The bidomain equations of cardiac electrophysiology

*Homogenization and stochastic forcing*

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Chapter 1

Introduction

The subject of this thesis is the bidomain model in cardiac electrophysiology [13, 35]. We derive the bidomain equations from a discrete cellular model using rigorous homogenization [11] in context of two-scale convergence [3]. We also provide a well-posedness theory for a stochastically forced [15, 33] bidomain model.

The bidomain equations is a quantitative model for propagation of electrical impulses in the tissue of the heart. Mathematically, it consists of a system of degenerate reaction-diffusion equations coupled to a system of ordinary differential equations [13,35].

There are two different perspectives on the cardiac bidomain model. In the cellular (or microscopic) model, the geometry of cardiac tissue is modeled at the cellular level. Cells are connected via gap junctions allowing electric currents to flow more freely in between them, see Figure 2.4. Therefore we think of the collective interior of all the cells of the heart as a single intracellular space [13]. The intra- and extracellular spaces are separated by the cell membrane. The cell membrane is electrically excitable, creating voltage across the membrane. This transmembrane potential [13] allows for electric signals to propagate along the tissue.

The high geometric complexity of the microscopic model makes it unsuitable for numerical computations, as well as for theoretical investigations. Therefore, a simplified macroscopic model is used in practice [35]. This is the one usually referred to as the bidomain model in the literature. Here the intra- and extracellular spaces are superimposed onto a single domain, i.e. they coexist at every point. The macroscopic model can be viewed as a phenomenological model [35], in the sense that it, although highly accurate in predicting experiments, lacks a clear physiological interpretation. The microscopic model, although never directly used in practice, is therefore important for heuristic understanding.
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This can be compared to how the heat equation can be used to model heat flow, without reference to the underlying molecular mechanisms. The derivation of the heat equation from Brownian motion of individual particles is however important for a physical interpretation of the heat equation [20].

Due to their respective limitations, it is of interest to establish a connection between the micro- and macroscopic bidomain models. This is the subject of homogenization [11,24].

Homogenization is practiced at different levels of rigor. Classically, one assumes that the solutions to the sequence of discrete models allows for an asymptotic expansion [11], and then uses this expansion to deduce the properties of the macroscopic model. This is the celebrated multiple scale method [11]. From a mathematical point of view on the other hand, it is not clear that such an asymptotic expansion is valid. For this reason, there has been developed the theoretical framework of two-scale convergence [3] and unfolding operators [10], to rigorously study the limits of discrete models. Not only do these theories rigorously justify the formal multiple scale expansion, but they can also explain interesting non-local phenomena in the limiting models [11].

The first topic of this thesis is to apply the theory of two-scale convergence to give a rigorous derivation of the bidomain model, starting from the cellular model. Thereby, a closer connection between the two is established. This has to our knowledge not been done before, although there has been some formal derivations based on the multiple scale method, see [14,24].

The difficulty of the homogenization problem for the bidomain equations has two main sources. The first difficulty is due to the degenerate structure of the equations, in combination with the highly oscillating underlying geometry. As a consequence, standard parabolic a priori estimates are not immediately available [13].

Secondly, the (nonlinear) dynamics of the cellular model takes place on the cell membrane, and this is a wildly oscillating geometric object. It is for instance not clear what a proper notion of ”strong convergence” of functions should be in this context. Some kind of strong convergence is however necessary in order to pass to the limit in the nonlinear equations.

As a second topic, we consider the macroscopic bidomain model with stochastic forcing. On the molecular level, the gating of ionic channels exhibits some random fluctuations [17]. It is reasonable to believed that the effect of this, on a macroscopic level, can be accurately modeled by space-time white noise [2].

While stochastic semilinear parabolic equations is by now a mature subject [15], the macroscopic bidomain system is again degenerate, so standard techniques cannot directly be applied. By using the bidomain oper-
ator [8], we reformulate the system as a single non-degenerate parabolic pseudo-differential equation [21]. The theory of analytic semigroups [32] is then applied to obtain regularity results. While we only consider a simple variant of the bidomain equations, it is a first step in obtaining a stochastic theory in this context.

1.1 Physical derivation of the cellular model

The cardiac tissue $U$ is split into two non-intersecting open sets, corresponding to the intra and extracellular spaces (see Figure 2.4),

$$U = U_i \cup U_e \cup \Gamma,$$

$$U_i \cap U_e = \emptyset,$$

with $\Gamma$ representing the cell membrane,

$$\Gamma = \partial U_i \cap \partial U_e.$$  

Under the quasi-static assumption that

$$\nabla \times E_j = 0, \quad j = i, e,$$

the electrical fields $E_i$ and $E_e$ in the intra and extracellular spaces are given by

$$E_j = \nabla u_j \quad j = i, e,$$

for some potentials $u_j$, $j = i, e$. According to Ohm’s law, the electric currents satisfy

$$J_j = -\Sigma_j \nabla u_j, \quad j = i, e,$$

where $\Sigma_i$ and $\Sigma_e$ are the conductivity tensors of the intra and extracellular media. It is conventional to regard electrical current as pointing from negative to positive, hence the minus sign.

Let $I_i$ and $I_e$ denote the applied currents, $(I_i, I_e) := I_{\text{app}}$, in the intra and extracellular domains respectively. By Gauss law, the intra and extracellular potentials $u_i$ and $u_e$ satisfy Poisson’s equation

$$-\text{div} (\Sigma_j \nabla u_j) = I_j \quad \text{in } U_j, \quad j = i, e.$$  

Let $\nu$ be the outward pointing unit normal of the intracellular space. By Kirchhoff’s law of conservation of currents, the current fluxes $\nu \cdot J_i$ and $\nu \cdot J_e$ are equal on the cell membrane:

$$\nu \cdot J_e = \nu \cdot J_i := I_m \quad \text{on } \Gamma.$$
where $I_m$ is the transmembrane current per unit area. The transmembrane potential $v$ is by definition the difference of $u_i$ and $u_e$ restricted to the cell membrane

$$v := u_i|_\Gamma - u_e|_\Gamma.$$  

It is known that the cell membrane actively transports charge by a variety of ions, resulting in an ionic current [17] across the membrane.

There are numerous cellular models, of varying degree of complexity, for describing the dynamics of these ion channels. One of the basic mechanism is that of voltage gated ionic channels [17], resulting in an action potential propagating along the tissue.

The first successful description of propagating action potentials, in terms of voltage gated ionic channels, is the celebrated Hodgkin-Huxley model [25]. It describes nerve signal propagation in squid giant axons.

We also mention its precursor, the FitzHugh–Nagumo model [18], which is a less detailed model for general threshold phenomena in excitable media.

In this thesis we will consider an even simpler model where the ionic current depends solely on the transmembrane potential,

$$I_{ion} = h(v).$$

It can be noted that this corresponds to the FitzHugh–Nagumo without a recovery variable.

In addition, we model the cell membrane as an ideal capacitor. That is, we assume that there is a linear relation

$$Q = C_m v,$$

where $Q$ is the charge across the cell membrane and the constant $C_m$ is the capacitance per unit area. Taking the time derivative of this relation, the capacitive current density takes the form

$$I_c = C_m \partial_t v.$$  

Adding the two current densities, the total transmembrane flux, $I_m$, takes the form

$$I_m = I_c + I_{ion} = C_m \partial_t v + h(v).$$

Assuming that the heart is embedded in an insulating medium, we impose the Neumann boundary conditions

$$\nu \cdot \nabla u_j = 0 \text{ on } \partial U.$$
Finally, an initial datum $v_0$ for $v$ is added to complete the system. The full system for $(u_i, u_e, v)$ thus reads

$$\begin{align*}
-\nu \cdot \Sigma_i \nabla u_i &= C_m \partial_t v + h(v) \quad \text{on } \Gamma, \quad j = i, e, \\
-\text{div} (\Sigma_j \nabla u_j) &= I_{\text{app}j} \quad \text{in } U_j, \quad j = i, e, \\
\nu \cdot \nabla u_j &= 0 \quad \text{on } \partial U \quad j = i, e, \\
v(0, \cdot) &= v_0 \quad \text{on } \{t = 0\} \times \Gamma.
\end{align*}$$

(1.1)

### 1.2 Dimensional analysis

In order to better understand the interaction of cellular and macroscopic scales, we carry out a dimensional analysis. The dimensionless equations thus obtained are more suitable to mathematical analysis.

The physical variables in the model are the following:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Dimension</th>
<th>Physical significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>L</td>
<td>Position</td>
</tr>
<tr>
<td>$t$</td>
<td>T</td>
<td>Time</td>
</tr>
<tr>
<td>$u_i, u_e, v$</td>
<td>JR</td>
<td>Electric potential energy.</td>
</tr>
<tr>
<td>$I_{\text{ion}}$</td>
<td>$JL^{-2}$</td>
<td>Ionic current across the membrane.</td>
</tr>
<tr>
<td>$I_{\text{app}i}, I_{\text{app}e}$</td>
<td>$JL^{-3}$</td>
<td>Applied current densities.</td>
</tr>
<tr>
<td>$l_c$</td>
<td>L</td>
<td>Diameter of a typical cell.</td>
</tr>
<tr>
<td>$\Sigma_i$</td>
<td>$R^{-1}L^{-1}$</td>
<td>Conductivity tensor.</td>
</tr>
<tr>
<td>$R_m$</td>
<td>$RL^2$</td>
<td>Area resistance of the membrane at resting potential level.</td>
</tr>
<tr>
<td>$C_m$</td>
<td>$R^{-1}L^{-2}T$</td>
<td>Capacity of the cell membrane per unit area.</td>
</tr>
</tbody>
</table>

Here we use the dimensional basis:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$:</td>
<td>length</td>
</tr>
<tr>
<td>$T$:</td>
<td>time</td>
</tr>
<tr>
<td>$R$:</td>
<td>resistance</td>
</tr>
<tr>
<td>$J$:</td>
<td>current</td>
</tr>
</tbody>
</table>

Let $\Lambda$ be a spatial scale to be determined later. Scaling the spatial variable $x$ by

$$\hat{x} = \frac{x}{\Lambda},$$

the first equation in (1.1) through the chain rule becomes

$$\nu \cdot \Sigma_{i,e} \nabla \hat{x} u_{i,e} = \Lambda C_m \partial_t v + \Lambda I_{\text{ion}}.$$  

(1.2)
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We continue by normalizing the conductivity tensors. Set

$$\bar{\mu} = \sum_{k=1}^{3} \lambda_{i,k} + \sum_{k=1}^{3} \lambda_{e,k},$$

where $\lambda_{j,k}$ are the eigenvalues of $\Sigma_j$. The dimensionless tensors are given by

$$\sigma_j = \frac{\Sigma_j}{\bar{\mu}},$$

which substituted into (1.2) yields

$$-\nu \cdot \sigma_j \nabla_\hat{x} u_j = \frac{\Lambda C_m}{\bar{\mu}} \partial_t v + \frac{\Lambda}{\bar{\mu}} I_{ion}, \quad j = i, e. \quad (1.4)$$

Next, we normalize the capacitance of the cell membrane $C_m$ by multiplying and dividing by $R_m$:

$$\nu \cdot \sigma_j \nabla_\hat{x} u_j = \frac{\Lambda R_m C_m}{R_m \bar{\mu}} \partial_t v + \frac{\Lambda}{\bar{\mu}} I_{ion}. \quad (1.5)$$

Observing that $R_m C_m$ has the dimension of time, we eliminate $R_m C_m$ from the second term by the time scaling

$$\hat{t} = \frac{t}{R_m C_m}.$$ 

This inserted into (1.5) and another application of the chain rule results in

$$\nu \cdot \sigma_j \nabla_\hat{x} u_j = \frac{\Lambda}{R_m \bar{\mu}} \partial_t \hat{v} + \frac{\Lambda}{\bar{\mu}} I_{ion}. \quad (1.6)$$

Let $\varepsilon$ be the the dimensionless quantity

$$\varepsilon = \frac{\Lambda}{R_m \bar{\mu}}.$$ 

The final independent variable to normalize is the cellular scale $l_c$. By setting the spatial scale $\Lambda$ to

$$\Lambda = \frac{l_c}{\varepsilon},$$

we normalize $\frac{l_c}{\varepsilon}$ to 1. Solving for $\Lambda$ yields the spatial unit

$$\Lambda = \sqrt{l_c R_m \bar{\mu}}.$$
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Denoting \( \hat{I}_{ion} := \frac{\Lambda}{\mu} I_{ion} \), we arrive at the dimensionless equation
\[
\hat{I}_m = \nu \cdot \sigma_j \nabla \hat{u}_j = \varepsilon \left( \partial_t \hat{v} + \hat{I}_{ion} \right), \quad j = i, e.
\] (1.7)

Similarly, by rescaling the applied currents \( \hat{I}_j = \frac{\Lambda^2}{\mu} I_j, \quad j = i, e \),
the second equation in (1.1) becomes
\[
-\text{div} \mathbf{u} \left( \sigma_j \nabla \hat{u}_j \right) = \hat{I}_j, \quad j = i, e.
\] (1.8)

For notational convenience we will from here on drop the hats.

Remark 1.1. The dimensionless parameter \( \varepsilon \) represents the ratio of the macroscopic and microscopic scales of the problem.

1.3 Assumptions of periodicity

Let \( Y \) be a reference unit cell
\[
Y = [0, 1]^3,
\]
and let \( Y_i \) and \( Y_e \) be the intra and extracellular spaces in the unit cell
\[
Y_j \subset Y \quad j = i, e, \quad Y_i \cup Y_e = Y.
\]
The cell membrane in the unit reference cell is denoted by \( \Gamma \),
\[
\Gamma \subset Y.
\]

As a simplifying assumption for homogenization, the intra and extracellular spaces are assumed \( \varepsilon \)-periodic in the sense that they are dilatations of the unit cell extended periodically (see Figure 1.1):
\[
U^e_j = U \cap \bigcup_{k \in \mathbb{Z}^3} \varepsilon (k + Y_j) \quad j = i, e.
\]

The assumption of a perfectly rectangular periodic pattern is of course not realistic, since in reality the (approximately) periodic pattern is curved along
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Figure 1.1: Schematic picture the idealized periodic domain. Each square (black lines) represents a rescaled translation of the reference cell. The intracellular space $U^\varepsilon_i$ (pink) and the extracellular space $U^\varepsilon_e$ (blue) separated by the cell membrane $\Gamma^\varepsilon$ (green). In reality cells are not rectangularly organized, but this is a simplifying assumption for homogenization.

the muscle fibers, see Figure 1.2. It is however necessary to make some simplifications to obtain a tractable mathematical problem.

We also set

$$\Gamma^\varepsilon = U \cap \bigcup_{k \in \mathbb{Z}^3} \varepsilon(k + \Gamma).$$

We assume that the dimensionless conductivity tensors takes the form

$$\sigma^\varepsilon_j(x) = \sigma_j\left(x, \frac{x}{\varepsilon}\right),$$

for some tensors

$$\sigma_j(x, y),$$

being 1-periodic in the second variable.

The full dimensionless cellular problem parametrized by $\varepsilon$ reads

$$-\text{div} (\sigma_j \nabla u_j) = I_j \quad \text{in} \ U^\varepsilon_j, \quad j = i, e,$$

$$-\nu \cdot \sigma_j \nabla u_j = \varepsilon \left( \partial_v v + h(v) \right) \quad \text{on} \ \Gamma^\varepsilon, \quad j = i, e,$$

$$\nu \cdot \nabla u_j = 0 \quad \text{on} \ \partial U, \quad j = i, e,$$

$$v(0, \cdot) = v_0 \quad \text{on} \ \Gamma^\varepsilon.$$

(1.9)
1.3. ASSUMPTIONS OF PERIODICITY

Figure 1.2: Real cardiac muscle cells seen from a microscope [27]. The dark dots are the nuclei, red area is intracellular and white is extracellular.
Chapter 2

Wellposedness of the cellular problem

In this section we consider the wellposedness of the cellular problem introduced in the previous chapter. The main difficulty comes from the degenerate structure of the temporal derivatives. Wellposedness has previously been established in [14, 38]. Both authors employ the change of variable \( v = u_i - u_e \) to convert (1.9) into non-degenerate abstract parabolic equation. In [14] the authors then appeals to theory of abstract variational inequalities, whereas [38] applies the Schauder fixed point theorem to show existence of a solution.

We use a similar change of variable but interpret the resulting variational problems in terms of concrete boundary value problems. We thereafter use Galerkin approximations and the compactness method to construct solutions. This approach leads to a series of a priori estimates which will be used in Chapter 3, where we rigorously derive the bidomain equations in the context of two-scale convergence. This makes our existence and uniqueness discussion somewhat more cumbersome since we have to keep track of how various constants depend on \( \varepsilon \).

2.1 Notation and mathematical preliminaries

The purpose of this section is to recall some basic definitions and results of functional analysis, measure theory and Sobolev spaces. In addition, it serves to establish some notation. This material is based on [12, 16, 21, 28, 36].

**Functional analysis.** The duality pairing between a Banach space \( X \) and its dual \( X' \) will be denoted by \( \langle \cdot, \cdot \rangle_{X',X} \), and we write

\[
\ell(x) = \langle \ell, x \rangle_{X',X} \quad x \in X, \; \ell \in X'.
\]
or just \( \langle \ell, x \rangle \) if there is no confusion which space we are referring to. A subset \( Y \) of a Banach space \( X \) is called *weak sequentially compact* if every sequence \( \{x_n\} \) in \( Y \) has a subsequence \( \{x_{n_k}\} \) such that

\[
\lim_{k \to \infty} \langle \ell, x_{n_k} \rangle = \langle \ell, x \rangle,
\]

for some \( x \in X \) and all \( \ell \in X' \). Every bounded set in a reflexive Banach space is weak sequentially compact \([28]\). In particular, bounded sequences in Hilbert spaces have weakly convergent subsequences.

Recall that the *weak topology* on \( X \) is the coarsest topology in which all \( \ell \in X' \) are continuous. As a consequence of the Hahn-Banach theorem, convex norm-closed sets are weakly closed \([28]\). Let \( p : X \to \mathbb{R} \) be convex and strongly lower semicontinuous. Thus the level sets \( \{x \mid p(x) \leq r\} \) are convex and norm-closed. According to the previous statement, these level sets are also weakly closed. This implies that \( p \) is weakly lower semicontinuous.

If \( Z \) is the dual of a Banach space \( X \), i.e. \( Z = X' \), and \( \{z_n\} \) is a sequence in \( Z \) we say that \( \{z_n\} \) is *weak-* convergent to \( z \in Z \) if

\[
\lim_{n \to \infty} \langle z_n, x \rangle = \langle z, x \rangle, \quad \forall x \in X.
\]

A subset \( B \) of \( Z \) is called *weak-* sequentially compact if any sequence in \( B \) has a subsequence that is weak-* convergent to some element in \( Z \). If \( X \) is separable then it is well known that bounded sets in \( X' \) are weak-* sequentially compact \([28]\).

Let \( X \) and \( Y \) be Banach spaces, and suppose that \( M \) is a surjective bounded linear map \( M : X \to Y \). The open mapping theorem asserts that \( M \) maps open sets in \( X \) to open sets in \( Y \) \([28]\). As a consequence, bounded, bijective, linear maps between Banach spaces have bounded inverses.

A linear map \( K : X \to Y \) between Banach spaces is compact if \( K \) maps bounded sets into precompact sets. The Fredholm alternative \([28]\) asserts that for a compact map \( K : X \to X \), the following dichotomy holds: *Either* the equation

\[(K + I)u = f,\]

is uniquely solvable for all \( f \in X \), *or* the equation

\[(K + I)u = 0,\]

has nonzero solutions. In the latter case, \((K + I)u = f\) is solvable precisely when

\[
f \in \ker(K^* + I)^\perp := \{f \in X \mid \langle (K^* + I)\ell, f \rangle = 0, \forall \ell \in X'\},
\]
where $K^*$ is the adjoint of $K$. For a topological space $X$ with a subspace $Y$ we write

$$Y \subset\subset X$$

if $Y$ is compactly contained in $X$.

**Measure theory.** We will assume familiarity with the basic results of measure theory, which can be found in [12]. These include:

- Construction of the $L^p$ spaces and their various properties such as density results, reflexivity and completeness.
- Convergence theorems such as Fatou’s lemma, Egoroff’s theorem and the dominated convergence theorem, as well as the relation between different modes of convergence.
- The Riesz-Kakutani representation theorem which characterizes the dual of $C_0(X)$, where $X$ is a locally compact Hausdorff space.

Let $(\Omega, \mu)$ be a measure space and let $X$ be a Banach space. A simple function is a function $g : \Omega \to X$ on the form

$$g = \sum_{j=1}^{n} x_j \chi_{A_j},$$

where $x_j \in X$ and the sets $A_j$ are measurable with respect to the Borel $\sigma$-algebra on $X$. A function $f : \Omega \to X$ is called *strongly measurable* if it is the pointwise limit of simple functions. The Bochner space [12] of strongly measurable functions with values in $X$ having finite $L^p$ norm will be denoted by $L^p(\Omega; X)$, with the norm

$$\|u\|_{L^p(\Omega; X)} = \int_{\Omega} \|u(t)\|_X^p \, d\mu.$$  

The Banach spaces of continuous functions from $[0, T]$ to $X$ is denoted by $C(0, T; X)$ with the norm

$$\|u\|_{C(0, T; X)} = \max_{t \in [0, T]} \|u(t)\|_X.$$  

**Distributions and Sobolev spaces.** The space of $k$ (for $k$ integer) times continuously differentiable functions in $U$ is denoted by

$$C^k(U),$$

with the special case $C(U) := C^0(U)$. The subspace where the $k$’th order derivatives extends continuously to the closure $\overline{U}$ of $U$, is denoted by $C^k(\overline{U})$. These are separable Banach spaces under the norm

$$\|u\|_{C^k} = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^\infty}.$$
Here we use the multi-index notation \( \partial^\alpha u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u \) and \( |\alpha| = \sum_j \alpha_j \).

We set
\[
C^\infty(U) = \bigcap_{k \in \mathbb{N}} C^k(U),
\]
and
\[
C_0^\infty(U) = \{ \varphi \in C^\infty(U) \mid \varphi \text{ has compact support in } U \}.
\]
The dual space of the latter (under a certain topology which we defer from specifying, but see [21, 26]) is the space of distributions and will be denoted by \( \mathcal{D}'(U) \). For an accessible introduction to the theory of distributions, cf [34].

For a non-negative integer \( k \), the Sobolev space \( W^{k,p}(U) \) is the space of distributions \( u \) such that \( \partial^\alpha u \in L^p \) for \( |\alpha| \leq k \). The space \( W^{k,p}(U) \) is given the norm
\[
\|u\|_{W^{k,p}(U)} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(U)}.
\]
Of particular interest is the case when \( p = 2 \), in which case we employ the notation \( H^k(U) := W^{k,2}(U) \). The letter "\( H \)" emphasizes the fact that these are Hilbert spaces.

For basic facts about Sobolev spaces on open sets, such as completeness, density results, Sobolev embeddings, the Poincaré-Wirtinger inequality and the Rellich-Kondrachov compactness, we refer to [16].

**Sobolev spaces on boundary surfaces.** Let \( U \) be an open bounded subset of \( \mathbb{R}^n \) and set \( M := \partial U \). We say that \( U \) is of class \( C^k \) if, for every point \( x \in M \), there exists an open neighborhood \( V \) of \( x \) in \( \mathbb{R}^n \) and a \( C^k \)-diffeomorphism [21] \( \kappa : V \to B(0,1) \subset \mathbb{R}^n \) such that \( \kappa(x) = 0 \) and
\[
\kappa(U \cap V) = \{ y = (y', y_n) \in B(0,1) \mid y_n > 0 \},
\]
see Figure 2.1.

The definition entails two statements. First, in the language of differential geometry, the boundary \( M \) is a compact, \( (n-1) \)-dimensional \( C^k \)-manifold [26]. This means that there exists a finite open covering \( \{V_j\}_{j=1}^J \) where \( V_j \subset M \) with corresponding diffeomorphisms \( \kappa_j : V_j \to \tilde{V}_j \subset \mathbb{R}^{n-1} \) such that \( \kappa_j \circ \kappa_k^{-1} \in C^k(\tilde{V}_j \cap \tilde{V}_k) \).

Secondly, the definition also formalizes the notion of \( U \) being "locally on one side of \( M \)”, which excludes some pathological examples where \( U \) and \( M \) are badly mixed up. In particular, the outward pointing unit normal \( \nu \) is well defined.

A domain is called smooth if it is \( C^k \) for every \( k \in \mathbb{N} \).

The Euclidean surface measure on \( M \) is denoted by \( S \). This measure can be constructed from Riesz-Kakutani’s representation theorem via the
2.1. NOTATION AND MATHEMATICAL PRELIMINARIES

Figure 2.1: A neighborhood of the boundary is mapped into the half-space.

Riemann integral for continuous functions. If $f$ has support in the coordinate patch $V_j \subset M$, then

$$
\int_M f \, dS = \int_{\tilde{V}_j} f(\kappa_j^{-1}(y)) |\kappa_j^{-1}| \, dy,
$$

where $|\kappa_j^{-1}|$ is the Jacobian determinant of $\kappa_j^{-1}$. The Lebesgue spaces $L^p(M)$ should from here on always be understood with respect to the Euclidean surface measure.

When $M$ is smooth, the Euclidean surface measure provides a natural injection

$$
C^\infty(M) \hookrightarrow C^\infty(M)',
$$

given by

$$
\langle \varphi, \psi \rangle = \int_M \varphi \psi \, dS,
$$

thus we can unambiguously associate locally integrable functions with distributions on $M$. There are many ways of defining the Sobolev spaces on surfaces (or more general manifolds). For instance, they can be defined local coordinates [21, 26], by difference quotients or as trace spaces. These typically yields equivalent norms under some modest regularity assumptions of the surface.

Since we will be dealing with oscillating domains and need uniform estimates, we will use an explicit norm provided by the Riesz potential [11]. For $s \in (0, 1)$ the fractional Sobolev space $H^s(M)$ is the set of (locally integrable) distributions on $M$ for which

$$
\|u\|^2_{H^s(M)} := \int_M |u|^2 \, dS + \int_M \frac{|u(x) - u(y)|^2}{|x - y|^{n/2 + s}} \, dS(x)dS(y) < \infty.
$$
It is well known that $H^s(M) \subset \subset H^r(M)$ if $s > r$, see for example [21].

The dual space of $H^s(M)$ will be denoted by $H^{-s}(M)$, and $L^2(M)$ is naturally embedded into $H^{-s}(M)$ by

$$\langle u, v \rangle = \int_M uv\,dS, \quad u \in L^2(M), \, v \in H^s(M).$$

For $s \in (-1, 1)$, $H^s(M)$ is a separable Hilbert space.

The homogeneous fractional seminorm is

$$|u|_{H^s_0(M)} := \int_M \int_M \frac{|u(x) - u(y)|^2}{|x - y|^{n/2 + s}}\,dS(x)dS(y). \quad (2.1)$$

Some miscellaneous function spaces.

The quotient space of $H^1(U)$, under the equivalence relation

$$u \sim v \iff u - v \text{ is constant},$$

is a Hilbert space and will be denoted by

$$H^1(U)/\mathbb{R}.$$

Let $Y$ be the unit cube $[0, 1]^n$ and $U$ be an open set in $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called $Y$-periodic (or just periodic) if $f(x + k) = f(x)$ for all $k \in \mathbb{Z}^n$. The space of smooth periodic functions is denoted by $C^\infty_#(Y)$, and the closure of this space under the $H^1$ norm will be denoted by $H^1_#(Y)$.

The functions in this space have the same trace on opposite sides of the unit cube [11, Proposition 3.49].

If $M$ is an open set of $\mathbb{R}^n$ or a surface, the open time-space cylinder is denoted by

$$M_T := (0, T) \times M.$$

For spaces of time dependent functions, the Aubin-Lions theorem [6,29] provides the compact embedding

$$\{u \in L^2(0, T; H^{1/2}(M)) \mid \partial_t u \in L^2(0, T; H^{-1/2}(M))\} \subset \subset L^2(M_T), \quad (2.2)$$

and the continuous embedding

$$\{u \in L^2(0, T; H^{1/2}(M)) \mid \partial_t u \in L^2(0, T; H^{-1/2}(M))\} \hookrightarrow C(0, T; L^2(M)). \quad (2.3)$$

If $U$ is a Lipschitz domain, then $H^{1/2}(\partial U)$ is well defined [10]. A function $u$ in $H^1(U)$ has trace in $H^{1/2}(\partial U)$ which will be denoted by

$$u|_{\partial U},$$
whenever explicit reference to the boundary value is needed. The trace satisfies the bound
\[ \|u|_{\partial U}\|_{H^{1/2}(\partial U)} \leq C\|u\|_{H^1(U)}, \]
for some constant independent of \( u \). We also have an inverse trace inequality; for any function \( g \in H^{1/2}(\partial U) \) there exists an extension \( Kg \in H^1(U) \) with
\[ \|Kg\|_{H^1(U)} \leq C\|g\|_{H^{1/2}(\partial U)}, \]
such that \( K \) is a right inverse of the trace operator:
\[ (Kg)|_{\partial U} = g, \quad \forall g \in H^{1/2}(\partial U). \]

The operator, \( K : H^{1/2}(\partial U) \to H^1(U) \), is called a lifting operator, and can be explicitly constructed via the Fourier transform [21] in local coordinates.

The subspace of \( H^1(U) \) for which the trace vanishes is denoted by \( H^1_0(U) \) and coincides with the closure of \( C_0^\infty(U) \) in \( H^1(U) \).

**Generalized normal trace.** Let \( U \) be an open bounded subset of \( \mathbb{R}^N \) of class \( C^1 \). Define the unbounded operator
\[ \text{div}_{\max} : (L^2(U))^N \to L^2(U), \]
with domain
\[ D(\text{div}_{\max}) = \{ F \in (L^2(U))^N \mid \text{div} F \in L^2(U) \}, \]
and the action in the sense of distributions. We define the generalized normal trace
\[ \gamma_\nu : D(\text{div}_{\max}) \to H^{-1/2}(\partial U), \]
by
\[ \langle \gamma_\nu F, v \rangle_{H^{-1/2}(\partial U), H^{1/2}(\partial U)} = \langle \text{div} F, Kv \rangle_{L^2(U)} + \langle F, \nabla Kv \rangle_{(L^2(U))^N}, \]
where \( K \) is a lifting operator. This definition is independent of the choice of lifting operator and extends the usual normal trace [21]. We will often write \( \nu \cdot F \) instead of \( \gamma_\nu F \) even for the extension. A celebrated theorem of De Rham [39] states that a distribution \( u \in \mathcal{D}'(U)^N \) is a gradient if
\[ \langle u, \varphi \rangle = 0, \quad \forall \varphi \in C_0^\infty(U)^N, \quad \text{div} \varphi = 0. \]

From this result one can deduce the following orthogonal decomposition of \( L^2(U)^N \): Let \( V \) be the subspace of gradients in \( (L^2(U))^N \):
\[ V = \{ v \in (L^2(U))^N \mid v = \nabla u, \ u \in H^1(U) \}. \]
Then $V$ is the orthogonal complement of the divergence-free functions with vanishing normal trace:

$$V = (\ker \text{div}_{\text{max}} \cap \ker \gamma_{\nu})^\perp.$$

(2.8)

Let $A$ be a second order elliptic operator in divergence form

$$Au = \text{div} (a \nabla u),$$

where $a(x) \in C^1(U; \mathbb{R}^{n \times n})$ satisfies, for some $c > 0$,

$$\xi^t a(x) \xi \geq c|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, x \in U.$$

The formal adjoint $\tilde{A}$ of $A$ is given by

$$\tilde{A}u = -\text{div} (a^t \nabla u),$$

where $a^t$ is the transpose of $a$. Let $A$ act on distributions by

$$\langle Au, \varphi \rangle := \left\langle u, \tilde{A}\varphi \right\rangle, \quad u \in \mathcal{D}'(U), \varphi \in C_0^\infty(U).$$

Define the unbounded operator $A_{\text{max}}$ on $L^2(U)$ with domain

$$D(A_{\text{max}}) = \{u \in L^2(U) \mid Au \in L^2(U)\},$$

where the action is in the sense of distributions. Greens formula extends to $H^1(U)$ in the following way: Let $u \in D(A_{\text{max}}) \cap H^1(U)$, then for all $v \in H^1(U)$

$$\langle \gamma_{\nu} a \nabla u, v \rangle_{H^{-1/2}(\partial U), H^{1/2}(\partial U)} = \langle Au, v \rangle_{L^2(U)} + \int_U a \nabla u \cdot \nabla v \, dx.$$  

(2.9)

### 2.2 Some definitions and results related to periodic structures

In this section we give precise definitions and assumptions of the cellular geometry. There is a number of technical issues to address.

The first one is how to treat the cells that intersect the boundary of $U$. To avoid certain technical issues, only cells completely contained in $U$ will be included in the intracellular region. As a consequence, the cell membrane will never intersect the boundary of $U$.

Secondly, we modify the boundary of $U^\varepsilon$ in the cells bordering to the boundary cells so that the boundary remains smooth.
Finally, we will make assumptions ensuring that the perforated domain is sufficiently connected, which is manifested in terms of a uniform Poincaré constant.

Let $U$ be a bounded, convex, smooth subset of $\mathbb{R}^3$, and let $Y_i$ be an open subset of the unit cube $Y$,

$$Y_i \subset Y := [0,1]^3.$$

For each $\varepsilon > 0$, let $K^\varepsilon$ be the set of indices such that the translated rescaled unit cube is completely contained in $U$:

$$K^\varepsilon := \{ k \in \mathbb{Z}^3 \mid \varepsilon(k + Y_i) \subset U \}.$$

The unmodified periodic intracellular space $\widetilde{U}^\varepsilon_i$ is

$$\widetilde{U}^\varepsilon_i := \bigcup_{k \in K^\varepsilon} \varepsilon(k + Y_i). \quad (2.10)$$

We then close the boundary of $\widetilde{U}^\varepsilon$ such that the boundary is compact and smooth. This can be done by fixing $2^3$ special boundary cells (where the
shapes depend on how many sides are connected to the bulk) once and for all, and then rescaling them, see Figure 2.3.

This modification will be denoted $U^\varepsilon_i$, see Figure 2.4.

We then denote the cell membrane by

$$\Gamma^\varepsilon = \partial U^\varepsilon_i,$$

and the extracellular space

$$U^\varepsilon_e := U \setminus U^\varepsilon_i.$$

Hence the cells intersecting the boundary of $U$ are considered to belong to the extracellular space.

We assume that the surface $\Gamma^\varepsilon$ thus obtained, is smooth, connected and compact. We also assume that $U^\varepsilon_i$ and $U^\varepsilon_e$ are connected. In addition, we assume that $\partial Y_j$ is Lipschitz, for $j = i, e$. Note that this does not follow from the assumption that $\Gamma^\varepsilon$ is smooth, since there in general will be corners where two cells meet, cf Figure 2.2.

The following uniform Poincaré inequality can be proved with piecewise linear interpolation (cf. [10] for details). It can also be obtained via an extension operator $E : H^1(U^\varepsilon_j) \rightarrow H^1(U)$, but this approach puts more restrictive conditions on the geometry. In fact, our philosophy throughout this thesis will be to avoid extension operators altogether.

**Lemma 2.1** (Uniform Poincaré inequality). There exist a constant $C$, independent of $\varepsilon$, such that,

$$\left\| u - \frac{1}{|U^\varepsilon_j|} \int_{U^\varepsilon_j} u \, dx \right\|_{L^2(U^\varepsilon_j)} \leq C \| \nabla u \|_{L^2(U^\varepsilon_j)},$$

for all $u \in H^1(U^\varepsilon_j)$ and all $\varepsilon > 0$, $j = i, e$. 

![Figure 2.3: A modified boundary cell.](image)
2.2. SOME DEFINITIONS AND RESULTS RELATED TO PERIODIC STRUCTURES

Figure 2.4: The periodic domain, $U_i^\varepsilon$ (pink), $U_e^\varepsilon$ (blue) and $\Gamma^\varepsilon$ (green). Notice how only cells strictly included in $U$ contribute to the intracellular space $U_i^\varepsilon$. In general, we assume that both $U_i^\varepsilon$ and $U_e^\varepsilon$ are connected, although that is difficult to illustrate in a 2-dimensional image.

Lemma 2.1 can be interpreted as "uniform connectedness" of the perforated domains.

The following trace inequality follows by scaling each $\varepsilon$-cell and can be found in [5]. Note that the proof relies on the Lipschitz assumption of $\partial Y_i$, $\partial Y_e$.

Lemma 2.2 (Trace). There exists a constant $C$, independent of $\varepsilon$, such that

$$
\varepsilon \|u\|_{L^2(\Gamma^\varepsilon)}^2 \leq C \left( \|u\|_{L^2(U_j^\varepsilon)}^2 + \varepsilon^2 \|\nabla u\|_{L^2(U_j^\varepsilon)} \right),
$$

(2.11)

for all $u \in H^1(U_j^\varepsilon)$, $j = i, e$.

Proof. We estimate each $\varepsilon$-cube separately. Let the diffeomorphism

$$
\kappa : (k + \Gamma) \to \varepsilon(k + \Gamma),
$$
be defined by $\kappa : x \mapsto \varepsilon x$. The Jacobian determinant, as a mapping between surfaces, is $|J_\kappa| = \varepsilon^2$. Hence,

$$\int_{\varepsilon(k+\Gamma)} |\varphi(x)|^2 dS(x) = \varepsilon^2 \int_{k+\Gamma} |\varphi(\varepsilon x)|^2 dS(x). \quad (2.12)$$

Obviously, since $\Gamma \subset \partial Y_j$ (cf. Figure 2.2),

$$\|\varphi\|_{L^2(\Gamma)} \leq \|\varphi \rvert_{\partial Y_j}\|_{L^2(\partial Y_j)}.$$

Since $\partial Y_j$ is Lipschitz, there is according to (2.4) a constant $C$, independent of $\varepsilon$, such that

$$\varepsilon^2 \int_{k+\Gamma} |\varphi(\varepsilon x)|^2 dS(x) \leq \varepsilon^2 C \left( \int_{k+Y_j} |\varphi(x)|^2 dx + \int_{k+Y_j} |\nabla \varphi(\varepsilon x)|^2 dx \right).$$

By the chain rule,

$$\int_{k+Y_j} |\nabla \varphi(\varepsilon x)|^2 dx = \varepsilon^2 \int_{k+Y_j} |\nabla \varphi(\varepsilon x)|^2 dx.$$

Using that the 3-D Lebesgue measure scales like $dx(rz) = r^3 dx(z)$, we obtain

$$\varepsilon^2 \int_{\varepsilon(k+\Gamma)} |\varphi(x)|^2 dS(x) \leq C \left( \frac{1}{\varepsilon} \int_{\varepsilon(k+Y_j)} |\varphi|^2 dx + \varepsilon \int_{\varepsilon(k+Y_j)} |\nabla \varphi|^2 dx \right).$$

multiplying by $\varepsilon$ and summing over $k$ the statement of the lemma follows. \qed

The following lemma will be used to estimate how well the local averages approximates a function in $H^{1/2}$. The main ingredient of the proof is the compact embedding $H^{1/2}(M) \hookrightarrow L^2(M)$, valid for sufficiently regular surfaces. Although we have not defined precisely what it means for a surface to have a regular boundary, we assume that the compact embedding holds for the cell membrane $\Gamma$ in the reference cell.

Recall that

$$|u|_{H^{1/2}_0(M)}^2 = \int_M \int_M \frac{|u(x) - u(y)|^2}{|x-y|^{(n+1)/2}} dS(x)dS(y).$$

Lemma 2.3 (Fractional Poincaré-Wirtinger inequality on surfaces). Let $M$ be a smooth connected surface in $\mathbb{R}^N$ of finite measure. Then there exists a constant $C$ such that

$$\left\| u - \frac{1}{|M|} \int_M u dS \right\|_{L^2(M)} \leq C |u|_{H^{1/2}_0(M)}, \quad (2.13)$$

for all $u \in H^{1/2}(M)$. 

2.3. ASSUMPTIONS ON THE DATA

Proof. We argue by contradiction. Let \( u_k \) be sequence in \( H^{1/2}(\Gamma) \) such that
\[
\left\| u_k - \frac{1}{|M|} \int_M u_k \right\|_{L^2(M)} > k |u_k|_{H^{1/2}_0(M)}.
\] (2.14)

By normalizing we can assume that
\[
\frac{1}{|M|} \int_M u_k dS = 0, \quad \|u_k\|_{L^2(M)} = 1, \quad |u_k|_{H^{1/2}_0(M)} < \frac{1}{k}.
\] (2.15)

Hence \( u_k \) is a bounded sequence in \( H^{1/2}(M) \), so by weak sequential compactness there is a subsequence (which we do not bother to relabel) such that
\[ u_k \rightharpoonup u \text{ weakly in } H^{1/2}(M). \] (2.16)

Since the embedding \( H^{1/2}(M) \hookrightarrow L^2(M) \) is compact, \( u_k \) converges strongly in \( L^2(M) \). Hence the limit satisfies
\[
\|u\|_{L^2(M)} = 1, \quad \frac{1}{|M|} \int_M udS = 0.
\] (2.17)

Since the seminorm \( |\cdot|_{H^{1/2}_0(M)} \) is convex and (obviously) continuous with respect to the norm topology in \( H^{1/2}(M) \), it is also weakly lower semicontinuous. Hence,
\[
|u|_{H^{1/2}_0(M)} \leq \liminf_{k \to \infty} |u_k|_{H^{1/2}_0(M)} = 0.
\] (2.18)

Since \( M \) is connected, (2.18) implies that \( u \) is constant. But this clearly violates (2.17), and this contradiction proves the lemma. \( \Box \)

Remark 2.4. The surface \( M \) is not assumed to be compact, i.e. it can have a nonempty smooth boundary. In particular it applies to the cell membrane \( \Gamma \) in the unit cell, see Figure 2.2.

2.3 Assumptions on the data

We assume that the conductivity tensors are given by
\[
\sigma_j^e = \sigma_j \left( x, \frac{x}{\varepsilon} \right), \quad j = i, e,
\] (2.19)

for some smooth bounded functions
\[
\sigma_j(x, y) \in C^\infty(\overline{U}; C^\infty(\overline{Y}_j)), \quad j = i, e.
\]
They are assumed to satisfy the usual ellipticity conditions, i.e., there exists $\alpha > 0$ such that

$$\eta \cdot \sigma_j \eta \geq \alpha |\eta|^2, \quad \forall (x,y) \in U \times Y_j, \forall \eta \in \mathbb{R}^3.$$  

We also assume symmetry

$$\sigma^T_j = \sigma_j,$$

where $\sigma^T_j$ denotes the transpose of $\sigma_j$. The ionic current $h$ is assumed to be continuous with $h(0) = 0$, and satisfy the monotonicity condition

$$\frac{h(u) - h(v)}{u-v} \geq -C_h,$$

for some positive constant $C_h$ and all $u,v \in \mathbb{R}, u \neq v$.

As a consequence we have

$$h(u)u + C_h u^2 \geq 0.$$  

(2.21)

We also require that $h$ grows like a third degree polynomial at infinity

$$0 < \liminf_{|z| \to \infty} \frac{h(z)}{z^3} \leq \limsup_{|z| \to \infty} \frac{h(z)}{z^3} < \infty,$$

thus there exist constants such that

$$C_1 |z|^3 \leq |h(z)| \leq C_2 (1 + |z|^3).$$

(2.22)

Indeed, the typical example is a third degree polynomial of the form

$$h(z) = z(z-a_1)(z-a_2)$$

for some positive constants $a_1 < a_2$, see Figure 2.5. The associated ODE

$$\partial_t x = -h(x)$$

then has two stable fixedpoints at $x = 0$ and $x = a_2$, and one unstable fixedpoint at $x = a_1$. Such an ODE can be thought of as a switch, since $x$ stabilizes either at 0 or at $a_2$ depending on weather it is pushed above the threshold $a_1$ or not.

We assume that the initial datum $v^\varepsilon_0$ is in $L^2(\Gamma^\varepsilon)$, with

$$\varepsilon^{1/2}\|v^\varepsilon_0\|_{L^2(\Gamma^\varepsilon)} \leq C.$$  

The scaling factor is there since the surface measure $|\Gamma^\varepsilon|$ of $\Gamma^\varepsilon$ grows proportionally to $\varepsilon$. Thus if for example $v^\varepsilon_0$ is the restriction to $\Gamma^\varepsilon$ of a fixed smooth function on $U$, then the $L^2$-norm of $v^\varepsilon_0$ is proportional to $1/\sqrt{\varepsilon}$. 

Furthermore, we assume that there is an (artificial) decomposition of the initial datum as

$$v^\varepsilon_0 = u^\varepsilon_{0,i}\big|_{\Gamma^\varepsilon} - u^\varepsilon_{0,e}\big|_{\Gamma^\varepsilon},$$

where

$$(u^\varepsilon_{0,i}, u^\varepsilon_{0,e}) \in H^1(U^\varepsilon_i) \times H^1(U^\varepsilon_e)$$

and

$$\sum_{j=i,e} \|u^\varepsilon_{0,j}\|_{H^1(U^\varepsilon_j)} \leq C,$$

for some constant $C$ independent of $\varepsilon$. For technical reasons we will also require that

$$\varepsilon^{1/4}\|v^\varepsilon_0\|_{L^4(\Gamma^\varepsilon)} \leq C.$$

This may seem superfluous in view of the Sobolev embedding

$$H^{1/2}(\Gamma^\varepsilon) \hookrightarrow L^4(\Gamma^\varepsilon),$$

but we have however not shown that the implicit constant in this embedding can be chosen independently of $\varepsilon$.

The applied currents are also assumed to be uniformly bounded

$$\|I^\varepsilon_j\|_{L^2(U^\varepsilon_j)} \leq C, \quad j = i, e,$$
2.4 Functional-analytic setting

In the rest of this section we will consider \( \varepsilon \) fixed but arbitrary. Although everything will depend implicitly on \( \varepsilon \), we will drop this symbol wherever possible to simplify notation. Thus \( U_i \) is short for \( U_i^\varepsilon \), \( \Gamma \) means \( \Gamma^\varepsilon \) etc.

Suppose that \( u_e \) is a \( C^2 \) solution of the cellular problem (1.9). Let \( \nu \) be the inward pointing unit normal to \( U_e \). By Green’s formula,

\[
\int_{U_e} \sigma_e \nabla u_e \cdot \nabla \varphi \, dx = \int_{U_e} (-\text{div}(\sigma_e \nabla u_e)) \varphi \, dx - \int_{\partial U_e} \nu \cdot (\sigma_e \nabla u_e) \varphi \, dS,
\]

for all \( \varphi \in H^1(U_e) \). Due to the Neumann boundary condition, \( \nu \cdot (\sigma \nabla u_e) \) vanishes on \( \partial U_e \setminus \Gamma \). Thus \( u_e \) satisfies the equation

\[
\int_{U_e} \sigma_e \nabla u_e \cdot \nabla \varphi \, dx = \int_{U_e} I_e \varphi \, dx + \int_{\Gamma} I_m \varphi \, dS, \quad \forall \varphi \in H^1(U_e),
\]

where \( I_m = \varepsilon (\partial_t v + h(v)) \). Arguing similarly for \( u_i \), we obtain the equation

\[
\int_{U_i} \sigma_i \nabla u_i \cdot \nabla \varphi \, dx = \int_{U_i} I_i \varphi \, dx - \int_{\Gamma} I_m \varphi \, dS, \quad \forall \varphi \in H^1(U_i),
\]

where \( \nu \) this time is the outward pointing unit normal to \( U_i \), which explains the sign difference in the last term.

We are now in a position to state the weak formulation of the cellular problem (1.9) introduced in Chapter 1.

**Definition 2.5** (Weak formulation). A weak solution of (1.9) is a pair \( (u_i, u_e) \in L^2(0, T; H^1(U_i) \times H^1(U_e)) \), with \( v := u_i|_{\Gamma} - u_e|_{\Gamma} \), such that \( \partial_t v \in L^2(0, T; H^{-1/2}(\Gamma)) \), \( v(0) = v_0 \), and

\[
\varepsilon \langle \partial_t v, \varphi_i \rangle + \int_{U_i} \sigma_i \nabla u_i \cdot \nabla \varphi_i \, dx + \varepsilon \int_{\Gamma} h(v) \varphi_i \, dS = \int_{U_i} I_i \varphi_i \, dx, \quad (2.25)
\]
\[ \varepsilon \langle \partial_t v, \varphi_e \rangle - \int_{U_e} \sigma_e \nabla u_e \cdot \nabla \varphi_e \, dx + \varepsilon \int_{\Gamma} h(v) \varphi_e \, dS = - \int_{U_e} I_e \varphi_e \, dx, \quad (2.26) \]

for all \( \varphi_j \) in \( L^2(0, T; H(U_j)) \), \( j = i, e \) and a.e. \( t \in (0, T) \). Here the duality pairing \( \langle \cdot, \cdot \rangle \) means \( \langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \).

**Remark 2.6.** By the embedding (2.3), \( v \in C(0, T; L^2(\Gamma)) \), so the pointwise evaluation \( v(0) \) is well defined.

**Remark 2.7.** By (4.39) and the Sobolev embedding \( H^{1/2}(\Gamma) \hookrightarrow L^4(\Gamma) \), the ionic current \( h(v) \) belongs to \( L^{4/3}(\Gamma) \). Consequently, by Hölder’s inequality,

\[
\int_{\Gamma} |h(v)(\varphi_i - \varphi_e)| \, dS \leq \|h(v)\|_{L^{4/3}(\Gamma)} \|\varphi_i - \varphi_e\|_{L^4(\Gamma)} \\
\leq C (1 + \|v\|_{L^4}) \|\varphi_i - \varphi_e\|_{L^4(\Gamma)} \\
\leq C \left( 1 + \|v\|_{H^{1/2}(\Gamma)} \right) \|\varphi_i - \varphi_e\|_{H^{1/2}(\Gamma)},
\]

so the integral does make sense.

### 2.4.1 Orthogonal decomposition

By subtracting the equation (2.26) for \( u_e \) from equation (2.25) for \( u_i \), we obtain the equivalent formulation

\[
\int_{\Gamma} I_m(\varphi_i - \varphi_e) \, dS + \sum_{j=i,e} \int_{U_j} \sigma_j \nabla u_j \cdot \nabla \varphi_j \, dx \\
= \sum_{j=i,e} \int_{U_j} I_j \varphi_j \, dx, \quad \forall (\varphi_i, \varphi_e) \in H^1(U_i) \times H^1(U_e). \quad (2.27)
\]

It is a well known phenomenon of physics that potential energy is only well-defined up to additive constants. Indeed, testing with \( \varphi = (c, c) \) in (2.27) for some constant \( c \in \mathbb{R} \), we see that it is necessary to introduce the compatibility condition

\[
\sum_{j=i,e} \int_{U_j} I_j \, dx = 0.
\]

It is thus natural to look for a solution in the quotient space

\[ W := H^1(U_i) \times H^1(U_e) / \{c, c\}, \]

which, through the scalar product on \( H^1(U_i) \times H^1(U_e) \), we canonically identify with the orthogonal complement of the constant functions, namely the functions of zero mean value:

\[ W = \left\{ w = (w_i, w_e) \in H^1(U_i) \times H^1(U_e) \mid \sum_{j=i,e} \int_{U_j} w_j \, dx = 0 \right\}. \]
We now recall that $\sigma_j$ is symmetric, so that the bilinear form $a(\cdot, \cdot)$ on $W$ given by
\[ a(u, v) = \sum_{j=i,e} \int_{U_j} \sigma_j \nabla u_j \cdot \nabla v_j \, dx, \] (2.28)
is also symmetric. Let $B$ be the boundary operator
\[ B : H^1(U_i) \times H^1(U_e) \to L^2(\Gamma), \]
defined by
\[ Bu = u_i|_\Gamma - u_e|_\Gamma, \quad u = (u_i, u_e). \]
The following lemma is a direct consequence of the uniform Poincaré inequality (2.1).

**Lemma 2.8.** The bilinear form
\[ \varepsilon(B(\cdot), B(\cdot))_{L^2(\Gamma)} + a(\cdot, \cdot) : W \times W \to \mathbb{R} \]
induces an equivalent norm on $W$, uniformly in $\varepsilon$. That is, there exist positive constants $C_1, C_2$, independent of $\varepsilon$, such that
\[ C_1 \left( \varepsilon \|Bu\|_{L^2(\Gamma)}^2 + a(u, u) \right) \leq \sum_{j=i,e} \left( \|u_j\|_{L^2(U_j)}^2 + \|\nabla u_j\|_{L^2(U_j)}^2 \right) \]
\[ \leq C_2 \left( \varepsilon \|Bu\|_{L^2(\Gamma)}^2 + a(u, u) \right), \] (2.29)
for all $u = (u_i, u_e) \in W$.

**Proof.** Decompose $u := (u_i, u_e)$ into
\[ u = (\bar{u}_i, \bar{u}_e) + (\tilde{u}_i, \tilde{u}_i), \]
where $\bar{u}_j$ is constant in $U_j$ and $\tilde{u}_j$ has zero mean in $U_j$. We have by orthogonality,
\[ \|u_j\|_{L^2(U_j)}^2 = \|\bar{u}_j\|_{L^2(U_j)}^2 + \|\tilde{u}_j\|_{L^2(U_j)}^2. \]
In view of the uniform Poincaré inequality (2.1), it remains to bound
\[ \sum_{j=i,e} \|\bar{u}_j\|_{L^2(U_j)}^2. \]
Now, since $u \in W$,
\[ \bar{u}_i + \bar{u}_e = 0. \]
It follows that
\[ (\bar{u}_i - \bar{u}_e)^2 = \bar{u}_i^2 + \bar{u}_e^2. \]
Since the surface measure of $\Gamma^\varepsilon$ satisfies
\[ \varepsilon |\Gamma^\varepsilon| \geq C, \]
we have
\[ \|\tilde{u}_i - \tilde{u}_e\|_{L^2(\Gamma^\varepsilon)}^2 = \|\tilde{u}_i\|_{L^2(\Gamma^\varepsilon)}^2 + \|\tilde{u}_e\|_{L^2(\Gamma^\varepsilon)}^2 \geq C \left( \tilde{u}_i^2 + \tilde{u}_e^2 \right). \]
This inequality proves the lemma.

Since the trace is continuous from $H^1(U_j)$ to $L^2(\Gamma)$, the subspace
\[ \ker B := \{ u \in W \mid Bu = 0 \}, \]
is closed in $W$ (with respect to the norm topology). Let $V$ be the orthogonal complement of $\ker B$ with respect to $a(\cdot, \cdot)$:
\[ V := \{ u \in W \mid a(u, v) = 0, \forall v \in \ker B \}. \]
We are thus able to decompose $W$ orthogonally as
\[ W = \ker B \oplus V. \]

The trace theorem (2.4) and the inverse trace theorem 2.5 show that $B$ is bounded and surjective from $W$ to $H^{1/2}(\Gamma)$. Since $B$ is obviously injective from $W/\ker B$ to $H^{1/2}(\Gamma)$ (and hence invertible), it follows from the open mapping theorem that the inverse
\[ B^{-1} : H^{1/2}(\Gamma) \rightarrow W/\ker B, \]
is bounded. Identifying $W/\ker B$ with $V$ in the usual way, we obtain an isomorphism (still denoted by $B$) from $V$ to $H^{1/2}(\Gamma)$.

By Lemma 2.8, $a(\cdot, \cdot)$ is coercive on $\ker B$.

With the notation just introduced, the variational formulation (2.27) takes the form
\[ \begin{cases} \text{Find } u \in W \text{ such that} \\ (I_m, B\varphi)_{L^2(\Gamma)} + a(u, \varphi) \\ = \sum_{j=i,e} \int_{U_j} I_j \varphi_j \, dx \quad \forall \varphi = (\varphi_i, \varphi_e) \in W. \end{cases} \]

(2.31)

where we recall that $I_m = \varepsilon (\partial_t v + h(v))$. Now, consider the decomposition,
\[ u = u^0 + u^h \]
(2.32)
where \( u^0 \in \text{ker} \, B \) and \( u^h \in V \). Observe that
\[
v = Bu = Bu^h,
\]
thus \( I_m \) is independent of \( u^0 \). Similarly we write \( \varphi = \varphi^h + \varphi^0 \) for any \( \varphi \in W \), and insert this decomposition into (2.31):
\[
(I_m, B\varphi)_{L^2(\Gamma)} + a(u^h + u^0, \varphi^h + \varphi^0)
= (I_m(u^h), B\varphi^h)_{L^2(\Gamma)} + a(u^h, \varphi^h) + a(u^0, \varphi^0)
= \sum_{j=i,e} \int_{U_j} I_j \varphi_j^h \, dx + \sum_{j=i,e} \int_{U_j} I_j \varphi_j^0 \, dx, \quad \forall \varphi = (\varphi_i, \varphi_e) \in W,
\]
where we used that \( a(u^h, \varphi^0) = a(u^0, \varphi^h) = 0 \). Thus to solve (2.31), it suffices to consider the following two variational problems:
\[
\begin{cases}
\text{Find } u^0 \in \text{ker} \, B \text{ such that } \sum_{j=i,e} \int_{U_j} I_j \varphi_j^0 = 0, \forall \varphi \in \text{ker} \, B, \\
a(u^0, \varphi) = \sum_{j=i,e} \int_{U_j} I_j \varphi_j^0, \forall \varphi \in \text{ker} \, B.
\end{cases} \tag{V^1}
\]
and
\[
\begin{cases}
\text{Find } u^h \in V \text{ such that } \sum_{j=i,e} \int_{U_j} I_j \varphi_j^0 = 0, \forall \varphi \in V.
\end{cases} \tag{V^2}
\]
At this point it is a good idea to identify the boundary conditions encoded in these two variational equations. Start by observing that
\[
C^\infty_0(U_i) \times C^\infty_0(U_e) \subset \text{ker} \, B,
\]
thus
\[
-\text{div}(\sigma_j \nabla u^0_j) = I_j \quad \text{in } U_j,
\]
in the sense of distributions. In the generalized sense of (2.6), it holds that
\[
a(u^0, \varphi) = \langle \partial_{\nu_i} u^0_i - \partial_{\nu_e} u^0_e, \varphi \rangle + \sum_{j=i,e} \int_{U_j} I_j \varphi_j^0 \, dx,
\]
where \( \partial_{\nu_j} u^0_j := \nu \cdot \sigma_j \nabla u^0_j \). But it then follows from (V^1) that
\[
\partial_{\nu_i} u^0_i - \partial_{\nu_e} u^0_e = 0 \quad \text{on } \Gamma.
\]
Thus (V^1) encodes the boundary value problem
\[
\begin{cases}
-\text{div}(\sigma_j \nabla u^0_j) = I_j \quad \text{in } U_j, \; j = i, e \\
u^0_i - u^0_e = 0 \quad \text{on } \Gamma \\
\partial_{\nu_i} u^0_i - \partial_{\nu_e} u^0_e = 0 \quad \text{on } \Gamma.
\end{cases} \tag{2.34}
\]
The boundary value problem implied by \((\mathcal{V}^2)\) is more mysterious. For all functions \(v = (v_i, v_e) \in V\) we have
\[-\text{div}(\sigma_j \nabla v_j^h) = 0, \quad \text{in } U_j, \quad j = i, e,\]
in the sense of distributions, and
\[\partial_{\nu_i} v_i^h - \partial_{\nu_e} v_e^h = 0\]
in the sense of (2.6). But how should then the source term \(I_j\) be understood? The following lemma addresses this issue. The proof relies on standard elliptic theory, see [16].

**Lemma 2.9.** Suppose
\[I := (I_i, I_e) \in L^2(U_i^T) \times L^2(U_e^T)\]
satisfies the compatibility condition
\[\sum_{j=i,e} \int_{U_j} I_j \varphi_j dx = 0, \quad \text{for a.e. } t \in (0, T).\]
Then the problem \((\mathcal{V}^1)\) admits a unique solution,
\[u^0 \in L^2((0, T; \ker B \cap (H^2(U_i) \times H^2(U_e))) .\]
The solution \(u^0\) satisfies the bound
\[\sum_{j=i,e} \|(u^0)^j\|_{H^1(U_j^T)} \leq C \sum_{j=i,e} \|I_j\|_{L^2(U_j^T)}, \quad (2.35)\]
uniformly in \(\varepsilon\).

**Remark 2.10.** It follows from the \(H^2\) regularity of \(u^0\) that
\[\partial_{\nu_i} u_i^0 - \partial_{\nu_e} u_e^0 = 0 \quad \text{on } \Gamma\]
holds in the usual trace sense.

We can now interpret the source term in \((\mathcal{V}^2)\) as a boundary condition. Apply Green’s formula with \(\varphi \in V\),
\[\sum_{j=i,e} \int_{U_j} I_j \varphi_j dx = \sum_{j=i,e} \int_{U_j} -\text{div}(\sigma_j \nabla u_j^0) \varphi_j dx = (\partial_\nu u_i^0, B\varphi)_{L^2(\Gamma)} + a(u^0, \varphi) = (\partial_\nu u_i^0, B\varphi)_{L^2(\Gamma)},\]
where \( \partial_{\nu} u^0 := \partial_{\nu^i} u^0 = \partial_{\nu^e} u^0 \) and \( a(u^0, \varphi) = 0 \) by orthogonality. Thus, we identify \((V^2)\) as the boundary value problem

\[
\begin{cases}
- \text{div}(\sigma_j \nabla u^h_i) = 0 & \text{in } U_j \\
- \partial_{\nu^i} u^h_i = - \partial_{\nu^e} u^h_e = I_m - \partial_{\nu^0} u^0 & \text{on } \Gamma \\
u^h_i - u^h_e = v & \text{on } \Gamma.
\end{cases}
\] (2.36)

Denote by \( K_j \) the solution operator to the Dirichlet problem,

\[
\begin{cases}
A_j u = 0 & \text{in } U_j, \\
u \cdot a_i \nabla K_i v, \\
u \cdot a_e \nabla K_e v.
\end{cases}
\] (2.37)

It is well known [9] that this operator satisfy the the estimate

\[
\|K_j \varphi\|_{L^2(U_j)} \leq C\|\varphi\|_{L^2(\Gamma)}.
\]

We introduce the Dirichlet-Neumann operators [21], \( P_j : H^s(\Gamma) \rightarrow H^{s-1}(\Gamma) \) defined by

\[
\begin{cases}
P_i v = \nu \cdot a_i \nabla K_i v, \\
P_e v = -\nu \cdot a_e \nabla K_e v.
\end{cases}
\] (2.38)

These are positive, symmetric, elliptic pseudodifferential operators of order 1 on \( \Gamma \) [21]. In particular \( P_i + P_e \) is invertible (at least modulo constants) and satisfy the mapping property,

\[
(P_i + P_e)^{-1} : H^{s-1}(\Gamma) \rightarrow H^s(\Gamma), \quad \forall s \in \mathbb{R}.
\]

Using this, we can explicitly express,

\[
\begin{align}
u^h_i |_{\Gamma} &= (P_i + P_e)^{-1} P_e v, \\
u^h_e |_{\Gamma} &= -(P_i + P_e)^{-1} P_i v.
\end{align}
\] (2.39) (2.40)

Thus the inverse of \( B \) takes the explicit form,

\[
B^{-1} v = \left( K_i (P_i + P_e)^{-1} P_e v, -K_e (P_i + P_e)^{-1} P_i v \right).
\] (2.41)

An import consequence of this characterization is the \( L^2 \)-bound,

\[
\sum_{j=i,e} \|B^{-1} v\|_{L^2(U_j)} \leq C\|v\|_{L^2(\Gamma)},
\]

which can be deduced from the mapping properties of the individual operators. Hence \( B^{-1} \) extends by continuity from \( H^{1/2}(\Gamma) \) to \( L^2(\Gamma) \). Unfortunately we have not been able obtain a uniform (in \( \varepsilon \)) estimate of \( B^{-1} \).
2.5 Galerkin approximations

We intend to construct approximate solutions to \((\nabla^2)^p\) by solving a sequence of equations projected on finite-dimensional subspaces of \(V\). Through the isomorphism \(B\) we can think of \(V\) as \(H^{1/2}(\Gamma)\). For this purpose we construct an orthonormal basis of \(L^2(\Gamma)\).

2.5.1 A basis for the Galerkin method

Consider the boundary value problem

\[
\begin{aligned}
-\text{div}(\sigma_j \nabla u^h_j) &= 0 & \text{in } U_j, \ j = i, e, \\
\partial_{\nu_i} \varphi_i &= \partial_{\nu_e} \varphi_e = \eta & \text{on } \Gamma.
\end{aligned}
\] (2.42)

Let \(G\) be the Neumann-Dirichlet operator [21, Definition 9.24]:

\[
G : \eta \mapsto B\varphi,
\]

which formally takes Neumann data to Dirichlet data. We define this operator rigorously by duality: Set \(X = H^{1/2}(\Gamma)/\mathbb{R}\). We identify \(X\) with

\[
X = \left\{ v \in H^{1/2}(\Gamma) \mid \int_\Gamma v \, dS = 0 \right\}.
\]

Let \(\tilde{a}(\cdot, \cdot)\) be the pullback of \(a(\cdot, \cdot)\) from \(V\) to \(X\):

\[
\tilde{a}(\cdot, \cdot) = a \left( B^{-1}(\cdot), B^{-1}(\cdot) \right).
\]

Since \(B^{-1}\) is an isomorphism from \(H^{1/2}(\Gamma)\) to \(V\), it is easy to see that \(\tilde{a}\) is an equivalent scalar product on \(X\). Fix \(\eta \in X'\), where

\[
X' = \left\{ u \in H^{-1/2}(\Gamma) \mid \langle u, 1 \rangle = 0 \right\},
\]

and set

\[
\langle \eta, \varphi \rangle = \tilde{a}(G\eta, \varphi), \quad \forall \varphi \in H^{1/2}(\Gamma),
\]

which is well-defined by the Riesz representation theorem. Extend \(G\) to all of \(H^{-1/2}(\Gamma)\) by

\[
G1 = 1.
\]

Clearly \(G\) maps \(H^{-1/2}(\Gamma)\) into \(H^{1/2}(\Gamma)\) (it is in fact also onto but that is less important to us). The restriction of \(G\) to \(L^2(\Gamma)\) is

i) compact,
ii) symmetric and
iii) positive.

Thus by the spectral theorem there exists an orthonormal basis of eigenfunctions \( \{ e_k \}_{k=1}^{\infty} \), corresponding to a decreasing sequence of positive eigenvalues \( \{ \lambda_k \}_{k=1}^{\infty} \). These functions diagonalize \( a \) in the following sense

\[
a \left( B^{-1} e_k, B^{-1} e_l \right) = \frac{1}{\lambda_k} a \left( B^{-1} G e_k, B^{-1} e_l \right) = \frac{1}{\lambda_k} (e_k, e_l).
\]

By (2.39) it follows that

\[
G^{-1} = P := P_i (P_i + P_e)^{-1} P_e,
\]

i.e., \( G \) is the inverse of a Dirichlet-Neumann-type operator. It is well-known that the Dirichlet-Neumann operator can be characterized as a first order elliptic pseudo-differential operator on \( \Gamma \) [21]. By elliptic regularity it then follows that the eigenfunctions \( e_k \) are smooth, and in particular sufficiently regular to allow for the calculations carried out in the next subsections.

### 2.5.2 Building approximate solutions

Define the finite-dimensional subspace \( V^n \) of \( L^2(\Gamma) \) by

\[
V^n := \text{span}\{ e_1, ..., e_n \}.
\]

We equip \( V^n \) with the scalar product

\[
(u, v) = \int_{\Gamma} uv \, dS.
\]

As an approximation to the initial datum in \( V^n \) we take orthogonal projection:

\[
v^n_0 = \sum_{k=1}^{n} \left( \int_{\Gamma} v_0 e_k \, dS \right) e_k.
\]

The approximate problem in \( V^n \) is now:

**Definition 2.11** (Galerkin approximation). A Galerkin approximation to the solution of (\( V^2 \)) in \( V^n \) is a function

\[
v^n \in C^1(0, T; V^n)
\]

satisfying the initial condition

\[
v^n_0 = v^n_0 \quad \text{on} \, \Gamma,
\]
and the equation

\[\varepsilon (\partial_t v^n, \varphi) + a(B^{-1}v^n, B^{-1}\varphi) + \varepsilon (h(v^n), \varphi) = \sum_{j=i,e} \int_{U_j} I_j(B^{-1}\varphi) dx,\]

(2.45)

for all \(\varphi\) in \(V^n\).

A generic element \(v\) in \(C^1(0,T; V^n)\) can be expressed as

\[v(t, x) = \sum_{k=1}^n d_k(t) e_k(x),\]

for some continuously differentiable coefficients \(d = (d_1, ..., d_n) \in C^1(0,T; \mathbb{R}^n)\). Insert this expression into (2.45) and let \(\varphi\) range over \(\{e_k\}_{k=1}^n\) to obtain the explicit ODE

\[
\partial_t d_k = -\frac{1}{\lambda_k} d_k - \left( h \left( \sum_{k=1}^n d_k(t) e_k(x) \right), e_k \right) + \sum_{j=i,e} \int_{U_j} I_j (B^{-1}e_k) dx \\
:= F_k(d).
\]

(2.46)

We claim that \(F = (F_1, ..., F_n) : \mathbb{R}^n \to \mathbb{R}^n\) is continuous (the choice of norm on \(\mathbb{R}^n\) is does not matter since all norms on finite dimensional vector spaces are topologically equivalent). Since the first and last term in the definition of \(F\) are linear, it suffices to show continuity of

\[d \mapsto \left( h \left( \sum_{l=1}^n d_l(t) e_l \right), e_k \right) \in L^2(\Gamma).\]

So suppose \(d^j \to d\) in \(\mathbb{R}^n\). This implies that

\[\lim_{j \to \infty} \|v_j - v\|^2_{L^\infty(\Gamma)} = 0,
\]

where \(v_j = \sum_{k=1}^n d^j_k e_k\). By continuity of \(h\) it follows that

\[\lim_{j \to \infty} \|h(v_j) - h(v)\|_{L^2(\Gamma)} = 0,
\]

which in combination with the Cauchy-Schwarz inequality proves the desired continuity.
We now appeal to Peano’s existence theorem for ODEs \cite{37} to conclude that for each \( n \) there exists a maximal interval \( I^n = [0, \rho^n) \subset [0, T) \) and a (not necessarily unique) \( C^1 \) function \( v^n \)

\[ v^n : [0, \rho^n) \to V^n \]

that solves (2.45) on \( I^n \). Taking \( u^n = B^{-1}v^n \) we see that \( u^n \) satisfies (V2) for all \( \varphi \in B^{-1}(V^n) \).

### 2.5.3 Energy estimates

In order to pass to the limit in (2.45) as \( n \to \infty \), we need some a priori estimates on the approximate solutions. These estimates will also play a decisive role in the section on homogenization. We will therefore be careful to not let any constant depend on \( \varepsilon \). In these estimates we fix \( n \) and let \( T \) be an arbitrary time in the maximal interval of existence \([0, \rho^n)\).

**Lemma 2.12.** The Galerkin solution \( u^n = B^{-1}v^n \) satisfies the following estimates

\[
\begin{align*}
\varepsilon^{1/2}\|v^n\|_{L^\infty(0,T;L^2(\Gamma))} & \leq c_1, \quad (2.47) \\
\|u^n\|_{L^2(0,T;(H^1(U_i) \times H^1(U_e)))} & \leq c_2, \quad (2.48) \\
\varepsilon^{1/2}\|\partial_t v^n\|_{L^2(\Gamma_T^\varepsilon)} & \leq c_3, \quad (2.49) \\
\varepsilon^{3/4}\|h(v^n)\|_{L^{1/3}(\Gamma_T)} & \leq c_4, \quad (2.50)
\end{align*}
\]

for some constants \( c_1, c_2, c_3, c_4 \) independent of \( n \) and \( \varepsilon \).

**Remark 2.13.** The presence of powers of \( \varepsilon \) in the estimates on the boundary terms is natural, since the measure of \( \Gamma^\varepsilon \) grows proportionally to \( 1/\varepsilon \). In particular we do not expect any pathological behavior as \( \varepsilon \to 0 \).

**Remark 2.14.** Even though \( H^{1/2}(\Gamma) \hookrightarrow L^4(\Gamma) \) we will establish the bound (2.50) separately. This is since neither the trace \( H^1(U_i^\varepsilon) \to H^{1/2} \), nor the Sobolev embedding \( H^{1/2}(\Gamma) \hookrightarrow L^4(\Gamma) \) are uniformly bounded in \( \varepsilon \).

**Proof.** We introduce the notation,

\[
I_{\text{app}}(x) = \begin{cases} I_i(x) & \text{if } x \in U_i \\ I_e(x) & \text{if } x \in U_e, \end{cases} \\
u^n(x) = \begin{cases} u^n_i(x) & \text{if } x \in U_i \\ u^n_e(x) & \text{if } x \in U_e, \end{cases}
\]
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hence \( \|I_{\text{app}}\|_{L^2(U)} = \sum_{j=i,e} \|I_j\|_{L^2(U_j)} \), and similarly for \( u^n \). Take \( \varphi = v^n \) in (2.45) to obtain

\[
\frac{\varepsilon}{2} \partial_t \|v^n\|_{L^2(\Gamma)}^2 + a(u^n, u^n) + \varepsilon \int_{\Gamma} h(v^n) v^n \, dS = \sum_{j=i,e} \int_{U_j} I_j u^n_j \, dx. \tag{2.51}
\]

We use the monotonicity assumption on \( h \) (2.20), to estimate the third term in (2.51) as follows:

\[
\varepsilon \int_{\Gamma} h(v^n) v^n \, dS \geq -\varepsilon C h \|v^n\|_{L^2(\Gamma)}^2. \tag{2.52}
\]

By Lemma 2.8, there exists a constant \( C_p \) independent of \( \varepsilon \) such that

\[
\|w\|_{H^1(U)} \leq C_p \left( \varepsilon \|Bw\|^2 + a(w, w) \right), \quad \forall w \in V. \tag{2.53}
\]

We estimate the last term in (2.51) using the Cauchy-Schwarz and Cauchy inequalities:

\[
\left| \sum_{j=i,e} \int_{U_j} I_j u^n_j \, dx \right| \leq \|I_{\text{app}}\|_{L^2(U)} \|u^n\|_{L^2(U)} \leq \frac{\delta^2}{2} \|u^n\|_{L^2(U)}^2 + \frac{1}{2\delta^2} \|I_{\text{app}}\|_{L^2(U_j)}^2. \tag{2.54}
\]

Take \( \delta \leq \sqrt{C_p} \) to deduce that for a.e. \( t \in (0, T) \),

\[
\varepsilon \frac{1}{2} \partial_t \|v^n\|_{L^2(\Gamma)}^2 + \sum_{j=i,e} \| \nabla u^n_j \|_{L^2(U_j)}^2 \leq C \left( \varepsilon \|v^n\|_{L^2(\Gamma)}^2 + \|I_{\text{app}}\|_{L^2(U_j)}^2 \right). \tag{2.55}
\]

Integrate the inequality (2.55) on \((0, T)\) to deduce that

\[
\varepsilon \|v^n(T)\|_{L^2(\Gamma)}^2 + \sum_{j=i,e} \| \nabla u^n_j \|_{L^2(U_j)}^2 \leq C \varepsilon \|v_0\|_{L^2(\Gamma)}^2 + \varepsilon C \int_0^T \|v^n\|_{L^2(\Gamma)}^2 \, dt + C \|I_{\text{app}}\|_{L^2(U_T)}^2. \tag{2.56}
\]

Apply Grönwall’s inequality to arrive at

\[
\varepsilon \|v^n(T)\|_{L^2(\Gamma)}^2 \leq \left( \varepsilon \|v_0\|_{L^2(\Gamma)}^2 + C \|I_{\text{app}}\|_{L^2(U_T)}^2 \right) (1 + C Te^{CT}),
\]
and consequently
\[ \varepsilon \| v^n \|_{L^\infty(0,T;L^2(\Gamma))}^2 \leq c_1, \] (2.57)
and, by (2.55),
\[ \sum_{j=1,e} \| \nabla u^n_j \|_{L^2(U_j')}^2 \leq c_2. \] (2.58)

Recall that \( \Gamma_T = (0,T) \times \Gamma \). Going back to (2.51) we deduce that
\[ \varepsilon \int_{\Gamma_T} |h(v^n)v^n| \, dS \, dt \leq C, \] (2.59)
which in view of (2.20) implies
\[ \varepsilon \| v^n \|_{L^4(\Gamma_T)}^4 \leq C, \]
and consequently by (2.22),
\[ \varepsilon \| h(v^n) \|_{L^{4/3}(\Gamma_T)}^{4/3} \leq c_4. \] (2.60)

Next, we take \( \varphi = \partial_t v^n \) in (2.45), which results in
\[
\varepsilon \| \partial_t v^n \|_{L^2(\Gamma)}^2 + a(u^n, \partial_t u^n) + \varepsilon \int_{\Gamma} h(v^n) \partial_t v^n \, dS = \sum_{j=1,e} \int_{U_j} I_j \partial_t u^n_j \, dx. \] (2.61)

Define \( H \) to be the primitive of \( h \):
\[ H(z) = \int_0^z h(\rho) \, d\rho, \] (2.62)
which, due to the assumptions on \( h \) satisfies
\[ H(z) + \frac{C_m}{2} z^2 \geq 0, \quad |H(z)| \leq C (|z| + |z|^4). \] (2.63)

Using this we can integrate equation (2.61) in time to obtain
\[
\varepsilon \int_0^T \| \partial_t v^n \|_{L^2(\Gamma)}^2 \, dt + \frac{1}{2} a(u^n(T), u^n(T)) + \varepsilon \int_{\Gamma} H(v^n(T)) \, dS = \frac{1}{2} a(u^n(0), u^n(0)) + \varepsilon \int_{\Gamma} H(v^n(0)) \, dS + \int_{U_T} I_{app} \partial_t u^n \, dx \, dt. \] (2.64)
Integration by parts in the last term yields

\[
\int_{U_T} I_{app} \partial_t u^n dxdt = \int_U I_{app}(T) u^n(T) - I_{app}(0) u^n(0) dx \\
- \int_{U_T} \partial_t I_{app} u^n dxdt,
\]

(2.65)

which we can bound by

\[
\delta \left( \| u^n(0) \|^2_{L^2(U)} + \| u^n(T) \|^2_{L^2(U)} + \| u^n(0) \|^2_{L^2(U_T)} \right) + C_d \| \partial_t I_{app} \|^2.
\]

(2.66)

The term \( \varepsilon | \int_{\Gamma} H(v^n(0)) dS| \) is bounded by \( \varepsilon \| v_0 \|^4_{L^4(\Gamma)} \), and due to (2.63),

\[
\int_{\Gamma} H(v^n(T)) dS + \frac{C_h}{2} \int_{\Gamma} (v^n(T))^2 dS \geq 0.
\]

(2.67)

Combining these estimates with (2.64) and taking \( \delta < \frac{C_p}{4} \), we obtain

\[
\varepsilon \int_0^T \| \partial_t v^n \|^2_{L^2(\Gamma)} dt + \frac{1}{4} a(u^n(T), u^n(T)) \\
\leq C \left( \varepsilon \| v_0 \|^4_{L^4(\Gamma)} + \varepsilon \| v^n(T) \|^2_{L^2(\Gamma_T)} + a(u^n(0), u^n(0)) + \int_0^T a(u^n(t), u^n(t)) dt \right),
\]

(2.68)

from which it follows, via (2.57) and (2.58), that

\[
\varepsilon \| \partial_t v^n \|^2_{L^2(\Gamma_T)} \leq c_3.
\]

(2.69)

\[\square\]

**Remark 2.15.** Instead of requiring \( \partial_t I_{app} \in L^2 \), we could have used an estimate of the form

\[
\| B^{-1} v \|_{L^2(U)} \leq \varepsilon^{1/2} C \| v \|_{L^2(\Gamma^\varepsilon)},
\]

(2.70)

to obtain the \( L^2 \)-bound on \( \partial_t v^n \). To establish (2.70) would, in view of the characterization (2.41) of \( B^{-1} \), involve obtaining uniform \( L^2 \) bounds on the Dirichlet-Neumann operators, in addition to the solution operator Dirichlet problem with \( L^2 \) boundary data.

This seems reasonable since singularities of solutions to elliptic equations do not propagate, so local methods should suffice to establish (2.70). However, due to the highly oscillating behavior of \( \Gamma^\varepsilon \), we have for technical reasons not been able to obtain an estimate of the form (2.70).
2.5.4 Existence

In view of (2.47), the approximate solution \( v^n(t) \) is uniformly bounded in \( V^n \) on the maximal interval of existence \([0, \rho^n)\). Hence, by standard ODE theory [37], the maximal interval of existence coincides with \([0, T)\) for every \( n \). We are now in a position to let \( n \to \infty \), and thus conclude the existence of solutions.

**Theorem 2.16.** For every \( \varepsilon > 0 \) the variational problem (\( V^2 \)) possesses a unique solution \( u^\varepsilon \) in \( L^2(0, T; V^\varepsilon) \) with \( v^\varepsilon \in L^2(0, T; H^{1/2}(\Gamma^\varepsilon)) \) and \( \partial_t v^\varepsilon \in L^2(\Gamma_T^\varepsilon) \). The solution \( u^\varepsilon \) satisfies the bounds of Lemma 2.12.

**Proof.** Due to the uniform bounds on the sequences \( u^n \) and \( v^n \) provided by (2.12), we can appeal to weak sequential compactness in \( L^2(0, T; V) \), \( L^2(0, T; H^{1/2}(\Gamma)) \) and the compact embedding

\[
L^2(0, T; H^{1/2}(\Gamma)) \cap H^1(0, T; L^2(\Gamma)) \hookrightarrow L^2(\Gamma_T),
\]

(2.71)
to extract subsequences (still indexed by \( n \)) with the following properties:

\[
\begin{align*}
    u^n &\rightharpoonup u \text{ weakly in } L^2(0, T; V), \\
v^n &\rightharpoonup v \text{ weakly in } L^2(0, T; H^{1/2}(\Gamma)), \\
v^n &\to v \text{ strongly in } L^2(\Gamma), \\
v^n &\to v \text{ a.e. on } \Gamma_T, \\
\partial_t v^n &\rightharpoonup \partial_t v \text{ weakly in } L^2(\Gamma_T), \\
h(v^n) &\rightharpoonup h(v) \text{ weakly in } L^{4/3}(\Gamma_T),
\end{align*}
\]

(2.72)

By the lower semi-continuity of norms with respect to weak convergence, the limit \( u \) satisfies all the bounds of Lemma 2.12.

To prove the convergence (2.72), start by observing that integration by parts shows that \( \lim_{n \to \infty} \partial_t v^n = \partial_t v \), in the sense of distributions. By the continuity and linearity of \( B \), we also have \( v = Bu \). To show that

\[
\xi := \lim_{n \to \infty} h(v^n) = h(v),
\]

we argue by contradiction. Suppose that the set

\[
E = \{ x \in \Gamma_T \mid \xi < h(v) \},
\]

has strictly positive measure. Since \( v^n \) converges a.e we can, by Egorov’s theorem, if necessary by taking a smaller \( E \), assume that

\[
v^n \to v \text{ uniformly on } E.
\]
It follows by continuity of \( h \) that
\[
\lim_{n \to \infty} \int_E h(v^n) \, dS = \int_E h(v) \, dS.
\]
But since the functional
\[
f \mapsto \int_E f \, dS,
\]
is bounded on \( L^{4/3}(\Gamma_T) \), we have by weak convergence
\[
0 = \lim_{n \to \infty} \int_E (h(v^n) - \xi) \, dS = \int_E (h(v) - \xi) \, dS.
\]
But this clearly contradicts \( \xi < h(v) \) on \( E \) unless \( E \) has measure zero. Arguing similarly for the set
\[
\{ x \in \Gamma_T \mid \xi > h(v) \},
\]
we conclude that
\[
\xi = h(v) \quad \text{a.e.,}
\]
which concludes the proof of (2.72).

Next, we show that the limit \( u \) actually solves the PDE (V2). Fix \( \varphi \) in \( C^1(0,T;V^m) \) having the form
\[
\varphi = \sum_{l=1}^m b_l(t) e_l(x). \tag{2.73}
\]
For \( n \geq m \), and for fixed \( t \), we have by construction
\[
\int_{\Gamma} \partial_t v^n \varphi \, dS + a(u^n, B^{-1} \varphi) + \int_{\Gamma} h(v^n) \varphi \, dS = \sum_{j=i,e} I_j (B^{-1} \varphi)_j \, dx. \tag{2.74}
\]
Integrating this identity in time yields
\[
\int_0^T \left( \int_{\Gamma} \partial_t v^n \varphi \, dS + a(u^n, B^{-1} \varphi) + \int_{\Gamma} h(v^n) \varphi dS \right) dt = \sum_{j=i,e} \int_{U_j^T} I_j (B^{-1} \varphi)_j \, dx dt. \tag{2.75}
\]
Sending $n \to \infty$ and using the convergence in (2.72) we obtain
\[
\int_0^T \left( \int_\Gamma \partial_t v \varphi \, dS + a(u, B^{-1} \varphi) + \int_\Gamma h(v) \varphi \, dS \right) \, dt = \sum_{j=i,e} \int_{U_j^I} I_j (B^{-1} \varphi)_j \, dxdt.
\] (2.76)

By density in $L^2(0,T; H^{1/2})$ of finite linear combinations (2.73), this identity holds for all $\varphi \in L^2(0,T; H^{1/2})$. Hence
\[
\int_\Gamma \partial_t v \varphi \, dS + a(u, B^{-1} \varphi) + \int_\Gamma h(v) \varphi \, dS = \sum_{j=i,e} \int_{U_j^I} I_j (B^{-1} \varphi)_j \, dxdt,
\] (2.77)

for all $\varphi$ in $H^{1/2}(\Gamma)$ for a.e. $t \in (0,T)$.

We continue to show that the initial condition is fulfilled, $v(0) = v_0$. For any test function $\varphi \in C^1(0,T; H^{1/2})$ with $\varphi(T) = 0$, integration by parts shows that
\[
\int_0^T \left( - \int_\Gamma v \partial_t \varphi \, dS + a(u, B^{-1} \varphi) + \int_\Gamma h(v) \varphi \, dS \right) \, dt = \int_\Gamma v(0) \varphi \, dS + \sum_{j=i,e} \int_{U_j^I} I_j (B^{-1} \varphi)_j \, dxdt.
\] (2.78)

For fixed $\varphi$ of the form (2.73) with $m \leq n$, the Galerkin approximation $v^n$ satisfies
\[
\int_0^T \left( - \int_\Gamma v^n \partial_t \varphi \, dS + a(u^n, B^{-1} \varphi) + \int_\Gamma h(v^n) \varphi \, dS \right) \, dt = \int_\Gamma v^n(0) \varphi \, dS + \sum_{j=i,e} \int_{U_j^I} I_j (B^{-1} \varphi)_j \, dxdt = \int_\Gamma v_0 \varphi(0) \, dS + \sum_{j=i,e} \int_{U_j^I} I_j (B^{-1} \varphi)_j \, dxdt,
\] (2.79)

since $v^n(0)$ is the orthogonal projection of $v_0$. Taking the limit as $n \to \infty$ and again using the convergence (2.72), we deduce that
\[
\int_0^T \left( - \int_\Gamma v \partial_t \varphi \, dS + a(u, B^{-1} \varphi) + \int_\Gamma h(v) \varphi \, dS \right) \, dt = \int_\Gamma v_0 \varphi(0) \, dS + \sum_{j=i,e} \int_{U_j^I} I_j (B^{-1} \varphi)_j \, dxdt,
\] (2.80)
and thus once again by density
\[ v(0) = v_0. \]

It remains to show uniqueness. Suppose that \( v_1 \) and \( v_2 \) solve (2.5) with initial conditions \( v_0^1 \) and \( v_0^2 \) in \( L^2(\Gamma) \). Their difference \( v := v_1 - v_2 \) satisfies the following equation:

\[
\varepsilon \int_{\Gamma} \partial_t v v dS + a \left( B^{-1} v, B^{-1} v \right) + \varepsilon \int_{\Gamma} (h(v_1) - h(v_2))(v_1 - v_2) dS = 0.
\]

Using the monotonicity (2.20) of \( h \), we obtain

\[
\frac{1}{2} \partial_t \|v\|^2_{L^2(\Gamma)} \leq C_h \|v\|^2_{L^2(\Gamma)},
\]

which establishes the uniqueness and concludes the proof of the wellposedness.
CHAPTER 2. WELLPOSEDNESS OF THE CELLULAR PROBLEM
Chapter 3

Homogenization

3.1 Formal asymptotic expansion

Before going into the rigorous theory of homogenization, we provide a formal argument based on the multiple scale method [11]. This is a standard exercise for elliptic problems, and the coupling to the dynamic boundary condition does not make any significant difference in this context. Indeed, homogenization of the bidomain model based on the multiple scale method has been carried out several times before, see [14, 24].

We are interested in the limit of $u^\varepsilon$ as $\varepsilon \to 0$. In particular we want to show that the weak limit of $u^\varepsilon$ is the unique solution of an averaged system of equations. To get some intuition we will assume momentarily that the involved functions are smooth. We look for an asymptotic expansion of $u^\varepsilon$ of the form

$$u^\varepsilon(t, x) \sim u_0\left(t, x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(t, x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(t, x, \frac{x}{\varepsilon}\right) + \mathcal{O}(\varepsilon^3),$$ (3.1)

where each $u_k$ is $Y := [0, 1]^3$-periodic in the second variable. We start by considering the intracellular potential $u := u_i$, $\sigma := \sigma_i$. For a function of the form

$$\psi(x) = \psi_1\left(x, \frac{x}{\varepsilon}\right) := \psi_1(x, y) \circ \left(x, \frac{x}{\varepsilon}\right)$$

the chain rule yields

$$\partial_x \psi = (\partial_x \psi_1)\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon} (\partial_y \psi_1)\left(x, \frac{x}{\varepsilon}\right),$$

where $\partial_x$ and $\partial_y$ means differentiation with respect to the first and second variable, respectively. A second differentiation and application of the chain
rule yields
\[ \partial_{x_k x_j} \psi = (\partial_{x_k x_j} \psi_1) \left( x, \frac{x}{\varepsilon} \right) \]
\[ + \frac{1}{\varepsilon} \left[ (\partial_{x_k y_j} \psi_1) \left( x, \frac{x}{\varepsilon} \right) + (\partial_{x_k y_j} \psi_1) \left( x, \frac{x}{\varepsilon} \right) \right] + \frac{1}{\varepsilon^2} (\partial_{y_k y_j} \psi_1) \left( x, \frac{x}{\varepsilon} \right). \]

By collecting terms with equal powers in \( \varepsilon \), the operator \( A = -\text{div} \left( \sigma \left( x, \frac{x}{\varepsilon} \right) \nabla (\cdot) \right) \) can be expressed as
\[ A = \frac{1}{\varepsilon^2} A_0 + \frac{1}{\varepsilon} A_1 + A_2, \] (3.2)
where
\[ \begin{cases} 
A_0 = -\text{div}_y (\sigma \nabla_y (\cdot)) \\
A_1 = -\text{div}_y (\sigma \nabla_x (\cdot)) - \text{div}_x (\sigma \nabla_y (\cdot)) \\
A_2 = -\text{div}_x (\sigma \nabla_x (\cdot)). 
\end{cases} \] (3.3)

Plugging the formal expansion (3.1) into (1.9) we obtain
\[ \frac{1}{\varepsilon^2} A_0 u_0 + \frac{1}{\varepsilon} (A_0 u_1 + A_1 u_0) + (A_0 u_2 + A_1 u_1 + A_2 u_0) + O(\varepsilon) = I_i, \quad \text{in } Y_i, \]
and

\[ \begin{align*}
\nu \cdot \sigma \nabla u^\varepsilon &= \frac{1}{\varepsilon} \left( \nu \cdot \sigma \nabla_y u_0 \right) + \left( \nu \cdot \sigma \nabla_x u_0 + \nu \cdot \sigma \nabla_y u_1 \right) \\
&\quad + \varepsilon \left( \nu \cdot \sigma \nabla_x u_1 + \nu \cdot \sigma \nabla_y u_2 \right) + O(\varepsilon^2) \\
&= \varepsilon \left( \partial_t v^\varepsilon + h(v^\varepsilon) \right) \\
&= \varepsilon \left( \partial_t v_0 + h(v_0) \right) + O(\varepsilon^2),
\end{align*} \]
on \( \Gamma \). Here we used that \( h(v_j) = O(\varepsilon) \) for \( j > 0 \), since \( h(0) = 0 \).

Equating the first three powers of \( \varepsilon \) we obtain
\[ \begin{cases} 
A_0 u_0 = 0 & \text{in } Y_i, \\
\nu \cdot \sigma \nabla_y u_0 = 0 & \text{on } \Gamma, \\
A_0 u_1 = A_1 u_0 & \text{in } Y_i, \\
\nu \cdot \sigma \nabla_y u_1 = \nu \cdot \sigma \nabla_x u_0 & \text{on } \Gamma, \\
A_0 u_2 = A_1 u_1 + A_2 u_0 + I_i & \text{in } Y_i, \\
\nu \cdot \sigma \nabla_y u_2 = \nu \cdot \sigma \nabla_x u_1 + \partial_t v_0 + h(v_0) & \text{on } \Gamma.
\end{cases} \] (3.4) (3.5) (3.6)

Here \( v_0 \) refers to the first term, \( u_0^\varepsilon - u_0^0 \), in the asymptotic expansion of \( v^\varepsilon = u_1^\varepsilon - u_0^\varepsilon \), and not the initial datum. Observe that only the \( y \)-derivative occurs on the left in these three equations, so that when solved successively, \( x \) appears as a parameter. We impose periodic boundary conditions, i.e., \( u_j(t, x, \cdot) \in H^1_\#(Y) \).
**Definition 3.1.** Let $A = \text{div} (a \nabla (\cdot))$ be symmetric and elliptic, and fix $\gamma \in \mathbb{R}$. We say that $u \in H^1_\#(Y)$ is a weak solution to

$$Au + \gamma u = f, \quad f \in L^2(Y),$$

provided

$$\int_Y a \nabla u \cdot \nabla \varphi dy + \gamma \int_Y u \varphi dy = \int_Y f \varphi dy,$$

for all $\varphi \in H^1_\#(Y)$.

Existence and uniqueness of weak solutions when $\gamma > 0$ follows from an application of the Riesz representation theorem. When $\gamma = 0$, we have existence and uniqueness modulo constants. A proof in the non-periodic setting is given in [16].

**Lemma 3.2.** Let $A$ be as in Definition 3.1. The equation

$$Au = f, \quad f \in L^2(Y),$$

has a weak solution $u \in H^1_\#(Y)$ if and only if

$$\int_Y f \, dy = 0.$$

The solution, when it exists, is unique up to additive constants.

**Proof.** Let $K : L^2(Y) \to L^2(Y)$ be the resolvent $K = (A + I)^{-1}$, which takes data ($f$ in Definition 3.1 with $\gamma = 1$) to weak solutions in $H^1_\#(Y)$ ($u$ in Definition 3.1). Since the embedding $H^1_\#(Y) \hookrightarrow L^2(Y)$ is compact and $A$ is symmetric, $K$ is compact and symmetric. By the Fredholm alternative, the equation

$$(K - I)u = Kf$$

is uniquely solvable in $L^2(Y)/\ker(K - I)$ if and only if

$$Kf \perp \ker(K - I).$$

(3.7)

But if $v \in \ker(K - I)$ then $Kv = v$. By the symmetry of $K$,

$$(Kf, v) = (f, Kv) = (f, v),$$

thus (3.7) is equivalent to $f \perp \ker(K - I)$. On the other hand, it is easy to see that $v \in \ker(K - I)$ if and only if $v$ is a weak solution to $Av = 0$. But if

$$\int_Y a \nabla v \cdot \nabla v \, dy = 0,$$

then $v$ is constant. The statement of the lemma follows. \qed
The first equation (3.4) shows, in view of Lemma 3.2, that \( u_0 \) is independent of \( y \):

\[
u_0(x, y) = u_0(x).
\]

The second equation (3.5) is uniquely solvable in \( H_1^\#(Y)/\mathbb{R} \) if and only if

\[
\int_{Y_i} A_1 u_0 \ dy + \int_{\Gamma} \nu(y) \cdot \sigma \nabla_x u_0 \ dS(y) = 0.
\]

To show that the sum vanishes recall that

\[
\partial Y_i = (\partial Y_i \cap \partial Y) \cup \Gamma,
\]

so that by the divergence theorem and the periodicity of \( \sigma(x, \cdot) \):

\[
\int_{Y_i} A_1 u_0 \ dy = \int_{Y_i} -\text{div}_y (\sigma \nabla_x u_0) \ dy
= -\int_{\partial Y_i} \nu(y) \cdot \sigma \nabla_x u_0 \ dS(y)
= \int_{\partial Y_i \cap \partial Y} \nu(y) \cdot \sigma \nabla_x u_0 \ dS(y)
+ \int_{\Gamma} \nu(y) \cdot \sigma \nabla_x u_0 \ dS(y)
= -\int_{\Gamma} \nu(y) \cdot \sigma \nabla_x u_0 \ dS(y).
\]

Here

\[
\int_{\partial Y_i \cap \partial Y} \nu(y) \cdot \sigma \nabla_x u_0 \ dS(y) = 0,
\]

since functions in \( H_1^\#(Y) \) have equal trace on opposite sides of the unit cube. Thus the second equation possesses, for each \( x \), a unique solution in \( H_1^\#(Y) \), up to additive constants. We will determine in more detail the form of \( u_1 \).

We look for a solution of the form

\[
u_1 = \chi(x, y) \cdot \nabla_x u_0,
\]

for some (row) vector valued function \( \chi \). This ansatz put into (3.5) yields

\[
(A_0 \chi) \cdot \nabla u_0 = -\text{div}_y (\sigma \nabla_x u) = -\text{div}_y (\sigma) \cdot u_0, \quad \text{in} \ Y_i
\]

and

\[
(\nu \cdot \sigma \nabla_y \chi) \cdot \nabla_x u_0 = \nu \cdot \sigma \nabla_x u_0 \quad \text{on} \ \Gamma,
\]
3.1. FORMAL ASYMPTOTIC EXPANSION

which is satisfied if

\[
\begin{cases}
A_0 \chi = -\text{div}_y \sigma & \text{in } Y_i, \\
\nu \cdot \sigma \nabla_y \chi = \nu \cdot \sigma & \text{on } \Gamma,
\end{cases}
\]

which is again (component wise) uniquely solvable in \(H^1_\#(Y)/\mathbb{R}\). One calls \(\chi\) the first order corrector.

The third equation (3.6) is again uniquely solvable if the right hand side annihilates all constants in \(H^1_\#(Y)\). Thus, the condition becomes

\[
\int_{Y_i} (A_1 u_1 + A_2 u_0 + I_i) \, dy + \int_{\Gamma} (\partial_t v_0 + h(v_0)) \, dS(y) = 0 \quad (3.10)
\]

Since \((u_0, v_0, I_i)\) are independent of \(y\) this means that

\[
\int_{Y_i} \text{div}_x (\sigma \nabla_y u_1) + \text{div}_x (\sigma \nabla_x u_0) \, dy = |\Gamma|(\partial_t v_0 + h(v_0)) - |Y_i| I_i, \quad (3.11)
\]

where \(|\Gamma|\) denotes the surface measure of \(\Gamma\) and \(|Y_i|\) is the volume of \(Y_i\). Using the representation (3.8) for \(u_1\) (3.11) can be written as

\[
\text{div}_x \left[ \left( \int_{Y_i} \sigma - \nabla_y \chi \, dy \right) \cdot \nabla_x u_0 \right] = |\Gamma|(\partial_t u_0 + h(u_0)) - |Y_i| I_i. \quad (3.12)
\]

Denoting \(\int_{Y_i} \sigma - \nabla_y \chi_i \, dy\) by \(M_i(x)\), this reads

\[
\text{div}_x (M_i \nabla_x u_0^i) = |\Gamma|(\partial_t u_0 + h(u_0)) - |Y_i| I_i. \quad (3.13)
\]

For the extracellular potential \(u_e\) we can proceed completely analogously to arrive at

\[
\text{div}_x (M_e \nabla_x u_0^e) = -|\Gamma|(\partial_t v_0 + h(v_0)) - |Y_e| I_e, \quad (3.14)
\]

with a difference in sign since \(\nu\) is the inner normal to \(U_e\). Thus we have derived the averaged equation for \(u_0\).

**Remark 3.3.** The averaged equations (3.13) and (3.14) are defined in all of \(U\), with the intra and extracellular potentials coexisting at every point. The system (3.13)-(3.14) is referred to in the literature as the bidomain equations [13].

**Remark 3.4.** We have assumed for convenience that \(Y = [0, 1]^3\), but more general cuboids can be considered [24]. In general, the averaged equation involves a factor which represents the ratio of cell membrane area to cell volume.
3.2 Two-scale Convergence

We want to rigorously prove that the solution $u^\varepsilon$ of the cell problem (1.9) converges "in some sense" to the solution $u = (u_i, u_e)$ of the averaged equations (3.13)-(3.14). Since the functions $u^\varepsilon_j$ live on sets varying with $\varepsilon$ it is not clear what "convergence" means in this context. A minimal requirement should be that $u^\varepsilon$ converges to $u$ in the following sense of distributions:

$$
\varepsilon \int_{\Gamma^\varepsilon} (\partial_t v^\varepsilon + h(v^\varepsilon)) \varphi \, dS dt \to |\Gamma| \int_{U_T} (\partial_t v + h(v)) \varphi \, dx dt
$$

$$
\int_{(U_i^\varepsilon)_T} u^\varepsilon_i \varphi \, dx dt \to |Y_i| \int_{U_T} u_i \varphi \, dx dt
$$

$$
\int_{(U_e^\varepsilon)_T} u^\varepsilon_e \varphi \, dx dt \to |Y_e| \int_{U_T} u_e \varphi \, dx dt
$$

(3.15)

In this type of convergence, however, the microscopic scale is "blurred out" by the smoothness of $\varphi$. By instead taking test functions that oscillates, a more precise statement can be made about the convergence of $u^\varepsilon$ to $u$. Thus, we are able to show that the averaged model (3.12) is not just "any equation" satisfied by "some weak limit of $u^\varepsilon$", but rather that the solution of the averaged equation captures the macroscopic behavior of $u^\varepsilon$, for small $\varepsilon$.

There is of course also the practical difficulty of actually showing (3.15), and a direct approach seems overwhelmingly difficult to the author. Instead we will use compactness to create a limiting object, and then show that the limiting object satisfies (4.1-4.2). In order to pass to the limit in the weak formulation it is important that the convergence is sufficiently strong, and here convergence in the sense of distributions is to weak.

Instead we will rely on two-scale convergence and its various generalizations. It was introduced by Nguetseng in [31], and developed further by Allaire in [3] and [5]. Subsequently there has been a somewhat different, but in many ways equivalent, formulation in terms of unfolding operators [10]. Although we will only use these results in dimension $N = 3$ we state them in the general case.

3.2.1 Two-scale convergence in fixed domains

The theory of two-scale convergence is often presented in a time independent context, but most proofs remain identical when a time variable is added. In the rest of this section $U$ is an arbitrary open set in $\mathbb{R}^N$, $U_T$ is the time-space...
cylinder $U_T = (0, T) \times U$ and $Y$ is the unit cube $Y = [0, 1]^N$. Recall that $L^2_\#(Y)$ consists of functions in $L^2(Y)$ periodically extended to $\mathbb{R}^N$.

**Definition 3.5.** A function $\psi \in L^2(U_T; L^2_\#(Y))$ is an admissible test function for two-scale convergence if

$$
\lim_{\varepsilon \to 0} \int_0^T \int_U \left| \psi \left( t, x, \frac{x}{\varepsilon} \right) \right|^2 dxdt = \int_0^T \int_{U \times Y} |\psi(t, x, y)|^2 dxdydt.
$$

**Remark 3.6.** Sufficient conditions for $\psi$ to be admissible is $\psi \in L^2_\#(Y; C(U_T))$ or $\psi \in L^2(U_T; C_\#(Y))$, see [3]. Necessary conditions are however not known; indeed, counterexamples can be found in $L^\infty(U_T; L^\infty_\#(Y))$ [3].

**Definition 3.7.** We say $u^\varepsilon \in L^2(U_T)$ two-scale converges to $u \in L^2(U_T \times Y)$ and write $u^\varepsilon \rightharpoonup u$, if

$$
\int_0^T \int_U u^\varepsilon(t, x) \varphi \left( t, x, \frac{x}{\varepsilon} \right) dxdt \to \int_0^T \int_U \int_Y u(t, x, y) \varphi(t, x, y) dxdydt,
$$

for all admissible $\varphi$.

**Remark 3.8.** By choosing $\varphi$ independent of $y$ one sees that two-scale convergence implies $L^2$-weak convergence. Furthermore, if $u^\varepsilon \rightharpoonup u^0$, then the weak limit $u$ equals

$$
u(t, x) = \int_Y u^0(t, x, y) dy.
$$

In particular, if the two scale limit is independent of $y$, then the two limits coincide.

**Example 3.9.** i) Let $u \in C^\infty_\#(Y)$ and set $u^\varepsilon(t, x) := u(x/\varepsilon)$. Note that $u^\varepsilon$ converges weakly in $L^2(U)$ to the mean of $u$. The two scale limit on the other hand satisfies, $u^\varepsilon \rightharpoonup u^0$ where

$$u^0(t, x, y) = u(y).
$$

Thus the profile of $u$ is preserved under the two-scale limit. This is in stark contrast to the usual weak limit, where almost all information about the sequence is lost.

ii) Suppose $u^\varepsilon$ converges strongly to $u$ in $L^2(U_T)$. Then $u^\varepsilon \rightharpoonup u^0$ where $u^0(t, x, y) = u(t, x)$. 

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The following compactness result is the fundamental theorem of two-scale convergence [3,11].

**Theorem 3.10.** Let \( u_\varepsilon \) be a bounded sequence in \( L^2(U_T) \), \( \| u_\varepsilon \|_{L^2(U_T)} \leq C \). Then there exists a subsequence \( \varepsilon' \) and a function \( u_0 \in L^2(U_T;L^2(Y)) \) such that \( u_\varepsilon \) two-scale converges to \( u_0 \), as \( \varepsilon' \to 0 \),

\[
\int_{U_T} u_\varepsilon'(t,x)\varphi\left(t,x,\frac{x}{\varepsilon}\right)\,dxdt \to \int_{U_T} \int_Y u_0(t,x,y)\varphi(t,x,y)\,dy\,dxdt,
\]

for all admissible \( \varphi \in L^2(U_T;L^2_#(Y)) \).

**Proof.** Let \( u_\varepsilon \) be a bounded sequence in \( L^2(U_T) \). Consider the Banach space \( X := C(U_T;C_#(Y)) \). Define the functional \( F_\varepsilon \in X' \) by

\[
F_\varepsilon(\varphi) = \int_{U_T} u_\varepsilon\varphi\left(t,x,\frac{x}{\varepsilon}\right)\,dxdt.
\]

The sequence \( \{F_\varepsilon\}_\varepsilon \) is uniformly bounded in \( X' \):

\[
|F_\varepsilon(\varphi)| = \left| \int_{U_T} u_\varepsilon(t,x)\varphi\left(t,x,\frac{x}{\varepsilon}\right)\,dxdt \right| \leq C\|u_\varepsilon\|_{L^2}\|\varphi\|_X, \tag{3.16}
\]

where we used the Cauchy-Schwarz inequality and the estimate

\[
\|\varphi\|_{L^2(U_T)} \leq |U_T|^{1/2} \max_{(t,x) \in U_T} |\varphi(t,x)|.
\]

Since \( X \) is separable, there is, by weak-* sequential compactness, a subsequence \( \varepsilon' \) and functional \( F^0 \in X' \) such that

\[
F_\varepsilon(\varphi) \to F^0(\varphi), \quad \forall \varphi \in X. \tag{3.17}
\]

But since \( \varphi \) is an admissible test function for two-scale convergence,

\[
\int_{U_T} \left| \varphi\left(t,x,\frac{x}{\varepsilon}\right) \right|^2\,dxdt \to \|\varphi\|_{L^2(U_T \times Y)}^2.
\]

It follows from (3.16) that

\[
|F^0(\varphi)| \leq C\|\varphi\|_{L^2(U_T \times Y)}, \quad \forall \varphi \in X.
\]

By density of \( X \) in \( L^2(U_T \times \Gamma) \), \( F_0 \) has a unique continuous extension,

\[
\tilde{F}_0 : L^2(U_T \times Y) \to \mathbb{R}.
\]

By the Riesz representation theorem there exists \( u^0 \in L^2(U_T \times Y) \) such that

\[
\tilde{F}_0(\varphi) = \int_{U_T \times Y} u^0\varphi\,dxdydt, \quad \forall \varphi \in L^2(U_T \times Y).
\]

By (3.17), \( u_\varepsilon \) two-scale converges to \( u^0 \).
3.2. TWO-SCALE CONVERGENCE

For a bounded sequence in $L^2(0, T; H^1(U))$, the gradient two-scale converges to the gradient of the two-scale limit in the following sense [3,11]:

**Theorem 3.11.** Let $u^\varepsilon$ be a weakly convergent sequence in $L^2(0, T; H^1(U))$ with a limit $u$. Then $u^\varepsilon$ two-scale converges to its weak limit

$$u^\varepsilon \rightharpoonup u^0(t, x, y) = u(t, x).$$

Furthermore, there exists a subsequence $\varepsilon'$ and a function $u_1 \in L^2(U_T; H^1_#(Y))$ such that

$$\nabla u^\varepsilon' \rightharpoonup \nabla_x u + \nabla_y u_1.$$

**Remark 3.12.** The two-scale convergence of the vector valued function $\nabla u^\varepsilon$ should be understood component wise.

**Proof.** In view of Theorem 3.10,

$$u^\varepsilon \rightharpoonup u^0 \in L^2(U_T \times Y),$$

$$\nabla u^\varepsilon \rightharpoonup \xi \in L^2(U_T \times Y)^N,$$

along some subsequence (still denoted by $\varepsilon$). Fix $\psi \in C^\infty_0(U_T; C^\infty_#(Y))^N$. The two-scale convergence implies that

$$\lim_{\varepsilon \to 0} \int_{U_T} \nabla u^\varepsilon(t, x) \cdot \psi(t, x, \frac{x}{\varepsilon}) \, dt \, dx = \int_{U_T \times Y} \xi(x, y) \cdot \psi(x, y) \, dt \, dx \, dy. \tag{3.18}$$

By the definition of distributional derivative,

$$\int_{U_T} \nabla u^\varepsilon(t, x) \cdot \psi(t, x, \frac{x}{\varepsilon}) \, dt \, dx$$

$$= - \int_{U_T} \left( \text{div}_x \psi \left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon} \text{div}_y \psi \left(x, \frac{x}{\varepsilon}\right) \right) \, dt \, dx. \tag{3.19}$$

We multiply (3.19) by $\varepsilon$ and pass to the two scale limit to arrive at

$$\int_{U_T \times Y} u^0(t, x, y) \text{div}_y \psi(t, x, y) \, dt \, dx \, dy = 0. \tag{3.20}$$

Since $\psi$ is arbitrary,

$$\nabla_y u^0 = 0, \tag{3.21}$$

in the sense of distributions, thus $u^0(t, x, y) = u^0(t, x)$ is independent of $y$. By Remark 3.8, $u^0 = u$, where $u$ is the usual weak limit of $u^\varepsilon$ in $L^2(0, T; H^1(U))$. 

Next, fix $\Psi \in C_0^\infty(U_T; C_#^\infty(Y))^N$ with the additional requirement that
\[
\text{div}_y \Psi = 0.
\]

We insert $\Psi$ into (3.19) and pass to the two scale limit (this time without multiplying with $\varepsilon$):

\[
\lim_{\varepsilon \to 0} \int_{U_T} \nabla u^\varepsilon \cdot \Psi \left(t, x, \frac{x}{\varepsilon}\right) \ dtdx = -\lim_{\varepsilon \to 0} \int_{U_T} u^\varepsilon \text{div}_y \Psi \left(x, \frac{x}{\varepsilon}\right) \ dtdx
\]

\[
= -\int_{U_T \times Y} u \text{div}_y \Psi(t, x, y) \ dtdxdy
\]

\[
= \int_{U_T \times Y} \nabla u \cdot \Psi(t, x, y) \ dtdxdy,
\]

where we used (3.21) to obtain the last equality. This identity combined with (3.18) yields

\[
\int_{U_T \times Y} (\nabla u - \xi) \cdot \Psi(t, x, y) \ dtdxdy = 0. \quad (3.22)
\]

It follows that $F(y) := \nabla u(x) - \xi(x, y)$ is orthogonal to all divergence free functions in $C_#^\infty(Y)^N$ for a.e. $(t, x) \in U_T$. Thus there exists $u_1 \in L^2(U_T; H_#^1(Y))$ such that $\xi = \nabla u + \nabla_y u_1$, and this completes the proof of the theorem. \qed

Remark 3.13. Two show that the limit was regular we used the fact that it coincided with the weak limit in $L^2(0, T, L^2(U_T))$, and hence was equal to the weak limit in $L^2(0, T, H^1)$.

Remark 3.14. In the proof above, we encounter an instance where the dependence on time does make a difference. In the time independent case, we could also have inferred (3.21) from the compact embedding $H^1 \hookrightarrow L^2$ and hence strong converge in $L^2$. But since $L^2(0, T, H^1)$ is not compactly embedded in $L^2(U_T)$, this type of argument could not be used here.

### 3.2.2 Two-scale convergence on periodic surfaces and perforated domains

In this subsection we will consider a generic perforated domain,

\[
U^\varepsilon := \left\{ x \in U \mid \frac{x}{\varepsilon} \in k + Y^* \text{ for some } k \in \mathbb{Z}^N \right\}, \quad (3.23)
\]

where $U$ is a bounded smooth subset of $\mathbb{R}^N$, and $Y^*$ is a subset of the unit cube such that the periodic extension to $\mathbb{R}^N$ is smooth and connected. Hence,
we do not pay any particular attention to how cells intersect \( \partial U \). Let \( \Gamma \) be the part of \( \partial Y^* \) contained in the interior of \( Y \), i.e.,
\[
\Gamma = \partial Y^* \setminus \partial Y.
\]
Set
\[
\Gamma^\varepsilon := \left\{ x \in U \mid \frac{x}{\varepsilon} \in k + \Gamma \text{ for some } k \in \mathbb{Z}^N \right\}, \tag{3.24}
\]
which is precisely the part of \( \partial U^\varepsilon \) contained in \( U \).

The problems of homogenization are complicated by the fact that one has to consider sequences of functions defined on these kind of oscillating domains. If for example \( u^\varepsilon \) is a bounded sequence in \( H^1(U^\varepsilon) \), then Theorem 3.11 cannot directly be applied, since \( u^\varepsilon \) is not defined on all of \( U \). It is not even clear what “\( u^\varepsilon \to u \)” should mean in this context. Another problem is that, even though we may obtain some kind of strong convergence, it is in general impossible to extract a pointwise convergent subsequence.

There are currently three main approaches in the literature for dealing with homogenization in perforated domains:

i) bounded extensions in \( H^1(U) \) [1];

ii) extension by zero [3];

iii) the use of unfolding operators [10].

We will here take the second approach, extending the solution by zero. Let \( \tilde{\cdot} \) denote the extension by zero from \( U^\varepsilon \) to \( U \). If \( u^\varepsilon \in H^1(U^\varepsilon) \) then \( \tilde{u}^\varepsilon \) is generally not in \( H^1(U) \), due to the potential jump discontinuity along \( \partial U^\varepsilon \). Thus we cannot use the usual weak compactness in \( H^1(U) \) to deduce regularity of the two-scale limit. Nonetheless, we will show that the limiting function is differentiable in \( x \), and that the discontinuity in \( y \) has a very simple structure.

This is the content of the following lemma which can be found in [3].

A difference in this setting compared to Lemma 3.11 is that it takes some effort to show \( H^1(U) \)-regularity of the limit. This is because there is no weak compactness to start with.

**Lemma 3.15.** Let \( U^\varepsilon \) be as in (3.23) and assume that \( u^\varepsilon \) is a uniformly bounded sequence in \( L^2(0,T;H^1(U^\varepsilon)) \). Then there exists a subsequence (still denoted by \( \varepsilon \)) and functions \( u \in L^2(0,T;H^1(U)) \), \( u_1 \in L^2(U_T;H^1_0(Y)) \) such that
\[
\begin{align*}
\tilde{u}^\varepsilon &\to \chi(y)u(t,x) \in L^2(U_T;L^2(Y)), \\
\tilde{\nabla} u^\varepsilon &\to \chi(y) (\nabla_x u(t,x) + \nabla_y u_1(t,x,y)) \in L^2(U_T;L^2(Y))^N. \tag{3.25}
\end{align*}
\]
Here $\chi(y)$ is the characteristic function of $Y^*$:

$$\chi(y) = \begin{cases} 
1 & \text{if } y \in Y^* \\
0 & \text{if } y \notin Y^*. 
\end{cases} \quad (3.26)$$

**Proof.** The sequences $\tilde{u}^\varepsilon$ and $\nabla \tilde{u}^\varepsilon$ are bounded in $L^2(U_T)$ and $L^2(U_T)^N$, respectively. Thus, by Theorem 3.10 they two-scale converge, up to subsequences, to some functions

$$u^0(t, x, y) \in L^2(U_T, L^2(Y)),$$

$$\xi^0(t, x, y) \in L^2(U_T, L^2(Y))^N.$$ 

Note that these functions are zero in $Y \setminus Y^*$, for if $\varphi \in C_0^\infty(U_T; C_#^\infty(Y))$ with $\text{supp}(\varphi) \subset U_T \times (Y \setminus Y^*)$, then

$$\int_{U_T} \tilde{u}^\varepsilon(t, x) \varphi(t, x, \frac{x}{\varepsilon}) \, dxdt = 0,$$

hence in the limit

$$\int_{U_T \times Y} u^0(t, x) \varphi(t, x, y) \, dx dy dt = 0,$$

so $u^0$ is indeed zero in $Y \setminus Y^*$. We express this by writing

$$u^0(t, x, y) = u^0(t, x, y)\chi(y), \quad (3.27)$$

and similarly

$$\xi^0 = \xi^0\chi(y). \quad (3.28)$$

In order to further investigate the properties of $u^0, \xi^0$, take $\varphi \in C_0^\infty(U_T; C_#^\infty(Y))$ and $\Psi \in C_0^\infty(U_T; C_#^\infty(Y))^N$, both vanishing on $\partial Y^*$. It follows that the supports are contained in the perforated domain:

$$\varphi \left(t, x, \frac{x}{\varepsilon}\right) \in C_0^\infty(U_T^\varepsilon), \quad \Psi \left(t, x, \frac{x}{\varepsilon}\right) \in C_0^\infty(U_T^\varepsilon)^N. \quad (3.29)$$

By two-scale convergence,

$$\int_{U_T} u^\varepsilon(t, x) \varphi(t, x, \frac{x}{\varepsilon}) \, dxdt \to \int_{U_T} \int_{Y^*} u^0(t, x, y) \varphi(t, x, y) \, dx dy dt$$

$$\int_{U_T} \nabla u^\varepsilon \cdot \Psi \left(t, x, \frac{x}{\varepsilon}\right) \, dxdt \to \int_{U_T} \int_{Y^*} \xi^0(t, x, y) \cdot \Psi(t, x, y) \, dx dy dt. \quad (3.30)$$
Integrating by parts before passing to the two-scale limit, we get

\[ \int_{U_T} \nabla u^\varepsilon \cdot \Psi \left( t, x, \frac{x}{\varepsilon} \right) \, dxdt = - \int_{U_T} u^\varepsilon \div_x \Psi \left( t, x, \frac{x}{\varepsilon} \right) \, dxdt - \frac{1}{\varepsilon} \int_{U_T} u^\varepsilon \div_y \Psi \left( t, x, \frac{x}{\varepsilon} \right) \, dxdt, \]

where we use (3.29) to get rid of the boundary terms. Multiplying (3.31) by \( \varepsilon \), and by passing to the two scale limit, we get

\[ 0 = - \int_{U_T} \int_Y u^0(t, x, y) \div_y \Psi(t, x, y) \, dydxdydt. \]

It follows that \( u^0(t, x, \cdot) \) is constant in \( Y^* \). Thus, in view of (3.27),

\[ u^0(t, x, y) = u(t, x)\chi(y), \]

for some \( u \in L^2(U_T) \).

Assume now additionally that \( \div_y \Psi = 0 \) in (3.31). Upon passing to the two scale limit, this time we obtain

\[ \int_{U_T} \int_{Y^*} \xi^0(t, x, y) \cdot \Psi(t, x, y) \, dxdydt = - \int_{U_T} \int_{Y^*} u(t, x) \div_x \Psi(t, x, y) \, dxdydt. \]

For a given \( \Psi \), we can form the integral

\[ \theta(t, x) := \int_{Y^*} \Psi(t, x, y) \, dy. \]

We claim that every \( \theta \in C_0^\infty(U_T)^N \) can be obtained in this way for some function \( \Psi \), such that the integration by parts (3.31) and passing to the limit (3.30) remains valid. Indeed, we can take

\[ \Psi(t, x, y) = A(y)\theta(t, x), \]

where \( A \) is an \( N \times N \) matrix, with columns, \( A_j \) satisfying

\[ \int_{Y^*} A_j \, dy = e_j, \quad \div A_j = 0, \quad \nu \cdot A_j = 0 \text{ on } \partial Y^* \setminus \partial Y, \]

and the normal component of \( A_j \) is continuous across the boundary of adjacent cells. Introducing the potential \( A_j = \nabla v_j \), we consider the Neumann problem

\[
\begin{align*}
\Delta v_j &= 0 \quad \text{in } Y^* \\
\nu \cdot \nabla v_j &= g_j \quad \text{on } \partial Y^*. 
\end{align*}
\]
As boundary data we take some smooth function \( g_j \in C^\infty(\partial Y^*) \) satisfying
\[
\text{supp} \ g_j \subset (\{ y_j = 0 \} \cup \{ y_j = 1 \}), \quad \int_{\{ y_j = 1 \}} g_j \, dS = 1, \quad g_j\big|_{y_j = 0} = -g_j\big|_{y_j = 1}.
\]
It follows that
\[
\int_{\partial Y^*} g_i \, y_j \, dS = \delta_{i,j}.
\]
This is obvious if \( i = j \). If \( i \neq j \), then the integral on the two opposing sides cancels:
\[
\int_{\partial Y^* \cap \{ y_i = 0 \}} g_i(y) y_j \, dS(y) + \int_{\partial Y^* \cap \{ y_i = 1 \}} g_i(y) y_j \, dS(y) = 0.
\]
By Green’s formula, the solution \( v_i \) satisfies
\[
\int_{\partial Y^*} \nabla v_i \cdot e_j \, dy = \int_{\partial Y^*} g_j y_i \, dS = \delta_{i,j}.
\]
Extending \( v_j \) periodically, we conclude that \( A_j = \nabla v_j \) does the trick.

Going back to (3.31), we see that for all \( \theta \in C_0^\infty(U_T) \),
\[
\int_{U_T \times Y^*} \xi^0(t,x,y) \cdot A(y) \theta(t,x) \, dydxdt
= \int_{U_T} \left( \int_{Y^*} \xi^0(t,x,y) A^T(y) \, dy \right) \cdot \theta(t,x) \, dxdt
= -\int_{U_T} u \, \text{div}_x \left( \int_{Y^*} A(y) \theta(x,t) \, dy \right) \, dxdt,
= -\int_{U_T} u \, \text{div}_x \theta \, dxdt.
\]
Hence,
\[
\nabla u = \int_{Y^*} \xi^0(t,x,y) A^T(y) \, dy, \quad (3.37)
\]
in the sense of distributions. But since this function belongs to \( L^2(U_T) \) we conclude that \( u \in L^2(0,T,H^1(U)) \).

Integrating by parts in (3.31) yields
\[
\int_{U_T} \int_{Y^*} (\nabla u - \xi) \cdot \Psi \, dydxdt = 0. \quad (3.38)
\]
Recall that the set of divergence free vector fields is the orthogonal complement to the gradients in \( L^2_\#(Y^*)^N \). More precisely, suppose \( F \in L^2_\#(Y^*)^N \) satisfies
\[
\int_{Y^*} F \cdot \Psi \, dy = 0,
\]
3.2. TWO-SCALE CONVERGENCE

for all $\Psi \in C^\infty_\#(Y^*)^N$ such that $\text{div} \, \Psi = 0$ and $\Psi = 0$ on $\partial Y^* \setminus \partial Y$. Then $F = \nabla p$ for some $p \in H^1_\#(Y^*)$.

Applying this result to (3.38) for a.e. $(t, x) \in U_T$ we conclude that

$$\chi(y)(\xi^0 - \nabla u) = \chi(y)\nabla_y u_1,$$

for some $u_1 \in L^2(U_T; H^1_\#(Y^*))$. Thus we have identified the two-scale limit of $\tilde{\nabla} u^\varepsilon$ as

$$\xi^0 = \chi(y)(\nabla u + \nabla_y u_1).$$

\[\square\]

\textbf{Remark 3.16.} The construction of $A$ above depends crucially on the assumption that the perforated domain is connected, so that the boundary $\partial Y^* \cap \partial Y$ is nonempty on every side of the unit cube. Indeed, it is not hard to envision a scenario where $u^\varepsilon$ is wildly oscillating and yet uniformly bounded in $H^1(U^\varepsilon)$, if $U^\varepsilon$ is spectacularly disconnected. Note that in the current periodic setting, the perforated set $U^\varepsilon$ necessarily has a certain “uniform connectedness” in the sense that the ratio of $\partial Y^* \cap \partial Y$ and $\partial Y^*$ is fixed. This is also reflected in the fact that we have a uniform Poincaré inequality, see Lemma 2.1.

Since boundary values play an essential role in (1.9), we need a concept of convergence for sequences of functions on periodic surfaces. It turns out that there is an extension of two-scale convergence to this setting [5].

\textbf{Definition 3.17.} Let $\Gamma^\varepsilon$ be a periodic surface in an open bounded set $U$. A sequence $v^\varepsilon$ in $L^2(\Gamma_T^\varepsilon)$ is said to two-scale converge to $v_0 \in L^2(U_T; L^2(\Gamma))$ and we write

$$v^\varepsilon \xrightarrow{\text{2-scale}} v_0,$$

if

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma_T^\varepsilon} v^\varepsilon(t, x) \varphi \left( t, x, \frac{x}{\varepsilon} \right) \, dS(x) \, dt$$

$$= \int_U \int_{\Gamma_T} v_0(t, x, y) \varphi(t, x, y) \, dx \, dS(y) \, dt,$$

for all $\varphi \in C(\overline{U_T}; C_\#(Y))$.

\textbf{Remark 3.18.} Note the somewhat stricter regularity assumption,

$$\varphi \in C(\overline{U_T}; C_\#(Y)).$$
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compared to the usual two-scale convergence in the sense of Definition 3.7. Certainly, if \( \varphi \) is merely assumed to be in \( L^2(U; C_\#(Y)) \), the trace is not well defined. We could however allow test functions in \( H^1(U, C_\#(Y)) \), piecewise continuous functions, etc. Indeed, analogous to Definition 3.7 the main property we require from a test function, in addition to having a well defined trace, is that

\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma_\varepsilon} \left| \varphi \left( t, x, \frac{x}{\varepsilon} \right) \right|^2 dS(x) dt = \int_{U_T} \int_{\Gamma} |\varphi(t, x, y)|^2 dS(y) dx dt,
\]

and the class of functions satisfying these two criteria is not restricted to \( C(\overline{U}; C_\#(Y)) \), but we choose to work with this class for convenience.

Lemma 3.19. Let \( \varphi \) be a function in \( C \left( \overline{U}; C_\#(Y) \right) \). Then

\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma_\varepsilon} \left| \varphi \left( t, x, \frac{x}{\varepsilon} \right) \right|^2 dS(x) dt = \int_{U_T} \int_{\Gamma} |\varphi(t, x, y)|^2 dS(y) dx dt,
\]

Proof. For notational convenience we provide a proof in the case when \( \varphi \) does not depend of time.

We approximate \( \varphi \) with a function \( \varphi_\varepsilon \) which is piecewise constant in \( x \). Let \( x_k(\varepsilon) \) be the midpoint of the cube \( \varepsilon(k + Y) \). Define

\[
\varphi_\varepsilon(x, y) = \begin{cases} \varphi(x_k, y) & \text{if } x \in \varepsilon(k + Y) \\ 0 & \text{if } x \in U \setminus \bigcup_{k \in K_\varepsilon} \varepsilon(k + Y). \end{cases} \quad (3.39)
\]

By uniform continuity,

\[
\max_{x \in \bigcup_{k \in K_\varepsilon} \varepsilon(k + Y)} \max_{y \in Y} |\varphi_\varepsilon(x, y) - \varphi(x, y)| \to 0, \quad \text{as } \varepsilon \to 0.
\]

Thus, if \( V \subset \subset U \) is compactly contained in \( U \), \( \varphi_\varepsilon \) converges uniformly on \( V \times Y \). By the dominated convergence theorem,

\[
\int_{U \times \Gamma} |\varphi_\varepsilon(x, y) - \varphi(x, y)|^2 dS(y) dx \to 0, \quad \text{as } \varepsilon \to 0. \quad (3.40)
\]

Next, we claim that

\[
\left| \varepsilon \int_{\Gamma_\varepsilon} \varphi_\varepsilon \left( x, \frac{x}{\varepsilon} \right) dS(x) \right| - \varepsilon \int_{\Gamma_\varepsilon} \varphi \left( x, \frac{x}{\varepsilon} \right) dS(x) \to 0, \quad (3.41)
\]
as \( \varepsilon \to 0 \). Let \( \delta > 0 \). By uniform continuity there is \( r > 0 \) such that \( \max_{y \in Y} |\varphi(x_1, y) - \varphi(x_2, y)|^2 < \delta \), if \( |x_1 - x_2| < r \). Thus, if \( \varepsilon < r \), we can estimate (3.41) by

\[
\varepsilon \int_{\Gamma_\varepsilon} \left| \varphi^\varepsilon \left( x, \frac{x}{\varepsilon} \right) - \varphi \left( x, \frac{x}{\varepsilon} \right) \right|^2 dS(x) \leq \varepsilon \int_{\Gamma_\varepsilon} \delta dS \leq C\delta,
\]

which proves (3.41).

Finally, a calculation yields the identity,

\[
\varepsilon \int_{\Gamma_\varepsilon} \left| \varphi^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \right|^2 dS(x) = \sum_{k \in K} \varepsilon \int_{\varepsilon (k+Y)} \left| \varphi \left( x_k, \frac{x}{\varepsilon} \right) \right|^2 dS(x)
\]

\[
= \sum_{k \in K} \varepsilon^N \int_{\Gamma} |\varphi(x_k, y)|^2 dS(y)
\]

\[
= \sum_{k \in K} \int_{\varepsilon (k+Y)} \left( \int_{\Gamma} |\varphi(x_k, y)|^2 dS(y) \right) dx
\]

\[
= \int_{U \times \Gamma} |\varphi^\varepsilon(x, y)|^2 dx dS(y),
\]

where we use the change of variable \( y = x/\varepsilon \), and note that \( dS(\varepsilon x) = \varepsilon^{N-1} dS(x) \). We also used that \( N \)-dimensional Lebesgue measure scales according to \( |\varepsilon (k+Y)| = \varepsilon^N \). In view of (3.40) and (3.41), this identity delivers the lemma.

\[ \square \]

Remark 3.20. The technique of ”approximation by piecewise constant” employed above will be used extensively in later sections. It is closely related to unfolding operator techniques, in particular when the two-scale limit is independent of \( y \). Since the transmembrane potential \( v^\varepsilon \) of the bidomain equations turns out to have a trivial (\( y \)-independent) two-scale limit, convergence in the sense of measures suffices to answer most questions. The effort to develop the full machinery of unfolding operators would thus be largely wasted (although the cumbersome calculations above could have been avoided).

With the chosen class of test functions, the analogue of Theorem 3.10 holds for sequences of bounded \( L^2 \) functions on periodic surfaces [5].

Theorem 3.21. Let \( \Gamma^\varepsilon \) be a periodic surface in an open bounded domain \( U \subset \mathbb{R}^N \). If \( v^\varepsilon \) is a sequence in \( L^2 (\Gamma^\varepsilon) \) such that

\[
\varepsilon \int_{\Gamma^\varepsilon} |v^\varepsilon|^2 dS dt \leq C,
\]

as \( \varepsilon \to 0 \). Let \( \delta > 0 \). By uniform continuity there is \( r > 0 \) such that \( \max_{y \in Y} |\varphi(x_1, y) - \varphi(x_2, y)|^2 < \delta \), if \( |x_1 - x_2| < r \). Thus, if \( \varepsilon < r \), we can estimate (3.41) by

\[
\varepsilon \int_{\Gamma_\varepsilon} \left| \varphi^\varepsilon \left( x, \frac{x}{\varepsilon} \right) - \varphi \left( x, \frac{x}{\varepsilon} \right) \right|^2 dS(x) \leq \varepsilon \int_{\Gamma_\varepsilon} \delta dS \leq C\delta,
\]

which proves (3.41).

Finally, a calculation yields the identity,

\[
\varepsilon \int_{\Gamma_\varepsilon} \left| \varphi^\varepsilon \left( x, \frac{x}{\varepsilon} \right) \right|^2 dS(x) = \sum_{k \in K} \varepsilon \int_{\varepsilon (k+Y)} \left| \varphi \left( x_k, \frac{x}{\varepsilon} \right) \right|^2 dS(x)
\]

\[
= \sum_{k \in K} \varepsilon^N \int_{\Gamma} |\varphi(x_k, y)|^2 dS(y)
\]

\[
= \sum_{k \in K} \int_{\varepsilon (k+Y)} \left( \int_{\Gamma} |\varphi(x_k, y)|^2 dS(y) \right) dx
\]

\[
= \int_{U \times \Gamma} |\varphi^\varepsilon(x, y)|^2 dx dS(y),
\]

where we use the change of variable \( y = x/\varepsilon \), and note that \( dS(\varepsilon x) = \varepsilon^{N-1} dS(x) \). We also used that \( N \)-dimensional Lebesgue measure scales according to \( |\varepsilon (k+Y)| = \varepsilon^N \). In view of (3.40) and (3.41), this identity delivers the lemma. \[ \square \]

Remark 3.20. The technique of ”approximation by piecewise constant” employed above will be used extensively in later sections. It is closely related to unfolding operator techniques, in particular when the two-scale limit is independent of \( y \). Since the transmembrane potential \( v^\varepsilon \) of the bidomain equations turns out to have a trivial (\( y \)-independent) two-scale limit, convergence in the sense of measures suffices to answer most questions. The effort to develop the full machinery of unfolding operators would thus be largely wasted (although the cumbersome calculations above could have been avoided).

With the chosen class of test functions, the analogue of Theorem 3.10 holds for sequences of bounded \( L^2 \) functions on periodic surfaces [5].

Theorem 3.21. Let \( \Gamma^\varepsilon \) be a periodic surface in an open bounded domain \( U \subset \mathbb{R}^N \). If \( v^\varepsilon \) is a sequence in \( L^2 (\Gamma^\varepsilon) \) such that

\[
\varepsilon \int_{\Gamma^\varepsilon} |v^\varepsilon|^2 dS dt \leq C,
\]
then there exists a function \( v^0 \in L^2(U_T, L^2(\Gamma)) \) such that, up to the extraction of a subsequence,

\[ v^\varepsilon \overset{\text{w}^*}{\rightharpoonup} v^0 \in L^2(U_T; L^2(\Gamma)). \]

**Proof.** The Banach space \( X = C(U_T; C_0(Y)) \) is separable and dense in \( L^2(U_T; L^2(\Gamma)) \). This follows from the well-known density of \( C_0^\infty(\Gamma) \) in \( L^2(\Gamma) \) [21], and obviously every smooth function with compact support can be periodically extended as a continuous function.

Define the functional \( F^\varepsilon \) on \( X \) by

\[ F^\varepsilon(\varphi) := \varepsilon \int_{\Gamma_T} v^\varepsilon(t, x, \frac{x}{\varepsilon}) \, dS(x) \, dt. \]

By Cauchy-Schwarz inequality, the family \( \{F^\varepsilon\}_{\varepsilon > 0} \) is uniformly bounded:

\[
|F^\varepsilon(\varphi)| = \left| \varepsilon \int_{\Gamma_T} v^\varepsilon(t, x, \frac{x}{\varepsilon}) \, dS(x) \, dt \right| \\
\leq \left( \varepsilon \int_{\Gamma_T} |v^\varepsilon(t, x)|^2 \, dS(x) \, dt \right)^{1/2} \left( \int_{\Gamma_T} |\varphi(t, x, \frac{x}{\varepsilon})|^2 \, dS(x) \, dt \right)^{1/2} \\
\leq C \|\varphi\|_X,
\]

where we used the estimate

\[
\varepsilon \int_{\Gamma_T} |\varphi(t, x, \frac{x}{\varepsilon})|^2 \, dS(x) \, dt \leq \varepsilon |\Gamma_T|^2 \max_{(t,x,y) \in U_T \times Y} |\varphi(t, x, y)|^2 \\
\leq C \|\varphi\|_X^2.
\]

By weak-* sequential compactness there exist a subsequence \( \varepsilon' \) and a functional \( F^0 \in X' \) such that

\[ F^{\varepsilon'}(\varphi) \to F^0(\varphi) \quad \forall \varphi \in X. \]

Now, apply Lemma 3.19 and take the limsup in the second line of (3.43) to conclude that

\[ |F^0(\varphi)| \leq C \|\varphi\|_{L^2(U_T; L^2(\Gamma))}, \quad \forall \varphi \in X. \]

Since \( X \) is dense in \( L^2(U_T; L^2(\Gamma)) \), there exists a unique continuous extension \( F^0 \in [L^2(U_T; L^2(\Gamma))]^* \). By the Riesz representation theorem there exists \( v^0 \in L^2(U_T; L^2(\Gamma)) \), such that

\[ F^0(\varphi) = \int_{U_T \times \Gamma} v^0(t, x, y)\varphi(t, x, y) \, dtdxdS(y), \quad \forall \varphi \in L^2(U_T \times \Gamma). \]

This concludes the proof. \( \square \)
3.2. TWO-SCALE CONVERGENCE

We can now combine Lemma 3.15 with Theorem 3.21 to characterize the two-scale limit of traces of bounded sequences in \( H^1(U^\varepsilon) \) [5]. Recall that \( \tilde{\cdot} \) denotes extension by zero from \( U^\varepsilon \) to \( U \).

**Lemma 3.22.** Let \( u^\varepsilon \) be a bounded sequence in \( L^2(0,T;H^1(U^\varepsilon)) \) with trace \( g^\varepsilon := u^\varepsilon|_{\Gamma^\varepsilon} \in L^2(\Gamma^\varepsilon_T) \), where \( \Gamma^\varepsilon = \partial U^\varepsilon \setminus \partial U \). Then, up to a subsequence, the two-scale limit (in the sense of Definition 3.17) \( g \) is the trace of \( u \):

\[
g^\varepsilon \overset{2-S}{\to} g = u|_{\Gamma},
\]

where \( u \in L^2(0,T;H^1(U)) \) satisfies

\[
\tilde{\tilde{u}}^\varepsilon \overset{2}{\to} \chi(y)u(t,x) \in L^2(U_T;L^2(Y)).
\]

**Proof.** By Theorem 3.15 there is a subsequence such that

\[
\tilde{\tilde{u}}^\varepsilon \overset{2}{\to} \chi(y)u(t,x),
\]

for some \( u \in L^2(0,T;H^1(U)) \). According to the trace inequality, Lemma 2.4,

\[
e \int_{\Gamma^\varepsilon} |g^\varepsilon|^2 \, dS(x) \leq C\|u^\varepsilon\|_{H^1(U^\varepsilon)}^2 \leq C,
\]

therefore Theorem 3.21 asserts that, possibly along another subsequence,

\[
g^\varepsilon \overset{2-S}{\to} g,
\]

for some \( g \in L^2(U_T;L^2(\Gamma)) \).

Now, fix a smooth vector valued function \( \Psi \) and consider the following integration by parts:

\[
e \int_{U^\varepsilon_T} \nabla u^\varepsilon \cdot \Psi \left( t, x, \frac{x}{\varepsilon} \right) \, dxd t
\]

\[
= -e \int_{U^\varepsilon_T} u^\varepsilon \text{div}_x \Psi \left( t, x, \frac{x}{\varepsilon} \right) \, dxd t
\]

\[
- \int_{U^\varepsilon_T} u^\varepsilon \text{div}_y \Psi \left( t, x, \frac{x}{\varepsilon} \right) \, dxd t
\]

\[
+ e \int_{\Gamma^\varepsilon_T} u^\varepsilon \nu \cdot \Psi \left( t, x, \frac{x}{\varepsilon} \right) \, dS(x)dt.
\]

Passing to the two-scale limit in (3.44), the first two terms vanish. We are thus left with

\[
\int_{U_T} \int_{Y^*} u \text{div}_y \Psi (x,y) \, dxd y = \int_{U_T} \int_{\Gamma} g \nu \cdot \Psi(x,y) \, dxdS(y)dt.
\]
Integrate by parts in the first term and recall that, by Lemma 3.15, $u$ is independent of $y$. The result is

$$\int_{U_T} \int_{\Gamma} u \nu \cdot \Psi(t, x, y) dS(y) dx dt = \int_{U_T} \int_{\Gamma} g \nu \cdot \Psi(t, x, y) dS(y) dx dt.$$

Since $\Psi$ is arbitrary we have $u|_{\Gamma} = g$ on $\Gamma$.

**Remark 3.23.** Since as $u$ is independent of $y$ (by Lemma 3.15) so is $g$. Thus we can identify $g = u$, as elements in $L^2(0, T; H^1(U))$. Note however that this identification is somewhat arbitrary up to multiplicative constants, and depends on $|Y^*|$ and $|\Gamma|$. More precisely, the functionals on $C^\infty_0(U_T)$ given by

$$\lim_{\varepsilon \to 0} \int_{\Gamma_T} g^\varepsilon \varphi dS(x) dt = |\Gamma| \int_{U_T} u \varphi dx dt$$

and

$$\lim_{\varepsilon \to 0} \int_{U_T} u^\varepsilon \varphi dx dt = |Y^*| \int_{U_T} u \varphi dx dt,$$

differ by the factor of $|Y^*|/|\Gamma|$.

### 3.3 Two-scale convergence of the bidomain equation

**3.3.1 Assumptions**

We assume that $\tilde{I}_j$ converges weakly to some function $I_j \in L^2(U_T)$,

$$\tilde{I}_j \rightharpoonup I_j; \quad \text{weakly in } L^2(U_T).$$

We also assume that the initial datum $v_0^\varepsilon$ converges strongly to $v_0 \in H^1(U)$ in the sense that

$$\lim_{\varepsilon \to 0} \varepsilon^{1/2} \|v_0^\varepsilon - v_0|_{\Gamma^*}\|_{L^2(\Gamma^*)} = 0.$$

The scaling of $\varepsilon^{1/2}$ can be motivated in the following way: If $\{\varphi_j\}_{j=1}^\infty$ is a sequence of smooth functions in $U_T$ with uniformly bounded derivatives, then by the mean value theorem

$$\|\varphi_j\|_{L^2(U_T)} \to 0, \quad \text{as } j \to \infty.$$
3.3. TWO-SCALE CONVERGENCE OF THE BIDOMAIN EQUATION

if and only if

\[ \varepsilon^{1/2} \| \varphi_j \|_{L^2(\Gamma)} \to 0, \text{ uniformly in } \varepsilon \text{ as } j \to \infty. \]

Moreover, we assume that \( v_0^\varepsilon \) is the trace of a uniformly bounded sequence in \( H^1(U) \). The main point is that we do not allow \( v_0^\varepsilon \) oscillate wildly in the limit, in particular we do not consider initial data of the form \( v_0^\varepsilon = v_0(x/\varepsilon) \).

Recall the assumption in Section 2.2 that \( \sigma_j^\varepsilon \) is of the form \( \sigma_j^\varepsilon = \sigma_j(x, y/\varepsilon) \) for some functions \( \sigma_j(x, y) \in C^\infty(U \times Y_j) \). According to the remark after Definition 3.5, \( \sigma_j^\varepsilon \) satisfies

\[
\lim_{\varepsilon \to 0} \int_{U_j} |\sigma_j^\varepsilon|^2 \, dx = \int_{U \times Y_j} |\sigma_j(x, y)|^2 \, dx \, dy, \quad j = i, e.
\]

This means, according to Definition 3.5, that \( \sigma_j(x, y) \) qualify as test functions for two-scale convergence.

3.3.2 Extracting two-scale limits

Recall that \( \tilde{\cdot} \) denotes extension by zero. Using the a priori estimates provided by Lemma 2.16, we can apply Lemmas 3.15 and 3.22 to extract subsequences such that

\[
\begin{align*}
\tilde{u}_i^\varepsilon & \overset{2-\text{S}}{\rightharpoonup} \chi_{Y_i}(y) u_i(t, x) \\
\tilde{u}_e^\varepsilon & \overset{2-\text{S}}{\rightharpoonup} \chi_{Y_e}(y) u_e(t, x) \\
\tilde{\nabla} u_i^\varepsilon & \overset{2-\text{S}}{\rightharpoonup} \chi_{Y_i}(y) \left( \nabla_x u_i(t, x) + \nabla_y u_1^i(t, x, y) \right) \\
\tilde{\nabla} u_e^\varepsilon & \overset{2-\text{S}}{\rightharpoonup} \chi_{Y_e}(y) \left( \nabla_x u_e(t, x) + \nabla_y u_1^e(t, x, y) \right) \\
v^\varepsilon & \overset{2-\text{S}}{\rightharpoonup} v(t, x) = u_i - u_e \\
\partial_t v^\varepsilon & \overset{2-\text{S}}{\rightharpoonup} \partial_t v(t, x) \\
h(v^\varepsilon) & \overset{2-\text{S}}{\rightharpoonup} \xi,
\end{align*}
\]

for some functions \( u_i, u_e \) in \( L^2(0, T; H^1(U)) \) and \( u_1^i, u_1^e \) in \( L^2(U_T; H^1_\#(Y)) \).

Here we identify

\[ v = u_i - u_e \]

as elements in \( L^2(0, T; H^1(U)) \), according to Remark 3.23. Since nonlinear functions are not in general continuous with respect to weak convergence, we can not immediately conclude that \( \xi = h(v) \). For example \( \sin(2\pi nx) \) converges weakly to 0 in \( L^2(0, 1) \) as \( n \to \infty \), but \( (\sin(2\pi nx))^2 \to 1/2 \neq 0^2 \).
3.3.3 Strong convergence of the transmembrane potential

In order to show that $\xi = h(v)$, we prove that $v^\epsilon$ converges in a suitable strong sense. This is done by approximating $\{v^\epsilon\}$ by a sequence of piecewise constant functions $\{\tilde{v}^\epsilon\}$, defined on the fixed set $U_T$. We then apply Kolmogorov’s compactness criterion to $\{\tilde{v}^\epsilon\}$ in $L^2_{\text{loc}}(U_T)$. In a somewhat similar problem [4], strong convergence via local averages was obtained by using boundedness in $H^1$ alone. Because of trace issues, we lack uniform bounds on the spatial derivative of $v^\epsilon$. Instead we will actively use the differential equation to estimate

$$\tau_z v^\epsilon - v^\epsilon := v^\epsilon(\cdot - z) - v^\epsilon(\cdot).$$

A complication that arises here is that these translations are only welldefined for certain values of $z$, since $v^\epsilon$ lives on a periodic surface. What we will do to overcome this obstacle is to “compare solutions on neighboring cells”.

Homogenization of reaction diffusion equations on periodic surfaces have previously been studied in [19], where a similar approach was taken to obtain strong convergence. The difference in our setting is that we do not possess any a priori estimates of the tangential gradient $\nabla_t v^\epsilon$. This leads to the problem of how to estimate $\|v^\epsilon - \tilde{v}^\epsilon\|$, i.e., how well the local mean approximates $v^\epsilon$. To overcome this obstacle we will introduce a special type of local seminorm, which can be controlled in terms of the trace of $u_i$ and $u_e$.

To identify $\xi$ as $h(v)$ it suffices consider sets compactly contained in $U_T$.

**Lemma 3.24.** Passing if necessary to a subsequence, the transmembrane potential $v^\epsilon$ converges strongly locally, in the sense that

$$\lim_{\epsilon \to 0} \int_{\Gamma^\epsilon \cap W} |v^\epsilon - v| |\nu^\epsilon|^2 dS d\tau = 0,$$

for every $W$ compactly contained in $U_T$. As a consequence,

$$\xi = h(v),$$

where $\xi$ is the limit in measure of $h(v^\epsilon)$.

**Remark 3.25.** With limit in measure we mean that

$$\frac{\epsilon}{|\Gamma|} \int_{\Gamma^\epsilon} h(v^\epsilon) \varphi dS \to \int_U \xi \varphi dx, \quad \forall \varphi \in C(\overline{U}).$$

This is equivalent to two-scale convergence in the surface sense, if the two-scale limit is independent of $y$. Note however, that we do not know whether $\xi$
3.3. TWO-SCALE CONVERGENCE OF THE BIDOMAIN EQUATION

is independent of $y$, even if $v$ is. It turns out (see next subsection) that this is not a problem, since convergence in measure suffices to derive the bidomain equations.

The remaining part of this subsection is devoted to the proof of Lemma 3.46. The proof consists of several steps:

Step 1. We show that $v_\varepsilon$ is well approximated in $L^2(\Gamma_\varepsilon)$ by its local averages. This relies on $v_\varepsilon$ being the difference of traces of uniformly bounded sequences in $H^1(U_\varepsilon^i) \times H^1(U_\varepsilon^e)$, and a rather simple but technical scaling argument.

Step 2. In the second step we apply Kolmogorov’s compactness criterion in $L^2_{loc}(U_T)$ to the sequence of averages, \( \bar{v}_\varepsilon \). Here we actively use structure of the differential equation satisfied by $v_\varepsilon$ to estimate spatial and temporal translations.

Step 3. In the third step we identify the strong limit of $\bar{v}_\varepsilon$ with the two-scale limit of $v_\varepsilon$.

Step 4. In final step use the strong convergence to apply a Minty-type argument, which finally shows that $\xi = h(v)$.

Proof. Step 1: We start by recalling the following notation for the decomposition of $U$ and $\Gamma_\varepsilon$ into $\varepsilon$-cells (cf. Figure 2.4):

\[
Y_\varepsilon^k = \left\{ x \in U \mid \frac{x}{\varepsilon} \in Y + k \right\}, \\
\Gamma_\varepsilon^k = \left\{ x \in \Gamma_\varepsilon \mid \frac{x}{\varepsilon} \in Y + k \right\}, \\
K_\varepsilon = \left\{ k \in \mathbb{Z}^3 \mid (Y + k) \subset U \right\}.
\]

Note that $K_\varepsilon$ only include cells fully contained in $U$. Consider the piecewise constant function $\bar{v}_\varepsilon$ in $L^2(U_T)$ defined by the local averages of $v_\varepsilon$:

\[
\bar{v}_\varepsilon := \begin{cases} 
\frac{1}{|\Gamma_\varepsilon^k|} \int_{\Gamma_\varepsilon^k} v_\varepsilon dS & \text{in } Y_\varepsilon^k, \ k \in K_\varepsilon \\
0 & \text{in } U_T \setminus \bigcup Y_\varepsilon^k
\end{cases}.
\]

(3.47)

Of course, we can also regard $\bar{v}_\varepsilon$ as a function on $\Gamma_\varepsilon$. For all $\varepsilon$-piecewise constant functions we have the ”surface to space” integral conversion formula:

\[
\int_U \bar{v}_\varepsilon dx = \frac{\varepsilon}{|\Gamma|} \int_{\Gamma^s} \bar{v}_\varepsilon dS
\]

(3.48)

This is valid since the surface and volume measures have the scaling properties

\[
\frac{\varepsilon}{|\Gamma|}|\varepsilon(\Gamma + k)| = \varepsilon^3 = |\varepsilon(Y + k)|.
\]
Consequently, we also have

$$\|\tilde{v}\|_{L^2(U)} = \sqrt{\frac{\varepsilon}{|\Gamma|}} \|\tilde{v}\|_{L^2(\Gamma^e)}.$$  \hspace{1cm} (3.49)

In the unit cell there is, by Lemma 2.3, a constant C such that

$$\left\| v - \frac{1}{|\Gamma|} \int_{\Gamma} v \, dS \right\|_{L^2(\Gamma)} \leq C |u|_{H^{1/2}_0(\Gamma^e)},$$ \hspace{1cm} (3.50)

for all $u \in H^{1/2}(\Gamma)$. Recall that

$$|u|_{H^{1/2}_0(\Gamma)} = \int_{\Gamma^e} \int_{\Gamma^e} \frac{|u(x) - u(y)|^2}{|x - y|^3} \, dS(x) \, dS(y).$$

Rescaling the Poincaré-Wirtinger inequality (3.50) in each $\varepsilon$-cell, we obtain

$$\int_{\varepsilon(\Gamma+k)} |v(x) - \tilde{v}|^2 \, dS(x) = \varepsilon^2 \int_{\Gamma+k} |v(\varepsilon x) - \tilde{v}|^2 \, dS(x)$$

$$\leq C \varepsilon^2 \int_{\Gamma} \int_{(\Gamma+k)^2} \frac{|v(\varepsilon x) - v(\varepsilon y)|^2}{|x - y|^3} \, dS(x) \, dS(y)$$

$$= C \varepsilon^5 \int_{\Gamma} \int_{(\Gamma+k)^2} \frac{|v(\varepsilon x) - v(\varepsilon y)|^2}{|\varepsilon x - \varepsilon y|^3} \, dS(x) \, dS(y)$$

$$= C \varepsilon \int_{\varepsilon(\Gamma+k)} \int_{(\varepsilon(\Gamma+k))^2} \frac{|v(x) - v(y)|^2}{|x - y|^3} \, dS(x) \, dS(y).$$

Hence, for a.e. $t$,

$$\|v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\|_{L^2(\Gamma_k^e)}^2 \leq \varepsilon \|v^\varepsilon(t)\|_{H^{1/2}_0(\varepsilon(\Gamma+k)^e)}^2 \forall k \in K^e.$$ \hspace{1cm} (3.51)

This motivates the following definition.

**Definition 3.26.** The seminorm $|\cdot|_{\varepsilon, 1/2}$ is

$$|\varphi|_{\varepsilon, 1/2} = \sum_{k \in K^e} |\varphi|_{H^{1/2}_0(\varepsilon(k+\Gamma))^e}.$$

**Remark 3.27.** The seminorm $|\cdot|_{\varepsilon, 1/2}$ only measures the variation within each cell. Hence, a function can vary greatly between different cells yet be small in $|\cdot|_{\varepsilon, 1/2}$-norm. In particular, all piecewise constant functions are annihilated by this seminorm.
3.3. TWO-SCALE CONVERGENCE OF THE BIDOMAIN EQUATION

The advantage of this new seminorm is that it is uniformly controlled in terms of trace.

Lemma 3.28 (Homogeneous trace inequality). There exists a constant $C$, independent of $\varepsilon$, such that

$$|u|_{\Gamma^\varepsilon,\frac{1}{2}} \leq C \|\nabla u\|_{L^2(U^\varepsilon_j)},$$

for all $u \in H^1(U^\varepsilon_j)$, $j = i, e$.

Proof. Let $j \in \{i, e\}$ be fixed. In the unit cell $Y$, the trace inequality (2.4) reads

$$|u|_{H^{1/2}(\Gamma)}^2 \leq C_\gamma \left( \|u\|^2_{L^2(Y_j)} + \|\nabla u\|^2_{L^2(Y_j)} \right), \quad (3.52)$$

where $|u|_{H^{1/2}(\Gamma)}^2$ is defined in (2.1). Observe that the 2-dimensional Euclidean surface measure scales like $dS(\varepsilon x) = \varepsilon^2 dS(x)$. Scaling each $\varepsilon$-cell, $\varepsilon(k+Y)$, we find that

$$\int_{\varepsilon(k+Y)} \int_{\varepsilon(k+Y)} \frac{|u(x) - u(y)|^2}{|x - y|^3} dS(x) dS(y)$$

$$= \varepsilon^4 \int_{(k+Y)} \int_{(k+Y)} \frac{|u(\varepsilon x) - u(\varepsilon y)|^2}{|\varepsilon x - \varepsilon y|^3} dS(x) dS(y) \quad (3.53)$$

$$= \varepsilon \int_{(k+Y)} \int_{(k+Y)} \frac{|u(\varepsilon x) - u(\varepsilon y)|^2}{|x - y|^3} dS(x) dS(y).$$

The trace inequality in the unit cell (3.52) applied to $u(\varepsilon(\cdot))$, shows that (3.53) is bounded by

$$\varepsilon C_\gamma \left( \int_{k+Y} |u(\varepsilon x)|^2 dx + \int_{k+Y} |\nabla (u(\varepsilon x))|^2 dx \right) \quad (3.54)$$

By the chain rule, (3.54) equals

$$\varepsilon C_\gamma \int_{k+Y} |u(\varepsilon x)|^2 dx + \varepsilon^3 C_\gamma \int_{k+Y} |(\nabla u)(\varepsilon x)|^2 dx. \quad (3.55)$$

Observe that the seminorm $|\cdot|_{H^{1/2}(\varepsilon(k+Y))}$ is invariant under additive constants,

$$|u - c|_{H^{1/2}(\varepsilon(k+Y))} = |u|_{H^{1/2}(\varepsilon(k+Y))}, \quad c \in \mathbb{R}.$$

Inserting the mean value, $\bar{u}_k$, of $u(\varepsilon(\cdot))$ over $k + Y$ in (3.55), we arrive at

$$|u|_{H^{1/2}(\varepsilon(k+Y))}^2 \leq \varepsilon C_\gamma \int_{k+Y} |u(\varepsilon x) - \bar{u}_k|^2 dx + \varepsilon^3 C_\gamma \int_{k+Y} |(\nabla u)(\varepsilon x)|^2 dx. \quad (3.56)$$
Using Poincaré-Wirtinger’s inequality [16, Section 5.8.1] in the unit cell we find that
\[
\varepsilon C_\gamma \int_{k+Y} |u(\varepsilon x) - \bar{u}_k|^2 dx \leq \varepsilon C_\gamma C_p \int_{k+Y} |\nabla (u(\varepsilon x))|^2 dx
\]
\[
= \varepsilon^3 C_\gamma C_p \int_{k+Y} |(\nabla u)(\varepsilon x)|^2 dx,
\]
where \(C_p\) is the Poincaré constant in \(Y\).

Scaling back to \(\varepsilon(k+Y)\), and using the fact that the 3-dimensional Lebesgue measure scales like \(d(rx) = r^3 d(x)\), we obtain
\[
\varepsilon^3 C_\gamma C_p \int_{k+Y} |(\nabla u)(\varepsilon x)|^2 dx = C_\gamma \int_{\varepsilon(k+Y)} |(\nabla u)(x)|^2 dx,
\]

hence,
\[
|u|_{H^{1/2}(\varepsilon(k+Y))}^2 \leq \left(1 + C_p\right) C_\gamma \|\nabla u\|_{L^2(\varepsilon(k+Y))}^2.
\]

By suming over \(k\) we obtain the statement of the lemma with \(C = \sqrt{(C_p + 1)C_\gamma}\).

Equipped with Lemma 3.28, we return to (3.51) where summation over \(k\) and integration over \((0, T)\) yields
\[
\|v^\varepsilon - \bar{v}^\varepsilon\|_{L^2(\Gamma_T^\varepsilon)}^2 \leq \varepsilon \int_0^T |v^\varepsilon|_{L^1(\varepsilon, 1/2)}^2 dt \leq \varepsilon C \sum_{j=i,e} \int_0^T \|\nabla u_j^\varepsilon\|_{L^2(\varepsilon_j^\varepsilon)} dt.
\]

Since by Lemma 2.12, \(\|u_j^\varepsilon\|_{L^2(0, T; H^1(\varepsilon_j^\varepsilon))}\) is bounded uniformly in \(\varepsilon\), it follows that
\[
\lim_{\varepsilon \to 0} \|v^\varepsilon - \bar{v}^\varepsilon\|_{L^2(\Gamma_T^\varepsilon)} = 0.
\]
This shows that the local averages approximates \(v^\varepsilon\) well.

**Step 2:** We now continue to the second step, estimating the large-scale oscillations of \(v^\varepsilon\). To extract a strongly convergent subsequence of \(\{\bar{v}^\varepsilon\}\), we recall the following version of Kolmogorov’s compactness criterion [23]:

**Lemma 3.29 (Kolmogorov).** Let \(U\) be an open bounded set in \(\mathbb{R}^N\), and let \(\{f_n\}_{n=1}^\infty\) be a bounded sequence in \(L^p(U)\). Denote \(\tau_z f(\cdot) := f(\cdot - z)\). Suppose that for every open set \(W \subset U\), the sequence \(\{f_n\}\) satisfies
\[
\|\tau_z f_n - f_n\|_{L^p(W)} \leq a(z) + b(n),
\]
where \(a\) and \(b\) are functions such that
\[
\lim_{z \to 0} a(z) = 0, \quad \lim_{n \to \infty} b(n) = 0.
\]
Then the sequence \(\{f_n\}\) is relatively compact in \(L^p_{\text{loc}}(U)\).
Let us show that \( \{\bar{v}^\varepsilon\}_{\varepsilon>0} \) satisfies the hypothesis of Lemma 3.29 in \( L^2_{\text{loc}}(U_T) \). We consider spatial translations first. Fix open sets \( W \subset \subset W' \subset \subset U \), a standard basis vector \( e_i \in \mathbb{R}^3 \), and let \( z \in \mathbb{R} \), with \( |z| < \text{dist}(W', \partial U) \), be fixed but arbitrary. Let \( \tau_z \) denote the spatial translation along \( e_i \)

\[
\tau_z u(t, x) = u(t, x - ze_i).
\]

A complication is that the translations \( \tau_z u_j^\varepsilon, \tau_z v^\varepsilon \) are only well defined when \( z = \varepsilon l \) for some \( l \in \mathbb{Z} \), since \( u_j^\varepsilon, v^\varepsilon \) are defined on \( \varepsilon \)-periodic domains. We claim however that

\[
\varepsilon \|\tau_{\varepsilon l} v^\varepsilon - v^\varepsilon\|_{L^2(\Gamma \cap W)}^2 \leq C \varepsilon l,
\]

for \( l \in \mathbb{Z} \) such that \( \varepsilon l \leq |z| \).

To prove the claim, take \( \varphi \in H^1(U_i^\varepsilon) \times H^1(U_e^\varepsilon) \), with \( \text{supp} \varphi \subset W \). Insert \( \tau_{-z} \varphi = (\tau_{-z} \varphi_i, \tau_{-z} \varphi_e) \) as a test function in the variational formulation (2.31). In the resulting equation, use the change of variable, \( x \mapsto x - ze_i \), to obtain

\[
\int_{\Gamma^\varepsilon} \partial_t (\tau_z v^\varepsilon)(\varphi_i - \varphi_e) dS + \sum_{j=i,e} \int_{U_j^\varepsilon \cap W} (\tau_z \sigma^\varepsilon) \nabla (\tau_z u^\varepsilon) \cdot \nabla \varphi_j dx
\]

\[
+ \int_{\Gamma^\varepsilon} h(\tau_z v^\varepsilon)(\varphi_i - \varphi_e) dS = \sum_{j=i,e} \int_{U_j^\varepsilon \cap W} (\tau_z I_j) \varphi_j dx,
\]

which is valid for all \( \varphi \in H^1(U_i^\varepsilon) \times H^1(U_e^\varepsilon) \) for which \( \text{supp} \varphi \subset W' \). Now, let \( \eta \in C_0^\infty(W') \) be a cutoff function for \( W \):

\[
\eta = \begin{cases} 
1 & \text{in } W \\
0 & \text{in } U \setminus W',
\end{cases}
\]

and take

\[
\varphi = \eta^2(\tau_z u^\varepsilon - u^\varepsilon).
\]

Plugging this choice of \( \varphi \) into the variational equation for \( \tau_z u^\varepsilon - u^\varepsilon \) we obtain

\[
\frac{d}{dt} \int_{\Gamma^\varepsilon \cap W} |\tau_z v^\varepsilon - v^\varepsilon|^2 dS
\]

\[
+ \sum_{j=i,e} \int_{U_j^\varepsilon \cap W} ((\tau_z \sigma^\varepsilon) \nabla (\tau_z u^\varepsilon_j) - \sigma^\varepsilon \nabla u^\varepsilon_j) \cdot (\nabla (\tau_z u^\varepsilon_j) - \nabla u^\varepsilon_j) dx
\]

\[
+ \varepsilon \int_{\Gamma^\varepsilon \cap W} (h(\tau_z v^\varepsilon) - h(v^\varepsilon)(\tau_z v^\varepsilon - v^\varepsilon) dS
\]

\[
= \sum_{j=i,e} \int_{U_j^\varepsilon \cap W} (\tau_z I_j - I_j)(\tau_z u^\varepsilon_j - u^\varepsilon_j) dx
\]
We estimate the second term in (3.63) as follows:

\[
\sum_{j=i.e} \int_{U_j^e \cap W} ((\tau_z \sigma_j^e) \nabla (\tau_z u_j^e) - \sigma_j^e \nabla u_j^e) \cdot (\nabla (\tau_z u_j^e) - \nabla u_j^e) \, dx
\]

\[
= \sum_{j=i.e} \int_{U_j^e \cap W} \sigma_j^e (\nabla (\tau_z u_j^e) - \nabla u_j^e) \cdot (\nabla (\tau_z u_j^e) - \nabla u_j^e) \, dx
\]

\[
+ \sum_{j=i.e} \int_{U_j^e \cap W} (\tau_z \sigma_j^e - \sigma_j^e) \nabla (\tau_z u_j^e) \cdot (\nabla (\tau_z u_j^e) - \nabla u_j^e) \, dx \quad (3.64)
\]

\[
\geq 0 - C \sum_{j=i.e} \|\tau_z \sigma_j^e - \sigma_j^e\|_{L^2(U_j^e \cap W)} \|\nabla u_j\|_{L^2(U_j^e)}
\]

\[
\geq -\varepsilon l C.
\]

Here we used the periodicity,

\[
\tau_{(\varepsilon l)} \sigma_j^e(x) = \sigma_j \left( x - \varepsilon l, \frac{x - \varepsilon l}{\varepsilon} \right) = \sigma_j \left( x - \varepsilon l, \frac{x}{\varepsilon} \right),
\]

and the mean value theorem, to obtain the estimate,

\[
\sum_{j=i.e} \|\tau_z \sigma_j^e - \sigma_j^e\|_{L^2(U_j^e \cap W)} \leq \varepsilon l \sum_{j=i.e} \|\nabla_x \sigma_j\|_{L^2(U_j \cap W)} \|\nabla u_j\|_{L^2(U_j)} \leq \varepsilon l C.
\]

By monotonicity, we can estimate the third term in (3.63) from below:

\[
\varepsilon \int_{\Gamma^e \cap W} (h(\tau_z v^e) - h(v^e))(\tau_z v^e - v^e) \, dS
\]

\[
\geq -\varepsilon C_h \|\tau_z v^e - v^e\|_{L^2(\Gamma^e)}^2. \quad (3.65)
\]

The source term in (3.63) can be estimated as follows:

\[
\left| \sum_{j=i.e} \int_{U_j^e \cap W} (\tau_z I_j - I_j)(\tau_z u_j - u_j) \, dx \right|
\]

\[
\leq 2 \sum_{j=i.e} \|I_j\|_{L^2(U_j)} \|\nabla u_j^e\|_{L^2(U_j)} \|\nabla u_j^e\|_{L^2(U_j)} \quad (3.66)
\]

\[
\leq C \varepsilon l,
\]

where we again used the mean value theorem to estimate,

\[
\|\tau_z u_j - u_j\|_{L^2(U_j^e \cap W)} \leq C \varepsilon l \|\nabla u_j^e\|_{L^2(U_j^e)}.
\]
3.3. TWO-SCALE CONVERGENCE OF THE BIDOMAIN EQUATION

Putting all these estimates together and integrating in time, we obtain,

\[ \varepsilon \parallel \tau_z v^\varepsilon(t) - v^\varepsilon(t) \parallel^2 \leq C\varepsilon l + C_h \varepsilon \int_0^t \parallel \tau_z v^\varepsilon - v^\varepsilon \parallel^2 ds. \]  

(3.67)

By Grönewall’s inequality,

\[ \varepsilon \parallel \tau_z v^\varepsilon(t) - v^\varepsilon(t) \parallel_{L^2(\Gamma^c \cap W)}^2 \leq e^{C_h t}(C\varepsilon l + \varepsilon \parallel \tau_z v^\varepsilon_0 - v^\varepsilon_0 \parallel_{L^2(\Gamma^c \cap W)}^2) \leq C\varepsilon l, \]

where we in the second inequality estimated

\[ \varepsilon \parallel \tau_z v^\varepsilon_0 |_{\Gamma^c} - v^\varepsilon_0 |_{\Gamma^c} \parallel_{L^2(\Gamma^c \cap W)}^2 \leq \parallel \tau_z v^\varepsilon_0 - \tau_z v^\varepsilon_0 \parallel_{L^2(U \cap W)}^2 + \varepsilon^2 \parallel \nabla (\tau_z v^\varepsilon_0 - v^\varepsilon_0) \parallel_{L^2(U \cap W)}^2 \leq C\varepsilon l \parallel \nabla v^\varepsilon_0 \parallel_{L^2(U)}. \]

Here we use the trace inequality for periodic surfaces (Lemma 2.2) and the assumption that \( v^\varepsilon_0 \) is the trace of a uniformly bounded sequence in \( H^1(U) \).

The claim (3.60) is proved.

We now estimate the translations of \( \bar{v}^\varepsilon \) by comparing with the estimates for \( v^\varepsilon \). For \( z = \varepsilon l \), combining (3.49), (3.60), (3.58) and the triangle inequality yields

\[ \parallel \tau_z \bar{v}^\varepsilon - \bar{v}^\varepsilon \parallel_{L^2(U)} = \sqrt{\frac{\varepsilon}{|\Gamma|}} \parallel \tau_z \bar{v}^\varepsilon - \bar{v}^\varepsilon \parallel_{L^2(\Gamma)} \]

\[ \leq C \varepsilon \parallel \nabla v^\varepsilon_0 \parallel_{L^2(U)} \]

\[ \leq C \sqrt{\varepsilon l}. \]

Now, for every \( \varepsilon \), the translation by \( z \) can be decomposed as

\[ z = \varepsilon l(\varepsilon) + \delta(\varepsilon), \]

where \( l \in \mathbb{Z}, |\delta| < \varepsilon \), and since \( \bar{v}^\varepsilon \) is piecewise constant,

\[ \parallel \tau_z \bar{v}^\varepsilon - \bar{v}^\varepsilon \parallel \leq \max_{\omega \in \{-1,0,1\}} \parallel \tau_{(\varepsilon l + \omega)\varepsilon} \bar{v}^\varepsilon - \bar{v}^\varepsilon \parallel \leq C \sqrt{\varepsilon (l + 1)}. \]

Thus we conclude that for any \( z \in \mathbb{R} \) with \( |z| < \text{dist}(W', \partial U) \) the inequality

\[ \parallel \tau_z \bar{v}^\varepsilon - \bar{v}^\varepsilon \parallel_{L^2(0,T;L^2(W))}^2 \leq C(|z| + \varepsilon), \]  

(3.68)

holds if \( \varepsilon \) is sufficiently small.

Next, we show that the temporal translations

\[ \tau_s u(t) := u(t - s), \]
CHAPTER 3. HOMOGENIZATION

satisfies
\[ \| \tau_s \bar{v}^\varepsilon - \bar{v}^\varepsilon \|_{L^2(q, T - q; L^2(U))} \leq C(|s| + \varepsilon), \tag{3.69} \]
where \( q \) is a small positive number and \( |s| < q \). By (3.48), (3.58) and the triangle inequality,
\[
\sqrt{\Gamma} \| \tau_s \bar{v}^\varepsilon - \bar{v}^\varepsilon \|_{L^2(U)} = \varepsilon^{1/2} \| \tau_s \bar{v}^\varepsilon - \bar{v}^\varepsilon \|_{L^2(\Gamma^\varepsilon)} \\
\leq \varepsilon^{1/2} \| \tau_s \bar{v}^\varepsilon - \tau_s v^\varepsilon \|_{L^2(\Gamma^\varepsilon)} + \varepsilon^{1/2} \| \tau_s v^\varepsilon - v^\varepsilon \|_{L^2(\Gamma^\varepsilon)} \\
+ \varepsilon^{1/2} \| v^\varepsilon - \bar{v}^\varepsilon \|_{L^2(\Gamma^\varepsilon)} \\
\leq C\varepsilon^{1/2} + \varepsilon^{1/2} \| \tau_s v^\varepsilon - v^\varepsilon \|_{L^2(\Gamma^\varepsilon)}. \tag{3.70} \]

By the fundamental theorem of calculus in Banach spaces [16, Section 9.5.2],
\[ v(t) - v(s) = \int_s^t \partial_t v \, dr, \]
if \( \partial_t v \in L^2(0, T; X) \). Applying the Cauchy-Schwarz inequality,
\[ \| v(t) - v(s) \| \leq |t - s|^{1/2} \left( \int_s^t \| \partial_t v \|^2 \, dr \right)^{1/2}. \]

Using this result, and recalling that \( \varepsilon^{1/2} \| \partial_t v^\varepsilon \|_{L^2(0, T; L^2(\Gamma^\varepsilon))} \) is bounded (Lemma 2.12), we can estimate
\[ \varepsilon^{1/2} \| \tau_s v^\varepsilon(t) - v^\varepsilon(t) \|_{L^2(\Gamma^\varepsilon)} \leq \varepsilon^{1/2} |s|^{1/2} \left( \int_t^{t-s} \| \partial_t v^\varepsilon(r) \|^2_{L^2(\Gamma^\varepsilon)} \, dr \right)^{1/2} \leq C |s|^{1/2}, \]
and (3.68) follows.

By (3.68) and (3.69), the sequence \( \bar{v}^\varepsilon \) fulfills Kolmogorov’s compactness criterion (Lemma 3.29). Hence there is an element \( \bar{v} \in L^2(U_T) \) and a subsequence such that
\[ \lim_{\varepsilon \to 0} \| \bar{v}^\varepsilon - \bar{v} \|_{L^2(W)} = 0, \]
where \( W \subset \subset U_T \).

\textbf{Step 3:} We claim that \( \bar{v} = v \) where \( v \) is the two-scale limit of \( v^\varepsilon \). By (3.59), for all \( \varphi \in C_0^\infty(U_T) \) and \( W \subset \subset U_T \),
\[ \lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma^\varepsilon \cap W} \bar{v}^\varepsilon \varphi \, dS = \lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma^\varepsilon \cap W} v^\varepsilon \varphi \, dS. \tag{3.71} \]

We claim that
\[ \lim_{\varepsilon \to 0} \frac{\varepsilon}{|\Gamma^\varepsilon|} \int_{\Gamma^\varepsilon \cap W} \bar{v}^\varepsilon \varphi \, dS = \lim_{\varepsilon \to 0} \frac{\varepsilon}{|W|} \int_W \bar{v}^\varepsilon \varphi \, dx, \tag{3.72} \]
indeed this follows from the argument in the proof of Lemma 3.19. Hence, by two-scale convergence of $v^\varepsilon$,

$$
\int_W \tilde{v} \varphi \, dx = \lim_{\varepsilon \to 0} \frac{\varepsilon}{|\Gamma|} \int_{\Gamma^\varepsilon \cap W} \tilde{v}^\varepsilon \varphi \, dS = \lim_{\varepsilon \to 0} \frac{\varepsilon}{|\Gamma|} \int_{\Gamma^\varepsilon \cap W} v^\varepsilon \varphi \, dS = \int_W v \varphi \, dx.
$$

(3.73)

Since $W$ and $\varphi$ are arbitrary it follows that $\tilde{v}^\varepsilon = v^\varepsilon$.

Inspecting the proof of Lemma 2.4, we see that for a.e. $t \in (0, T)$,

$$
\varepsilon \int_{\Gamma^\varepsilon \cap W} |\tilde{v}^\varepsilon - v|^2 \, dS \leq C \int_{U \cap W} |\tilde{v}^\varepsilon - v|^2 \, dx + \varepsilon^2 \int_{U \cap W} \|\nabla v\|^2 \, dx,
$$

since $\nabla \tilde{v}^\varepsilon = 0$ in $\varepsilon(k + Y)$.

Integration in time yields

$$
\varepsilon \int_{\Gamma^\varepsilon \cap W} |\tilde{v}^\varepsilon - v|^2 \, dS dt \leq C \|\tilde{v}^\varepsilon - v\|^2_{L^2(U \cap W)} + \varepsilon^2 \|\nabla v\|^2_{L^2(U \cap W)}
$$

$$
\to 0, \quad \text{as } \varepsilon \to 0.
$$

Finally, the triangle inequality yields

$$
\lim_{\varepsilon \to 0} \varepsilon \|v^\varepsilon - v\|^2_{L^2(U \cap W)} \leq \lim_{\varepsilon \to 0} \varepsilon \|\tilde{v}^\varepsilon - v^\varepsilon\|^2_{L^2(U \cap W)} + \lim_{\varepsilon \to 0} \varepsilon \|\tilde{v}^\varepsilon - v\|^2_{L^2(U \cap W)} = 0
$$

(3.74)

and the proof of Lemma 3.46 is complete.

The strong convergence allows us to pass to weak limits in products. The proof is a simple application of the triangle inequality and Cauchy-Schwarz inequality.

**Lemma 3.30.** Let $w^\varepsilon, v^\varepsilon$ be sequences in $L^2(\Gamma^\varepsilon)$ where such that

$$
\frac{\varepsilon}{|\Gamma|} \int_{\Gamma^\varepsilon} w^\varepsilon \varphi \, dS \to \int_U w \varphi \, dx, \quad \forall \varphi \in C(\overline{U}),
$$

and $v^\varepsilon$ converges strongly in the sense of Lemma 3.46. Then the product $v^\varepsilon w^\varepsilon$ converges to the limit of the products:

$$
\lim_{\varepsilon \to 0} \frac{\varepsilon}{|\Gamma|} \int_{\Gamma^\varepsilon} v^\varepsilon w^\varepsilon \varphi \, dS = \int_U v w \varphi \, dx, \quad \forall \varphi \in C(\overline{U}).
$$
Step 4: Even though $v^\varepsilon$ converges strongly in the sense of Lemma 3.46, there is no notion of pointwise convergence. Thus we cannot for instance immediately apply Fatou’s lemma to conclude that $\xi = h(v)$. Instead we proceed with a Minty-type argument [16], which relies heavily on the ability to compute weak limits of products.

Fix $w \in L^2(0,T;H^1(U))$, and an open set $W \subset \subset U_T$. By monotonicity (2.20)

$$\varepsilon \int_{\Gamma_{T,T}^{\varepsilon} \cap W} (h(v^\varepsilon) - h(w))(v^\varepsilon - w) + C_h(v^\varepsilon - w)^2 dS dt \geq 0. \quad (3.75)$$

Since $v^\varepsilon$ converges strongly in the sense of Lemma 3.46 we can, using Lemma 3.30, pass to the limit in the products in (3.75), obtaining

$$\int_{U_T \cap W} (\xi - h(w))(v - w) + C_h(v - w)^2 dS dt \geq 0. \quad (3.76)$$

Take $w = v - \tau \varphi$ for some $\tau > 0$ and $\varphi \in C^\infty(U_T)$ in (3.76) to find that

$$\int_{U_T \cap W} (\xi - h(v - \tau))(\tau \varphi) + C_h(\tau \varphi)^2 dS dt \geq 0. \quad (3.77)$$

Divide by $\tau$ and take the limit as $\tau \to 0$ to conclude that

$$\int_{U_T \cap W} (\xi - h(v)) \varphi dS dt \geq 0. \quad (3.78)$$

By replacing $\varphi$ with $-\varphi$ we also deduce that

$$\int_{U_T \cap W} (\xi - h(v)) \varphi dS dt \leq 0. \quad (3.79)$$

Since $\varphi$ and $W$ are arbitrary, it follows that $h(v) = \xi$. This finishes the proof of Lemma 3.24.

3.3.4 Deriving the averaged equations

We continue to show that the two-scale limits $(u_i, u_e, v)$ satisfy the averaged bidomain equations (3.13-3.14). Since this system possesses a unique solution we can thereafter conclude that the whole sequence converges. Just like in the formal derivation of Section 3.1, we first consider the intra and extracellular potentials separately. Take

$$\varphi^\varepsilon(t,x) = \varphi_1(t,x) + \varepsilon \varphi_2 \left(t,x,\frac{x}{\varepsilon}\right),$$
3.3. TWO-SCALE CONVERGENCE OF THE BIDOMAIN EQUATION

where \( \varphi_1 \in C_0^\infty(U_T) \) and \( \varphi_2 \in C_0^\infty(U_T; C_0^\infty(Y)) \). Since \( \varphi^\varepsilon \in L^2(0,T; H^1(U^-)) \), the intracellular potential \( u_i^- \) satisfies

\[
\varepsilon \int_{\Gamma_T^-} \partial_t v^\varepsilon \varphi^\varepsilon dS dt + \int_{(U_i^-)_T} \sigma_i^- \nabla u^\varepsilon \cdot \nabla \varphi^\varepsilon dx dt + \varepsilon \int_{\Gamma_T^-} h(u^\varepsilon) \varphi^\varepsilon dS dt = \int_{(U_i^-)_T} I_i \varphi^\varepsilon dx dt.
\]

Rewrite the first integral as

\[
\varepsilon \int_{\Gamma_T^-} \partial_t v^\varepsilon \left[ \varphi_1(t,x) + \varepsilon \varphi_2 \left( t, x, \frac{x}{\varepsilon} \right) \right] dS(x) dt
= \varepsilon \int_{\Gamma_T^-} \partial_t v^\varepsilon \varphi_1(t,x) dS(x) dt + \varepsilon^2 \int_{\Gamma_T^-} \partial_t v^\varepsilon \varphi_2 \left( t, x, \frac{x}{\varepsilon} \right) dS(x) dt. \tag{3.81}
\]

We estimate the second term of (3.81) as follows:

\[
\left| \varepsilon^2 \int_{\Gamma_T^-} \partial_t v^\varepsilon \varphi_2 \left( t, x, \frac{x}{\varepsilon} \right) dS(x) dt \right|
\leq \varepsilon \left( \varepsilon^{1/2} \| \partial_t v^\varepsilon \|_{L^2(\Gamma_T)} \right) \left( \varepsilon^{1/2} \| \varphi_2 \|_{L^2(\Gamma_T)} \right) \leq \varepsilon C.
\]

Thus, by the two-scale convergence (3.45),

\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma_T^-} \partial_t v^\varepsilon \varphi^\varepsilon dS dt = |\Gamma| \int_{U_T} \partial_t v \varphi_1 dx dt.
\]

Similar arguments show that

\[
\lim_{\varepsilon \to 0} \int_{(U_i^-)_T} I_i \varphi^\varepsilon dx dt = |Y_i| \int_{U_T} I_i \varphi_1 dx dt.
\]

To show that

\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma_T^-} h(v^\varepsilon) \varphi^\varepsilon dS dt = |\Gamma| \int_{U_T} h(v) \varphi_1 dx dt,
\]

\[
\int_{(U_i^-)_T} I_i \varphi^\varepsilon dx dt.
\]
we use the Hölder inequality, with \( p = 4/3 \) and \( p' = 1/4 \). In the second term of (3.80), the chain rule yields

\[
\int_{(U^\varepsilon_t)^T} \sigma_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla \left[ \varphi_1(t, x) + \varepsilon \varphi_2 \left( t, x, \frac{x}{\varepsilon} \right) \right] \, dxdt
\]

\[
= \int_{(U^\varepsilon_t)^T} \sigma_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla \varphi_1(t, x) \, dxdt
\]

\[
+ \varepsilon \int_{(U^\varepsilon_t)^T} \sigma_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla_x \varphi_2 \left( t, x, \frac{x}{\varepsilon} \right) \, dxdt
\]

\[
+ \int_{(U^\varepsilon_t)^T} \sigma_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla_y \varphi_2 \left( t, x, \frac{x}{\varepsilon} \right) \, dxdt.
\]

The second last term is estimated as follows:

\[
\left| \varepsilon \int_{(U^\varepsilon_t)^T} \sigma_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla_x \varphi_2 \left( t, x, \frac{x}{\varepsilon} \right) \, dxdt \right|
\]

\[
\leq \varepsilon C \left( \| \nabla u_i^\varepsilon \|_{L^2((U^\varepsilon_t)^T)} \right) \left( \| \nabla_x \varphi_2 \|_{L^2((U^\varepsilon_t)^T)} \right) \leq \varepsilon C.
\]

Thus, by passing to the two-scale limit (3.82), we arrive at

\[
\lim_{\varepsilon \to 0} \int_{(U^\varepsilon_t)^T} \sigma_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla \varphi \, dxdt
\]

\[
= \int_{U_T \times Y_1} \sigma_i(x, y) \left( \nabla_x u_i + \nabla_y u_i^1 \right) \cdot \nabla_x \varphi_1(t, x) \, dxdydt
\]

\[
+ \int_{U_T \times Y_1} \sigma_i(x, y) \left( \nabla_x u_i + \nabla_y u_i^1 \right) \cdot \nabla_y \varphi_2(t, x, y) \, dxdydt.
\]

Here we used that \( \sigma_i^T(x, y) \varphi_j(x, y) \) is an admissible test function for two-scale convergence (Definition 3.5). The limit when \( \varepsilon \to 0 \) of (3.80) thus becomes

\[
|\Gamma| \int_{U_T} \partial_t v_\varphi_1 \, dxdt + |\Gamma| \int_{U_T} h(v) \varphi_1 \, dxdt
\]

\[
+ \int_{U_T \times Y_1} \sigma_i(x, y) \left( \nabla_x u_i + \nabla_y u_i^1 \right) \cdot \left( \nabla_x \varphi_1(t, x) + \nabla_y \varphi_2(t, x, y) \right) \, dxdydt
\]

\[
= |Y_1| \int_{U_T} I_i \varphi_1 \, dxdt.
\]

(3.84)
valid for all \( \varphi_1 \in C_0^\infty(U_T) \) and \( \varphi_2 \in C_0^\infty(U_T; C_0^\infty(Y)) \).

Take \( \varphi_1 = 0 \) in (3.84) to deduce that for a.e. \( t \in (0, T) \),

\[
\int_{U \times Y_i} \sigma_i(x, y) \left( \nabla_x u_i + \nabla_y u_i \right) \cdot \nabla_y \varphi_2(x, y) dxdy = 0. \tag{3.85}
\]

We rewrite this equation as

\[
\int_{U \times Y_i} \sigma_i \nabla_y u_i \cdot \nabla_y \varphi(x, y) dxdy = - \int_{U \times Y_i} \text{div}_y(\sigma_i) \cdot \nabla_x \varphi dxdy + \int_{U \times \Gamma} \sigma_i \nabla_x \varphi dxdS(y), \tag{3.86}
\]

which is precisely the weak formulation of (3.5). Thus

\[
u^1_i = \chi_i \cdot \nabla_x u_i, \tag{3.87}\]

where \( \chi_i(x, y) \) is the first order corrector defined in (3.9).

Now, set \( \varphi_2 = 0 \) in (3.84). The representation (3.87) of \( u^1_i \) inserted into the second term yields

\[
\int_{U \times Y_i} \left( \sigma_i(x, y) + \nabla_y \chi_i^T(x, y) \right) \nabla_x u_i \cdot \nabla_x \varphi_1(x, y) dxdy = 0. \tag{3.88}
\]

Thus, denoting

\[
M_i(x) = \int_{Y_i} \sigma_i(x, y) + \nabla_y \chi_i^T(x, y) dy, \tag{3.89}\]

we see that (3.80), with \( \varphi_2 = 0 \), is precisely the weak formulation of the first bidomain equation:

\[
|\Gamma| \partial_t v - \text{div}(M_i \nabla u_i) + |\Gamma| h(v) = |Y_i| I_i. \tag{3.90}\]

Similarly, if we set \( \chi_e(x, y) \) to be the unique solution of

\[
\int_{U \times Y_e} \sigma_e \nabla_y \chi_e \cdot \nabla_y \varphi(x, y) dxdy = - \int_{U \times Y_e} \text{div}_y(\sigma_e) \varphi(x, y) dxdy + \int_{U \times \Gamma} \nu \cdot \sigma_e \varphi(x, y) dS(y) dx, \quad \forall \varphi \in L^2(U; H^1(#(Y_e))/\mathbb{R}), \tag{3.91}\]

then \( u^1_e \) is given by

\[
u^1_e = \chi_e \cdot \nabla u_e.\]
Thus, with
\[ M_\varepsilon(x) := \int_{Y_\varepsilon} \sigma_\varepsilon(x,y) + \nabla_y \chi_\varepsilon(x,y) \, dy, \tag{3.92} \]
we conclude that \( u_\varepsilon \) satisfies the equation
\[ |\Gamma| \partial_t v + \text{div}(M_\varepsilon \nabla u_\varepsilon) + |\Gamma|h(v) = -|Y_\varepsilon| I_\varepsilon. \tag{3.93} \]

The system of (3.90) and (3.93) is well studied in the literature, the following well-posedness result is proved in [7,8,14].

\textbf{Lemma 3.31.} Under the same assumptions on \( v_0, \ h, \ \sigma_j \ \text{and} \ I_j \) as given in Section (2.3), the system of (3.90) and (3.93) possesses a unique solution \( u_i, \ u_\varepsilon \ \text{and} \ v \) in \( L^2(0,T; H^1(U)) \), with \( \partial_t v \in L^2(U_T) \).

It follows that every convergent subsequence of \( u^\varepsilon \) converges to the same limit, namely the unique solution of (3.90) and (3.93). Thus the whole sequence two-scale converges, in the sense of (3.45), to the solution of the bidomain equations.

\textbf{Remark 3.32.} The two-scale convergence was not really for the boundary terms, only convergence in measure.
Chapter 4

Bidomain equations with stochastic forcing

The aim of this chapter is to develop a streamlined wellposedness theory for the bidomain equations with stochastic forcing. We will consider the system

\[ \partial_t v + h(v) - \text{div} (M_i \nabla u_i) = \xi, \quad \text{in } U_T, \quad (4.1) \]

\[ \partial_t v + h(v) + \text{div} (M_e \nabla u_e) = \xi, \quad \text{in } U_T, \quad (4.2) \]

where \( v = u_i - u_e \). Here \( \xi \) is Gaussian space-time white noise, formally defined by \( \mathbb{E}\xi(s, x)\xi(t, y) = \delta(s-t)\delta(x-y) \) i.e., \( \xi \) is a stochastic process uncorrelated in time and space. We will give a precise meaning white noise as \( dW \) where \( W \) is a Wiener process with values \( L^2(U) \). For technical reasons however, we cannot take \( \xi \) completely uncorrelated in space, instead will assume that the noise is "colored". We complete the system with homogeneous Dirichlet boundary conditions

\[ u_j = 0 \quad \text{on } \partial U, \]

and a deterministic initial datum,

\[ v\big|_{t=0} = v_0. \]

The rigorous study of stochastic PDE's (SPDE's) in full generality requires extensive amounts of probability theory, most of which is beyond the scope of this text. We will therefore take advantage any particularities of the problem at hand and simplify definitions and proofs whenever possible. As a consequence, many of the definitions and theorems will not be stated in their most general form.

Due to the lack of temporal (and spatial) regularity of white noise, solutions cannot be understood classically, or even weakly (in the usual sense).
This leads to numerous generalized solution concepts, most notably strong, weak and mild solutions, see [33]. There is also the notion of Martingale solutions which is a very weak probabilistic solution concept. These solutions are so rough that they cannot be understood in the pathwise sense. It turns out that the amount of spatial correlation in the white noise one is prepared to accept determines the level of difficulty for the problem. In many applications it is interesting to have no correlation at all.

The deterministic bidomain equations have been widely studied from a wellposedness perspective. Just like for the cellular model, the main obstacle is the degenerate structure of the temporal derivatives. There has been a variety of strategies for dealing with this problem.

One of the first wellposedness results where obtained in [14], where an abstract change of variable reformulated the problem into an abstract variational inequality. Standard methods of nonlinear semigroups where then applied, although considerable effort was needed to interpret the solution thus obtained. As a consequence, wellposedness results were only obtained under strong temporal regularity assumptions on the forcing term.

Another approach where presented in [8], where a certain non-local “bidomain operator” were introduced. Using this operator they where able to reduce the degenerate system into a single non-degenerate parabolic pseudodifferential equation which could then be solved with standard methods.

A third approach where developed in [7] where a generalization to monotone operators (p-Laplacians) where treated. Using a parabolic regularization technique they were able to produce a sequence on non-degenerate problems, which in turn could be solved by standard Galerkin techniques. Using a number of a priori bounds obtained with the Galerkin method, a compactness argument was then used to deduce existence of a solution.

To deal with the temporal and spatial roughness of the noise term, the approach in [14] do not look very fruitful, since abstract setting makes it hard to analyze how singularities propagate and how parabolic smoothing effects comes into play. On the other hand, the lack of temporal temporal regularity of the noise makes some of the a priori bounds used in [7] unattainable. (But see a forthcoming paper [30])

The approach in [8] on the other hand is concrete enough to obtain good regularity results, while at the same time being considerably more “operational” than [7]. In particular we obtain an analytic semigroup, were the generator can be analyzed with some precision. This allow us to formulate (4.1- 4.2) in the context of stochastic convolutions with analytic semigroups, which is one of the most accessible approaches for dealing with semilinear parabolic SPDE’s [22].

The strategy is to fix a realization of the Wiener process and then work
strictly pathwise. This allows for a treatment where the amount of probability theory required is minimal. In the end however, we are able to recover useful asymptotic estimates of the variance of the solution. Although we for technical reasons only treat relatively "nice" noise terms (the underlying Wiener process takes values in $L^2$) we believe that current approach can be extended to treat more rough noises.

In order to fit the bidomain system into the framework of stochastic convolutions, the first step is to reduce (4.1-4.2) into a non-degenerate parabolic pseudo-differential equation [21, 26],

$$
\partial_t v + h(v) + Av = \xi, \quad v(0) = v_0,
$$

(4.3)

where the operator $A$ generates an analytic semigroup $\{S_t\}_{t \geq 0}$ on $L^2(U)$ [32]. The operator $A$ is the bidomain operator introduced in [8]. A stochastic process $v$ is then said to be a mild solution of (4.1-4.2) if it satisfies the stochastic convolution equation

$$
v(t) = S(t)v_0 - \int_0^t S(t-s)h(v(s))ds + \int_0^t S(t-s)dW(s),
$$

where $dW$ is the stochastic differential of a colored Wiener process $W$. Much of the material of this chapter is inspired by [22], with [15, 33] as complementing sources.

4.1 Reduction to non-degenerate pseudo-differential equation

We use an orthogonal decomposition technique, similar to the one in Chapter 2, to derive the bidomain operator. Set $L^2 := L^2(U)$, $H^1_0 := H^1_0(U)$ and $H^{-1} := (H^1_0(U))^\prime$, and use $\langle \cdot, \cdot \rangle$ for any duality pairing extending the $L^2$-inner product. Let $a_j$ be the bilinear form on $H^1_0 \times H^1_0$ given by

$$
a_j(u, v) = \int_U M_j \nabla u \cdot \nabla v \, dx, \quad j = i, e.
$$

(4.4)

On the product space, set

$$
a : (H^1_0 \times H^1_0) \times (H^1_0 \times H^1_0) \to \mathbb{R},
$$

(4.5)

defined by

$$
a((u_i, u_e), (v_i, v_e)) = a_i(u_i, v_i) + a_e(u_e, v_e).
$$

(4.6)
The analogue of the boundary operator from Chapter 2 is the following operator:
\[ B : H^1_0 \times H^1_0 \to H^1_0, \quad B(u_i, u_e) = u_i - u_e. \] (4.7)

Clearly \( B \) is continuous and its kernel is the functions \( \varphi = (\varphi_i, \varphi_e) \) such that \( \varphi_i = \varphi_e \).

Multiplying (4.1) and (4.2) with test functions and subtracting them from each other we obtain
\[ \langle \partial_t v, B\varphi \rangle + a(u, \varphi) + \langle h(v), B\varphi \rangle = \langle \xi, B\varphi \rangle, \] (4.8)

where \( \varphi = (\varphi_i, \varphi_e) \). Thus the problem reduces to a deterministic elliptic problem when \( \varphi \in \ker B \).

We will now invert \( B \) on a suitable subdomain of \( H^1_0 \times H^1_0 \). Define the closed subspace
\[ V = \{ u \in H^1_0 \times H^1_0 \mid a(u, \varphi) = 0 \quad \forall \varphi \in \ker B \}. \] (4.9)

Denote \( A_j = \text{div}(M_j \nabla (\cdot)) \) for \( j = i, e \). We regard these either as weak operators \( H^1_0 \to H^{-1} \), or as strong operators \( H^2 \cap H^1_0 \to L^2 \). In both cases they represent the homogeneous Dirichlet boundary condition, and in both cases they are invertible.

For all \( u \in H^1_0 \times H^1_0 \) and \( \varphi \in C^\infty_0(U) \),
\[ a((u_i, u_e), (\varphi, \varphi)) = \langle A_i u_i + A_e u_e, \varphi \rangle. \] (4.10)

By density of \( C^\infty_0 \) in \( H^1_0 \), it follows that \( V \) is precisely the subspace of functions \( u = (u_i, u_e) \), such that \( A_i u_i + A_e u_e = 0 \) in the weak sense. To find the inverse \( B^{-1} : H^1_0 \to V \) we solve the system
\[ \begin{cases} u_i - u_e = v \\ A_i u_i + A_e u_e = 0. \end{cases} \] (4.11)

The solution is
\[ B^{-1} v = ((A_i + A_e)^{-1} A_e v, -(A_i + A_e)^{-1} A_i v). \] (4.12)

Here \( (A_i + A_e)^{-1} = R \) maps elements in \( H^{-1} \) to the unique weak solution of the Dirichlet problem in \( H^1_0 \). By standard elliptic regularity theory we also have the mapping property \( R : L^2 \to H^2 \cap H^1_0 \), and \( R \) restricted to \( L^2 \) is compact, symmetric and positive. Inserting (4.12) into (4.8) we obtain the equation
\[ \partial_t v + h(v) + Av = \xi, \quad \text{in} \ U_T, \] (4.13)
4.1. REDUCTION TO NON-DEGENERATE PSEUDO-DIFFERENTIAL EQUATION

where \( A : H^1_0 \to H^{-1} \) is the bidomain operator \([8]\) given by

\[
A = A_i(A_i + A_e)^{-1}A_e.
\]

(4.14)

This is not a usual differential operator but a pseudo-differential operator \([21]\), a fact we however will make little use of. On an abstract level it shares many features with standard elliptic operators. In particular, the mapping properties,

\[
A : H^{s+2} \cap H^1_0 \to H^s \quad \text{bijectively } \forall s \geq -1,
\]

(4.15)

is deduced from the individual factors. We also note that all of these realizations are symmetric. A little algebra shows that the bidomain operator is equivalently represented by

\[
A_i(A_i + A_e)^{-1}A_e = A_e(A_i + A_e)^{-1}A_i.
\]

It is also straightforward to compute,

\[
\tilde{a}(v, w) := a(B^{-1}v, B^{-1}w) = \langle Av, w \rangle.
\]

(4.16)

The following proposition is an immediate consequence of the continuity of \( B \) and \( B^{-1} \) \([8]\).

**Proposition 4.1.** The bilinear form \( \tilde{a} \) satisfies

\[
|\tilde{a}(v, w)| \leq C_1 \|v\|_{H^1} \|w\|_{H^1},
\]

and

\[
\tilde{a}(v, v) \geq C_2 \|v\|_{H^1}^2,
\]

for all \( v, w \in H^1_0 \).

The strong realization of \( A \), with domain \( H^2 \cap H^1_0 \), is a closed positive self-adjoint unbounded operator on \( L^2 \). Since the resolvent is compact we obtain the following spectral decomposition:

**Proposition 4.2.** There exists an orthonormal basis of eigenvectors \( \{e_j\}_{j=1}^{\infty} \) and sequence of positive eigenvalues \( \{\lambda_j\}_{j=1}^{\infty} \) such that

\[
Ae_j = \lambda_j e_j.
\]

Equipped with this spectral decomposition we define the analytic semigroup \([32]\) \( \{S(t)\}_{t \geq 0} \) associated to \( A \) by

\[
S(t)u = \sum_j e^{-\lambda_j t}c_je_j,
\]

(4.17)
where \( u = \sum_j c_j e_j \). The name "semigroup" comes from the fact that
\[
S(s + t) = S(s)S(t), \quad s, t \geq 0.
\]
For \( \alpha \in \mathbb{R} \) we also define the fractional powers \( A^\alpha \) by
\[
A^\alpha u := \sum_j \lambda_j^\alpha c_j e_j, \quad (4.18)
\]
with domain,
\[
D(A^\alpha) = \left\{ u = \sum_k c_k e_k \in L^2 \mid \sum_k \lambda_k^{2\alpha} c_k^2 < \infty \right\}.
\]
if \( \alpha \geq 0 \) and
\[
D(A^\alpha) = \text{closure of } L^2 \text{ with respect to the norm } \|A^\alpha u\|,
\]
if \( \alpha < 0 \). It follows from the elementary inequality
\[
\lambda^\alpha e^{-t\lambda} \leq C \alpha t^{-\alpha}, \quad \forall t, \lambda > 0,
\]
that for all \( \alpha \in \mathbb{R} \)
\[
\|A^\alpha S(t)\| \leq C \alpha t^{-\alpha}. \quad (4.19)
\]
As a consequence \( S \) leaves \( D(A^\alpha) \) invariant for all \( \alpha \in \mathbb{R} \). We shall also make use of the estimate,
\[
\|S(t)u - u\| \leq C t^\alpha \|A^\alpha u\|, \quad \forall t > 0, \quad (4.20)
\]
which follows along the same line of reasoning. Of particular interest will be the case \( \alpha = 1/2 \). We can then characterize the interpolation space \([32]\]
\[
D(A^{1/2}) = H^1_0
\]
as follows: For all \( \varphi \in C_0^\infty \) we have the equality
\[
\|A^{1/2} \varphi\|^2 = (A^{1/2} \varphi, A^{1/2} \varphi) = (A \varphi, \varphi) = \tilde{a}(\varphi, \varphi).
\]
By Proposition 4.1 the rightmost expression is equivalent to the \( H^1 \) norm. In fact, it is possible to show that \( \|A^{s/2} \varphi\| \) is comparable to \( \|\varphi\|_{H^s} \) for all \( s \in \mathbb{R} \) \([21]\), but we do not go give details. In particular, in three spatial dimensions
\[
D(A^\alpha) \hookrightarrow L^\infty, \quad \text{if } \alpha > 3/4,
\]
as a consequence of the Sobolev embedding theorem. It follows that we also have the embedding
\[
L^1 \hookrightarrow D(A^{-\alpha}), \quad \text{for } \alpha > 3/4. \quad (4.21)
\]
4.2 Wiener process in Hilbert space

Let $H$ be a real separable Hilbert space with scalar product $\langle \cdot , \cdot \rangle$. A Gaussian measure $\mu$ on $H$ is a measure on the Borel sigma algebra $\mathcal{B}(H)$ such that for all $x \in H$ the push forward measure

$$x^*\mu(A) = \mu(\{y \in H | \langle y, x \rangle \in A\}) \quad A \in \mathcal{B}(\mathbb{R}),$$

has a centered normal distribution. The properties of push forward measure implies that

$$\int_H g(\langle x, y \rangle) d\mu(y) = \frac{1}{\sqrt{2\pi\sigma^2(x)}} \int_{\mathbb{R}} g(z) e^{-z^2/2\sigma^2(x)} dz,$$

where $dz$ is the usual one-dimensional Lebesgue measure and $g$ is any function on $\mathbb{R}$ such that the right integral is well defined. The following celebrated theorem is due to Fernique, the proof of which can be found in [15].

**Proposition 4.3** (Fernique). Let $\mu$ be a Gaussian measure on a separable Hilbert space $H$. Then the moments of $\mu$ can be bounded in terms its second moment

$$\int_H \|x\|^{2m} d\mu(x) \leq C_m \left( \int_H \|x\|^2 d\mu(x) \right)^m,$$

for some universal constants $C_m$. Here universal means that the same constant works for all Gaussian measures.

The covariance $C_\mu : H \times H \to \mathbb{R}$ associated to a Gaussian measure $\mu$ is

$$C_\mu(x, y) = \int_H (x, z)(y, z) d\mu(z).$$

The corresponding operator $Q : H \to H$ is defined by

$$(Qx, y) = C_\mu(x, y). \quad (4.22)$$

Clearly $Q$ is positive and symmetric. Furthermore the next proposition shows it is of trace class [33]. Recall an operator $T$ on $H$ is said to be of trace class [28] if for some orthonormal basis $\{e_k\}$,

$$\text{tr}(T) := \sum \langle Te_k, e_k \rangle < \infty.$$  

It is well known that $\text{tr}(T)$ is independent of the choice of orthonormal basis and that $\text{tr}(T)$, when finite, is the sum of the eigenvalues of $T$. This obvious from the spectral theorem in the symmetric case [28]. That it holds also for non symmetric operators is a nontrivial theorem due to Lidskii [28]. It is clear from the definition that

$$\text{tr}(T) = \text{tr}(T^*).$$
Proposition 4.4. The covariance operator $Q$ associated to $\mu$ is a symmetric trace class operator. Conversely, every symmetric trace class operator on $\mathcal{H}$ is the covariance of some Gaussian measure on $\mathcal{H}$.

Thus the study of Gaussian measures on Hilbert spaces reduces to the study of trace class operators. We associate a random variable to a measure in the standard way [20].

Definition 4.5. Let $X$ be a random variable on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ taking values in a Hilbert space $\mathcal{H}$. We say that $X$ is a Gaussian (or normal) random variable with distribution $\mu$, if $X^*\mathbb{P}$ is a Gaussian measure with covariance $C_\mu$.

Given a Gaussian measure $\mu$ with covariance $Q$ we can always construct a random variable $X$ with $\mu$ as its distribution. Let $\{b_k\}_{k=1}^\infty$ be a sequence of i.i.d. random variables with standard normal distribution $\mathcal{N}(0,1)$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\{\lambda_k\}_{k=1}^\infty$ be the decreasing sequence of (positive) eigenvalues of $Q$ with corresponding eigenvectors $\{e_k\}_{k=1}^\infty$. Define the random variable

$$X := \sum_{k=1}^\infty \sqrt{\lambda_k} b_k e_k,$$

with convergence in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathcal{H})$, since $\mathbb{E} \sum_{k=1}^\infty \|\sqrt{\lambda_k} b_k e_k\|^2 = \sum_{k=1}^\infty \lambda_k$. We show that $X$ has the right covariance. For fixed $y, z \in \mathcal{H}$, we have

$$\int_\mathcal{H} \langle x, y \rangle \langle z, x \rangle \, dX^*\mathbb{P} = \int_\Omega \langle X, y \rangle \langle z, X \rangle \, d\mathbb{P}$$

$$= \sum_{k=1}^\infty \left( \int_\Omega b_k^2 \, d\mathbb{P} \right) \lambda_k \langle e_k, y \rangle \langle z, e_k \rangle$$

$$= \langle Qy, z \rangle.$$

We now want to construct Brownian motion with values in $\mathcal{H}$. We first recall the properties of Brownian motion with scalar values [20]. A real valued standard Brownian motion is a stochastic process $\{B_t\}_{t \geq 0}$ with values in $\mathbb{R}$ such that

- $B$ starts at the origin: $B(0) = 0$ a.s.;
- $B$ has independent increments: $B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$ are independent random variables;
4.2. WIENER PROCESS IN HILBERT SPACE

- $B$ has continuous paths:
  \( t \mapsto B(t, \omega) \) is continuous a.s.;

- $B$ has normal increments:
  \( B(t_2) - B(t_1) \sim \mathcal{N}(0, t_2 - t_1) \) for \( t_1 < t_2 \).

It is well known that such a process exists, and the following classical theorem shows that there even exists a Hölder 1/2 continuous version of it [15]. Recall that $Y$ is a version of $X$ if for every $t$, $X(t) = Y(t)$ a.s.

**Theorem 4.6 (Kolmogorov continuity).** Suppose that $X$ is stochastic process with values in $\mathcal{H}$ such that

\[
\mathbb{E}\|X(t) - X(s)\|^\delta \leq C\|t - s\|^{1+\varepsilon}.
\]

Then there exists a version $Y$ of $X$ such that for every $\alpha < \varepsilon/\delta$, $Y \in C^\alpha(0, T; \mathcal{H})$ a.s.

Combining this theorem with the Fernique theorem we obtain the following useful criterion:

**Corollary 4.7.** Let $X$ be a Gaussian process with values in $\mathcal{H}$ and suppose that there is $\gamma \in (0, 1)$ such that

\[
\mathbb{E}\|X(t) - X(s)\|^2 \leq C|t - s|^{\gamma}, \quad \forall s, t \geq 0.
\]

Then there exists a version of $X$ that is $\alpha$-Hölder continuous for $\alpha < \frac{\gamma}{2}$

**Proof.** According to the Fernique theorem, the moments of $X(t) - X(s)$ are bounded by the powers of the second moment. Thus, for ever $m \in \mathbb{N}$ there is a constant $C_m$, working for all $s$ and $t$, such that

\[
\mathbb{E}\|X(t) - X(s)\|^{2m} \leq C_m \left( \mathbb{E}\|X(t) - X(s)\|^2 \right)^m \leq CC_m|t - s|^{\gamma m}.
\]

Applying the Kolmogorov continuity test, we conclude that there is a $\alpha$-Hölder continuous version of $X$ for every $\alpha < \frac{m\gamma - 1}{2m}$. Thus, if $\alpha < \frac{\gamma}{2}$ we obtain $\alpha$-Hölder continuity if $m$ is sufficiently large.

The covariance of Brownian motion is given by $s \land t := \min\{s, t\}$ since if $s < t$ we have

\[
\mathbb{E}B(t)B(s) = \mathbb{E}[(B(t) - B(s))B(s) + B^2(s)]
\]

\[
= \mathbb{E}[B(t) - B(s)]\mathbb{E}B(s) + \mathbb{E}B^2(s)
\]

\[
= 0 + 0 + s. \quad \text{(4.23)}
\]
where we used independence of increments to move expectation inside the product.

Recall that a filtration $\{F_t\}_{t \geq 0}$ of a $\sigma$-algebra $\mathcal{F}$ is an increasing family of $\sigma$-algebras that generates $\mathcal{F}$ in the sense that

$$F_s \subset F_t, \quad \text{if } s < t,$$

and

$$\mathcal{F} = \bigcup_{t \geq 0} F_t.$$

Recall that the $\sigma$-algebra, $\sigma(X)$, generated by a random variable $X$ is the smallest $\sigma$-algebra such that $X$ is measurable. A random variable $X$ is independent of a sigma algebra $\mathcal{G}$ if $\sigma(X)$ and $\mathcal{G}$ are independent. This simply means that for all events in $A \in \sigma(X)$ and $B \in \mathcal{G}$,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B).$$

If this is the case, a consequence is that $\mathbb{E} \langle X, Y \rangle = \langle \mathbb{E}X, \mathbb{E}Y \rangle$ for all $Y$ that are $\mathcal{G}$-measurable.

We say that $X$ is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ if $X(t)$ is $\mathcal{F}_t$-measurable for every $t$. Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and suppose the $\mathcal{G}$ is a sub sigma-algebra of $\mathcal{F}$. The conditional expectation $\mathbb{E}[X | \mathcal{G}]$ of $X$ given $\mathcal{G}$ is the orthogonal projection of $X$ onto $L^2(\Omega, \mathcal{G}, \mathbb{P})$ [20].

**Definition 4.8.** A (square integrable) martingale [33] with respect to a filtration $\{\mathcal{F}_t\}$ with values in $\mathcal{H}$ is a stochastic process $M$ with $M(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$ such that $M$ is $\mathcal{F}_t$-adapted and satisfies the martingale property

$$\mathbb{E}[M(t) | \mathcal{F}_s] = M(s), \quad \forall s \leq t.$$

Thus $M(s)$ is the ”best guess” of $M(t)$ at time $s < t$. The most important consequence of the martingale property is that the variance of the maximum process is estimated by the variance at the end time [33]. We state it in the $L^2$ case only, where the result is also known as Doob’s inequality.

**Lemma 4.9** (Burkholder-Davies). Let $M$ be a continuous martingale with values in $\mathcal{H}$. Then

$$\mathbb{E} \left( \sup_{t \in [0,T]} \|M(t)\|^2 \right) \leq 4 \mathbb{E}\|M(T)\|^2.$$

Another way of stating this theorem is that $\mathbb{E} \left( \sup_{t \in [0,T]} \|M(t)\|^2 \right)$ and $\sup_{t \in [0,T]} \mathbb{E}\|M(t)\|^2$ are equivalent norms for continuous martingales.

We are now in a position to consider $\mathcal{H}$-valued Brownian motion.
Definition 4.10 (Q-Wiener process). An $\mathcal{F}_t$-adapted stochastic process $\{W_t|t \geq 0\}$ with values in $\mathcal{H}$ is called a Q-Wiener process on $\mathcal{H}$ if

$$W(t) - W(s) \text{ is } \mathcal{F}_s \text{- independent for } t > s,$$

and

$$\mathbb{E} \langle W(s), x \rangle \langle W(t), y \rangle = s \wedge t \langle Qx, y \rangle$$

for all $x, y \in \mathcal{H}$ and $W(\cdot, \omega) \in C([0, T]; \mathcal{H})$ a.s. Here $s \wedge t = \min\{s, t\}$.

It is a fact that not all stochastic processes can be realized as functions. For example there does not exist (in the usual sense) a process with the properties of white noise. When the covariance operator is of trace class the question of existence does however have an affirmative answer [33].

To explicitly construct the Wiener process on $\mathcal{H}$ let $\{B_k(t)\}$ be a sequence of independent standard Brownian motions on $[0, T]$, and let $\{e_k\}$ be an orthonormal basis of $\mathcal{H}$ such that $Qe_k = \lambda_k e_k$. Define

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} B_k(t) e_k,$$

where we claim convergence of the series in $L^2(\Omega, \mathcal{A}, \mathbb{P}; C([0, T]; \mathcal{H}))$. We have, using Doob’s inequality,

$$\mathbb{E} \sup_{t \in [0, T]} \left\| \sum_{k=n+1}^{\infty} \sqrt{\lambda_k} b_k(t) e_k \right\|^2 = \sum_{k=n+1}^{\infty} \lambda_k \mathbb{E} \sup_{t \in [0, T]} |B_k(t)|^2 \leq \sum_{k=n+1}^{\infty} \lambda_k 4 \sup_{t \in [0, T]} \mathbb{E}|B_k(t)|^2 = 4T \sum_{k=n+1}^{\infty} \lambda_k \to 0,$$

as $n \to \infty$. Furthermore, the Hölder continuity is inherited:

$$W \in L^2(\Omega, \mathcal{A}, \mathbb{P}; C^\alpha([0, T]; \mathcal{H})).$$

This is seen from the estimate,

$$\mathbb{E} \|W(s) - W(t)\|^2 = \sum_{k=1}^{\infty} \lambda_k \mathbb{E}|B_k(s) - B_k(t)|^2 \leq |s - t| \sum_{k=1}^{\infty} \lambda_k.$$
and appealing to Kolmogorov’s continuity criterion (Lemma 4.7), the claim follows.

When the covariance is not of trace class (for example if \( Q = \text{id} \)) there are still ways to construct a Wiener process with the given covariance by defining it as a process on a larger space space \( \mathcal{H}' \) such that the inclusion \( \iota : \mathcal{H} \rightarrow \mathcal{H}' \) is trace class. This is usually called a cylindrical Wiener process on \( \mathcal{H} \) even though it does not strictly speaking take values in \( \mathcal{H} \) [15,33]. We will however not consider such constructions here. In the rest of this chapter we will by Wiener process mean a \( Q \)-Wiener process instead of the more conventional cylindrical notion.

### 4.3 Stochastic integration

We will now define the integral with respect to a \( Q \)-Wiener process, giving a rigorous meaning to \( \int_0^t S(t-s) dW(s) \) [15]. This expression is also referred to as a stochastic convolution. Observe that the function we are hoping to integrate is operator valued and that the integral itself is a stochastic process with values in a Hilbert space. Let \( \mathcal{L}(\mathcal{H}) \) denote the algebra of bounded operators on \( \mathcal{H} \).

In general one can also consider stochastic integrands \( S : (0,T) \times \Omega \rightarrow \mathcal{L}(\mathcal{H}) \), which is necessary when considering SPDE’s with multiplicative noise. Since we here restrict attention to additive noise, we will only define the stochastic integral for deterministic integrands. This makes the definition of the stochastic integral somewhat simpler.

A function \( S : [0,T] \rightarrow \mathcal{L}(\mathcal{H}) \) is called elementary if there is a partition \( 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T \) and elements \( S_j \in \mathcal{L}(\mathcal{H}), \ j = 1, \ldots, n \) such that

\[
S(t) = \sum_{j=1}^n S_j \chi_{(t_j,t_{j+1}]}.
\]  

For elementary functions we define the stochastic integral \( I \) by

\[
I[S](t) := \sum_{j=1}^n S_j \left( W_{t_{j+1} \wedge t} - W_{t_j \wedge t} \right)
\]

In order to extend the definition to continuous semigroups, we show that the integral defined above is an isometry with respect to certain norms.

The relevant norm in this context is the Hilbert-Schmidt norm for operators, given by

\[
\|T\|_{\text{HS}} := \text{tr}(T^*T).
\]
An operator $T$ is said to be Hilbert-Schmidt if $\|T\|_{HS}$ is finite. Thus if $K$ is a symmetric positive trace class operator, then $K^{1/2}$ is Hilbert-Schmidt. It follows from the definition that $\|T^*\|_{HS} = \|T\|_{HS}$, and that $\|AB\|_{HS} \leq \|A\|_{HS} \|B\|$. (4.27)

Hence, the Hilbert-Schmidt operators form an ideal in the algebra of bounded operators [28]. This simply means that you can multiply a Hilbert-Schmidt operator by a bounded operator, and the product remains Hilbert-Schmidt.

The relevance of the Hilbert-Schmidt norm is the Ito isometry, which is the content of the following theorem [15,33].

**Theorem 4.11 (Ito isometry).** Let $S : [0, T] \to \mathcal{L}(\mathcal{H})$ be an elementary function. Then $I[S]$ is a continuous $\mathcal{F}_t$-martingale and we have the isometry

$$
\mathbb{E} \left\| \int_0^t S dW \right\|^2 = \int_0^t \|S(s)Q^{1/2}\|_{HS}^2 \, ds.
$$

(4.28)

**Proof.** We first show the martingale property. Set $\eta_j := W(t_{j+1}) - W(t_j)$ and let $n < m$ with $t_m \leq T$. Then

$$
\mathbb{E}[I(S)(t_m) \mid \mathcal{F}_n] = \mathbb{E} \sum_{j=1}^{n-1} S_j \eta_j + \mathbb{E} \left[ \sum_{j=n}^m S_j \eta_j \mid \mathcal{F}_n \right],
$$

since $\eta_j$ is $\mathcal{F}_n$-adapted for $j < n$. Since $W$ has independent increments, $\eta_j$ is independent of $\mathcal{F}_n$ for $j \geq n$. Hence,

$$
\mathbb{E} \left[ \sum_{j=n}^m S_j \eta_j \mid \mathcal{F}_n \right] = \sum_{j=n}^m S_j \mathbb{E} \eta_j = 0,
$$

and it follows that $I(S)$ is an $\mathcal{F}_t$-martingale. To prove the isometry, let $t = t_m \leq T$. We have

$$
\mathbb{E} \|I[S](t)\|^2 = \mathbb{E} \left\| \sum_{j=1}^{m-1} S_j \eta_j \right\|^2 \\
= \mathbb{E} \sum_{j=1}^{m-1} \|S_j \eta_j\|^2 + 2 \mathbb{E} \sum_{1 \leq i < j \leq m-1} (S_j \eta_j, S_i \eta_i),
$$
where we simply multiplied out the terms. We show that the second sum is zero. Note that \( \eta_j \) and \( \eta_i \) are independent when \( i < j \) (\( \eta_j \) is \( \mathcal{F}_{t_j} \)-independent and \( \eta_i \) is \( \mathcal{F}_{t_j} \)-measurable since \( t_i < t_j \)). We thus have

\[
E \langle S_j \eta_j, S_i \eta_i \rangle = E \langle S_j \eta_j, E S_i \eta_i \rangle = 0,
\]

where we used independence to move expectation inside the scalar product.

It remains to show that

\[
E \sum_{j=1}^{m-1} \| S_j \eta_j \|^2 = \sum_{j=1}^{m-1} \| S_j Q^{1/2} \|^2_{HS} \tag{4.29}
\]

Let \( e_k \) be the eigenvectors of \( Q \). Using that \( Q^{1/2} S_j^* = (S_j Q^{1/2})^* \) we have

\[
E \| S_j \eta_j \|^2 = E \sum_k (S_j \eta_j, e_k)^2
\]

\[
= \sum_k E \langle \eta_j, S_j^* e_k \rangle^2
\]

\[
= (t_{j+1} - t_j) \sum_k \langle QS_j^* e_k, S_j^* e_k \rangle
\]

\[
= (t_{j+1} - t_j) \| Q^{1/2} S_j^* \|^2_{HS}
\]

\[
= (t_{j+1} - t_j) \| S_j Q^{1/2} \|^2_{HS},
\]

where we used the dominated convergence theorem to interchange sum and integral. Summing over \( j \) we obtain the desired equality. \( \square \)

Using the Ito isometry we extend the stochastic integral to all functions which can be approximated by simple functions in the norm

\[
S \mapsto \int_0^T \| S Q^{1/2} \|^2_{HS} dt.
\]

By (4.27), this class includes \( L^2(0, T; \mathcal{L}(\mathcal{H})) \). Depending on the properties of \( Q \), we may also integrate some unbounded operators [15].

It follows from the Ito isometry that this extension satisfies

\[
I : L^2(0, T; \mathcal{L}(\mathcal{H})) \to C(0, T; L^2(\Omega; \mathcal{H})). \tag{4.30}
\]

By the Burkholder-Davies inequality (Lemma 4.9) we have

\[
I(S) \in L^2(\Omega; C(0, T; \mathcal{H})) ,
\]

also for the extension.
4.4 Regularity of the stochastic convolution

As a first step in obtaining existence and uniqueness, we study the regularity properties of the solution to the linear equation

\[
\begin{aligned}
\frac{du}{dt} + Au &= dW, \\
u(0) &= 0.
\end{aligned}
\]  

(4.31)

The mild solution is by definition given by the stochastic convolution

\[
u(t) = \int_0^t S(t-s) dW(s) =: W_A(t),
\]  

(4.32)

where \( S \) is the semigroup generated by \( A \) (4.17). The following theorem shows how spatial regularity, in terms of interpolation spaces (4.18) of \( A \), are related to the covariance operator \( Q \).

**Theorem 4.12.** Suppose that \( \| A^\beta Q^{1/2} \|_{HS} < \infty \) for some \( \beta \geq 0 \). Then we have a.s.,

\[
W_A(t) \in D(A^\alpha), \quad \forall \alpha < 1/2 + \beta.
\]

**Proof.** We will show that

\[
\mathbb{E} \left\| A^\alpha \int_0^t S(t-s) dW(s) \right\|^2 < \infty,
\]

from which the theorem follows. Since \( A^\alpha \) is a closed operator we can (using the definition of the stochastic integral) move it inside the integral

\[
A^\alpha W_A(t) = \int_0^t A^\alpha S(t-s) dW(s).
\]

By the Ito isometry,

\[
\mathbb{E} \left\| \int_0^t A^\alpha S(t-s) dW(s) \right\|^2 = \int_0^t \left\| A^\alpha S(t-s) Q^{1/2} \right\|^2_{HS} ds
\]

\[
= \int_0^t \left\| A^\alpha S(r) Q^{1/2} \right\|^2_{HS} dr.
\]

Using (4.27) and (4.19) the integrand is estimated as follows:

\[
\left\| A^\alpha S(r) Q^{1/2} \right\|^2_{HS} \leq \left\| A^{\alpha-\beta} S(r) A^\beta Q^{1/2} \right\|^2_{HS}
\]

\[
\leq \left\| A^\beta Q^{1/2} \right\|^2_{HS} \left\| A^{\alpha-\beta} S(r) \right\|^2
\]

\[
\leq C r^{-2\alpha+2\beta}.
\]

where we use that \( A^\alpha \) and \( S(t) \) commutes in the domain of \( A^\alpha \) [32]. This expression is integrable near zero precisely when \( \alpha < 1/2 + \beta \). \( \square \)
Similar to the situation for deterministic parabolic PDE’s there is a close link between temporal and spatial regularity. More precisely, we can obtain a higher Hölder continuity in time, at the expense of measuring the spatial regularity in a lower interpolation space, and vice versa. The amount of room to play with is ultimately limited by the smoothness of the covariance operator.

**Theorem 4.13.** Suppose \( \| A^\beta Q^{1/2} \|_{HS} < \infty \) for some \( \beta \geq 0 \) and fix \( \gamma < 1/2 + \beta \). Then \( W_A \) is \( \alpha \)-Hölder continuous in \( D(A^\gamma) \) for every \( \alpha < 1/2 + \beta - \gamma \),

\[ W_A \in C^\alpha(0, T; D(A^\gamma)) \quad \text{a.s.} \quad (4.33) \]

**Remark 4.14.** In spatial dimension \( n = 3 \), we can take \( Q = I - \Delta_{\text{Dir}}^2 \), where \( \Delta_{\text{Dir}} \) is the Dirichlet realization of \( \Delta \). This operator is of trace class since it is well known that the Sobolev embedding \( H^s \hookrightarrow L^2 \) is trace class if \( s > n \) \([21]\). It follows that \( \| A^\beta Q \|_{HS} < \infty \) for \( \beta < 1/2 \). Choosing \( \gamma = 1/2 \) we have for every \( \alpha < 1/2 \),

\[ W_A \in C^\alpha(0, T; H^1_0) \]

**Proof.** According to Kolmogorov’s continuity theorem and Corollary 4.7, it suffices to check that

\[ \mathbb{E} \| A^\gamma W_A(t) - A^\gamma W_A(s) \|^2 \leq C |t - s|^{2\alpha}. \quad (4.34) \]

Let \( s < t \). Using the semigroup property, we rewrite

\[ \int_0^t S(t - r) \, dW(r) = S(t - s) \int_0^s S(s - r) \, dW(r) + \int_s^t S(t - r) \, dW(r), \]

where the \( \mathcal{F}_s \)-independence of \( W(r) - W(s) \) for \( r > s \) implies that the two terms are independent. It follows that

\[ \mathbb{E} \left< S(t - s) A^\gamma W_A(s) - A^\gamma W_A(s), \int_s^t A^\gamma S(t - r) \, dW(r) \right> \]

\[ = \mathbb{E} \left< S(t - s) A^\gamma W_A(s) - A^\gamma W_A(s), \int_s^t A^\gamma S(t - r) \, dW(r) \right> = 0, \]

since a Gaussian integral has zero mean. Thus the left-hand side of (4.34) equals

\[ \mathbb{E} \| S(t - s) A^\gamma W_A(s) - A^\gamma W_A(s) \|^2 + \mathbb{E} \left\| \int_s^t A^\gamma S(t - r) \, dW(r) \right\|^2. \quad (4.35) \]

The second term is estimated as in the proof of Theorem 4.12 by

\[ C \int_0^{t-s} r^{2(\beta - \gamma)} \leq C(t - s)^{2\alpha} \left( \int_0^{t-s} r^{2(\beta - \gamma) - 2\alpha} \, dr \right)^{1 - 2\alpha}, \]
where we used Hölder's inequality with \( p = 1/2\alpha, \ p' = 1/(1 - 2\alpha) \). The function \( r^{\frac{2(\beta-\gamma)}{1+2\alpha}} \) is clearly integrable if \( \alpha \leq \beta - \gamma + 1/2 \).

Next, we estimate the first term in (4.35). By (4.20) we have

\[
\mathbb{E} \|S(t - s)A^\gamma W_A(s) - A^\gamma W_A(s)\|^2 \leq C(t - s)^{2\alpha} \mathbb{E} \|A^{\gamma + \alpha} W_A(s)\|^2.
\]

But according to Theorem 4.12, we have \( A^\delta W_A(s) \in \mathcal{H} \) for \( \delta < 1/2 + \beta \), with a uniform bound on \( \mathbb{E} \|A^\delta W_A(s)\|^2 \) on \([0,T]\).

\[ \square \]

4.5 Existence and uniqueness

We now proceed with a Duhamel iteration argument to construct local solutions to the nonlinear problem. With local we mean a sufficiently short time frame. In the stochastic context this statement needs some clarification.

**Definition 4.15.** A stopping time \( \tau \) with respect to a filtration, \( \{\mathcal{F}_t\}_{t \geq 0} \), is a random variable such that the event \( \tau \leq t \) is \( \mathcal{F}_t \)-measurable for every \( t \geq 0 \).

**Definition 4.16.** We say \( u \in L^1((0,T); L^3(U)) \) is a local mild solution to (4.3) if there exists a stopping time \( \tau > 0 \) almost surely, such that

\[
u(t) = S(t)u_0 - \int_0^t S(t - s)h(u(s))\,ds + \int_0^t S(t - s)dW(s), \quad (4.36)
\]

holds almost surely for every \( t \leq \tau \).

**Remark 4.17.** Observe that the integral

\[
\int_0^t S(t - s)h(u(s))\,ds,
\]

is well defined as a Bochner integral in \( L^1(0,T; L^1) \) under the cubic growth assumption on \( h \). This is since \( S \) leaves \( L^1 \) invariant according to (4.21).

There is a related concept of weak (or variational) solutions. Recall from Section 4.1 the definition (4.16)

\[
\tilde{a}(u,v) = a(B^{-1}u, B^{-1}v).
\]
Definition 4.18. We say $u \in L^2(0, T; H^1_0(U)$ is a local weak solution to (4.3) if there exists a stopping time $\tau > 0$ almost surely, such that

$$
(u(t), \varphi) + \int_0^t \tilde{a}(u(s), \varphi) \, ds + \int_0^t \langle h(u(s)), \varphi \rangle \, ds = (u_0, \varphi) + (W(t), \varphi) \quad \forall \varphi \in C^\infty_0(U),
$$

holds almost surely for every $t \leq \tau$.

The following proposition shows that mild solutions generalizes weak solutions.

Proposition 4.19. If $u$ is a mild solution and $u \in L^2(0, T; H^1_0(U)$, then $u$ is a weak solution.

Proof. Let $\varphi \in C^\infty_0(U)$. We assume that first that $u_0 = 0$. Split $u$ into

$$
v + W_A := -\int_0^t S(t-s)h(u(s)) \, ds + \int_0^t S(t-s)dW(s),
$$

and set $f(s) := -h(u(s))$. Recall that $\langle Au, \varphi \rangle = \langle u, A\varphi \rangle$, in the sense of distributions and in $H^1_0$, and that also $S$ is symmetric on $L^2$. We show that

$$
\langle W_A(t), \varphi \rangle = \int_0^t \langle W_A(s), A\varphi \rangle \, ds + \langle W(t), \varphi \rangle,
$$

and that

$$
\langle v(t), \varphi \rangle = \int_0^t \langle v(s), A\varphi \rangle \, ds + \int_0^t \langle f(s), \varphi \rangle \, ds,
$$

from which the theorem follows by adding the two identities. Let $W_A$ act on $A\varphi$ and integrate from 0 to $T$:

$$
\int_0^t \langle W_A, A\varphi \rangle \, ds = \int_0^t \left( \int_0^s S(s-r)dW(r), A\varphi \right) \, ds
= \int_0^t \int_0^s \langle S(s-r)dW(r), A\varphi \rangle \, ds
= \int_0^t \int_0^s \langle dW(r), S(s-r)A\varphi \rangle \, ds.
$$
This is well defined since \( x \mapsto \langle x, S(s-r)A\varphi \rangle \) is a bounded deterministic operator in \( \mathcal{L}(\mathcal{H}; \mathbb{R}) \). Using Fubini’s theorem to change the order of integration, this equals

\[
\int_0^t \int_0^s \langle dW(r), S(s-r)A\varphi \rangle \, ds = \int_0^t \left( \int_0^t S(s-r)A\varphi \, ds \right) \, dW(r).
\]

Using Fubini’s theorem to change the order of integration, this equals

\[
\int_0^t \langle dW(r), S(t-r)\varphi - \varphi \rangle \, dr = \int_0^t \langle S(t-r)dW(r), \varphi \rangle + \langle \int_0^t dW(r), \varphi \rangle = \langle W_A(t), \varphi \rangle - \langle W(t), \varphi \rangle.
\]

Here we used the identity \( \partial_s S(s-r)\varphi = S(s-r)A\varphi \), valid for all \( \varphi \in D(A) = H^2 \cap H_0^1 \). Similarly,

\[
\int_0^t \langle v, A\varphi \rangle \, dr = \int_0^t \left( \int_0^t S(s-r)f(r) \, dr \right) A\varphi \, drds = \int_0^t \langle f(r), S(s-r)A\varphi \rangle \, drds = \int_0^t \langle f(r), S(t-r)\varphi - \varphi \rangle \, dr = \langle v(t), \varphi \rangle - \int_0^t \langle f(r), \varphi \rangle \, dr.
\]

If \( u(t) \in H_0^1 \) we can integrate by parts to obtain \( \langle u, A\varphi \rangle = \tilde{a}(u, \varphi) \). The case when \( u_0 \neq 0 \) follows by considering \( \tilde{u} = u(t) - S(t)u_0 \).

**Theorem 4.20.** Let \( U \subset \mathbb{R}^3 \) be bounded a smooth and suppose \( Q \) is a symmetric trace class operator satisfying \( \|\Delta^s Q^{1/2}\|_{HS} < \infty \) for some \( s > 0 \). Assume \( h \) satisfies the growth estimates

\[
|h(z)| \leq C(1 + |z|^3), \quad |h'(z)| \leq C(1 + |z|^2). \tag{4.39}
\]

Then there exists a unique local mild solution to (4.3).

**Proof.** Fix \( \omega \in \Omega \) such that \( W_A(\omega) \in C^\alpha(0, T; H_0^1) \) for some \( \alpha > 0 \), and set \( g(t) = S(t)u_0 + W_A(t) \). We will use the Banach fixed point theorem in the space \( C(0, T; H_0^1) \). Define the mapping

\[
\Phi : u \mapsto \Phi u : (\Phi u)(t) = \int_0^t S(t-s)h(u(s)) \, ds + g(t). \tag{4.40}
\]
We first show that
\[ \Phi : B_T(g, \delta) \rightarrow B_T(g, \delta), \] (4.41)
for sufficiently small \( T, \delta > 0 \). Here \( B_T(g, \delta) \) is the "tube" around \( g \) in \( H^1_0 \):
\[ B_T(g, \delta) = \left\{ v \in C(0,T; H^1_0) \mid \max_{t \in [0,T]} \| g(t) - v(t) \|_{H^1} \leq \delta \right\}. \]

By the cubic growth of \( h \) and the Sobolev embedding \( H^1 \hookrightarrow L^6 \) we have
\[ h : H^1 \rightarrow L^2, \] continuously. (4.42)

Set
\[ M_{\delta,T} = \max \left\{ \| h(v(t)) \|_{L^2} \mid v \in B_T(g, \delta) \right\}, \]
where continuity guaranties that the maximum is obtained. We estimate
\[
\| \Phi u(t) - g(t) \|_{H^1} \leq C \int_0^t \| A^{1/2} S(t-s) h(u(s)) \|_{L^2} \, ds
\leq C \int_0^t |t-s|^{-1/2} \| h(u(s)) \|_{L^2} \, ds
\leq C \int_0^t |t-s|^{-1/2} M_{\delta,T} \, ds
\leq C t^{1/2} M_{\delta,T}.
\]

Here the constant \( C \) is different from line to line but independent of \( \delta, t, g \).

Choosing
\[ T_0 = \min \left\{ T, \left( \frac{\delta}{CM_\delta} \right)^2 \right\}, \]
we obtain (4.41). Note that \( T_0 \) a priori depends on the size of the initial datum and the realization of \( W_A \).

Next, we show that \( \Phi \) is a contraction on \( \overline{B_{T_0}(g, \delta)} \). Observe that
\[ h : H^1 \rightarrow L^2, \] under the growth assumptions on the derivative (4.39), is locally Lipschitz continuous:
\[
\| h(u) - h(v) \|_{L^2}^2 = \int_U |h(u) - h(v)|^2 \, dx
\leq C \int_U |u - v|^2 (1 + |u|^{4} + |v|^{4}) \, dx
\leq C \left( \int_U |u - v|^6 \, dx \right)^{1/3} \left( 1 + \int_U |u|^6 + |v|^6 \, dx \right)^{2/3}
\leq C \| u - v \|_{H^1}^2 (1 + \| u \|_{H^1} + \| v \|_{H^1})^2.
\]
Thus if \( u, v \in \overline{B_{T_0}(g, \delta)} \) there is a Lipschitz constant \( L \leq CM_{\delta, T_0} \). Repeating the calculations above, we find that
\[
\| \Phi(u) - \Phi(v) \|_{C(0, T_0; H^1)} < \| u - v \|_{C(0, T_0; H^1)},
\]
if \( T_0 \) is sufficiently small. \( \square \)

Observe that we in the argument above did not pay any attention to the sign of the nonlinearity. Since cubic nonlinearities with positive sign in general leads to blow ups in finite time, something different is needed to obtain global solutions.

Inspecting the proof of Theorem 4.20 we see that, once \( W_A \) is fixed, there is a lower bound on \( T_0 \) only depending on the size of \( u_0 \) in \( H^1_0 \). Since we can (pathwise) paste together solutions on different intervals just like for ODE’s [22], we conclude the following lemma:

**Lemma 4.21.** Suppose that the maximal time of existence \( T^* \) is finite. Then
\[
\| v(t) \|_{H^1} \to \infty, \quad \text{as } t \uparrow T^*.
\]

Thus it suffices to show that \( \sup_{t \in (0, T^*)} \| u(t) \|_{H^1} < \infty \), in order to obtain global solutions. Since a mild solution in \( C(0, T; H^1_0) \) is also a weak solution, we can use the characterization (2.5) to obtain energy estimates similar to those of Section 2. This will also yield stability with respect to initial data. However, due to the lack of temporal regularity in the stochastic forcing term, we cannot simply copy the calculations from Section 2. Similar to the situation for stochastic ODE’s, the unbounded variation (in time) of \( W \) leads to a correction term in the chain rule, a phenomenon known as Ito’s formula.

Calculating formally, we multiply the SPDE
\[
du + (Au + h(u))dt = dW,
\]
with \( u \) and integrate in time to obtain
\[
\frac{1}{2} \| u(t) \|^2 = \int_0^t \langle A(u) + h(u), u \rangle \, ds + \int_0^t \langle dW(s), u(s) \rangle \, ds.
\]
From here we estimate
\[
\left| \int_0^t \langle dW(s), u(s) \rangle \, ds \right| \leq \int_0^t \| Q \|_{HS} \| u(s) \| \, ds,
\]
after which an application of Grönwall’s inequality yields the desired energy estimate. Not only is the this calculation unjustified but it is also wrong. It can however be corrected (see [33]) to
\[
\frac{1}{2} \| u(t) \|^2 = \int_0^t \langle A(u) + h(u), u \rangle \, ds + \int_0^t \langle dW(s), u(s) \rangle \, ds + t \| Q^{1/2} \|_{HS}^2,
\]
but this relies on some elements of stochastic integration beyond the scope of this text. Instead we will stick to the pathwise paradigm and make the following observation:

Under the monotonicity assumption (2.20)
\[(h(v + w) - h(w))v \geq -C_h v^2,\]

hence,
\[h(v + w)v \geq h(w)v - C_h v^2 \geq -|h(w)||v| - C_h v^2 \geq -C \left(1 + |w|^6 + v^2\right). \tag{4.43}\]

Estimates involving the derivative, \(\partial_t v\), are more difficult to obtain. Let us assume that \(w \in L^\infty\). In the model case when \(h(z) = z^3\), we have
\[\int_0^T (v + w)^3 \partial_t v \, dt = \int_0^T v^3 \partial_t v \, dt + \int_0^T (3v^2 w + 3vw^2 + w^3) \partial_t v \, dt \geq \frac{1}{4} v^4(T) - \frac{1}{4} v^4(0) - \left| \int q(v, w) \partial_t v \, dt \right|,\]

where \(q(v, w) = 3v^2 w + 3vw^2 + w^3\). If \(w \in L^\infty\) then we can obtain an estimate of the form
\[
\left| \int q(v, w) \partial_t v \, dt \right| \leq \delta \int_0^T |\partial_t v|^2 \, dt + C(\delta)\|w(t)\|_{L^\infty(0,T)}^2 \left(1 + \int_0^T v^4 \, dt\right) \tag{4.44}
\]

Here we used the Cauchy inequality with an \(\epsilon\). This motivates the following assumption on \(h\):

**Assumption.** We say that \(h\) satisfies assumption \(A_1, A_2\), provided there exists constants \(C'_h, \delta_h > 0\), such that
\[|h(v + w) - h(v)| \leq C'_h (1 + v^2) (1 + w^2), \quad \forall v, w \in \mathbb{R}. \tag{A_1}\]

and
\[\int_0^v h(z) \, dz \geq \delta_h v^4 - C (v^2 + 1). \tag{A_2}\]

The following lemma follows from the calculations above.

**Lemma 4.22.** Suppose \(h\) satisfies \(A_1, A_2\), and that \(w \in L^\infty(U_T)\). Then there exists a constant \(C\) such that
\[
\int_0^T \int_U h(v + w) \partial_t v \geq \delta_h \|v(T)\|_{L^4}^2 - \frac{1}{2} \int_0^T \|\partial_t v\|_{L^2}^2 dt - C \left(1 + \|w\|_{L^\infty(U_T)}^2 \right) \left(\int_0^T \|v\|_{L^4}^4 \, dt + \|v\|_{L^2(U_T)}^2 + \|v(T)\|_{L^2}^2 + \|v(0)\|_{L^4}^4 + 1\right), \tag{4.45}
\]
for all sufficiently regular functions \( v : U_T \to \mathbb{R} \).

Equipped with this lemma, we can establish some a priori estimates on the local solution. Note that Lemma 4.33 assures that assumption of \( W_A \in L^\infty \) is fulfilled if \( Q = (-\Delta_{\text{Dir}})^{-2} \).

**Theorem 4.23.** The local solution satisfies the priori estimate

\[
\|u(t)\|_{L^2} \leq e^{tC} (\|W_A(t)\|_{L^6}^3 + \|u_0\|_{L^2}),
\]

for all \( t < T^* \). Here \( W_A \) is the linear stochastic convolution (4.32), and \( C \) is a constant independent of \( t, u_0, \omega \).

If in addition \( W_A \in L^\infty(U_T) \) and \( h \) satisfies \((A_1)\) and \((A_2)\), then

\[
\|u(t)\|_{H^1} \leq C',
\]

for some constant \( C' \) independent of \( t < T^* \).

**Remark 4.24.** The consequences of this theorem are twofold. First of all, the pathwise \( H^1 \) bound yields global existence of solutions. Secondly, it shows stability with respect to \( L^2 \) data. By taking expectations, it also shows stability with respect to stochastic perturbations.

**Proof.** Fix \( t < T^* \), where \( T^*(\omega) \) is the maximal time of existence. Set \( v = u - W_A \), i.e.

\[
v(t) = S(t)u_0 + \int_0^t S(t-s)h(v(s) + W_A(s)) \, ds.
\]

Thanks to the regularity of the stochastic convolution (Lemma 4.33), we know that \( \sup_{t \in [0,T]} \|W_A(t)\|_{H^1} < \infty \) for all \( T < \infty \). Thus it suffices to bound \( v(t) \). From (4.48) it follows that \( v \) is weakly differentiable in \( t \), with

\[
\partial_t v = Av + h(v + W_A), \quad v(0) = u_0.
\]

Since \( v, W_A \in L^2(0,t;H^1) \), we have \( h(v + W_A) \in L^2(U_t) \) by (4.42). Since also \( u_0 \in H^1_0 \) it follows from standard parabolic regularity theory \( \partial_t v, Av \in L^2(U_t) \). Thus (4.48) can be understood in the strong \( (L^2) \) sense. Taking the scalar product of (4.48) and \( v \), we get

\[
\frac{1}{2} \|v(t)\|^2 + \langle Av, v \rangle + \langle h(v + W_A), v \rangle = 0.
\]

Recall that \( \langle Av, v \rangle \geq 0 \). Using (4.43) we obtain the bound

\[
\langle h(v + W_A), v \rangle \geq -C \left( \|v\|_{L^2}^2 + \|W_A\|_{L^6}^6 + 1 \right).
\]
Applying the differential version of Grönwall’s lemma, we deduce

\[ \|v(t)\|_{L^2}^2 \leq e^{Ct} \left( \|u_0\|_{L^2}^2 + \int_0^t \|W_A\|_{L^6}^6 \, ds \right). \]  
(4.49)

Returning to (4.48), multiplication by \( \partial_t v \) and integration in time yields

\[ \int_0^t \|\partial_s v\|^2 \, ds + \langle Av(t), v(t) \rangle - \langle Av(0), v(0) \rangle = -\int_0^t \langle h(v + W_A), \partial_s v \rangle \, ds. \]  
(4.50)

Recall that \( \langle Au, u \rangle \) is comparable to \( \|u\|_{H^1}^2 \). Using Lemma 4.22 to estimate the righthand side, we obtain

\[
\|v(t)\|_{H^1}^2 + \frac{1}{2} \int_0^t \|\partial_s v\|^2 \, ds + \delta_t \|v(t)\|_{L^4}^4 \\
\leq C \left( 1 + \|W_A\|_{L^\infty(U_T)}^2 \right) \left( \int_0^t \|v(s)\|_{L^4}^4 \, ds + \|u_0\|_{H^1}^2 + \|u_0\|_{L^4}^4 + \|v(t)\|_{L^2}^2 + 1 \right). \]
(4.51)

In view of (4.49), there is a constant \( C \), such that

\[ \|v(t)\|_{L^2}^4 \leq C \left( \int_0^t \|v(s)\|_{L^4}^4 \, ds + 1 \right). \]

Applying Grönwall’s lemma, we find that \( \|v(t)\|_{L^2}^4 \) is uniformly bounded for \( t < T^* \). Going back to (4.51, we conclude that there is a constant, independent of \( t \), such that

\[ \|v(t)\|_{H^1}^2 \leq C. \]
Bibliography


