EQUILIBRIUM STATES ON RIGHT LCM SEMIGROUP C^*-ALGEBRAS

ZAHRA AFSAR, NATHAN BROWNLOWE, NADIA S. LARSEN, AND NICOLAI STAMMEIER

Abstract. We determine the structure of equilibrium states for a natural dynamics on the boundary quotient diagram of C^*-algebras for a large class of right LCM semigroups. The approach is based on abstract properties of the semigroup and covers the previous case studies on $\mathbb{N} \rtimes \mathbb{N}^\times$, dilation matrices, self-similar actions, and Baumslag–Solitar monoids. At the same time, it provides new results for right LCM semigroups associated to algebraic dynamical systems.

1. Introduction

Equilibrium states have been studied in operator algebras starting with quantum systems modelling ensembles of particles, and have been the means of describing properties of physical models with C^*-algebraic tools, cf. [BR97]. Specifically, given a quantum statistical mechanical system, which is a pair formed of a C^*-algebra A that encodes the observables and a one-parameter group of automorphisms σ of A interpreted as a time evolution, one seeks to express equilibrium of the system via states with specific properties. A KMS_β-state for (A, σ) is a state on A that satisfies the KMS_β-condition (for Kubo-Martin-Schwinger) at a real parameter β bearing the significance of an inverse temperature. Gradually, it has become apparent that the study of KMS-states for systems that do not necessarily have physical origins brings valuable insight into the structure of the underlying C^*-algebra, and uncovers new directions of interplay between the theory of C^*-algebras and other fields of mathematics. A rich supply of examples is by now present in the literature, see [BC95, LR10, LRRW14, CDL13, Nes13], to mention only some.

This article is concerned with the study of KMS-states for systems whose underlying C^*-algebras are Toeplitz-type algebras modelling a large class of semigroups. While our initial motivation was to classify KMS-states for a specific class of examples, as had been done previously, we felt that by building upon the deep insight developed in the case-studies already present in the literature, the time was ripe for proposing a general framework that would encompass our motivating example and cover all these case-studies. We believe it is a strength of our approach that we can identify simply phrased conditions at the level of the semigroup which govern the structure of KMS-states, including uniqueness of the KMS_β-state for β in the critical interval. In all the examples we treat, these conditions admit natural interpretations. Moreover, they are often subject to feasible verification.

To be more explicit, the interest in studying KMS-states on Toeplitz-type C^*-algebras gained new momentum with the work of Laca and Raeburn [LR10] on the Toeplitz

Date: 4 February 2017.

algebra of the $ax+b$-semigroup over the natural numbers $\mathbb{N} \rtimes \mathbb{N}^\times$. Further results on the KMS-state structure of Toeplitz-type C^*-algebras include the work on Exel crossed products associated to dilation matrices [LRR11], C^*-algebras associated to self-similar actions [LRRW14], and C^*-algebras associated to Baumslag–Solitar monoids [CaHR16]. In all these cases the relevant Toeplitz-type C^*-algebra can be viewed as a semigroup C^*-algebra in the sense of [Li12]. Further, the semigroups in question are right LCM semigroups, see [BLS17] for dilation matrices, and [BRRW14] for all the other cases. In [BLS17, BLS], the last three named authors have studied C^*-algebras of right LCM semigroups associated to algebraic dynamical systems. These C^*-algebras often admit a natural dynamics that is built from the underlying algebraic dynamical system. The original motivation for our work was to classify KMS-states in this context.

Although there is a common thread in the methods and techniques used to prove the KMS-classification results in [LR10, LRR11, LRRW14, CaHR16], one cannot speak of a single proof that runs with adaptations. Indeed, in each of these papers a careful and rather involved analysis is carried out using concrete properties of the respective setup. One main achievement of this paper is a general theory of KMS-classification which both unifies the classification results from [LR10, LRR11, LRRW14, CaHR16], and also considerably enlarges the class of semigroup C^*-algebras for which KMS-classification is new and interesting, see Theorem 4.3 and Section 5.

We expect that our work may build a bridge to the analysis of KMS-states via groupoid models and the result of Neshveyev [Nes13, Theorem 1.3]. At this point we would also like to mention the recent analysis of equilibrium states on C^*-algebras associated to self-similar actions of groupoids on graphs that generalises [LRRW14], see [LRRW]. It would be interesting to explore connections to this line of development.

One of the keys to establishing our general theory is the insight that it pays off to work with the boundary quotient diagram for right LCM semigroups proposed by the fourth-named author in [Sta17]. The motivating example for this diagram was first considered in [BaHLR12], where it was shown to give extra insight into the structure of KMS-states on $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$. Especially the *core subsemigroup* S_c, first introduced in [Star15], and motivated by the quasi-lattice ordered situation in [CL07], and the *core irreducible elements* S_{ci} introduced in [Sta17] turn out to be central for our work. Recall that S_c consists of all the elements whose principal right ideal intersects all other principal right ideals and S_{ci} of all $s \in S \setminus S_c$ such that every factorisation $s = ta$ with $a \in S_c$ forces a to be invertible in S. Via so-called (accurate) foundation sets, S_c and S_{ci} give rise to quotients $\mathcal{Q}_c(S)$ and $\mathcal{Q}_p(S)$ of $C^*(S)$ of S, respectively. Together with $C^*(S)$ and the boundary quotient $\mathcal{Q}(S)$ from [BRRW14], they form a commutative diagram:

$$\begin{array}{ccc} C^*(S) & \xrightarrow{\ \pi_p\ } & \mathcal{Q}_p(S) \\ \big\downarrow{\scriptstyle \pi_c} & & \big\downarrow{\scriptstyle \pi_c'} \\ \mathcal{Q}_c(S) & \xrightarrow{\ \pi_p'\ } & \mathcal{Q}(S) \end{array} \tag{1.1}$$

We refer to [Sta17] for examples and further details, but remark that if $S = S_{ci}^1 S_c$ holds (with $S_{ci}^1 = S_{ci} \cup \{1\}$), a property we refer to as S being *core factorable*, then the quotient maps π_p' and π_c' are induced by the corresponding relations for π_p and π_c, respectively, see [Sta17, Proposition 2.10]. Another relevant condition in this context is the property that any right LCM in S of a pair in S_{ci} belongs to S_{ci} again. If this holds, then we say that $S_{ci} \subset S$ is *$\cap$-closed*.

In this paper we work with what we call *admissible* right LCM semigroups, which are core factorable right LCM semgroups with $S_{ci} \subset S$ being $\cap$-closed that admit a suitable homomorphism $N\colon S \to \mathbb{N}^\times$. We refer to such a map N as a *generalised scale* to stress the analogy to the *scale* appearing in [Lac98]. The dynamics σ of $\mathbb{R}$ on $C^*(S)$ that we consider for a generalised scale N is natural in the sense that $\sigma_x(v_s) = N_s^{ix} v_s$ on the canonical generators of $C^*(S)$. In each of the examples we consider in Section 5 there is a natural choice for a generalised scale. Every generalised scale on a right LCM semigroup S gives rise to a ζ-function on $\mathbb{R}$. The *critical inverse temperature* β_c for the dynamics σ is defined to be the least bound in $\mathbb{R} \cup \{\infty\}$ so that $\zeta_S(\beta)$ is finite for all β above β_c. The uniqueness of KMS$_\beta$-states for β inside the *critical interval* ($[1, \beta_c]$ for finite β_c, and $[1, \infty)$ otherwise) has become an intriguing topic in the field.

Our main result Theorem 4.3 classifies the KMS-state structure on (1.1) for an admissible right LCM semigroup S with regards to the dynamics σ arising from a generalised scale N. There are no KMS$_\beta$-states for $\beta < 1$, and for $\beta > \beta_c$ there is an affine homeomorphism between KMS$_\beta$-states on $C^*(S)$ and normalised traces on $C^*(S_c)$. For β in the critical interval there exists at least one KMS$_\beta$-state, and two sufficient criteria for the KMS$_\beta$-state to be the unique one are established. Both require faithfulness of a natural action α of S_c on S/S_c arising from left multiplication. The first criterion demands the stronger hypothesis that α is almost free and works for arbitrary β_c. In contrast, the second criterion only works for $\beta_c = 1$, but relies on the seemingly more modest, yet elusive condition that S has what we call *finite propagation*. The ground states on $C^*(S)$ are affinely homeomorphic to the states on $C^*(S_c)$. Moreover, if $\beta_c < \infty$, then a ground state is a KMS$_\infty$-state if and only if it corresponds to a trace on $C^*(S_c)$. With regards to the boundary quotient diagram, it is proven that every KMS$_\beta$-state and every KMS$_\infty$-state factors through $\pi_c\colon C^*(S) \to \mathcal{Q}(S_c)$. In contrast, a KMS$_\beta$-state factors through $\pi_p\colon C^*(S) \to \mathcal{Q}_p(S)$ if and only if $\beta = 1$. Last but not least, there are no ground states on $\mathcal{Q}_p(S)$, and hence none on the boundary quotient $\mathcal{Q}(S)$.

As every result requiring abstract hypotheses ought to be tested for applicability to concrete examples, we make an extensive effort to provide a wide range of applications. The results on KMS-classification for self-similar actions, (subdynamics of) $\mathbb{N} \rtimes \mathbb{N}^\times$, and Baumslag-Solitar monoids from [LRRW14],[LR10] and [BaHLR12], and [CaHR16] are recovered in sections 5.3, 5.4, and 5.5, respectively; we remark that in order to make the similarities between these examples more apparent we make a different choice of time evolution leading to different critical temperatures. Our analysis is achieved by first establishing reduction results for general Zappa-Szép products $S = U \bowtie A$ of right LCM semigroups U and A, see Corollary 5.5 and Corollary 5.6. These in turn rely on an intriguing result concerning the behaviour of S_c and S_{ci} under the Zappa-Szép product construction $S = U \bowtie A$, which we expect to be of independent interest, see Theorem 5.3. As a simple application of the reduction results, we also cover the case of right-angled Artin monoids that are direct products of a free monoid by an abelian monoid. We then investigate admissibility for right LCM semigroups arising from algebraic dynamical systems, see 5.6. This yields plenty of examples for which no KMS-classification was known before, and we obtain the KMS-classification results from [LRR11] as a corollary, see Example 5.14. In addition, we give an example of an admissible right LCM semigroup with finite propagation for which the action α is faithful but not almost free by means of standard restricted wreath products of finite groups by $\mathbb{N}$, see Example 5.13.

Apart from this variety of examples, there is also a structural link to be mentioned here: Via a categorical equivalence from [Law99] based on Clifford's work [Cli53], right LCM semigroups correspond to 0-bisimple inverse monoids, which constitute a well-studied and rich class of inverse semigroups, see for instance [McA74] and the references therein. The results of this work may therefore be seen as a contribution to the study of 0-bisimple inverse monoids, and the relevance of the core structures exhibited in right LCM semigroups for the respective 0-bisimple inverse monoids are yet to be investigated.

The paper is organised as follows: A brief description of basic notions and objects related to right LCM semigroups and their C^*-algebras, as well as to KMS-state of a C^*-algebra is provided in Section 2. In Section 3, the class of admissible right LCM semigroups is introduced. Section 4 is a small section devoted to stating our main theorem and to providing an outline of the proof. Before proving our main theorem, we apply Theorem 4.3 to a wide range of examples in Section 5. Sections 6–9 constitute the essential steps needed in the proof of Theorem 4.3: In Section 6 algebraic characterisations for the existence of KMS-states are provided. These yield bounds on the kinds of KMS-states that can appear. In Section 7 a reconstruction formula in the spirit of [LR10, Lemma 10.1] is provided; and in Section 8 KMS-states are constructed, where Theorem 8.4 provides a means to construct representations for $C^*(S)$ out of states on $C^*(S_c)$, which we believe to be of independent interest. Even more so, since it only requires S to be core factorable and $S_{ci} \subset S$ to be $\cap$-closed. Section 9 addresses the existence of a unique KMS_β-state for β within the critical interval, and Section 10 is an attempt at identifying some challenges for future research.

Acknowledgements: This research was supported by the Australian Research Council. Parts of this work were carried out when N.L. visited Institut Mittag-Leffler (Sweden) for the program "Classification of operator algebras: complexity, rigidity, and dynamics", and the University of Victoria (Canada), and she thanks Marcelo Laca and the mathematics department at UVic for the hospitality extended to her. N.S. was supported by ERC through AdG 267079, by DFG through SFB 878 and by RCN through FRIPRO 240362. This work was initiated during a stay of N.S. with N.B. at the University of Wollongong (Australia), and N.S. thanks the work group for its great hospitality.

2. Background

2.1. **Right LCM semigroups.** A discrete, left cancellative semigroup S is right LCM if the intersection of principal right ideals is empty or another principal right ideal, see [Nor14, Law12, BRRW14]. If $r, s, t \in S$ with $sS \cap tS = rS$, then we call r a *right LCM* of s and t. In this work, all semigroups will admit an identity element (and so are monoids) and will be countable. We let S^* denote the subgroup of invertible elements in S. Two subsets of a right LCM semigroup S are used in [Sta17]: the *core subsemigroup*

$$S_c := \{a \in S \mid aS \cap sS \neq \emptyset \text{ for all } s \in S\},$$

which was first considered in [Star15] in the context of right LCM semigroups, and the set S_{ci} of *core irreducible elements*, where $s \in S \setminus S_c$ is core irreducible if every factorisation $s = ta$ with $a \in S_c$ forces $a \in S^*$. The latter is a subsemigroup of S, at least under moderate conditions, see Proposition 3.4. The core subsemigroup S_c contains S^*, and induces an equivalence relation on S defined by

$$s \sim t :\Leftrightarrow sa = tb \text{ for some } a, b \in S_c.$$

We call $\sim$ the *core relation* and say that s and t are *core equivalent* if $s \sim t$. The equivalence class of $s \in S$ under $\sim$ is denoted $[s]$. We write $s \perp t$ if $s, t \in S$ have disjoint right ideals. One can verify that if $s \sim s'$ for some $s, s' \in S$, then $s \perp t$ if and only if $s' \perp t$ for all $t \in S$. By a transversal $\mathcal{T}$ for $A/_\sim$ with $A \subset S$, we mean a subset of A that forms a minimal complete set of representatives for $A/_\sim$.

Recall from [BRRW14] that a finite subset F of S is called a *foundation set* if for every $s \in S$ there is $f \in F$ with $sS \cap fS \neq \emptyset$. The second and fourth author coined the refined notion of *accurate foundation set* in [BS16], where a foundation set F is accurate if $fS \cap f'S = \emptyset$ for all $f, f' \in F$ with $f \neq f'$. A right LCM semigroup S has the *accurate refinement property*, or property (AR), if for every foundation set F there is an accurate foundation set F_a such that $F_a \subseteq FS$, in the sense that for every $f_a \in F_a$ there is $f \in F$ with $f_a \in fS$.

2.2. **C^*-algebras associated to right LCM semigroups.** The *full semigroup C^*-algebra* $C^*(S)$ of a right LCM semigroup S is the C^*-algebra of S as defined by Li [Li12, Definition 2.2]. It can be shown that $C^*(S)$ is the universal C^*-algebra generated by an isometric representation v of S satisfying

$$v_s^* v_t = \begin{cases} v_{s'} v_{t'}^* & \text{if } sS \cap tS = ss'S,\ ss' = tt' \\ 0 & \text{if } s \perp t, \end{cases}$$

for all $s, t \in S$. For each $s \in S$, we write e_{sS} for the range projection $v_s v_s^*$. The *boundary quotient* C^*-algebra $\mathcal{Q}(S)$ is defined in [BRRW14] to be the quotient of $C^*(S)$ by the relation $\prod_{f \in F}(1 - e_{fS}) = 0$ for all foundation sets $F \subseteq S$. However, if S has property (AR), then in $\mathcal{Q}(S)$ this relation reduces to

$$\sum_{f \in F} e_{fS} = 1 \quad \text{for all accurate foundation sets } F \subseteq S. \tag{2.1}$$

In [Sta17], the fourth author introduced two intermediary quotients. The *core boundary quotient* $\mathcal{Q}_c(S)$ is the quotient of $C^*(S)$ by the relation $v_s v_s^* = 1$ for all $s \in S_c$. Note that this amounts to the quotient by the relation (2.1) for accurate foundation sets contained in the core S_c. The second intermediary quotient depends on the notion of a *proper foundation set*, which is a foundation set contained in the core irreducible elements S_{ci}. The *proper boundary quotient* $\mathcal{Q}_p(S)$ is the quotient of $C^*(S)$ by (2.1) for all proper accurate foundation sets. As first defined in [Sta17, Definition 2.9], the *boundary quotient diagram* is the commutative diagram from (1.1) which encompasses the C^*-algebras $C^*(S)$, $\mathcal{Q}_c(S)$, $\mathcal{Q}_p(S)$ and $\mathcal{Q}(S)$.

2.3. **KMS-states.** We finish the background section with the basics of the theory of KMS-states. A much more detailed presentation can be found in [BR97]. Suppose σ is an action of $\mathbb{R}$ by automorphisms of a C^*-algebra A. An element $a \in A$ is said to be *analytic* for σ if $t \mapsto \sigma_t(a)$ is the restriction of an analytic function $z \mapsto \sigma_z(a)$ from $\mathbb{C}$ into A. For $\beta > 0$, a state ϕ of A is called a *KMS_β-state* of A if it satisfies

$$\phi(ab) = \phi(b\sigma_{i\beta}(a)) \quad \text{for all analytic } a, b \in A. \tag{2.2}$$

We call (2.2) the *KMS_β-condition*. To prove that a given state of A is a KMS$_\beta$-state, it suffices to check (2.2) for all a, b in any σ-invariant set of analytic elements that spans a dense subspace of A.

A *KMS_∞-state* of A is a weak* limit of KMS$_{\beta_n}$-states as $\beta_n \to \infty$, see [CM06]. A state ϕ of A is a *ground state* of A if $z \mapsto \phi(a\sigma_z(b))$ is bounded on the upper-half plane for all analytic $a, b \in A$.

3. The setting

Given a right LCM semigroup S and its C^*-algebra $C^*(S)$, we wish to find a natural dynamics for which we can calculate the KMS-state structure. We then ask: what conditions on S are needed in order to define such a dynamics? The answer that first comes to mind is the existence of a nontrivial homomorphism of monoids $N\colon S \to \mathbb{N}^\times$, which allows us to set $\sigma_x(v_s) := N_s^{ix} v_s$ for $s \in S$ and $x \in \mathbb{R}$. However, it soon becomes apparent that the mere existence of a homomorphism N on S seems far from sufficient to give rise to KMS-states on $C^*(S)$, yet alone describe them all. As announced in the introduction, we shall take into account the boundary quotient diagram for $C^*(S)$. The ingredients that are at work behind the scenes to get this diagram are the core S_c and the core irreducible elements S_{ci} of S. Looking for relationships between these and the homomorphism $N\colon S \to \mathbb{N}^\times$ will provide clues as to what conditions S ought to satisfy.

Before we introduce our main concept of admissibility we fix the following notation: if P is a subset of positive integers, we let $\langle P \rangle$ be the subsemigroup of $\mathbb{N}^\times$ generated by P. An element of $\langle P \rangle$ is *irreducible* if $m = nk$ in $\langle P \rangle$ implies n or k is 1. We denote by $\mathrm{Irr}(\langle P \rangle)$ the set of irreducible elements of $\langle P \rangle$. We are now ready to present the outcome of our efforts of distilling a minimal sufficient set of conditions for the aim of defining a suitable dynamics σ:

Definition 3.1. A right LCM semigroup S is called *admissible* if it satisfies the following conditions:

- (A1) $S = S_{ci}^1 S_c$.
- (A2) Any right LCM in S of a pair of elements in S_{ci} belongs to S_{ci}.
- (A3) There is a nontrivial homomorphism of monoids $N\colon S \to \mathbb{N}^\times$ such that
  - (a) $|N^{-1}(n)/_\sim| = n$ for all $n \in N(S)$ and
  - (b) for each $n \in N(S)$, every transversal of $N^{-1}(n)/_\sim$ is an accurate foundation set for S.
- (A4) The monoid $N(S)$ is the free abelian monoid in $\mathrm{Irr}(N(S))$.

We refer to the property in (A1) as saying that S is *core factorable*, and when S satisfies (A2) we say that $S_{ci} \subset S$ is *$\cap$-closed*. We call a homomorphism N satisfying (A3) a *generalised scale*. This name is inspired by the notion of a scale on a semigroup originally defined in [Lac98, Definition 9].

To state and prove our results on the KMS structure of the C^*-algebras of admissible right LCM semigroups, we also need to introduce a number of properties of these semigroups, and to prove a number of consequences of admissibility. The rest of this section is devoted to establishing these properties and thereby also explaining the meaning of the conditions (A1)–(A3). Our first observation explains the meaning of (A2) in the presence of (A1).

Lemma 3.2. *Suppose S is a core factorable right LCM semigroup. Then for each $s \in S$ the set $[s] \cap S_{ci}^1$ is non-empty, and the following conditions are equivalent:*

- *(i) The equivalence classes for $\sim$ have minimal representatives, that is, $[s] = tS_c$ holds for every $t \in [s] \cap S_{ci}^1$ and all $s \in S$.*
- *(ii) If $s, t \in S_{ci}^1$ satisfy $s \sim t$, then $s \in tS^*$.*

Moreover, if $S_{ci} \subset S$ is $\cap$-closed, then (i) and (ii) hold.

Proof. Suppose S has (A2). The set $[s] \cap S_{ci}^1$ is non-empty for all $s \in S$ due to (A1). Let $t \in [s]$, and write $s = s'a, t = t'b$ with $s', t' \in S_{ci}^1, a, b \in S_c$ using (A1). Fix $c, d \in S_c$ with $sc = td$. Then $s' \not\perp t'$, so (A2) implies $s'S \cap t'S = rS$ for some $r \in S_{ci}^1$, say $r = s'p$ for a suitable $p \in S$. As $sc = td$, we have $s'ac = re = s'pe$ for some $e \in S$. Since $ac \in S_c$, we deduce that $p, e \in S_c$. But then $r \in S_{ci}^1$ forces $p \in S^*$ so that $r \in s'S^*$. The same argument applied to t in place of s shows $r \in t'S^*$. Thus we have shown $s' \in t'S^*$, and hence $[s] = s'S_c$. This establishes (i).

If (i) holds and $s \sim t$ for $s, t \in S_{ci}^1$, then $s \in [t] = tS_c$ forces $s \in tS^*$ as s is core irreducible. Therefore, (i) implies (ii). Conversely, suppose (ii) holds. Now assume (iii) holds true. For $s \in S, s', t \in [s]$ with $t \in S_{ci}^1$, condition (A1) gives some $t' \in S_{ci}^1, a \in S_c$ with $s' = t'a$ so that $t \sim t'$. By (ii), there is $x \in S^*$ such that $t' = tx$, and hence $s' = txa \in tS_c$, proving (i). $\square$

We proceed with a natural notion of homomorphisms between right LCM semigroups that appeared in [BLS, Theorem 3.3], though without a name.

Definition 3.3. Let $\varphi\colon S \to T$ be a monoidal homomorphism between two right LCM semigroups S and T. Then φ is called a *homomorphism of right LCM semigroups* if

$$\varphi(s_1)T \cap \varphi(s_2)T = \varphi(s_1S \cap s_2S)T \quad \text{for all } s_1, s_2 \in S. \tag{3.1}$$

In particular, (3.1) implies that $\varphi(r)$ is a right LCM for $\varphi(s_1)$ and $\varphi(s_2)$ whenever $r \in S$ is a right LCM for s_1 and s_2 in S. This notion allows us to recast the condition of $S_{ci} \subset S$ being $\cap$-closed using the semigroup $S'_{ci} := S_{ci} \cup S^*$:

Proposition 3.4. *Let S be a right LCM semigroup. Then the following hold:*

- *(i) The natural embedding of S_c into S is a homomorphism of right LCM semigroups.*
- *(ii) Suppose S is core factorable. Then $S_{ci} \subset S$ is $\cap$-closed if and only if S'_{ci} is a right LCM semigroup and the natural embedding of S'_{ci} into S is a homomorphism of right LCM semigroups.*

Proof. In both cases, the right hand side of (3.1) is contained in the left hand side. By the definition of S_c, any $c \in S$ satisfying $aS \cap bS = cS$ for some $a, b \in S_c$ belongs to S_c, and we have $aS_c \cap bS_c \neq \emptyset$ for all $a, b \in S_c$. This proves (i).

Suppose (A1) holds. Assuming (A2), we must prove that S'_{ci} is a semigroup. The non-trivial case is to show that $st \in S'_{ci}$ when s, t are core irreducible. So let $s, t \in S_{ci}$

and $st = ra$ for some $r \in S \setminus S_c, a \in S_c$. As S is core factorable, we can assume $r \in S_{ci}$. Then $sS \cap rS = r'S$ for some $r' \in S$. Since s, r are in S_{ci}, so is r' by our assumption. Since $ra \in r'S \subset rS$, it follows that $r' \sim r$. By Lemma 3.2, (A2) yields $r \in r'S^* \subset sS$. Therefore $r = ss'$ for some $s' \in S$, and left cancellation leads to $t = s'a$. By core irreducibility of t, we conclude that $a \in S^*$. Thus $st \in S_{ci}$.

For the other claim in the forward direction of (ii), it suffices to look at intersections $sS'_{ci} \cap tS'_{ci}$ versus $sS \cap tS$ with $s, t \in S_{ci}$ because when $s \in S^*$ we are left with tS'_{ci} and tS, respectively. Since S'_{ci} is a subsemigroup of S, we have $sS'_{ci} \cap tS'_{ci} \subset sS \cap tS$. In particular, we can assume that $sS \cap tS = rS$ for some $r \in S$. By (A2), $r \in S_{ci}$. Then any $s', t' \in S$ with $ss' = r = tt'$ belong to S'_{ci}. Therefore $sS'_{ci} \cap tS'_{ci} = rS'_{ci}$, showing that the intersection of principal right ideals in S'_{ci} may be computed in S. Hence, S'_{ci} is right LCM because S is, and the embedding is a homomorphism of right LCM semigroups.

Conversely, let $s, t \in S_{ci}$ such that $sS \cap tS = rS$ for some $r \in S$. Then $r \in S \setminus S_c$. But the right LCM property for the embedding of S'_{ci} into S gives $r \in S'_{ci}$, so $r \in (S \setminus S_c) \cap S'_{ci} = S_{ci}$, which is (A2). □

Lemma 3.5. *Suppose S is a core factorable right LCM semigroup, and $a \in S_c$, $s \in S_{ci}$ satisfy $aS \cap sS = atS$ and $at = sb$ for some $b, t \in S$. Then $b \in S_c$. Further, if $S_{ci} \subset S$ is $\cap$-closed, then t belongs to S_{ci}.*

Proof. To see that $b \in S_c$, let $s' \in S$. Since $a \in S_c$, we have $aS \cap ss'S \neq \emptyset$. Hence left cancellation gives

$$sbS \cap ss'S = (aS \cap sS) \cap ss'S \neq \emptyset \implies bS \cap s'S \neq \emptyset \implies b \in S_c.$$

Since $s \in S_{ci}$, we must have $t \in S \setminus S_c$. We use (A1) to write $t = rc$ for $r \in S_{ci}$ and $c \in S_c$. Then $ar \sim s$, and due to (A2) we know from Lemma 3.2 that $ar \in sS_c \subseteq sS$. Hence $ar \in aS \cap sS = atS$, and so $arS \subseteq atS$. Since $atS = arcS \subseteq arS$, we have $arS = atS$. So $rS = tS$. This implies that $t = rx$ for some $x \in S^*$, and thus $t \in S_{ci}$ as $r \in S_{ci}$. □

We now prove some fundamental results about right LCM semigroups that admit a generalised scale.

Proposition 3.6. *Let N be a generalised scale on a right LCM semigroup S.*

(i) We have $\ker N = S_c$, and S_c is a proper subsemigroup of S.
(ii) If $s, t \in S$ satisfy $N_s = N_t$, then either $s \sim t$ or $s \perp t$.
(iii) Let F be a foundation set for S with $F \subset N^{-1}(n)$ for some $n \in N(S)$. If $|F| = n$ or F is accurate, then F is a transversal for $N^{-1}(n)/_\sim$.
(iv) The map N is a homomorphism of right LCM semigroups.
(v) Every foundation set for S admits an accurate refinement by a transversal for $N^{-1}(n)/_\sim$ for some $n \in N(S)$. In particular, S has the accurate refinement property.

Proof. For (i), suppose $s \in \ker N$. Then (A3) implies that $\{s\}$ is a foundation set for S, which says exactly that $s \in S_c$. On the other hand, if $s \in S$ with $N_s > 1$, then (A3) implies that s is part of an accurate foundation set of cardinality N_s. Thus $\{s\}$ is not a foundation set, and hence $s \notin S_c$. The second claim holds because N is nontrivial.

For (ii), we first claim that if $s, s' \in S$ satisfy $s \sim s'$, then $s \perp t$ if and only $s' \perp t$ for all $t \in S$. Indeed, for $sa = s'b$ with $a, b \in S_c$ we have

$$sS \cap tS \neq \emptyset \overset{a \in S_c}{\Longrightarrow} saS \cap tS \neq \emptyset \overset{sa=s'b}{\Longrightarrow} s'S \cap tS \neq \emptyset.$$

Now suppose we have $s, t \in S$ with $N_s = N_t$. Choose a transversal for $N^{-1}(N_s)/_\sim$ and let s' and t' be the unique elements of the transversal satisfying $s \sim s'$ and $t \sim t'$. If $s' = t'$, then $s \sim t$. If $s' \neq t'$, then since we know from (A3) that the transversal is an accurate foundation set for S, we have $s' \perp t'$. It now follows from the claim that $s \perp t$.

For (iii), let $\mathcal{T}_n$ be a transversal for $N^{-1}(n)/_\sim$. We know that $\mathcal{T}_n$ is accurate because of (A3)(b). By (ii), we know that for every $f \in F$, there is precisely one $t_f \in \mathcal{T}_n$ with $f \sim t_f$. So $f \mapsto t_f$ is a well-defined map from F to $\mathcal{T}_n$, which is surjective because F is a foundation set. The map is injective if F is accurate or has cardinality $|\mathcal{T}_n| = n$.

For (iv), let $s, t, r \in S$ with $sS \cap tS = rS$. Then $N_r \in N_sN(S) \cap N_tN(S)$. We need to show that N_r is the least common multiple of N_s and N_t inside $N(S)$. Suppose there are $m, n \in N(S)$ with $N_sm = N_tn$. For $k = N_s, N_t, m, n$ pick transversals $\mathcal{T}_k$ for $N^{-1}(k)/_\sim$ such that $s \in \mathcal{T}_{N_s}$ and $t \in \mathcal{T}_{N_t}$. Since $s \not\perp t$, and $\mathcal{T}_m$ is a foundation set, there exists $s' \in \mathcal{T}_m$ such that $ss' \not\perp t$. The same argument for n shows that there is $t' \in \mathcal{T}_n$ with $ss' \not\perp tt'$. Now $\mathcal{T}_{N_s}\mathcal{T}_m$ and $\mathcal{T}_{N_t}\mathcal{T}_n$ are accurate foundation sets contained in $N^{-1}(N_sm)$. By (iii), they are transversals for $N^{-1}(N_sm)/_\sim$. In particular, $ss' \not\perp tt'$ is equivalent to $ss' \sim tt'$. Thus there are $a, b \in S_c$ with $ss'a = tt'b \in sS \cap tS = rS$, which results in $N_sm = N_{ss'a} \in N_rN(S)$ due to (i). Thus N_r is the least common multiple of N_s and N_t.

For (v), let F be a foundation set for S. Define n to be the least common multiple of $\{N_f \mid f \in F\}$ inside $N(S)$, and set $n_f := n/N_f \in N(S)$ for each $f \in F$. Choose a transversal $\mathcal{T}_f$ for $N^{-1}(n_f)/_\sim$ for every $f \in F$. Then $F' := \bigcup_{f \in F} f\mathcal{T}_f$ is a foundation set as F is a foundation set and each $\mathcal{T}_f$ is a foundation set. Moreover, we have $F' \subset N^{-1}(n)$. Thus by (ii), for $f, f' \in F'$, we either have $f \sim f'$ or $f \perp f'$. If we now choose a maximal subset F_a of F' with elements having mutually disjoint principal right ideals, F_a is necessarily an accurate foundation set that refines F. In fact, F_a is a transversal for $N^{-1}(n)/_\sim$. In particular, this shows that S has the accurate refinement property. □

The next result is an immediate corollary of Proposition 3.6 (ii) and Lemma 3.2.

Corollary 3.7. *Let S be a core factorable right LCM semigroup such that $S_{ci} \subset S$ is $\cap$-closed. If N is a generalised scale on S, then $N_s = N_t$ for $s, t \in S_{ci}^1$ forces either $s \perp t$ or $s \in tS^*$.*

In particular, if $\mathcal{T}_n = \{f_1, \ldots, f_n\}$ and $\mathcal{T}'_n = \{f'_1, \ldots, f'_n\}$ are two transversals for $N^{-1}(n)/_\sim$, respectively, such that both are contained in S_{ci}, then there are $x_1, \ldots, x_n$ in S^* and a permutation ρ of $\{1, \ldots, n\}$ so that $f'_i = f_{\rho(i)}x_i$ for $i = 1, \ldots, n$.

Lemma 3.2 allows us to strengthen the factorisation $S = S_{ci}^1S_c$ in the following sense.

Lemma 3.8. *Let S be a core factorable right LCM semigroup such that $S_{ci} \subset S$ is $\cap$-closed. Then there are a transversal $\mathcal{T}$ for $S/_\sim$ with $\mathcal{T} \subset S_{ci}^1$ and maps $i\colon S \to \mathcal{T}$, $c\colon S \to S_c$ such that $s = i(s)c(s)$ for all $s \in S$. For every family $(x_t)_{t \in T} \subset S^*$ with $x_1 = 1$, the set $\mathcal{T}' := \{tx_t \mid t \in \mathcal{T}\}$ defines a transversal for $S/_\sim$ with $\mathcal{T}' \subset S_{ci}^1$, and every transversal for $S/_\sim$ contained in S_{ci}^1 is of this form.*

Proof. By (A1), $S/_\sim$ admits a transversal $\mathcal{T}$ with $\mathcal{T} \subset S_{ci}^1$. For each $s \in S$ we define $i(s)$ to be the single element in $\mathcal{T} \cap [s]$, and $c(s)$ to be the unique element in S_c such that $s = i(s)c(s)$, which exists by virtue of Lemma 3.2 due to the presence of (A2). This gives us the maps i and c.

As $S_{ci}S^* = S_{ci}$ and $S^* \cap S^1_{ci} = \{1\}$, the remaining claims follow from Lemma 3.2. □

In our investigations we have found that uniqueness of the KMS$_\beta$-state for β in a critical interval, see Theorem 4.3, is closely related to properties of a natural action α of S_c on $S/_\sim$ arising from left multiplication. By an action α of a semigroup T on a set X we mean a map $T \times X \to X, (t,x) \mapsto \alpha_t(x)$ such that $\alpha_{st}(x) = \alpha_s(\alpha_t(x))$ for all $s,t \in T$ and $x \in X$. For later use, we denote $\mathrm{Fix}(\alpha_t)$ the set $\{x \in X \mid \alpha_t(x) = x\}$.

Lemma 3.9. *Let S be a right LCM semigroup. Then left multiplication defines an action $\alpha\colon S_c \curvearrowright S/_\sim$ by bijections $\alpha_a([s]) = [as]$. Every generalised scale N on S is invariant under this action. If S is admissible and $\mathcal{T} \subset S^1_{ci}$ is a transversal for $S/_\sim$, then there is a corresponding action of S_c on $\mathcal{T}$, still denoted α, which satisfies*

$$aS \cap tS = a\alpha_a^{-1}(t)S \quad \textit{for all } a \in S_c, t \in \mathcal{T}.$$

Proof. For every $a \in S_c$, the map α_a is well-defined as left multiplication preserves the core equivalence relation. If $as \sim at$ for some $s,t \in S$, then $s \sim t$ by left cancellation. Hence α_a is injective. On the other hand, Lemma 3.5 states that for $s \in S$ we have $aS \cap sS = atS, at = sb$ for some $b \in S_c$ and $t \in S$. Thus $\alpha_a([t]) = [at] = [s]$, and we conclude that α_a is a bijection.

For a generalised scale N, invariance follows from $\ker N = S_c$, see Proposition 3.6 (i).

Finally, let $\mathcal{T}$ be a transversal for $S/_\sim$ contained in S^1_{ci}. For $a \in S_c$ and $t \in \mathcal{T}$, it is immediate that $a\alpha_a^{-1}(t) = tc(a\alpha_a^{-1}(t))$. Assume $t \neq 1$ (else the claim about the interesection is trivial), and use Lemma 3.5 to write $aS \cap tS = asS$ for some $s \in S_{ci}$. By the choice of $\mathcal{T}$, there is a unique $s \in \mathcal{T}$ with this property. Proposition 3.6 (i) and (iv) show that $N_{\alpha_a(s)} = N_{as} = N_t$, so that $t = \alpha_a(s)$ by Corollary 3.7. □

Recall that an action $\gamma : T \curvearrowright X$ is *faithful* if, for all $s,t \in T, s \neq t$, there is $x \in X$ such that $\gamma_s(x) \neq \gamma_t(x)$.

Corollary 3.10. *If S is a right LCM semigroup for which $S_c \curvearrowright S/_\sim$ is faithful, then S_c is right cancellative, and hence a right Ore semigroup. In particular, S_c is group embeddable.*

Proof. By Lemma 3.9, we have a monoidal homomorphism from S_c to the bijections on $S/_\sim$, which is injective due to faithfulness of the action. □

Remark 3.11. Recall that an action $\gamma\colon T \curvearrowright X$ of a monoid T on a set X is called *almost free* if the set $\{x \in X \mid \gamma_s(x) = \gamma_t(x)\}$ is finite for all $s,t \in T, s \neq t$. Clearly, if X is infinite, then every almost free action γ on X is faithful. In particular, this is the case for $\alpha\colon S_c \curvearrowright S/_\sim$ for every admissible right LCM semigroup S.

Remark 3.12. If T is a group, then almost freeness states that every γ_t fixes only finitely many points in X. The same conclusion holds if T is totally ordered with respect to $s \geq t :\Leftrightarrow s \in tT$ and acts by injective maps: We then have $T^* = \{1\}$, and $s = tr$ $(s \geq t)$ gives $\gamma_t^{-1}\gamma_s = \gamma_r$, while $t = sr$ $(t \geq s)$ yields $\gamma_s^{-1}\gamma_t = \gamma_r$.

Almost freeness is known to be too restrictive a condition for some of the semigroups we wish to study, for instance for $S = X^* \bowtie G$ arising from a self-similar action (G, X)

that admits a word $w \in X^*$ such that $g(w) = w$ for some $g \in G \setminus \{1\}$ (see 5.3 for more on self-similar actions). To study the uniqueness of KMS$_\beta$-states with β in the critical interval, we will thus also elaborate on the approach used in [LRRW14] for self-similar actions. But we phrase their ideas in a more abstract setting which gives us a potentially useful extra degree of freedom: The condition we work with is a localised version of the finite state condition used in [LRRW14].

Definition 3.13. Let S be an admissible right LCM semigroup, and $\mathcal{T}$ a transversal for $S/_\sim$ with $\mathcal{T} \subset S_{ci}^1$. For $a \in S_c$, let $C_a := \{c(af) \mid f \in \mathcal{T}\}$. The transversal $\mathcal{T}$ is said to be a *witness of finite propagation* of S at $(a, b) \in S_c \times S_c$ if C_a and C_b are finite. The right LCM semigroup S is said to have *finite propagation* if there is a witness of finite propagation at every pair $(a, b) \in S_c \times S_c$.

Remark 3.14. (a) If S is such that S^* is infinite, then for any prescribed $a \in S_c \setminus \{1\}$ there exists a transversal $\mathcal{T}$ for which C_a is infinite, just take $\{xc(a) \mid x \in S^*\}$. Thus, the choice of $\mathcal{T}$ is crucial in Definition 3.13.

(b) Note that S has finite propagation if and only if for every transversal $\mathcal{T}$ for $S/_\sim$ with $\mathcal{T} \subset S_{ci}^1$ (with corresponding maps i, c) and for all $a, b \in S_c$, there is $(x_f)_{f \in \mathcal{T}} \subset S^*$ such that $\{x_f c(af) \mid f \in \mathcal{T}\}$ and $\{x_f c(bf) \mid f \in \mathcal{T}\}$ are finite. Indeed, assuming that S has finite propagation, let $\mathcal{T}$ be a transversal contained in S_{ci}^1 and let $a, b \in S_c$. Then there is a transversal $\mathcal{T}'$ which is a witness of finite propagation at (a, b). However, by Corollary 3.7, there are $x_f \in S^*$ for all $f \in \mathcal{T}$ such that $\mathcal{T}' = \{fx_f \mid f \in \mathcal{T}\}$. Hence $C_a = \{x_{i(af')}c'(af') \mid f' \in \mathcal{T}'\}$ is finite, and similarly for C_b. Reversing the argument gives the converse assertion.

4. The main theorem

If S is a right LCM semigroup admitting a generalised scale N, then standard arguments using the universal property of $C^*(S)$ show that there is a strongly continuous action $\sigma : \mathbb{R} \to \operatorname{Aut} C^*(S)$, where

$$\sigma_x(v_s) = N_s^{ix} v_s \quad \text{for each } x \in \mathbb{R} \text{ and } s \in S.$$

It is easy to see that $\{v_s v_t^* \mid s, t \in S\}$ is a dense family of analytic elements. Noting that σ is the identity on ker π_c and ker π_p, hence also on ker π, the action σ drops to strongly continuous actions of $\mathbb{R}$ on $\mathcal{Q}_c(S), \mathcal{Q}_p(S)$ and $\mathcal{Q}(S)$. Thus we may talk of KMS-states for the corresponding dynamics on $\mathcal{Q}_c(S)$ and $\mathcal{Q}_p(S)$.

Remark 4.1. Since the canonical embedding of S_c into S is a homomorphism of right LCM semigroups, see Proposition 3.4 (i), the universal property of semigroup C^*-algebras guarantees existence of a $*$-homomorphism $\varphi : C^*(S_c) \to C^*(S), w_a \mapsto v_a$, where w_a denotes the generating isometry for $a \in S_c$ in $C^*(S_c)$.

In our analysis of KMS-states on $C^*(S)$ and its boundary quotients under the dynamics σ, a key role is played by a ζ-function for (parts of) S:

Definition 4.2. Let $I \subset \operatorname{Irr}(N(S))$. For each $n \in I$, let $\mathcal{T}_n$ be a transversal for $N^{-1}(n)/_\sim$, which by (A3)(b) is known to be an accurate foundation set. Then the formal series

$$\zeta_I(\beta) := \sum_{n \in \langle I \rangle} \sum_{f \in \mathcal{T}_n} N_f^{-\beta} = \sum_{n \in \langle I \rangle} n^{-(\beta - 1)},$$

where $\beta \in \mathbb{R}$, is called the *I-restricted ζ-function* of S. For $I = \mathrm{Irr}(N(S))$, we write simply ζ_S for $\zeta_{\mathrm{Irr}(N(S))}$ and call it the *ζ-function* of S. The *critical inverse temperature* $\beta_c \in \mathbb{R} \cup \{\infty\}$ is the smallest value so that $\zeta_S(\beta) < \infty$ for all $\beta \in \mathbb{R}$ with $\beta > \beta_c$. The *critical interval* for S is given by $[1, \beta_c]$ if β_c is finite, and $[1, \infty)$ otherwise.

Recall the action α introduced in Lemma 3.9. The statement of our main result, which is the following theorem, makes reference to a trace ρ on $C^*(S_c)$ constructed in Proposition 9.2 under the assumption that $\beta_c = 1$.

Theorem 4.3. *Let S be an admissible right LCM semigroup and let σ be the one-parameter group of automorphisms of $C^*(S)$ given by $\sigma_x(v_s) = N_s^{ix} v_s$ for $x \in \mathbb{R}$ and $s \in S$. The* KMS*-state structure with respect to σ on the boundary quotient diagram* (1.1) *has the following properties:*

- *(1) There are no* KMS_β*-states on $C^*(S)$ for $\beta < 1$.*
- *(2a) If α is almost free, then for each β in the critical interval there is a unique* KMS_β*-state ψ_β given by $\psi_\beta(v_s v_t^*) = N_s^{-\beta} \delta_{s,t}$ for $s, t \in S$.*
- *(2b) If $\beta_c = 1$, α is faithful, and S has finite propagation, then there is a unique* KMS_β*-state ψ_β determined by the trace ρ on $C^*(S_c)$.*
- *(3) For $\beta > \beta_c$, there is an affine homeomorphism between* KMS_β*-states on $C^*(S)$ and normalised traces on $C^*(S_c)$.*
- *(4) Every* KMS_β*-state factors through π_c.*
- *(5) A* KMS_β*-state factors through π_p if and only if $\beta = 1$.*
- *(6) There is an affine homeomorphism between ground states on $C^*(S)$ and states on $C^*(S_c)$. In case that $\beta_c < \infty$, a ground state is a* KMS_∞*-state if and only if it corresponds to a normalised trace on $C^*(S_c)$ under this homeomorphism.*
- *(7) Every* KMS_∞*-state factors through π_c. If $\beta_c < \infty$, then all ground states on $\mathcal{Q}_c(S)$ are* KMS_∞*-states if and only if every ground state on $C^*(S)$ that factors through π_c corresponds to a normalised trace on $C^*(S_c)$ via the map from (6).*
- *(8) No ground state on $C^*(S)$ factors through π_p, and hence none factor through π.*

Proof. Assertion (1) will follow directly from Proposition 6.1. For the existence claim in (2a) and (2b), we apply Proposition 8.8 starting from the canonical trace τ on $C^*(S_c)$ given by $\tau(w_s w_t^*) = \delta_{s,t}$ for all $s, t \in S_c$. The uniqueness assertion will follow from Proposition 9.1 for (2a), and from Proposition 9.2 for (2b).

For assertion (3), Proposition 8.8 shows the existence of a continuous, affine parametrisation $\tau \mapsto \psi_{\beta,\tau}$ that produces a KMS_β-state $\psi_{\beta,\tau}$ on $C^*(S)$ out of a trace τ on $C^*(S_c)$. This map is surjective by Corollary 8.9 and injective by Proposition 8.10.

To prove (4), it suffices to show that a KMS_β-state ϕ on $C^*(S)$ for $\beta \in \mathbb{R}$ vanishes on $1 - v_a v_a^*$ for all $a \in S_c$. But $S_c = \ker N$ implies $\sigma_{i\beta}(v_a) = v_a$ so $\phi(v_a v_a^*) = \phi(v_a^* v_a) = \phi(1) = 1$ since v_a is an isometry. Assertion (5) is the content of Proposition 6.6.

Towards proving (6), Proposition 8.6 provides an affine assignment $\rho \mapsto \psi_\rho$ of states ρ on $C^*(S_c)$ to ground states ψ_ρ of $C^*(S)$ such that $\psi_\rho \circ \varphi = \rho$, for the canonical map $\varphi\colon C^*(S_c) \to C^*(S)$ from Remark 4.1. Thus $\rho \mapsto \psi_\rho$ is injective, and surjectivity follows from Proposition 6.2. If $\beta_c < \infty$, then (3) and Proposition 6.5 show that KMS_∞-states correspond to normalised traces on $C^*(S_c)$.

The first claim in statement (7) follows from (4). For the second part of (7), we observe that $\phi \circ \pi_c$ is a ground state on $C^*(S)$ for every ground state ϕ on $\mathcal{Q}_c(S)$. Suppose first that all ground states on $\mathcal{Q}_c(S)$ are KMS$_\infty$-states. If ϕ' is a ground state on $C^*(S)$ such that $\phi' = \phi \circ \pi_c$ for some ground state ϕ on $\mathcal{Q}_c(S)$, then ϕ is a KMS$_\infty$-state by assumption. This readily implies that ϕ' is also a KMS$_\infty$-state, and hence corresponds to a trace as $\beta_c < \infty$, see (6). Conversely, suppose that all ground states on $C^*(S)$ that factor through π_c correspond to traces under the map from (6). Then every ground state ϕ on $\mathcal{Q}_c(S)$ corresponds to a trace via $\phi \circ \pi_c$. Hence $\phi \circ \pi_c$ is a KMS$_\infty$-state by (6) as $\beta_c < \infty$, and (4) implies that ϕ is a KMS$_\infty$-state on $\mathcal{Q}_c(S)$ as well.

Finally, statement (8) is proved in Proposition 6.7. □

Remark 4.4. The idea behind the trace ρ on $C^*(S_c)$ in Theorem 4.3 (2b) stems from [LRRW14]. If S is right cancellative, then ρ is the canonical trace, i.e. $\rho(w_a w_b^*) = \delta_{a,b}$ for $a, b \in S_c$. If α is almost free then α is faithful by Remark 3.11, and S_c is right cancellative by Corollary 3.10. Now if S_c is right cancellative, then the value $\rho(w_a w_b^*)$ describes the asymptotic proportion of elements in $S/_\sim \cap N^{-1}(n)$ that are fixed under $\alpha_a \alpha_b^{-1}$ as $n \to \infty$ in $N(S)$. Thus, if $\beta_c = 1$, and S has finite propagation, then almost freeness forces ρ to be the canonical trace so that (2a) and (2b) are consistent.

Remark 4.5. In [CDL13], the KMS-state structure for $S = R \rtimes R^\times$, where R is the ring of integers in a number field was considered for a dynamics of the same type as we discuss here. S is right LCM if and only if the class group of R is trivial, and it is quite intriguing to see how the results of [CDL13] relate to Theorem 4.3: We have $\beta_c = 2$ and the unique KMS$_\beta$-state for $\beta \in [1, 2]$ in [CDL13, Theorem 6.7] is analogous to the one in Theorem 4.3 (2a). Since the action $\alpha\colon S_c \curvearrowright S/_\sim$ is almost free, we may expect this uniqueness result to hold outside the class of right LCM semigroups.

On the other hand, the parametrisation of KMS$_\beta$-states for $\beta > 2$ from [CDL13, Theorem 7.3] is by traces on $\bigoplus_\gamma C^*(J_\gamma \rtimes R^*)$, where γ ranges over the class group along with a fixed reference ideal $J_\gamma \in \gamma$. If S is right LCM, then this coincides with Theorem 4.3 (3) as $S_c = S^* = R \rtimes R^*$ and we can take $J_1 = R$. So [CDL13] predicts that we can expect to see a more complicated trace simplex beyond the right LCM case that reflects the finer structure of $\alpha\colon S_c \curvearrowright S/_\sim$. In connection with the ideas behind Theorem 4.3 (6). this might also yield a new perspective on the description of ground states provided in [CDL13, Theorem 8.8].

Remark 4.6. The phase transition result for ground states for the passage from $C^*(S)$ to $\mathcal{Q}_c(S)$ in Theorem 4.3 (7) raises the question whether there is an example of an admissible right LCM semigroup S for which π_c is nontrivial and not all ground states on $\mathcal{Q}_c(S)$ are KMS$_\infty$-states. In all the examples we know so far, see Section 5, we either have $\pi_c = \mathrm{id}$, for instance for the case of self-similar actions or algebraic dynamical systems, or the core subsemigroup S_c is abelian so that all ground states that factor through π_c correspond to traces on $C^*(S_c)$.

5. Examples

Before proving our main theorem on the KMS structure of admissible right LCM semigroups, we use this section to prove that a large number of concrete examples from the literature are admissible. Indeed, for some of the requirements for admissibility ((A1) and (A2)), we do not know any right LCM semigroups that fail to possess them. We break this section into subsections, starting with reduction results for Zappa-Szép products of right LCM semigroups that will be applied to easy right-angled Artin monoids 5.2, self-similar actions 5.3, subdynamics of $\mathbb{N} \rtimes \mathbb{N}^\times$ 5.4, and Baumslag-Solitar monoids 5.5. However, the case of algebraic dynamical systems requires different considerations, see 5.6.

5.1. **Zappa-Szép products.** Let U, A be semigroups with identity e_U, e_A. Suppose there exist maps $(a,u) \mapsto a(u) : A \times U \to U$ and $(a,u) \mapsto a|_u : A \times U \to A$ such that

(ZS1) $e_A(u) = u$;	(ZS5) $e_A\|_u = e_A$;
(ZS2) $(ab)(u) = a(b(u))$;	(ZS6) $a\|_{uv} = (a\|_u)\|_v$;
(ZS3) $a(e_U) = e_U$;	(ZS7) $a(uv) = a(u)a\|_u(v)$; and
(ZS4) $a\|_{e_U} = a$;	(ZS8) $(ab)\|_u = a\|_{b(u)}b\|_u$.

Following [Bri05, Lemma 3.13(xv)] and [BRRW14], the external Zappa-Szép product $U \bowtie A$ is the cartesian product $U \times A$ endowed with the multiplication $(u,a)(v,b) = (ua(v), a|_v b)$ for all $a, b \in A$ and $u, v \in U$. Given $a \in A, u \in U$, we call $a(u)$ the action of a on u, and $a|_u$ the restriction of a to u. For convenience, we write 1 for both e_A and e_U.

In this subsection, we derive an efficient way of identifying admissible Zappa-Szép products $S = U \bowtie A$ among those which share an additional intersection property: For this class, the conditions (A1)–(A4) mostly boil down to the corresponding statements on U. Suppose that $S = U \bowtie A$ is a right LCM semigroup and that A is also right LCM. Then U is necessarily a right LCM semigroup. Let us remark that $S^* = U^* \bowtie A^*$. Moreover, for $(u,a), (v,b) \in S$, it is straightforward to see that $(u,a) \not\perp (v,b)$ implies $u \not\perp v$. We are interested in a strong form of the converse implication:

$$(\cap) \qquad \text{If } uU \cap vU = wU, \text{ then } (u,a)S \cap (v,b)S = (w,c)S \text{ for some } c \in A.$$

We say that $(\cap)$ holds (for S) if it is valid for all $(u,a), (v,b) \in S$.

Remark 5.1. The property $(\cap)$ is inspired by [BRRW14, Remark 3.4]. If a Zappa-Szép product $U \bowtie A$ satisfies the hypotheses of [BRRW14, Lemma 3.3], that is,

(a) U is a right LCM semigroup,
(b) A is a left cancellative monoid whose constructible right ideals are totally ordered by inclusion, and
(c) $u \mapsto a(u)$ is a bijection of U for each $a \in A$,

then [BRRW14, Remark 3.4] shows that the right LCM semigroup $U \bowtie A$ satisfies $(\cap)$.

While surprisingly many known examples fit into the setup of [BRRW14, Lemma 3.3], the following easy example shows that they are not necessary to ensure that $U \bowtie A$ is right LCM or that $(\cap)$ holds.

Example 5.2. Consider the standard restricted wreath product $S := \mathbb{N} \wr \mathbb{N} = (\bigoplus_{\mathbb{N}} \mathbb{N}) \rtimes \mathbb{N}$, where the endomorphism appearing in the semidirect product is the shift. Then one can check that $A := S_c = \{(m,0) \mid m \in \bigoplus_{\mathbb{N}} \mathbb{N}\}$ and $U := S_{ci}^1 = \{(m,n) \mid n \in \mathbb{N}, m \in \bigoplus_{\mathbb{N}} \mathbb{N} : m_k = 0 \text{ for all } k \geq n\}$ with action and restriction determined by $(m,0)(m',n) = (m+m',n) = (m'',n)((m_{\ell+n})_{\ell \in \mathbb{N}}, 0)$ with $m''_k = \chi_{\{0,\dots,n-1\}}(k)\ (m_k + m'_k)$ yields a Zappa-Szép product description $S = U \bowtie A$. It is also straightforward to check that $U \bowtie A$ is right LCM and satisfies $(\cap)$. However, the constructible ideals of A are directed but not totally ordered by inclusion, and the action of A on U is by non-surjective injections for all $a \in A \setminus \{0\}$. Note that the same treatment applies to $S' = T \wr \mathbb{N}$ for every nontrivial, left cancellative, left reversible monoid T.

Theorem 5.3. *If $S = U \bowtie A$ is a right LCM semigroup with A right LCM such that $(\cap)$ holds, then*

(i) $A_c = A$, i.e. A is left reversible,
(ii) S_c equals $U_c \bowtie A$,
(iii) $(u,a) \sim (v,b)$ if and only if $u \sim v$,
(iv) S_{ci} is given by $U_{ci} \bowtie A^*$,
(v) S is core factorable if and only U is core factorable, and
(vi) $S_{c_i} \subset S$ is $\cap$-closed if and only if $U_{c_i} \subset U$ is $\cap$-closed.

Proof. Part (i) follows from $(\cap)$ as $(1,a)S \cap (1,b)S = (1,c)S$ for some $c \in A$, so that $c \in aA \cap bA$. Similarly, $(\cap)$ forces $S_c = U_c \times A$ as $A = A_c$. In addition, to get a Zappa-Szép product as claimed in (ii), we note that $a(u) \in U_c$ for all $a \in A, u \in U_c$ because $(a(u), a|_u) = (1,a)(u,1) \in S_c = U_c \times A$.

Part (iii) is an easy consequence of (ii): $(u,a) \sim (v,b)$ forces $u \sim v$ since $a(w) \in U_c$ for all $a \in A, w \in U_c$. Conversely, let $uw = vw'$ for some $w, w' \in U_c$ and let $a, b \in A$. Then $(1,a)(w,1), (1,b)(w',1) \in S_c$ and hence there are $s, s', t, t' \in S_c$ such that $(1,a)s' = ws$ and $(1,b)t' = w't$. Likewise, there are $s'', t'' \in S_c$ such that $ss'' = tt''$. This leads us to

$$(u,1)(1,a)s's'' = (u,1)(w,1)ss'' = (uw,1)tt'' = (vw',1)tt'' = (v,b)t't'',$$

which proves $(u,a) \sim (v,b)$ as $s's'', tt'' \in S_c$.

For (iv), we first show that $S_{ci} = U_{ci} \times A^*$. Suppose we have $(u,a) \in S_{ci}$. If $a \in A \setminus A^*$, then $(u,a) = (u,1)(1,a)$ together with (ii) shows that (u,a) is not core irreducible. Thus we must have $a \in A^*$. Similarly, if $u \notin U_{ci}$, then $u \in U^*$ or there exist $v \in U, w \in U_c \setminus U^*$ such that $u = vw$. In the first case, we get $(u,a) \in S^*$ and hence $(u,a) \notin S_{ci}$. The latter case yields a factorization $(u,a) = (v,1)(w,a)$ with $(w,a) \in S_c \setminus S^*$. Altogether, this shows $S_{ci} \subset U_{ci} \times A^*$. To prove the reverse containment, suppose $(u,a) \in U_{ci} \times A^*$ and $(u,a) = (v,b)(w,c) = (vb(w), b|_w c)$ with $(v,b) \in S, (w,c) \in S_c = U_c \bowtie A$. Then $b|_w, c \in A^*$ as $a \in A^*$, and the Zappa-Szép product structure on S_c implies $b(w) \in U_c$. Since $u \in U_{ci}$, we conclude that $b(w) \in U^*$. But then $(1,b)(w,1) = (b(w), b|_w) \in U^* \bowtie A^* = S^*$ forces $(1,b), (w,1) \in S^*$. In particular, we have $w \in U^*$ and hence $(w,c) \in S^*$. Thus $(u,a) \in S_{ci}$ and we have shown that $S_{ci} = U_{ci} \times A^*$.

Next we prove that $a(u) \in U_{ci}$ and $a|_u \in A^*$ whenever $a \in A^*, u \in U_{ci}$. The second part is easily checked with $a|_u^{-1} = a^{-1}|_{a(u)}$. So suppose $a(u) = vw$ with $w \in U_c$. Then we get $u = a^{-1}(v)a^{-1}|_v(w)$, and $S_c = U_c \bowtie A$ yields $a^{-1}|_v(w) \in U_c$. Since u is core irreducible, we get $a^{-1}|_v(w) \in U^*$, and then $w \in U^*$ due to $S^* = U^* \bowtie A^*$. Therefore, $a(u) \in U_{ci}$ and we have proven (iv).

For (v), let $U = U_{ci}^1 U_c$. Observe that (ii) and (iv) yield $S_{ci}^1 S_c = (U_{ci} \bowtie A^*)^1 (U_c \bowtie A) \supseteq U_{ci}^1 U_c \times A \supseteq S$. Thus S is core factorable. Conversely, suppose that $S = S_{ci}^1 S_c$. For $u \in U$, take $(u,1) = (v,a)(w,b) = (va(w), a|_w b)$ with $(v,a) \in S_{ci}^1 = (U_{ci} \bowtie A^*)^1$ and $(w,b) \in S_c = U_c \bowtie A$. For $u \in U \setminus U_c$ we get $v \in U_{ci}$. Combining this with $a(w) \in U_c$ using (ii), we arrive at $u = va(w) \in U_{ci} U_c$. Thus $U = U_{ci}^1 U_c$ holds.

For (vi), suppose that $S_{ci} \subset S$ is $\cap$-closed. Let $u, v \in U_{ci}$ such that $uU \cap vU = wU$ for some $w \in U$. Therefore $(u,1)S \cap (v,1)S = (w,1)S$, which by (iv) means that $w \in U_{ci}$.

Conversely, suppose that $U_{ci} \subset U$ is $\cap$-closed. As $S_{ci} = U_{ci} \bowtie A^*$ by (iv) and $sxS = sS$ for all $s \in S, x \in S^*$, it suffices to consider $(u,1)S \cap (v,1)S = (w,1)S$ for $u, v \in U_{ci}$. If this holds, then $uU \cap vU = w'U$ for some $w' \in U$ with $w \in w'U$. As $U_{ci} \subset U$ is $\cap$-closed, $w' \in U_{ci}$. Moreover, ($\cap$) implies that $(w,1)S = (u,1)S \cap (v,1)S = (w',a)S$ for some $a \in A$, which amounts to $a \in A^*$ and, more importantly, $w \in w'U^* \subset U_{ci}$. Thus $(w,1) \in S_{ci}$ by (iv). □

Proposition 5.4. *Let $S = U \bowtie A$ be a right LCM semigroup with A right LCM such that ($\cap$) holds. The restriction $N \mapsto N|_U$ defines a one-to-one correspondence between*

(a) generalised scales $N\colon S \to \mathbb{N}^\times$; and

(b) generalised scales $N'\colon U \to \mathbb{N}^\times$ satisfying $N'_{a(u)} = N'_u$ for all $a \in A, u \in U$.

Moreover, $N(S) = N|_U(U)$, so that (A4) holds for N if and only if it holds for $N|_U$.

Proof. For every generalised scale N on S, the restriction $N|_U\colon U \to \mathbb{N}^\times$ defines a non-trivial homomorphism. Since $S_c = \ker N$, Theorem 5.3 (ii) implies that $N(S) = N|_U(U)$ as $N((u,a)) = N((u,1))N((1,a)) = N((u,1))$ for all $(u,a) \in S$, and $N|_U(a(u)) = N((a(u),1)) = N((a(u),a|_u)) = N((1,a))N((u,1)) = N|_U(u)$ for all $a \in A, u \in U$.

By virtue of Theorem 5.3 (iii), (A3)(a) passes from N to $N|_U$. Similarly, if F' is a transversal for $N|_U^{-1}(n)/_\sim$ for $n \in N|_U(U)$, then Theorem 5.3 (iii) implies that $F := \{(u,1) | u \in F'\}$ is a transversal for $N^{-1}(n)/_\sim$. Due to (A3)(b) for N, we know that F is an accurate foundation set for S. But then F' is an accurate foundation set because of ($\cap$). Thus we get (A3)(b) for $N|_U$.

Conversely, suppose that N' is one of the generalised scales in (b). Then $N\colon S \to \mathbb{N}^\times, (u,a) \mapsto N'(u)$ defines a non-trivial homomorphism on S with $N|_U = N'$ because $N((u,a)(v,b)) = N((ua(v), a|_v b)) = N'(ua(v)) = N'(u)N'(v) = N((u,a))N((v,b))$ for $(u,a),(v,b) \in S$ by the invariance of N' under the action of A. Similar to the first part, Theorem 5.3 (iii) shows that (A3)(a) passes from N' to N. Now if F is a transversal of $N^{-1}(n)/_\sim$ for some $n \in N(S)$, then $F_U := \{u \in U \mid (u,a) \in F \text{ for some } a \in A\}$ is a transversal for $N'^{-1}(n)/_\sim$, by Theorem 5.3 (iii). Since F_U is an accurate foundation set for U due to (A3)(b) for N', ($\cap$) implies that F is an accurate foundation set for S, i.e. (A3)(b) holds for N. □

Corollary 5.5. *Let $S = U \bowtie A$ be a right LCM semigroup with A right LCM such that ($\cap$) holds. Then S is admissible if and only if U is admissible with a generalised scale that is invariant under the action of A on U*

Proof. This follows from Theorem 5.3 (v),(vi) and Proposition 5.4. □

Observe that for $S = U \bowtie A$ with $S_c = U_c \bowtie A$, the action of A on U induces a well-defined action $\gamma\colon A \times U/_\sim \to U/_\sim, (a,[u]) \mapsto [a(u)]$.

Corollary 5.6. *Suppose that $S = U \bowtie A$ is a right LCM semigroup with U right LCM satisfying Theorem 5.3 (ii),(iii). If U_c is trivial, then the following statements hold:*

(i) The action $\alpha\colon S_c \curvearrowright S/_\sim$ is faithful if and only if $\gamma\colon A \curvearrowright U/_\sim$ is faithful.

(ii) The action α is almost free if and only if γ is almost free.

Proof. We start by noting that $\alpha_{(1,a)}[(u,c)] = [(a(u), a|_u c)] = [b(u), b|_u c)] = \alpha_{(1,b)}[(u,c)]$ if and only if $\gamma_a[u] = [a(u)] = [b(u)] = \gamma_b[u]$ for all $a,b,c \in A, u \in U$ by Theorem 5.3 (ii) and (iii). If U_c is trivial the canonical embedding $A \to S_c = U_c \bowtie A$ is an isomorphism which is equivariant for α and γ. □

5.2. **Easy right-angled Artin monoids.** For $m, n \in \mathbb{N} \cup \infty$, consider the direct product $S := \mathbb{F}_m^+ \times \mathbb{N}^n$, where $\mathbb{F}_m^+$ denotes the free monoid in m generators and $\mathbb{N}^n$ denotes the free abelian monoid in n generators. The monoid S is a very simple case of a right-angled Artin monoid arising from the graph $\Gamma = (V, E)$ with vertices $V = V_m \sqcup V_n$, where $V_m := \{v_k \mid 1 \leq k \leq m\}$ and $V_n := \{v'_k \mid 1 \leq k \leq n\}$, and edge set $E = \{(v, w) \mid v \in V_n \text{ or } w \in V_n\}$. In particular, $(S, \mathbb{F}_m \times \mathbb{Z}^n)$ is a quasi lattice-ordered group, see [CL07], so S is right LCM.

We note that $\mathbb{F}_m^+$ admits a generalised scale if and only if m is finite, and thus $\mathbb{F}_m^+$ is admissible if and only if $2 \leq m < \infty$.

Proposition 5.7. *For $2 \leq m < \infty, 0 \leq n \leq \infty$, the right LCM semigroup $S := \mathbb{F}_m^+ \times \mathbb{N}^n$ has the following properties:*

(i) S is admissible with $\beta_c = 1$.
(ii) The action α is faithful if and only if α is almost free if and only if $n = 0$.
(iii) S has finite propagation.

Proof. By appealing to Corollary 5.5, $S = \mathbb{F}_m^+ \bowtie \mathbb{N}^n$ with trivial action and restriction is admissible for the generalised scale N induced by the invariant generalised scale $N' \colon \mathbb{F}_m^+ \to \mathbb{N}^\times, w \mapsto m^{\ell(w)}$, where $\ell(w)$ denotes the word length of w with respect to the canonical generators of $\mathbb{F}_m^+$. In particular, $\mathrm{Irr}(N(S)) = \{m\}$, so $\beta_c = 1$.

Due to Corollary 5.6, we further know that α is faithful (almost free) if and only if $\gamma \colon \mathbb{N}^n \curvearrowright \mathbb{F}_m^+/_\sim$ is faithful (almost free). But γ is trivial as the action of $\mathbb{N}^n$ on $\mathbb{F}_m^+$ is trivial. Thus we get faithfulness if only if $S_c \cong \mathbb{N}^n$ is trivial. In this case, the action is of course also almost free.

The only transversal $\mathcal{T} \subset S_{ci}^1 = \mathbb{F}_m^+$ for $S/_\sim$ is $\mathbb{F}_m^+$. For every $a \in S_c = \mathbb{N}^n$, we get $c(aw) = c(wa) = a$ so that $C_a = \{a\}$ is finite. Hence S has finite propagation. □

In the case of $S := \mathbb{F}_m^+ \times \mathbb{N}^n$ with $2 \leq m < \infty$, it is easy to see that KMS_1-states correspond to traces on $C^*(\mathbb{N}^n)$, or, equivalently, Borel probability measures on $\mathbb{T}^n$ via $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$. Thus we cannot expect uniqueness unless $n = 0$. In fact, this example describes a scenario where α is degenerate. However, the situation is much less clear for general right-angled Artin monoids.

5.3. **Self-similar actions.** Given a finite alphabet X, let X^n be the set of words with length n. The set $X^* := \cup_{n=0}^\infty X^n$ is a monoid with concatenation of words as the operation and empty word $\varnothing$ as the identity. A self-similar action (G, X) is an action of a group G on X^* such that for every $g \in G$ and $x \in X$ there is a uniquely determined element $g|_x \in G$ satisfying

$$g(xw) = g(x)g|_x(w) \quad \text{for all } w \in X^*.$$

We refer to [Nek05] for a thorough treatment of the subject. It is observed in [Law08] and [BRRW14, Theorem 3.8] that the Zappa-Szép product $X^* \bowtie G$ with $(v, g)(w, h) := (vg(w), g|_w h)$ defines a right LCM semigroup that satisfies $(\cap)$.

Proposition 5.8. *Suppose that (G, X) is a self-similar action. Then the Zappa-Szép product $S := X^* \bowtie G$ has the following properties:*

(i) S is admissible with $\beta_c = 1$.

(ii) *The action α is faithful (almost free) if and only if $G \curvearrowright X^*$ is faithful (almost free).*

(iii) *S has finite propagation if and only if for all $g^{(1)}, g^{(2)} \in G$ there exists a family $(h_w)_{w\in X^*} \subset G$ such that $\{h^{-1}_{g^{(i)}(w)} g^{(i)}|_w h_w \mid w \in X^*\}$ is finite for $i = 1, 2$. In particular, S has finite propagation if (G, X) is finite state.*

Proof. For (i), note that S is a right LCM semigroup that satisfies ($\cap$). Then by Corollary 5.5, it suffices to show that X^* is admissible with an invariant generalised scale $N'\colon X^* \to \mathbb{N}^\times$. Since the core subsemigroup of X^* consists of the empty word $\varnothing$ only, the equivalence relation $\sim$ is trivial on X^*. It follows that X^* satisfies the conditions (A1) and (A2). It is apparent that the map $N'\colon X^* \to \mathbb{N}^\times, w \mapsto |X|^{\ell(w)}$ is a generalised scale, where $\ell(w)$ denotes the *word length* of w in the alphabet X. Since for each $g \in G$, $w \mapsto g(w)$ is a bijection on X^n (see for example [BRRW14, Lemma 3.7]), N' is invariant under $G \curvearrowright X^*$. Finally, (A4) follows from $\mathrm{Irr}(N'(X^*)) = \{|X|\}$. Thus N' induces a generalised scale N on S that makes S admissible. Moreover, we have $\beta_c = 1$ as $\mathrm{Irr}(N'(X^*)) = \{|X|\}$ is finite.

Part (ii) is an immediate consequence of Corollary 5.6.

For (iii), we observe that an arbitrary transversal $\mathcal{T} \subset S^1_{ci} = (X^* \setminus \{\varnothing\} \bowtie G)^1$ for $S/_\sim$ is given by $((w, h_w))_{w\in X^*}$, where $(h_w)_{w\in X^*} \subset G$ satisfies $h_\varnothing = 1$. For a given $g \in G \cong S_c$, we get $C_g := C_{(\varnothing,g)} = \{h^{-1}_{g(w)} g|_w h_w \mid w \in X^*\}$ as $(\varnothing, g)(w, h_w) = (g(w), h_{g(w)})(\varnothing, h^{-1}_{g(w)} g|_w h_w) \in \mathcal{T} S_c$. This establishes the characterisation of finite propagation. So if (G, X) is finite state, i.e. $\{g|_w \mid w \in X^*\}$ is finite for all $g \in G$, then $h_w = 1$ for all $w \in X^*$ yields finite propagation for S. □

Faithfulness of $G \curvearrowright X^*$ is assumed right away in [LRRW14, Section 2]. Our approach shows that this is unnecessary unless we want a unique KMS_1-state.

5.4. **Subdynamics of** $\mathbb{N} \rtimes \mathbb{N}^\times$**.** Consider the semidirect product $\mathbb{N} \rtimes \mathbb{N}^\times$ with

$$(m, p)(n, q) = (m + np, pq) \text{ for all } m, n \in \mathbb{N} \text{ and } p, q \in \mathbb{N}^\times.$$

Let $\mathcal{P} \subset \mathbb{N}^\times$ be a family of relatively prime numbers, and P the free abelian submonoid of $\mathbb{N}^\times$ generated by $\mathcal{P}$. Then we get a right LCM subsemigroup $S := \mathbb{N} \rtimes P$ of $\mathbb{N} \rtimes \mathbb{N}^\times$. As observed in [Sta17, Example 3.1], S can be displayed as the internal Zappa-Szép product $S^1_{ci} \bowtie S_c$, where $S_c = \mathbb{N} \times \{1\}$ and $S^1_{ci} = \{(n, p) \in S \mid 0 \leq n \leq p - 1\}$. The action and restriction maps satisfy

$$(m, 1)\big((n, p)\big) = \big((m + n) \,(\mathrm{mod}\ p), p\big), \text{ and}$$
$$(m, 1)|_{(n,p)} = \Big(\frac{m + n - \big((m + n) \,(\mathrm{mod}\ p)\big)}{p}, 1\Big).$$

Both S^1_{ci} and S_c are right LCM. Also the argument of the paragraph after [BRRW14, Lemma 3.5] shows that $S^1_{ci} \bowtie S_c$ satisfies the hypothesis of [BRRW14, Lemma 3.3]. Therefore, $S^1_{ci} \bowtie S_c$ is a right LCM semigroup satisfying ($\cap$), see Remark 5.1.

Proposition 5.9. *Suppose that $\mathcal{P} \subset \mathbb{N}^\times$ is a nonempty family of relatively prime numbers. Then the right LCM semigroup $S = \mathbb{N} \rtimes P$ has the following properties:*

(i) *S is admissible, and if $\mathcal{P}$ is finite, then $\beta_c = 1$.*

(ii) *α is faithful and almost free.*

(iii) S has finite propagation.

Proof. Since S is isomorphic to $S_{ci}^1 \bowtie S_c$, (i) follows if S_{ci}^1 is admissible with an invariant generalised scale, see Corollary 5.5. To see this, first note that the core subsemigroup for S_{ci}^1 is trivial, hence so is the relation $\sim$ for S_{ci}^1. Thus (A1) and (A2) hold for S_{ci}^1. We claim that $N' \colon S_{ci}^1 \to \mathbb{N}^\times, (n,p) \mapsto p$ defines a generalised scale on S_{ci}^1. The only transversal for $(N')^{-1}(p)/_\sim$ for $p \in N'(S_{ci}^1)$ is $(N')^{-1}(p) = \{(m,p) \mid 0 \leq m \leq p-1\}$. This gives condition (A3)(a). For (A3)(b), we observe that

$$(m,p) \perp (n,q) \text{ if and only if } m-n \notin \gcd(p,q)\mathbb{Z} \text{ for all } (m,p),(n,q) \in S_{ci}^1.$$

It follows $(N')^{-1}(p)$ is an accurate foundation set. Thus N' is a generalised scale on S_{ci}^1. Since $S_c = \mathbb{N} \times \{1\}$, the action of S_c on S_{ci}^1 fixes the second coordinate. Therefore, N' is invariant under the action of S_c on S_{ci}^1. Finally, note that the set $\operatorname{Irr}(N'(S_{ci}^1))$ is given by $\mathcal{P}$, which is assumed to consist of relatively prime elements. Hence (A4) holds. Moreover, if $\mathcal{P}$ is finite, then $\beta_c = 1$ by the product formula for ζ_S from Remark 7.4.

For (ii), note that Corollary 5.6 applies. As $S_c \cong \mathbb{N} \times \{1\}$ is totally ordered, it suffices to examine the fixed points in $S/_\sim$ for γ_m with $m \geq 1$, see Remark 3.12. But $\gamma_m[(n,p)] = [(r(m+n),p)]$, where $0 \leq r(m+n) \leq p-1, m+n \in r(m+n)+p\mathbb{N}$, so that $[(n,p)]$ is fixed by γ_m if and only if $m \in p\mathbb{N}$. As $\{p \in P \mid m \in p\mathbb{N}\}$ is finite, γ is almost free. In particular, γ is faithful because $S_{ci}^1/_\sim$ is infinite, see Remark 3.11. Hence α is faithful and almost free.

Part (iii) follows from the observation that the transversal $\mathcal{T} = S_{ci}^1$ for $S/_\sim$ satisfies $C_{(m,1)} = \{(m,1)\} \cup \{(n,1) \mid n \in \{k,k+1\} \text{ with } kp \leq m < (k+1)p \text{ for some } p \in P\setminus\{1\}\}$ for $m \geq 1$. Thus the cardinality of $C_{(m,1)}$ does not exceed $m+1$, and S has finite propagation. ☐

By Proposition 5.9, Theorem 4.3 applies to $\mathbb{N} \rtimes \mathbb{N}^\times$. This recovers the essential results on KMS-states from [LR10, BaHLR12].

5.5. **Baumslag-Solitar monoids.** Let $c,d \in \mathbb{N}$ with $cd > 1$. The *Baumslag-Solitar monoid* $BS(c,d)^+$ is the monoid in two generators a,b subject to the relation $ab^c = b^d a$. It is known that $S := BS(c,d)^+$ is quasi-lattice ordered, hence right LCM, see [Spi12, Theorem 2.11]. It is observed in [Sta17, Example 3.2] and [BRRW14, Subsection 3.1] that the core subsemigroup S_c is the free monoid in b, whereas S_{ci}^1 is the free monoid in $\{b^j a \mid 0 \leq j \leq d-1\}$. In addition, we have $S_{ci}^1 \cap S_c = \{1\}$ and $S = S_{ci}^1 S_c$, so that the two subsemigroups of S give rise to an internal Zappa-Szép product description $S_{ci}^1 \bowtie S_c$. The action and restriction map (on the generators) are given by

$$b(b^j a) = \begin{cases} b^{j+1}a & \text{if } j < d-1, \\ a & \text{if } j = d-1, \end{cases} \quad \text{and} \quad b|_{b^j a} = \begin{cases} 1 & \text{if } j < d-1, \\ b^c & \text{if } j = d-1. \end{cases}$$

Moreover, $S_{ci}^1 \bowtie S_c$ satisfies the hypothesis of [BRRW14, Lemma 3.3] and therefore $S_{ci}^1 \bowtie S_c$ is a right LCM satisfying ($\cap$), see Remark 5.1.

Proposition 5.10. *Let $c,d \in \mathbb{N}$ such that $cd > 1$. Then the Baumslag-Solitar monoid $S = BS(c,d)^+$ is admissible with $\beta_c = 1$, and the following statements hold:*

(i) The action α is almost free if and only if α is faithful if and only if $c \notin d\mathbb{N}$.
(ii) S has finite propagation if and only if $c \leq d$.

Proof. Since $S \cong S_{ci}^1 \bowtie S_c$, we can invoke Corollary 5.5. As S_{ci}^1 is the free monoid in d generators $X := \{w_j := b^j a \mid 0 \leq j \leq d-1\}$, it is admissible for the generalised scale $N'\colon S_{ci}^1 \to \mathbb{N}^\times, w \mapsto d^{\ell(w)}$, where $\ell(w)$ denotes the word length of w with respect to X, compare Proposition 5.8. The action of S_c on S_{ci}^1 preserves ℓ, and hence N' is invariant. Thus N' induces a generalised scale N on S that makes S admissible. Due to $\mathrm{Irr}(N(S)) = \{d\}$, we get $\beta_c = 1$.

The quotient $S/_\sim$ is infinite, almost freeness implies faithfulness by Remark 3.11. As $S \cong S_{ci}^1 \bowtie S_c$ satisfies $(\cap)$, we can appeal to Corollary 5.6, and thus consider $\gamma\colon \mathbb{N} \curvearrowright S_{ci}^1/_\sim$ in place of α, where we identify the quotient by $\sim$ with S_{ci}^1 because $\sim$ is trivial on S_{ci}^1. As $S_c \cong \mathbb{N}$ is totally ordered, it suffices to consider fixed points for γ_n with $n \geq 1$, see Remark 3.12. Next, we observe that

$$\gamma_n(w) = w \text{ for } w \in S_{ci}^1 \text{ if and only if } n(c/d)^{k-1} \in d\mathbb{N} \text{ for all } 1 \leq k \leq \ell(w).$$

So if $c \in d\mathbb{N}$, then $\gamma_d = \mathrm{id}$, so that γ is not faithful. Hence α is not faithful as well. On the other hand, if $c \notin d\mathbb{N}$, then $e := \gcd(c,d)^{-1}d \geq 2$ and γ_n fixes $w \in S_{ci}^1$ if and only if $n \in e^\ell(w)\mathbb{N}$. So as soon as $\ell(w)$ is large enough, w cannot be fixed by γ_n. Since the alphabet X is finite, we conclude that γ (and hence α) is almost free if $c \notin d\mathbb{N}$.

For (ii), we remark that S_{ci}^1 is the only transversal for $S/_\sim$ that is contained in S_{ci}^1. For $m \in \mathbb{N}$, we let $m = kd + r$ with $k, r \in \mathbb{N}$ and $0 \leq r \leq d-1$. Then we get

$$b^m w_i = w_j b^n \text{ for } 0 \leq i \leq d-1 \text{ with } n = \begin{cases} kc, & \text{if } 0 \leq i < d-r, \\ (k+1)c, & \text{if } d-r \leq i \leq d-1, \end{cases} \tag{5.1}$$

with a suitable $0 \leq j \leq d-1$. If $c = d$, then we get $C_{b^m} = \{b^m\}$ for $m \in d\mathbb{N}$ and $C_{b^m} = \{b^m, b^{m+d-r}, b^{m-r}\}$ for $m \notin d\mathbb{N}$. Thus S has finite propagation in this case. For $c < d$, it follows from (5.1) that every expression $b^m v = w b^n$ for given $v \in S_{ci}^1$ and $m \in \mathbb{N}$ satisfies $n \leq (k+1)d$. Hence C_{b^m} is finite and S has finite propagation. Now suppose $c > d$, and let $m \geq d$, say $m = kd + r$ with $0 \leq r \leq d-1, k \geq 1$. If $r = 0$, (5.1) gives $b^m w_i = w_i b^n$ with $n = kc > m$ for all i. If $r > 0$, then there is $d - r \leq i \leq d-1$, and we have $b^m w_i = w_j b^n$ with $n = (k+1)c > m$ for all such i. Thus C_{b^m} contains arbitrarily large powers of b, and hence must be infinite. Therefore S does not have finite propagation for $c \geq d$. □

We note that for this example, finite propagation can be regarded as a consequence of faithfulness of α. The condition that d is not a divisor of c also appeared in [CaHR16]. More precisely, it was shown in [CaHR16, Corollary 5.3, Proposition 7.1, and Example 7.3] that this condition is necessary and sufficient for the KMS_1-state to be unique. Let us remark here that we do not need to request that d is not a divisor of c in order to obtain the parametrisation of the KMS_β-states for $\beta > 1$, compare Theorem 4.3 and Proposition 5.10 with [CaHR16, Theorem 6.1].

5.6. **Algebraic dynamical systems.** Suppose (G, P, θ) is an algebraic dynamical system, that is, an action θ of a right LCM semigroup P on a countable, discrete group G by injective endomorphisms such that $pP \cap qP = rP$ implies $\theta_p(G) \cap \theta_q(G) = \theta_r(G)$. The semidirect product $G \rtimes_\theta P$ is the semigroup $G \times P$ equipped with the operation $(g,p)(h,q) = (g\theta_p(h), pq)$. It is known that injectivity and the intersection condition, known as *preservation of order*, are jointly equivalent to $G \rtimes_\theta P$ being a right LCM semigroup, see [BLS17, Proposition 8.2].

In order to have an admissible semidirect product $G \rtimes_\theta P$, we need to consider more constraints on (G, P, θ):

(a) (G, P, θ) is *finite type*, that is, the index $N_p := [G : \theta_p(G)]$ is finite for all $p \in P$.
(b) There exists $p \in P$ for which θ_p is not an automorphism of G.
(c) If θ_p is an automorphism of G, then $p \in P^*$.
(d) If $p, q \in P$ have the same index, i.e. $N_p = N_q$, then $p \in qP^*$. That is to say, there is an automorphism ϑ of G such that θ_p and θ_q are conjugate via ϑ, and $\vartheta = \theta_x$ for some $x \in P^*$.

Conditions (a) and (b) are mild assumptions. Also it is natural to assume that p is invertible in P if θ_p is an automorphism of G. For otherwise, we can replace P by the acting semigroup of endomorphisms of G. But (d) is significantly more complex, and we will see that it incorporates deep structural consequences for $G \rtimes_\theta P$.

Proposition 5.11. *Let (G, P, θ) be an algebraic dynamical system satisfying the above conditions (a)–(d). Then $S := G \rtimes_\theta P$ has the following properties:*

(i) S is admissible, and $\beta_c = 1$ if $Irr(P/_\sim)$ is finite.
(ii) The action α is faithful if and only if $\bigcap_{(h,q)\in S} h\theta_q(G)h^{-1} = \{1\}$, and

$$\bigcap_{(h,q)\in S} h\theta_q(G)\theta_p(h)^{-1} = \emptyset \text{ for all } p \in P^* \setminus \{1\}.$$

(iii) The action α is almost free if and only if P^* is trivial and the set $\{(g, p) \in S \mid h \in g\theta_p(G)g^{-1}\}$ is finite for all $h \in G \setminus \{1\}$.

Proof. For (i), note that $S_c = S^* = G \rtimes P^*$, see [Sta17, Example 3.2]. So that $S_{ci} = S\setminus S^*$ and therefore S satisfies (A1) and (A2). Also note that

$$(g, p) \sim (h, q) \Leftrightarrow (g, p)S = (h, q)S \Leftrightarrow p \in qP^* \text{ and } g^{-1}h \in \theta_p(G)\ (= \theta_q(G)). \tag{5.2}$$

Then the natural candidate for a homomorphism $N\colon S \to \mathbb{N}^\times$ is given by $N(g, p) := N_p$. Properties (a) and (b) imply that N is a nontrivial homomorphism from S to $\mathbb{N}^\times$. The condition (A3)(a) follows from (d) together with (5.2). Since (G, P, θ) is finite type, (A3)(b) holds if and only if P is directed with regards to $p \geq q :\Leftrightarrow p \in qP$, see [Sta17, Example 3.2]. This follows immediately from (d): Given $p, q \in P$, we have $N_{pq} = N_pN_q = N_{qp}$ so that $pqx = qp \in pP \cap qP$ for a suitable $x \in P^*$. Thus N is a generalised scale on S.

Now we are looking at (A4). First note that (d) implies $P^*p \subset pP^*$ for all $p \in P$. Thus the restriction of $\sim$ to P given by $p \sim' q :\Leftrightarrow p \in qP^*$ defines a congruence on P. If $\overline{P} := P/_{\sim'}$ denotes the resulting monoid, then $\overline{P}$ is necessarily abelian as $N_{pq} = N_{qp}$. The monoid $\overline{P}$ is generated by $\{[p] \in \overline{P} \mid N_p \in \mathrm{Irr}(N(S))\}$. This set is minimal as a generating set for $\overline{P}$ and coincides with the irreducible elements of $\overline{P}$. So to prove (A4), it suffices to show that $\overline{P}$ is in fact freely generated by $\mathrm{Irr}(\overline{P})$.

To see this, let $\prod_{1\leq i\leq k}[p_i]^{m_i} = \prod_{1\leq j\leq \ell}[q_j]^{n_j}$ with $[p_i], [q_j] \in \mathrm{Irr}(\overline{P}), m_i, n_j \in \mathbb{N}$, where we assume that the $[p_i]$ are mutually distinct, as are the $[q_j]$. Then we get a corresponding equation $\prod_{1\leq i\leq k} N_{p_i}^{m_i} = \prod_{1\leq j\leq \ell} N_{q_j}^{n_j}$ in $N(S)$. Since $N(S)$ has the unique factorization property (inherited from $\mathbb{N}^\times$), and $N_{p_i}, N_{q_j} \in \mathrm{Irr}(N(S))$ for all i, j, we get $k = \ell$ and a permutation σ of $\{1,\dots,k\}$ such that $N_{p_i} = N_{q_{\sigma(i)}}$ and $m_i = n_{\sigma(i)}$. Now $N_{p_i} = N_{q_{\sigma(i)}}$ forces $q_{\sigma(i)} \in p_i P^*$, so $[q_{\sigma(i)}] = [p_i]$. Thus $\overline{P}$ is the free abelian monoid in $\mathrm{Irr}(\overline{P})$.

We clearly have $\beta_c = 1$ whenever $\mathrm{Irr}(\overline{P})$ is finite, as these are in one-to-one correspondence with $\mathrm{Irr}(N(G \rtimes_\theta P))$ and $\zeta(\beta) = \prod_{n \in \mathrm{Irr}(N(G\rtimes_\theta P))}(1-n^{-(\beta-1)})^{-1}$, see Remark 7.4.

For (ii), since $S_c = S^*$ is a group, faithfulness of α amounts to the following: For every $(g,p) \in S^* \setminus \{1\}$ there exists $(h,q) \in S$ such that $h^{-1}g\theta_p(h) \notin \theta_q(G)$. This is equivalent to $\bigcap_{(h,q)\in S} h\theta_q(G)h^{-1} = \{1\}$ and $\bigcap_{(h,q)\in S} h\theta_q(G)\theta_p(h)^{-1} = \emptyset$ for all $p \in P^* \setminus \{1\}$.

For (iii), since $S_c = S^* = G \rtimes_\theta P^*$ is a group, by Remark 3.12, it suffices to look at the fixed points of the map $\alpha_{(h,q)}$ for each element $(h,q) \in S_c \setminus \{1\}$. Since $P^*p \subset pP^*$ for all $p \in P$, the equation (5.2) implies that an equivalence class $[(g,p)] \in S/_\sim$ is fixed by $\alpha_{(h,q)}$ if and only if $g^{-1}h\theta_q(g) \in \theta_p(G)$. Now suppose that $P^* \neq \{1\}$. Fix $q \in P^* \setminus \{1\}$ and take $h = 1$. As $qp = pq'$ for some $q' \in P^*$, we get $\theta_q(\theta_p(G)) = \theta_p(G)$ for all $p \in P$. Thus the element $[(g,p)] \in S/_\sim$ with $g \in \theta_p(G)$ is fixed by $\alpha_{(1,q)}$ for every $p \in P$. By assumption, $\overline{P}$ is infinite as there is at least one $p \in P$ for which θ_p is not an automorphism of G. This tells us that $(1,q)$ violates the almost freeness for the action α. It follows that if α is almost free, then P^* has to be trivial. Now suppose that $P^* = \{1\}$. Then (5.2) allows us to conclude that α is almost free if and only if the set $\{(g,p) \in S \mid h \in g\theta_p(G)g^{-1}\}$ is finite for all $h \in G \setminus \{1\}$. □

Remark 5.12. Let (G,P,θ) be an algebraic dynamical system and $S = G \rtimes_\theta P$. Then the conditions for α to be faithful are closely related to the conditions (2) and (3) in [BS16, Corollary 4.10] which are needed for the simplicity of $\mathcal{Q}(S)$. In addition to that, the requirement $P^*p \subset pP^*$ for all $p \in P$ coming from property (d) in the Proposition 5.11 is needed there as well. This intimate relation between conditions for simplicity of $\mathcal{Q}(S)$ and uniqueness of the KMS$_\beta$-state for $\beta \in [1,\beta_c]$ seems quite intriguing, and may very well deserve a thorough study. To the best of our knowledge, no explanation for this phenomenon is known so far.

We also note that the characterisation of almost freeness of α for the case of trivial P^* we get here is a natural strengthening of the condition for faithfulness of α.

The next example describes a class of algebraic dynamical systems for which $\beta_c = 1$, S has finite propagation, and α is faithful but not almost free.

Example 5.13. Let H be a finite group with $|H| > 2$. Then there exists a non-trivial subgroup $P_0 \subset \mathrm{Aut}(H)$. Consider the standard restricted wreath product $G := H \wr \mathbb{N} = \bigoplus_{n\in\mathbb{N}} H$. Each $p \in P_0$ corresponds to an automorphism of G that commutes with the right shift Σ on G, and we denote by $P := \langle\Sigma\rangle \times P_0$ the corresponding semigroup of endomorphisms of G. Then P is lattice ordered, hence right LCM, and the natural action $\theta\colon P \curvearrowright G$ respects the order so that (G,P,θ) forms an algebraic dynamical system for which we can consider the right LCM semigroup $S = G \rtimes_\theta P$. We note the following features:

(a) (G,P,θ) is of finite type because H is finite.
(b) P^*p equals pP^* for all $p \in P$ since the chosen automorphisms of G commute with Σ.
(c) The action α is faithful.
(d) As P_0 is nontrivial, α is not almost free, see Proposition 5.11.

(e) Take $\mathcal{T} = \bigcup_{n\in\mathbb{N}} \mathcal{T}_n$ with $\mathcal{T}_n := \{((h_k)_{k\geq 0}, \Sigma^n) \in S \mid h_k = 1_H \text{ for all } k > n\}$. This is a natural transversal for $S/_\sim$ that witnesses finite propagation (for all pairs $(s,t) \in S \times S$). Indeed, for $s = (g,p) \in S_c$ we get $|C_s| \leq \ell(g) + 1 < \infty$, where $\ell(g)$ denotes the smallest $n \in \mathbb{N}$ with $h_m = 1_H$ for all $m > n$.

(f) $\beta_c = 1$ as $\mathrm{Irr}(P/_\sim) = \{[\Sigma]\}$.

Thus we conclude that S is an admissible right LCM semigroup that has a unique KMS_1-state by virtue of Theorem 4.3 (2b), while the condition in Theorem 4.3 (2a) does not apply.

As a byproduct of our treatment of algebraic dynamical systems, we recover the results for a dilation matrix from [LRR11]:

Example 5.14 (Dilation matrix). Let $d \in \mathbb{N}$ and $A \in M_d(\mathbb{Z})$ an integer matrix with $|\det A| > 1$, and consider $(G,P,\theta) = (\mathbb{Z}^d, \mathbb{N}, \theta)$ with $\theta_n(m) := A^n m$ for $n \in \mathbb{N}, m \in \mathbb{Z}^d$. Then (G,P,θ) constitutes an algebraic dynamical system that satisfies (a)–(d) from Proposition 5.11 as

$$N_1 = \mathrm{coker}\theta_1 = |\mathbb{Z}^d/\mathrm{im}A| = |\det A| \in \mathbb{N} \setminus \{1\}.$$

Also, we have $\beta_c = 1$ because $\mathrm{Irr}(N(\mathbb{Z}^d \rtimes_\theta \mathbb{N})) = \{|\det A|\}$. Since $\mathbb{Z}^d$ is abelian, the action α is faithful if and only if $\bigcap_{n\in\mathbb{N}} A^n\mathbb{Z}^d = \{0\}$, see Proposition 5.11 (ii). As $P \cong \mathbb{N}$, we deduce that this is also equivalent to almost freeness of α, see Proposition 5.11 (iii).

Example 5.15. Let $A, B \subset \mathbb{N}^\times$ be non-empty disjoint sets such that $A \cup B$ consists of relatively prime numbers. Then the free abelian submonoid P of $\mathbb{N}^\times$ generated by A acts on the ring extension $G := \mathbb{Z}[1/q \mid q \in B]$ via multiplication by injective group endomorphisms θ_p in an order preserving way, that is, (G,P,θ) is an algebraic dynamical system. We observe that $N_p = [G : pG] = p$ (as for $B = \emptyset$) so that the system is of finite type, and it is easy to check that the requirements (a)–(d) for admissibility of $S = G \rtimes_\theta P$ hold. It thus follows from Proposition 5.11 (i) that S is admissible and $\beta_c = 1$ if A is finite. Since G is abelian and P^* is trivial, parts (ii) and (iii) imply that faithfulness and almost freeness of α are both equivalent to A being non-empty. The behaviour of the KMS-state structure on this semigroup resembles very much the one of $\mathbb{N} \rtimes \mathbb{N}^\times$, except for the fact that the parametrising space corresponds to Borel probability measures on the solenoid $\mathbb{T}[1/q \mid q \in B]$ instead of the torus $\mathbb{T}$.

Example 5.16. For a finite field F, consider the polynomial ring $F[t]$ as an additive group G. Via multiplication, every $f \in F[t] \setminus F$ gives rise to an injective, non-surjective group endomorphism θ_f of $F[t]$, whose index is $N_f = |F|^{\deg f}$. For every such f, the semigroup $S := F[t] \rtimes_f \mathbb{N}$ is an admissible right LCM semigroup. By Proposition 5.11, we further have $\beta_c = 1$, and α is almost free, hence faithful. In the simplest case of $f = t$, the semigroup $F[t] \rtimes_f \mathbb{N}$ is canonically isomorphic to $(F \wr \mathbb{N}) \rtimes \mathbb{N}$ from Example 5.13.

6. Algebraic constraints

This section presents some considerations on existence of KMS_β and ground states, and in particular identifies some necessary conditions for existence. Throughout this section, we shall assume that S is a right LCM semigroup that admits a generalised scale $N\colon S \to \mathbb{N}^\times$, and denote by σ the time evolution on $C^*(S)$ given by $\sigma_x(v_s) = N_s^{ix} v_s$ for all $x \in \mathbb{R}$ and $s \in S$.

Proposition 6.1. *There are no* KMS_β*-states on* $C^*(S)$ *for* $\beta < 1$.

Proof. Let $\beta \in \mathbb{R}$ and suppose that ϕ is a KMS_β-state for $C^*(S)$. By Proposition 3.6 (iii), there exists an accurate foundation set F for S with $|F| = n = N_f$ for all $f \in F$ and some $n > 1$. The KMS_β-condition gives that $\phi(v_f v_f^*) = N_f^{-\beta}\phi(v_f^* v_f) = n^{-\beta}$. Hence, $1 = \phi(1) \geq \sum_{f\in F} \phi(e_{fS}) = n^{1-\beta}$, so necessarily $\beta \geq 1$. □

Proposition 6.2. *A state* ϕ *on* $C^*(S)$ *is a ground state if and only if* $\phi(v_s v_t^*) \neq 0$ *for* $s, t \in S$ *implies* $s, t \in S_c$*. In particular, a ground state on* $C^*(S)$ *vanishes outside* $\varphi(C^*(S_c))$.

Proof. Let ϕ be a state on $C^*(S)$ and $y_i := v_{s_i} v_{t_i}^*$, with $s_i, t_i \in S$ for $i = 1, 2$, be generic elements of the spanning family for $C^*(S)$. The expression

$$\phi(y_2\sigma_{x+iy}(y_1)) = \left(\tfrac{N_{s_1}}{N_{t_1}}\right)^{-y+ix} \phi(y_2y_1) \tag{6.1}$$

is bounded on $\{x + iy \mid x \in \mathbb{R}, y > 0\}$ if and only if either $N_{s_1} \geq N_{t_1}$ or $N_{s_1} < N_{t_1}$ and $\phi(y_2y_1) = 0$. Now assume ϕ is a ground state and suppose $\phi(v_s v_t^*) \neq 0$ for some $s, t \in S$. Choosing $s_2 = s, t_1 = t, s_1 = t_2 = 1$ in (6.1), we get $N_{t_1} \leq N_{s_1} = 1$. Thus $N_t = 1$, and hence $t \in S_c$ by Proposition 3.6 (i). Taking adjoints shows that $s \in S_c$ as well.

Conversely, let ϕ be a state on $C^*(S)$ for which $\phi(v_s v_t^*) \neq 0$ implies $s, t \in S_c$. By the Cauchy-Schwartz inequality for y_1 and y_2 we have

$$|\phi(y_2\sigma_{x+iy}(y_1))|^2 = \left(\tfrac{N_{s_1}}{N_{t_1}}\right)^{-2y} |\phi(y_2y_1)|^2 \leq \left(\tfrac{N_{s_1}}{N_{t_1}}\right)^{-2y} \phi(y_2y_2^*)\phi(y_1^*y_1).$$

As $y_1^*y_1 = e_{t_1S}$ and $y_2y_2^* = e_{s_2S}$, the expression on the right-hand side of the inequality vanishes unless $t_1, s_2 \in S_c$. So if it does not vanish we must have $N_{t_1} = 1$. Therefore using that $N_{s_1} \geq 1$, it follows that $|\phi(y_2\sigma_{x+iy}(y_1))|^2 \leq 1$ for $y > 0$. Thus ϕ is a ground state on $C^*(S)$. □

Remark 6.3. For the purposes of the next result we need the following observation: if τ is a trace on $C^*(S_c)$, then $\tau(yw_aw_a^*z) = \tau(yz)$ for all $y, z \in C^*(S_c)$ and $a \in S_c$. The reason is that the image of the isometry w_a in the GNS-representation for τ is a unitary.

Proposition 6.4. *For* $\beta \in [1, \infty)$*, consider the following conditions:*

(i) ϕ *is a* KMS_β*-state.*
(ii) ϕ *is a state on* $C^*(S)$ *such that* $\phi \circ \varphi$ *is a normalised trace on* $C^*(S_c)$ *and for all* $s, t \in S$

$$\phi(v_s v_t^*) = \begin{cases} 0 & \text{if } s \not\sim t, \\ N_s^{-\beta}\phi \circ \varphi(w_{r'}w_r^*) & \text{if } sr = tr' \text{ with } r, r' \in S_c. \end{cases} \tag{6.2}$$

Then (i) implies (ii). Assume moreover that S *is core factorable and* $S_{ci} \subset S$ *is* $\cap$*-closed, and fix maps* c *and* i *as in Lemma 3.8. If* ϕ *is a state on* $C^*(S)$ *such that* $\phi \circ \varphi$ *is a trace, then* (6.2) *is equivalent to*

$$\phi(v_s v_t^*) = \begin{cases} 0 & \text{if } i(s) \neq i(t), \\ N_s^{-\beta}\phi \circ \varphi(w_{c(s)}w_{c(t)}^*) & \text{if } i(s) = i(t), \end{cases} \tag{6.3}$$

and (ii) implies (i).

Proof. First, let ϕ be a KMS$_\beta$-state on $C^*(S)$. Since $\sigma = \mathrm{id}$ on $\varphi(C^*(S_c))$, the state $\phi \circ \varphi$ is tracial. The KMS-condition gives

$$\phi(v_s v_t^*) = N_s^{-\beta}\ \phi(v_t^* e_{tS} e_{sS} v_s) = N_s^{-\beta} N_t^{\beta}\ \phi(v_s v_t^*).$$

Thus $\phi(v_s v_t^*) \neq 0$ implies both $N_s = N_t$ and $s \not\perp t$. Proposition 3.6 (ii) implies that $s \sim t$. Suppose first that $sS \cap tS = saS = tbS$ with $sa = tb$ for some $a, b \in S_c$. Since $e_{saS} \leq e_{sS}$, we have $v_t^* v_s = v_b v_a^*$. Using the KMS$_\beta$ property of ϕ we obtain (6.2) in the case that we may choose a right LCM of s and t to be of the form $sa = tb$. The general case $sr = tr'$ displayed in (6.2) is then of the form $r = ac, r' = bc$ with arbitrary $c \in S_c$. Thus we get $\phi \circ \varphi(w_{r'} w_r^*) = \phi \circ \varphi(w_b w_c w_c^* w_a^*) = \phi \circ \varphi(w_b w_a^*)$ by Remark 6.3.

Now suppose that (A1) and (A2) are satisfied as well, and that ϕ is a state on $C^*(S)$ for which $\phi \circ \varphi$ is tracial. Note that $s \sim t$ happens precisely when $i(s) = i(t)$ for all $s, t \in S$. The equivalence of (6.2) and (6.3) will follow if show that for all $r, r' \in S_c$ with $sr = tr'$ we have $\phi \circ \varphi(w_{r'} w_r^*) = \phi \circ \varphi(w_{c(s)} w_{c(t)}^*)$ if $i(s) = i(t)$. Fix therefore a pair r, r' in S such that $sr = tr'$ and $i(s) = i(t)$. We saw in the first part of the proof that $\phi \circ \varphi(w_{r'} w_r^*) = \phi \circ \varphi(w_b w_a^*)$ whenever a, b in S_c satisfy $sS \cap tS = saS, sa = tb$. By left cancellation and $i(s) = i(t)$, we reduce this to $c(s)S_c \cap c(t)S_c = c(s)aS_c, c(s)a = c(t)b$. But then the trace property of $\phi \circ \varphi$ yields

$$\phi \circ \varphi(w_{c(s)} w_{c(t)}^*) = \phi \circ \varphi(w_{c(t)}^* w_{c(s)}) = \phi \circ \varphi(w_b w_a^*) = \phi \circ \varphi(w_{r'} w_r^*).$$

It remains to prove that if ϕ is a state on $C^*(S)$ for which $\phi \circ \varphi$ is tracial and (6.3) holds, then ϕ is a KMS$_\beta$-state. As the family of analytic elements $\{v_s v_t^* \mid s, t \in S\}$ spans a dense $*$-subalgebra of $C^*(S)$, ϕ is a KMS$_\beta$-state if and only if

$$\phi(v_{s_1} v_{t_1}^* v_{s_2} v_{t_2}^*) = N_{s_1}^{-\beta} N_{t_1}^{\beta} \phi(v_{s_2} v_{t_2}^* v_{s_1} v_{t_1}^*) \quad \text{for all } s_i, t_i \in S, i = 1, 2. \tag{6.4}$$

We will first show that (6.4) is valid for $t_1 = e$, i.e. we show that

$$\phi(v_s v_t v_r^*) = N_s^{-\beta} \phi(v_t v_r^* v_s) \text{ for all } r, s, t \in S. \tag{6.5}$$

Before we prove this identity, we note that (6.3) implies

$$\phi(v_s \varphi(a) v_s^*) = N_s^{-\beta} \phi \circ \varphi(a) \text{ for every } a \in C^*(S_c), \tag{6.6}$$

since $(w_t w_r^*)_{t,r \in S_c}$ form a spanning family for $C^*(S_c)$. Now let $r, s, t \in S$ and compute, using the trace property of $\phi \circ \varphi$ in the second equality, that

$$\begin{aligned} \phi(v_s v_t v_r^*) &\overset{(6.3)}{=} N_{st}^{-\beta}\ \delta_{i(st),i(r)} \phi \circ \varphi(w_{c(si(t))} w_{c(t)} w_{c(r)}^*) \\ &= N_{st}^{-\beta}\ \delta_{i(st),i(r)} \phi \circ \varphi(w_{c(t)} w_{c(r)}^* w_{c(si(t))}) \\ &\overset{(6.6) \text{ for } i(t)}{=} N_s^{-\beta}\ \delta_{i(st),i(r)} \phi(v_t v_{c(r)}^* v_{c(si(t))} v_{i(t)}^*). \end{aligned}$$

Suppose that $i(st) = i(r)$. We claim that $v_{c(si(t))} v_{i(t)}^* = v_{i(r)}^* v_s$. To see this, we first show that $sS \cap i(r)S = si(t)S$. Since $si(t) = i(si(t))c(si(t)) = i(st)c(si(t))$, we have $si(t)S \subseteq sS \cap i(r)S$. For the reverse containment, let ss' be a right LCM for s and $i(r)$. Then by Proposition 3.6 (iv) we have $N_s N_s' = \mathrm{lcm}(N_s, N_{i(r)}) = \mathrm{lcm}(N_s, N_s N_t) = N_s N_t$. Hence $N_s' = N_t$, and by Proposition 3.6 (ii) we know that $i(s') = i(t)$ or $s' \perp t$. We cannot have $s' \perp t$ because this would contradict that $ss' \in i(st)S \subseteq stS$. Hence $i(s') = i(t)$, and so $ss' \in si(t)S$. This gives the reverse containment, and thus we have

$sS \cap i(r)S = si(t)S$. Hence $si(t) = i(si(t))c(si(t)) = i(st)c(si(t)) = i(r)c(si(t))$ is a right LCM for s and $i(r)$, and so gives the claim with the standard argument. Therefore,

$$\phi(v_s v_t v_r^*) = N_s^{-\beta}\ \delta_{i(st),i(r)}\phi(v_t v_{c(r)}^* v_{i(r)}^* v_s) = N_s^{-\beta}\ \delta_{i(st),i(r)}\phi(v_t v_r^* v_s),$$

and to prove (6.5) it remains to show that $\phi(v_t v_r^* v_s) = 0$ if $i(st) \neq i(r)$. Suppose $\phi(v_t v_r^* v_s) \neq 0$. Then $sS \cap rS \neq \emptyset$, so we can assume $v_r^* v_s = v_{s'} v_{r'}^*$ for some $s', r' \in S$ with $rs' = sr'$. According to (6.3), $\phi(v_t v_r^* v_s) = \phi(v_{ts'} v_{r'}^*)$, so our assumption that this is nonzero implies $i(ts') = i(r')$. But then $i(sts') = i(sr') = i(rs')$, and in particular $N_{st} = N_r$ and $st \not\perp r$. Thus, by the definition of the map i, we have $i(st) = i(r)$, showing the desired converse that establishes (6.5). To get $\phi(v_s^* v_t v_r^*) = N_s^{\beta}\phi(v_t v_r^* v_s^*)$, we use the case just established to get

$$\phi(v_t v_r^* v_s^*) = \overline{\phi(v_s v_r v_t^*)} = N_s^{-\beta}\overline{\phi(v_r v_t^* v_s)} = N_s^{-\beta}\phi(v_s^* v_t v_r^*).$$

Now suppose we have $s_1, s_2, t_1, t_2 \in S$. Then $\phi(v_{s_1} v_{t_1}^* v_{s_2} v_{t_2}^*)$ vanishes unless $t_1 S \cap s_2 S = t_1 t_3 S, t_1 t_3 = s_2 s_3$ for some $s_3, t_3 \in S$. In this case, we get

$$\phi(v_{s_1} v_{t_1}^* v_{s_2} v_{t_2}^*) = \phi(v_{s_1} v_{t_3} v_{t_2 s_3}^*) = N_{s_1}^{-\beta}\phi(v_{t_3} v_{t_2 s_3}^* v_{s_1}) = N_{s_1}^{-\beta}\phi(v_{t_1}^* v_{s_2} v_{t_2}^* v_{s_1})$$

by (6.5). Proceeding in the same manner with $v_{t_1}^*$ against $v_{s_2} v_{t_2}^* v_{s_1}$, the term vanishes unless $t_2 S \cap s_1 S \neq \emptyset$, in which case the adjoint version of (6.5) gives

$$\phi(v_{s_1} v_{t_1}^* v_{s_2} v_{t_2}^*) = N_{s_1}^{-\beta}\phi(v_{t_1}^* v_{s_2} v_{t_2}^* v_{s_1}) = N_{s_1}^{-\beta} N_{t_1}^{\beta}\phi(v_{s_2} v_{t_2}^* v_{s_1} v_{t_1}^*).$$

As a final step we remark that an analogous argument with s_2 and t_2 in place of s_1 and t_1 shows that $\phi(v_{s_2} v_{t_2}^* v_{s_1} v_{t_1}^*)$ also vanishes unless $t_1 S \cap s_2 S \neq \emptyset$ and $t_2 S \cap s_1 S \neq \emptyset$. This concludes the proof that every state ϕ on $C^*(S)$ with $\phi \circ \varphi$ tracial and such that (6.3) holds is a KMS$_\beta$-state. □

We continue this section with some results describing which KMS-states factor through the maps appearing in the *boundary quotient diagram* (1.1). Recall that the kernel of the homomorphism π_c is generated by $1 - v_a v_a^*$ with $a \in S_c$.

Proposition 6.5. *Suppose that for each normalised trace τ on $C^*(S_c)$ there exists a* KMS$_\beta$*-state ϕ on $C^*(S)$ with $\phi \circ \varphi = \tau$ for arbitrarily large $\beta \in \mathbb{R}$. Then a ground state ψ on $C^*(S)$ is a* KMS$_\infty$*-state if and only if $\psi \circ \varphi$ is tracial.*

Proof. If ψ is a KMS$_\infty$-state, then $\psi \circ \varphi$ is the weak* limit of normalised traces $\phi_n \circ \varphi$ for some sequence of KMS$_{\beta_n}$-states ϕ_n with $\beta_n \to \infty$, see Proposition 6.4. Hence $\psi \circ \varphi$ is also a normalised trace.

Now let ψ be a ground state on $C^*(S)$ such that $\psi \circ \varphi$ is tracial. By assumption, there is a sequence of KMS$_{\beta_n}$-states ϕ_n on $C^*(S)$ with $\beta_n \to \infty$ and $\phi_n \circ \varphi = \psi \circ \varphi$. Weak* compactness of the state space of $C^*(S)$ yields a subsequence of $(\phi_n)_n$ converging to a KMS$_\infty$-state ϕ' which necessarily satisfies $\phi' \circ \varphi = \psi \circ \varphi$. Therefore ψ and ϕ' agree on $\varphi(C^*(S_c))$, hence $\psi = \phi'$ according to Remark 4.1, and ψ is thus a KMS$_\infty$-state.

□

Proposition 6.6. *A* KMS$_\beta$*-state ϕ on $C^*(S)$ factors through $\pi\colon C^*(S) \to \mathcal{Q}(S)$ if and only if $\beta = 1$. If S is also core factorable, then π can be replaced by $\pi_p\colon C^*(S) \to \mathcal{Q}_p(S)$.*

Proof. Assume that $\phi = \phi' \circ \pi$ for a KMS$_\beta$-state ϕ' on $\mathcal{Q}(S)$. As in the proof of Proposition 6.1, there is an accurate foundation set F with $|F| = N_f = n$ for all $f \in F$ and some $n > 1$. Thus

$$1 = \phi'(1) = \phi'(\sum_{f\in F} \pi(e_{fS})) = \sum_{f\in F} \phi(e_{fS}) = n^{1-\beta}, \tag{6.7}$$

so necessarily $\beta = 1$.

Conversely, suppose that ϕ is a KMS$_1$-state. We must show that $\ker \pi \subset \ker \phi$. Recall from Proposition 3.6 (v) that S has the accurate refinement property from [BS16], thus the kernel of π is generated by differences $1 - \sum_{f\in F} e_{fS}$ for F running over accurate foundation sets. By Proposition 3.6, it suffices to show that $1 - \sum_{f\in F} e_{fS}$ is in $\ker \phi$ for F running over accurate foundation sets that are transversals for $N^{-1}(n)/_\sim$ as $n \in N(S)$. However, for such F, the computations in (6.7) done backwards prove that $\phi(1 - \sum_{f\in F} e_{fS}) = 0$, as needed.

If S is core factorable, then F can be chosen as a subset of S^1_{ci} so that we need to have $\beta = 1$. As $\ker \pi_p \subset \ker \pi$, the second part is clear. □

For the proof of the next result we refer to [Sta17, Theorem 4.1].

Proposition 6.7. *There are no ground states on $\mathcal{Q}(S)$. If S is core factorable, then $\mathcal{Q}_p(S)$ does not admit any ground states.*

7. The reconstruction formula

As in the previous section, we shall assume that S is a right LCM semigroup that admits a generalised scale $N: S \to \mathbb{N}^\times$, and denote by σ the time evolution on $C^*(S)$ given by $\sigma_x(v_s) = N_s^{ix} v_s$ for all $x \in \mathbb{R}$ and $s \in S$. Under the additional assumptions that make S an admissible semigroup, we obtain a formula to reconstruct KMS$_\beta$-states on $C^*(S)$ from conditioning on subsets I of $\mathrm{Irr}(N(S))$ for which $\zeta_I(\beta)$ is finite, see Lemma 7.5 (iv).

Lemma 7.1. *Suppose that S is core factorable and admits a generalised scale N. For $n \in N(S)$ and a transversal $\mathcal{T}_n$ for $N^{-1}(n)/_\sim$ with $\mathcal{T}_n \subset S^1_{ci}$, let $d_n := 1 - \sum_{f\in\mathcal{T}_n} e_{fS}$. Then d_n is a projection in $C^*(S)$, and if $S_{ci} \subset S$ is $\cap$-closed, then d_n is independent of the choice of $\mathcal{T}_n$.*

Proof. For $n = 1$, we have $\mathcal{T}_1 = \{1\}$ and hence $d_1 = 0$. So let $n \in N(S) \setminus \{1\}$. As S is core factorable, there exists a transversal $\mathcal{T}_n$ for $N^{-1}(n)/_\sim$ with $\mathcal{T}_n \subset S_{ci}$. By (A3)(b), $\mathcal{T}_n$ is an accurate foundation set. Therefore $\{e_{fS} \mid f \in \mathcal{T}_n\}$ are mutually orthogonal projections in $C^*(S)$, so d_n is a projection. If $S_{ci} \subset S$ is $\cap$-closed and $\mathcal{T}'_n$ is another transversal for $N^{-1}(n)/_\sim$ contained in S^1_{ci}, then Corollary 3.7 implies that $\sum_{f\in\mathcal{T}_n} e_{fS} = \sum_{f'\in\mathcal{T}'_n} e_{f'S}$. □

Lemma 7.2. *Suppose that S is core factorable, $S_{ci} \subset S$ is $\cap$-closed, and S admits a generalised scale N. Then for each $n \in N(S)$ and $a \in S_c$, the isometry v_a commutes with the projection d_n.*

Proof. Fix $n \in N(S)$ and $t \in S_c$. We can assume $n > 1$. Since v_a is an isometry, the claim will follow once we establish that

$$v_a d_n v_a^* = e_{aS} - \sum_{f \in \mathcal{T}_n} e_{afS} \stackrel{!}{=} e_{aS} - \sum_{f \in \mathcal{T}_n} e_{aS \cap fS} = d_n v_a v_a^*,$$

where the second equality is the one requiring a proof. To do so, observe that since $a \in S_c$ and $\mathcal{T}_n = \{f_1, \dots, f_n\} \subset S_{ci}$, we have $aS \cap f_i S = af_i' S$ for some $f_i' \in S$ with $N_{f_i'} = N_{f_i} = n$. By Lemma 3.5, we have $\mathcal{T}_n' := \{f_1', \dots, f_n'\} \subset S_{ci}$. Moreover, $f_i' \perp f_j'$ for all $i \neq j$ because the same is true of f_i and f_j. Thus $\mathcal{T}_n$ and $\mathcal{T}_n'$ are two transversals for $N^{-1}(n)/_\sim$ that are contained in S_{ci}^1, and so Corollary 3.7 implies that $\sum_{f' \in F_n'} e_{f'S} = \sum_{f \in F_n} e_{fS}$, showing the desired equality. □

The next lemma is crucial to what follows. Roughly, it says the following: for an admissible semigroup, condition (A4) allows that a transversal corresponding to a finite intersection of principal right ideals $f_n S$ with $N_{f_n} = n$ and $n \in \text{Irr}(N(S))$ can be taken in the form of a transversal for the inverse image, under the scale N, of the product of all $n \in N(S)$ that appear in the intersection.

Lemma 7.3. *Let S be an admissible right LCM semigroup and $m, n \in Irr(N(S)), m \neq n$. If $\mathcal{T}_m = \{f_1, \dots, f_m\}$ and $\mathcal{T}_n = \{g_1, \dots, g_n\}$ are transversals for $N^{-1}(m)/_\sim$ and $N^{-1}(n)/_\sim$, respectively, then $f_i S \cap g_j S = h_{i+m(j-1)} S$ for some $h_{i+m(j-1)} \in S$ for all i, j, and $\mathcal{T}_{mn} := \{h_1, \dots, h_{mn}\}$ is a transversal for $N^{-1}(mn)/_\sim$.*

Proof. By (A3)(b), $\mathcal{T}_m$ and $\mathcal{T}_n$ are accurate foundation sets. Let $\mathcal{T}_{mn}$ consist of the $h_{i+m(j-1)}$ for which $f_i \not\perp g_j$, where $i = 1, \dots, m$ and $j = 1, \dots, n$. We will show that $f_i \not\perp g_j$ for all i and j. Clearly $\mathcal{T}_{mn}$ is an accurate foundation set. Since the generalised scale N is a homomorphism of right LCM semigroups by Proposition 3.6 (iv) and m and n are distinct irreducibles in $N(S)$, we have $\mathcal{T}_{mn} \subset N^{-1}(mn)$. Then Proposition 3.6 (iii) implies that $\mathcal{T}_{mn}$ must be a transversal for $N^{-1}(mn)/_\sim$. Thus, $|\mathcal{T}_{mn}| = mn$ by (A3)(a) which gives the claim. □

Remark 7.4. We record another consequence of condition (A4). Suppose $I \subset \text{Irr}(N(S))$. We claim that the I-restricted ζ-function ζ_I from Definition 4.2 admits a product description $\zeta_I(\beta) = \prod_{n \in I}(1 - n^{-(\beta-1)})^{-1}$. Indeed, all summands in the series defining $\zeta_I(\beta)$ are positive, so we are free to rearrange their order. Condition (A4) implies that the submonoid $\langle I \rangle$ of $\mathbb{N}^\times$ is the free abelian monoid in I, and therefore

$$\zeta_I(\beta) = \sum_{n \in \langle I \rangle} n^{-(\beta-1)} = \prod_{n \in I} (1 - n^{-(\beta-1)})^{-1}.$$

The next result is an abstract version of [LR10, Lemma 10.1], proved for the semigroup C^*-algebra of $\mathbb{N} \rtimes \mathbb{N}^\times$. Laca and Raeburn had anticipated back then that the result is "likely to be useful elsewhere". Here we confirm this point by showing that the techniques of proof, when properly updated, are valid in the abstract setting of our admissible semigroups.

We recall that the $*$-homomorphism $\varphi : C^*(S_c) \to C^*(S)$ was defined in Remark 4.1. In the following, we let $\mathcal{T}$ be a transversal for $S/_\sim$ with $\mathcal{T} \subset S_{ci}^1$ and $\mathcal{T}_I := \bigcup_{n \in \langle I \rangle} \mathcal{T}_n = N^{-1}(\langle I \rangle) \cap \mathcal{T}$ for $I \subset \text{Irr}(N(S))$.

Lemma 7.5. *Let S be an admissible right LCM semigroup. If ϕ is a KMS_β-state on $C^*(S)$ for some $\beta \in \mathbb{R}$, denote by $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ its GNS-representation and by $\tilde{\phi} = \langle \cdot\ \xi_\phi, \xi_\phi\rangle$ the vector state extension of ϕ to $\mathcal{L}(\mathcal{H}_\phi)$. For every subset I of $\mathrm{Irr}(N(S))$, $Q_I := \prod_{n\in I} \pi_\phi(d_n)$ defines a projection in $\pi_\phi(C^*(S))''$. If $\zeta_I(\beta) < \infty$, then Q_I has the following properties:*

(i) $\tilde{\phi}(Q_I) = \zeta_I(\beta)^{-1}$.

(ii) The map $y \mapsto \zeta_I(\beta)\ \tilde{\phi}(Q_I\pi_\phi(y)Q_I)$ defines a state ϕ_I on $C^*(S)$ for which $\phi_I \circ \varphi$ is a trace on $C^*(S_c)$.

(iii) The family $(\pi_\phi(v_s)Q_I\pi_\phi(v_s^*))_{s\in\mathcal{T}_I}$ consists of mutually orthogonal projections, and $Q^I := \sum_{s\in\mathcal{T}_I} \pi_\phi(v_s)Q_I\pi_\phi(v_s^*)$ defines a projection such that $\tilde{\phi}(Q^I) = 1$.

(iv) There is a reconstruction formula for ϕ given by

$$\phi(y) = \zeta_I(\beta)^{-1} \sum_{s\in\mathcal{T}_I} N_s^{-\beta}\ \phi_I(v_s^*yv_s) \quad \text{for all } y \in C^*(S).$$

Proof. If I is finite, then Q_I belongs to $\pi_\phi(C^*(S))$. If I is infinite, then we obtain Q_I as a weak limit of projections Q_{I_n} with $|I_n| < \infty$ and $I_n \nearrow I$ because the d_n commute. Hence Q_I belongs to the double commutant of $\pi_\phi(C^*(S))$ inside $\mathcal{L}(\mathcal{H}_\phi)$.

Let us first assume that I is finite. An iterative use of Lemma 7.3 then implies

$$\prod_{n\in I} d_n = \sum_{A\subset I}(-1)^{|A|} \sum_{(f_1,\dots,f_{|A|})\in \prod_{n\in A}\mathcal{T}_n} e_{\cap_{k=1}^{|A|} f_kS} = \sum_{A\subset I}(-1)^{|A|} \sum_{f\in\mathcal{T}_{n(A)}} e_{fS}. \tag{7.1}$$

Applying $\tilde{\phi}\circ\pi_\phi = \phi$ to (7.1), using that ϕ is a KMS_β-state on $C^*(S)$ as in the last equality of (6.7) and invoking the product formula from Remark 7.4 yields

$$\tilde{\phi}(Q_I) = \sum_{A\subset I}(-1)^{|A|}{n_A}^{-(\beta-1)} = \prod_{n\in I}(1-n^{-(\beta-1)}) = \zeta_I(\beta)^{-1}.$$

This proves part (i) in the case that I is finite. The general case follows from the finite case by taking limits $I_n \nearrow I$ with I_n finite, using continuity as $\tilde{\phi}$ is a vector state and the fact that Q_I is the weak limit of $(Q_{I_n})_{n\geq 1}$.

For (ii), we note that ϕ_I is a state on $C^*(S)$ by i), and it remains to prove that it is tracial on $\varphi(C^*(S_c))$. Let I be finite and $r_i \in S_c, i = 1,\dots,4$. According to Lemma 7.2 and the fact that ϕ is a KMS_β-state (keeping in mind that $N_{r_1} = N_{r_2} = 1$), we have

$$\begin{aligned}
\phi_I(v_{r_1}v_{r_2}^*v_{r_3}v_{r_4}^*) &= \zeta_I(\beta)\ \phi(\prod_{m\in I} d_m\ v_{r_1}v_{r_2}^*v_{r_3}v_{r_4}^* \prod_{n\in I} d_n)\\
&= \zeta_I(\beta)\ \phi(v_{r_1}v_{r_2}^* \prod_{m\in I} d_m\ v_{r_3}v_{r_4}^* \prod_{n\in I} d_n)\\
&= \zeta_I(\beta)\ \phi(\prod_{m\in I} d_m\ v_{r_3}v_{r_4}^* \prod_{n\in I} d_n\ v_{r_1}v_{r_2}^*)\\
&= \zeta_I(\beta)\ \phi(\prod_{m\in I} d_m\ v_{r_3}v_{r_4}^*\ v_{r_1}v_{r_2}^* \prod_{n\in I} d_n) = \phi_I(v_{r_3}v_{r_4}^*v_{r_1}v_{r_2}^*).
\end{aligned}$$

For general I, the state ϕ_I is the weak*-limit of ϕ_{I_n} where I_n is finite and $I_n \nearrow I$, so that the trace property is preserved. This completes (ii).

Towards proving (iii), note first that by Lemma 7.1 the family of projections appearing in the assertion is independent of the choice of the transversals as n varies in $\langle I\rangle$. Let $s \neq t$ with s in a transversal $\mathcal{T}_m$ for $N^{-1}(m)/_\sim$ and t in a transversal $\mathcal{T}_n$ for $N^{-1}(n)/_\sim$ for some $m,n \in \langle I\rangle$. If $m = n$, then s and t are distinct elements belonging to a transversal that is also a (proper) accurate foundation set, so $s \perp t$, which implies $v_s^* v_t = 0$ and therefore shows that the projections $\pi_\phi(v_s)Q_I\pi_\phi(v_s^*)$ and $\pi_\phi(v_t)Q_I\pi_\phi(v_t^*)$ are orthogonal. Suppose $m \neq n$. Now either $s \perp t$, which implies the claimed orthogonality as above, or s and t have a right LCM in S. In the latter case, using that N is a homomorphism of right LCM semigroups, it follows that there are $s', t' \in S$ with $ss' = tt'$ and $N_{s'} = m^{-1}\mathrm{lcm}(m,n), N_{t'} = n^{-1}\mathrm{lcm}(m,n)$. Since $s,t \in S_{ci}$, $sS \cap tS \neq \emptyset$ and $S_{ci} \subset S$ is $\cap$-closed, also $ss', tt' \in S_{ci}$. This forces $s', t' \in S_{ci}$. The assumption $m \neq n$ implies that $N_{s'} > 1$ or $N_{t'} > 1$. Suppose $N_{s'} > 1$. Note $N_{s'} \in \langle I\rangle$ because this monoid is free abelian by (A4). Take $n' \in I$ such that $N_{s'}$ is a multiple of n', and pick a corresponding proper accurate foundation set $F_{n'}$. Then $fS \cap s'S \neq \emptyset$ for some $f \in F_{n'}$. Using that $S_{ci} \subset S$ is $\cap$-closed, we can write $fS \cap s'S = s''S$ for some $s'' \in S_{ci}$ for which $N_{s''} = \mathrm{lcm}(N_{s'}, n') = N_{s'}$. Then s' and s'' differ by multiplication on the right with an element of S^* according to Corollary 3.7, so $fS \cap s'S = s'S$. Moreover, there is a unique $f \in F_{n'}$ with this property because $F_{n'}$ is accurate. Now $s'S \subset fS$ implies that $d_{n'}e_{s'S} = 0$, and this in turn yields

$$Q_I\pi_\phi(v_s^*v_t)Q_I = Q_{I\setminus\{n'\}}\pi_\phi(d_{n'}e_{s'S}v_{s'}v_{t'}^*)Q_I = 0,$$

from which the desired orthogonality follows. The case $N_{t'} > 1$ is analogous. Therefore, we get mutually orthogonal projections whenever $s \in \mathcal{T}_m$ and $t \in \mathcal{T}_n$ are distinct. In particular, Q^I is a projection in $\pi_\phi(C^*(S))''$. For a finite set I, an application of the KMS$_\beta$-condition at the second equality shows that

$$\tilde\phi(Q^I) = \sum_{s\in\mathcal{T}_I} \phi(v_s \prod_{m\in I} d_m\, v_s^*) = \sum_{s\in\mathcal{T}_I} N_s^{-\beta}\tilde\phi(Q_I) = \tilde\phi(Q_I)\zeta_I(\beta).$$

The last term is 1 by part (i), and (iii) follows. For arbitrary I, we invoke continuity as was done in the proof of part i).

Finally, to prove (iv), note that $\tilde\phi(Q^I) = 1$ implies $\tilde\phi(\pi_\phi(y)) = \tilde\phi(Q^I\pi_\phi(y)Q^I)$ for all $y \in C^*(S)$. For finite I, appealing to (iii) and using the KMS$_\beta$-condition three times, we obtain

$$\begin{aligned}
\phi(y) \;=\; \tilde\phi(\pi_\phi(y)) = \tilde\phi(Q^I\pi_\phi(y)Q^I) &= \sum_{s,t\in\mathcal{T}_I} \phi(v_s \prod_{k\in I} d_k\, v_s^* y v_t \prod_{\ell\in I} d_\ell\, v_t^*)\\
&= \sum_{s,t\in\mathcal{T}_I} N_s^{-\beta}\, \phi(\prod_{k\in I} d_k\, v_s^* y v_t \prod_{\ell\in I} d_\ell\, v_t^* v_s \prod_{k'\in I} d_{k'})\\
&= \sum_{s,t\in\mathcal{T}_I} \tilde\phi(Q_I\pi_\phi(v_s^*yv_t)Q_I\pi_\phi(v_t^*v_s)Q_I)\\
&= \zeta_I(\beta)^{-1}\sum_{s\in\mathcal{T}_I} N_s^{-\beta}\phi_I(v_s^*yv_s).
\end{aligned}$$

This proves the reconstruction formula in the case of finite I. For general $I \subset \mathrm{Irr}(N(S))$ the claim follows by continuity arguments. □

8. Construction of KMS-states

In this section we show that the constraints obtained in Section 6 are optimal. To begin with, we assume that S is a core factorable right LCM semigroup for which $S_{ci} \subset S$ is $\cap$-closed. We then fix a transversal $\mathcal{T}$ for $S/_\sim$ with $\mathcal{T} \subset S_{ci}^1$ according to Lemma 3.8; thus $\mathcal{T}$ gives rise to maps $i\colon S \to \mathcal{T}$ and $c\colon S \to S_c$ determined by $s = i(s)c(s)$ for all

$s \in S$. Recall from Lemma 3.9 that we denote by α both the action $S_c \curvearrowright S/_\sim$ and the corresponding action $S_c \curvearrowright \mathcal{T}$. Recall further from Remark 4.1 that the standard generating isometries for $C^*(S_c)$ are w_a for $a \in S_c$.

Remark 8.1. Consider $C^*(S_c)$ as a right Hilbert module over itself with inner product $\langle y, z\rangle = y^*z$ for $y, z \in C^*(S_c)$. For every $a \in S_c$, the map $y \mapsto w_a y$ defines an isometric endomorphism of $C^*(S_c)$ as a right Hilbert $C^*(S_c)$ module. If $a \in S^*$, then the map is an isometric isomorphism. Indeed, the map is linear and compatible with the right module structures. It also preserves the inner product as w_a is an isometry, and hence $\langle w_a y, w_a z\rangle = y^* w_a^* w_a z = y^*z$. Thus, the map is isometric. If $a \in S^*$, then the map is surjective because $w_a(w_a^* y) = y$ for all $y \in C^*(S_c)$.

We start with a brief lemma that allows us to recover t and $c(st)$ from s and $i(st)$ for $s \in S$ and $t \in \mathcal{T}$, followed by a technical lemma that makes the proof of Theorem 8.4 more accessible. Both results are further elaborations on the factorisation arising from Lemma 3.8.

Lemma 8.2. *For $s \in S$, the map $\mathcal{T} \to \mathcal{T}, t \mapsto i(st)$ is injective on $\mathcal{T}$, and there is a bijection $i(s\mathcal{T}) \to \{(t, c(st)) \mid t \in \mathcal{T}\}$.*

Proof. Suppose $s \in S, t_1, t_2 \in \mathcal{T}$ satisfy $i(st_1) = i(st_2)$. Then $st_1 \sim st_2$, and left cancellation implies $t_1 \sim t_2$, so that $t_1 = t_2$ as $t_1, t_2 \in \mathcal{T}$. Thus, we can recover $t \in \mathcal{T}$ from $i(st)$, and as $st = i(st)c(st)$, we then recover $c(st)$. □

Lemma 8.3. *Suppose there are $s_1, \dots, s_4 \in S, t_1, t_2, r \in \mathcal{T}$ such that $s_1S \cap s_2S = s_1s_3S$, $s_1s_3 = s_2s_4$, $t_1 = i(s_4r)$, and $t_2 = i(s_3r)$. Then $c(s_1t_2)c(s_3r) = c(s_2t_1)c(s_4r)$ and $c(s_1t_2)S_c \cap c(s_2t_1)S_c = c(s_1t_2)c(s_3r)S_c$.*

Proof. The equation

$$s_1s_3r = s_1t_2c(s_3r) = i(s_1t_2)c(s_1t_2)c(s_3r) = i(s_2t_1)c(s_2t_1)c(s_4r) = s_2t_1c(s_4r) = s_2s_4r$$

yields $i(s_1t_2) \sim i(s_2t_1)$, and hence $i(s_1t_2) = i(s_2t_1)$. Thus left cancellation in S gives $c(s_1t_2)c(s_3r) = c(s_2t_1)c(s_4r)$. For the second part of the claim, the first part implies $c(s_1t_2)S_c \cap c(s_2t_1)S_c \supset c(s_1t_2)c(s_3r)S_c$. Moreover, it suffices to show $c(s_1t_2)S \cap c(s_2t_1)S \subset c(s_1t_2)c(s_3r)S$ since the embedding of S_c into S is a homomorphism of right LCM semigroups, see Proposition 3.4. So suppose $c(s_1t_2)S \cap c(s_2t_1)S = c(s_1t_2)aS, c(s_1t_2)a = c(s_2t_1)b$ for some $a, b \in S_c$. Then $c(s_3r) = ad$ for some $d \in S_c$, and we need to show that $d \in S^*$. As

$$s_1t_2a = i(s_1t_2)c(s_1t_2)a = i(s_2t_1)c(s_2t_1)b = s_2t_1b \in s_1S \cap s_2S = s_1s_3S,$$

left cancellation forces $t_2a \in s_3S$, i.e. $t_2a = s_3r'$ for some $r' \in S$. Therefore, we get

$$s_3r = t_2c(s_3r) = t_2ad = s_3r'd.$$

By left cancellation, this yields $r = i(r')c(r')d$. But now $r \in \mathcal{T}$, so $i(r') = r$, and hence $1 = i(r')d$, i.e. $d \in S^*$. □

The next result is inspired by [LRRW14, Propositions 3.1 and 3.2], where the authors construct a representation of the Toeplitz algebra for a Hilbert bimodule associated to a self-similar action. We recall that when X is a right-Hilbert module over a C^*-algebra A and π is a representation of A on a Hilbert space H, then $X \otimes_A H$ is the Hilbert space obtained as the completion of the algebraic tensor product with respect to the inner-product characterised by $\langle x\otimes h, y\otimes k\rangle = \langle \pi(\langle y,x\rangle)h,k\rangle$, and where the tensor product is balanced over the left action on H given by π, see for instance [RW98]. If X is a direct sum Hilbert module $\oplus_{j\in J}A$ of the standard Hilbert module A, then $X\otimes_A H$ can be identified with the infinite direct sum of Hilbert spaces $A\otimes_A H$. In case A has an identity 1, then with ξ_j denoting the image of $1\in A$ in the j-th copy of A, we may identify $\xi_j a\otimes h$ for $a\in A$ as the element in $\oplus_{j\in J}(A\otimes_A H)$ that has $a\otimes h$ in the j-th place and zero elsewhere. From now on, we will freely use this identification.

Theorem 8.4. *Let S be a core factorable right LCM semigroup such that $S_{ci}\subset S$ is $\cap$-closed. Let $M := \bigoplus_{t\in\mathcal{T}} C^*(S_c)$ be the direct sum of the standard right Hilbert $C^*(S_c)$-module. Suppose ρ is a state on $C^*(S_c)$ with GNS-representation $(\pi'_\rho,\mathcal{H}'_\rho,\xi_\rho)$. Form the Hilbert space $\mathcal{H}_\rho := M\otimes_{C^*(S_c)}\mathcal{H}'_\rho$. The linear operator $V_s\colon \mathcal{H}_\rho\to\mathcal{H}_\rho$ given by*

$$V_s(\xi_t y\otimes\xi_\rho) := \xi_{i(st)}w_{c(st)}y\otimes\xi_\rho$$

for $t\in\mathcal{T}, y\in C^*(S_c)$ is adjointable for all $s\in S$. The family $(V_s)_{s\in S}$ gives rise to a representation $\pi_\rho\colon C^*(S)\to\mathcal{L}(\mathcal{H}_\rho)$.

Proof. It is clear that $\mathcal{H}_\rho$ is a Hilbert space, so we start by showing that the operator V_s is adjointable. For $s\in S$ and $t_1,t_2\in\mathcal{T}, y,z\in C^*(S_c)$, we have

$$\begin{aligned}\langle V_s(\xi_{t_1}y\otimes\xi_\rho),\xi_{t_2}z\otimes\xi_\rho\rangle &= \langle\xi_\rho,\pi'_\rho(\langle\xi_{i(st_1)}w_{c(st_1)}y,\xi_{t_2}z\rangle)\xi_\rho\rangle\\ &= \langle\xi_\rho,\pi'_\rho(\delta_{i(st_1),t_2}y^*w^*_{c(st_1)}z)\xi_\rho\rangle\\ &= \delta_{i(st_1),t_2}\ \rho(z^*w_{c(st_1)}y).\end{aligned}$$

Define the linear operator V'_s on $\mathcal{H}_\rho$ by

$$V'_s(\xi_t y\otimes\xi_\rho) := \chi_{i(s\mathcal{T})}(t)\xi_{t'}w^*_{c(st')}y\otimes\xi_\rho,$$

where $t'\in\mathcal{T}$ is uniquely determined by $i(st')=t$ in the case where $t\in i(s\mathcal{T})$, see Lemma 8.2. It is then straightforward to show that

$$\langle\xi_{t_1}y\otimes\xi_\rho, V'_s(\xi_{t_2}z\otimes\xi_\rho)\rangle = \delta_{i(st_1),t_2}\ \rho(z^*w_{c(st_1)}y),$$

so that V_s is adjointable with $V_s^*=V'_s$. It is also clear that each V_s is an isometry. Now let $s_1,s_2\in S$. To show that $V_{s_1}V_{s_2}=V_{s_1s_2}$, note that when computing $V_{s_1}V_{s_2}(\xi_t y\otimes\xi_\rho)$ for some $t\in\mathcal{T}$ we encounter the term $\xi_{i(s_1i(s_2t))}w_{c(s_1i(s_2t))}w_{c(s_2t)}y$, which is the same as $\xi_{i(s_1s_2t)}w_{c(s_1s_2t)}y$ entering $V_{s_1s_2}(\xi_t y\otimes\xi_\rho)$ because

$$i(s_1i(s_2t))c(s_1i(s_2t))c(s_2t) = s_1i(s_2t)c(s_2t) = s_1s_2t = i(s_1s_2t)c(s_1s_2t)$$

by Lemma 8.2. So it remains to prove that we have

$$V_{s_1}^*V_{s_2} = \begin{cases} V_{s_3}V_{s_4}^* & \text{if } s_1S\cap s_2S = s_1s_3S, s_1s_3=s_2s_4,\\ 0 & \text{if } s_1\perp s_2.\end{cases}$$

We start by observing that since

$$\begin{aligned}\langle V_{s_1}^*V_{s_2}(\xi_{t_1}y\otimes\xi_\rho),\xi_{t_2}z\otimes\xi_\rho\rangle &= \langle V_{s_2}(\xi_{t_1}y\otimes\xi_\rho),V_{s_1}\xi_{t_2}z\otimes\xi_\rho\rangle\\ &= \langle\xi_\rho,\pi'_\rho(\langle\xi_{i(s_2t_1)}w_{c(s_2t_1)}y,\xi_{i(s_1t_2)}w_{c(s_1t_2)}z\rangle)\xi_\rho\rangle\\ &= \delta_{i(s_2t_1),i(s_1t_2)}\ \rho(z^*w^*_{c(s_1t_2)}w_{c(s_2t_1)}y),\end{aligned}\tag{8.1}$$

the inner product on the left can only be nonzero if $s_1 \not\perp s_2$. Indeed, if $s_1 \perp s_2$, then $i(s_2t_1) \neq i(s_1t_2)$ for all $t_1, t_2 \in \mathcal{T}$ as $c(s_2t_1), c(s_1t_2) \in S_c$. So suppose there are $s_3, s_4 \in S$ such that $s_1S \cap s_2S = s_1s_3S, s_1s_3 = s_2s_4$. If $t_1, t_2 \in \mathcal{T}$ and $y, z \in C^*(S_c)$ are such that

$$\langle V_{s_3}V_{s_4}^*(\xi_{t_1}y \otimes \xi_\rho), \xi_{t_2}z \otimes \xi_\rho\rangle \neq 0,$$

then there must be $t_3 \in \mathcal{T}$ with the property that $t_1 = i(s_4t_3)$ and $t_2 = i(s_3t_3)$. In this case, we have

$$\langle V_{s_3}V_{s_4}^*(\xi_{t_1}y \otimes \xi_\rho), \xi_{t_2}z \otimes \xi_\rho\rangle = \rho(z^* w_{c(s_3t_3)} w_{c(s_4t_3)}^* y).$$

In addition, we get $s_2t_1c(s_4t_3) = s_2s_4t_3 = s_1s_3t_3 = s_1t_2c(s_3t_3)$ so that $i(s_2t_1) = i(s_1t_2)$, which by (8.1) also determines when $V_{s_1}^*V_{s_2}$ is nonzero. Now $w_{c(s_1t_2)}^* w_{c(s_2t_1)} = w_{r_1}w_{r_2}^*$ for all $r_1, r_2 \in S_c$ satisfying $c(s_1t_2)S_c \cap c(s_2t_1)S_c = c(s_1t_2)r_1S_c$ and $c(s_1t_2)r_1 = c(s_2t_1)r_2$. Due to Lemma 8.3, $c(s_3t_3)$ and $c(s_4t_3)$ have this property. Thus we conclude that $V_{s_1}^*V_{s_2} = V_{s_3}V_{s_4}^*$ for $s_1 \not\perp s_2$. This shows that $(V_s)_{s\in S}$ defines a representation π_ρ of $C^*(S)$ by the universal property of $C^*(S)$. □

From now on we assume that S is a core factorable right LCM semigroup, $S_{ci} \subset S$ is $\cap$-closed, and S admits a generalised scale.

Lemma 8.5. *For every state ρ on $C^*(S_c)$ the following assertions hold:*

(i) If $s, r \in S$ are such that $\langle V_s\, V_r^*(\xi_t \otimes \xi_\rho), \xi_t \otimes \xi_\rho\rangle \neq 0$ for some $t \in \mathcal{T}$, then $s \sim r$.
(ii) Given $a, b \in S_c$ and $t \in \mathcal{T}$, then for all $y, z \in C^*(S)$ we have

$$\langle V_a\, V_b^*(\xi_t y \otimes \xi_\rho), \xi_t z \otimes \xi_\rho\rangle = \chi_{Fix(\alpha_a\alpha_b^{-1})}(t)\ \rho(z^* w_{c(a\alpha_b^{-1}(t))} w_{c(b\alpha_b^{-1}(t))}^* y).$$

Proof. For (i), suppose r, s and t are given as in the first assertion of the lemma. Then there is $t' \in \mathcal{T}$ with $i(st') = t = i(rt')$. Thus we have $st' \sim rt'$. Hence, $N_s = N_r$ and $sS \cap rS \neq \emptyset$. By Proposition 3.6 (ii), we have $s \sim r$.

To prove (ii), note that straightforward calculations using Lemma 3.9 show that

$$V_a(\xi_t y \otimes \xi_\rho) = \xi_{\alpha_a(t)} w_{c(a\alpha_a(t))} y \otimes \xi_\rho \text{ and } V_b^*(\xi_t y \otimes \xi_\rho) = \xi_{\alpha_b^{-1}(t)}(w_{c(b\alpha_b^{-1}(t))}^* y \otimes \xi_\rho).$$

This immediately gives the claim on $V_a\, V_b^*$. □

Proposition 8.6. *For every state ρ on $C^*(S_c)$, the map*

$$\psi_\rho(y) := \langle \pi_\rho(y)(\xi_1 \otimes \xi_\rho), \xi_1 \otimes \xi_\rho\rangle$$

defines a ground state on $C^*(S)$ with $\psi_\rho \circ \varphi = \rho$.

Proof. The map ψ_ρ is a state on $C^*(S)$ with $\psi_\rho \circ \varphi = \rho$, see Theorem 8.4. By Lemma 8.5, we see that

$$\langle V_t^*(\xi_1 \otimes \xi_\rho), V_s^*(\xi_1 \otimes \xi_\rho)\rangle \neq 0$$

forces $1 \in i(s\mathcal{T}) \cap i(t\mathcal{T})$ for $s, t \in S$. This implies $s \sim 1 \sim t$, and hence $s, t \in S_c$, see Lemma 3.2. Thus ψ_ρ is a ground state by virtue of Proposition 6.2. □

Before we construct KMS$_\beta$-states on $C^*(S)$ from traces τ on $C^*(S_c)$ with the help of the representation π_τ, we make note of an intermediate step which produces states on $C^*(S)$ from traces on $C^*(S_c)$. In terms of notation, note that the finite subsets of $\mathrm{Irr}(N(S))$ form a (countable) directed set when ordered by inclusion.

Lemma 8.7. *Let $\beta \geq 1$, τ a trace on $C^*(S_c)$, and $I \subset Irr(N(S))$ with $\zeta_I(\beta) < \infty$. Then*

$$\psi_{\beta,\tau,I}(y) \ := \ \zeta_I(\beta)^{-1} \sum_{t \in \mathcal{T}_I} N_t^{-\beta} \langle \pi_\tau(y)(\xi_t \otimes \xi_\tau), \xi_t \otimes \xi_\tau \rangle$$

defines a state on $C^*(S)$ such that $\psi_{\beta,\tau,I} \circ \varphi$ is tracial on $C^*(S_c)$.

Proof. From the construction in Theorem 8.4, it is clear that $\psi_{\beta,\tau,I}$ is a state on $C^*(S)$. We claim that

$$\sum_{t \in \mathcal{T}_n} \langle V_a V_b^* V_d V_e^* (\xi_t \otimes \xi_\tau), \xi_t \otimes \xi_\tau \rangle = \sum_{t \in \mathcal{T}_n} \langle V_d V_e^* V_a V_b^* (\xi_t \otimes \xi_\tau), \xi_t \otimes \xi_\tau \rangle \tag{8.2}$$

holds for all $a, b, d, e \in S_c$ and $n \in \langle I \rangle$. Since $bS \cap dS \neq \emptyset$, Lemma 8.5 (ii) implies that

$$\langle V_a V_b^* V_d V_e^* (\xi_t \otimes \xi_\tau), \xi_t \otimes \xi_\tau \rangle$$

vanishes for $t \in \mathcal{T}_n$, unless t belongs to

$$\mathcal{T}_n^{(a,b),(d,e)} := \mathcal{T}_n \cap \mathrm{Fix}(\alpha_a \alpha_b^{-1} \alpha_d \alpha_e^{-1});$$

just write $bb' = dd'$ and invoke the inner product formula for $V_{ab'} V_{ed'}^*$ to decide when it vanishes. The bijection $f := \alpha_d \alpha_e^{-1}$ of $\mathcal{T}$ restricts to a bijection from $\mathcal{T}_n^{(a,b),(d,e)}$ to $\mathcal{T}_n^{(d,e),(a,b)}$ as $t \in \mathcal{T}_n^{(a,b),(d,e)}$ implies

$$f(t) = \alpha_d \alpha_e^{-1}(\alpha_a \alpha_b^{-1} \alpha_d \alpha_e^{-1}(t)) = \alpha_d \alpha_e^{-1} \alpha_a \alpha_b^{-1}(f(t)) \in \mathcal{T}_n^{(d,e),(a,b)}.$$

For $t \in \mathcal{T}_n^{(a,b),(d,e)}$, Lemma 8.5 (ii) yields

$$\langle V_a V_b^* V_d V_e^* (\xi_t \otimes \xi_\tau), \xi_t \otimes \xi_\tau \rangle = \tau(w_{c(ar)} w_{c(br)}^* w_{c(ds)} w_{c(es)}^*),$$

where $s, r \in \mathcal{T}_n$ are given by $s = \alpha_e^{-1}(t)$ and $r = \alpha_b^{-1} \alpha_d \alpha_e^{-1}(t) = \alpha_a^{-1}(t)$. Likewise, we have

$$\langle V_d V_e^* V_a V_b^* (\xi_{f(t)} \otimes \xi_\tau), \xi_{f(t)} \otimes \xi_\tau \rangle = \tau(w_{c(ds')} w_{c(es')}^* w_{c(ar')} w_{c(br')}^*),$$

where $s', r' \in \mathcal{T}_n$ are given by

$$r' = \alpha_b^{-1} f(t) = \alpha_b^{-1} \alpha_d \alpha_e^{-1}(t) = \alpha_a^{-1}(t) = r,$$

and

$$s' = \alpha_e^{-1} \alpha_a \alpha_b^{-1} f(t) = \alpha_e^{-1} \alpha_a \alpha_b^{-1} \alpha_d \alpha_e^{-1}(t) = \alpha_e^{-1}(t) = s.$$

Since τ is a trace, we thus have

$$\begin{aligned}
\langle V_d V_e^* V_a V_b^* (\xi_{f(t)} \otimes \xi_\tau), \xi_{f(t)} \otimes \xi_\tau \rangle &= \tau(w_{c(ds)} w_{c(es)}^* w_{c(ar)} w_{c(br)}^*) \\
&= \tau(w_{c(ar)} w_{c(br)}^* w_{c(ds)} w_{c(es)}^*) \\
&= \langle V_a V_b^* V_d V_e^* (\xi_t \otimes \xi_\tau), \xi_t \otimes \xi_\tau \rangle
\end{aligned}$$

as claimed. This establishes (8.2), and hence that $\psi_{\beta,\tau,I} \circ \varphi$ is a trace on $C^*(S_c)$. □

Proposition 8.8. *For $\beta \geq 1$ and every trace τ on $C^*(S_c)$, there is a sequence $(I_k)_{k \geq 1}$ of finite subsets of $Irr(N(S))$ such that ψ_{β,τ,I_k} weak*-converges to a KMS_β-state $\psi_{\beta,\tau}$ on $C^*(S)$ as $I_k \nearrow Irr(N(S))$. For $\beta \in (\beta_c, \infty)$, the state $\psi_{\beta,\tau}$ is given by $\psi_{\beta,\tau,IrrN(S)}$.*

Proof. Due to weak*-compactness of the state space on $C^*(S)$, there is a sequence $(I_k)_{k\geq 1}$ of finite subsets of $\mathrm{Irr}(N(S))$ with $I_k \nearrow \mathrm{Irr}(N(S))$ such that $(\psi_{\beta,\tau,I_k})_{k\geq 1}$ obtained from Lemma 8.7 converges to some state $\psi_{\beta,\tau}$ in the weak* topology. By Proposition 6.4, $\psi_{\beta,\tau}$ is a KMS_β-state if and only if $\psi_{\beta,\tau}\circ\varphi$ defines a trace on $C^*(S_c)$ and (6.3) holds. Note that $\psi_{\beta,\tau}\circ\varphi$ is tracial on $C^*(S_c)$ because each of the ψ_{β,τ,I_k} has this property by Lemma 8.7. It remains to prove (6.3). For fixed $s,r\in S$, Lemma 8.5 (i) shows that $\langle V_s V_r^*(\xi_t\otimes\xi_\tau),\xi_t\otimes\xi_\tau\rangle$ vanishes for all $t\in\mathcal{T}$, unless $s\sim r$. Therefore $\psi_{\beta,\tau}(v_s v_r^*)=0$ if $s\not\sim r$. Now fix $s,r\in S$ such that $s\sim r$. Write $s=t'a$ and $r=t'b$ for some $t'\in\mathcal{T}, a,b\in S_c$. Then $\langle V_s V_r^*(\xi_t\otimes\xi_\tau),\xi_t\otimes\xi_\tau\rangle$ can only be nonzero if $t\in i(t'\mathcal{T})\cap\mathcal{T}$. Note that for $t''\in\mathcal{T}$ such that $t=i(t't'')$ we have $tc(t't'')=t't''$, so $N_t=N_{t'}N_{t''}$. For large enough k, we have $t'\in\mathcal{T}_{I_k}$. Hence, as we sum over $t\in\mathcal{T}_{I_k}$, for k large enough also $t''\in\mathcal{T}_{I_k}$. Thus

$$\begin{aligned}
\psi_{\beta,\tau}(v_s v_r^*) &= \lim_{k\to\infty}\zeta_{I_k}(\beta)^{-1}\sum_{t\in\mathcal{T}_{I_k}} N_t^{-\beta}\langle V_s V_r^*(\xi_t\otimes\xi_\tau),\xi_t\otimes\xi_\tau\rangle\\
&= \lim_{k\to\infty}\zeta_{I_k}(\beta)^{-1}\sum_{t\in\mathcal{T}_{I_k}} N_t^{-\beta}\langle V_a V_b^* V_{t'}^*(\xi_t\otimes\xi_\tau),V_{t'}^*(\xi_t\otimes\xi_\tau)\rangle\\
&= \lim_{k\to\infty}\zeta_{I_k}(\beta)^{-1}\sum_{t''\in\mathcal{T}_{I_k}} N_{i(t't'')}^{-\beta}\langle V_a V_b^*(\xi_{t''}w_{c(t't'')}^*\otimes\xi_\tau),\xi_{t''}w_{c(t't'')}^*\otimes\xi_\tau\rangle\\
&= \lim_{k\to\infty}N_{t'}^{-\beta}\zeta_{I_k}(\beta)^{-1}\sum_{t''\in\mathcal{T}_{I_k}} N_{t''}^{-\beta}\langle V_a V_b^*(\xi_{t''}\otimes\xi_\tau),\xi_{t''}\otimes\xi_\tau\rangle\\
&= \lim_{k\to\infty}N_s^{-\beta}\psi_{\beta,\tau,I_k}(v_a v_b^*),
\end{aligned}$$

where we used Lemma 8.5 (ii) and the trace property of τ in the penultimate equality. This shows that the limit $\psi_{\beta,\tau}$ satisfies (6.3), and hence is a KMS_β-state. The claim for $\beta\in(\beta_c,\infty)$ follows immediately from $I_k\nearrow\mathrm{Irr}(N(S))$ because the formula from Lemma 8.7 makes sense for $I=\mathrm{Irr}(N(S))$. □

Corollary 8.9. *Let S be an admissible right LCM semigroup. If ϕ is a KMS_β-state on $C^*(S)$ for $\beta\in(\beta_c,\infty)$, then $\psi_{\beta,\phi_{\mathrm{Irr}(N(S))}\circ\varphi}=\phi$.*

Proof. As $\phi_{\mathrm{Irr}(N(S))}\circ\varphi$ is a trace on $C^*(S_c)$ by Proposition 6.4, $\psi_{\beta,\phi_{\mathrm{Irr}(N(S))}\circ\varphi}$ is a KMS_β-state, see Proposition 8.8. Due to Proposition 6.4, it suffices to show that $\psi_{\beta,\phi_{\mathrm{Irr}(N(S))}\circ\varphi}\circ\varphi=\phi\circ\varphi$. For $a,b\in S_c$, Lemma 8.5 (ii) and $\varphi(w_aw_b^*)=v_av_b^*$ give

$$\begin{aligned}
\psi_{\beta,\phi_{\mathrm{Irr}(N(S))}\circ\varphi}(v_av_b^*) &= \zeta_S(\beta)^{-1}\sum_{t\in\mathcal{T}} N_t^{-\beta}\langle V_a V_b^*(\xi_t\otimes\xi_{\phi_{\mathrm{Irr}(N(S))}\circ\varphi}),\xi_t\otimes\xi_{\phi_{\mathrm{Irr}(N(S))}\circ\varphi}\rangle\\
&= \zeta_S(\beta)^{-1}\sum_{n\in N(S)}\sum_{t\in\mathcal{T}_n} n^{-\beta}\chi_{\mathrm{Fix}(\alpha_a\alpha_b^{-1})}(t)\ \phi_{\mathrm{Irr}(N(S))}\circ\varphi(w_{c(a\alpha_b^{-1}(t))}w^*_{c(b\alpha_b^{-1}(t))}).
\end{aligned}$$

We claim that $\chi_{\mathrm{Fix}(\alpha_a\alpha_b^{-1})}(t)\ \varphi(w_{c(a\alpha_b^{-1}(t))}w^*_{c(b\alpha_b^{-1}(t))})=v_t^*v_av_b^*v_t$. Indeed, the expression $v_t^*v_av_b^*v_t$ vanishes unless $t\in\mathrm{Fix}(\alpha_a\alpha_b^{-1})$. But if $t\in\mathrm{Fix}(\alpha_a\alpha_b^{-1})$, then $a\alpha_b^{-1}(t)=tc(a\alpha_b^{-1}(t))$ and $b\alpha_b^{-1}(t)=tc(b\alpha_b^{-1}(t))$, see the proof of Lemma 3.9, hence

$$\begin{aligned}
v_t^*v_av_b^*v_t &= v_t^*e_{a\alpha_b^{-1}(t)S}v_av_b^*e_{b\alpha_b^{-1}(t)S}v_t\\
&= v_t^*v_{tc(a\alpha_b^{-1}(t))}v^*_{a\alpha_b^{-1}(t)}v_av_b^*v_{b\alpha_b^{-1}(t)}v^*_{tc(b\alpha_b^{-1}(t))}v_t\\
&= v_{c(a\alpha_b^{-1}(t))}v^*_{\alpha_b^{-1}(t)}v_{\alpha_b^{-1}(t)}v^*_{c(b\alpha_b^{-1}(t))}\\
&= v_{c(a\alpha_b^{-1}(t))}v^*_{c(b\alpha_b^{-1}(t))}.
\end{aligned}$$

Therefore we conclude that $\psi_{\beta,\phi_{\mathrm{Irr}(N(S))}\circ\varphi} = \phi$ by appealing to Lemma 7.5 (iv). □

Corollary 8.9 shows that every KMS$_\beta$-state on $C^*(S)$ for $\beta \in (\beta_c, \infty)$ arises from a trace on $C^*(S_c)$. In other words, we have obtained a surjective parametrisation. We now show that this mapping is also injective.

Proposition 8.10. *Let S be an admissible right LCM semigroup. For every $\beta \in (\beta_c, \infty)$ and every trace τ on $C^*(S_c)$, let $\psi_{\beta,\tau}$ be the* KMS$_\beta$*-state obtained in Proposition 8.8. Then $(\psi_{\beta,\tau})_{Irr(N(S))} \circ \varphi = \tau$.*

Proof. Recall from Lemma 7.2 that $\varphi(y)d_n = d_n\varphi(y)$ for all $y \in C^*(S_c)$ and $n \in N(S)$. Using Lemma 7.5, Proposition 8.8 and Lemma 8.7, we have

$$\begin{aligned}
&(\psi_{\beta,\tau})_{\mathrm{Irr}(N(S))} \circ \varphi(y)\\
&\qquad = \zeta_S(\beta)\tilde{\psi}_{\beta,\tau}(Q_{\mathrm{Irr}(N(S))}\pi_{\psi_{\beta,\tau}}(\varphi(y))Q_{\mathrm{Irr}(N(S))})\\
&\qquad = \zeta_S(\beta)\tilde{\psi}_{\beta,\tau}(\pi_{\psi_{\beta,\tau}}(\varphi(y))\prod_{n\in\mathrm{Irr}(N(S))}\pi_{\psi_{\beta,\tau}}(d_n))\\
&\qquad = \zeta_S(\beta)\lim_{\substack{I_k\nearrow\mathrm{Irr}(N(S))\\ |I_k|<\infty}}\psi_{\beta,\tau}(\varphi(y)\prod_{n\in I_k}d_n)\\
&\qquad = \lim_{\substack{I_k\nearrow\mathrm{Irr}(N(S))\\ |I_k|<\infty}}\sum_{m\in N(S)} m^{-\beta}\sum_{t\in\mathcal{T}_m}\langle\pi_\tau(\varphi(y)\prod_{n\in I_k}d_n)(\xi_t\otimes\xi_\tau),\xi_t\otimes\xi_\tau\rangle.
\end{aligned}$$

Noting that $\mathcal{T}_1 = \{1\}$ and $\pi_\tau(\prod_{n\in I_k} d_n)(\xi_1\otimes\xi_\tau) = \xi_1\otimes\xi_\tau$ as $1 \notin tS$ for all $t \in \mathcal{T}_n$, where $n \in I_k \subset \mathrm{Irr}(N(S))$, the term at $m = 1$ is given by

$$\langle\pi_\tau(\varphi(y)\prod_{n\in I_k}d_n)(\xi_1\otimes\xi_\tau),\xi_1\otimes\xi_\tau\rangle \;=\; \langle\pi_\tau(\varphi(y))(\xi_t\otimes\xi_\tau),\xi_t\otimes\xi_\tau\rangle \;=\; \tau(y).$$

For $m \in N(S)$ with $m > 1$, there is k so that there is $n \in I_\ell$ with $m \in nN(S)$ for all $\ell \geq k$. We claim that $\pi_\tau(d_n)(\xi_t\otimes\xi_\tau) = 0$ for all $t \in \mathcal{T}_m$. To prove the claim, we recall that $d_n = 1 - \sum_{f\in\mathcal{T}_n} e_{fS}$ and that $\pi_\tau(e_{fS})(\xi_t\otimes\xi_\tau) = \chi_{i(fS)}(t)\ \xi_t\otimes\xi_\tau$. Let $n' \in N(S)$ such that $m = nn'$. Let $\mathcal{T}_{n'}$ be any transversal for $N^{-1}(n')/_\sim$ contained in S_{ci}^1. According to Corollary 3.7, we either have $ff' \perp t$ or $ff' \in tS^*$ for every pair $(f, f') \in \mathcal{T}_n \times \mathcal{T}_{n'}$ as $\mathcal{T}_n\mathcal{T}_{n'} \subset N^{-1}(m) \cap S_{ci}$. However, since $\mathcal{T}_n\mathcal{T}_{n'}$ is an accurate foundation set, there is exactly one such pair (f, f') satisfying $ff' \in tS^*$. This forces $\pi_\tau(d_n)(\xi_t\otimes\xi_\tau) = 0$ and thus $\langle\pi_\tau(\varphi(y)\prod_{n\in I_\ell} d_n)(\xi_t\otimes\xi_\tau),\xi_t\otimes\xi_\tau\rangle = 0$ for all $\ell \geq k$. Thus we get $(\psi_{\beta,\tau})_{\mathrm{Irr}(N(S))} \circ \varphi(y) = \tau(y)$. □

9. Uniqueness within the critical interval

This section deals with the proof of the uniqueness assertions in Theorem 4.3 (2). For the purpose of this section, we assume throughout that S is an admissible right LCM semigroup, and write τ for the canonical trace $\tau(w_s w_t^*) = \delta_{s,t}$ on $C^*(S_c)$. We begin with the case that α is an almost free action, where uniqueness will be an application of the reconstruction formula in Lemma 7.5 (iv):

Proposition 9.1. *Let S be an admissible right LCM semigroup, and $\beta \in [1, \beta_c]$. If $\alpha\colon S_c \curvearrowright S/_\sim$ is almost free, then the only* KMS$_\beta$*-state on $C^*(S)$ is $\psi_\beta := \psi_{\beta,\tau}$ determined by $\psi_\beta \circ \varphi = \tau$.*

Proof. By Proposition 8.8, ψ_β is a KMS$_\beta$-state. Let ϕ be an arbitrary KMS$_\beta$-state. Due to Proposition 6.4, it suffices to show that $\psi_\beta \circ \varphi = \phi \circ \varphi$ on $C^*(S_c)$. Since $\phi \circ \varphi$ is a trace, we have $\psi_\beta \circ \varphi(w_a w_a^*) = 1$ for all $a \in S_c$. Let $a, b \in S_c$ with $a \neq b$. Lemma 7.5 (iv) gives

$$\phi(v_a v_b^*) \;\; = \;\; \zeta_I(\beta)^{-1} \sum_{t \in \mathcal{T}_I} N_t^{-\beta} \; \phi_I(v_t^* v_a v_b^* v_t)$$

for every finite subset I of $\mathrm{Irr}(N(S))$. Note that $|\phi_I(v_t^* v_a v_b^* v_t)| \leq 1$. We claim that a summand can only be nonzero if $\alpha_a \alpha_b^{-1}(t) = t$. Indeed, $v_t^* v_a = v_{a_1} v_{a_2}^* v_{\alpha_a^{-1}(t)}^*$ for some $a_1, a_2 \in S_c$, and similarly for $v_b^* v_t$. Thus a summand is nonzero if $v_{\alpha_a^{-1}(t)}^* v_{\alpha_b^{-1}(t)} \neq 0$, which is equivalent to $\alpha_a^{-1}(t) = \alpha_b^{-1}(t)$, that is $t \in \mathrm{Fix}(\alpha_a \alpha_b^{-1})$.

As α is almost free, the set $\mathrm{Fix}(\alpha_a \alpha_b^{-1}) \subset \mathcal{T}$ is finite, so that

$$\Big| \sum_{t \in F_I} N_t^{-\beta} \; \phi_I(v_t^* v_a v_b^* v_t) \Big| \leq |\mathrm{Fix}(\alpha_a \alpha_b^{-1})|$$

while $\beta \in [1, \beta_c]$ forces $\zeta_I(\beta) \to \infty$ as $I \nearrow \mathrm{Irr}(N(S))$. Thus we conclude that $\phi(v_a v_b^*) = 0$ for all $a, b \in S_c$ with $a \neq b$. In particular, there is only one such state ψ_β. □

The case where $\beta_c = 1$, α is faithful, and S has finite propagation requires more work. The underlying strategy is based on [LRRW14, Section 7]. Given $n \in N(S)$, let $\mathcal{T}_n$ be a transversal for $N^{-1}(n)/_\sim$ contained in S_{ci}^1. Note that we may take $\mathcal{T}_n$ of the form $\mathcal{T} \cap N^{-1}(n)$, where $\mathcal{T}$ is a transversal for $S/_\sim$ contained in S_{ci}^1. For $a, b \in S_c$, let

$$\mathcal{T}_n^{a,b} := \{ f \in \mathcal{T}_n \mid af = bf \} \quad \text{and} \quad \kappa_{a,b,n} := |\mathcal{T}_n^{a,b}|/n.$$

Note that by Corollary 3.7, any other choice of transversal for $N^{-1}(n)/_\sim$ contained in S_{ci}^1 will give the same $\kappa_{a,b,n}$. If moreover $m \in N(S)$, then since $i(\mathcal{T}_m \mathcal{T}_n) = \mathcal{T}_{mn}$ and since $f \sim f'$ for core irreducible elements f, f' implies $f \in f'S^*$, see Lemma 3.2, we have $\mathcal{T}_{mn}^{a,b} \supset i(\mathcal{T}_m^{a,b} \mathcal{T}_n)$ so that $(\kappa_{a,b,n})_{n \in N(S)}$ is increasing for the natural partial order $n' \geq n :\Leftrightarrow n' \in nN(S)$. As $N(S)$ is directed, we get a limit $\kappa_{a,b} = \lim_{n \in N(S)} \kappa_{a,b,n} \in [0,1]$ for all $a, b \in S_c$.

Proposition 9.2. *Let S be an admissible right LCM semigroup and $\beta_c = 1$. Then there exists a* KMS$_1$*-state ψ_1 characterized by $\psi_1 \circ \varphi = \rho$, where $\rho(w_a w_b^*) = \kappa_{a,b}$ for $a, b \in S_c$. In particular, ρ is a trace on $C^*(S_c)$. If α is faithful and S has finite propagation, then ψ_1 is the only* KMS$_1$*-state on $C^*(S)$.*

The idea of the proof is as follows: Take $(\beta_k)_{k \geq 1}$ with $\beta_k > 1$ and $\beta_k \searrow 1$. Then a subsequence of the sequence of KMS$_{\beta_k}$-states $(\psi_{\beta_k, \tau})_{k \geq 1}$ obtained from Proposition 8.8 weak*-converges to a state ψ_1 on $C^*(S)$. For simplicity, we relabel and then assume that the original sequence converges. It is well-known that ψ_1 is a KMS$_1$-state in this case, see for example [BR97, Proposition 5.3.23]. We then use Proposition 6.6 to deduce that every KMS$_1$-state ϕ satisfies $\phi \circ \varphi = \rho$.

We start with two preparatory lemmas that will streamline the proof of Proposition 9.2.

Lemma 9.3. *Suppose S is an admissible right LCM semigroup, $\beta_c = 1$, and $\beta_k \searrow 1$. Then $\zeta_S(\beta_k)^{-1} \sum_{n \in N(S)} n^{-\beta_k + 1} \kappa_{a,b,n}$ converges to $\kappa_{a,b}$ as $k \to \infty$ for all $a, b \in S_c$.*

Proof. Let $a,b \in S_c$ and $\varepsilon > 0$. As $(\kappa_{a,b,n})_{n\in N(S)}$ is increasing and converges to $\kappa_{a,b}$, and $N(S) \cong \bigoplus_{n\in \mathrm{Irr}(N(S))} \mathbb{N}$, there are $C \in \mathbb{N}, n_1,\dots,n_C \in \mathrm{Irr}(N(S)), m_1,\dots,m_C \geq 1$ such that $n \in \tilde{n}N(S)$ implies $0 \leq \kappa_{a,b} - \kappa_{a,b,n} < \varepsilon/2$, where $\tilde{n} := \prod_{1\leq i\leq C} n_i^{m_i}$. Next, note that

a) $N(S) \setminus \tilde{n}N(S) \subset \bigcup\limits_{1\leq i\leq C} \bigcup\limits_{0\leq m\leq m_i-1} n_i^m\{n \in N(S) \mid n \notin n_iN(S)\}$, and

b) $\zeta_S(\beta_k) = \prod\limits_{n\in \mathrm{Irr}(N(S))} (1-n^{-\beta_k+1})^{-1}$.

In particular, b) implies that

$$\begin{aligned}\zeta_S(\beta_k)^{-1} \sum_{\substack{n\in N(S)\\ n\notin n_iN(S)}} n^{-\beta_k+1} &= (1-n_i^{-\beta_k+1})^{-1}\zeta_{\mathrm{Irr}(N(S))\setminus\{n_i\}}(\beta_k)^{-1} \sum_{n\in\langle \mathrm{Irr}(N(S))\setminus\{n_i\}\rangle} n^{-\beta_k+1} \\ &= (1-n_i^{-\beta_k+1})^{-1} \overset{k\to\infty}{\longrightarrow} 0\end{aligned}$$

for each $1 \leq i \leq C$ because $\beta_k \searrow 1$. Now choose ℓ large enough so that

c) $\zeta_S(\beta_k)^{-1} \sum_{n\in N(S)\setminus n_iN(S)} n^{-\beta_k+1} \leq \varepsilon/2Cm_i$ for all $1 \leq i \leq C$ and $k \geq \ell$.

By definition, $\zeta_S(\beta_k) = \sum_{n\in N(S)} n^{-\beta_k+1}$, so we get

$$\begin{aligned}&\Big|\kappa_{a,b} - \zeta_S(\beta_k)^{-1} \sum_{n\in N(S)} n^{-\beta_k+1}\kappa_{a,b,n}\Big| \\ &\overset{\kappa_{a,b}\geq\kappa_{a,b,n}}{=} \zeta_S(\beta_k)^{-1} \sum_{n\in N(S)} n^{-\beta_k+1}(\kappa_{a,b}-\kappa_{a,b,n}) \\ &\overset{a)}{\leq} \zeta_S(\beta_k)^{-1} \sum_{n\in \tilde{n}N(S)} n^{-\beta_k+1} \underbrace{(\kappa_{a,b}-\kappa_{a,b,n})}_{<\varepsilon/2} \\ &\quad + \sum_{i=1}^{C}\sum_{m=0}^{m_i-1} \zeta_S(\beta_k)^{-1} \sum_{\substack{n\in N(S)\\ n\notin n_iN(S)}} n^{-\beta_k+1} \underbrace{n_i^{m(-\beta_k+1)}(\kappa_{a,b}-\kappa_{a,b,nn_i^m})}_{\in[0,1]} \\ &\overset{c)}{<} \tfrac{\varepsilon}{2} + \sum_{i=1}^{C}\sum_{m=0}^{m_i-1} \tfrac{\varepsilon}{2Cm_i} = \varepsilon\end{aligned}$$

for all $k \geq \ell$. □

Remark 9.4. In analogy to $\mathcal{T}_n^{a,b} = \{f \in \mathcal{T}_n \mid af = bf\} = \{f \in \mathcal{T}_n \mid i(af) = i(bf) \text{ and } c(af) = c(bf)\}$, let $G_n^{a,b} := \{f \in \mathcal{T}_n \mid i(af) = i(bf)\}$ for $n \in N(S)$ and $a,b \in S_c$. Note that $i(af) = i(bf)$ is equivalent to $\alpha_a([f]) = \alpha_b([f])$. By Corollary 3.7, the number $|G_n^{a,b}|$ does not depend on $\mathcal{T}_n$. Also, for $m,n \in N(S)$, we have $\mathcal{T}_{mn} = i(\mathcal{T}_m\mathcal{T}_n)$ and $aff' = i(af)i(c(af)f')c(c(af)f')$. By using Corollary 3.7, this allows us to deduce

$$G_{mn}^{a,b} \setminus \mathcal{T}_{mn}^{a,b} = \{i(ff') \mid f \in G_m^{a,b} \setminus \mathcal{T}_m^{a,b}, f' \in G_n^{c(af),c(bf)} \setminus \mathcal{T}_n^{c(af),c(bf)}\}. \tag{9.1}$$

Lemma 9.5. *Let S be an admissible right LCM semigroup. If α is faithful and S has finite propagation, then $(|G_n^{a,b} \setminus \mathcal{T}_n^{a,b}|/n)_{n\in N(S)}$ converges to 0 for all $a,b \in S_c$.*

Proof. Let $a,b \in S_c$. If $a = b$, then $G_n^{a,b} = \mathcal{T}_n^{a,b} = \mathcal{T}_n$ for all $n \in N(S)$, so there is nothing to show. Thus, we may suppose $a \neq b$. Since S has finite propagation, there exists a transversal $\mathcal{T}$ for $S/_\sim$ that witnesses finite propagation for (a,b), i.e. the sets $C_a = \{c(at) \mid t \in \mathcal{T}\}$ and C_b are finite. In particular, the set $C_{a,b} := \{(d,e) \in C_a \times C_b \mid$

$d \neq e\} \subset S_c \times S_c$ is finite. We claim that for each $(d,e) \in C_{a,b}$, faithfulness of α gives $n(d,e) \in N(S)$ such that $|G^{d,e}_{n(d,e)}| \leq n(d,e) - 1$. Indeed, we get $r \in S$ with $dr \not\sim er$. This forces $n(d,e) := N_r > 1$ and $di(r) \perp ei(r)$ by Proposition 3.6 (ii). In particular, we have $i(di(r)) \neq i(ei(r))$.

With $n := \prod_{(d,e)\in C_{a,b}} n(d,e)$ and $\gamma := \max_{(d,e)\in C_{a,b}}(n(d,e)-1)/n(d,e) \in (0,1)$, we claim that $n^{-k}|G^{a,b}_{n^k} \setminus \mathcal{T}^{a,b}_{n^k}| \leq \gamma^k$ holds for all $k \geq 1$. We proceed by induction. For $k = 1$, we use $(a,b) \in C_{a,b}$ and (9.1) for $n(a,b)$ to get

$$n^{-1}|G^{a,b}_n \setminus \mathcal{T}^{a,b}_n| \leq \tfrac{(n(a,b)-1)}{n(a,b)} \leq \gamma.$$

For $k \mapsto k+1$, let us consider (9.1) for $m := n^k$ and $m' = n$. The induction hypothesis gives $|G^{a,b}_{n^k} \setminus \mathcal{T}^{a,b}_{n^k}| \leq (\gamma n)^k$, so it suffices to have $|G^{c(af),c(bf)}_n \setminus \mathcal{T}^{c(af),c(bf)}_n| \leq \gamma n$ for every $f \in G^{a,b}_{n^k} \setminus \mathcal{T}^{a,b}_{n^k}$. As before, if $c(af) = c(bf)$, then the set in question is empty. But if $c(af) \neq c(bf)$, then the choice of n guarantees that $|G^{c(af),c(bf)}_n \setminus \mathcal{T}^{c(af),c(bf)}_n| \leq \gamma n$ by applying (9.1) to $m = n(c(af),c(bf))$ and $m' = n(c(af),c(bf))^{-1}n)$. This establishes the claim. In turn, $n^{-k}|G^{a,b}_{n^k} \setminus \mathcal{T}^{a,b}_{n^k}| \leq \gamma^k$ for all $k \geq 1$ forces $|G^{a,b}_m \setminus \mathcal{T}^{a,b}_m|/m \to 0$ as $m \to \infty$ in $N(S)$ with respect to the natural partial order. □

Proof of Proposition 9.2. We know that ψ_1 is a KMS$_1$-state, so we start by showing that $\psi_1 \circ \varphi(w_a w_b^*) = \rho(w_a w_b^*) = \kappa_{a,b}$ for $a,b \in S_c$. Note that $\psi_{\beta_k,\tau}(v_a v_b^*) = \psi_{\beta_k,\tau}(v_b^* v_a)$ since $\psi_{\beta_k,\tau} \circ \varphi$ is tracial. The reconstruction formula from Lemma 7.5 (iv) and Proposition 8.10 imply that

$$\begin{aligned}
\psi_{\beta_k,\tau}(v_b^* v_a) &= \zeta_S(\beta_k)^{-1} \sum_{n\in N(S)} n^{-\beta_k} \sum_{f\in\mathcal{T}_n} (\psi_{\beta_k,\tau})_{\mathrm{Irr}(N(S))}(v_{bf}^* v_{af}) \\
&= \zeta_S(\beta_k)^{-1} \sum_{n\in N(S)} n^{-\beta_k} \sum_{f\in\mathcal{T}_n} \delta_{i(af),i(bf)} (\psi_{\beta_k,\tau})_{\mathrm{Irr}(N(S))}(v_{c(bf)}^* v_{c(af)}) \\
&\overset{8.10}{=} \zeta_S(\beta_k)^{-1} \sum_{n\in N(S)} n^{-\beta_k} \sum_{f\in\mathcal{T}_n} \delta_{i(af),i(bf)} \tau(w_{c(bf)}^* w_{c(af)}) \\
&= \zeta_S(\beta_k)^{-1} \sum_{n\in N(S)} n^{-\beta_k} |\mathcal{T}^{a,b}_n|.
\end{aligned}$$

Since $n^{-\beta_k}|\mathcal{T}^{a,b}_n| = n^{-\beta_k+1}\kappa_{a,b,n}$, Lemma 9.3 and $\psi_{\beta_k,\tau} \to \psi_1$ jointly imply $\psi_1 \circ \varphi(w_a w_b^*) = \kappa_{a,b} = \rho(w_a w_b^*)$. According to Proposition 6.4, we conclude that ρ is a trace on $C^*(S_c)$.

Now suppose ϕ is any KMS$_1$-state on $C^*(S)$. Again appealing to Proposition 6.4, it suffices to show that $\phi \circ \varphi = \rho$ on $C^*(S_c)$. Since ϕ is a KMS$_1$-state, it factors through $\mathcal{Q}_p(S)$ (or $\mathcal{Q}_(S)$), see Proposition 6.6. As $\pi_p(\sum_{f\in\mathcal{T}_n} v_f v_f^*) = 1$ for all $n \in N(S)$, we get

$$\begin{aligned}
\phi(v_a v_b^*) = \sum_{f\in\mathcal{T}_n} \phi(v_{af} v_{bf}^*) &= \sum_{f\in\mathcal{T}_n} n^{-1}\delta_{i(af)i(bf)}\phi(v_{c(af)} v_{c(bf)}^*) \\
&= \kappa_{a,b,n} + n^{-1} \sum_{f\in G^{a,b}_n \setminus \mathcal{T}^{a,b}_n} \phi(v_{c(af)} v_{c(bf)}^*).
\end{aligned}$$

Since $|\phi(v_{c(af)} v_{c(bf)}^*)| \leq 1$, Lemma 9.5 shows that $\phi(v_a v_b^*) = \lim_{n\in N(S)} \kappa_{a,b,n} = \rho(w_a w_b^*)$. □

While Proposition 9.1 and Proposition 9.2 provide sufficient criteria for uniqueness of KMS$_\beta$-states for β in the critical interval, we will now prove that faithfulness of α

is necessary, at least under the assumption that S_c is abelian. If S_c is abelian, it is also right cancellative, and hence a right Ore semigroup. It therefore embeds into a group $G_c = S_c S_c^{-1}$, and the action $\alpha\colon S_c \curvearrowright S/_\sim$ induces an action $\alpha'\colon G_c \curvearrowright S/_\sim$ by $\alpha'_{ab^{-1}} = \alpha_a\alpha_b^{-1}$ for all $a, b \in S_c$. We let $G_c^\alpha := \{ab^{-1} \mid \alpha_a = \alpha_b\}$ be the subgroup of G_c for which α' is trivial, and we write u_g for the generating unitaries in $C^*(G_c^\alpha)$.

Proposition 9.6. *Let S be an admissible right LCM semigroup with S_c abelian. Suppose that there exist $\beta \geq 1$ and distinct states ϕ, ϕ' on $C^*(G_c^\alpha)$ such that there exist $ab^{-1} \in G_c^\alpha$ and a sequence $(I_k)_{k\in\mathbb{N}} \subset Irr(N(S))$ with I_k finite and $I_k \nearrow Irr(N(S))$ satisfying*

$$\lim_{k\to\infty} \zeta_{I_k}(\beta)^{-1} \sum_{t\in\mathcal{T}_{I_k}} N_t^{-\beta}\phi(u_{c(at)c(bt)^{-1}}) \neq \lim_{k\to\infty} \zeta_{I_k}(\beta)^{-1} \sum_{t\in\mathcal{T}_{I_k}} N_t^{-\beta}\phi'(u_{c(at)c(bt)^{-1}}). \tag{9.2}$$

Then $C^*(S)$ has at least two distinct KMS_β*-states. If, for any given pair of distinct states on $C^*(G_c^\alpha)$, there exists such a sequence $(I_k)_{k\in\mathbb{N}}$, then there exists an affine embedding of the state space of $C^*(G_c^\alpha)$ into the* KMS_β*-states on $C^*(S)$. In particular, $C^*(S)$ does not have a unique* KMS_β*-state unless α is faithful under these assumptions.*

Proof. Since S_c is abelian, so are the groups G_c and G_c^α. In particular, G_c and G_c^α are amenable, so that $C^*(G_c^\alpha) \subset C^*(G_c)$ as a unital subalgebra, and states are traces on $C^*(G_c)$. So if ϕ is a state on $C^*(G_c^\alpha)$, it extends to a state on $C^*(G_c)$, which then corresponds to a trace τ on $C^*(S_c)$. Note that we have $\tau(w_a w_b^*) = \phi(u_{ab^{-1}})$ for all $ab^{-1} \in G_c^\alpha$ by construction.

Now let $\beta \geq 1$, and ϕ, ϕ' distinct states on $C^*(G_c^\alpha)$ such that there exists a sequence $(I_k)_{k\in\mathbb{N}} \subset \mathrm{Irr}(N(S))$ with I_k finite, $I_k \nearrow \mathrm{Irr}(N(S))$, and (9.2). Denote by $\tilde{\tau}$ and τ' the corresponding traces on $C^*(S_c)$. Following the proof of Proposition 8.8, we obtain KMS_β-states $\psi_{\beta,\phi} := \psi_{\beta,\tilde{\tau}}$ and $\psi_{\beta,\phi'} := \psi_{\beta,\tau'}$ by passing to subsequences $(I_{i(k)})_{k\in\mathbb{N}}$ and $(I_{j(k)})_{k\in\mathbb{N}}$ of $(I_k)_{k\in\mathbb{N}}$, if necessary. The state $\psi_{\beta,\phi}$ is given by

$$\begin{aligned}\psi_{\beta,\phi}(v_a v_b^*) &= \lim_{k\to\infty} \zeta_{I_{i(k)}}(\beta)^{-1} \sum_{t\in\mathcal{T}_{I_{i(k)}}} N_t^{-\beta}\langle V_a V_b^*(\xi_t\otimes\xi_{\tilde{\tau}}), \xi_t\otimes\xi_{\tilde{\tau}}\rangle \\ &\overset{8.5\,(b)}{=} \lim_{k\to\infty} \zeta_{I_{i(k)}}(\beta)^{-1} \sum_{t\in\mathcal{T}_{I_{i(k)}}} \chi_{\mathrm{Fix}(\alpha_a\alpha_b^{-1})}(t) N_t^{-\beta}\tilde{\tau}(w_{c(a\alpha_b^{-1}(t))} w^*_{c(b\alpha_b^{-1}(t))})\end{aligned}$$

for $a, b \in S_c$. But if $ab^{-1} \in G_c^\alpha$, then this reduces to

$$\psi_{\beta,\phi}(v_a v_b^*) = \lim_{k\to\infty} \zeta_{I_{i(k)}}(\beta)^{-1} \sum_{t\in\mathcal{T}_{I_{i(k)}}} N_t^{-\beta}\phi(u_{c(at)c(bt)^{-1}})$$

The analogous computation applies to $\psi_{\beta,\phi'}$. Since (9.2) implicitly assumes existence of the displayed limits, passing to subsequences leaves the values invariant, so that $\psi_{\beta,\phi}(v_a v_b^*) \neq \psi_{\beta,\phi'}(v_a v_b^*)$ for some $ab^{-1} \in G_c^\alpha$. This shows that ϕ and ϕ' yield distinct KMS_β-states $\psi_{\beta,\phi}$ and $\psi_{\beta,\phi'}$. The map $\phi \mapsto \psi_{\beta,\phi}$ is affine, continuous, and injective by the assumption. Hence it is an embedding. Finally, the state space of $C^*(G_c^\alpha)$ is a singleton if and only if G_c^α is trivial, which corresponds to faithfulness of α. □

10. Open questions

The present work naturally leads to some questions for further research, and we would like to take the opportunity to indicate a few directions we find intriguing.

Questions 10.1. Can the prerequisites for Theorem 4.3 be weakened? This leads to:

(a) Are there right LCM semigroups that are not core factorable?
(b) Are there right LCM semigroups S for which $S_{ci} \subset S$ is not $\cap$-closed?
(c) How does a violation of (A4) affect the KMS-state structure?

Question 10.2. Suppose S is an admissible right LCM semigroup. Faithfulness of $\alpha\colon S_c \curvearrowright S/_\sim$ is assumed in both parts of Theorem 4.3 (2), see also Remark 3.11, and it is necessary under additional assumptions, see Proposition 9.6. The extra assumptions in both directions might be artifacts of the proofs, so it is natural to ask: Is faithfulness of α necessary or sufficient for uniqueness of KMS$_\beta$-states for β in the critical interval?

If the answer to Question 10.2 is affirmative for both parts, then, for right cancellative S_c, the group G_c^α would correspond to the periodicity group PerΛ for strongly connected finite k-graphs Λ, see [aHLRS15].

Using self-similar actions (G, X), see Proposition 5.8, one can give examples $S = X^* \bowtie G$ of admissible right LCM semigroups with $\beta_c = 1$, α faithful, but not almost free, and without finite propagation. The following problem seems to be more difficult:

Question 10.3. Is there an example of a self-similar action (G, X) that is not finite state such that $S = X^* \bowtie G$ has finite propagation?

In [aHLRS15, Section 12], the authors showed that their findings can be obtained via [Nes13, Theorem 1.3] since their object of study admits a canonical groupoid picture.

Questions 10.4. As indicated in [Sta17], the boundary quotient diagram ought to admit a reasonable description in terms of C^*-algebras associated to groupoids. Assuming that this gap can be bridged, we arrive at the following questions:

(a) Does [Nes13, Theorem 1.3] apply to the groupoids related to the boundary quotient diagram for (admissible) right LCM semigroups?
(b) Which groupoid properties do (A1)–(A4) correspond to?
(c) What is the meaning of $\alpha\colon S_c \curvearrowright S/_\sim$ for the associated groupoids? Which groupoid properties do faithfulness and almost freeness of α correspond to?
(d) Which groupoid property does finite propagation correspond to?

References

[BC95] Jean-Benoît Bost and Alain Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) **1** (1995), 411–457.

[BR97] Ola Bratteli and Derek W. Robinson, *Operator algebras and quantum statistical mechanics. Equilibrium states. Models in quantum statistical mechanics. 2nd edition*, Texts and Monographs in Physics, vol. II, Springer-Verlag, Berlin, 1997.

[Bri05] Matthew G. Brin, *On the Zappa-Szép product*, Comm. Algebra **33** (2005), no. 2, 393–424, DOI 10.1081/AGB-200047404.

[BaHLR12] Nathan Brownlowe, Astrid an Huef, Marcelo Laca, and Iain Raeburn, *Boundary quotients of the Toeplitz algebra of the affine semigroup over the natural numbers*, Ergodic Theory Dynam. Systems **32** (2012), no. 1, 35–62, DOI 10.1017/S0143385710000830.

[BLS17] Nathan Brownlowe, Nadia S. Larsen, and Nicolai Stammeier, *On C^*-algebras associated to right LCM semigroups*, Trans. Amer. Math. Soc. **369** (2017), no. 1, 31–68, DOI 10.1090/tran/6638.

[BLS] ______, *C^*-algebras of algebraic dynamical systems and right LCM semigroups*, to appear in Indiana Univ. Math. J. preprint available under IUMJ/Preprints/7527.

[BRRW14] Nathan Brownlowe, Jacqui Ramagge, David Robertson, and Michael F. Whittaker, *Zappa-Szép products of semigroups and their C^*-algebras*, J. Funct. Anal. **266** (2014), no. 6, 3937–3967, DOI 10.1016/j.jfa.2013.12.025.

[BS16] Nathan Brownlowe and Nicolai Stammeier, *The boundary quotient for algebraic dynamical systems*, J. Math. Anal. Appl. **438** (2016), no. 2, 772–789, DOI 10.1016/j.jmaa.2016.02.015.

[CaHR16] Lisa Orloff Clark, Astrid an Huef, and Iain Raeburn, *Phase transition on the Toeplitz algebra of Baumslag-Solitar semigroups*, Indiana Univ. Math. J. **65** (2016), 2137–2173, DOI 10.1512/iumj.2016.65.5934.

[Cli53] A. H. Clifford, *A class of d-simple semigroups*, Amer. J. Math. **75** (1953), 547–556, DOI 10.2307/2372503.

[CM06] Alain Connes and Matilde Marcolli, *$\mathbb{Q}$-lattices: quantum statistical mechanics and Galois theory*, J. Geom. Phys. **56** (2006), no. 1, 2–23, DOI 10.1016/j.geomphys.2005.04.010.

[CL07] John Crisp and Marcelo Laca, *Boundary quotients and ideals of Toeplitz C^*-algebras of Artin groups*, J. Funct. Anal. **242** (2007), no. 1, 127–156, DOI 10.1016/j.jfa.2006.08.001.

[CDL13] Joachim Cuntz, Christopher Deninger, and Marcelo Laca, *C^*-algebras of Toeplitz type associated with algebraic number fields*, Math. Ann. **355** (2013), no. 4, 1383–1423, DOI 10.1007/s00208-012-0826-9.

[aHLRS15] Astrid an Huef, Marcelo Laca, Iain Raeburn, and Aidan Sims, *KMS states on the C^*-algebra of a higher-rank graph and periodicity in the path space*, J. Funct. Anal. **268** (2015), no. 7, 1840–1875, DOI 10.1016/j.jfa.2014.12.006.

[Lac98] Marcelo Laca, *Semigroups of $*$-endomorphisms, Dirichlet series, and phase transitions*, J. Funct. Anal. **152** (1998), no. 2, 330–378, DOI 10.1006/jfan.1997.3166.

[LR10] Marcelo Laca and Iain Raeburn, *Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers*, Adv. Math. **225** (2010), no. 2, 643–688, DOI 10.1016/j.aim.2010.03.007.

[LRR11] Marcelo Laca, Iain Raeburn, and Jacqui Ramagge, *Phase transition on Exel crossed products associated to dilation matrices*, J. Funct. Anal. **261** (2011), no. 12, 3633–3664, DOI 10.1016/j.jfa.2011.08.015.

[LRRW14] Marcelo Laca, Iain Raeburn, Jacqui Ramagge, and Michael F. Whittaker, *Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions*, J. Funct. Anal. **266** (2014), no. 11, 6619–6661, DOI 10.1016/j.jfa.2014.03.003.

[LRRW] ______, *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs.* preprint, arxiv.org:1610.00343.

[Law99] Mark V. Lawson, *The structure of 0-E-unitary inverse semigroups. I. The monoid case*, Proc. Edinburgh Math. Soc. (2) **42** (1999), no. 3, 497–520, DOI 10.1017/S0013091500020484.

[Law08] ______, *A correspondence between a class of monoids and self-similar group actions. I*, Semigroup Forum **76** (2008), no. 3, 489–517, DOI 10.1007/s00233-008-9052-x.

[Law12] ______, *Non-commutative Stone duality: inverse semigroups, topological groupoids and C^*-algebras*, Internat. J. Algebra Comput. **22** (2012), no. 6, 1250058, 47 pp.

[Li12] Xin Li, *Semigroup C^*-algebras and amenability of semigroups*, J. Funct. Anal. **262** (2012), no. 10, 4302–4340, DOI 10.1016/j.jfa.2012.02.020.

[McA74] D. B. McAlister, *0-bisimple inverse semigroups*, Proc. London Math. Soc. (3) **28** (1974), 193–221, DOI 10.1112/plms/s3-28.2.193.

[Nek05] Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, American Mathematical Society, Providence, RI, 2005.

[Nes13] Sergey Neshveyev, *KMS states on the C^*-algebras of non-principal groupoids*, J. Operator Theory **70** (2013), no. 2, 513–530, DOI 10.7900/jot.2011sep20.1915.

[Nor14] Magnus Dahler Norling, *Inverse semigroup C^*-algebras associated with left cancellative semigroups*, Proc. Edinb. Math. Soc. (2) **57** (2014), no. 2, 533–564, DOI 10.1017/S0013091513000540.

[RW98] Iain Raeburn and Dana P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, Providence, RI, 1998.

[Spi12] Jack Spielberg, *C^*-algebras for categories of paths associated to the Baumslag-Solitar groups*, J. Lond. Math. Soc. (2) **86** (2012), no. 3, 728–754, DOI 10.1112/jlms/jds025.

[Sta17] Nicolai Stammeier, *A boundary quotient diagram for right LCM semigroups*, Semigroup Forum, posted on 2017, 1–16, DOI 10.1007/s00233-017-9850-0.

[Star15] Charles Starling, *Boundary quotients of C^*-algebras of right LCM semigroups*, J. Funct. Anal. **268** (2015), no. 11, 3326–3356, DOI 10.1016/j.jfa.2015.01.001.

School of Mathematics and Applied Statistics, University of Wollongong, Australia
E-mail address: `za.afsar@gmail.com`

School of Mathematics and Statistics, University of Sydney, Australia
E-mail address: `nathan.brownlowe@gmail.com`

Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, NO-0316 Oslo, Norway
E-mail address: `nadiasl@math.uio.no, nicolsta@math.uio.no`