wrong sign to the spatial gradient terms thus yield imaginary ω. In conclusion, theories involving gradient instabilities are not healthy. Even while considering the scalar theory as an effective field theory where the rate of variation of the background is well below the UV-cutoff scale Λ, the rate of variation of the instabilities extend up to Λ, causing an unstable situation.

5.1.2 Tachyonic instability

$U > 0$, $V > 0$, $W < 0$

As we found above, when the dispersion relation yields imaginary roots it is usually not a good sign. For sufficiently low momentum, $V\mathbf{p}^2 < W$, the dispersion relation (244) also cause imaginary ω which may lead to tachyonic instabilities. Simply put, the instabilities is a result of negative sign to the mass squared term. With the same argument we used for gradient instabilities, $k << m$ at low momentum result result in imaginary ω. It is a case which stability is difficult to analyse intuitively. Out of the high-momentum regime the time-dependence of $U(t)$, $V(t)$ and $W(t)$ cannot be neglected. At short time scales the background is stable, but at long time scales it is difficult to predict and a more careful analyse is necessary. However, we can declare that tachyonic instabilities does not indicate any clear pathology to the theory such as other instabilities. If we focus on high momentum regimes, the theory is insensitive to the fact that system is unstable.

5.1.3 Ghost

$U < 0$, $V < 0$

In the classical regime the background in this case is stable, the dispersion relation give real ω. However, in the quantum mechanic realm it is unstable. Unlike the gradient instability, this involves the wrong sign of the temporal terms as well. The negativity of U and V give rise to negative energy in equation (245), which in turn result in ghost instability. Naturally, the sign of the kinetic terms is merely a topic of convention. However trouble arise if there exists a coupling to another field ϕ whose kinetic terms have the right sign. The quantized χ particles will have negative energies, causing an unstable vacuum. While energy conservation indeed allows for creation of particles with negative energies together with normal particles, in this case the process cost zero energy and thus the pair production happen at an infinite rate. For a sufficiently low UV-cutoff scale Λ it could be accepted. Otherwise, which is generally the case, ghosts are unwanted and considered pathologies. Theories like the ghost condensate model in the k-essence class, which we discussed earlier, revolve around this kind of negative kinetic terms.

5.1.4 Stable background

For $U > 0$, $V > 0$, $W \geq 0$ we have a seemingly stable background. Even though instabilities manifest themselves in the effective theory we consider, there might be pathologies of a more subtle nature. Most commonly is the presence of superluminality. We can find the propagation speed of the field by

$$c_s = \frac{V}{U}. \tag{248}$$

Hence if $V > U$ the χ-waves are superluminal. This could be problematic since it breaks causality, and in doing so prevents the theory to be in the effective field theory regime. If $U = V$ the χ-waves travel at the speed of light, which is also not completely safe, since it opens the possibility that there may be other backgrounds in the near surroundings about which the perturbations are superluminal. The reliable case would be when $U > V > 0$, at which the perturbations are subluminal. The special case is when $U > 0$ and $V = 0$, where higher order corrections usually suppressed by the effective field theory framework cannot be neglected. The reason is that only these corrections give rise to the spatial gradients for the perturbations in the Lagrangian. This does however results in a Lagrangian on a different form which dominant terms involves higher derivatives. Considering what we discussed in the above sections we can conclude that in order to maintain a safe and stable theory we need that the following condition hold:

$$U > V > 0 \tag{249}$$

and the absence of tachyons at low momenta depend on positive definiteness of W.

5.2 NEC violation with scalar field theories

Above we identified some scenarios which lead to pathologies we would like to avoid. It can help us to find out if certain solutions we find prove to be a healthy solutions, even if they violate NEC. To illustrate the violation of NEC in practice, consider a scalar field theory with the general Lagrangian $\mathcal{L} = P(X)$ where $X = \partial_\mu \phi \partial^\mu \phi$ and the energy momentum tensor is

$$T_{\mu\nu} = P_{,X} \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}. \tag{250}$$

In order to see which scenarios that obey this inequality it is useful to derive the quadratic Lagrangian for perturbations of the field. Considering perturbations about the scalar $\phi \to \bar{\phi} + \delta\phi$ gives

$$\begin{aligned} \Rightarrow X &= \partial_\mu (\bar{\phi} + \delta\phi) \delta^\mu (\bar{\phi} + \delta\phi) \\ &= (\partial_\mu \bar{\phi} + \partial_\mu \delta\phi)(\partial^\mu \bar{\phi} + \partial^\mu \delta\phi) \\ &= \partial_\mu \bar{\phi} \partial^\mu \bar{\phi} + 2\partial_\mu \bar{\phi} \partial^\mu \delta\phi + \partial_\mu \delta\phi \partial^\mu \delta\phi \\ &= \bar{X} + \delta X. \end{aligned} \tag{251}$$

Expanding the perturbed Lagrangian $\mathcal{L}$ to second order gives

$$\mathcal{L} = P(\bar{X} + \delta X) = \bar{P} + \bar{P}_{,X} \delta X + \frac{1}{2} \bar{P}_{,XX} \delta X^2. \tag{252}$$

where we have $\delta X = 2\partial_\mu \bar{\phi} \partial^\mu \delta\phi + \partial_\mu \delta\phi \partial^\mu \delta\phi$ and

$$\begin{aligned} \delta X^2 &= (2\partial_\mu \bar{\phi} \partial^\mu \delta\phi + \partial_\mu \delta\phi \partial^\mu \delta\phi)^2 \\ &= 4\partial_\mu \bar{\phi} \partial^\mu \delta\phi \partial_\nu \bar{\phi} \partial^\nu \delta\phi \\ &\quad + 4\partial_\mu \bar{\phi} \partial^\mu \delta\phi \partial_\nu \delta\phi \partial^\nu \delta\phi \\ &\quad + \partial_\mu \delta\phi \partial^\mu \delta\phi \partial_\nu \delta\phi \partial^\nu \delta\phi \\ &\approx 4\partial_\mu \bar{\phi} \partial^\mu \delta\phi \partial_\nu \bar{\phi} \partial^\nu \delta\phi. \end{aligned} \tag{253}$$

The approximation that the two latter terms can be ignored is due to the higher order of the perturbations. When $\delta\phi$ is higher than second order it is small enough and can be neglected. The expanded Lagrangian now reads

$$\mathcal{L} = \bar{P} + \bar{P}_{,X}(2\partial_\mu\bar{\phi}\partial^\mu\delta\phi + \partial_\mu\delta\phi\partial^\mu\delta\phi) + 2\bar{P}_{,XX}\partial_\mu\bar{\phi}\partial^\nu\partial_\nu\bar{\phi}\partial^\nu\delta\phi. \quad (254)$$

A closer look at the parenthesis reveals that it can be simplified further. In the unperturbed case we would have that the variation of the Lagrangian equals $\delta\mathcal{L} = \delta P(X) = P_{,X}\delta X$ which is zero by the principle of least action. We can identify the first term in the parenthesis as exactly the variation of the unperturbed Lagrangian, or rather the equation of motion in the unperturbed case. Hence this term can be neglected. The expanded Lagrangian can thus be written as

$$\mathcal{L} = \bar{P} + \bar{P}_{,X}\partial_\mu\delta\phi\partial^\mu\delta\phi + 2\bar{P}_{,XX}\partial_\mu\bar{\phi}\partial^\nu\partial_\nu\bar{\phi}\partial^\nu\delta\phi. \quad (255)$$

The quadratic Lagrangian now takes the form

$$\mathcal{L}^{(2)} = (\bar{P}_{,X}g_{\mu\nu} + 2\bar{P}_{,XX}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi})\partial^\mu\delta\phi\partial^\nu\delta\phi. \quad (256)$$

In terms of components the equation becomes

$$\mathcal{L}^{(2)} = (\bar{P}_{,X}g_{00} + 2\bar{P}_{,XX}\dot{\bar{\phi}}^2)\dot{\delta\phi}^2 + \bar{P}_{,X}|\nabla\delta\phi|^2. \quad (257)$$

Because we assume a homogeneous background, the derivatives of the spatial background components vanishes. From equation (235) we have that NEC is violated if $T_{\mu\nu}n^\mu n^\nu < 0$. Converted into the energy momentum tensor in question (from equation (250)), we have that NEC is violated if

$$T_{\mu\nu}n^\mu n^\nu = P_{,X}\dot{\phi}^2(n^0)^2 < 0 \quad (258)$$

which translates into the condition $P_{,X} < 0$. If we take a look at the quadratic Lagrangian (257) we see that a negative $P_{,X}$ gives a negative sign in front of the gradient term. Hence it will cause gradient instability, and we find that the violation of NEC in this case is not healthy.

5.3 Stability of vector field theories

In the following we will analyse the stability of a vector field theory, identify the conditions that violate NEC and see if they indeed prove to be stable solutions.

Consider a Lagrangian $\mathcal{L} = K(Y)$ where K is a function of $Y = F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}$ is the electromagnetic field tensor. Considering perturbations about the background $Y \rightarrow \bar{Y} + \delta Y$ gives

$$Y + \delta Y = (\bar{F}_{\mu\nu} + \delta F_{\mu\nu})(\bar{F}^{\mu\nu} + \delta F^{\mu\nu}) = \bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + 2\bar{F}_{\mu\nu}F^{\mu\nu} + \delta F_{\mu\nu}\delta F^{\mu\nu} \quad (259)$$

where we identify $\delta Y = 2\bar{F}_{\mu\nu}\delta F^{\mu\nu} + \delta F_{\mu\nu}\delta F^{\mu\nu}$. Expanding the Lagrangian to second order,

$$\mathcal{L} = K(\bar{Y} + \delta Y) = \bar{K} + \bar{K}_Y \delta Y + \frac{1}{2}\bar{K}_{YY}\delta Y^2 \quad (260)$$

where $\bar{K}_Y = d\bar{K}/dY$ and $\bar{K}_{YY} = d\bar{K}^2/d^2Y$. The perturbation squared δY^2 can be approximated to

$$\delta Y^2 = (2\bar{F}_{\mu\nu}\delta F^{\mu\nu} + \delta F_{\mu\nu}\delta F^{\mu\nu})^2 \approx 4\bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}\delta F^{\mu\nu}\delta F^{\alpha\beta}. \quad (261)$$

Plugging this into the expanded Lagrangian gives

$$\mathcal{L} = \bar{K} + \bar{K}_Y(2\bar{F}_{\mu\nu}\delta F^{\mu\nu} + \delta F_{\mu\nu}\delta F^{\mu\nu}) + 2\bar{K}_{YY}\bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}\delta F^{\mu\nu}\delta F^{\alpha\beta}. \quad (262)$$

The first term in the parenthesis vanishes by the same argument as in the scalar field case. In the unperturbed case the variation of the action gives $\delta\mathcal{L} = \delta(K(Y)) = K_Y 2F_{\mu\nu}\delta F^{\mu\nu}$ which is zero by the principle of least action. Hence the term is zero. The Lagrangian can be rewritten as

$$\mathcal{L} = \bar{K} + \bar{K}_Y \delta F_{\mu\nu}\delta F^{\mu\nu} + 2\bar{K}_{YY}\bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}\delta F^{\mu\nu}\delta F^{\alpha\beta}. \quad (263)$$

The quadratic Lagrangian thus takes the form

$$\mathcal{L}^{(2)} = \bar{K}_Y \delta F_{\mu\nu}\delta F^{\mu\nu} + 2\bar{K}_{YY}\bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}\delta F^{\mu\nu}\delta F^{\alpha\beta} \quad (264)$$

which written in components is reduced to

$$\Rightarrow \mathcal{L}^{(2)} = \bar{K}_Y(2\delta F_{0i}\delta F^{0i} + \delta F_{ij}\delta F^{ij}) \\ +2\bar{K}_{YY}(2\bar{F}_{0i}\bar{F}_{0k}\delta F^{0i}\delta F^{0k} \\ +\bar{F}_{ij}\bar{F}_{kl}\delta F^{ij}\delta F^{kl}) \tag{265}$$

where $F_{0i} = E_i$ and $F_{ij} = \varepsilon_{ijk}B_k$. We can separate between two cases. In the first case the background is dominated by the electric field, $\bar{F}_{\mu\nu} \to \bar{F}_{0i} = E_i$, and the background magnetic field is neglected, $\bar{F}_{ij} = 0$,

$$\Rightarrow \mathcal{L}^{(2)} = \bar{K}_Y(2\delta F_{0i}\delta F^{0i} + \delta F_{ij}\delta F^{ij}) + 4\bar{K}_{YY}\bar{F}_{0i}\bar{F}_{0k}\delta F^{0i}\delta F^{0k} \\ = \bar{K}_Y(-2(\delta\mathbf{E})^2 + 2(\delta\mathbf{B})^2) + 4\bar{K}_{YY}\mathbf{E}^2(\delta\mathbf{E})^2 cos^2\theta \tag{266}$$

where θ is the angle between the perturbed field and the background. In the other case the background is dominated by the magnetic field, $\bar{F}_{\mu\nu} \to \bar{F}_{ij} = \varepsilon_{ijk}B_k$, and hence the background electric field vanishes, $\bar{F}_{0i} = 0$,

$$\mathcal{L}^{(2)} = \bar{K}_Y(2\delta F_{0i}\delta F^{0i} + \delta F_{ij}\delta F^{ij}) + 2\bar{K}_{YY}\bar{F}_{ij}\bar{F}_{kl}\delta F^{ij}\delta F^{kl}) \\ = \bar{K}_Y(-2(\delta\mathbf{E})^2 + 2(\delta\mathbf{B})^2) + 2\bar{K}_{YY}(\mathbf{B}\times\delta\mathbf{B})^2. \tag{267}$$

Here we have used that $\varepsilon_{ijk}\varepsilon_{ijl} = 2\delta^k_l$, and the definition of the Levi-Civita symbol to manipulate the vector product,

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\varepsilon_{ijk}\mathbf{B}_j\delta\mathbf{B}_k = (\mathbf{B}\times\delta\mathbf{B})_i. \tag{268}$$

Now that we have calculated the quadratic Lagrangian for the perturbations, it remains to derive the energy momentum tensor in order to find out which solutions violate NEC. Suppose we have $\mathcal{L} = -(1/4)F_{\mu\nu}F^{\mu\nu}$ in flat Minkowski space, with the equation of motion $\partial_\mu F^{\mu\nu}\delta A_\nu = 0$. From Noether's theorem [29] we have the energy momentum tensor

$$T^\mu_\nu = -\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\lambda)}\partial_\nu A_\lambda + \mathcal{L}\delta^\mu_\nu \tag{269}$$

Expanding the Lagrangian into

$$K(Y) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\nu A^\mu) \tag{270}$$

we see that

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\lambda)} = -F^{\mu\lambda}. \tag{271}$$

Thus equation (270) turns into

$$T^{\mu\nu} = F^{\mu\lambda}\partial^\nu A_\lambda - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}. \tag{272}$$

This energy momentum tensor is obviously not symmetric in the indices μ and ν, as it should be. In order to make it so, we improve the equation by adding a term, $\partial_\kappa f^{\kappa\mu\nu}$, where $f^{\kappa\mu\nu}$ is antisymmetric in its first two indices, and hence is divergenceless. We choose $f^{\kappa\mu\nu} = F^{\kappa\mu}A^\nu$ such that the energy momentum tensor takes the form

$$T^{\mu\nu} = F^{\mu\lambda}F^\nu_\lambda - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \tag{273}$$

which is the energy momentum tensor for electromagnetism. However, in our case we would like to be more general, with $\mathcal{L} = K(Y)$. The energy momentum tensor we will use therefore takes the form

$$T_{\mu\nu} = -K_Y F_{\mu\alpha}F^\alpha_\nu + K g_{\mu\nu} \tag{274}$$

If we recall the condition (235) we see that NEC is violated if

$$T_{\mu\nu}n^\mu n^\nu < 0. \tag{275}$$

The null vectors are by definition vectors which when multiplied with the metric give zero. Hence when multiplying equation (275) with the null vectors we are left with

$$
\begin{aligned}
T_{\mu\nu}n^{\mu}n^{\nu} &= -K_Y F_{\mu\alpha}F^{\alpha}_{\nu}n^{\mu}n^{\nu} \\
&= -K_Y(F_{0i}F^{i}_{0}n^0n^0 + F_{i0}F^{0}_{j}n^in^j + F_{ik}F^{k}_{j}n^in^j) \\
&= -K_Y(F_{0i}F^{0i}g_{00}n^0n^0 + F_{i0}F^{i0}g_{ij}n^in^j + F_{ik}F^{ik}g_{ij}n^in^j) \\
&= -K_Y(\mathbf{E}^2(n^0)^2 - \mathbf{E}^2n^2cos^2\theta + (\mathbf{B}\times n)^2).
\end{aligned} \tag{276}
$$

Because it is a null vector, the time component $(n^0)^2$ is equal in magnitude to its spatial components n^2. Thus the only way for this to be negative, and for NEC to be violated, is if $K_Y > 0$.

In order to see if the violation of NEC is healthy or if it produce pathologies we look at the quadratic Lagrangian and see the consequences this criteria gives. In the quadratic Lagrangian (267) we can associate $(\delta\mathbf{E})^2$ with the photon energy of the perturbed field. We find that with a positive K_Y gives negative sign to $(\delta\mathbf{E})^2$ in the first term. Hence it is likely to produce ghosts. One might argue that if $K_{YY} >> K_Y$, the ghost terms will be suppressed. However since the K_{YY} term depend on $cos\theta$, there will be waves propagating in directions such that $cos\theta$ is small. The term with K_{YY} could therefore in fact always be suppressed by θ. Thus in these vector field theories the violation of NEC is not healthy.

5.3.1 Stability of screened solutions

To finish of this section we will further analyse the stability of the screened solutions we derived earlier in order to ensure their viability. Although we have shown that while bounded by NEC we avoid ghosts, it is imperative to investigate if the introduced screening mechanism is affected by other kinds of instabilities. As such we must require that the propagation speed of the perturbation $c_s \geq 0$ in order to avoid Laplacian instabilities. Furthermore to prevent superluminal propagation we also need $c_s \leq 1$. For the electric background we have the propagation speed

$$
c_s^2 = \frac{2\bar{K}_{,Y}}{2\bar{K}_{,Y} + 4\bar{K}_{YY}\mathbf{E}^2cos^2\theta} = \frac{1}{1 + 2\frac{\bar{K}_{YY}}{\bar{K}_{,Y}}\mathbf{E}^2}. \tag{277}
$$

Since we are interested in the radial direction of the source the perturbation propagates parallel to the background field, hence the angle between them is zero. Recall that far away from the source the electric field is proportional to r^{-2}. Since r is large, the second term in the denominator of (278) is suppressed. Thus $c_s^2 \sim 1$ and the perturbation will propagate as a free electromagnetic field.

In the close vicinity of the source where the screening mechanism kicks inn, the electric field behave as

$$E \propto \left(\frac{1}{r}\right)^{\frac{2}{3(\alpha-1)}} \tag{278}$$

with $\alpha > 1$. When inserting this, equation (278) read

$$c_s^2 = \frac{1}{1 + 2\frac{\bar{K}_{YY}}{K_{,Y}}\left(\frac{1}{r}\right)^{\frac{2}{3(\alpha-1)}}}. \tag{279}$$

Recall that the crossover scale between the two regimes is ~ 1. Thus when the screening mechanism kicks in, $r << 1$ will force the propagation speed towards zero. However, since $K_{,Y}$ is negative there is a significant risk of encountering Laplacian instabilities. In fact, unless $K_{,YY}$ is negative as well then $c_s^2 < 0$. For instance, the more specific model from which we derived the screening solutions where

$$K(Y) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{\Lambda_\nu^4}(F_{\mu\nu}F^{\mu\nu})^2 \tag{280}$$

gives $K_{,YY} = 2c_1/\Lambda_\nu^4$. Hence in order to avoid Laplacian instability we must require $c_1 < 0$.

In conclusion, although they avoid superluminal propagation since both asymptotic regimes satisfy $c_s^2 \leq 1$, the constraint from NEC force the negative definiteness of $K_{,YY}$ in order to prevent Laplacian instabilities.

6 Conclusion

Screening mechanisms have previously been implemented in some scalar field theories. The procedures we have reviewed here includes the chameleon, symmetron, k-mouflage and the Vainshtein mechanism. The intention of this thesis has been to investigate the potential of applying these methods to certain vector field theories. In detail we have analysed the k-mouflage mechanism originally intended for scalar field theories with non-canonical kinetic terms, and attempted to apply the same principles to general vector theories of the form $f(F^2)$ and $f(F^2, F\tilde{F})$. We found that the conversion was successful in the simple case of a static, spherically symmetric object, and the result resemble that of the scalar field.

Further, we proceeded to analyse the stability of the vector theories in question. We found that the condition for the absence of ghosts is the same as the requirement of satisfying NEC. Thus we cannot escape the presence of ghosts in solutions where NEC is violated and can conclude that the violation is not healthy. When analysing the stability around the obtained screened solutions, we found that the condition for NEC also implies a restriction on $K_{,YY}$. Since NEC indicate $K_{,Y} < 0$ we showed that the propagation speed takes negative values and hence produce Laplacian instabilities below the crossover scale, unless we require $K_{,YY} < 0$. As for superluminal propagation, both asymptotic cases are within the boundaries. In the far away regime, $c_s^2 \to 1$ the wave propagate freely, while close to the source $c_s^2 \to 0$ indicating the effect of the screening mechanism.

Related conversions of screening mechanisms between scalar field and vector field theories have been attempted in the past. In [24] a vector screening mechanism is inspired by the symmetron mechanism we have discussed, where the screening occurs due to a spontaneous symmetry breaking. The model successfully give rise to screening of the force mediated by the vector field. In addition the mechanism also effectively screen Lorentz violations in high dens regions, which is a distinctive signature with respect to scalar fields. This result could indicate unique opportunities revealed from vector field theories. In general they offer interesting development in the field of cosmology. For instance, the intriguing possibility of the potential existence of anisotropies in the CMB that could point to the presence of a preferred direction in the Universe.

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