

A BOUNDARY QUOTIENT DIAGRAM FOR RIGHT LCM SEMIGROUPS

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ABSTRACT. We propose a boundary quotient diagram for right LCM semigroups with property (AR) that generalizes the boundary quotient diagram for $\mathbb{N} \rtimes \mathbb{N}^\times$. Our approach focuses on two important subsemigroups: the core subsemigroup and the semigroup of core irreducible elements. The diagram is then employed to unify several case studies on KMS-states, and we end with a discussion on K -theoretical aspects of the diagram motivated by recent findings for integral dynamics.

1. INTRODUCTION

A countable discrete semigroup S is called *right LCM* if it is left cancellative and the intersection of two principal right ideals in S is either empty or another principal right ideal. The terminology alludes to the existence of right least common multiples given the existence of any common right multiple. It is known that a left cancellative semigroup S is right LCM if and only if Li's family of constructible right ideals $\mathcal{J}(S)$ is given by \emptyset and the principal right ideals in S , see [Li12, BLS17]. We shall assume that the semigroup S is not only right LCM and unital, but also has property (AR), which is explained in Section 2. But we remark here that no example of a right LCM semigroup without this property is known so far, see [BS16].

A classical example of a right LCM semigroup S is $\mathbb{N} \rtimes \mathbb{N}^\times$. Within the operator-algebraic context, this example is treated in great detail in the celebrated work [LR10], where Laca and Raeburn studied the Toeplitz algebra for the quasi-lattice ordered pair $(\mathbb{Q} \rtimes \mathbb{Q}_+^\times, \mathbb{N} \rtimes \mathbb{N}^\times)$, in particular with regards to its KMS-state structure for a natural dynamics. They also considered the boundary quotient $\mathcal{Q}(\mathbb{N} \rtimes \mathbb{N}^\times)$ of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ in the sense of Crisp and Laca [CL07]. Using a suitable presentation for $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ by generators and relations, they show that $\mathcal{Q}(\mathbb{N} \rtimes \mathbb{N}^\times)$ is obtained from $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ by imposing two extra relations:

- (a) The isometry corresponding to $(1, 1) \in \mathbb{N} \rtimes \mathbb{N}^\times$ is a unitary.
- (m) For every prime $p \in \mathbb{N}^\times$, the isometries for $\{(m, p) \mid 0 \leq m \leq p - 1\}$ form a Cuntz family, that is, their range projections sum up to one.

As a straightforward consequence, $\mathcal{Q}(\mathbb{N} \rtimes \mathbb{N}^\times)$ coincides with Cuntz's $\mathcal{Q}_{\mathbb{N}}$ from [Cun08].

The results of [LR10] were extended in [BaHLR12] by introducing and analysing two complementary intermediate quotients between $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ and $\mathcal{Q}(\mathbb{N} \rtimes \mathbb{N}^\times)$: the additive boundary quotient $\mathcal{T}_{\text{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$, obtained by imposing (a), and the multiplicative

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boundary quotient $\mathcal{T}_{\text{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times)$, obtained by imposing (m), see [BaHLR12, Proposition 3.3]. Altogether, these form the *boundary quotient diagram* for $\mathbb{N} \rtimes \mathbb{N}^\times$:

$$\begin{array}{ccc} \mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times) & \longrightarrow & \mathcal{T}_{\text{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times) \\ \downarrow & & \downarrow \\ \mathcal{T}_{\text{add}}(\mathbb{N} \rtimes \mathbb{N}^\times) & \longrightarrow & \mathcal{Q}(\mathbb{N} \rtimes \mathbb{N}^\times) \end{array}$$

This diagram was shown to exhibit interesting features with respect to KMS-states, see [BaHLR12, Section 4].

By now, several works on KMS-state structures on Toeplitz type algebras and their quotients have been influenced, if not very much inspired by the approach in [LR10], see for instance [LRR11, LRRW14, CaHR]. Somewhat intriguingly, the treatment for Baumslag-Solitar monoids $BS(c, d)^+$ features a boundary quotient diagram in disguise, see [CaHR, Corollary 5.3]. For both $\mathbb{N} \rtimes \mathbb{N}^\times$ and $BS(c, d)^+$, the choices of the intermediate quotients are natural, yet based on the particular presentation of the semigroup. In addition, each of the aforementioned accounts on KMS-state structures remains an isolated case study for a specific family of right LCM semigroups, even though the similarities with regards to results and methods of proof are apparent.

One central aim of this work is to overcome this deficiency by introducing a boundary quotient diagram (2.2) for every right LCM semigroup with property (AR) that allows us to display the results on KMS-states from [LR10, BaHLR12, LRR11, LRRW14, CaHR] in a unified manner, see Theorem 4.1.

A convenient framework for this is provided through Li's theory [Li12] of full semigroup C^* -algebras $C^*(S)$ and the notion of a boundary quotient $\mathcal{Q}(S)$ for right LCM semigroups S from [BRRW14], as these constructions generalize the corresponding ones for quasi-lattice ordered groups. Thus the task reduces to identifying two natural intermediate quotients that complement each other in a suitable sense. To this end, we recall that $\mathcal{Q}(S)$ is obtained from $C^*(S)$ by imposing the boundary relation $\sum_{f \in F} e_{fs} = 1$ for all accurate foundation sets F , see Section 2 for details. The singleton foundation sets play a special role: They are given by the elements of the *core subsemigroup* $S_c = \{s \in S \mid sS \cap tS \neq \emptyset \text{ for all } t \in S\}$, and the boundary relation turns the corresponding generating isometries $v_s \in C^*(S)$ into unitaries. This will serve as our defining relation for the *core boundary quotient* $\mathcal{Q}_c(S)$, which generalizes $\mathcal{T}_{\text{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$.

The naive approach of defining the analogue of $\mathcal{T}_{\text{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ as the quotient of $C^*(S)$ by the boundary relation for every accurate foundation set F with $|F| \geq 2$, or equivalently $F \subset S \setminus S_c$, is bound to fail. Indeed, it is easy to see that we get nothing but $\mathcal{Q}(S)$ in this case. Therefore, we propose a slightly more elaborate version: We call an element $s \in S \setminus S_c$ *core irreducible* if every factorization $s = tr$ with $r \in S_c$ satisfies $r \in S^*$, where S^* denotes the subgroup of invertible elements in S . Since S is assumed left cancellative, the core irreducible elements form a subsemigroup S_{ci} of S . We say that a foundation set F is *proper* if F consists of core irreducible elements, and then define the *proper boundary quotient* $\mathcal{Q}_p(S)$ as the quotient of $C^*(S)$ by the boundary relation for all proper accurate foundation sets. Thus, the conditions

- (c) For every $s \in S_c$, the isometry v_s is a unitary.
- (p) For every proper accurate foundation set F , the Toeplitz-Cuntz family of isometries $(v_f)_{f \in F}$ is a Cuntz family.

replace (a) and (m) from $\mathbb{N} \rtimes \mathbb{N}^\times$ for a general right LCM semigroup with property (AR), and thus giving rise to the *boundary quotient diagram*

$$\begin{array}{ccc}
 C^*(S) & \longrightarrow & \mathcal{Q}_p(S) \\
 \downarrow & & \downarrow \\
 \mathcal{Q}_c(S) & \longrightarrow & \mathcal{Q}(S)
 \end{array}$$

We then show that, under mild assumptions, $\mathcal{Q}_c(S)$ and $\mathcal{Q}_p(S)$ are complementary quotients in between $C^*(S)$ and $\mathcal{Q}(S)$ in the sense that (c) and (p) together yield $\mathcal{Q}(S)$, see Proposition 2.10. As a continuation of considerations from [BLS], we then describe sufficient conditions under which semigroup homomorphisms between right LCM semigroups give rise to $*$ -homomorphisms between corresponding corners of the boundary quotient diagrams, see Remark 2.14.

In order to demonstrate the utility of our approach, various examples are discussed in Section 3. In particular, our boundary quotient diagram is shown to explain the appearance of the two intermediate quotients $C_A^*(U \rtimes A)$ and $C_U^*(U \rtimes A)$ in [BRRW14, Remark 5.4] for rather special Zappa-Szép products $U \rtimes A$ of right LCM semigroups, see Example 3.4. More importantly, our perspective indicates that the focus on the two components U and A is somewhat misleading: The reason why the two quotients agree with $\mathcal{Q}_c(U \rtimes A)$ and $\mathcal{Q}_p(U \rtimes A)$ is that the prescribed conditions U and A need to satisfy force $(U \rtimes A)_c = U^* \rtimes A_c$ and $(U \rtimes A)_{ci} = U_{ci} \rtimes A^*$, so that proper accurate foundation sets of $U \rtimes A$ are essentially determined by proper accurate foundation sets of U .

The structure of the subsemigroup of *core irreducible* elements S_{ci} is of independent interest. It appears to contain vital information on the semigroup S itself, see Remark 2.13. With regards to the associated C^* -algebras, we provide evidence that S_{ci} plays an important role in the quest for the K -theory of $\mathcal{Q}(S)$ and $\mathcal{Q}_p(S)$. This is discussed in Section 5 in the context of integral dynamics [BOS15] and Baumslag-Solitar monoids [Spi12].

We suspect the boundary quotient diagram to admit an elegant and equally useful description in the language of groupoids. In particular, this might be the key to studying a boundary quotient diagram for countable discrete left cancellative semigroups and to obtaining a vast generalization of Theorem 4.1. In this direction, the work of Laca and Neshveyev [LN11] may be crucial, and its appendix indicates the existence of a common theme for KMS-state structures of this kind. Having said that, we will not address this here for the sake of an elementary and brief exposition.

The paper is organized as follows: The boundary quotient diagram for right LCM semigroups with property (AR) is constructed in Section 2. The particular form of the boundary quotient diagram for a variety of examples is discussed in Section 3. As an application of the diagram, a unifying statement for the results on KMS-states from the

four different case studies [LR10, BaHLR12, LRR11, LRRW14, CaHR] is established in Section 4. In the final Section 5, these results are contrasted by K -theoretical considerations, where we review the torsion subalgebra for integral dynamics in order to present two candidates for an analogue of this subalgebra for other right LCM semigroups.

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2. CONSTRUCTION OF THE BOUNDARY QUOTIENT DIAGRAM

In [BRRW14], a boundary quotient $\mathcal{Q}(S)$ was introduced for right LCM semigroups S as the quotient of the full semigroup C^* -algebra $C^*(S)$ by the relation $\prod_{f \in F} (1 - e_{fS}) = 0$ for all foundation sets F for S . Recall that a finite subset F of S is called a *foundation set*, if, for every $t \in S$, there is $s \in F$ such that $sS \cap tS \neq \emptyset$. For convenience, let us denote the set of all foundation sets for S by $\mathcal{F}(S)$. This approach to defining $\mathcal{Q}(S)$ was inspired by the work of Crisp and Laca in the setting of quasi lattice-ordered groups [CL07]. Shortly thereafter, it was observed in [BS16] that a broad class of right LCM semigroups has the *accurate refinement property*, or property (AR) for short: For every $F \in \mathcal{F}(S)$, there is $F' \in \mathcal{F}(S)$ such that

- a) F' is *accurate* (fS and $f'S$ are disjoint for $f, f' \in F', f \neq f'$), and
- b) F' refines F (for every $f' \in F'$ there is $f \in F$ with $f' \in fS$).

If S has property (AR), then the boundary relation for $\mathcal{Q}(S)$ reduces to

$$(2.1) \quad \sum_{f \in F} e_{fS} = 1$$

for every accurate foundation set F , the collection of which we shall denote by $\mathcal{F}_a(S)$.

A simple, but important observation from [Star15] is the relevance of the *core subsemigroup*

$$S_c := \{s \in S \mid sS \cap tS \neq \emptyset \text{ for all } t \in S\},$$

whose origin can again be traced back to [CL07]. The semigroup S_c contains the group of units S^* , and forms a right reversible semigroup, that is, finite intersections of nonempty right ideals are nonempty. Hence S_c is a right Ore semigroup provided that it has right cancellation. With regards to the boundary quotient $\mathcal{Q}(S)$, we note that every generating isometry $v_s \in C^*(S)$ with $s \in S_c$ is turned into a unitary when passing to $\mathcal{Q}(S)$. This motivates the definition of the first intermediate quotient between $C^*(S)$ and $\mathcal{Q}(S)$.

Definition 2.1. The *core boundary quotient* $\mathcal{Q}_c(S)$ is the quotient of $C^*(S)$ by the relation $v_s v_s^* = 1$ for all $s \in S_c$.

One may be tempted to define the second intermediate quotient as the quotient of $C^*(S)$ by (2.1) restricted to (accurate) foundation sets that do not contain any element from the core S_c . However, this yields nothing but $\mathcal{Q}(S)$ as we shall now see. The starting point are the following two basic observations whose straightforward proofs are left to the reader.

Lemma 2.2. For $F_1, F_2 \subset S$, the set $F_1 \cdot F_2 := \{st \mid s \in F_1, t \in F_2\} \in \mathcal{F}(S)$ is an accurate foundation set if and only if F_1 and F_2 are accurate foundation sets.

If $F_i = \{s\}$ for some $s \in S$, we shall simply write $s \cdot F_2$ or $F_1 \cdot s$, respectively.

Lemma 2.3. Let $F_1, F_2 \in \mathcal{F}_a(S)$. Then the boundary relation (2.1) for both F_1 and F_2 is equivalent to the boundary relation (2.1) for $F_1 \cdot F_2$.

As a direct consequence of Lemma 2.2 and Lemma 2.3, we get:

Corollary 2.4. Let $F \in \mathcal{F}_a(S)$ and $s \in S_c$. Then $s \cdot F, F \cdot s \in \mathcal{F}_a(S)$, and the boundary relation (2.1) for $s \cdot F$ or $F \cdot s$ is equivalent to the boundary relation (2.1) for F and $v_s v_s^* = 1$.

By virtue of Corollary 2.4, we see that imposing the boundary relation (2.1) on $C^*(S)$ for all $F \in \mathcal{F}_a(S)$ with $F \cap S_c = \emptyset$ still yields $\mathcal{Q}(S)$. Thus we need a more sophisticated approach, to this end, recall that a non-invertible element s of a monoid S is said to be *irreducible* if $s \notin S^*$ and any decomposition $s = tr$ in T satisfies $t \in S^*$ or $r \in S^*$.

Definition 2.5. An element $s \in S \setminus S_c$ is called *core irreducible* if $s = tr$ for $t \in S$ and $r \in S_c$ implies $r \in S^*$. The set of core irreducible elements in S is denoted by S_{ci} .

Note that the core irreducible elements are minimal representatives of the equivalence classes in S/\sim , where $s \sim t$ if there are $r, r' \in S_c$ such that $sr = tr'$. In addition, let us remark that S_{ci} is a semigroup (without identity) as S is left cancellative, and we denote its unitization by S_{ci}^1 .

Definition 2.6. A foundation set F for S is called *proper* if $F \subset S_{ci}$. The set of accurate proper foundation sets is denoted by $\mathcal{F}_a^{(p)}(S)$.

Definition 2.7. The *proper boundary quotient* $\mathcal{Q}_p(S)$ is the quotient of $C^*(S)$ by the boundary relation (2.1) for all proper accurate foundation sets F .

We remark that Definition 2.7 does not cater for cases of type $s \cdot F$ with $s \in S_c \setminus S^*$ and $F \in \mathcal{F}_a^{(p)}(S)$ from Corollary 2.4 explicitly. The reason is that we always get $s \cdot F = F' \cdot s'$ for some $s' \in S_c \setminus S^*$ and $F' \in \mathcal{F}_a^{(p)}(S)$ in all the examples that we considered, see Section 3. This raises the question whether the definition of $\mathcal{Q}_p(S)$ ought to be modified:

Question 2.8. Is there a right LCM semigroup S (with property (AR)) for which there are $s \in S_c \setminus S^*$ and $F \in \mathcal{F}_a^{(p)}(S)$ with $s \cdot F \subset S_{ci}$, i.e. such that $s \cdot F \in \mathcal{F}_a^{(p)}(S)$?

With Definition 2.1 and Definition 2.7 at hands, we are ready for the main definition.

Definition 2.9. The *boundary quotient diagram* of a right LCM semigroup S is given by:

$$(2.2) \quad \begin{array}{ccc} C^*(S) & \xrightarrow{\pi_p} & \mathcal{Q}_p(S) \\ \pi_c \downarrow & & \downarrow \\ \mathcal{Q}_c(S) & \xrightarrow{\quad} & \mathcal{Q}(S) \end{array}$$

It is a natural question whether $\mathcal{Q}(S)$ can be obtained by imposing the relations for $\mathcal{Q}_p(S)$ on $\mathcal{Q}_c(S)$, and vice versa. The next proposition shows that this is indeed the case, given that all elements in S admit a factorization into a core irreducible and a core element. This holds true whenever S satisfies the *ascending chain condition* with respect to $\mathcal{J}(S)$, i.e. every ascending sequence of constructible right ideals becomes stationary. More precisely, this is true if the binary relation $s \rightarrow t :\Leftrightarrow s \in t(S_c \setminus S^*)$ is *terminating* as discussed in [Bri05, Subsection 2.5]: There is no infinite sequence $(s_n)_{n \geq 1}$ with $s_n \rightarrow s_{n+1}$, $s_n \neq s_{n+1}$ for all n . Clearly, if \rightarrow is terminating, then $S = S_{ci}^1 S_c$.

Proposition 2.10. *Let S be a right LCM semigroup with $S = S_{ci}^1 S_c$. Then $\mathcal{Q}(S)$ is the quotient of $\mathcal{Q}_p(S)$ by the relation $\pi_p(v_s v_s^*) = 1$ for all $s \in S_c$. Equivalently, $\mathcal{Q}(S)$ is the quotient of $\mathcal{Q}_c(S)$ by the relation $\sum_{f \in F} \pi_c(e_{fS}) = 1$ for all accurate proper foundation sets F . In particular, this holds true if the relation \rightarrow is terminating.*

Proof. Let $\mathcal{Q}'(S)$ be the quotient of $\mathcal{Q}_p(S)$ obtained by imposing $\pi_p(v_s v_s^*) = 1$ for all $s \in S_c$. Then $\mathcal{Q}(S)$ is a quotient of $\mathcal{Q}'(S)$. Hence it suffices to show that (2.1) holds for every accurate foundation set in $\mathcal{Q}'(S)$. Let $\pi'_c: \mathcal{Q}_p(S) \rightarrow \mathcal{Q}'(S)$ denote the quotient map and suppose F is an accurate foundation set. If $F \cap S_c \neq \emptyset$, then necessarily $F = \{s\}$ for some $s \in S_c$ due to accuracy. But in this case, there is nothing to show. So let $F \subset S \setminus S_c$. By assumption, each $f \in F$ can be written as $f = f_i f_c$ with $f_i \in S_{ci}$ and $f_c \in S_c$. Noting that $F_i := \{f_i \mid f \in F\}$ is an accurate proper foundation set, and $\pi'_c(\pi_p(e_{f_i S})) = \pi'_c(\pi_p(e_{f S}))$ as $e_{f_i S} - e_{f S} = v_{f_i}(1 - e_{f_c S})v_{f_i}^*$, we get

$$\sum_{f \in F} \pi'_c(\pi_p(e_{f S})) = \sum_{f_i \in F_i} \pi'_c(\pi_p(e_{f_i S})) = 1.$$

Thus $\mathcal{Q}'(S)$ coincides with $\mathcal{Q}(S)$. □

Given that $S = S_{ci}^1 S_c$, Proposition 2.10 shows that (2.2) takes the form

$$(2.3) \quad \begin{array}{ccc} C^*(S) & \xrightarrow{\pi_p} & \mathcal{Q}_p(S) \\ \pi_c \downarrow & & \downarrow \pi'_c \\ \mathcal{Q}_c(S) & \xrightarrow{\pi'_p} & \mathcal{Q}(S) \end{array}$$

where π'_p and π'_c are induced by π_p and π_c , respectively. As we shall see in Section 3, all our examples satisfy $S = S_{ci}^1 S_c$. This motivates the following two questions:

Question 2.11. Is there a right LCM semigroup S that does not satisfy $S = S_{ci}^1 S_c$?

Question 2.12. Is there a right LCM semigroup S for which (2.3) does not hold?

Remark 2.13. Let S be right LCM with $S^* = \{1\}$. Then $S = S_{ci}^1 S_c$ is precisely what is needed to display S as the internal Zappa-Szép product $S_{ci}^1 \bowtie S_c$, see [Bri05]. If $S^* \neq \{1\}$, but S_{ci}^1 admits a transversal $T \subset S_{ci}^1$ for the right action of S^* which forms a semigroup, then S is the internal Zappa-Szép product $T \bowtie S_c$. This is for instance the case for self-similar actions, see Example 3.5.

Let us now examine under which conditions a semigroup homomorphism $\phi: S \rightarrow T$ between two right LCM semigroups S and T induces maps of the quotients appearing

in the boundary quotient diagram (2.2). This constitutes a natural continuation of [BLS, Section 3] leads to new applications, see Section 3. To begin with, observe that ϕ is necessarily unital because T is assumed left cancellative and thus the only idempotent in T is 1_T . Next, recall from [BLS, Theorem 3.3] that ϕ induces a $*$ -homomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ if and only if

$$(2.4) \quad \phi(s_1)T \cap \phi(s_2)T = \phi(s_1S \cap s_2S)T \quad \text{for all } s_1, s_2 \in S.$$

Remark 2.14. Let S and T be right LCM semigroups. A semigroup homomorphism $\phi: S \rightarrow T$ satisfying (2.4) induces a $*$ -homomorphism

- a) $\varphi_c: \mathcal{Q}_c(S) \rightarrow \mathcal{Q}_c(T)$ if $\phi(S_c)$ is a subsemigroup of T_c ,
- b) $\varphi_p: \mathcal{Q}_p(S) \rightarrow \mathcal{Q}_p(T)$ if ϕ maps $\mathcal{F}_a^{(p)}(S)$ to $\mathcal{F}_a^{(p)}(T)$, and
- c) $\varphi_q: \mathcal{Q}(S) \rightarrow \mathcal{Q}(T)$ if ϕ maps $\mathcal{F}(S)$ to $\mathcal{F}(T)$.

If (2.2) is given by (2.3), e.g. if $S = S_{ci}^1 S_c$, then condition $\phi(\mathcal{F}(S)) \subset \mathcal{F}(T)$ from c) is equivalent to

$$c') \quad \phi(S_c) \subset T_c \text{ and } \phi \text{ maps } \mathcal{F}_a^{(p)}(S) \text{ to } \mathcal{F}_a^{(p)}(T).$$

Question 2.15. Are the conditions presented in Remark 2.14 a)–c') necessary?

3. EXAMPLES

Within this section we discuss the boundary quotient diagram for a selection of right LCM semigroups encompassing integral dynamics 3.1, Baumslag-Solitar monoids 3.2, algebraic dynamical systems 3.3, Zappa-Szép products 3.4, and self-similar actions 3.5. In addition, we mention right Ore semigroups in Example 3.6, and indicate obstructions to induced maps for the core boundary quotient for inclusions of right LCM subsemigroups in Example 3.7.

Example 3.1. Let $P \subset \mathbb{N}^\times$ be a monoid generated by a family \mathcal{P} of relatively prime numbers and consider $S := \mathbb{N} \rtimes P \subset \mathbb{N} \rtimes \mathbb{N}^\times$. Then S is right LCM, S^* is trivial, and $S_c = \mathbb{N} \times \{1\}$. An element $(n, p) \in S$ is core irreducible if and only if $0 \leq n \leq p-1$, and it is irreducible if, in addition, p is irreducible in P . We note that S^* is trivial and $S = S_{ci}^1 S_c$ so that $S = S_{ci}^1 \bowtie S_c$, a description that appeared already in [BRRW14, Subsection 3.2].

As P is directed, a finite set $F \subset S$ is a foundation set if and only if it is refined by the *elementary foundation set* $F' := \{(n, p_F) \mid 0 \leq n \leq p_F - 1\}$, where p_F is the least common multiple of $\{p \mid (n, p) \in F \text{ for some } n \in \mathbb{N}\}$. A proof of this observation can be obtained along the lines of [BS16, Lemma 3.2 and Lemma 3.3]. Note that elementary foundation sets are accurate. As a consequence, it suffices to impose (2.1) for proper elementary foundation sets in order to form $\mathcal{Q}_p(S)$ (and $\mathcal{Q}(S)$).

Let $p = p_1 \cdots p_n$ be a factorization of $p \in P$ with $p_i \in \mathcal{P}$ for all i . If F_i denotes the elementary foundation set for p_i , then each F_i is a proper accurate foundation set and the elementary foundation set for p is given by $F_1 \cdots F_n$. By Lemma 2.3 and Proposition 2.10, we get that:

- (a) $\mathcal{Q}_c(S)$ is the quotient by $v_{(1,1)} v_{(1,1)}^* = 1$.
- (b) $\mathcal{Q}_p(S)$ is the quotient by $\sum_{0 \leq k \leq p-1} v_{(k,p)} v_{(k,p)}^* = 1$ for all $p \in \mathcal{P}$.
- (c) $\mathcal{Q}(S)$ is the quotient by $v_{(1,1)} v_{(1,1)}^* = 1$ and $\sum_{0 \leq k \leq p-1} v_{(k,p)} v_{(k,p)}^* = 1$ for all $p \in \mathcal{P}$.

In particular, Definition 2.9 recovers the boundary quotient of [BaHLR12] in the case where \mathcal{P} is the set of all primes, see [BaHLR12, Proposition 3.3].

Example 3.2. Consider the Baumslag-Solitar monoid $S = BS(c, d)^+ := \langle a, b \mid ab^c = b^d a \rangle$ for $c, d \in \mathbb{N}^\times$ with $cd > 1$. According to [Spi12, Theorem 2.11], S is quasi-lattice ordered, hence right LCM. By [Spi12, Proposition 2.3], every $s \in S$ admits a unique normal form $s = w_1 w_2 \cdots w_m b^i$ with $w_k \in F_d := \{b^\ell a \mid 0 \leq \ell \leq d-1\}$ and $i \in \mathbb{N}$. In particular, $s \mapsto m$ gives a homomorphism $\ell: S \rightarrow \mathbb{N}$ and we call $\ell(s)$ the *length* of s .

Next we observe that for $s = v_1 v_2 \cdots v_m b^i, t = w_1 w_2 \cdots w_n b^j \in S$ with $m \leq n$, the intersection of sS and tS is non-empty if and only if $v_k = w_k$ for all $1 \leq k \leq m$, in which case $sS \cap tS = tb^\ell S$ for a suitable $\ell \in \mathbb{N}$. Thus $S_c = \langle b \rangle \cong \mathbb{N}$ and $s = w_1 w_2 \cdots w_m b^i$ is core irreducible if and only if $i = 0$. The element s is irreducible if and only if $m = 1$. Thus we see that $S_{ci}^1 = \langle F_d \rangle \cong \mathbb{F}_d^+$, the free monoid in d generators. In particular, every accurate proper foundation set F for S is of the form $F = (F_d)^k$ for some $k \geq 1$, i.e. all words in F_d of length k . As for Example 3.1, S^* is trivial, $S = S_{ci}^1 S_c$, and thus $S = S_{ci}^1 \rtimes S_c$. Thus by Proposition 2.10, the boundary quotient diagram is characterized as follows:

- (a) $\mathcal{Q}_c(S)$ is the quotient by $v_b v_b^* = 1$.
- (b) $\mathcal{Q}_p(S)$ is the quotient by $\sum_{0 \leq k \leq d-1} v_{b^k a} v_{b^k a}^* = 1$.
- (c) $\mathcal{Q}(S)$ is the quotient by $v_b v_b^* = 1$ and $\sum_{0 \leq k \leq d-1} v_{b^k a} v_{b^k a}^* = 1$.

Implicitly, $\mathcal{Q}_c(S)$ and $\mathcal{Q}_p(S)$ have already appeared in [CaHR, Corollary 5.3(a) and (c)].

Example 3.3. For an algebraic dynamical system (G, P, θ) , that is, a right LCM semigroup P acting on a discrete group G by injective group endomorphisms θ_p so that $pP \cap qP = rP$ forces $\theta_p(G) \cap \theta_q(G) = \theta_r(G)$, we consider the right LCM semigroup $S = G \rtimes_\theta P$, see [BLS] for details. Let $N_p := [G : \theta_p(G)]$ for $p \in P$. We will assume that $N_p = 1$ implies $p \in P^*$, and that P is directed with respect to $p \geq q : \Leftrightarrow p \in qP$. Then $S_c = S^* = G \rtimes_\theta P^*$ since for every $(g, p) \in S$ with $p \notin P^*$, we have $N_p \geq 2$, so there is $h \in G$ with $h^{-1}g \notin \theta_p(G)$ and hence $(g, p)S \cap (h, p)S = \emptyset$. Therefore, $C^*(S) = \mathcal{Q}_c(S)$ and every element in $S \setminus S^*$ is core irreducible.

Let $P^{fin} := \{p \in P \mid N_p < \infty\}$ denote the subsemigroup of *finite* elements in P . By [BS16, Proposition 3.9], every foundation set for S can be refined by an elementary foundation set in the sense of [BS16, Definition 3.7] provided that every foundation set F for P with $F \subset P^{fin}$ admits an accurate refinement F' that also satisfies $F' \subset P^{fin}$. In particular, S has property (AR) in this case. Let us assume that this holds true. Then a proper elementary foundation set is of the form $F_p := \{(g_1, p), (g_2, p), \dots, (g_{N_p}, p)\}$, where $2 \leq N_p < \infty$ and $\{g_1, \dots, g_{N_p}\}$ forms a transversal for $G/\theta_p(G)$. If $p = p_1 \cdots p_n$ is a factorization into irreducible elements $p_i \in P$, then $F_{p_1} \cdots F_{p_n}$ forms an elementary foundation set for p , and each F_{p_i} is a proper elementary foundation set. Thus, if every $p \in P$ admits a factorization into a product of finitely many irreducible elements, then it suffices to consider elementary foundation sets of irreducibles in P , thanks to Lemma 2.3, and therefore $\mathcal{Q}_p(S) \cong \mathcal{Q}(S)$ is the quotient of $C^*(S)$ by $\sum_{\bar{g} \in G/\theta_p(G)} v_{(g,p)} v_{(g,p)}^* = 1$ for all irreducible $p \in P$.

Example 3.4. Let $S = U \rtimes A$ be a Zappa-Szép product with U and A right LCM semigroups, $\mathcal{J}(A)$ totally ordered, and $U \rightarrow U, u \mapsto a \cdot u$ bijective for all $a \in A$. According to [BRRW14, Lemma 3.3], S is a right LCM semigroup. Then [BRRW14,

Remark 3.4] gives $S^* = U^* \rtimes A^*$ and $(u, a)S \cap (v, b)S \neq \emptyset \iff uU \cap vU \neq \emptyset$, from which we deduce:

- (a) The natural inclusions $\phi_U: U \rightarrow S$ and $\phi_A: A \rightarrow S$ satisfy (2.4).
- (b) $S_c = U_c \times A$, and hence $\phi_U(U_c), \phi_A(A) \subset S_c$.
- (c) A finite set $F \subset S$ is an accurate proper foundation set if and only if $\{u \mid (u, a) \in F \text{ for some } a \in A\}$ is an accurate proper foundation set for U . In particular, ϕ maps $\mathcal{F}_a^{(p)}(U)$ to $\mathcal{F}_a^{(p)}(S)$. We have $A = A_c$ because $\mathcal{J}(A)$ is totally ordered, so A does not have any proper foundation sets.

Hence we get a commutative diagram

$$(3.1) \quad \begin{array}{ccccc} C^*(U) & \longrightarrow & C^*(S) & \longleftarrow & C^*(A) \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow \\ & \mathcal{Q}_p(U) & \longrightarrow & \mathcal{Q}_p(S) & \longleftarrow & C^*(A) \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathcal{Q}_c(U) & \longrightarrow & \mathcal{Q}_c(S) & \longleftarrow & \mathcal{Q}(A) & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathcal{Q}(U) & \longrightarrow & \mathcal{Q}(S) & \longleftarrow & \mathcal{Q}(A) & \downarrow \\ & & & & & \mathcal{Q}(A) \end{array}$$

Moreover, we also have $S_{ci} = U_{ci} \rtimes A^*$ and $S = S_{ci}^1 S_c$ if and only if $U = U_{ci}^1 U_c$ because $A^* U_c = U_c A^*$. In particular, we recover [BRRW14, Theorem 5.2] and get a conceptual approach to the intermediate quotients $C_A^*(U \rtimes A)$ and $C_U^*(U \rtimes A)$ from [BRRW14, Remark 5.4]. Note that the quotient $\mathcal{Q}_c(U \rtimes A)$ is likely to be different from $C_A^*(U \rtimes A)$ as soon as $U_c \neq U^*$. More importantly, our approach provides candidates for intermediate quotients in the case where A_c is a proper subsemigroup of A , i.e. outside the realm of [BRRW14, Lemma 3.3].

Example 3.5. Let (G, X) be a self-similar action and denote by X^* the free monoid in the alphabet X . Then $S = X^* \rtimes G$ is a right LCM semigroup which fits into the setup of Example 3.4 according to [BRRW14, Theorem 3.8], which is in fact a result due to Lawson, see [Law08, Proposition 3.5 and 3.6]. As $S_c = S^* = G$, $C^*(G) = \mathcal{Q}(G)$, $\mathcal{Q}_c(X^*) = C^*(X^*)$, and $\mathcal{Q}_p(X^*) = \mathcal{Q}(X^*) \cong \mathcal{O}_{|X|}$, the diagram (3.1) simplifies to:

$$(3.2) \quad \begin{array}{ccccc} C^*(X^*) & \longrightarrow & C^*(S) & \longleftarrow & C^*(G) \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow \\ & \mathcal{O}_{|X|} & \longrightarrow & \mathcal{Q}(S) & \longleftarrow & C^*(G) \end{array}$$

Example 3.6. Suppose S is a right LCM semigroup that satisfies the right Ore condition, that is, there is an embedding of S into a group G such that $G = SS^{-1}$. Thanks to well-known results of Ore and Dubreil, the right Ore condition is equivalent to cancellation and left reversibility ($sS \cap tS \neq \emptyset$ for all $s, t \in S$). Under this assumption, we get $S_c = S$, and thus $C^*(S) = \mathcal{Q}_p(S)$ as well as $\mathcal{Q}_c(S) = \mathcal{Q}(S) \cong C^*(G)$.

Examples 3.7. Choose a right LCM subsemigroup S of a right LCM semigroup T and let $\phi: S \rightarrow T$ be the natural inclusion. If the equation (2.4) is satisfied by ϕ , then $S \cap T_c$ is a subsemigroup of S_c . If $S \cap T_c$ coincides with S_c , then ϕ induces a map $\varphi_c: \mathcal{Q}_c(S) \rightarrow \mathcal{Q}_c(T)$, see Remark 2.14. Note that $S \cap T_c$ can be a proper subsemigroup

of S_c , e.g. if S is abelian but not contained in T_c . For instance, take T to be the free monoid in two generators a and b , and let S the free abelian submonoid generated by a .

4. TOWARDS A UNIFIED TREATMENT FOR KMS STATES

In this section, we use the boundary quotient diagram (2.2) to recast the essential results concerning KMS-states

- (a) for $\mathbb{N} \rtimes \mathbb{N}^\times$ from [LR10, BaHLR12],
- (b) for $\mathbb{Z}^d \rtimes_A \mathbb{N}$ with $A \in M_d(\mathbb{Z}), |\det A| > 1$ from [LRR11],
- (c) for $X^* \rtimes G$, where (G, X) is a self-similar action with $|X| > 1$, from [LRRW14], and
- (d) for Baumslag-Solitar monoids $BS(c, d)^+$ with $c, d \in \mathbb{N}^\times, d > 1$ from [CaHR].

Note that we appeal to the dual picture of $\mathbb{T}^d \rtimes_A \mathbb{N}$ in (b) as opposed to the original treatment in [LRR11]. We remark that if A is invertible in $M_d(\mathbb{Z})$ in case (b), X is a singleton in case (c), or $d = 1$ in case (d), the study of KMS-states on the boundary quotient diagram essentially reduces to the study of traces on group C^* -algebras, compare [LRR11]. These cases will be excluded from our considerations as we intend to focus on proper semigroups.

In all the cases (a)-(d), the semigroup S features a natural homomorphism $N: S \rightarrow \mathbb{N}^\times, s \mapsto N_s$ arising from a scaling factor $\kappa \in \mathbb{R}_{>0}$ and a length function $\ell: S \rightarrow \mathbb{N}$ as $N_s := \kappa^{\ell(s)}$. The map N satisfies $N^{-1}(1) = S_c$ and yields a natural dynamics σ of \mathbb{R} on $C^*(S)$ via $\sigma_x(v_s) := N_s^{ix} v_s$ for $x \in \mathbb{R}$.

Define the ζ -function for S to be the formal series $\zeta_S(\beta) := \sum_{\bar{s} \in S/S_c} N_s^{-\beta}$ for $\beta \in \mathbb{R}$. Note that ζ_S converges for all β above a so-called *critical inverse temperature* β_c . The key to uniqueness of KMS_β -states on $C^*(S)$ for β within a critical interval $[1, \beta_c]$ is a notion of minimality for the semigroups in (a)-(d). This data is displayed in the following table:

type	S_c	κ	$\ell: S \rightarrow \mathbb{N}$	β_c	minimality
(a)	\mathbb{N}	1	$(m, p) \mapsto \log p$	2	$\bigcap_{p \in \mathbb{N}^\times} p\mathbb{N} = \{0\}$
(b)	\mathbb{Z}^d	$ \det A $	$(g, n) \mapsto n$	1	$\bigcap_{n \in \mathbb{N}} A^n(\mathbb{Z}^d) = \{0\}$
(c)	G	$ X $	$(w, g) \mapsto \ell'(w)$	1	$\forall g \in G: \{g _w \mid w \in X^*\} < \infty$
(d)	\mathbb{N}	d	length from 3.2	1	$\bigcap_{n \in \mathbb{N}} \left(\frac{d}{c}\right)^n \cdot \mathbb{N} = \{0\}$

Apparently, $\mathbb{N} \rtimes \mathbb{N}$ is minimal. As opposed to [LRR11], we do not require that A is a *dilation matrix*, that is, all its eigenvalues need to be larger than one in absolute value, because this feature is only used in [LRR11, Lemma 5.7], where the condition $\bigcap_{n \in \mathbb{N}} A^n(\mathbb{Z}^d) = \{0\}$ is then established. The point is that the latter condition is much more natural for the dynamical system $A: \mathbb{N} \curvearrowright \mathbb{Z}^d$ as it expresses minimality of the dual system. The minimality condition for (d) is equivalent to $c \notin d\mathbb{N}$.

To distinguish between elements in $C^*(S)$ and $C^*(S_c)$, let the standard generating isometries for $C^*(S_c)$ be denoted by w_s .

Theorem 4.1. *Suppose S is a right LCM semigroup of type (a), (b), (c) or (d). Then the KMS-state structure on $C^*(S)$ with respect to the dynamics σ given by $\sigma_x(v_s) := N_s^{ix} v_s$ is characterized by:*

- (i) There are no KMS_β -states for $\beta < 1$.
- (ii) For $\beta \in [1, \beta_c]$, there is a KMS_β -state ψ_β given by $\psi_\beta(v_s v_t^*) = \delta_{st} N_s^{-\beta}$ for all $s, t \in S$. If S is minimal, then ψ_β is the only KMS_β -state.
- (iii) For $\beta \in (\beta_c, \infty)$, there is an affine homeomorphism $\tau \mapsto \psi_{\beta, \tau}$ between the tracial states on $C^*(S_c)$ and the KMS_β -states given by

$$\psi_{\beta, \tau}(v_s v_t^*) = \begin{cases} N_s^{-\beta} \tau(w_y w_x^*) & \text{if } sS \cap tS = sxS, sx = ty \text{ with } x, y \in S_c, \\ 0 & \text{otherwise.} \end{cases}$$

- (iv) There is a one-to-one correspondence $\phi \mapsto \psi_\phi$ between states on $C^*(S_c)$ and ground states on $C^*(S)$ given by $\psi_\phi(v_s v_t^*) = \chi_{S_c}(s) \chi_{S_c}(t) \phi(w_s w_t^*)$. A ground state is a KMS_∞ -state if and only if it comes from a tracial state on $C^*(S_c)$.

With regards to the boundary quotient diagram (2.2) the following statements hold:

- (v) All the KMS_β -states for $\beta \in [1, \infty)$ factor through π_c .
- (vi) A ground state factors through π_c if and only if it is a KMS_∞ -state. In particular, every ground state on $\mathcal{Q}_c(S)$ is a KMS_∞ -state.
- (vii) If a KMS_β -state factors through π_p , then $\beta = 1$. In particular, $\mathcal{Q}_p(S)$ and $\mathcal{Q}(S)$ have a unique KMS_β -state corresponding to ψ_1 if S is minimal.
- (viii) There are no ground states on $\mathcal{Q}_p(S)$, and hence none on $\mathcal{Q}(S)$.

Proof. Using the descriptions obtained in Example 3.1, Example 3.3, Example 3.5, and Example 3.2 for (a)-(d), we embark on a reference chase, and leave it as an exercise to identify the semigroup C^* -algebra $C^*(S)$ with the C^* -algebra of Toeplitz type considered in the respective reference. We will prove part (viii) simultaneously for all cases at the end.

For $S = \mathbb{N} \rtimes \mathbb{N}^\times$, the claims follow from [LR10, Theorem 7.1] and [BaHLR12, Section 4]. For $S = \mathbb{Z}^d \rtimes_A \mathbb{N}$, we note that $\pi_c = \text{id}$ and hence $\pi = \pi_p$, so that [LRR11, Theorem 1.1] yields (i)-(vii).

Next, let $S = X^* \rtimes G$. Part (i) is [LRRW14, Proposition 4.1(1)], and (iii) is [LRRW14, Theorem 6.1]. As $C^*(S) = \mathcal{Q}_c(S)$, claim (v) is trivial. Also, (iv) and (vi) merge to a single statement that corresponds to [LRRW14, Proposition 5.3]: Ground states on $C^*(S)$ are KMS_∞ -states, and they are given by tracial states on $C^*(S_c) \cong C^*(G)$. The claims (ii) and (vii) are proven in [LRRW14, Proposition 7.1 and Theorem 7.3].

Now let $S = BS(c, d)^+$. Then [CaHR, Corollary 5.3(a) and (b)] gives (i) and (v). The combination of [CaHR, Corollary 5.3(c)] with [CaHR, Proposition 7.1] yields (ii) and (vii). Statement (iv) corresponds to [CaHR, Theorem 8.1], and (vi) is an immediate consequence of this. Claim (iii) is essentially provided by [CaHR, Theorem 6.1] except for the fact that the authors assume minimality of S in order to show injectivity of the parametrization $\tau \mapsto \psi_{\beta, \tau}$. Thus we need to strengthen this result slightly. To this end,

we observe that the formula [CaHR, (6.1)] becomes

$$\begin{aligned}
\zeta_S(\beta)\psi_{\beta,\tau}(v_{b^n}) &= \tau(v_{b^n}) + \sum_{\substack{k \geq 1: n \in d(d/c)^j \mathbb{N} \\ \text{for } 0 \leq j \leq k-1}} d^{k(1-\beta)} \tau(v_{b^{n(c/d)^k}}) \\
&= \tau(v_{b^n}) + \chi_{d\mathbb{N}}(n) d^{1-\beta} \sum_{\substack{k \geq 0: (c/d)n \in d(d/c)^j \mathbb{N} \\ \text{for } 0 \leq j \leq k-1}} d^{k(1-\beta)} \tau(v_{b^{n(c/d)^{k+1}}}) \\
&= \tau(v_{b^n}) + \chi_{d\mathbb{N}}(n) d^{1-\beta} \psi_{\beta,\tau}(v_{b^{n(c/d)}})
\end{aligned}$$

for every $n \in \mathbb{N}$ within our notation. The analogous formula holds for $v_{b^n}^*$. Hence, if τ and ρ are tracial states on $C^*(\mathbb{N})$ with $\psi_{\beta,\tau} = \psi_{\beta,\rho}$, then $\tau = \rho$, without assuming that S is minimal, as in [LRR11, LRRW14]. This completes (iii).

In all cases, (viii) is a consequence of (iv) and the existence of accurate proper foundation sets, as indicated in [BaHLR12, End of Section 4]: Since π_p is a $*$ -homomorphism, every ground state ϕ on $\mathcal{Q}_p(S)$ lifts to a ground state on $C^*(S)$. We then conclude by (iv) that ϕ vanishes on all projections $\pi_p(e_{sS})$ with $s \in S \setminus S_c$. So if there is an accurate proper foundation set F , then $1 = \phi(1) = \phi(\sum_{f \in F} \pi_p(e_{fS})) = 0$. Thus there are no ground states on $\mathcal{Q}_p(S)$ and $\mathcal{Q}(S)$. \square

Assuming S_c to be right cancellative and hence right Ore, let G_c denote the group $S_c S_c^{-1}$. As the traces on $C^*(S_c)$ correspond to traces on $C^*(G_c)$, it may seem more natural to use the group C^* -algebra in Theorem 4.1. However, we emphasize the core subsemigroup S_c here because we like to think of the KMS_β -states as arising from the $*$ -homomorphism $\varphi: C^*(S_c) \rightarrow C^*(S)$ induced by the inclusion $S_c \subset S$.

The parallels between the results in Theorem 4.1 for the different types (a)-(d) extend beyond their mere statements, for which the boundary quotient diagram provides a unifying framework. Indeed, there are striking analogies in the method of proof. It thus seems natural to ask whether there is a unified treatment for KMS -state structures on the proposed boundary quotient diagram (2.2) for general right LCM semigroups. An important step towards achieving precisely this has been accomplished recently in [ABLS], and we emphasize that the two subsemigroups S_c and S_{ci} are at the heart of this approach.

5. A FIRST LOOK AT THE K-THEORY

In Section 4 we learned that the left column of the boundary quotient diagram (2.2) is of much greater interest with respect to KMS -states than the right column. With regards to K -theory the situation is almost the opposite: If S is a left Ore right LCM semigroup whose enveloping group $S^{-1}S$ is amenable, then $K_*(C^*(S)) \cong K_*(C^*(S^*))$, see [BLS, Theorem 5.3], which is an application of the magnificent work in [CEL14].

For instance, this applies to $S = \mathbb{N} \rtimes P$ from Example 3.1. In this case, $\mathcal{Q}_c(S) \cong C^*(\mathbb{Z} \rtimes P)$, and $\mathbb{Z} \rtimes P$ is also covered by [BLS, Theorem 5.3]. Thus we get $K_*(C^*(S)) \cong K_*(\mathbb{C})$ and $K_*(\mathcal{Q}_c(S)) \cong K_*(C^*(\mathbb{Z}))$ in these cases. But the computation of $K_*(\mathcal{Q}_p(S))$ and $K_*(\mathcal{Q}(S))$ turns out to be harder, reveals more interesting K -groups with connections to $K_*(\mathcal{Q}(S_{ci}^1))$, and is still incomplete, see [LN16, Subsection 6.3] and [BOS15]. We will now present a brief exposition of the matter with the (boundary quotient of the) subsemigroup S_{ci}^1 in mind.

In [LN16, Subsection 6.3], Li and Norling determined $K_*(\mathcal{Q}_p(S))$ for $|\mathcal{P}| \leq 2$ as well as $K_1(\mathcal{Q}_p(S))$ for $|\mathcal{P}| = 3$. In joint work with Barlak and Omland [BOS15], we obtained similar formulas for $K_*(\mathcal{Q}(S))$ in the case of $|\mathcal{P}| \leq 2$ or $g_{\mathcal{P}} = 1$, where $g_{\mathcal{P}}$ is the greatest common divisor of $\mathcal{P} - 1 \subset \mathbb{N}^\times$, i.e. $g_{\mathcal{P}} := \gcd(\{p - 1 \mid p \in \mathcal{P}\})$. In addition, we proved a number of structural results about $K_*(\mathcal{Q}(S))$ that can be phrased in terms of the boundary quotient $\mathcal{Q}(S_{ci}^1)$ for the right LCM semigroup $S_{ci}^1 = \{(n, p) \in S \mid 0 \leq n \leq p - 1\}$ as $\mathcal{Q}(S_{ci}^1)$ is canonically isomorphic with the *torsion subalgebra* $\mathcal{A}(S)$ of $\mathcal{Q}(S)$, see [BOS15, Proposition 5.5]:

- a) There are isomorphisms $K_i(\mathcal{Q}(S)) \cong \mathbb{Z}^{2^{|\mathcal{P}|-1}} \oplus K_i(\mathcal{Q}(S_{ci}^1))$ for $i = 0, 1$, see [BOS15, Theorem 6.1].
- b) $\mathcal{Q}(S_{ci}^1)$ is a unital UCT Kirchberg algebra. In particular, $\mathcal{Q}(S_{ci}^1)$ embeds canonically into both $\mathcal{Q}_p(S)$ and $\mathcal{Q}(S)$, see [BOS15, Corollary 5.2 and Corollary 5.4].
- c) $\mathcal{Q}(S_{ci}^1)$ is isomorphic to $\bigotimes_{p \in \mathcal{P}} \mathcal{O}_p$ for $|\mathcal{P}| \leq 2$, and for $|\mathcal{P}| \geq 3$, the order of every element in $K_*(\mathcal{Q}(S_{ci}^1))$ is a divisor of $g_{\mathcal{P}}$, see [BOS15, Theorem 6.4,]. In particular, $K_*(\mathcal{Q}(S_{ci}^1))$ is a torsion group. For $\mathcal{Q}(S)$, the embedding from b) is an isomorphism between $K_*(\mathcal{Q}(S_{ci}^1))$ and the torsion part of $K_*(\mathcal{Q}(S))$. At least for $|\mathcal{P}| \leq 2$, the results of [LN16] imply that the embedding of $\mathcal{Q}(S_{ci}^1)$ into $\mathcal{Q}_p(S)$ is an isomorphism in K -theory.

In view of the results for $S = \mathbb{N} \rtimes P$ from Example 3.1, it is reasonable to investigate the role of S_{ci}^1 with regards to the K -theory of $\mathcal{Q}_p(S)$ and $\mathcal{Q}(S)$ for other right LCM semigroups S . The first thing to note is that we need to request that S_{ci}^1 is indeed right LCM. While this is always true for S_c , S_{ci}^1 may fail to have this property, and one should rather consider the subsemigroup $S'_{ci} := S_{ci} \cup S^*$, see [ABLS, Proposition 3.3]. However, we have $S'_{ci} = S_{ci}^1$ for the examples we consider here, namely Baumslag-Solitar monoids $S = BS(c, d)^+$ with $c, d \geq 2$. Thanks to the efforts of Spielberg, the K -theory for $\mathcal{Q}(S)$ is known:

Theorem 5.1 ([Spi12, Theorem 4.8]). *For $c, d \geq 2$ and $S = BS(c, d)^+$, the K -theory of $\mathcal{Q}(S)$ is given by $K_0(\mathcal{Q}(S)) \cong \mathbb{Z}/(d-1)\mathbb{Z}$, $K_1(\mathcal{Q}(S)) \cong \mathbb{Z}/(c-1)\mathbb{Z}$, where $[1]_0$ generates $\mathbb{Z}/(d-1)\mathbb{Z}$.*

Note that $c = d = 1$ gives $S = \mathbb{N}^2$ so that $\mathcal{Q}_c(S) = \mathcal{Q}(S) \cong C^*(\mathbb{Z}^2)$. Let us focus on the case where $c, d \geq 2$. By Example 3.2, $S_{ci}^1 \cong \mathbb{F}_d^+$, so that $\mathcal{Q}(S_{ci}^1) \cong \mathcal{O}_d$ and it embeds into $\mathcal{Q}(S)$ in the canonical way. More importantly, Theorem 5.1 implies that this map is injective in K -theory. To explain the parts of $K_*(\mathcal{Q}(S))$ related to c , we first remark that S is amenable and hence $C^*(S)$ is isomorphic to the concrete C^* -algebra $C_\lambda^*(S)$ given by the left regular representation λ of S on $\ell^2(S)$. In addition, it is true for arbitrary cancellative semigroups T that the right regular representation ρ of T on $\ell^2(T)$ satisfies $C_\rho^*(T) \cong C_\lambda^*(T^{\text{opp}})$ for the opposite semigroup T^{opp} of T . While the C^* -algebras $C_\lambda^*(T)$ and $C_\rho^*(T)$ have quite distinct properties, there are various situations where their K -theories agree, see [CEL13, Subsections 6.3, 6.4, and 6.5].

In the case of Baumslag-Solitar monoids $S = BS(c, d)^+$, S^{opp} coincides with $BS(d, c)^+$, so that $(S^{\text{opp}})_{ci}^1 \cong \mathbb{F}_c^+$ and $\mathcal{Q}((S^{\text{opp}})_{ci}^1) \cong \mathcal{O}_c$ whenever $c \geq 2$. This leads us to the following natural question:

Question 5.2. Suppose $c, d > 1$. Is there a $*$ -homomorphism from the suspension of \mathcal{O}_c to $\mathcal{Q}(BS(c, d)^+)$ that is injective in K -theory?

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