EQUILIBRIUM STATES ON RIGHT LCM SEMIGROUP C*-ALGEBRAS

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Abstract. We determine the structure of equilibrium states for a natural dynamics on the boundary quotient diagram of C*-algebras for a large class of right LCM semigroups. The approach is based on abstract properties of the semigroup and covers the previous case studies on \( \mathbb{N} \times \mathbb{N} \times \), dilation matrices, self-similar actions, and Baumslag–Solitar monoids. At the same time, it provides new results for right LCM semigroups associated to algebraic dynamical systems.

1. Introduction

Equilibrium states have been studied in operator algebras starting with quantum systems modelling ensembles of particles, and have been the means of describing properties of physical models with C*-algebraic tools, cf. [BR97]. Specifically, given a quantum statistical mechanical system, which is a pair formed of a C*-algebra \( A \) that encodes the observables and a one-parameter group of automorphisms \( \sigma \) of \( A \) interpreted as a time evolution, one seeks to express equilibrium of the system via states with specific properties. A KMS\(_\beta\)-state for \((A, \sigma)\) is a state on \( A \) that satisfies the KMS\(_\beta\)-condition (for Kubo-Martin-Schwinger) at a real parameter \( \beta \) bearing the significance of an inverse temperature. Gradually, it has become apparent that the study of KMS-states for systems that do not necessarily have physical origins brings valuable insight into the structure of the underlying C*-algebra, and uncovers new directions of interplay between the theory of C*-algebras and other fields of mathematics. A rich supply of examples is by now present in the literature, see [BC95,LR10,LRRW14,CDL13,Nes13], to mention only some.

This article is concerned with the study of KMS-states for systems whose underlying C*-algebras are Toeplitz-type algebras modelling a large class of semigroups. While our initial motivation was to classify KMS-states for a specific class of examples, as had been done previously, we felt that by building upon the deep insight developed in the case-studies already present in the literature, the time was ripe for proposing a general framework that would encompass our motivating example and cover all these case-studies. We believe it is a strength of our approach that we can identify simply phrased conditions at the level of the semigroup which govern the structure of KMS-states, including uniqueness of the KMS\(_\beta\)-state for \( \beta \) in the critical interval. In all the examples we treat, these conditions admit natural interpretations. Moreover, they are often subject to feasible verification.

To be more explicit, the interest in studying KMS-states on Toeplitz-type C*-algebras gained new momentum with the work of Laca and Raeburn [LR10] on the Toeplitz
algebra of the $ax + b$-semigroup over the natural numbers $\mathbb{N} \rtimes \mathbb{N}^\times$. Further results on the KMS-state structure of Toeplitz-type $C^*$-algebras include the work on Exel crossed products associated to dilation matrices [LRR11], $C^*$-algebras associated to self-similar actions [LRRW14], and $C^*$-algebras associated to Baumslag–Solitar monoids [CaHR16]. In all these cases the relevant Toeplitz-type $C^*$-algebra can be viewed as a semigroup $C^*$-algebra in the sense of [Li12]. Further, the semigroups in question are right LCM semigroups, see [BLS17] for dilation matrices, and [BRRW14] for all the other cases. In [BLS17, BLS], the last three named authors have studied $C^*$-algebras of right LCM semigroups associated to algebraic dynamical systems. These $C^*$-algebras often admit a natural dynamics that is built from the underlying algebraic dynamical system. The original motivation for our work was to classify KMS-states in this context.

Although there is a common thread in the methods and techniques used to prove the KMS-classification results in [LR10, LRR11, LRRW14, CaHR16], one cannot speak of a single proof that runs with adaptations. Indeed, in each of these papers a careful and rather involved analysis is carried out using concrete properties of the respective setup. One main achievement of this paper is a general theory of KMS-classification which both unifies the classification results from [LR10, LRR11, LRRW14, CaHR16], and also considerably enlarges the class of semigroup $C^*$-algebras for which KMS-classification is new and interesting, see Theorem 4.3 and Section 5.

We expect that our work may build a bridge to the analysis of KMS-states via groupoid models and the result of Neshveyev [Nes13, Theorem 1.3]. At this point we would also like to mention the recent analysis of equilibrium states on $C^*$-algebras associated to self-similar actions of groupoids on graphs that generalises [LRRW14], see [LRRW]. It would be interesting to explore connections to this line of development.

One of the keys to establishing our general theory is the insight that it pays off to work with the boundary quotient diagram for right LCM semigroups proposed by the fourth-named author in [Sta17]. The motivating example for this diagram was first considered in [BaHLR12], where it was shown to give extra insight into the structure of KMS-states on $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$. Especially the core subsemigroup $S_c$, first introduced in [Star15], and motivated by the quasi-lattice ordered situation in [CL07], and the core irreducible elements $S_{ci}$ introduced in [Sta17] turn out to be central for our work. Recall that $S_c$ consists of all the elements whose principal right ideal intersects all other principal right ideals and $S_{ci}$ of all $s \in S \setminus S_c$ such that every factorisation $s = ta$ with $a \in S_c$ forces $a$ to be invertible in $S$. Via so-called (accurate) foundation sets, $S_c$ and $S_{ci}$ give rise to quotients $Q_c(S)$ and $Q_{ci}(S)$ of $C^*(S)$ of $S$, respectively. Together with $C^*(S)$ and the boundary quotient $Q(S)$ from [BRRW14], they form a commutative diagram:

\[
\begin{array}{ccc}
C^*(S) & \xrightarrow{\pi_p} & Q_p(S) \\
\downarrow{\pi_c} & & \downarrow{\pi_p'} \\
Q_c(S) & \xrightarrow{\pi_c'} & Q(S)
\end{array}
\]

We refer to [Sta17] for examples and further details, but remark that if $S = S_{ci}^1 S_c$ holds (with $S_{ci}^1 = S_{ci} \cup \{1\}$), a property we refer to as $S$ being core factorable, then the quotient
With regards to the boundary quotient diagram, it is proven that every KMS\(\beta\) then a ground state is a KMS\(\infty\) modest, yet elusive condition that contrast, the second criterion only works for omorphism between KMS\(\beta\) on the canonical generators of \(C\alpha\) demands the stronger hypothesis that semigroup there is a natural choice for a generalised scale. Every generalised scale on a right LCM \(Q\) no ground states on factors through \(\pi\). In each of the examples we consider in Section 5 there is a natural choice for a generalised scale. Every generalised scale on a right LCM semigroup \(S\) gives rise to a \(\xi\)-function on \(R\). The critical inverse temperature \(\beta_c\) for the dynamics \(\sigma\) is defined to be the least bound in \(R \cup \{\infty\}\) so that \(\xi_S(\beta)\) is finite for all \(\beta > \beta_c\). The uniqueness of KMS\(\beta\)-states for \(\beta\) inside the critical interval \([1, \beta_c]\) for finite \(\beta_c\), and \([1, \infty)\) otherwise) has become an intriguing topic in the field.

Our main result Theorem 4.3 classifies the KMS-state structure on (1.1) for an admissible right LCM semigroup \(S\) with regards to the dynamics \(\sigma\) arising from a generalised scale \(N\). There are no KMS\(\beta\)-states for \(\beta < 1\), and for \(\beta > \beta_c\) there is an affine homeomorphism between KMS\(\beta\)-states on \(C^*(S)\) and normalised traces on \(C^*(S_c)\). For \(\beta\) in the critical interval there exists at least one KMS\(\beta\)-state, and two sufficient criteria for the KMS\(\beta\)-state to be the unique one are established. Both require faithfulness of a natural action \(\alpha\) of \(S_c\) on \(S/S_c\) arising from left multiplication. The first criterion demands the stronger hypothesis that \(\alpha\) is almost free and works for arbitrary \(\beta_c\). In contrast, the second criterion only works for \(\beta_c = 1\), but relies on the seemingly more modest, yet elusive condition that \(S\) has what we call finite propagation. The ground states on \(C^*(S)\) are affinely homeomorphic to the states on \(C^*(S_c)\). Moreover, if \(\beta_c < \infty\), then a ground state is a KMS\(\infty\)-state if and only if it corresponds to a trace on \(C^*(S_c)\). With regards to the boundary quotient diagram, it is proven that every KMS\(\beta\)-state and every KMS\(\infty\)-state factors through \(\pi_p: C^*(S) \to Q(S_c)\). In contrast, a KMS\(\beta\)-state factors through \(\pi_p: C^*(S) \to Q_p(S)\) if and only if \(\beta = 1\). Last but not least, there are no ground states on \(Q_p(S)\), and hence none on the boundary quotient \(Q(S)\).

As every result requiring abstract hypotheses ought to be tested for applicability to concrete examples, we make an extensive effort to provide a wide range of applications. The results on KMS-classification for self-similar actions, (subdynamics of) \(\mathbb{N} \rtimes \mathbb{N}^e\), and Baumslag-Solitar monoids from \(\text{LRRW14, LR10}\) and \(\text{BaHLR12}\), and \(\text{CaHR16}\) are recovered in sections 5.3, 5.4 and 5.5, respectively; we remark that in order to make the similarities between these examples more apparent we make a different choice of time evolution leading to different critical temperatures. Our analysis is achieved by first establishing reduction results for general Zappa-Szép products \(S = U \bowtie A\) of right LCM semigroups \(U\) and \(A\), see Corollary 5.5 and Corollary 5.6. These in turn rely on an intriguing result concerning the behaviour of \(S_c\) and \(S_{ci}\) under the Zappa-Szép product construction \(S = U \bowtie A\), which we expect to be of independent interest, see Theorem 5.3. As a simple application of the reduction results, we also cover the case of right-angled Artin monoids that are direct products of a free monoid by an abelian monoid. We then investigate admissibility for right LCM semigroups arising
from algebraic dynamical systems, see 5.6. This yields plenty of examples for which no KMS-classification was known before, and we obtain the KMS-classification results from [LRR11] as a corollary, see Example 5.14. In addition, we give an example of an admissible right LCM semigroup with finite propagation for which the action $\alpha$ is faithful but not almost free by means of standard restricted wreath products of finite groups by $\mathbb{N}$, see Example 5.13.

Apart from this variety of examples, there is also a structural link to be mentioned here: Via a categorical equivalence from [Law99] based on Clifford’s work [Cli53], right LCM semigroups correspond to 0-bisimple inverse monoids, which constitute a well-studied and rich class of inverse semigroups, see for instance [McA74] and the references therein. The results of this work may therefore be seen as a contribution to the study of 0-bisimple inverse monoids, and the relevance of the core structures exhibited in right LCM semigroups for the respective 0-bisimple inverse monoids are yet to be investigated.

The paper is organised as follows: A brief description of basic notions and objects related to right LCM semigroups and their $C^*$-algebras, as well as to KMS-state of a $C^*$-algebra is provided in Section 2. In Section 3 the class of admissible right LCM semigroups is introduced. Section 4 is a small section devoted to stating our main theorem and to providing an outline of the proof. Before proving our main theorem, we apply Theorem 4.3 to a wide range of examples in Section 5. Sections 6–9 constitute the essential steps needed in the proof of Theorem 4.3. In Section 6 algebraic characterisations for the existence of KMS-states are provided. These yield bounds on the kinds of KMS-states that can appear. In Section 7 a reconstruction formula in the spirit of [LR10, Lemma 10.1] is provided; and in Section 8 KMS-states are constructed, where Theorem 8.4 provides a means to construct representations for $C^*(S)$ out of states on $C^*(S_c)$, which we believe to be of independent interest. Even more so, since it only requires $S$ to be core factorable and $S_c \subseteq S$ to be $\cap$-closed. Section 9 addresses the existence of a unique KMS$_\beta$-state for $\beta$ within the critical interval, and Section 10 is an attempt at identifying some challenges for future research.

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2. **Background**

2.1. **Right LCM semigroups.** A discrete, left cancellative semigroup $S$ is right LCM if the intersection of principal right ideals is empty or another principal right ideal, see [Nor14, Law12, BRRW14]. If $r, s, t \in S$ with $sS \cap tS = rS$, then we call $r$ a right LCM of $s$ and $t$. In this work, all semigroups will admit an identity element (and so are monoids) and will be countable. We let $S^*$ denote the subgroup of invertible elements in $S$. Two
subsets of a right LCM semigroup $S$ are used in [Sta17]: the core subsemigroup
\[ S_c := \{ a \in S \mid aS \cap sS \neq \emptyset \text{ for all } s \in S \}, \]
which was first considered in [Star15] in the context of right LCM semigroups, and
the set $S_{ci}$ of core irreducible elements, where $s \in S \setminus S_c$ is core irreducible if every
decomposition $s = ta$ with $a \in S_c$ forces $a \in S^*$. The latter is a subsemigroup of $S$,
at least under moderate conditions, see Proposition 3.4. The core subsemigroup $S_c$
contains $S^*$, and induces an equivalence relation on $S$ defined by
\[ s \sim t :\iff sa = tb \text{ for some } a, b \in S_c. \]
We call $\sim$ the core relation and say that $s$ and $t$ are core equivalent if $s \sim t$. The
equivalence class of $s \in S$ under $\sim$ is denoted $[s]$. We write $s \perp t$ if $s, t \in S$ have disjoint
right ideals. One can verify that if $s \sim s'$ for some $s, s' \in S_c$, then $s \perp t$ if and only if
$s' \perp t$ for all $t \in S$. By a transversal $T$ for $A/\sim$, with $A \subset S$, we mean a subset of $A$
that forms a minimal complete set of representatives for $A/\sim$.

Recall from [BRRW14] that a finite subset $F$ of $S$ is called a foundation set if for
every $s \in S$ there is $f \in F$ with $sS \cap fS \neq \emptyset$. The second and fourth author coined the
refined notion of accurate foundation set in [BS16], where a foundation set $F$ is accurate
if $fS \cap f'S = \emptyset$ for all $f, f' \in F$ with $f \neq f'$. A right LCM semigroup $S$ has the accurate
refinement property, or property (AR), if for every foundation set $F$ there is an accurate
foundation set $F_a$ such that $F_a \subseteq FS$, in the sense that for every $f_a \in F_a$ there is $f \in F$
with $f_a \in fS$.

2.2. C*-algebras associated to right LCM semigroups. The full semigroup C*-algebra $C^*(S)$ of a right LCM semigroup $S$ is the C*-algebra of $S$ as defined by Li [Li12, Definition 2.2]. It can be shown that $C^*(S)$ is the universal C*-algebra generated
by an isometric representation $v$ of $S$ satisfying
\[ v^*_s v_t = \begin{cases} v_{s'} v^*_{t'} & \text{if } ss' = ts' = tt' \\ 0 & \text{if } s \perp t, \end{cases} \]
for all $s, t \in S$. For each $s \in S$, we write $e_s$ for the range projection $v^*_s v_s$. The boundary quotient C*-algebra $Q(S)$ is defined in [BRRW14] to be the quotient of $C^*(S)$
by the relation $\prod_{f \in F} (1 - e_{fs}) = 0$ for all foundation sets $F \subseteq S$. However, if $S$ has property (AR), then in $Q(S)$ this relation reduces to
\[ \sum_{f \in F} e_{fs} = 1 \text{ for all accurate foundation sets } F \subseteq S. \]
In [Sta17], the fourth author introduced two intermediary quotients. The core boundary quotient $Q_c(S)$ is the quotient of $C^*(S)$ by the relation $v^*_s v^*_s = 1$ for all $s \in S_c$. Note
that this amounts to the quotient by the relation (2.1) for accurate foundation sets
contained in the core $S_c$. The second intermediary quotient depends on the notion of a proper foundation set, which is a foundation set contained in the core irreducible elements $S_{ci}$. The proper boundary quotient $Q_p(S)$ is the quotient of $C^*(S)$ by (2.1)
for all proper accurate foundation sets. As first defined in [Sta17, Definition 2.9], the boundary quotient diagram is the commutative diagram from (1.1) which encompasses
the C*-algebras $C^*(S)$, $Q_c(S)$, $Q_p(S)$ and $Q(S)$. 
2.3. KMS-

states. We finish the background section with the basics of the theory of

KMS-states. A much more detailed presentation can be found in [BR97]. Suppose \( \sigma \) is an action of \( \mathbb{R} \) by automorphisms of a \( C^* \)-algebra \( A \). An element \( a \in A \) is said to be analytic for \( \sigma \) if \( t \mapsto \sigma_t(a) \) is the restriction of an analytic function \( z \mapsto \sigma_z(a) \) from \( \mathbb{C} \) into \( A \). For \( \beta > 0 \), a state \( \phi \) of \( A \) is called a KMS\( _\beta \)-state of \( A \) if it satisfies

\[
(2.2) \quad \phi(ab) = \phi(b \sigma_\beta(a)) \quad \text{for all analytic} \ a, b \in A.
\]

We call (2.2) the KMS\( _\beta \)-condition. To prove that a given state of \( A \) is a KMS\( _\beta \)-state, it suffices to check (2.2) for all \( a, b \) in any \( \sigma \)-invariant set of analytic elements that spans a dense subspace of \( A \).

A KMS\(_\infty\)-state of \( A \) is a weak* limit of KMS\(_\beta_n\)-states as \( \beta_n \to \infty \), see [CM06]. A state \( \phi \) of \( A \) is a ground state of \( A \) if \( z \mapsto \phi(a \sigma_z(b)) \) is bounded on the upper-half plane for all analytic \( a, b \in A \).

3. The setting

Given a right LCM semigroup \( S \) and its \( C^* \)-algebra \( C^*(S) \), we wish to find a natural dynamics for which we can calculate the KMS-state structure. We then ask: what conditions on \( S \) are needed in order to define such a dynamics? The answer that first comes to mind is the existence of a nontrivial homomorphism \( N : S \to \mathbb{N}^\times \), which allows us to set \( \sigma_x(v_s) := N_s^x v_s \) for \( s \in S \) and \( x \in \mathbb{R} \). However, it soon becomes apparent that the mere existence of a homomorphism \( N \) on \( S \) seems far from sufficient to give rise to KMS-states on \( C^*(S) \), yet alone describe them all. As announced in the introduction, we shall take into account the boundary quotient diagram for \( C^*(S) \). The ingredients that are at work behind the scenes to get this diagram are the core \( S_c \) and the core irreducible elements \( S_{ci} \) of \( S \). Looking for relationships between these and the homomorphism \( N : S \to \mathbb{N}^\times \) will provide clues as to what conditions \( S \) ought to satisfy.

Before we introduce our main concept of admissibility we fix the following notation: if \( P \) is a subset of positive integers, we let \( \langle P \rangle \) be the subsemigroup of \( \mathbb{N}^\times \) generated by \( P \). An element of \( \langle P \rangle \) is irreducible if \( m = nk \) in \( \langle P \rangle \) implies \( n \) or \( k \) is 1. We denote by \( \text{Irr}(\langle P \rangle) \) the set of irreducible elements of \( \langle P \rangle \). We are now ready to present the outcome of our efforts of distilling a minimal sufficient set of conditions for the aim of defining a suitable dynamics \( \sigma \):

**Definition 3.1.** A right LCM semigroup \( S \) is called admissible if it satisfies the following conditions:

(A1) \( S = S_{ci} S_c \).

(A2) Any right LCM in \( S \) of a pair of elements in \( S_{ci} \) belongs to \( S_{ci} \).

(A3) There is a nontrivial homomorphism of monoids \( N : S \to \mathbb{N}^\times \) such that

(a) \( |N^{-1}(n)/\sim| = n \) for all \( n \in N(S) \) and

(b) for each \( n \in N(S) \), every transversal of \( N^{-1}(n)/\sim \) is an accurate foundation set for \( S \).

(A4) The monoid \( N(S) \) is the free abelian monoid in \( \text{Irr}(N(S)) \).

We refer to the property in (A1) as saying that \( S \) is core factorable, and when \( S \) satisfies (A2) we say that \( S_{ci} \subset S \) is \( \cap \)-closed. We call a homomorphism \( N \) satisfying (A3) a generalised scale. This name is inspired by the notion of a scale on a semigroup originally defined in [Lac98, Definition 9].
To state and prove our results on the KMS structure of the $C^*$-algebras of admissible right LCM semigroups, we also need to introduce a number of properties of these semigroups, and to prove a number of consequences of admissibility. The rest of this section is devoted to establishing these properties and thereby also explaining the meaning of the conditions (A1) (A3). Our first observation explains the meaning of (A2) in the presence of (A1).

**Lemma 3.2.** Suppose $S$ is a core factorable right LCM semigroup. Then for each $s \in S$ the set $[s] \cap S_{ci}^1$ is non-empty, and the following conditions are equivalent:

(i) The equivalence classes for $\sim$ have minimal representatives, that is, $[s] = tS_c$ holds for every $t \in [s] \cap S_{ci}^1$ and all $s \in S$.

(ii) If $s, t \in S_{ci}^1$ satisfy $s \sim t$, then $s \in tS^*$.

Moreover, if $S_{ci} \subset S$ is $\cap$-closed, then (i) and (ii) hold.

**Proof.** Suppose $S$ has (A2). The set $[s] \cap S_{ci}^1$ is non-empty for all $s \in S$ due to (A1). Let $t \in [s]$, and write $s = sa, t = tb$ with $s', t' \in S_{ci}^1, a, b \in S_c$ using (A1). Fix $c, d \in S_c$ with $sc = td$. Then $s' \not\sim t'$, so (A2) implies $s'S \cap t'S = rS$ for some $r \in S_{ci}^1$, say $r = sp$ for a suitable $p \in S$. As $sc = td$, we have $s'ac = re = s'pe$ for some $e \in S$. Since $ac \in S_c$, we deduce that $p, e \in S_c$. But then $r \in S_{ci}^1$ forces $p \in S^*$ so that $r \in s'S^*$. The same argument applied to $t$ in place of $s$ shows $r \in t'S^*$. Thus we have shown $s' \in t'S^*$, and hence $[s] = s'S_c$. This establishes (i).

If (i) holds and $s \sim t$ for $s, t \in S_{ci}^1$, then $s \in [t] = tS_c$ forces $s \in tS^*$ as $s$ is core irreducible. Therefore, (i) implies (ii). Conversely, suppose (ii) holds. Now assume (iii) holds true. For $s \in S, s', t \in [s]$ with $t \in S_{ci}^1$, condition (A1) gives some $t' \in S_{ci}^1, a \in S_c$ with $s' = t'a$ so that $t \sim t'$. By (ii), there is $x \in S^*$ such that $t' = tx$, and hence $s' = txa \in tS_c$, proving (i). \hfill $\square$

We proceed with a natural notion of homomorphisms between right LCM semigroups that appeared in [BLS] Theorem 3.3, though without a name.

**Definition 3.3.** Let $\varphi : S \to T$ be a monoidal homomorphism between two right LCM semigroups $S$ and $T$. Then $\varphi$ is called a homomorphism of right LCM semigroups if

$$\varphi(s_1)T \cap \varphi(s_2)T = \varphi(s_1S \cap s_2S)T \quad \text{for all } s_1, s_2 \in S.$$  

(3.1)

In particular, (3.1) implies that $\varphi(r)$ is a right LCM for $\varphi(s_1)$ and $\varphi(s_2)$ whenever $r \in S$ is a right LCM for $s_1$ and $s_2$ in $S$. This notion allows us to recast the condition of $S_{ci} \subset S$ being $\cap$-closed using the semigroup $S_{ci}^1 := S_{ci} \cup S^*$.

**Proposition 3.4.** Let $S$ be a right LCM semigroup. Then the following hold:

(i) The natural embedding of $S_c$ into $S$ is a homomorphism of right LCM semigroups.

(ii) Suppose $S$ is core factorable. Then $S_{ci} \subset S$ is $\cap$-closed if and only if $S_{ci}^1$ is a right LCM semigroup and the natural embedding of $S_{ci}^1$ into $S$ is a homomorphism of right LCM semigroups.

**Proof.** In both cases, the right hand side of (3.1) is contained in the left hand side. By the definition of $S_c$, any $c \in S$ satisfying $aS \cap bS = cS$ for some $a, b \in S_c$ belongs to $S_c$, and we have $aS_c \cap bS_c \neq \emptyset$ for all $a, b \in S_c$. This proves (i).

Suppose (A1) holds. Assuming (A2), we must prove that $S_{ci}^1$ is a semigroup. The non-trivial case is to show that $st \in S_{ci}$ when $s, t$ are core irreducible. So let $s, t \in S_{ci}$
and \( st = ra \) for some \( r \in S \setminus S_c, a \in S_c \). As \( S \) is core factorable, we can assume \( r \in S_{ci} \). Then \( ss \cap rS = r'S \) for some \( r' \in S \). Since \( s, r \) are in \( S_{ci} \), so is \( r' \) by our assumption. Since \( ra \in r'S \cap rS \), it follows that \( r' \sim r \). By Lemma 3.2 ([A2]) yields \( r \in r'S^* \subset sS \). Therefore \( r = ss' \) for some \( s' \in S \), and left cancellation leads to \( t = s'a \). By core irreducibility of \( t \), we conclude that \( a \in S^* \). Thus \( st \in S_{ci} \).

For the other claim in the forward direction of (ii), it suffices to look at intersections \( sS' \cap tS' \) versus \( sS \cap tS \) with \( s, t \in S_{ci} \) because when \( s \in S^* \) we are left with \( tS' \) and \( tS \), respectively. Since \( S_{ci} \) is a subsemigroup of \( S \), we have \( sS' \cap tS' \subset sS \cap tS \). In particular, we can assume that \( sS \cap tS = rS \) for some \( r \in S \). By ([A2]) \( r \in S_{ci} \). Then any \( s', t' \in S \) with \( ss' = rt' \) belong to \( S_{ci}' \). Therefore \( sS' \cap tS' = rS_{ci} \), showing that the intersection of principal right ideals in \( S_{ci} \) may be computed in \( S \). Hence, \( S_{ci} \) is right LCM because \( S \) is, and the embedding is a homomorphism of right LCM semigroups.

Conversely, let \( s, t \in S_{ci} \) such that \( sS \cap tS = rS \) for some \( r \in S \). Then \( r \in S \setminus S_c \). But the right LCM property for the embedding of \( S_{ci}' \) into \( S \) gives \( r \in S_{ci} \), so \( r \in (S \setminus S_c) \cap S_{ci}' = S_{ci} \), which is ([A2]).

Lemma 3.5. Suppose \( S \) is a core factorable right LCM semigroup, and \( a \in S_c, s \in S_{ci} \) satisfy \( aS \cap ss' = atS \) and \( at = sb \) for some \( b, t \in S \). Then \( b \in S_c \). Further, if \( S_{ci} \subset S \) is \( \cap \)-closed, then \( t \) belongs to \( S_{ci} \).

**Proof.** To see that \( b \in S_c \), let \( s' \in S \). Since \( a \in S_c \), we have \( aS \cap ss'S \neq \emptyset \). Hence left cancellation gives

\[
ssbS \cap ss'S = (aS \cap ssS) \cap ss'S \neq \emptyset \quad \implies \quad bS \cap ss'S \neq \emptyset \quad \implies \quad b \in S_c.
\]

Since \( s \in S_{ci} \), we must have \( t \in S \setminus S_c \). We use ([A1]) to write \( t = rc \) for \( r \in S_{ci} \) and \( c \in S_c \). Then \( ar \sim s \), and due to ([A2]) we know from Lemma 3.2 that \( ar \in sS_c \subseteq sS \). Hence \( ar \in aS \cap ss = atS \), and so \( arS \subseteq atS \). Since \( atS = arcS \subseteq arS \), we have \( arS = atS \). So \( rS = tS \). This implies that \( t = rx \) for some \( x \in S^* \), and thus \( t \in S_{ci} \) as \( r \in S_{ci} \).

We now prove some fundamental results about right LCM semigroups that admit a generalised scale.

**Proposition 3.6.** Let \( N \) be a generalised scale on a right LCM semigroup \( S \).

(i) We have \( \ker N = S_c \), and \( S_c \) is a proper subsemigroup of \( S \).

(ii) If \( s, t \in S \) satisfy \( N_s = N_t \), then either \( s \sim t \) or \( s \perp t \).

(iii) Let \( F \) be a foundation set for \( S \) with \( F \subset N^{-1}(n) \) for some \( n \in N(S) \). If \( |F| = n \) or \( F \) is accurate, then \( F \) is a transversal for \( N^{-1}(n)/\sim \).

(iv) The map \( N \) is a homomorphism of right LCM semigroups.

(v) Every foundation set for \( S \) admits an accurate refinement by a transversal for \( N^{-1}(n)/\sim \) for some \( n \in N(S) \). In particular, \( S \) has the accurate refinement property.

**Proof.** For (i), suppose \( s \in \ker N \). Then ([A3]) implies that \( \{s\} \) is a foundation set for \( S \), which says exactly that \( s \in S_c \). On the other hand, if \( s \in S \) with \( N_s > 1 \), then ([A3]) implies that \( s \) is part of an accurate foundation set of cardinality \( N_s \). Thus \( \{s\} \) is not a foundation set, and hence \( s \notin S_c \). The second claim holds because \( N \) is nontrivial.
For (ii), we first claim that if \( s, s' \in S \) satisfy \( s \sim s' \), then \( s \perp t \) if and only \( s' \perp t \) for all \( t \in S \). Indeed, for \( sa = sb \) with \( a, b \in S_c \) we have

\[
sS \cap tS \neq \emptyset \quad \overset{\text{by \([(A1)])}}{\iff} \quad saS \cap tS \neq \emptyset \quad \overset{\text{by \([(A3)](b)]}}{\iff} \quad sS \cap tS \neq \emptyset.
\]

Now suppose we have \( s, t \in S \) with \( N_s = N_t \). Choose a transversal for \( N^{-1}(N_s)/\sim \) and let \( s' \) and \( t' \) be the unique elements of the transversal satisfying \( s \sim s' \) and \( t \sim t' \). If \( s' = t' \), then \( s \sim t \). If \( s' \neq t' \), then since we know from \([(A3)]\) that the transversal is an accurate foundation set for \( S \), we have \( s' \perp t' \). It now follows from the claim that \( s \perp t \).

For (iii), let \( T_n \) be a transversal for \( N^{-1}(n)/\sim \). We know that \( T_n \) is accurate because of \([(A3)](b)]\) By (ii), we know that for every \( f \in F \), there is precisely one \( t_f \in T_n \) with \( f \sim t_f \). So \( f \mapsto t_f \) is a well-defined map from \( F \) to \( T_n \), which is surjective because \( F \) is a foundation set. The map is injective if \( F \) is accurate or has cardinality \(|T_n| = n\).

For (iv), let \( s, t, r \in S \) with \( sS \cap tS = rS \). Then \( N_r = N_sN(S) \cap N_tN(S) \). We need to show that \( N_r \) is the least common multiple of \( N_s \) and \( N_t \) inside \( N(S) \). Suppose there are \( m, n \in N(S) \) with \( N_s m = N_t n \). For \( k = N_s, N_t, m, n \) pick transversals \( T_k \) for \( N^{-1}(k)/\sim \) such that \( s \in T_{N_s} \) and \( t \in T_{N_t} \). Since \( s \perp t \), and \( T_n \) is a foundation set, there exists \( s' \in T_{N_s} \) such that \( ss' \not\perp t \). The same argument for \( n \) shows that there is \( t' \in T_{N_t} \) with \( ss' \not\perp tt' \). Now \( T_{N_s}T_{N_t} \) and \( T_{N_s}T_{N_t} \) are accurate foundation sets contained in \( N^{-1}(N_{sa}) \).

By (iii), they are transversals for \( N^{-1}(N_{sa})/\sim \). In particular, \( ss' \not\perp tt' \) is equivalent to \( ss' \not\perp tt' \). Thus there are \( a, b \in S_c \) with \( ss' a = tt' b \in sS \cap tS = rS \), which results in \( N_{sa} N_{sb} \in N(S) \) due to (i). Thus \( N_r \) is the least common multiple of \( N_s \) and \( N_t \).

For (v), let \( F \) be a foundation set for \( S \). Define \( n \) to be the least common multiple of \( \{N_f \mid f \in F \} \) inside \( N(S) \), and set \( n_f := n/N_f \in N(S) \) for each \( f \in F \). Choose a transversal \( T_f \) for \( N^{-1}(n_f)/\sim \) for every \( f \in F \). Then \( F' := \bigcup_{f \in F} fT_f \) is a foundation set as \( F \) is a foundation set and each \( T_f \) is a foundation set. Moreover, we have \( F' \subset N^{-1}(n) \).

Thus by (ii), for \( f, f' \in F' \), we either have \( f \sim f' \) or \( f \perp f' \). If we now choose a maximal subset \( F_a \) of \( F' \) with elements having mutually disjoint principal right ideals, \( F_a \) is necessarily an accurate foundation set that refines \( F \). In fact, \( F_a \) is a transversal for \( N^{-1}(n)/\sim \). In particular, this shows that \( S \) has the accurate refinement property. \( \square \)

The next result is an immediate corollary of Proposition 3.6 (ii) and Lemma 3.2.

**Corollary 3.7.** Let \( S \) be a core factorable right LCM semigroup such that \( S_{c1} \subset S \) is \( \cap \)-closed. If \( N \) is a generalised scale on \( S \), then \( N_s = N_t \) for \( s, t \in S_{c1} \) forces either \( s \perp t \) or \( s \in tS^* \).

In particular, if \( T_n = \{f_1, \ldots, f_n\} \) and \( T_n' = \{f'_1, \ldots, f'_n\} \) are two transversals for \( N^{-1}(n)/\sim \), respectively, such that both are contained in \( S_{c1} \), then there are \( x_1, \ldots, x_n \) in \( S^* \) and a permutation \( \rho \) of \( \{1, \ldots, n\} \) so that \( f'_i = f_{\rho(i)}x_i \) for \( i = 1, \ldots, n \).

Lemma 3.2 allows us to strengthen the factorisation \( S = S_{c1}S_c \) in the following sense.

**Lemma 3.8.** Let \( S \) be a core factorable right LCM semigroup such that \( S_{c1} \subset S \) is \( \cap \)-closed. Then there are a transversal \( T \) for \( S/\sim \) with \( T \subset S_{c1} \) and maps \( i: S \to T \), \( c: S \to S_c \) such that \( s = i(s)c(s) \) for all \( s \in S \). For every family \( (x_t)_{t \in T} \subset S^* \) with \( x_1 = 1 \), the set \( T' := \{tx_t \mid t \in T \} \) defines a transversal for \( S/\sim \) with \( T' \subset S_{c1} \), and every transversal for \( S/\sim \) contained in \( S_{c1} \) is of this form.

**Proof.** By \([(A1)] \) \( S/\sim \) admits a transversal \( T \) with \( T \subset S_{c1} \). For each \( s \in S \) we define \( i(s) \) to be the single element in \( T \cap [s] \), and \( c(s) \) to be the unique element in \( S_c \) such
that \( s = i(s)c(s) \), which exists by virtue of Lemma 3.2 due to the presence of [A2]. This gives us the maps \( i \) and \( e \).

As \( S_{ci}S^* = S_{ci} \) and \( S^* \cap S_{ci}^1 = \{1\} \), the remaining claims follow from Lemma 3.2. □

In our investigations we have found that uniqueness of the KMS\(_\beta\)-state for \( \beta \) in a critical interval, see Theorem 4.3, is closely related to properties of a natural action \( \alpha \) of \( S_c \) on \( S/\sim \), arising from left multiplication. By an action \( \alpha \) of a semigroup \( T \) on a set \( X \) we mean a map \( T \times X \to X, (t, x) \mapsto \alpha_t(x) \) such that \( \alpha_{st}(x) = \alpha_s(\alpha_t(x)) \) for all \( s, t \in T \) and \( x \in X \). For later use, we denote \( \text{Fix}(\alpha_t) \) the set \( \{ x \in X \mid \alpha_t(x) = x \} \).

Lemma 3.9. Let \( S \) be a right LCM semigroup. Then left multiplication defines an action \( \alpha: S_c \curvearrowright S/\sim \) by bijections \( \alpha_s([s]) = [as] \). Every generalised scale \( N \) on \( S \) is invariant under this action. If \( S \) is admissible and \( T \subset S_{ci}^1 \) is a transversal for \( S/\sim \), then there is a corresponding action of \( S_c \) on \( T \), still denoted \( \alpha \), which satisfies

\[
\alpha S \cap tS = \alpha a^{-1}_s(t)S \quad \text{for all } a \in S_c, t \in T.
\]

Proof. For every \( a \in S_c \), the map \( \alpha_a \) is well-defined as left multiplication preserves the core equivalence relation. If \( as \sim at \) for some \( s, t \in S \), then \( s \sim t \) by left cancellation. Hence \( \alpha_a \) is injective. On the other hand, Lemma 3.5 states that for \( s \in S \) we have \( aS \cap sS = atS, at = sb \) for some \( b \in S_c \) and \( t \in S \). Thus \( \alpha_a([t]) = [at] = [s] \), and we conclude that \( \alpha_a \) is a bijection.

For a generalised scale \( N \), invariance follows from \( \ker N = S_c \), see Proposition 3.6 (i).

Finally, let \( T \) be a transversal for \( S/\sim \) contained in \( S_{ci}^1 \). For \( a \in S_c \) and \( t \in T \), it is immediate that \( aa^{-1}_a(t) = tc(aa^{-1}_a(t)) \). Assume \( t \neq 1 \) (else the claim about the intersection is trivial), and use Lemma 3.5 to write \( aS \cap tS = asS \) for some \( s \in S_{ci} \). By the choice of \( T \), there is a unique \( s \in T \) with this property. Proposition 3.6 (i) and (iv) show that \( N a(s) = N a = N_t \), so that \( t = \alpha_a(s) \) by Corollary 3.7. □

Recall that an action \( \gamma: T \curvearrowright X \) is faithful if, for all \( s, t \in T, s \neq t \), there is \( x \in X \) such that \( \gamma_s(x) \neq \gamma_t(x) \).

Corollary 3.10. If \( S \) is a right LCM semigroup for which \( S_c \curvearrowright S/\sim \) is faithful, then \( S_c \) is right cancellative, and hence a right Ore semigroup. In particular, \( S_c \) is group embeddable.

Proof. By Lemma 3.9 we have a monoidal homomorphism from \( S_c \) to the bijections on \( S/\sim \), which is injective due to faithfulness of the action. □

Remark 3.11. Recall that an action \( \gamma: T \curvearrowright X \) of a monoid \( T \) on a set \( X \) is called almost free if the set \( \{ x \in X \mid \gamma_s(x) = \gamma_t(x) \} \) is finite for all \( s, t \in T, s \neq t \). Clearly, if \( X \) is infinite, then every almost free action \( \gamma \) on \( X \) is faithful. In particular, this is the case for \( \alpha: S_c \curvearrowright S/\sim \) for every admissible right LCM semigroup \( S \).

Remark 3.12. If \( T \) is a group, then almost freeness states that every \( \gamma_t \) fixes only finitely many points in \( X \). The same conclusion holds if \( T \) is totally ordered with respect to \( s \geq t \) :\( s \in tT \) and acts by injective maps: We then have \( T^* = \{1\} \), and \( s = tr \) \( (s \geq t) \) gives \( \gamma_t^{-1} \gamma_s = \gamma_r \), while \( t = sv \) \( (t \geq s) \) yields \( \gamma_s^{-1} \gamma_t = \gamma_r \).

Almost freeness is known to be too restrictive a condition for some of the semigroups we wish to study, for instance for \( S = X^* \rtimes G \) arising from a self-similar action \((G, X)\).
that admits a word \( w \in X^* \) such that \( g(w) = w \) for some \( g \in G \setminus \{1\} \) (see \textsc{LRRW14} for more on self-similar actions). To study the uniqueness of KMS\( _\beta \)-states with \( \beta \) in the critical interval, we will thus also elaborate on the approach used in \textsc{LRRW14} for self-similar actions. But we phrase their ideas in a more abstract setting which gives us a potentially useful extra degree of freedom: The condition we work with is a localised version of the finite state condition used in \textsc{LRRW14}.

**Definition 3.13.** Let \( S \) be an admissible right LCM semigroup, and \( \mathcal{T} \) a transversal for \( S/\sim \) with \( \mathcal{T} \subset S^1_c \). For \( a \in S_c \), let \( C_a := \{ c(af) \mid f \in \mathcal{T} \} \). The transversal \( \mathcal{T} \) is said to be a witness of finite propagation of \( S \) at \( (a,b) \in S_c \times S_c \) if \( C_a \) and \( C_b \) are finite. The right LCM semigroup \( S \) is said to have finite propagation if there is a witness of finite propagation at every pair \( (a,b) \in S_c \times S_c \).

**Remark 3.14.** (a) If \( S \) is such that \( S^* \) is infinite, then for any prescribed \( a \in S_c \setminus \{1\} \) there exists a transversal \( \mathcal{T} \) for which \( C_a \) is infinite, just take \( \{ x \pi(c(a)) \mid x \in S^* \} \). Thus, the choice of \( \mathcal{T} \) is crucial in Definition 3.13.

(b) Note that \( S \) has finite propagation if and only if for every transversal \( \mathcal{T} \) for \( S/\sim \) with \( \mathcal{T} \subset S^1_c \) (with corresponding maps \( i, c \)) and for all \( a, b \in S_c \), there is \( (x_f)_f \in \mathcal{T} \subset S^* \) such that \( \{ x_f c(af) \mid f \in \mathcal{T} \} \) and \( \{ x_f c(bf) \mid f \in \mathcal{T} \} \) are finite. Indeed, assuming that \( S \) has finite propagation, let \( \mathcal{T} \) be a transversal contained in \( S^1_c \) and let \( a, b \in S_c \). Then there is a transversal \( \mathcal{T}' \) which is a witness of finite propagation at \( (a,b) \). However, by Corollary 3.7, there are \( x_f \in S^* \) for all \( f \in \mathcal{T} \) such that \( \mathcal{T}' = \{ f x_f \mid f \in \mathcal{T} \} \). Hence \( C_a = \{ x_{i(af)} c'(a f') \mid f' \in \mathcal{T}' \} \) is finite, and similarly for \( C_b \). Reversing the argument gives the converse assertion.

## 4. The main theorem

If \( S \) is a right LCM semigroup admitting a generalised scale \( N \), then standard arguments using the universal property of \( C^*(S) \) show that there is a strongly continuous action \( \sigma : \mathbb{R} \to \text{Aut} C^*(S) \), where

\[
\sigma_x(v_s) = N^{it} \pi_x v_s \quad \text{for each } x \in \mathbb{R} \text{ and } s \in S.
\]

It is easy to see that \( \{ v_s v_{s'}^* \mid s, t \in S \} \) is a dense family of analytic elements. Noting that \( \sigma \) is the identity on \( \ker \pi_c \) and \( \ker \pi_p \), hence also on \( \ker \pi_\tau \), the action \( \sigma \) drops to strongly continuous actions of \( \mathbb{R} \) on \( \mathcal{Q}_c(S), \mathcal{Q}_p(S) \) and \( \mathcal{Q}(S) \). Thus we may talk of KMS-states for the corresponding dynamics on \( \mathcal{Q}_c(S) \) and \( \mathcal{Q}_p(S) \).

**Remark 4.1.** Since the canonical embedding of \( S_c \) into \( S \) is a homomorphism of right LCM semigroups, see Proposition 3.4 (i), the universal property of semigroup \( C^*- \)algebras guarantees existence of a \( * \)-homomorphism \( \varphi : C^*(S_c) \to C^*(S), w_a \mapsto v_a \), where \( w_a \) denotes the generating isometry for \( a \in S_c \) in \( C^*(S_c) \).

In our analysis of KMS-states on \( C^*(S) \) and its boundary quotients under the dynamics \( \sigma \), a key role is played by a \( \zeta \)-function for (parts of) \( S \):

**Definition 4.2.** Let \( I \subset \text{Irr}(N(S)) \). For each \( n \in I \), let \( \mathcal{T}_n \) be a transversal for \( N^{-1}(n) \), which by \textsc{A3}[b] is known to be an accurate foundation set. Then the formal series

\[
\zeta_I(\beta) := \sum_{n \in I} \sum_{f \in \mathcal{T}_n} N_f^{-\beta} = \sum_{n \in I} n^{-(\beta-1)},
\]
where $\beta \in \mathbb{R}$, is called the $I$-restricted $\zeta$-function of $S$. For $I = \text{Irr}(N(S))$, we write simply $\zeta_S$ for $\zeta_{\text{Irr}(N(S))}$ and call it the $\zeta$-function of $S$. The critical inverse temperature $\beta_c \in \mathbb{R} \cup \{\infty\}$ is the smallest value so that $\zeta_S(\beta) < \infty$ for all $\beta \in \mathbb{R}$ with $\beta > \beta_c$. The critical interval for $S$ is given by $[1, \beta_c]$ if $\beta_c$ is finite, and $[1, \infty)$ otherwise.

Recall the action $\alpha$ introduced in Lemma 3.9. The statement of our main result, which is the following theorem, makes reference to a trace $\rho$ on $C^*(S_c)$ constructed in Proposition 9.2 under the assumption that $\beta_c = 1$.

**Theorem 4.3.** Let $S$ be an admissible right LCM semigroup and let $\sigma$ be the one-parameter group of automorphisms of $C^*(S)$ given by $\sigma_x(v_s) = N_x^sv_s$ for $x \in \mathbb{R}$ and $s \in S$. The KMS-state structure with respect to $\sigma$ on the boundary quotient diagram $\text{(1.1)}$ has the following properties:

1. There are no KMS$_\beta$-states on $C^*(S)$ for $\beta < 1$.
2. If $\alpha$ is almost free, then for each $\beta$ in the critical interval there is a unique KMS$_\beta$-state $\psi_\beta$ given by $\psi_\beta(v_s v_t^*) = N_s^{-\beta}\delta_{s,t}$ for $s, t \in S$.
3. For $\beta > \beta_c$, there is an affine homeomorphism between KMS$_\beta$-states on $C^*(S)$ and normalised traces on $C^*(S_c)$.
4. Every KMS$_\beta$-state factors through $\pi_c$.
5. A KMS$_\beta$-state factors through $\pi_p$ if and only if $\beta = 1$.
6. There is an affine homeomorphism between ground states on $C^*(S)$ and states on $C^*(S_c)$. In case that $\beta_c < \infty$, a ground state is a KMS$_\infty$-state if and only if it corresponds to a normalised trace on $C^*(S_c)$ under this homeomorphism.
7. Every KMS$_\infty$-state factors through $\pi_c$. If $\beta_c < \infty$, then all ground states on $Q_c(S)$ are KMS$_\infty$-states if and only if every ground state on $C^*(S)$ that factors through $\pi_c$ corresponds to a normalised trace on $C^*(S_c)$ via the map from (6).
8. No ground state on $C^*(S)$ factors through $\pi_p$, and hence none factor through $\pi$.

**Proof.** Assertion (1) will follow directly from Proposition 6.1. For the existence claim in (2a) and (2b), we apply Proposition 8.8 starting from the canonical trace $\tau$ on $C^*(S_c)$ given by $\tau(w_s w_t^*) = \delta_{s,t}$ for all $s, t \in S_c$. The uniqueness assertion will follow from Proposition 9.1 for (2a), and from Proposition 9.2 for (2b).

For assertion (3), Proposition 8.8 shows the existence of a continuous, affine parametrisation $\tau \mapsto \psi_{\beta, \tau}$ that produces a KMS$_\beta$-state $\psi_{\beta, \tau}$ on $C^*(S)$ out of a trace $\tau$ on $C^*(S_c)$. This map is surjective by Corollary 8.9 and injective by Proposition 8.10.

To prove (4), it suffices to show that a KMS$_\beta$-state $\phi$ on $C^*(S)$ for $\beta \in \mathbb{R}$ vanishes on $1 - v_a v_a^*$ for all $a \in S_c$. But $S_c = \ker \phi$ implies $\phi(v_a) = v_a$ so $\phi(v_a v_a^*) = \phi(v_a^* v_a) = \phi(1) = 1$ since $v_a$ is an isometry. Assertion (5) is the content of Proposition 6.6.

Towards proving (6), Proposition 8.6 provides an affine assignment $\rho \mapsto \psi_\rho$ of states $\rho$ on $C^*(S_c)$ to ground states $\psi_\rho$ of $C^*(S)$ such that $\psi_{\rho \circ \varphi} = \rho$, for the canonical map $\varphi: C^*(S_c) \to C^*(S)$ from Remark 4.1. Thus $\rho \mapsto \psi_\rho$ is injective, and surjectivity follows from Proposition 6.2. If $\beta_c < \infty$, then (3) and Proposition 6.5 show that KMS$_\infty$-states correspond to normalised traces on $C^*(S_c)$.

The first claim in statement (7) follows from (4). For the second part of (7), we observe that $\phi \circ \pi_c$ is a ground state on $C^*(S)$ for every ground state $\phi$ on $Q_c(S)$. Suppose first
that all ground states on $Q_c(S)$ are KMS$_\infty$-states. If $\phi'$ is a ground state on $C^*(S)$ such that $\phi' = \phi \circ \pi_c$ for some ground state $\phi$ on $Q_c(S)$, then $\phi$ is a KMS$_\infty$-state by assumption. This readily implies that $\phi'$ is also a KMS$_\infty$-state, and hence corresponds to a trace as $\beta_c < \infty$, see (6). Conversely, suppose that all ground states on $C^*(S)$ that factor through $\pi_c$ correspond to traces under the map from (6). Then every ground state $\phi$ on $Q_c(S)$ corresponds to a trace via $\phi \circ \pi_c$. Hence $\phi \circ \pi_c$ is a KMS$_\infty$-state by (6) as $\beta_c < \infty$, and (4) implies that $\phi$ is a KMS$_\infty$-state on $Q_c(S)$ as well.

Finally, statement (8) is proved in Proposition 6.7.

Remark 4.4. The idea behind the trace $\rho$ on $C^*(S_c)$ in Theorem 4.3 (2b) stems from [LRRW14]. If $S$ is right cancellative, then $\rho$ is the canonical trace, i.e. $\rho(w_aw_b^*) = \delta_{a,b}$ for $a,b \in S_c$. If $\alpha$ is almost free then $\alpha$ is faithful by Remark 3.11 and $S_c$ is right cancellative by Corollary 3.10. Now if $S_c$ is right cancellative, then the value $\rho(w_aw_b^*)$ describes the asymptotic proportion of elements in $S/\sim \cap N^{-1}(n)$ that are fixed under $\alpha_n\alpha_b^{-1}$ as $n \to \infty$ in $N(S)$. Thus, if $\beta_c = 1$, and $S$ has finite propagation, then almost freeness forces $\rho$ to be the canonical trace so that (2a) and (2b) are consistent.

Remark 4.5. In [CDL13], the KMS-state structure for $S = R \rtimes R^\times$, where $R$ is the ring of integers in a number field was considered for a dynamics of the same type as we discuss here. $S$ is right LCM if and only if the class group of $R$ is trivial, and it is quite intriguing to see how the results of [CDL13] relate to Theorem 4.3. We have $\beta_c = 2$ and the unique KMS$_\beta$-state for $\beta \in [1,2]$ in [CDL13] Theorem 6.7 is analogous to the one in Theorem 4.3 (2a). Since the action $\alpha : S_c \rtimes S/\sim$ is almost free, we may expect this uniqueness result to hold outside the class of right LCM semigroups.

On the other hand, the parametrisation of KMS$_\beta$-states for $\beta > 2$ from [CDL13 Theorem 7.3] is by traces on $\bigoplus_\gamma C^*(J_\gamma \rtimes R^\times)$, where $\gamma$ ranges over the class group along with a fixed reference ideal $J_\gamma \in \gamma$. If $S$ is right LCM, then this coincides with Theorem 4.3 (3) as $S_c = S^* = R \rtimes R^\times$ and we can take $J_1 = R$. So [CDL13] predicts that we can expect to see a more complicated trace simplex beyond the right LCM case that reflects the finer structure of $\alpha : S_c \rtimes S/\sim$. In connection with the ideas behind Theorem 4.3 (6), this might also yield a new perspective on the description of ground states provided in [CDL13] Theorem 8.8.

Remark 4.6. The phase transition result for ground states for the passage from $C^*(S)$ to $Q_c(S)$ in Theorem 4.3 (7) raises the question whether there is an example of an admissible right LCM semigroup $S$ for which $\pi_c$ is nontrivial and not all ground states on $Q_c(S)$ are KMS$_\infty$-states. In all the examples we know so far, see Section 5, we either have $\pi_c = \text{id}$, for instance for the case of self-similar actions or algebraic dynamical systems, or the core subsemigroup $S_c$ is abelian so that all ground states that factor through $\pi_c$ correspond to traces on $C^*(S_c)$.

5. Examples

Before proving our main theorem on the KMS structure of admissible right LCM semigroups, we use this section to prove that a large number of concrete examples from the literature are admissible. Indeed, for some of the requirements for admissibility (A1) and (A2), we do not know any right LCM semigroups that fail to possess them. We
break this section into subsections, starting with reduction results for Zappa-Szép products of right LCM semigroups that will be applied to easy right-angled Artin monoids. However, the case of algebra dynamical systems requires different considerations, see 5.6.

5.1. Zappa-Szép products. Let $U,A$ be semigroups with identity $e_U,e_A$. Suppose there exist maps $(a,u) \mapsto a(u) : A \times U \to U$ and $(a,u) \mapsto a|_{u} : A \times U \to A$ such that

\begin{align*}
(ZS1) \quad & e_A(u) = u; \\
(ZS2) \quad & (ab)(u) = a(b(u)); \\
(ZS3) \quad & a(e_U) = e_U; \\
(ZS4) \quad & a|_{e_U} = a; \quad (ZS5) \quad & e_A|_{u} = e_A; \\
(ZS6) \quad & a|_{uv} = (a|_{u})|_{v}; \quad (ZS7) \quad & a(uv) = a(u)a|_{u}(v); \quad (ZS8) \quad & a|_{vu} = a|_{b(u)b|_{u}}.
\end{align*}

Following [Bri05, Lemma 3.13(xv)] and [BRRW14], the external Zappa-Szép product $U \bowtie A$ is the cartesian product $U \times A$ endowed with the multiplication $(u,a)(v,b) = (ua(v),a|_{b})$ for all $a,b \in A$ and $u,v \in U$. Given $a \in A$, $u \in U$, we call $a(u)$ the action of $a$ on $u$, and $a|_{u}$ the restriction of $a$ to $u$. For convenience, we write $1$ for both $e_A$ and $e_U$.

In this subsection, we derive an efficient way of identifying admissible Zappa-Szép products $S = U \bowtie A$ among those which share an additional intersection property: For this class, the conditions (A1)–(A4) mostly boil down to the corresponding statements on $U$. Suppose that $S = U \bowtie A$ is a right LCM semigroup and that $A$ is also right LCM. Then $U$ is necessarily a right LCM semigroup. Let us remark that $S^{*} = U^{*} \bowtie A^{*}$. Moreover, for $(u,a),(v,b) \in S$, it is straightforward to see that $(u,a) \not\perp (v,b)$ implies $u \not\perp v$. We are interested in a strong form of the converse implication:

\[ uU \cap vU = wU, \text{ then } (u,a)S \cap (v,b)S = (w,c)S \text{ for some } c \in A. \]

We say that (A1) holds (for $S$) if it is valid for all $(u,a),(v,b) \in S$.

**Remark 5.1.** The property (A1) is inspired by [BRRW14] Remark 3.4]. If a Zappa-Szép product $U \bowtie A$ satisfies the hypotheses of [BRRW14, Lemma 3.3]], that is,

\begin{itemize}
  \item[(a)] $U$ is a right LCM semigroup,
  \item[(b)] $A$ is a left cancellative monoid whose constructible right ideals are totally ordered by inclusion, and
  \item[(c)] $u \mapsto a(u)$ is a bijection of $U$ for each $a \in A$,
\end{itemize}

then [BRRW14] Remark 3.4] shows that the right LCM semigroup $U \bowtie A$ satisfies (A1).

We have seen that a Zappa-Szép product $U \bowtie A$ satisfies (A1) if and only if $U \bowtie A$ is an external Zappa-Szép product. However, the constructible ideals $U \bowtie A$ play a crucial role in the study of dynamical systems and operator algebras. Therefore, we will focus on understanding the structure of these ideals in order to establish the corresponding results.
of $A$ are directed but not totally ordered by inclusion, and the action of $A$ on $U$ is by non-surjective injections for all $a \in A \setminus \{0\}$. Note that the same treatment applies to $S' = T \cup \mathbb{N}$ for every nontrivial, left cancellative, left reversible monoid $T$.

**Theorem 5.3.** If $S = U \bowtie A$ is a right LCM semigroup with $A$ right LCM such that \(\square\) holds, then

(i) $A_e = A$, i.e. $A$ is left reversible,

(ii) $S_c$ equals $U_c \bowtie A$,

(iii) $(u, a) \sim (v, b)$ if and only if $u \sim v$,

(iv) $S_{c_1}$ is given by $U_{c_1} \bowtie A^*$,

(v) $S$ is core factorable if and only $U$ is core factorable, and

(vi) $S_{c_1} \subset S$ is \(\cap\)-closed if and only if $U_{c_1} \subset U$ is \(\cap\)-closed.

**Proof.** Part (i) follows from \(\square\) as $$(1, a)S \cap (1, b)S = (1, c)S$$ for some $c \in A$, so that $c \in aA \cap bA$. Similarly, \(\square\) forces $S_c = U_c \times A$ as $A = A_e$. In addition, to get a Zappa-Szép product as claimed in (ii), we note that $$(1)$$

and (1) is a right LCM such that $$(2)$$

holds, then

Part (iii) is an easy consequence of (ii): $(u, a) \sim (v, b)$ forces $u \sim v$ since $a(w) \in U_c$ for all $a \in A$, $w \in U_c$. Conversely, let $uw = vw'$ for some $w, w' \in U_c$ and let $a, b \in A$. Then $(1, a)(w, 1), (1, b)(w', 1) \in S_c$ and hence there are $s, s', t, t' \in S_c$ such that $(1, a)s = ws$ and $(1, b)t = w't$. Likewise, there are $s'', t'' \in S_c$ such that $s'' = tt''$. This leads us to

$$(u, 1)(1, a)s'' = (u, 1)(w, 1)s'' = (uw, 1)t'' = (vw', 1)t'' = (v, b)t''t''$$

which proves $(u, a) \sim (v, b)$ as $s''t'' \in S_c$.

For (iv), we first show that $S_{c_1} = U_{c_1} \times A^*$. Suppose we have $(u, a) \in S_{c_1}$. If $a \in A \setminus A^*$, then $(u, a) = (u, 1)(1, a)$ together with (ii) shows that $(u, a)$ is not core irreducible. Thus we must have $a \in A^*$. Similarly, if $u \notin U_{c_1}$, then $u \in U^*$ or there exist $v \in U, w \in U_c \setminus U^*$ such that $u = vw$. In the first case, we get $(u, a) \in S^*$ and hence $(u, a) \notin S_{c_1}$. The latter case yields a factorization $(u, a) = (v, 1)(w, a)$ with $(v, w) \in S_c \setminus S^*$. Altogether, this shows $S_{c_1} \subset U_{c_1} \times A^*$. To prove the reverse containment, suppose $(u, a) \in U_{c_1} \times A^*$ and $(u, a) = (v, b)(w, c) = (vb(w), b_w)c$ with $(v, b) \in S, (w, c) \in S_c = U_c \bowtie A$. Then $b_w, c \in A^*$ as $a \in A^*$, and the Zappa-Szép product structure on $S_c$ implies $(b(w) \in U_c$. Since $u \in U_{c_1}$, we conclude that $(b(w) \in U^*$. But then $(1, b)(w, 1) = (b(w), b_w) \in U^* \bowtie A^* = S^*$ forces $(1, b)(w, 1) \in S^*$. In particular, we have $w \in U^*$ and hence $(w, c) \in S_c^*$. Thus $(u, a) \in S_{c_1}$ and we have shown that $S_{c_1} = U_{c_1} \times A^*$.

Next we prove that $a(u) \in U_{c_1}$ and $a|_{u} \in A^*$ whenever $a \in A^*, u \in U_{c_1}$. The second part is easily checked with $a|_{u} = a - 1|_{a}|_{u}$. So suppose $a(u) = vw$ with $w \in U_c$. Then we get $u = a - 1(v)a - 1|_{v}(w)$, and $S_c = U_c \bowtie A$ yields $a - 1|_{v}(w) \in U_c$. Since $u$ is core irreducible, we get $a - 1|_{v}(w) \in U^*$, and then $w \in U^*$ due to $S^* = U^* \bowtie A^*$. Therefore, $a(u) \in U_{c_1}$ and we have proven (iv).

For (v), let $U = U_{c_1}U_c$. Observe that (ii) and (iv) yield $S_{c_1}S_c = (U_c \bowtie A^*)^1(U_c \bowtie A) \supseteq U_{c_1}U_c \times S \supseteq S$ which is $\cap$-closed. Conversely, suppose that $S = S_{c_1}S_c$. For $u \in U$, take $(u, 1) = (v, a)(w, b) = (va(w), a|_{w}b)$ with $(v, a) \in S_{c_1} = (U_c \bowtie A^*)^1$ and $(w, b) \in S_c = U_c \bowtie A$. For $u \in U \setminus U_c$ we get $v \in U_{c_1}$. Combining this with $a(w) \in U_c$ using (ii), we arrive at $u = va(w) \in U_{c_1}U_c$. Thus $U = U_{c_1}U_c$ holds.

For (vi), suppose that $S_{c_1} \subset S$ is $\cap$-closed. Let $u, v \in U_{c_1}$ such that $wU \cap wU = wU$ for some $w \in U$. Therefore $(u, 1)S \cap (v, 1)S = (w, 1)S$, which by (iv) means that $w \in U_{c_1}$.
Conversely, suppose that $U_{ci} \subset U$ is $\cap$-closed. As $S_{ci} = U_{ci} \bowtie A^*$ by (iv) and $sxS = sS$ for all $s \in S, x \in S^*$, it suffices to consider $(u,1)S \cap (v,1)S = (w,1)S$ for $u, v \in U_{ci}$. If this holds, then $wU \cap vU = w'u'U$ for some $w', v' \in U$ with $w \in w'u'U$. As $U_{ci} \subset U$ is $\cap$-closed, $w' \in U_{ci}$. Moreover, (i) implies that $(w,1)S = (u,1)S \cap (v,1)S = (w',a)S$ for some $a \in A$, which amounts to $a \in A^*$ and, more importantly, $w \in w'U^* \cap U_{ci}$. Thus $(w,1) \in S_{ci}$ by (iv).

Proposition 5.4. Let $S = U \bowtie A$ be a right LCM semigroup with $A$ right LCM such that $\square$ holds. The restriction $N \mapsto N|_U$ defines a one-to-one correspondence between

(a) generalised scales $N: S \rightarrow \mathbb{N}^\times$; and

(b) generalised scales $N': U \rightarrow \mathbb{N}^\times$ satisfying $N'_a(a) = N_a$ for all $a \in A, u \in U$.

Moreover, $N(S) = N|_U(U)$, so that $\square$ holds for $N$ if and only if it holds for $N|_U$.

Proof. For every generalised scale $N$ on $S$, the restriction $N|_U(U) : U \rightarrow \mathbb{N}^\times$ defines a non-trivial homomorphism. Since $S_c = \ker N$, Theorem 5.3 (ii) implies that $N(S) = N|_U(U)$ for all $(u, a) \in S$ and $N|_U(u, a) = N((u, a)) \rightarrow N((u, 1)) = N((u, 1))$ for all $(u, a) \in S$. Theorem 5.3 (iv) implies that $F := \{(u, 1) | u \in F\}$ is a transversal for $N^{-1}(n)/\sim$ for $n \in N|_U(U)$, then Theorem 5.3 (iii) implies that $F := \{(u, 1) | u \in F\}$ is a transversal for $N^{-1}(n)/\sim$. Due to $\square$ for $N$, we know that $F$ is an accurate foundation set for $S$. Then $F'$ is an accurate foundation set because of $\square$. Thus we get $\square$ for $N|_U$. Conversely, suppose that $N'$ is one of the generalised scales in (b). Then $N: S \rightarrow \mathbb{N}^\times, (u, a) \mapsto N'(u)$ defines a non-trivial homomorphism on $S$ with $N|_U = N'$ because $N((u, a)(v, b)) = N((ua,v), a, b)) = N'(ua(v)) = N'(u)a(v) = N((u, a))N((v, b))$ for $(u, a), (v, b) \in S$ by the invariance of $N'$ under the action of $A$. Similarly to the first part, Theorem 5.3 (iii) shows that $\square$ holds for $N'$ to $N$. Now if $F$ is a transversal for $N^{-1}(n)/\sim$ for some $n \in N(S)$, then $F_U := \{(u, a) | u, a \in F\}$ is a transversal for $N^{-1}(n)/\sim$, by Theorem 5.3 (iii). Since $F_U$ is an accurate foundation set for $U$ due to $\square$, $\square$ implies that $F$ is an accurate foundation set for $S$, i.e. $\square$ holds for $N$.

Corollary 5.5. Let $S = U \bowtie A$ be a right LCM semigroup with $A$ right LCM such that $\square$ holds. Then $S$ is admissible if and only if $U$ is admissible with a generalised scale that is invariant under the action of $A$ on $U$.

Proof. This follows from Theorem 5.3 (v),(vi) and Proposition 5.4.

Observe that for $S = U \bowtie A$ with $S_c = U_c \bowtie A$, the action of $A$ on $U$ induces a well-defined action $\gamma: A \times U/\sim \rightarrow U/\sim, [a, [u]] \mapsto [a(u)]$.

Corollary 5.6. Suppose that $S = U \bowtie A$ is a right LCM semigroup with $A$ right LCM satisfying Theorem 5.3 (ii),(iii). If $U_c$ is trivial, then the following statements hold:

(i) The action $\alpha: S_c \cap S/\sim$ is faithful if and only if $\gamma: A \cap S/\sim$ is faithful.

(ii) The action $\alpha$ is almost free if and only if $\gamma$ is almost free.

Proof. We start by noting that $\alpha_{u,c}([u,c]) = [(a(u), a|_c)] = [b(u), b|_c] = \alpha_{(b,c)}([u,c])$ if and only if $\gamma_{u}([u]) = [b(u)] = \gamma_{c}([u])$ for all $a, b, c \in A, u \in U$ by Theorem 5.3 (ii) and (iii). If $U_c$ is trivial the canonical embedding $A \rightarrow S_c = U_c \bowtie A$ is an isomorphism which is equivariant for $\alpha$ and $\gamma$. □
5.2. Easy right-angled Artin monoids. For $m, n \in \mathbb{N} \cup \infty$, consider the direct product $S := F^+_m \times \mathbb{N}^n$, where $F^+_m$ denotes the free monoid in $m$ generators and $\mathbb{N}^n$ denotes the free abelian monoid in $n$ generators. The monoid $S$ is a very simple case of a right-angled Artin monoid arising from the graph $\Gamma = (V, E)$ with vertices $V = V_m \cup V_n$, where $V_m := \{v_k \mid 1 \leq k \leq m\}$ and $V_n := \{v'_k \mid 1 \leq k \leq n\}$, and edge set $E = \{(v, w) \mid v \in V_m \text{ or } w \in V_n\}$. In particular, $(S, F^+_m \times \mathbb{Z}^n)$ is a quasi lattice-ordered group, see [CL07], so $S$ is right LCM.

We note that $F^+_m$ admits a generalised scale if and only if $m$ is finite, and thus $F^+_m$ is admissible if and only if $2 \leq m < \infty$.

**Proposition 5.7.** For $2 \leq m < \infty, 0 \leq n \leq \infty$, the right LCM semigroup $S := F^+_m \times \mathbb{N}^n$ has the following properties:

(i) $S$ is admissible with $\beta_c = 1$.

(ii) The action $\alpha$ is faithful if and only if $\alpha$ is almost free if and only if $n = 0$.

(iii) $S$ has finite propagation.

**Proof.** By appealing to Corollary 5.5, $S = F^+_m \bowtie \mathbb{N}^n$ with trivial action and restriction is admissible for the generalised scale $N$ induced by the invariant generalised scale $N^c : F^+_m \to \mathbb{N}^c, w \mapsto \ell(w)$, where $\ell(w)$ denotes the word length of $w$ with respect to the canonical generators of $F^+_m$. In particular, $\text{Irr}(N(S)) = \{m\}$, so $\beta_c = 1$.

Due to Corollary 5.6, we further know that $\alpha$ is faithful (almost free) if and only if $\gamma : \mathbb{N}^n \bowtie F^+_m \sim$ is faithful (almost free). But $\gamma$ is trivial as the action of $\mathbb{N}^n$ on $F^+_m$ is trivial. Thus we get faithfulness if only if $S_c \cong \mathbb{N}^n$ is trivial. In this case, the action is of course also almost free.

The only transversal $T \subset S^+_c = F^+_m$ for $S/\sim$ is $F^+_m$. For every $a \in S_c = \mathbb{N}^n$, we get $c(aw) = c(wa) = a$ so that $C_a = \{a\}$ is finite. Hence $S$ has finite propagation. \qed

In the case of $S := F^+_m \times \mathbb{N}^n$ with $2 \leq m < \infty$, it is easy to see that KMS$_1$-states correspond to traces on $C^*(\mathbb{N}^n)$, or, equivalently, Borel probability measures on $\mathbb{T}^n$ via $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$. Thus we cannot expect uniqueness unless $n = 0$. In fact, this example describes a scenario where $\alpha$ is degenerate. However, the situation is much less clear for general right-angled Artin monoids.

5.3. Self-similar actions. Given a finite alphabet $X$, let $X^n$ be the set of words with length $n$. The set $X^* := \bigcup_{n=0}^{\infty} X^n$ is a monoid with concatenation of words as the operation and empty word $\emptyset$ as the identity. A self-similar action $(G, X)$ is an action of a group $G$ on $X^*$ such that for every $g \in G$ and $x \in X$ there is a uniquely determined element $g|_x \in G$ satisfying

$$g(xw) = g(x)g|_x(w) \quad \text{for all } w \in X^*.$$ 

We refer to [Nek05] for a thorough treatment of the subject. It is observed in [Law08] and [BRRW14, Theorem 3.8] that the Zappa-Szép product $X^* \bowtie G$ with $(v, g)(w, h) := (vg(w), g|_w h)$ defines a right LCM semigroup that satisfies $\Box$.

**Proposition 5.8.** Suppose that $(G, X)$ is a self-similar action. Then the Zappa-Szép product $S := X^* \bowtie G$ has the following properties:

(i) $S$ is admissible with $\beta_c = 1$. 
(ii) The action \( \alpha \) is faithful (almost free) if and only if \( G \acts X^* \) is faithful (almost free).

(iii) \( S \) has finite propagation if and only if for all \( g^{(1)}, g^{(2)} \in G \) there exists a family 
\[ (h_w)_{w \in X^*} \subset G \]\n
such that \( \{ h_w^{-1}g^{(i)(w)}h_w \mid w \in X^* \} \) is finite for \( i = 1, 2 \).
In particular, \( S \) has finite propagation if \( (G, X) \) is finite state.

\textbf{Proof.} For (i), note that \( S \) is a right LCM semigroup that satisfies \( \square \). Then by Corollary 5.5 it suffices to show that \( X^* \) is admissible with an invariant generalised scale \( N' : X^* = N^* \). Since the core subsemigroup of \( X^* \) consists of the empty word \( \emptyset \) only, the equivalence relation \( \sim \) is trivial on \( X^* \). It follows that \( X^* \) satisfies the conditions \((A1)\) and \((A2)\).

It is apparent that the map \( N' : X^* \rightarrow N^*, w \mapsto |X|^{\ell(w)} \) is a generalised scale, where \( \ell(w) \) denotes the word length of \( w \) in the alphabet \( X \). Since for each \( g \in G, w \mapsto g(w) \) is a bijection on \( X^n \) (see for example \cite[Lemma 3.7]{BRRW14}), \( N' \) is invariant under \( G \acts X^* \). Finally, \( \square \) follows from \( \text{Irr}(N'(X^*)) = \{|X|\} \). Thus \( N' \)

induces a generalised scale \( N \) on \( S \) that makes \( S \) admissible. Moreover, we have \( \beta_c = 1 \) as \( \text{Irr}(N'(X^*)) = \{|X|\} \) is finite.

Part (ii) is an immediate consequence of Corollary 5.6.

For (iii), we observe that an arbitrary transversal \( T \subset S^t_{ci} = (X^* \setminus \{ \emptyset \} \times G) \setminus \{(w, h_w)_{w \in X^*}, (h_w)_{w \in X^*} \subset G \text{ satisfies } h_{\emptyset} = 1 \} \) for \( S^t_{\sim} \) is given by \( (h_{\emptyset} g(w), h_w)_{w \in X^*} \) as \( (\emptyset, g)(w, h_w) = (g(w), h_g(w)) \). This establishes the characterisation of finite propagation. So if \( (G, X) \) is finite state, i.e. \( \{g(w) \mid w \in X^*\} \) is finite for all \( g \in G \), then \( h_w = 1 \) for all \( w \in X^* \) yields finite propagation for \( S \).

Faithfulness of \( G \acts X^* \) is assumed right away in \cite[Section 2]{LRRW14}. Our approach shows that this is unnecessary unless we want a unique KMS\( _1 \)-state.

\textbf{5.4. Subdynamics of} \( \mathbb{N} \times \mathbb{N}^* \). Consider the semidirect product \( \mathbb{N} \times \mathbb{N}^* \) with \( (m, p)(n, q) = (m + np, pq) \) for all \( m, n \in \mathbb{N} \) and \( p, q \in \mathbb{N}^* \).

Let \( \mathcal{P} \subset \mathbb{N}^* \) be a family of relatively prime numbers, and \( P \) the free abelian submonoid of \( \mathbb{N}^* \) generated by \( \mathcal{P} \). Then we get a right LCM subsemigroup \( S := \mathbb{N} \times P \) of \( \mathbb{N} \times \mathbb{N}^* \).

As observed in \cite[Example 3.1]{Sta17}, \( S \) can be displayed as the internal Zappa-Szép product \( S_{ci}^1 \bowtie S_c \), where \( S_c = \mathbb{N} \times \{ 1 \} \) and \( S_{ci}^1 = \{(n, p) \in S \mid 0 \leq n \leq p - 1 \} \). The action and restriction maps satisfy

\[ (m, 1)(n, p) = ((m + n) \mod p, p), \text{ and} \]

\[ (m, 1)|_{(n, p)} = \left( \frac{m + n - ((m + n) \mod p)}{p}, 1 \right). \]

Both \( S_{ci}^1 \) and \( S_c \) are right LCM. Also the argument of the paragraph after \cite[Lemma 3.5]{BRRW14} shows that \( S_{ci}^1 \bowtie S_c \) satisfies the hypothesis of \cite[Lemma 3.3]{BRRW14}. Therefore, \( S_{ci}^1 \bowtie S_c \) is a right LCM semigroup satisfying \( \square \), see Remark 5.1.

\textbf{Proposition 5.9.} Suppose that \( \mathcal{P} \subset \mathbb{N}^* \) is a nonempty family of relatively prime numbers. Then the right LCM semigroup \( S = \mathbb{N} \times P \) has the following properties:

(i) \( S \) is admissible, and if \( \mathcal{P} \) is finite, then \( \beta_c = 1 \).

(ii) \( \alpha \) is faithful and almost free.
Moreover, if $P$ is faithful and almost free. We claim that $N' : S^1 \rightarrow S^\infty, (n, p) \mapsto p$ defines a generalised scale on $S^1$. The only transversal for $(N')^{-1}(p)/\sim$ for $p \in N'(S^1)$ is $(N')^{-1}(p) = \{(m, p) \mid 0 \leq m \leq p - 1\}$. This gives condition \((A3)\) \((a)\). For \((A3)\) \((b)\) we observe that
\[
(m, p) \perp (n, q) \text{ if and only if } m - n \not\in \gcd(p, q)\mathbb{Z} \text{ for all } (m, p), (n, q) \in S^1.
\]
It follows $(N')^{-1}(p)$ is an accurate foundation set. Thus $N'$ is a generalised scale on $S^1$. Since $S_c = \mathbb{N} \times \{1\}$, the action of $S_c$ on $S^1$ fixes the second coordinate. Therefore, $N'$ is invariant under the action of $S_c$ on $S^1$. Finally, note that the set $\text{Irr}(N'(S^1))$ is given by $\mathcal{P}$, which is assumed to consist of relatively prime elements. Hence \((A4)\) holds. Moreover, if $\mathcal{P}$ is finite, then $\beta_c = 1$ by the product formula for $\zeta_S$ from Remark 7.4.

For \((ii)\), note that Corollary 5.6 applies. As $S_c = S^1$ is totally ordered, it suffices to examine the fixed points in $S/\sim$ for $\gamma_m$ with $m \geq 1$, see Remark 3.12. But $\gamma_m[(n, p)] = [(r(m + n), p)]$, where $0 \leq r(m + n) \leq p - 1, m + n \in r(m + n) + p\mathbb{N}$, so that $[(n, p)]$ is fixed by $\gamma_m$ if and only if $m \in p\mathbb{N}$. As $\{p \in P \mid m \in p\mathbb{N}\}$ is finite, $\gamma$ is almost free. In particular, $\gamma$ is faithful because $S^1_{ci}/\sim$ is infinite, see Remark 3.11. Hence $\alpha$ is faithful and almost free.

Part \((iii)\) follows from the observation that the transversal $\mathcal{V} = S^1_c$ for $S/\sim$ satisfies $C_{(m, 1)} = \{(m, 1) \cup (n, 1) \mid n \in \{k, k + 1\}$ with $kp \leq m < (k + 1)p$ for some $p \in P \setminus \{1\}$} for $m \geq 1$. Thus the cardinality of $C_{(m, 1)}$ does not exceed $m + 1$, and $S$ has finite propagation.

By Proposition 5.9, Theorem 4.3 applies to $\mathbb{N} \times \mathbb{N}^\infty$. This recovers the essential results on KMS-states from [LR10,BaHLR12].

5.5. Baumslag-Solitar monoids. Let $c, d \in \mathbb{N}$ with $cd > 1$. The Baumslag-Solitar monoid $BS(c, d)^+$ is the monoid in two generators $a, b$ subject to the relation $ab^c = b^ca$. It is known that $S := BS(c, d)^+$ is quasi-lattice ordered, hence right LCM, see [Spi12, Theorem 2.11]. It is observed in [Sta17, Example 3.2] and [BRRW14, Subsection 3.1] that the core subsemigroup $S_c$ is the free monoid in $b$, whereas $S^1_{ci}$ is the free monoid in $\{b^ja \mid 0 \leq j \leq d - 1\}$. In addition, we have $S^1_{ci} \cap S_c = \{1\}$ and $S = S^1_{ci} S_c$, so that the two subsemigroups of $S$ give rise to an internal Zappa-Szép product description $S^1_{ci} \rtimes S_c$. The action and restriction map (on the generators) are given by
\[
b(b^ja) = \begin{cases} b^{j+1}a & \text{if } j < d - 1, \\ a & \text{if } j = d - 1 \end{cases}, \quad b|_{b^ja} = \begin{cases} 1 & \text{if } j < d - 1, \\ b^c & \text{if } j = d - 1. \end{cases}
\]
Moreover, $S^1_{ci} \rtimes S_c$ satisfies the hypothesis of [BRRW14, Lemma 3.3] and therefore $S^1_{ci} \rtimes S_c$ is a right LCM satisfying \((A3)\), see Remark 5.1.

**Proposition 5.10.** Let $c, d \in \mathbb{N}$ such that $cd > 1$. Then the Baumslag-Solitar monoid $S = BS(c, d)^+$ is admissible with $\beta_c = 1$, and the following statements hold:

\begin{itemize}
  \item[(i)] The action $\alpha$ is almost free if and only if $\alpha$ is faithful if and only if $c \not\in d\mathbb{N}$.
  \item[(ii)] $S$ has finite propagation if and only if $c \leq d$.
\end{itemize}
Proof. Since $S \cong S^1_{cl} \bowtie S_c$, we can invoke Corollary 5.5. As $S^1_{cl}$ is the free monoid in $d$ generators $X := \{w_j := b^j a \mid 0 \leq j \leq d - 1\}$, it is admissible for the generalised scale $N'$: $S^1_{cl} \rightarrow \mathbb{N}^\times, w \mapsto d(\ell(w))$, where $\ell(w)$ denotes the word length of $w$ with respect to $X$, compare Proposition 5.8. The action of $S_c$ on $S^1_{cl}$ preserves $\ell$, and hence $N'$ is invariant. Thus $N'$ induces a generalised scale $N$ on $S$ that makes $S$ admissible. Due to $\text{Irr}(N(S)) = \{d\}$, we get $\beta_c = 1$.

The action $S/\sim$ is infinite, almost freeness implies faithfulness by Remark 3.11. As $S \cong S^1_{cl} \bowtie S_c$ satisfies (1), we can appeal to Corollary 5.6 and thus consider $\gamma: \mathbb{N} \curvearrowright S^1_{cl}/\sim$ in place of $\alpha$, where we identify the quotient by $\sim$ with $S^1_{cl}$ because $\sim$ is trivial on $S^1_{cl}$. As $S_c \cong \mathbb{N}$ is totally ordered, it suffices to consider fixed points for $\gamma_n$ with $n \geq 1$, see Remark 3.12. Next, we observe that

$$\gamma_n(w) = w \text{ for } w \in S^1_{cl} \text{ if and only if } n(c/d)^{k-1} \in \mathbb{dN} \text{ for all } 1 \leq k \leq \ell(w).$$

So if $c \in \mathbb{dN}$, then $\gamma_d = \text{id}$, so that $\gamma$ is not faithful. Hence $\alpha$ is not faithful as well. On the other hand, if $c \not\in \mathbb{dN}$, then $e := \gcd(c,d)^{-1}d \geq 2$ and $\gamma_n$ fixes $w \in S^1_{cl}$ if and only if $n \in \ell(w)\mathbb{N}$. So as soon as $\ell(w)$ is large enough, $w$ cannot be fixed by $\gamma_n$. Since the alphabet $X$ is finite, we conclude that $\gamma$ (and hence $\alpha$) is almost free if $c \not\in \mathbb{dN}$.

For (ii), we remark that $S^1_{cl}$ is the only transversal for $S/\sim$ that is contained in $S^1_{cl}$. For $m \in \mathbb{N}$, we let $m = kd + r$ with $k, r \in \mathbb{N}$ and $0 \leq r \leq d - 1$. Then we get

$$b^m w_i = w_j b^n \text{ for } 0 \leq i \leq d - 1 \text{ with } m = \begin{cases} kc, & \text{if } 0 \leq i < d - r, \\ (k+1)c, & \text{if } d - r \leq i \leq d - 1, \end{cases}$$

with a suitable $0 \leq j \leq d - 1$. If $c = d$, then we get $C_{b^n} = \{b^m\}$ for $m \in \mathbb{dN}$ and $C_{b^m} = \{b^m, b^m + d - r, b^m - r\}$ for $m \not\in \mathbb{dN}$. Thus $S$ has finite propagation in this case. For $c < d$, it follows from (5.1) that every expression $b^m v = wb^n$ for given $v \in S^1_{cl}$ and $m \in \mathbb{N}$ satisfies $n \leq (k+1)d$. Hence $C_{b^n}$ is finite and $S$ has finite propagation. Now suppose $c > d$, and let $m \geq d$, say $m = kd + r$ with $0 \leq r \leq d - 1, k \geq 1$. If $r = 0$, (5.1) gives $b^m w_i = w_j b^n$ with $n = kc > m$ for all $i$. If $r > 0$, then there is $d - r \leq r \leq d - 1$, and we have $b^m w_i = w_j b^n$ with $n = (k+1)c > m$ for all such $i$. Thus $C_{b^n}$ contains arbitrarily large powers of $b$, and hence must be infinite. Therefore $S$ does not have finite propagation for $c \geq d$.

We note that for this example, finite propagation can be regarded as a consequence of faithfulness of $\alpha$. The condition that $d$ is not a divisor of $c$ also appeared in [CaHR16]. More precisely, it was shown in [CaHR16, Corollary 5.3, Proposition 7.1, and Example 7.3] that this condition is necessary and sufficient for the KMS$_1$-state to be unique. Let us remark here that we do not need to request that $d$ is not a divisor of $c$ in order to obtain the parametrisation of the KMS$_\beta$-states for $\beta > 1$, compare Theorem 4.3 and Proposition 5.10 with [CaHR16 Theorem 6.1].

### 5.6. Algebraic dynamical systems

Suppose $(G, P, \theta)$ is an algebraic dynamical system, that is, an action $\theta$ of a right LCM semigroup $P$ on a countable, discrete group $G$ by injective endomorphisms such that $pP \cap qP = rP$ implies $\theta_p(G) \cap \theta_q(G) = \theta_r(G)$. The semidirect product $G \rtimes_P P$ is the semigroup $G \times P$ equipped with the operation $(g,p)(h,q) = (g \theta_p(h), pq)$. It is known that injectivity and the intersection condition,
known as preservation of order, are jointly equivalent to $G \rtimes \theta P$ being a right LCM semigroup, see [BLS17, Proposition 8.2].

In order to have an admissible semidirect product $G \rtimes \theta P$, we need to consider more constraints on $(G, P, \theta)$:

(a) $(G, P, \theta)$ is finite type, that is, the index $N_p := [G : \theta_p(G)]$ is finite for all $p \in P$.
(b) There exists $p \in P$ for which $\theta_p$ is not an automorphism of $G$.
(c) If $\theta_p$ is an automorphism of $G$, then $p \in P^*$. 
(d) If $p, q \in P$ have the same index, i.e. $N_p = N_q$, then $p \in qP^*$. That is to say, there is an automorphism $\vartheta$ of $G$ such that $\theta_p$ and $\theta_q$ are conjugate via $\vartheta$, and $\vartheta = \vartheta_x$ for some $x \in P^*$.

Conditions (a) and (b) are mild assumptions. Also it is natural to assume that $p$ is invertible in $P$ if $\theta_p$ is an automorphism of $G$. For otherwise, we can replace $P$ by the acting semigroup of endomorphisms of $G$. But (d) is significantly more complex, and we will see that it incorporates deep structural consequences for $G \rtimes \theta P$.

**Proposition 5.11.** Let $(G, P, \theta)$ be an algebraic dynamical system satisfying the above conditions (a)–(d). Then $S := G \rtimes \theta P$ has the following properties:

(i) $S$ is admissible, and $\beta_c = 1$ if $\text{Irr}(P/\sim)$ is finite.
(ii) The action $\alpha$ is faithful if and only if $\bigcap_{(h, q) \in S} h \theta_q(G)h^{-1} = \{1\}$, and

$$\bigcap_{(h, q) \in S} h \theta_q(G)\theta_p(h)^{-1} = \emptyset \text{ for all } p \in P^* \setminus \{1\}.$$ 

(iii) The action $\alpha$ is almost free if and only if $P^*$ is trivial and the set $\{(g, p) \in S \mid h \in g \theta_p(G)g^{-1}\}$ is finite for all $h \in G \setminus \{1\}$.

**Proof.** For (i), note that $S_c = S^{\ast} = G \rtimes P^*$, see [Sta17, Example 3.2]. So that $S_{ct} = S \setminus S^{\ast}$ and therefore $S$ satisfies (A1) and (A2). Also note that

$$\text{(5.2)} \quad (g, p) \sim (h, q) \iff (g, p)S = (h, q)S \iff p \in qP^* \text{ and } g^{-1}h \in \theta_p(G) = \theta_q(G)).$$

Then the natural candidate for a homomorphism $N : S \to \mathbb{N}^\times$ is given by $N(g, p) := N_p$.

Properties (a) and (b) imply that $N$ is a nontrivial homomorphism from $S$ to $\mathbb{N}^\times$. The condition (A3a) follows from (d) together with (5.2). Since $(G, P, \theta)$ is finite type, (A3b) holds if and only if $P$ is directed with regards to $\sim$. Thus (A3) follows immediately from (d): Given $p, q \in P$, we have $N_{pq} = N_pN_q = N_{qp}$ so that $pqx = qp \in pP \cap qP$ for a suitable $x \in P^*$. Thus $N$ is a generalised scale on $S$.

Now we are looking at (A4). First note that (d) implies $P^*p \subset pP^*$ for all $p \in P$. Thus the restriction of $\sim$ to $P$ given by $p \sim q :\iff p \in qP^*$ defines a congruence on $P$. If $\overline{P} := P/\sim$ denotes the resulting monoid, then $\overline{P}$ is necessarily abelian as $N_{pq} = N_{qp}$.

The monoid $\overline{P}$ is generated by $\{[p] \in \overline{P} \mid N_p \in \text{Irr}(N(S))\}$. This set is minimal as a generating set for $\overline{P}$ and coincides with the irreducible elements of $\overline{P}$. So to prove (A4) it suffices to show that $\overline{P}$ is in fact freely generated by $\text{Irr}(\overline{P})$.

To see this, let $\prod_{1 \leq i \leq k} [p_i]^{m_i} = \prod_{1 \leq j \leq \ell} [q_j]^{n_j}$ with $[p_i], [q_j] \in \text{Irr}(\overline{P})$, $m_i, n_j \in \mathbb{N}$, where we assume that the $[p_i]$ are mutually distinct, as are the $[q_j]$. Then we get a corresponding equation $\prod_{1 \leq i \leq k} N_{p_i}^{m_i} = \prod_{1 \leq j \leq \ell} N_{q_j}^{n_j}$ in $N(S)$. Since $N(S)$ has the unique factorization property (inherited from $\mathbb{N}^\times$), and $N_{p_i}, N_{q_j} \in \text{Irr}(N(S))$ for all $i, j$, we get $k = \ell$ and a
permutation $\sigma$ of $\{1, \ldots, k\}$ such that $N_{\sigma(i)} = N_{g_{\sigma(i)}}$ and $m_i = n_{\sigma(i)}$. Now $N_{\sigma(i)} = N_{g_{\sigma(i)}}$ forces $g_{\sigma(i)} \in p_iP^*$, so $[g_{\sigma(i)}] = [p_i]$. Thus $P$ is the free abelian monoid in $\operatorname{Irr}(P)$.

We clearly have $\beta_1 = 1$ whenever $\operatorname{Irr}(P)$ is finite, as these are in one-to-one correspondence with $\operatorname{Irr}(N(G \rtimes \theta P))$ and $\zeta(\beta) = \prod_{n \in \operatorname{Irr}(N(G \rtimes \theta P))} (1 - n^{-\beta_1})^{-1}$, see Remark 7.4.

For (ii), since $S_c = S^*$ is a group, faithfulness of $\alpha$ amounts to the following: For every $(g, p) \in S^* \setminus \{1\}$ there exists $(h, q) \in S$ such that $h^{-1}g\theta_p(h) \notin \theta_q(G)$. This is equivalent to $\bigcap_{(h,q) \in S} h\theta_q(G)h^{-1} = \{1\}$ and $\bigcap_{(h,q) \in S} h\theta_q(G)\theta_p(h)^{-1} = \emptyset$ for all $p \in P^* \setminus \{1\}$.

For (iii), since $S_c = S^* = G \rtimes \alpha P^*$ is a group, by Remark 5.12, it suffices to look at the fixed points of the map $\alpha(h,q)$ for each element $(h, q) \in S_c \setminus \{1\}$. Since $P^*p \subset pP^*$ for all $p \in P$, the equation (5.2) implies that an equivalence class $[(g, p)] \in S/\sim$ is fixed by $\alpha(h,q)$ if and only if $g^{-1}h\theta_q(g) \in \theta_p(G)$. Now suppose that $P^* \neq \{1\}$. Fix $q \in P^* \setminus \{1\}$ and take $h = 1$. As $qp = pq'$ for some $q' \in P^*$, we get $\theta_q(\theta_p(G)) = \theta_p(G)$ for all $p \in P$. Thus the element $[(g, p)] \in S/\sim$ with $g \in \theta_p(G)$ is fixed by $\alpha(1, q)$ for every $p \in P$. By assumption, $P$ is infinite as there is at least one $p \in P$ for which $\theta_p$ is not an automorphism of $G$. This tells us that $\beta_1 = 1$ violates the almost freeness for the action $\alpha$. It follows that if $\alpha$ is almost free, then $P^*$ has to be trivial. Now suppose that $P^* = \{1\}$. Then (5.2) allows us to conclude that $\alpha$ is almost free if and only if the set $\{(g, p) \in S \mid h \in g\theta_p(G)g^{-1}\}$ is finite for all $h \in G \setminus \{1\}$.

\begin{remark}
Let $(G, P, \theta)$ be an algebraic dynamical system and $S = G \rtimes \theta P$. Then the conditions for $\alpha$ to be faithful are closely related to the conditions (2) and (3) in [BS16, Corollary 4.10] which are needed for the simplicity of $Q(S)$. In addition to that, the requirement $P^*p \subset pP^*$ for all $p \in P$ coming from property (d) in the Proposition 5.11 is needed there as well. This intimate relation between conditions for simplicity of $Q(S)$ and uniqueness of the KMS$_{\beta_1}$-state for $\beta \in [1, \beta_c]$ seems quite intriguing, and may very well deserve a thorough study. To the best of our knowledge, no explanation for this phenomenon is known so far.

We also note that the characterisation of almost freeness of $\alpha$ for the case of trivial $P^*$ we get here is a natural strengthening of the condition for faithfulness of $\alpha$.
\end{remark}

The next example describes a class of algebraic dynamical systems for which $\beta_c = 1$, $S$ has finite propagation, and $\alpha$ is faithful but not almost free.

\begin{example}
Let $H$ be a finite group with $|H| > 2$. Then there exists a non-trivial subgroup $P_0 \subset \operatorname{Aut}(H)$. Consider the standard restricted wreath product $G := H \rtimes \mathbb{N} = \bigoplus_{n \in \mathbb{N}} H$. Each $p \in P_0$ corresponds to an automorphism of $G$ that commutes with the right shift $\Sigma$ on $G$, and we denote by $P := \langle \Sigma \rangle \rtimes P_0$ the corresponding semigroup of endomorphisms of $G$. Then $P$ is lattice ordered, hence right LCM, and the natural action $\theta : P \curvearrowright G$ respects the order so that $(G, P, \theta)$ forms an algebraic dynamical system for which we can consider the right LCM semigroup $S = G \rtimes \theta P$. We note the following features:

(a) $(G, P, \theta)$ is of finite type because $H$ is finite.
(b) $P^*p$ equals $pP^*$ for all $p \in P$ since the chosen automorphisms of $G$ commute with $\Sigma$.
(c) The action $\alpha$ is faithful.
(d) As $P_0$ is nontrivial, $\alpha$ is not almost free, see Proposition 5.11.
\end{example}
(e) Take \( \mathcal{T} = \bigcup_{n \in \mathbb{N}} \mathcal{T}_n \) with \( \mathcal{T}_n := \{ ((h_k)_{k \geq 0}, \Sigma^n) \in S \mid h_k = 1_H \text{ for all } k > n \} \). This is a natural transversal for \( S/\sim \) that witnesses finite propagation (for all pairs \((s, t) \in S \times S\)). Indeed, for \( s = (g, p) \in S_c \) we get \(|C_s| \leq \ell(g) + 1 < \infty \), where \( \ell(g) \) denotes the smallest \( n \in \mathbb{N} \) with \( h_m = 1_H \) for all \( m > n \).

(f) \( \beta_c = 1 \) as \( \text{Irr}(P/\sim) = \{[\Sigma]\} \).

Thus we conclude that \( S \) is an admissible right LCM semigroup that has a unique KMS\(_1\)-state by virtue of Theorem 4.3 (2b), while the condition in Theorem 4.3 (2a) does not apply.

As a byproduct of our treatment of algebraic dynamical systems, we recover the results for a dilation matrix from [LRR11]:

**Example 5.14** (Dilation matrix). Let \( d \in \mathbb{N} \) and \( A \in M_d(\mathbb{Z}) \) an integer matrix with \( |\det A| > 1 \), and consider \((G, P, \theta) = (\mathbb{Z}^d, \mathbb{N}, \theta) \) with \( \theta_n(m) := A^n m \) for \( n \in \mathbb{N}, m \in \mathbb{Z}^d \). Then \((G, P, \theta)\) constitutes an algebraic dynamical system that satisfies (a)–(d) from Proposition 5.11 as

\[
N_1 = \text{coker} \theta_1 = |\mathbb{Z}^d/\text{im} A| = |\det A| \in \mathbb{N} \setminus \{1\}.
\]

Also, we have \( \beta_c = 1 \) because \( \text{Irr}(N(\mathbb{Z}^d \rtimes_\theta \mathbb{N})) = \{ |\det A| \} \). Since \( \mathbb{Z}^d \) is abelian, the action \( \alpha \) is faithful if and only if \( \bigcap_{n \in \mathbb{N}} A^n \mathbb{Z}^d = \{0\} \), see Proposition 5.11 (ii). As \( P \cong \mathbb{N} \), we deduce that this is also equivalent to almost freeness of \( \alpha \), see Proposition 5.11 (iii).

**Example 5.15.** Let \( A, B \subset \mathbb{N}^\times \) be non-empty disjoint sets such that \( A \cup B \) consists of relatively prime numbers. Then the free abelian submonoid \( P \) of \( \mathbb{N}^\times \) generated by \( A \) acts on the ring extension \( G := \mathbb{Z}[1/q \mid q \in B] \) via multiplication by injective group endomorphisms \( \theta_p \) in an order preserving way, that is, \((G, P, \theta)\) is an algebraic dynamical system. We observe that \( N_p = [G : pG] = p \) (as for \( B = \emptyset \)) so that the system is of finite type, and it is easy to check that the requirements (a)–(d) for admissibility of \( S = G \rtimes_\theta P\) hold. It thus follows from Proposition 5.11 (i) that \( S \) is admissible and \( \beta_c = 1 \) if \( A \) is finite. Since \( G \) is abelian and \( P^* \) is trivial, parts (ii) and (iii) imply that faithfulness and almost freeness of \( \alpha \) are both equivalent to \( A \) being non-empty.

The behaviour of the KMS-state structure on this semigroup resembles very much the one of \( \mathbb{N} \rtimes \mathbb{N}^\times \), except for the fact that the parametrising space corresponds to Borel probability measures on the solenoid \( \mathbb{T}[1/q \mid q \in B] \) instead of the torus \( \mathbb{T} \).

**Example 5.16.** For a finite field \( F \), consider the polynomial ring \( F[t] \) as an additive group \( G \). Via multiplication, every \( f \in F[t] \setminus F \) gives rise to an injective, non-surjective group endomorphism \( \theta_f \) of \( F[t] \), whose index is \( N_f = |F|^{\deg f} \). For every such \( f \), the semigroup \( S := F[t] \rtimes_f \mathbb{N} \) is an admissible right LCM semigroup. By Proposition 5.11, we further have \( \beta_c = 1 \), and \( \alpha \) is almost free, hence faithful. In the simplest case of \( f = t \), the semigroup \( F[t] \rtimes_f \mathbb{N} \) is canonically isomorphic to \((F \setminus \mathbb{N}) \rtimes \mathbb{N}\) from Example 5.13.

6. **Algebraic constraints**

This section presents some considerations on existence of KMS\(_\beta\) and ground states, and in particular identifies some necessary conditions for existence. Throughout this section, we shall assume that \( S \) is a right LCM semigroup that admits a generalised scale \( N : S \to \mathbb{N}^\times \), and denote by \( \sigma \) the time evolution on \( C^*(S) \) given by \( \sigma_x(v_s) = N_s^{ix} v_s \) for all \( x \in \mathbb{R} \) and \( s \in S \).
Proposition 6.1. There are no KMS\(_{\beta}\)-states on \(C^*(S)\) for \(\beta < 1\).

Proof. Let \(\beta \in \mathbb{R}\) and suppose that \(\phi\) is a KMS\(_{\beta}\)-state for \(C^*(S)\). By Proposition 6.3 (iii), there exists an accurate foundation set \(F\) for \(S\) with \(|F| = n = N_f\) for all \(f \in F\) and some \(n > 1\). The KMS\(_{\beta}\)-condition gives that \(\phi(v_f v_f^*) = N_f^{-\beta} \phi(v_f^* v_f) = n^{-\beta}\). Hence, 
\[
1 = \phi(1) \geq \sum_{f \in F} \phi(e_f s) = n^{1-\beta},
\]
so necessarily \(\beta \geq 1\). \(\square\)

Proposition 6.2. A state \(\phi\) on \(C^*(S)\) is a ground state if and only if \(\phi(v_s v_t^*) \neq 0\) for \(s, t \in S\) implies \(s, t \in S_e\). In particular, a ground state on \(C^*(S)\) vanishes outside \(\varphi(C^*(S_e))\).

Proof. Let \(\phi\) be a state on \(C^*(S)\) and \(y_i := v_{s_i} v_{s_i}^*\), with \(s_i, t_i \in S\) for \(i = 1, 2\), be generic elements of the spanning family for \(C^*(S)\). The expression
\[
\phi(y_2 \sigma_{x+iy}(y_1)) = \left(\frac{N_{s_1}}{N_{t_1}}\right)^{-y+ix} \phi(y_2 y_1)
\]
is bounded on \(\{x + iy \mid x \in \mathbb{R}, y > 0\}\) if and only if either \(N_{s_1} \geq N_{t_1}\) or \(N_{s_1} < N_{t_1}\) and \(\phi(y_2 y_1) = 0\). Now assume \(\phi\) is a ground state and suppose \(\phi(v_s v_t^*) \neq 0\) for some \(s, t \in S\).
Choosing \(s_2 = s, t_1 = t, s_1 = t_2 = 1\) in (6.1), we get \(N_{t_1} \leq N_{s_1} = 1\). Thus \(N_t = 1\), and hence \(t \in S_e\) by Proposition 3.6 (i). Taking adjoints shows that \(s \in S_e\) as well.

Conversely, let \(\phi\) be a state on \(C^*(S)\) for which \(\phi(v_s v_t^*) \neq 0\) implies \(s, t \in S_e\). By the Cauchy-Schwartz inequality for \(y_1\) and \(y_2\), we have
\[
|\phi(y_2 \sigma_{x+iy}(y_1))|^2 \leq \left(\frac{N_{s_1}}{N_{t_1}}\right)^{-2y} \phi(y_2 y_1) \phi(y_2^* y_1).
\]
As \(y_1^* y_1 = e_{t_1} s\) and \(y_2 y_2^* = e_{s_2} s\), the expression on the right-hand side of the inequality vanishes unless \(t_1, s_2 \in S_e\). So if it does not vanish we must have \(N_{t_1} = 1\). Therefore using that \(N_{s_1} \geq 1\), it follows that \(|\phi(y_2 \sigma_{x+iy}(y_1))|^2 \leq 1\) for \(y > 0\). Thus \(\phi\) is a ground state on \(C^*(S)\). \(\square\)

Remark 6.3. For the purposes of the next result we need the following observation: if \(\tau\) is a trace on \(C^*(S_e)\), then \(\tau(y w_c w^*_a z) = \tau(y z)\) for all \(y, z \in C^*(S_e)\) and \(a \in S_e\). The reason is that the image of the isometry \(w_a\) in the GNS-representation for \(\tau\) is a unitary.

Proposition 6.4. For \(\beta \in [1, \infty)\), consider the following conditions:

(i) \(\phi\) is a KMS\(_{\beta}\)-state.

(ii) \(\phi\) is a state on \(C^*(S)\) such that \(\phi \circ \varphi\) is a normalised trace on \(C^*(S_e)\) and for all \(s, t \in S\)
\[
\phi(v_s v_t^*) = \begin{cases} 0 & \text{if } s \neq t, \cr N_s^{-\beta} \phi \circ \varphi(w_r w_r^*) & \text{if } s = t \text{ and } r, r' \in S_e. \end{cases}
\]

Then (i) implies (ii). Assume moreover that \(S\) is core factorable and \(S_{ci} \subset S\) is \(\cap\)-closed, and fix maps \(c\) and \(i\) as in Lemma 3.8. If \(\phi\) is a state on \(C^*(S)\) such that \(\phi \circ \varphi\) is a trace, then (6.2) is equivalent to
\[
\phi(v_s v_t^*) = \begin{cases} 0 & \text{if } i(s) \neq i(t), \cr N_s^{-\beta} \phi \circ \varphi(w_{c(s)} w_{c(t)}^*) & \text{if } i(s) = i(t), \end{cases}
\]
and (ii) implies (i).
Proof. First, let $\phi$ be a KMS$_\beta$-state on $C^*(S)$. Since $\sigma = \text{id}$ on $\varphi(C^*(S_c))$, the state $\phi \circ \varphi$ is tracial. The KMS-condition gives
\[
\phi(v_s v_s^*) = N_s^{-\beta} \phi(v_s^* e_s e_s v_s) = N_s^{-\beta} N_t^{\beta} \phi(v_s v_s^*).
\]
Thus $\phi(v_s v_s^*) \neq 0$ implies both $N_s = N_t$ and $s \not\sim t$. Proposition 3.6 (ii) implies that $s \sim t$. Suppose first that $s,t \in S$. Then $s \sim t$. Proposition 3.6 (ii) implies that $s,t \not\sim s$. Since $s,s \not\sim t$, we have $v_s v_s^* = v_s v_s^*$. Using the KMS$_\beta$ property of $\phi$ we obtain (6.2) in the case that we may choose a right LCM of $s$ and $t$ to be of the form $s,t = tb$. The general case $sr = tr'$ displayed in (6.2) is then of the form $r = ac, r' = bc$ with arbitrary $c \in S_c$. Thus we get $\phi \circ \varphi(w_r w_r^*) = \phi \circ \varphi(w_s w_s^*)$ by Remark 6.3.

Now suppose that [A1] and [A2] are satisfied as well, and that $\phi$ is a state on $C^*(S)$ for which $\phi \circ \varphi$ is tracial. Let $s,t \in S$. The equivalence of (6.2) and (6.3) will follow if show that for all $r,r' \in S_c$ such that $sr = tr'$ we have $\phi \circ \varphi(w_r w_r^*) = \phi \circ \varphi(w_s w_s^*)$ if $i(s) = i(t)$. Fix therefore a pair $r,r'$ in $S$ such that $sr = tr'$ and $i(s) = i(t)$. We saw in the first part of the proof that $\phi \circ \varphi(w_r w_r^*) = \phi \circ \varphi(w_s w_s^*)$ whenever $a,b$ in $S_c$ satisfy $s,t \not\sim s$. By left cancellation and $i(s) = i(t)$, we reduce this to $c(s)S_c \cap c(t)S_c = c(s)aS_c, c(s)a = c(t)b$. Then the trace property of $\phi \circ \varphi$ yields
\[
\phi \circ \varphi(w_{c(s)} w_{c(s)}^*) = \phi \circ \varphi(w_{c(t)} w_{c(t)}^*) = \phi \circ \varphi(w_b w_b^*) = \phi \circ \varphi(w_r w_r^*).
\]

It remains to prove that if $\phi$ is a state on $C^*(S)$ for which $\phi \circ \varphi$ is tracial and (6.3) holds, then $\phi$ is a KMS$_\beta$-state. As the family of analytic elements $\{ v_s v_s^* | s,t \in S \}$ spans a dense $*$-subalgebra of $C^*(S)$, $\phi$ is a KMS$_\beta$-state if and only if
\[
(6.4) \quad \phi(v_s v_t v_s^* v_t^*) = N_s^{-\beta} N_t^{\beta} \phi(v_s v_s^* v_t v_t^*) \quad \text{for all } s,t \in S, i = 1,2.
\]

We will first show that (6.4) is valid for $t = e$, i.e. we show that
\[
(6.5) \quad \phi(v_s v_t v_s^*) = N_s^{-\beta} \phi(v_s v_s^* v_t) \quad \text{for all } r,s,t \in S.
\]

Before we prove this identity, we note that (6.3) implies
\[
(6.6) \quad \phi(v_s (s \varphi(a)) v_s^*) = N_s^{-\beta} \phi \circ \varphi(a) \quad \text{for every } a \in C^*(S_c),
\]
since $(w_{t} w_{r}^*)_{t,r \in S_c}$ form a spanning family for $C^*(S_c)$. Now let $r,s,t \in S$ and compute, using the trace property of $\phi \circ \varphi$ in the second equality, that
\[
\phi(v_s v_t v_r^*) \overset{(6.3)}{=} N_s^{-\beta} \phi(v_s v_t v_s^* v_r^*) = N_s^{-\beta} \phi(v_s v_t v_s^* v_s^* v_r^*) = N_s^{-\beta} \phi(v_s v_t v_s^* v_r^*) \overset{(6.6)}{=} N_s^{-\beta} \phi(v_s v_t v_{c(t)i(s)} v_{c(t)i(s)}^*) v_{c(t)i(r)}^*).
\]

Suppose that $i(st) = i(r)$. We claim that $v_{c(s(t)i(s))} v_{c(s(t)i(s))}^* = v_{c(r)i(r)}^* v_{c(r)i(r)}$. To see this, we first show that $s,t \in S$. Since $s,t = i(st)c(s(t)) = i(st)c(s(t))$, we have $i(st)S \subseteq s \not\sim t$. For the reverse containment, let $ss' \in S$ and $i(r)$. Then by Proposition 3.6 (iv) we have $N_s N_r' = \text{lcm}(N_s, N_{i(r)}) = \text{lcm}(N_s, N_{i(s)}) = N_{s,t}$. Hence $N_s' = N_s$. By Proposition 3.6 (ii) we know that $i(s') = i(t)$ or $s' \not\sim t$. We cannot have $s' \not\sim t$ because this would contradict that $ss' \not\sim i(st)S \subseteq stS$. Hence $i(s') = i(t)$, and so $ss' \in S$. This gives the reverse containment, and thus we have
\[ sS \cap i(r)S = si(t)S. \] Hence \( si(t) = i(si(t))c(si(t)) = i(st)c(si(t)) = i(r)c(si(t)) \) is a right LCM for \( s \) and \( i(r) \), and so gives the claim with the standard argument. Therefore,
\[
\phi(v_s v_i v_s^*) = N_s^{-\beta} \delta_{i(st),i(r)} \phi(v_i v_{ci(r)} v_{i(r)} v_s) = N_s^{-\beta} \delta_{i(st),i(r)} \phi(v_i v_i^* v_s),
\]
and to prove (6.5) it remains to show that \( \phi(v_i v_i^* v_s) \neq 0 \) if \( i(st) \neq i(r) \). Suppose \( \phi(v_i v_i^* v_s) \neq 0 \). Then \( sS \cap rS \neq \emptyset \), so we can assume \( v_i^* v_s = v_s v_i^* = v_i^* v_{s'} \) for some \( s', r' \in S \) with \( rs' = sr' \). According to (6.3), \( \phi(v_i v_i^* v_s) = \phi(v_{i(r)} v_{r'\beta} v_s) \), so our assumption that this is nonzero implies \( i(ts') = i(r') \). But then \( i(sts') = i(sr') = i(rs') \), and in particular \( N_{st} = N_r \) and \( st \not\perp r \). Thus, by the definition of the map \( i \), we have \( i(st) = i(r) \), showing the desired converse that establishes (6.5). To get \( \phi(v_i^* v_i^* v_i^*) = N_1^2 \phi(v_i^* v_i^* v_i^*) \), we use the case just established to get
\[
\phi(v_i^* v_i^* v_i^*) = \frac{\phi(v_s v_i^* v_i^*)}{N_s^{-\beta}} = N_s^{-\beta} \phi(v_i^* v_i^* v_s) = N_s^{-\beta} \phi(v_i^* v_i^* v_i^*).
\]
Now suppose we have \( s_1, s_2, t_1, t_2 \in S \). Then \( \phi(v_{s_1} v_{t_1}^* v_{s_2} v_{t_2}^*) \) vanishes unless \( t_1 S \cap s_2 S = t_1 t_3 S, t_1 t_3 = s_2 s_3 \) for some \( s_3, t_3 \in S \). In this case, we get
\[
\phi(v_{s_1} v_{t_1}^* v_{s_2} v_{t_2}^*) = \phi(v_{s_1} v_{t_1}^* v_{t_2} v_{s_3}) = N_s^{-\beta} \phi(v_{t_1} v_{t_2} v_{s_3} v_s) = N_s^{-\beta} \phi(v_{t_1}^* v_{t_2} v_{s_3}^* v_{s_1})
\]
by (6.5). Proceeding in the same manner with \( v_{t_1}^* \) against \( v_{s_2}^* v_{t_2} v_{s_1} \), the term vanishes unless \( t_2 S \cap s_1 S \neq \emptyset \), in which case the adjoint version of (6.5) gives
\[
\phi(v_{s_1} v_{t_1}^* v_{s_2} v_{t_2}^*) = N_s^{-\beta} \phi(v_{i(t)}^* v_{i(s)} v_{i(s')}) = N_s^{-\beta} N_t^{-\beta} \phi(v_{i(s)}^* v_{i(s')} v_{i(t')} v_{i(t)}).
\]
As a final step we remark that an analogous argument with \( s_2 \) and \( t_2 \) in place of \( s_1 \) and \( t_1 \) shows that \( \phi(v_{s_2}^* v_{t_2}^* v_{s_1} v_{t_1}^*) \) also vanishes unless \( t_1 S \cap s_2 S \neq \emptyset \) and \( t_2 S \cap s_1 S \neq \emptyset \). This concludes the proof that every state \( \phi \) on \( C^*(S) \) with \( \phi \circ \varphi \) tracial and such that (6.3) holds is a KMS\(_\beta\)-state.

We continue this section with some results describing which KMS-states factor through the maps appearing in the boundary quotient diagram (1.1). Recall that the kernel of the homomorphism \( \pi_c \) is generated by \( 1 - v_a v_a^* \) with \( a \in S_c \).

**Proposition 6.5.** Suppose that for each normalised trace \( \tau \) on \( C^*(S_c) \) there exists a KMS\(_\beta\)-state \( \phi \) on \( C^*(S) \) with \( \phi \circ \varphi = \tau \) for arbitrarily large \( \beta \in \mathbb{R} \). Then a ground state \( \psi \) on \( C^*(S) \) is a KMS\(_\infty\)-state if and only if \( \psi \circ \varphi \) is tracial.

**Proof.** If \( \psi \) is a KMS\(_\infty\)-state, then \( \psi \circ \varphi \) is the weak* limit of normalised traces \( \phi_n \circ \varphi \) for some sequence of KMS\(_\beta\)-states \( \phi_n \) with \( \beta_n \to \infty \), see Proposition 6.3. Hence \( \psi \circ \varphi \) is also a normalised trace.

Now let \( \psi \) be a ground state on \( C^*(S) \) such that \( \psi \circ \varphi \) is tracial. By assumption, there is a sequence of KMS\(_\beta\)-states \( \phi_n \) on \( C^*(S) \) with \( \beta_n \to \infty \) and \( \phi_n \circ \varphi = \psi \circ \varphi \). Weak* compactness of the state space of \( C^*(S) \) yields a subsequence of \( (\phi_n)_n \) converging to a KMS\(_\infty\)-state \( \phi' \) which necessarily satisfies \( \phi' \circ \varphi = \psi \circ \varphi \). Therefore \( \psi \) and \( \phi' \) agree on \( \varphi(C^*(S_c)) \), hence \( \psi = \phi' \) according to Remark 1.1 and \( \psi \) is thus a KMS\(_\infty\)-state. □

**Proposition 6.6.** A KMS\(_\beta\)-state \( \phi \) on \( C^*(S) \) factors through \( \pi: C^*(S) \to Q(S) \) if and only if \( \beta = 1 \). If \( S \) is also core factorable, then \( \pi \) can be replaced by \( \pi_p: C^*(S) \to Q_p(S) \).
Lemma 7.1. Suppose that $\phi = \phi' \circ \pi$ for a KMS$_{\beta}$-state $\phi'$ on $\mathcal{Q}(S)$. As in the proof of Proposition 6.1, there is an accurate foundation set $F$ with $|F| = N_f = n$ for all $f \in F$ and some $n > 1$. Thus

$$1 = \phi'(1) = \phi'\left(\sum_{f \in F} \pi(e_f S)\right) = \sum_{f \in F} \phi(e_f S) = n^{1 - \beta},$$

so necessarily $\beta = 1$.

Conversely, suppose that $\phi$ is a KMS$_1$-state. We must show that $\ker \pi \subset \ker \phi$. Recall from Proposition 3.6 (v) that $S$ has the accurate refinement property from [BS16], thus the kernel of $\pi$ is generated by differences $1 - \sum_{f \in F} e_f S$ for $F$ running over accurate foundation sets. By Proposition 3.6, it suffices to show that $1 - \sum_{f \in F} e_f S$ is in $\ker \phi$ for $F$ running over accurate foundation sets that are transversals for $N^{-1}(n)/\sim$ as $n \in N(S)$. However, for such $F$, the computations in (6.7) done backwards prove that $\phi(1 - \sum_{f \in F} e_f S) = 0$, as needed.

If $S$ is core factorable, then $F$ can be chosen as a subset of $S_{cl}^1$ so that we need to have $\beta = 1$. As $\ker \pi_p \subset \ker \pi$, the second part is clear. □

For the proof of the next result we refer to [Sta17, Theorem 4.1].

Proposition 6.7. There are no ground states on $\mathcal{Q}(S)$. If $S$ is core factorable, then $\mathcal{Q}_p(S)$ does not admit any ground states.

7. The reconstruction formula

As in the previous section, we shall assume that $S$ is a right LCM semigroup that admits a generalised scale $N: S \to \mathbb{N}^\times$, and denote by $\sigma$ the time evolution on $C^*(S)$ given by $\sigma_x(v_s) = N_s^x v_s$ for all $x \in \mathbb{R}$ and $s \in S$. Under the additional assumptions that make $S$ an admissible semigroup, we obtain a formula to reconstruct KMS$_{\beta}$-states on $C^*(S)$ from conditioning on subsets $I$ of Irr($N(S)$) for which $\zeta_I(\beta)$ is finite, see Lemma 7.5 (iv).

Lemma 7.1. Suppose that $S$ is core factorable and admits a generalised scale $N$. For $n \in N(S)$ and a transversal $T_n$ for $N^{-1}(n)/\sim$ with $T_n \subset S_{cl}^1$, let $d_n := 1 - \sum_{f \in T_n} e_f S_f$. Then $d_n$ is a projection in $C^*(S)$, and if $S_{cl} \subset S$ is $\cap$-closed, then $d_n$ is independent of the choice of $T_n$.

Proof. For $n = 1$, we have $T_1 = \{1\}$ and hence $d_1 = 0$. So let $n \in N(S) \setminus \{1\}$. As $S$ is core factorable, there exists a transversal $T_n$ for $N^{-1}(n)/\sim$ with $T_n \subset S_{cl}$. By [A3][b], $T_n$ is an accurate foundation set. Therefore $\{e_f S_f \mid f \in T_n\}$ are mutually orthogonal projections in $C^*(S)$, so $d_n$ is a projection. If $S_{cl} \subset S$ is $\cap$-closed and $T'_n$ is another transversal for $N^{-1}(n)/\sim$ contained in $S_{cl}^1$, then Corollary 3.7 implies that $\sum_{f \in T_n} e_f S_f = \sum_{f \in T'_n} e_f S_f$. □

Lemma 7.2. Suppose that $S$ is core factorable, $S_{cl} \subset S$ is $\cap$-closed, and $S$ admits a generalised scale $N$. Then for each $n \in N(S)$ and $a \in S_{cl}$, the isometry $v_a$ commutes with the projection $d_n$. 
Proof. Fix \( n \in N(S) \) and \( t \in S_c \). We can assume \( n > 1 \). Since \( v_a \) is an isometry, the claim will follow once we establish that

\[
v_a d_nv_a^* = e_{aS} - \sum_{f \in T_n} e_{afS} = e_{aS} - \sum_{f \in T_n} e_{aS \cap fS} = d_nv_av_a^*,
\]

where the second equality is the one requiring a proof. To do so, observe that since \( a \in S_c \) and \( T_n = \{f_1, \ldots, f_n\} \subset S_c \), we have \( aS \cap f_iS = a_{f_i}S \) for some \( f_i' \in S \) with \( N_{f_i'} = N_{f_i} = n \). By Lemma 3.5 we have \( T_n' := \{f_1', \ldots, f_n'\} \subset S_c \). Moreover, \( f_i' \perp f_j' \) for all \( i \neq j \) because the same is true of \( f_i \) and \( f_j \). Thus \( T_n \) and \( T_n' \) are two transversals for \( N^{-1}(n)/_\sim \) that are contained in \( S_c^1 \), and so Corollary 3.7 implies that

\[
\sum_{f \in T_n} e_{fS} = \sum_{f \in T_n'} e_{fS},
\]

showing the desired equality. \( \square \)

The next lemma is crucial to what follows. Roughly, it says the following: for an admissible semigroup, condition \([A4]\) allows that a transversal corresponding to a finite intersection of principal right ideals \( f_nS \) with \( N_{f_n} = n \) and \( n \in \text{Irr}(N(S)) \) can be taken in the form of a transversal for the inverse image, under the scale \( N \), of the product of all \( n \in N(S) \) that appear in the intersection.

Lemma 7.3. Let \( S \) be an admissible right LCM semigroup and \( m, n \in \text{Irr}(N(S)), m \neq n \). If \( T_m = \{f_1, \ldots, f_m\} \) and \( T_n = \{g_1, \ldots, g_n\} \) are two transversals for \( N^{-1}(m)/_\sim \) and \( N^{-1}(n)/_\sim \), respectively, then \( f_iS \cap g_jS = h_{i+m-j-1}S \) for some \( h_{i+m-j-1} \in S \) for all \( i, j \), and \( T_{mn} := \{h_1, \ldots, h_{mn}\} \) is a transversal for \( N^{-1}(mn)/_\sim \).

Proof. By \([A3][b]\) \( T_m \) and \( T_n \) are accurate foundation sets. Let \( T_{mn} \) consist of the \( h_{i+m-j-1} \) for which \( f_i \not\perp g_j \), where \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \). We will show that \( f_i \not\perp g_j \) for all \( i \) and \( j \). Clearly \( T_{mn} \) is an accurate foundation set. Since the generalised scale \( N \) is a homomorphism of right LCM semigroups by Proposition 3.6(iv) and \( m \) and \( n \) are distinct irreducibles in \( N(S) \), we have \( T_{mn} \subset N^{-1}(mn) \). Then Proposition 3.6(iii) implies that \( T_{mn} \) must be a transversal for \( N^{-1}(mn)/_\sim \). Thus, \( |T_{mn}| = mn \) by \([A3][a]\) which gives the claim. \( \square \)

Remark 7.4. We record another consequence of condition \([A4]\). Suppose \( I \subset \text{Irr}(N(S)) \). We claim that the \( I \)-restricted \( \zeta \)-function \( \zeta_I \) from Definition 4.2 admits a product description \( \zeta_I(\beta) = \prod_{n \in I} (1 - n^{-\beta-1})^{-1} \). Indeed, all summands in the series defining \( \zeta_I(\beta) \) are positive, so we are free to rearrange their order. Condition \([A4]\) implies that the submonoid \( (I) \) of \( N^x \) is the free abelian monoid in \( I \), and therefore

\[
\zeta_I(\beta) = \sum_{n \in (I)} n^{-\beta-1} = \prod_{n \in I} (1 - n^{-\beta-1})^{-1}.
\]

The next result is an abstract version of \([LR10]\) Lemma 10.1, proved for the semigroup \( C^* \)-algebra of \( \mathbb{N} \times \mathbb{N}^x \). Laca and Raeburn had anticipated back then that the result is “likely to be useful elsewhere”. Here we confirm this point by showing that the techniques of proof, when properly updated, are valid in the abstract setting of our admissible semigroups.

We recall that the \(*\)-homomorphism \( \varphi : C^*(S_c) \to C^*(S) \) was defined in Remark 4.1. In the following, we let \( T \) be a transversal for \( S /_\sim \) with \( T \subset S_c^1 \) and \( T_I := \bigcup_{n \in (I)} T_n \) for \( I \subset \text{Irr}(N(S)) \).
Lemma 7.5. Let $S$ be an admissible right LCM semigroup. If $\phi$ is a KMS$_\beta$-state on $C^*(S)$ for some $\beta \in \mathbb{R}$, denote by $(\pi_\phi, H_\phi, \xi_\phi)$ its GNS-representation and by $\tilde{\phi} = \langle \cdot, \xi_\phi \rangle$ the vector state extension of $\phi$ to $\mathcal{L}(H_\phi)$. For every subset $I$ of Irr($N(S)$), $Q_I := \prod_{n \in I} \pi_\phi(d_n)$ defines a projection in $\pi_\phi(C^*(S))'$. If $\zeta_I(\beta) < \infty$, then $Q_I$ has the following properties:

(i) $\tilde{\phi}(Q_I) = \zeta_I(\beta)^{-1}$.

(ii) The map $y \mapsto \zeta_I(\beta) \tilde{\phi}(Q_I \pi_\phi(y)Q_I)$ defines a state $\phi_I$ on $C^*(S)$ for which $\phi_I \circ \varphi$ is a trace on $C^*(S)$. 

(iii) The family $(\pi_\phi(v_s)Q_I \pi_\phi(v_s^*)_{s \in T_I})$ consists of mutually orthogonal projections, and $Q_I' := \sum_{s \in T_I} \pi_\phi(v_s)Q_I \pi_\phi(v_s^*)$ defines a projection such that $\tilde{\phi}(Q_I') = 1$.

(iv) There is a reconstruction formula for $\phi$ given by 

$$
\phi(y) = \zeta_I(\beta)^{-1} \sum_{s \in T_I} N_s^{-\beta} \phi_I(v_s^* y v_s) \quad \text{for all } y \in C^*(S).
$$

Proof. If $I$ is finite, then $Q_I$ belongs to $\pi_\phi(C^*(S))$. If $I$ is infinite, then we obtain $Q_I$ as a weak limit of projections $Q_{I_n}$ with $|I_n| < \infty$ and $I_n \nearrow I$ because the $d_n$ commute. Hence $Q_I$ belongs to the double commutant of $\pi_\phi(C^*(S))$ inside $\mathcal{L}(H_\phi)$. 

Let us first assume that $I$ is finite. An iterative use of Lemma 7.3 then implies

$$
\prod_{n \in I} d_n = \sum_{A \subseteq I} (-1)^{|A|} \sum_{(f_1, \ldots, f_A) \in \prod_{n \in A} T_n} e_{f_1 \cdots f_A} S = \sum_{A \subseteq I} (-1)^{|A|} \sum_{f \in T_n(A)} e_{f S}.
$$

Applying $\tilde{\phi} \circ \pi_\phi = \phi$ to (7.1), using that $\phi$ is a KMS$_\beta$-state on $C^*(S)$ as in the last equality of (6.7), and invoking the product formula from Remark 7.4 yields

$$
\tilde{\phi}(Q_I) = \sum_{A \subseteq I} (-1)^{|A|} n_A^{-\beta-1} = \prod_{n \in I} (1 - n^{-\beta-1}) = \zeta_I(\beta)^{-1}.
$$

This proves part (i) in the case that $I$ is finite. The general case follows from the finite case by taking limits $I_n \nearrow I$ with $I_n$ finite, using continuity as $\phi$ is a vector state and the fact that $Q_I$ is the weak limit of $(Q_{I_n})_{n \geq 1}$.

For (ii), we note that $\phi_I$ is a state on $C^*(S)$ by i), and it remains to prove that it is tracial on $\varphi(C^*(S_c))$. Let $I$ be finite and $r_i \in S_c, i = 1, \ldots, 4$. According to Lemma 7.2 and the fact that $\phi$ is a KMS$_\beta$-state (keeping in mind that $N_{r_1} = N_{r_2} = 1$), we have

$$
\phi_I(v_{r_1} v_{r_2} v_{r_3} v_{r_4}) = \zeta_I(\beta) \phi(\prod_{m \in I} d_m v_{r_1} v_{r_2} v_{r_3} v_{r_4} \prod_{n \in I} d_n) = \zeta_I(\beta) \phi(v_{r_1} v_{r_2} \prod_{m \in I} d_m v_{r_3} v_{r_4} \prod_{n \in I} d_n) = \zeta_I(\beta) \phi(\prod_{m \in I} d_m v_{r_3} v_{r_4} \prod_{n \in I} d_n v_{r_1} v_{r_2}) = \zeta_I(\beta) \phi(\prod_{m \in I} d_m v_{r_3} v_{r_4} v_{r_1} v_{r_2} \prod_{n \in I} d_n) = \phi_I(v_{r_3} v_{r_4} v_{r_1} v_{r_2}).
$$

For general $I$, the state $\phi_I$ is the weak*-limit of $\phi_{I_n}$ where $I_n$ is finite and $I_n \nearrow I$, so that the trace property is preserved. This completes (ii).

Towards proving (iii), note first that by Lemma 7.1 the family of projections appearing in the assertion is independent of the choice of the transversals as $n$ varies in $(I)$. Let $s \neq t$ with $s$ in a transversal $T_m$ for $N^{-1}(m)/\sim$ and $t$ in a transversal $T_n$ for $N^{-1}(n)/\sim$ for
some $m, n \in \{I\}$. If $m = n$, then $s$ and $t$ are distinct elements belonging to a transversal that is also a (proper) accurate foundation set, so $s \perp t$, which implies $v^*_sv^*_t = 0$ and therefore shows that the projections $\pi_{\phi}(v^*_sv^*_t)Q_I\pi_{\phi}(v^*_sv^*_t)$ and $\pi_{\phi}(v^*_sv^*_t)Q_I\pi_{\phi}(v^*_sv^*_t)$ are orthogonal. Suppose $m \neq n$. Now either $s \perp t$, which implies the claimed orthogonality as above, or $s$ and $t$ have a right LCM in $S$. In the latter case, using that $N$ is a homomorphism of right LCM semigroups, it follows that there are $s', t' \in S$ with $ss' = tt'$ and $N_{s'} = m^{-1}\text{lcm}(m, n), N_{t'} = n^{-1}\text{lcm}(m, n)$. Since $s, t \in S_{ci}, sS \cap tS \neq \emptyset$ and $S_{ci} \subset S$ is $\cap$-closed, also $ss', tt' \in S_{ci}$. This forces $s', t' \in S_{ci}$. The assumption $m \neq n$ implies that $N_{s'} > 1$ or $N_{t'} > 1$. Suppose $N_{s'} > 1$. Note $N_{s'} \in \{I\}$ because this monoid is free abelian by (A4).

Take $n' \in I$ such that $N_{s'}$ is a multiple of $n'$, and pick a corresponding proper accurate foundation set $F_{n'}$. Then $fS \cap s' \not\in \emptyset$ for some $f \in F_{n'}$. Using that $S_{ci} \subset S$ is $\cap$-closed, we can write $fS \cap s' \not\in s''S$ for some $s'' \in S_{ci}$ for which $N_{s''} = \text{lcm}(N_{s'}, n') = N_{s'}$. Then $s'$ and $s''$ differ by multiplication on the right with an element of $S^*$ according to Corollary 3.7, so $fS \cap s' \not\in s''S$. Moreover, there is a unique $f \in F_{n'}$ with this property because $F_{n'}$ is accurate. Now $SS \subset fS$ implies that $d_{n'} e_{s''S} = 0$, and this in turn yields

$$Q_I\pi_{\phi}(v^*_sv^*_t)Q_I = Q_{f \cap \{n'\}^c}\pi_{\phi}(d_{n'} e_{s''S}v^*_sv^*_t)Q_I = 0,$$

from which the desired orthogonality follows. The case $N_{t'} > 1$ is analogous. Therefore, we get mutually orthogonal projections whenever $s \in T_m$ and $t \in T_n$ are distinct. In particular, $Q^I$ is a projection in $\pi_{\phi}(C^*(S))''$. For a finite set $I$, an application of the KMS$\beta$-condition at the second equality shows that

$$\tilde{\phi}(Q^I) = \sum_{s \in T_I} \phi(v_s \prod_{m \in I} d_m v^*_m) = \sum_{s \in T_I} N_{s}^{-\beta} \tilde{\phi}(Q_I) = \tilde{\phi}(Q_I)\zeta_I(\beta).$$

The last term is 1 by part (i), and (iii) follows. For arbitrary $I$, we invoke continuity as was done in the proof of part $i$.

Finally, to prove (iv), note that $\tilde{\phi}(Q^I) = 1$ implies $\tilde{\phi}(\pi_{\phi}(y)) = \tilde{\phi}(Q^I\pi_{\phi}(y)Q^I)$ for all $y \in C^*(S)$. For finite $I$, appealing to (iii) and using the KMS$\beta$-condition three times, we obtain

$$\phi(y) = \tilde{\phi}(\pi_{\phi}(y)) = \tilde{\phi}(Q^I\pi_{\phi}(y)Q^I) = \sum_{s,t \in T_I} \phi(v_s \prod_{k \in I} d_k v^*_s y v_t \prod_{l \in I} d_l v^*_l) = \sum_{s,t \in T_I} N^{-\beta}_s \phi(\prod_{k \in I} d_k v^*_s y v_t \prod_{l \in I} d_l v^*_l v_s \prod_{k \in I} d_{k'}) = \sum_{s,t \in T_I} \tilde{\phi}(Q_I\pi_{\phi}(v^*_sv_t)Q_I) = \zeta_I(\beta)^{-1} \sum_{s \in T_I} N^{-\beta}_s \tilde{\phi}(v^*_sv_s).$$

This proves the reconstruction formula in the case of finite $I$. For general $I \subset \text{Irr}(N(S))$ the claim follows by continuity arguments.

8. Construction of KMS-states

In this section we show that the constraints obtained in Section 6 are optimal. To begin with, we assume that $S$ is a core factorable right LCM semigroup for which $S_{ci} \subset S$ is $\cap$-closed. We then fix a transversal $T$ for $S_{ci}$, with $T \subset S^0_{ci}$ according to Lemma 3.8, thus $T$ gives rise to maps $i: S \to T$ and $c: S \to S_c$ determined by $s = i(s)c(s)$ for all
$s \in S$. Recall from Lemma 3.9 that we denote by $\alpha$ both the action $S_c \curvearrowright S/\sim$ and the corresponding action $S_c \curvearrowright T$. Recall further from Remark 4.1 that the standard generating isometries for $C^*(S_c)$ are $w_a$ for $a \in S_c$.

**Remark 8.1.** Consider $C^*(S_c)$ as a right Hilbert module over itself with inner product $\langle y, z \rangle = y^*z$ for $y, z \in C^*(S_c)$. For every $a \in S_c$, the map $y \mapsto w_ay$ defines an isometric endomorphism of $C^*(S_c)$ as a right Hilbert $C^*(S_c)$ module. If $a \in S^*$, then the map is an isometric isomorphism. Indeed, the map is linear and compatible with the right module structures. It also preserves the inner product as $w_a$ is an isometry, and hence $\langle w_ay, w_az \rangle = y^*w_a^*w_az = y^*z$. Thus, the map is isometric. If $a \in S^*$, then the map is surjective because $w_a(w_a^*y) = y$ for all $y \in C^*(S_c)$.

We start with a brief lemma that allows us to recover $t$ and $c(st)$ from $s$ and $i(st)$ for $s \in S$ and $t \in T$, followed by a technical lemma that makes the proof of Theorem 8.4 more accessible. Both results are further elaborations on the factorisation arising from Lemma 3.8.

**Lemma 8.2.** For $s \in S$, the map $T \to T, t \mapsto i(st)$ is injective on $T$, and there is a bijection $i(st) \to \{(t, c(st)) \mid t \in T\}$.

**Proof.** Suppose $s \in S, t_1, t_2 \in T$ satisfy $i(st_1) = i(st_2)$. Then $st_1 \sim st_2$, and left cancellation implies $t_1 \sim t_2$, so that $t_1 = t_2$ as $t_1, t_2 \in T$. Thus, we can recover $t \in T$ from $i(st)$, and as $st = i(st)c(st)$, we then recover $c(st)$. □

**Lemma 8.3.** Suppose there are $s_1, \ldots, s_4 \in S, t_1, t_2, r \in T$ such that $s_1S \cap s_2S = s_1s_3S, s_1s_3 = s_2s_4$, $t_1 = i(s_3r)$, and $t_2 = i(s_4r)$. Then $c(s_1t_2)c(s_3r) = c(s_2t_1)c(s_4r)$ and $c(s_1t_2)S_c \cap c(s_2t_1)S_c = c(s_1t_2)c(s_3r)S_c$.

**Proof.** The equation $s_1s_3r = s_1t_2(c(s_3r)) = i(s_1t_2)c(s_1t_2)c(s_3r) = i(s_2t_1)c(s_2t_1)c(s_4r) = s_2t_1(c(s_4r) = s_2s_4r$ yields $i(s_1t_2) \sim i(s_2t_1)$, and hence $i(s_1t_2) = i(s_2t_1)$. Thus left cancellation in $S$ gives $c(s_1t_2)c(s_3r) = c(s_2t_1)c(s_4r)$. For the second part of the claim, the first part implies $c(s_1t_2)S_c \cap c(s_2t_1)S_c \supset c(s_1t_2)c(s_3r)S_c$. Moreover, it suffices to show $c(s_1t_2)S_c \cap c(s_2t_1)S_c \subset c(s_1t_2)c(s_3r)S_c$ since the embedding of $S_c$ into $S$ is a homomorphism of right LCM semigroups, see Proposition 3.4. So suppose $c(s_1t_2)S_c \cap c(s_2t_1)S_c = c(s_1t_2)aS_c, c(s_1t_2)a = c(s_2t_1)b$ for some $a, b \in S_c$. Then $c(s_3r) = ad$ for some $d \in S_c$, and we need to show that $d \in S^*$. As $s_1t_2a = i(s_1t_2)c(s_1t_2)a = i(s_2t_1)c(s_2t_1)b = s_2t_1b \in s_1S \cap s_2S = s_1s_3S$, left cancellation forces $t_2a \in s_3S$, i.e. $t_2a = s_3r'$ for some $r' \in S$. Therefore, we get $s_3r = t_2(c(s_3r) = t_2ad = s_3r'd$. By left cancellation, this yields $r = i(r')c(r')d$. But now $r \in T$, so $i(r') = r$, and hence $1 = i(r')d$, i.e. $d \in S^*$. □

The next result is inspired by LRRW14, Propositions 3.1 and 3.2, where the authors construct a representation of the Toeplitz algebra for a Hilbert bimodule associated to a self-similar action. We recall that when $X$ is a right-Hilbert module over a $C^*$-algebra $A$ and $\pi$ is a representation of $A$ on a Hilbert space $H$, then $X \otimes_A H$ is the Hilbert
space obtained as the completion of the algebraic tensor product with respect to the inner-product characterised by \( \langle x \otimes h, y \otimes k \rangle = \langle \pi(y, x)h, k \rangle \), and where the tensor product is balanced over the left action on \( H \) given by \( \pi \), see for instance [RW98]. If \( X \) is a direct sum Hilbert module \( \oplus_{j \in J} A \) of the standard Hilbert module \( A \), then \( X \otimes A H \) can be identified with the infinite direct sum of Hilbert spaces \( A \otimes A H \). In case \( A \) has an identity 1, then with \( \xi_j \) denoting the image of 1 \( \in A \) in the \( j \)-th copy of \( A \), we may identify \( \xi_j a \otimes h \) for \( a \in A \) as the element in \( \oplus_{j \in J} (A \otimes A H) \) that has \( a \otimes h \) in the \( j \)-th place and zero elsewhere. From now on, we will freely use this identification.

**Theorem 8.4.** Let \( S \) be a core factorable right LCM semigroup such that \( S_{ci} \subset S \) is \( \cap \)-closed. Let \( M := \bigoplus_{t \in T} C^*(S_c) \) be the direct sum of the standard right Hilbert \( C^*(S_c) \)-module. Suppose \( \rho \) is a state on \( C^*(S_c) \) with GNS-representation \( (\pi', \mathcal{H}', \xi_\rho) \). Form the Hilbert space \( \mathcal{H}_\rho := M \otimes C^*(S_c) \mathcal{H}' \). The linear operator \( V_s : \mathcal{H}_\rho \to \mathcal{H}_\rho \) given by

\[
V_s(\xi t y \otimes \xi_\rho) := \xi i(st) w_i c(st)y \otimes \xi_\rho
\]

for \( t \in T, y \in C^*(S_c) \) is adjointable for all \( s \in S \). The family \((V_s)_{s \in S}\) gives rise to a representation \( \pi'_\rho : C^*(S) \to \mathcal{L}(\mathcal{H}_\rho) \).

**Proof.** It is clear that \( \mathcal{H}_\rho \) is a Hilbert space, so we start by showing that the operator \( V_s \) is adjointable. For \( s \in S \) and \( t_1, t_2 \in T, y, z \in C^*(S_c) \), we have

\[
\langle V_s(\xi t_1 y \otimes \xi_\rho), \xi t_2 z \otimes \xi_\rho \rangle = \langle \xi_\rho, \pi'_\rho(\langle \xi i(st_1) w_i c(st_1) y, \xi t_2 z \rangle) \xi_\rho \rangle
\]

\[
= \langle \xi_\rho, \pi'_\rho(\delta i(st_1), t_2 y^* w_i c(st_1) z) \xi_\rho \rangle
\]

\[
= \delta i(st_1), t_2 \rho(z^* w_i c(st_1) y).
\]

Define the linear operator \( V'_s \) on \( \mathcal{H}_\rho \) by

\[
V'_s(\xi t y \otimes \xi_\rho) := \chi i(st)(t) \xi s t y^* w_i c(st)y \otimes \xi_\rho,
\]

where \( t' \in T \) is uniquely determined by \( i(st') = t \) in the case where \( t \in i(sT) \), see Lemma 8.2. It is then straightforward to show that

\[
\langle \xi t_1 y \otimes \xi_\rho, V'_s(\xi t_2 z \otimes \xi_\rho) \rangle = \delta i(st_1), t_2 \rho(z^* w_i c(st_1) y),
\]

so that \( V_s \) is adjointable with \( V'_s = V_s \). It is also clear that each \( V_s \) is an isometry. Now let \( s_1, s_2 \in S \). To show that \( V_{s_1} V_{s_2} = V_{s_2} \), note that when computing \( V_{s_1} V_{s_2}(\xi t y \otimes \xi_\rho) \) for some \( t \in T \) we encounter the term \( \xi i(st c(s_2 t)) w_i c(s_2 t) y \) which is the same as \( \xi i(st c(s_2)) w_i c(s_2 t) y \) entering \( V_{s_2}(\xi t y \otimes \xi_\rho) \) because

\[
i(s_1 i(s_2 t)) c(s_1 i(s_2 t)) c(s_2 t) = s_1 i(s_2 t) c(s_2 t) = s_1 s_2 t = i(s_1 s_2 t) c(s_1 s_2 t)
\]

by Lemma 8.2. So it remains to prove that we have

\[
V^*_{s_1} V_{s_2} = \begin{cases} V^*_{s_3} V_{s_4} & \text{if } s_1 S \cap s_2 S = s_1 s_3 S, s_1 s_2 = s_2 s_4, \\ 0 & \text{if } s_1 \perp s_2. \end{cases}
\]

We start by observing that since

\[
\langle V^*_{s_1} V_{s_2}(\xi t_1 y \otimes \xi_\rho), \xi t_2 z \otimes \xi_\rho \rangle = \langle V_{s_2}(\xi t_1 y \otimes \xi_\rho), V_{s_1} \xi t_2 z \otimes \xi_\rho \rangle
\]

\[
= \langle \xi_\rho, \pi'_\rho(\langle \xi i(st_1) w_i c(st_1) y, \xi i(s_2 t) w_i c(s_2 t) z \rangle) \xi_\rho \rangle
\]

\[
= \delta i(s_2 t), i(st_1) \rho(z^* w_i c(s_2 t) w_i c(st_1) y),
\]

(8.1)
the inner product on the left can only be nonzero if \( s \perp s_2 \). Indeed, if \( s_1 \perp s_2 \), then \( i(s_2 t_1) \neq i(s_1 t_2) \) for all \( t_1, t_2 \in T \) as \( c(s_2 t_1), c(s_1 t_2) \in S_c \). So suppose there are \( s_3, s_4 \in S \) such that \( s_1 S \cap s_2 S = s_3 S, s_1 s_3 = s_2 s_4 \). If \( t_1, t_2 \in T \) and \( y, z \in C^*(S_c) \) are such that
\[
\langle V_{s_3} V_{s_4}^*(\xi_{t_1} y \otimes \xi_\rho), \xi_{t_2} z \otimes \xi_\rho \rangle \neq 0,
\]
then there must be \( t_3 \in T \) with the property that \( t_1 = i(s_3 t_3) \) and \( t_2 = i(s_3 t_3) \). In this case, we have
\[
\langle V_{s_3} V_{s_4}^*(\xi_{t_1} y \otimes \xi_\rho), \xi_{t_2} z \otimes \xi_\rho \rangle = \rho(z^* w_{c(s_3 t_3)} w_{c(s_3 t_3)}^* y).
\]
In addition, we get \( s_2 t_1 c(s_4 t_3) = s_2 s_3 t_3 = s_1 s_3 t_3 = s_1 t_2 c(s_4 t_3) \) so that \( i(s_2 t_1) = i(s_1 t_2) \), which by \( (8.1) \) also determines when \( V_{s_2}^* V_{s_2} \) is nonzero. Now \( w_{c(s_2 t_1)}^* w_{c(s_2 t_1)} = w_{r_1} w_{r_2}^* \) for all \( r_1, r_2 \in S_c \) satisfying \( c(s_2 t_2) S_c \cap c(s_2 t_2) S_c = c(s_2 t_2) r_1 S_c \) and \( c(s_2 t_2) r_1 = c(s_2 t_4) r_2 \). Due to Lemma \( 8.3 \) \( c(s_3 t_3) \) and \( c(s_4 t_3) \) have this property. Thus we conclude that \( V_{s_1}^* V_{s_2} = V_{s_3} V_{s_4}^* \) for \( s_1 \perp s_2 \). This shows that \( (V_s)_{s \in S} \) defines a representation \( \pi_\rho \) of \( C^*(S) \) by the universal property of \( C^*(S) \).

From now on we assume that \( S \) is a core factorable right LCM semigroup, \( S_{ci} \subset S \) is \( \cap \)-closed, and \( S \) admits a generalised scale.

**Lemma 8.5.** For every state \( \rho \) on \( C^*(S_c) \) the following assertions hold:

(i) If \( s, r \in S \) are such that \( (V_r V_r^*(\xi_t \otimes \xi_\rho), \xi_t \otimes \xi_\rho) \neq 0 \) for some \( t \in T \), then \( s \sim r \).

(ii) Given \( a, b \in S_c \) and \( t \in T \), then for all \( y, z \in C^*(S) \) we have
\[
\langle V_a V_b^*(\xi_{t} y \otimes \xi_\rho), \xi_{t} z \otimes \xi_\rho \rangle = \chi_{\text{Fix}(a_a a_b^{-1})}(t) \rho(z^* w_{c(a_a a_b^{-1})}^* w_{c(b b_a^{-1})}^* y).
\]

**Proof.** For (i), suppose \( r, s \) and \( t \) are given as in the first assertion of thelemma. Then there is \( t' \in T \) with \( i(st') = t = i(rt') \). Thus we have \( st' \sim rt' \). Hence, \( N_s = N_r \) and \( sS \cap rS \neq \emptyset \). By Proposition \( 3.6 \)(ii), we have \( s \sim r \).

To prove (ii), note that straightforward calculations using Lemma \( 3.9 \) show that
\[
V_a(\xi_t y \otimes \xi_\rho) = \xi_{a(t)} w_{c(a(t))} y \otimes \xi_\rho \text{ and } V_b^*(\xi_t y \otimes \xi_\rho) = \xi_{b(t)}^{-1} w_{c(b(t))}^* y \otimes \xi_\rho.
\]
This immediately gives the claim on \( V_a V_b^* \). \( \square \)

**Proposition 8.6.** For every state \( \rho \) on \( C^*(S_c) \), the map
\[
\psi_\rho(y) := \langle \pi_\rho(y)(\xi_1 \otimes \xi_\rho), \xi_1 \otimes \xi_\rho \rangle
\]
defines a ground state on \( C^*(S) \) with \( \psi_\rho \circ \varphi = \rho \).

**Proof.** The map \( \psi_\rho \) is a state on \( C^*(S) \) with \( \psi_\rho \circ \varphi = \rho \), see Theorem \( 8.4 \). By Lemma \( 8.5 \) we see that
\[
\langle V_r^*(\xi_1 \otimes \xi_\rho), V_r^*(\xi_1 \otimes \xi_\rho) \rangle \neq 0
\]
forces \( 1 \in i(sT) \cap i(tT) \) for \( s, t \in S \). This implies \( s \sim 1 \sim t \), and hence \( s, t \in S_{ci} \), see Lemma \( 3.2 \). Thus \( \psi_\rho \) is a ground state by virtue of Proposition \( 6.2 \). \( \square \)

Before we construct KMS\( _\beta \)-states on \( C^*(S) \) from traces \( \tau \) on \( C^*(S_c) \) with the help of the representation \( \pi_\tau \), we make note of an intermediate step which produces states on \( C^*(S) \) from traces on \( C^*(S_c) \). In terms of notation, note that the finite subsets of \( \text{Irr}(N(S)) \) form a (countable) directed set when ordered by inclusion.
Lemma 8.7. Let $\beta \geq 1$, $\tau$ a trace on $C^*(S_c)$, and $I \subset \text{Irr}(N(S))$ with $\zeta_I(\beta) < \infty$. Then

$$\psi_{\beta,\tau,I}(y) := \zeta_I(\beta)^{-1} \sum_{t \in T_I} N_t(\tau)(\xi_t \otimes \xi_t)$$

defines a state on $C^*(S)$ such that $\psi_{\beta,\tau,I} \circ \varphi$ is tracial on $C^*(S_c)$.

Proof. From the construction in Theorem 8.4, it is clear that

(8.2) \[ \sum_{t \in T_n} \langle V_a V_b V_d V_e^* (\xi_t \otimes \xi_t), \xi_t \otimes \xi_t \rangle = \sum_{t \in T_n} \langle V_d V_e^* V_a V_b^* (\xi_t \otimes \xi_t), \xi_t \otimes \xi_t \rangle \]

holds for all $a, b, d, e \in S_c$ and $n \in \langle I \rangle$. Since $bS \cap dS \neq \emptyset$, Lemma 8.5 (ii) implies that

$$\langle V_a V_b V_d V_e^* (\xi_t \otimes \xi_t), \xi_t \otimes \xi_t \rangle$$

vanishes for $t \in T_n$, unless $t$ belongs to

$$T_n^{(a,b),(d,e)} := T_n \cap \text{Fix}(\alpha_d \alpha_{a}^{-1} \alpha_d \alpha_{b}^{-1});$$

just write $bb' = dd'$ and invoke the inner product formula for $V_{ab} V_{ed}^*$ to decide when it vanishes. The bijection $f := \alpha_d \alpha_{e}^{-1}$ of $T$ restricts to a bijection from $T_n^{(a,b),(d,e)}$ to $T_n^{(d,e),(a,b)}$ as $t \in T_n^{(a,b),(d,e)}$ implies

$$f(t) = \alpha_d \alpha_{e}^{-1}(\alpha_a \alpha_b^{-1} \alpha_d \alpha_{e}^{-1}(t)) = \alpha_d \alpha_{e}^{-1} \alpha_a \alpha_b^{-1}(f(t)) \in T_n^{(d,e),(a,b)}.$$

For $t \in T_n^{(a,b),(d,e)}$, Lemma 8.5 (ii) yields

$$\langle V_a V_b V_d V_e^* (\xi_t \otimes \xi_t), \xi_t \otimes \xi_t \rangle = \tau(w_{c(ar)} w_{c(br)}^* w_{c(ds)} w_{c(es)}^*),$$

where $s, r \in T_n$ are given by $s = \alpha_e^{-1}(t)$ and $r = \alpha_b^{-1} \alpha_d \alpha_{e}^{-1}(t) = \alpha_a^{-1}(t)$. Likewise, we have

$$\langle V_d V_e^* V_a V_b^* (\xi_{f(t)} \otimes \xi_{f(t)}), \xi_{f(t)} \otimes \xi_{f(t)} \rangle = \tau(w_{c(ds)} w_{c(es)}^* w_{c(ar)} w_{c(br)}^*),$$

where $s', r' \in T_n$ are given by

$$r' = \alpha_b^{-1} f(t) = \alpha_b^{-1} \alpha_d \alpha_{e}^{-1}(t) = \alpha_a^{-1}(t) = r,$$

and

$$s' = \alpha_e^{-1} \alpha_a \alpha_{b}^{-1} f(t) = \alpha_e^{-1} \alpha_a \alpha_b^{-1} \alpha_d \alpha_{e}^{-1}(t) = \alpha_e^{-1}(t) = s.$$

Since $\tau$ is a trace, we thus have

$$\langle V_d V_e^* V_a V_b^* (\xi_{f(t)} \otimes \xi_{f(t)}), \xi_{f(t)} \otimes \xi_{f(t)} \rangle = \tau(w_{c(ds)} w_{c(es)}^* w_{c(ar)} w_{c(br)}^*),$$

as claimed. This establishes (8.2), and hence that $\psi_{\beta,\tau,I} \circ \varphi$ is a trace on $C^*(S_c)$. \qed

Proposition 8.8. For $\beta \geq 1$ and every trace $\tau$ on $C^*(S_c)$, there is a sequence $(I_k)_{k \geq 1}$ of finite subsets of $\text{Irr}(N(S))$ such that $\psi_{\beta,\tau,I_k}$ weak*-converges to a KMS$_\beta$-state $\psi_{\beta,\tau}$ on $C^*(S)$ as $I_k \nrightarrow \text{Irr}(N(S))$. For $\beta \in (\beta_c, \infty)$, the state $\psi_{\beta,\tau}$ is given by $\psi_{\beta,\tau,I_k,N(S)}$. 

Proof. Due to weak*-compactness of the state space on $C^*(S)$, there is a sequence $(I_k)_{k \geq 1}$ of finite subsets of Irr$(N(S))$ with $I_k \not\subseteq$ Irr$(N(S))$ such that $(\psi_{\beta,\tau,I_k})_{k \geq 1}$ obtained from Lemma 8.7 converges to some state $\psi_{\beta,\tau}$ in the weak* topology. By Proposition 6.4, $\psi_{\beta,\tau}$ is a KMS$_\beta$-state if and only if $\psi_{\beta,\tau} \circ \varphi$ defines a trace on $C^*(S_c)$ and (6.3) holds. Note that $\psi_{\beta,\tau} \circ \varphi$ is tracial on $C^*(S_c)$ because each of the $\psi_{\beta,\tau,I_k}$ has this property by Lemma 8.7. It remains to prove (6.3). For fixed $s, r \in S$, Lemma 8.5 (i) shows that $\langle V_s V_r^* (\xi_t \otimes \xi_r), \xi_t \otimes \xi_r \rangle$ vanishes for all $t \in T$, unless $s \sim r$. Therefore $\psi_{\beta,\tau}(v_s v_r^*) = 0$ if $s \not\sim r$. Now fix $s, r \in S$ such that $s \sim r$. Write $s = t^a$ and $r = t^b$ for some $t^c \in T, a, b \in S_c$. Then $\langle V_s V_r^* (\xi_t \otimes \xi_r), \xi_t \otimes \xi_r \rangle$ can only be nonzero if $t \in i(t'T) \cap T$. Note that for $t'' \in T$ such that $t = i(t't'')$ we have $tc(t''t) = t't''$, so $N_t = N_{t't''}$. For large enough $k$, we have $t^c \in T_{I_k}$. Hence, as we sum over $t \in T_{I_k}$, for $k$ large enough also $t'' \in T_{I_k}$. Thus

$$\psi_{\beta,\tau}(v_s v_r^*) = \lim_{k \to \infty} \xi_{I_k}(\beta)^{-1} \sum_{t \in T_{I_k}} N_{t^{-\beta}} \langle V_s V_r^* (\xi_t \otimes \xi_r), \xi_t \otimes \xi_r \rangle$$

$$= \lim_{k \to \infty} \xi_{I_k}(\beta)^{-1} \sum_{t \in T_{I_k}} N_{t^{-\beta}} \langle V_s V_b^* V_r^* (\xi_t \otimes \xi_r), V_c^* (\xi_t \otimes \xi_r) \rangle$$

$$= \lim_{k \to \infty} \xi_{I_k}(\beta)^{-1} \sum_{t^c \in T_{I_k}} N_{t^{-\beta}} \langle V_s V_b^* (\xi_{t'^c} w_{c(t'^c)}^*) \otimes \xi_{t'^c}, \xi_{t'^c} \otimes \xi_{t'^c} \rangle$$

$$= \lim_{k \to \infty} N_{s^{-\beta}} \psi_{\beta,\tau,I_k}(v_s v_r^*)$$

where we used Lemma 8.5 (ii) and the trace property of $\tau$ in the penultimate equality. This shows that the limit $\psi_{\beta,\tau}$ satisfies (6.3), and hence is a KMS$_\beta$-state. The claim for $\beta \in (\beta_c, \infty)$ follows immediately from $I_k \not\subseteq$ Irr$(N(S))$ because the formula from Lemma 8.7 makes sense for $I = \text{Irr}(N(S))$. \qed

Corollary 8.9. Let $S$ be an admissible right LCM semigroup. If $\phi$ is a KMS$_\beta$-state on $C^*(S_c)$ for $\beta \in (\beta_c, \infty)$, then $\psi_{\beta,\text{Irr}(N(S))} \circ \varphi = \phi$.

Proof. As $\phi_{\text{Irr}(N(S))} \circ \varphi$ is a trace on $C^*(S_c)$ by Proposition 6.4, $\psi_{\beta,\text{Irr}(N(S))} \circ \varphi$ is a KMS$_\beta$-state, see Proposition 8.8. Due to Proposition 6.4 it suffices to show that $\psi_{\beta,\text{Irr}(N(S))} \circ \varphi = \phi \circ \varphi$. For $a, b \in S_c$, Lemma 8.5 (ii) and $\varphi(w_{a}w_{b}^*) = v_a v_b^*$ give

$$\psi_{\beta,\text{Irr}(N(S))} \circ \varphi(v_a v_b^*) = \xi_S(\beta)^{-1} \sum_{t \in T} N_t^{-\beta} \langle V_a V_b^* (\xi_t \otimes \xi_{\text{Irr}(N(S))} \circ \varphi), \xi_t \otimes \xi_{\text{Irr}(N(S))} \circ \varphi \rangle$$

$$= \xi_S(\beta)^{-1} \sum_{n \in N(S)} \sum_{t \in \Gamma_n} n^{-\beta} \chi_{\text{Fix}(a \alpha_n^{-1})}(t) \phi_{\text{Irr}(N(S))} \circ \varphi(w_{c(a \alpha_n^{-1})}(t) w_c^{*}(c(b \alpha_n^{-1})(t)).$$

We claim that $\chi_{\text{Fix}(a \alpha_n^{-1})}(t) \varphi(w_{c(a \alpha_n^{-1})}(t) w_c^{*}(c(b \alpha_n^{-1})(t)) = v_t v_d v_b^* v_t$. Indeed, the expression $v_t v_d v_b^* v_t$ vanishes unless $t \in \text{Fix}(a \alpha_n^{-1})$. But if $t \in \text{Fix}(a \alpha_n^{-1})$, then $a \alpha_n^{-1}(t) = tc(a \alpha_n^{-1}(t))$ and $b \alpha_n^{-1}(t) = tc(b \alpha_n^{-1}(t))$, see the proof of Lemma 3.9. Hence

$$v_t v_d v_b^* v_t = v_t e_{c \alpha_n^{-1}(t)} S v_d v_b^* e_{c b \alpha_n^{-1}(t)} S v_t$$

$$= v_t v_{tc(a \alpha_n^{-1}(t))} v_{a \alpha_n^{-1}(t)} v_b^* v_{b \alpha_n^{-1}(t)} v_c(b \alpha_n^{-1}(t)) v_t$$

$$= v_{c(a \alpha_n^{-1}(t))} v_{a \alpha_n^{-1}(t)} v_{b \alpha_n^{-1}(t)} v_c(b \alpha_n^{-1}(t))$$

$$= v_{c(a \alpha_n^{-1}(t))} v_{a \alpha_n^{-1}(t)} v_{b \alpha_n^{-1}(t)}.$$
Therefore we conclude that $\psi_{\beta, d_{\text{trac}}} \circ \varphi = \phi$ by appealing to Lemma 7.5 (iv).

Corollary 8.9 shows that every KMS$_{\beta}$-state on $C^*(S)$ for $\beta \in (\beta_c, \infty)$ arises from a trace on $C^*_\omega(S_c)$. In other words, we have obtained a surjective parametrisation. We now show that this mapping is also injective.

**Proposition 8.10.** Let $S$ be an admissible right LCM semigroup. For every $\beta \in (\beta_c, \infty)$ and every trace $\tau$ on $C^*_\omega(S_c)$, let $\psi_{\beta, \tau}$ be the KMS$_{\beta}$-state obtained in Proposition 8.8. Then $(\psi_{\beta, \tau})_{\text{trac}(N(S))} \circ \varphi = \tau$.

**Proof.** Recall from Lemma 7.2 that $\varphi(y)d_n = d_n\varphi(y)$ for all $y \in C^*_\omega(S_c)$ and $n \in N(S)$. Using Lemma 7.5, Proposition 8.8 and Lemma 8.7, we have

$$(\psi_{\beta, \tau})_{\text{trac}(N(S))} \circ \varphi(y) = \zeta_S(\beta) \psi_{\beta, \tau}(Q_{\text{trac}(N(S))} \pi_{\psi_{\beta, \tau}}(\varphi(y))Q_{\text{trac}(N(S)))}
= \zeta_S(\beta) \psi_{\beta, \tau}(\pi_{\psi_{\beta, \tau}}(\varphi(y)) \prod_{n \in \text{trac}(N(S))} \pi_{\psi_{\beta, \tau}}(d_n))
= \zeta_S(\beta) \lim_{|I_k| < \infty} \psi_{\beta, \tau}(\varphi(y) \prod_{n \in I_k} d_n)
= \lim_{|I_k| < \infty} \sum_{m \in N(S)} m^{-\beta} \langle \pi_\tau(\varphi(y) \prod_{n \in I_k} d_n)(\xi_\ell \otimes \xi_\tau), \xi_\ell \otimes \xi_\tau \rangle.

Noting that $T_1 = \{1\}$ and $\pi_\tau(\prod_{n \in I_k} d_n)(\xi_\ell \otimes \xi_\tau) = \xi_\ell \otimes \xi_\tau$ as $1 \notin tS$ for all $t \in T$, where $n \in I_k \subset \text{trac}(N(S))$, the term at $m = 1$ is given by

$$\langle \pi_\tau(\varphi(y) \prod_{n \in I_k} d_n)(\xi_\ell \otimes \xi_\tau), \xi_\ell \otimes \xi_\tau \rangle = \langle \pi_\tau(\varphi(y))(\xi_\ell \otimes \xi_\tau), \xi_\ell \otimes \xi_\tau \rangle = \tau(y).$$

For $m \in N(S)$ with $m > 1$, there is $k$ so that there is $n \in I_k$ with $m \in nN(S)$ for all $\ell \geq k$. We claim that $\pi_\tau(d_n)(\xi_\ell \otimes \xi_\tau) = 0$ for all $t \in T$. To prove the claim, we recall that $d_n = 1 - \sum_{f \in T} e_{fs}$ and that $\pi_\tau(e_{fs})(\xi_\ell \otimes \xi_\tau) = \chi_{(fs)}(t) \xi_\ell \otimes \xi_\tau$. Let $n' \in N(S)$ such that $m = mn'$. Let $T_{n'}$ be any transversal for $N^{-1}(n') \sim$ contained in $S_{\alpha}$ for Corollary 4.7. We either have $f f' \perp \ell$ or $f f' \in T$ for every pair $(f, f') \in T \times T_{n'}$. According to Corollary 4.7, there is exactly one such pair $(f, f')$ satisfying $f f' \in T^*$. This forces $\pi_\tau(d_n)(\xi_\ell \otimes \xi_\tau) = 0$ and thus $\pi_\tau(\varphi(y) \prod_{n \in I_k} d_n)(\xi_\ell \otimes \xi_\tau), \xi_\ell \otimes \xi_\tau) = 0$ for all $\ell \geq k$. Thus we get $(\psi_{\beta, \tau})_{\text{trac}(N(S))} \circ \varphi(y) = \tau(y)$.

9. **Uniqueness within the critical interval**

This section deals with the proof of the uniqueness assertions in Theorem 4.3 (2). For the purpose of this section, we assume throughout that $S$ is an admissible right LCM semigroup, and write $\tau$ for the canonical trace $\tau(w, w^*) = \delta_{s, t}$ on $C^*_\omega(S_c)$. We begin with the case that $\alpha$ is an almost free action, where uniqueness will be an application of the reconstruction formula in Lemma 7.5 (iv):

**Proposition 9.1.** Let $S$ be an admissible right LCM semigroup, and $\beta \in [1, \beta_c]$. If $\alpha : S_c \sim S/\sim$ is almost free, then the only KMS$_{\beta}$-state on $C^*_\omega(S)$ is $\psi_{\beta} := \psi_{\beta, \tau}$ determined by $\psi_{\beta} \circ \varphi = \tau$.
Proof. By Proposition 8.8, \( \psi_\beta \) is a KMS\( \beta \)-state. Let \( \phi \) be an arbitrary KMS\( \beta \)-state. Due to Proposition 6.4, it suffices to show that \( \psi_\beta \circ \varphi = \phi \circ \varphi \) on \( C^*(S_c) \). Since \( \phi \circ \varphi \) is a trace, we have \( \psi_\beta \circ \varphi(w_a w_a^*) = 1 \) for all \( a \in S_c \). Let \( a, b \in S_c \) with \( a \neq b \). Lemma 7.5(iv) gives

\[
\phi(v_a v_b^*) = \zeta_I(\beta)^{-1} \sum_{t \in F_I} \phi_I(v_t^* v_a v_b v_t)
\]

for every finite subset \( I \) of \( \text{Irr}(N(S)) \). Note that \( |\phi_I(v_t^* v_a v_b v_t)| \leq 1 \). We claim that a summand can only be nonzero if \( \alpha a \beta^-1(t) = t \). Indeed, \( v_t^* v_a = v_{a_1} v_{a_2} v_{a_1}^{-1}(t) \) for some \( a_1, a_2 \in S_c \), and similarly for \( v_t^* v_b \). Thus a summand is nonzero if \( v_{a_1}^{-1}(t) v_{a_2}^{-1}(t) \neq 0 \), which is equivalent to \( \alpha a_1^1(t) = \alpha b_1^1(t) \), that is \( t \in \text{Fix}(\alpha a_1^{-1}) \).

As \( \alpha \) is almost free, the set \( \text{Fix}(\alpha a_1^{-1}) \subset T \) is finite, so that

\[
\text{max} \left( \sum_{t \in F_I} \phi_I(v_t^* v_a v_b v_t) \right) \leq |\text{Fix}(\alpha a_1^{-1})| \]

while \( \beta \in [1, \beta_c] \) forces \( \zeta_I(\beta) \rightarrow \infty \) as \( I \not
\in \text{Irr}(N(S)) \). Thus we conclude that \( \phi(v_a v_b^*) = 0 \) for all \( a, b \in S_c \) with \( a \neq b \). In particular, there is only one such state \( \psi_\beta \). □

The case where \( \beta_c = 1, \alpha \) is faithful, and \( S \) has finite propagation requires more work. The underlying strategy is based on [LRRW14, Section 7]. Given \( n \in N(S) \), let \( T_n \) be a transversal for \( N^{-1}(n)/\sim \) contained in \( S_c^1 \). Note that we may take \( T_n \) of the form \( T \cap N^{-1}(n) \), where \( T \) is a transversal for \( S/\sim \) contained in \( S_c^1 \). For \( a, b \in S_c \), let

\[
T_n^{a,b} := \{ f \in T_n \mid a f = b f \} \quad \text{and} \quad \kappa_{a,b,n} := |T_n^{a,b}|/n.
\]

Note that by Corollary 3.7, any other choice of transversal for \( N^{-1}(n)/\sim \) contained in \( S_c^1 \) will give the same \( \kappa_{a,b,n} \). If moreover \( m \in N(S) \), then since \( i(T_n T_m) = T_{mn} \) and since \( f \sim f' \) for core irreducible elements \( f, f' \) implies \( f \in f'S^* \), see Lemma 3.2, we have \( T_{mn}^{a,b} \supset i(T_n^{a,b} T_m) \) so that \( \kappa_{a,b,n} \) is increasing for the natural partial order \( n' \geq n \Leftrightarrow n' \in n N(S) \). As \( N(S) \) is directed, we get a limit \( \kappa_{a,b} = \lim_{n \in N(S)} \kappa_{a,b,n} \in [0, 1] \) for all \( a, b \in S_c \).

Proposition 9.2. Let \( S \) be an admissible right LCM semigroup and \( \beta_c = 1 \). Then there exists a KMS\( 1 \)-state \( \psi_1 \) characterized by \( \psi_1 \circ \varphi = \rho \), where \( \rho(w_a w_b^*) = \kappa_{a,b} \) for \( a, b \in S_c \). In particular, \( \rho \) is a trace on \( C^*(S_c) \). If \( \alpha \) is faithful and \( S \) has finite propagation, then \( \psi_1 \) is the only KMS\( 1 \)-state on \( C^*(S) \).

The idea of the proof is as follows: Take \( (\beta_k)_{k \geq 1} \) with \( \beta_k > 1 \) and \( \beta_k \searrow 1 \). Then a subsequence of the sequence of KMS\( \beta_k \)-states \( (\psi_{a,\tau})_{k \geq 1} \) obtained from Proposition 8.8 weak* converges to a state \( \psi_1 \) on \( C^*(S) \). For simplicity, we relabel and then assume that the original sequence converges. It is well-known that \( \psi_1 \) is a KMS\( 1 \)-state in this case, see for example [BR97, Proposition 5.3.23]. We then use Proposition 6.6 to deduce that every KMS\( 1 \)-state \( \phi \) satisfies \( \phi \circ \varphi = \rho \).

We start with two preparatory lemmas that will streamline the proof of Proposition 9.2

Lemma 9.3. Suppose \( S \) is an admissible right LCM semigroup, \( \beta_c = 1 \), and \( \beta_k \searrow 1 \). Then \( \zeta_S(\beta_k)^{-1} \sum_{n \in N(S)} n^{-\beta_k+1} \kappa_{a,b,n} \) converges to \( \kappa_{a,b} \) as \( k \rightarrow \infty \) for all \( a, b \in S_c \).
Proof. Let \( a, b \in S_c \) and \( \varepsilon > 0 \). As \( (\kappa_{a,b,n})_{n \in \text{N}(S)} \) is increasing and converges to \( \kappa_{a,b} \), and \( \text{N}(S) \cong \bigoplus_{n \in \text{Irr}(\text{N}(S))} \mathbb{N} \), there are \( C \in \mathbb{N}, n_1, \ldots, n_C \in \text{Irr}(\text{N}(S)) \), \( m_1, \ldots, m_C \geq 1 \) such that \( n \in n_i \text{N}(S) \) implies \( 0 \leq \kappa_{a,b} - \kappa_{a,b,n} < \varepsilon/2 \), where \( n := \prod_{1 \leq i \leq C} n_i^{m_i} \). Next, note that

a) \( \text{N}(S) \setminus n_i \text{N}(S) \subset \bigcup_{1 \leq i \leq C} \bigcup_{0 \leq m \leq m_i - 1} n_i^m \{ n \in \text{N}(S) \mid n \not\in n_i \text{N}(S) \} \), and

b) \( \zeta_S(\beta_k) = \prod_{n \in \text{Irr}(\text{N}(S))} (1 - n^{-\beta_k+1})^{-1} \).

In particular, bi) implies that

\[
\zeta_S(\beta_k)^{-1} \sum_{n \in \text{N}(S)} n^{-\beta_k+1} = (1 - n_i^{-\beta_k+1})^{-1} \zeta_{\text{Irr}(\text{N}(S)) \setminus \{n_i\}}(\beta_k)^{-1} \sum_{n \in \{\text{Irr}(\text{N}(S)) \setminus \{n_i\}\}} n^{-\beta_k+1} \\
= (1 - n_i^{-\beta_k+1})^{-1} k \to \infty 0
\]

for each \( 1 \leq i \leq C \) because \( \beta_k \downarrow 1 \). Now choose \( \ell \) large enough so that

c) \( \zeta_S(\beta_k)^{-1} \sum_{n \in \text{N}(S) \setminus \text{N}(S)} n^{-\beta_k+1} \leq \varepsilon/2 C m_i \) for all \( 1 \leq i \leq C \) and \( k \geq \ell \).

By definition, \( \zeta_S(\beta_k) = \sum_{n \in \text{N}(S)} n^{-\beta_k+1} \), so we get

\[
\left| \kappa_{a,b} - \zeta_S(\beta_k)^{-1} \sum_{n \in \text{N}(S)} n^{-\beta_k+1} \kappa_{a,b,n} \right| \\
\leq \zeta_S(\beta_k)^{-1} \sum_{n \in \text{N}(S)} n^{-\beta_k+1} (\kappa_{a,b} - \kappa_{a,b,n}) \\
\leq \zeta_S(\beta_k)^{-1} \sum_{n \in \text{N}(S)} n^{-\beta_k+1} \left(\kappa_{a,b} - \kappa_{a,b,n} \right) \\
+ \sum_{i=1}^{C} \sum_{m=0}^{m_i-1} \zeta_S(\beta_k)^{-1} \sum_{n \in \text{N}(S) \setminus \{n_i\}} n^{-\beta_k+1} n_i^{m} (\kappa_{a,b} - \kappa_{a,b,n_i^m}) \\
\leq \varepsilon/2 + \sum_{i=1}^{C} \sum_{m=0}^{m_i-1} \varepsilon/2 C m_i = \varepsilon
\]

for all \( k \geq \ell \). \( \square \)

Remark 9.4. In analogy to \( T_n^{a,b} = \{ f \in T_n \mid a f = b f \} = \{ f \in T_n \mid i(a f) = i(b f) \text{ and } c(a f) = c(b f) \} \), let \( G_n^{a,b} := \{ f \in T_n \mid i(a f) = i(b f) \} \) for \( n \in \text{N}(S) \) and \( a, b \in S_c \). Note that \( i(a f) = i(b f) \) is equivalent to \( \alpha_a([f]) = \alpha_b([f]) \). By Corollary 3.7 the number \( |G_n^{a,b}| \) does not depend on \( T_n \). Also, for \( m, n \in \text{N}(S) \), we have \( T_{mn} = i(T_m T_n) \) and \( a f f' = i(a f) i(c(a f) f') c(c(a f) f') \). By using Corollary 3.7 this allows us to deduce

(9.1) \( G_m \setminus T_m^{a,b} = \{ i(f f') \mid f \in G_m^{a,b} \setminus T_m^{a,b}, f' \in G_n^{c(a f), c(b f)} \setminus T_n^{c(a f), c(b f)} \} \).

Lemma 9.5. Let \( S \) be an admissible right LCM semigroup. If \( \alpha \) is faithful and \( S \) has finite propagation, then \( (G_n^{a,b} \setminus T_n^{a,b})/n \) converges to 0 for all \( a, b \in S_c \).

Proof. Let \( a, b \in S_c \). If \( a = b \), then \( G_n^{a,b} = T_n^{a,b} = T_n \) for all \( n \in \text{N}(S) \), so there is nothing to show. Thus, we may suppose \( a \neq b \). Since \( S \) has finite propagation, there exists a transversal \( \mathcal{T} \) for \( S/\sim \) that witnesses finite propagation for \( (a,b) \), i.e. the sets \( C_a = \{ c(a t) \mid t \in \mathcal{T} \} \) and \( C_b \) are finite. In particular, the set \( C_{a,b} := \{(d,c) \in C_a \times C_b \mid \)
We know that $\psi$ is a KMS$_1$-state, so we start by showing that $\psi \circ \varphi(w_n w_n^*) = \rho(w_n w_n^*) = \kappa_{a,b}$ for $a, b \in S_c$. Note that $\psi_{\beta_k,\tau}(v_n^* v_a) = \psi_{\beta_k,\tau}(v_n^* v_a)$ since $\psi_{\beta_k,\tau} \circ \varphi$ is tracial. The reconstruction formula from Lemma 7.5 (iv) and Proposition 8.10 imply that
\[
\psi_{\beta_k,\tau}(v_n^* v_a) = \zeta_S(\beta_k)^{-1} \sum_{n \in N(S)} n^{-\beta_k} \sum_{f \in T_n} \delta_{i(af),i(bf)}(\psi_{\beta_k,\tau}(v_n^* v_a) \cdot \text{Irr}(N(S))(v_{\varphi(f)}^* v_{\varphi(f)}))
\]
= \zeta_S(\beta_k)^{-1} \sum_{n \in N(S)} n^{-\beta_k} \sum_{f \in T_n} \delta_{i(af),i(bf)}(w_{\varphi(f)}^* w_{\varphi(f)})(v_{\varphi(f)}^* v_{\varphi(f)}).
\]

Since $n^{-\beta_k} \lfloor T_n^a \rfloor = n^{-\beta_k + 1} \kappa_{a,b,n}$, Lemma 9.3 and $\psi_{\beta_k,\tau} \to \psi_1$ jointly imply $\psi_1 \circ \varphi(w_n w_n^*) = \kappa_{a,b} = \rho(w_n w_n^*)$. According to Proposition 6.4, we conclude that $\rho$ is a trace on $C^*(S_c)$. Now suppose $\phi$ is any KMS$_1$-state on $C^*(S)$. Again appealing to Proposition 6.4, it suffices to show that $\phi \circ \varphi = \rho$ on $C^*(S_c)$. Since $\phi$ is a KMS$_1$-state, it factors through $Q_p(S)$ (or $Q(S)$), see Proposition 6.6. As $\pi_p \sum_{f \in T_n} v_f^* v_f = 1$ for all $n \in N(S)$, we get
\[
\phi(v_n v_n^*) = \sum_{f \in T_n} \phi(v_{\varphi(f)}^* v_{\varphi(f)}) = \sum_{f \in T_n} n^{-1} \delta_{i(af),i(bf)} \phi(v_{\varphi(f)}^* v_{\varphi(f)}) = \sum_{f \in T_n} \phi(v_{\varphi(f)}^* v_{\varphi(f)}) = \kappa_{a,b,n} + n^{-1} \sum_{f \in G_n \cap T_n^a} \phi(v_{\varphi(f)}^* v_{\varphi(f)}^*).
\]

Since $|\phi(v_{\varphi(f)}^* v_{\varphi(f)})| \leq 1$, Lemma 9.5 shows that $\phi(v_n v_n^*) = \lim_{n \in N(S)} \kappa_{a,b,n} = \rho(w_n w_n^*)$.

While Proposition 9.1 and Proposition 9.2 provide sufficient criteria for uniqueness of KMS$_\beta$-states for $\beta$ in the critical interval, we will now prove that faithfulness of $\alpha$
is necessary, at least under the assumption that $S_c$ is abelian. If $S_c$ is abelian, it is 
also right cancellative, and hence a right Ore semigroup. It therefore embeds into 
a group $G_c = S_cS_c^{-1}$, and the action $\alpha: S_c \acts S/\sim$ induces an action $\alpha': G_c \acts S/\sim$ by 
$\alpha'_{ab^{-1}} = a\alpha_b^{-1}$ for all $a, b \in S_c$. We let $G_c^\alpha := \{ab^{-1} | \alpha_a = \alpha_b\}$ be the subgroup of $G_c$ 
for which $\alpha'$ is trivial, and we write $u_g$ for the generating unitaries in $C^*(G_c^\alpha)$.

**Proposition 9.6.** Let $S$ be an admissible right LCM semigroup with $S_c$ abelian. Suppose 
that there exist $\beta \geq 1$ and distinct states $\phi, \phi'$ on $C^*(G_c^\alpha)$ such that there exist $ab^{-1} \in G_c^\alpha$ 
and a sequence $(I_k)_{k \in \mathbb{N}} \subset \text{Irr}(N(S))$ with $I_k$ finite and $I_k \not\supset \text{Irr}(N(S))$ satisfying 

\[
\lim_{k \to \infty} \zeta_{I_k}(\beta)^{-1} \sum_{t \in I_{k}} N_t^{-\beta} \phi(u_{c(at)c(bt)^{-1}}) \neq \lim_{k \to \infty} \zeta_{I_k}(\beta)^{-1} \sum_{t \in I_{k}} N_t^{-\beta} \phi'(u_{c(at)c(bt)^{-1}}).\]

Then $C^*(S)$ has at least two distinct KMS-\beta-states. If, for any given pair of distinct states 
on $C^*(G_c^\alpha)$, there exists such a sequence $(I_k)_{k \in \mathbb{N}}$, then there exists an affine embedding 
of the state space of $C^*(G_c^\alpha)$ into the KMS-\beta-states on $C^*(S)$. In particular, $C^*(S)$ does 
not have a unique KMS-\beta-state unless $\alpha$ is faithful under these assumptions.

**Proof.** Since $S_c$ is abelian, so are the groups $G_c$ and $G_c^\alpha$. In particular, $G_c$ and $G_c^\alpha$ are 
amenable, so that $C^*(G_c^\alpha) \subset C^*(G_c)$ as a unital subalgebra, and states are traces on 
$C^*(G_c)$. So if $\tilde{\phi}$ is a state on $C^*(G_c^\alpha)$, it extends to a state on $C^*(G_c)$, which then 
corresponds to a trace $\tau$ on $C^*(S_c)$. Note that we have $\tau(w_{a\alpha}^{-1}u_a) = \phi(u_{ab^{-1}})$ for all 
$ab^{-1} \in G_c^\alpha$ by construction.

Now let $\beta \geq 1$, and $\phi, \phi'$ distinct states on $C^*(G_c^\alpha)$ such that there exists a sequence 
$(I_k)_{k \in \mathbb{N}} \subset \text{Irr}(N(S))$ with $I_k$ finite, $I_k \not\supset \text{Irr}(N(S))$, and (9.2). Denote by $\tilde{\tau}$ and $\tau'$ 
the corresponding traces on $C^*(S_c)$. Following the proof of Proposition 8.8 we obtain 
KMS-\beta-states $\psi_{\beta, \phi} := \psi_{\beta, \phi'}$ and $\psi_{\beta, \phi'} := \psi_{\beta, \tau'}$ by passing to subsequences 
$(I_{i(k)})_{k \in \mathbb{N}}$ and $(I_{j(k)})_{k \in \mathbb{N}}$ of $(I_k)_{k \in \mathbb{N}}$, if necessary. The state $\psi_{\beta, \phi}$ is given by 

\[
\psi_{\beta, \phi}(v_a v_b^*) = \lim_{k \to \infty} \zeta_{I_{i(k)}}(\beta)^{-1} \sum_{t \in I_{i(k)}} N_t^{-\beta} \langle V_a V_b^* (\xi_t \otimes \xi_{\tilde{\tau}}), \xi_t \otimes \xi_{\tilde{\tau}} \rangle 
\]

\[
= \lim_{k \to \infty} \zeta_{I_{i(k)}}(\beta)^{-1} \sum_{t \in I_{i(k)}} \chi_{\text{Fix}(a, a_{\beta^{-1}})}(t) N_t^{-\beta} \tilde{\tau}(w_{c(at)c(bt)^{-1}}) \langle w_a w_b^* \rangle 
\]

for $a, b \in S_c$. But if $ab^{-1} \in G_c^\alpha$, then this reduces to 

\[
\psi_{\beta, \phi}(v_a v_b^*) = \lim_{k \to \infty} \zeta_{I_{i(k)}}(\beta)^{-1} \sum_{t \in I_{i(k)}} N_t^{-\beta} \phi(u_{c(at)c(bt)^{-1}}).
\]

The analogous computation applies to $\psi_{\beta, \phi'}$. Since (9.2) implicitly assumes existence of 
the displayed limits, passing to subsequences leaves the values invariant, so that 
$\psi_{\beta, \phi}(v_a v_b^*) \neq \psi_{\beta, \phi'}(v_a v_b^*)$ for some $ab^{-1} \in G_c^\alpha$. This shows that $\phi$ and $\phi'$ yield distinct 
KMS-\beta-states $\psi_{\beta, \phi}$ and $\psi_{\beta, \phi'}$. The map $\phi \mapsto \psi_{\beta, \phi}$ is affine, continuous, and injective 
by the assumption. Hence it is an embedding. Finally, the state space of $C^*(G_c^\alpha)$ is a 
singleton if and only if $G_c^\alpha$ is trivial, which corresponds to faithfulness of $\alpha$. \qed

10. Open questions

The present work naturally leads to some questions for further research, and we would 
like to take the opportunity to indicate a few directions we find intriguing.
Questions 10.1. Can the prerequisites for Theorem 4.3 be weakened? This leads to:
(a) Are there right LCM semigroups that are not core factorable?
(b) Are there right LCM semigroups $S$ for which $S_{ci} \subset S$ is not $\cap$-closed?
(c) How does a violation of (A4) affect the KMS-state structure?

Question 10.2. Suppose $S$ is an admissible right LCM semigroup. Faithfulness of $\alpha: S_c \curvearrowright S/\sim$ is assumed in both parts of Theorem 4.3 (2), see also Remark 9.11 and it is necessary under additional assumptions, see Proposition 9.6. The extra assumptions in both directions might be artifacts of the proofs, so it is natural to ask: Is faithfulness of $\alpha$ necessary or sufficient for uniqueness of KMS$_\beta$-states for $\beta$ in the critical interval?

If the answer to Question 10.2 is affirmative for both parts, then, for right cancellative $S_c$, the group $G_\alpha^\ast$ would correspond to the periodicity group Per$\Lambda$ for strongly connected finite $k$-graphs $\Lambda$, see [aHLRS15].

Using self-similar actions $(G, X)$, see Proposition 5.8, one can give examples $S = X^* \bowtie G$ of admissible right LCM semigroups with $\beta_c = 1$, $\alpha$ faithful, but not almost free, and without finite propagation. The following problem seems to be more difficult:

Question 10.3. Is there an example of a self-similar action $(G, X)$ that is not finite state such that $S = X^* \bowtie G$ has finite propagation?

In [aHLRS15, Section 12], the authors showed that their findings can be obtained via [Nes13, Theorem 1.3] since their object of study admits a canonical groupoid picture.

Questions 10.4. As indicated in [Sta17], the boundary quotient diagram ought to admit a reasonable description in terms of $C^\ast$-algebras associated to groupoids. Assuming that this gap can be bridged, we arrive at the following questions:
(a) Does [Nes13, Theorem 1.3] apply to the groupoids related to the boundary quotient diagram for (admissible) right LCM semigroups?
(b) Which groupoid properties do (A1) (A4) correspond to?
(c) What is the meaning of $\alpha: S_c \curvearrowright S/\sim$ for the associated groupoids? Which groupoid properties do faithfulness and almost freeness of $\alpha$ correspond to?
(d) Which groupoid property does finite propagation correspond to?

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