Effects of the interest rate on intergenerational distribution

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May 10 2017
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http://www.duo.uio.no/

Publisher: Reprosentralen, University of Oslo
Preface

Thanks to my supervisor Asbjørn Rødseth for useful comments and discussions. I would also like to thank the Professorship in Macro and monetary Policy Issues for providing me a scholarship and an office space. I am also grateful for those who took their time to proofread, Sigrid Valberg, Thomas Jacobsen and my mother, Sigrun. All remaining errors are my own.

Kristin Celius

May 10 2017
Abstract

I present a three period overlapping generations model in order to study the scope of intergenerational redistribution when the real interest rate falls temporarily. Because of consumption smoothing, agents have different levels of wealth over the life cycle. While the initial effect of an interest rate fall is a change in financial wealth, the net financial position of the economy and asymmetric responses from the agents due to different remaining life length might affect macroeconomic aggregates such as the house price, which in turn can strengthen or dampen the initial redistribution effect. The direction of the effects are similar with and without a borrowing constraint, while the effects can be substantially different from a one-period fall to a two-period fall.
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1 Introduction

The real interest rate has historically been subject to large fluctuations. Estimates from Statistics Norway show that the real interest rate reached levels as high as 12 percent in 1992, while it was as low as -0.1 percent in 2016. On the other hand, the house prices have increased substantially. Whether this is beneficial or not for households depends on their initial levels of financial- and housing wealth, and how it affects their access to the housing market. Unestablished households face a trade-off between low interest costs on their mortgages and high prices on the housing market, and they have a limited amount of accumulated wealth.

This article investigates how a temporary fall in the real interest rate may have redistribution effects between generations, based on the assumption that old agents are lenders, middle-aged agents are borrowers, and young agents are unestablished and have not yet accumulated any financial wealth or housing capital.

My ambition is to offer an understanding of how being born into a period with a temporary low interest rate have implications for welfare both in the period of the fall and over the life cycle, and how it affects the welfare for generations born prior to the interest rate fall when they have accumulated wealth based on higher levels. As long as levels of accumulated wealth are related to age, how far the agent has come is in the life cycle can determine whether the interest rate fall is beneficial or not.

I analyze this in an overlapping generations model of a small open economy with an exogenous real interest rate. The agents consume housing and other consumption goods, and leave bequest to their children. They also weight housing consumption differently according to their age, meaning that some age groups have a higher influence on the price than other groups. The agents smooth consumption over the life cycle by borrowing and saving, which is an explanation for different levels of financial wealth among age groups. Young households expecting higher income in the future may borrow to consume more

2https://www.ssb.no/statistikkbanken/px-igraph/Mak Graph.asp?checked=true
when young. Thus, they are likely to be net borrowers when they become middle-aged. Middle-aged households expecting their income to decline when they become older might save, meaning that old households are likely to be net lenders. I look both at the case where the households face a borrowing constraint and the case where they do not.

Based on this framework, I will study how the temporary interest rate fall affects the welfare for all generations alive at that point in time when the following mechanisms arise: The immediate effect is a redistribution of financial wealth from lenders to borrowers. However, the household responses are not necessarily symmetrical, as younger agents have more remaining periods of their lives than the old agents. We therefore need to investigate how consumption of other goods changes for each generation individually and on the aggregate level. Increased demand may in turn affect the house prices, which yields extra wealth to agents holding housing capital. The house price relative to the lifetime wealth determine the agents' consumption of housing. Thus, the initial wealth redistribution might affect macroeconomic aggregates, and might strengthen or dampen the initial effect.

The model predicts that a one-period fall in the interest rate causes similar effects both for the borrowing constrained economy and the unconstrained, but of a higher magnitude for the latter. The middle-aged borrowers are unambiguously better off, as they experience a gain both in their financial wealth and their housing wealth. The old lenders are worse off as their increased housing wealth is not enough to compensate them for their loss in financial wealth. The implications for the young and unestablished are more ambiguous; as they do not have any accumulated wealth prior to the shock, they are unaffected in the first round. However, increased prices on the housing market causes their housing consumption to be lower compared to a situation of constant interest rates. The effects are less clear if the interest rate is low for two periods due to a higher impact on the savings behavior of the households.

The simplicity of the model makes it easy to study the mechanisms that matters for the total welfare effect for the generations. However, several extensions could give a deeper understanding of the topic. I do not look at the production side of the economy. A more
flexible housing supply would cause prices to behave differently in the long run. Another interesting extension would be to add uncertainty, as uncertainty about future interest rates, prices and income affect behaviour of the agents. Because I use log-utility, I will not consider how different values of the intertemporal elasticity of substitution, which could matter for the outcome because of the effect on the savings behaviour of the agents.

The agents can own or rent their home in my setup, nobody will rent due to a preference for ownership. By including for instance a minimum size of housing, an interesting topic could be whether the interest rate fall could cause young and unestablished agents to become renters in stead of home owners. The inclusion of the rental market is still useful in my setup, as it allows me to separate between housing as an object of investment and as a consumption good.

I do not consider endogenous labor supply. In a generational perspective, endogenous labor supply could have implications for welfare because young and middle-aged agents can adjust their labor supply while old, retired agents cannot.

The rest of the article is structured in the following way: Chapter 2 contains a brief discussion of existing literature on the topic. I present the model setup in chapter 3, and offer the solutions to the benchmark model in chapter 4. I then extend the model in chapter 5 with a borrowing constraint. In chapter 6, I provide a discussion of the mechanisms that cause redistribution and asymmetrical responses among the generations when the interest rate falls temporarily. In chapter 7, I discuss the simulation results from Matlab. I also add a brief discussion of how the outcome could be if the interest rate been increased instead.

2 Related literature

There are several contributions who have addressed the issue of monetary policy shocks in a life cycle model, but to my knowledge there are few articles who consider distribution in a generational perspective. Most of the contributions are based on data, and are not
purely theoretical.

In the context of the Great Depression, Fisher (1933) investigated in his debt-deflation theory how lower prices imposed a larger burden on the debtors, such as homeowners and corporations. The debt made out a larger proportion of their expenditures, causing households to default on their loans. This reduced their further access to credit. While the debtors struggled to pay back their debts, the creditors experienced a gain. As my model predicts as well, the population is not distributed between debtors and borrowers randomly. Households are debtors for good reasons, and they are typically young families acquiring homes before they have the income to pay outright (Tobin (1980)). The consequences are of a different nature in my article, as those who lose by the real interest rate fall do not struggle to pay back their loans because they will still have positive net wealth.

A paper that studies inflation as a redistribution shock from lenders to borrowers is the article by Doepke and Schneider (2006). They use a quantitative overlapping generations model of the U.S economy, and model an episode of inflation as an unanticipated shock to wealth distribution. They use cross-sectional data on U.S households, while my model is purely theoretical. As opposed to my model, they include endogenous labor supply and income differences in their model. Purchase prices of housing are also normalized to one, and they assume that the oldest households sell their apartment and live in rented housing in the last period of life. Rental houses are owned by other households as part of their assets, it is not a separate object of investment from financial assets.

Doepke, Schneider, and Selezneva (2015) address the distributive effects of monetary policy as well, with a similar model setup to the article described above. They look at an unexpected fall in nominal inflation, a change in the inflation target of the Central Bank, and an unexpected drop in the real interest rate. Their findings are by large the same as in the previous article, namely that middle-class middle-aged win on the expense of the wealthy retirees. They study the effects both in the case of fixed house endogenous house prices. In both articles, they separate housing types and do not measure housing as a divisible good as in my model. I aim to put a larger emphasis on how the shock
have implications for life cycle consumption of the young, due to the higher prices on the housing market.

I study the same mechanisms as in the two articles above, namely that the initial wealth change can affect macroeconomic aggregates due to asymmetric responses from households of different ages.

Auclert et al. (2015) evaluates the role of redistribution in the transmission mechanism of monetary policy to consumption. He presents three channels that affects aggregate spending when winners and losers have different marginal propensities to consume. These channels are the earnings heterogeneity channel from unequal income gains, the Fisher channel from unexpected inflation, and the interest rate exposure channel from real interest rate changes. The paper argues that redistribution is a channel through which monetary policy affects macroeconomic aggregates, not that redistribution is a side effect of monetary policy changes. Housing is not a part of the utility function for the households, only consumption goods and labor.

Liane (2013) do not mainly address the issue of redistribution among generations but why the savings tend to increase when the interest rate is cut, rather than significantly increasing consumption. However, the analysis is based on the same assumption as in my article, namely that the initial net financial position of the households have large implications for the effect of a temporary interest rate cut, and she use a life cycle model. Housing is not included in the model.

Eerola et al. (2012) look at borrowing constraints and house price dynamics in the case of large shocks. They use an overlapping generations model with borrowing constraints in the presence of large positive and large negative income shocks, such as changes to the interest rate. Like my article, they compare effects with and without borrowing constraints. They find that down payment constraints do not magnify the impact effect of adverse income or interest rate shocks. There is no bequest in their model, and they assume that the oldest cohort sell the house they lived in the previous period and move to for instance an institution, and the rental market is not included.
3 Model setup

In this setup, which is similar to that of Gary-Bobo et al. (2015), households live for three periods. Old agents die in the end of the third period of their life cycle and a new generation is born every period. During all three periods they derive utility from consuming housing services and other consumption goods. The old agents also derive utility from leaving bequest to their offspring, which is the next generation of old agents. All agents within a generation are identical and they sum up to one so the population size is equal to three at all times.

Denote the age of a household by \( i = y, m, o \), and the number of periods after \( t \) as \( j \). Housing services is denoted \( h^o_{i,t} \) for owner-occupied housing, and \( h^r_{i,t} \) for rented housing. The bequest is \( b_{o,t+3} \). A household born in period \( t \) have the following utility function

\[
U = \sum_{j=0}^{2} \beta^j \left\{ \log(c_{t+j}) + \gamma_i \log(h^r_{i,t+j} + \kappa h^o_{i,t+j}) \right\} + \beta^2 \alpha \log(b_{o,t+3})
\]

\( \beta \) is the discount factor. \( \gamma_i \) is the relative weight the households put on consuming housing services. I will further on assume that the size of the utility weights are in the following order

\[ \gamma_m > \gamma_y = \gamma_o \]

\( \kappa \) represents the relative preference the households have for ownership over renting, assumed to be larger than one. \( \alpha \) is the strength of the bequest motive. The bequest motive is a so-called warm glow bequest, in that the old derives utility from leaving bequest, but do not take the utility of the heirs into account.

In the beginning of each period of the life-cycle, the households receive an endowment \( w_i \) of a size according to their age

\[ w_o < w_y < w_m \]

I provide an argument for why the endowment size of the old is smaller than that of the young in the calibration in chapter 7.
The households can borrow and lend credit, $A_{t+j+1}$, in a credit market to the real interest rate $r_{t+j}$, which is exogenously given. $A_{i,t+j+1}$ is for the rest of the article interpreted as the level of financial assets that the household accumulate during one period.

The households can also purchase buy-to-let housing, $h^l_{i,t}$, and rent it out to other agents. Hence, housing is simultaneously a consumption good, a value of storage as it can be sold in the future, and an object of investment due to rents. The purchase price per unit of housing is $p_t$, and the rental price is $q_t$. The net cost of purchasing buy-to-let housing becomes $p_t - q_t$. Depreciation of housing is assumed away, and there are no transaction costs of moving or changing the size of housing capital.

The bequest an old household leaves at time $t$ consists both of housing capital and financial wealth

$$b_{o,t+1} = h^o_{o,t} + h^l_{o,t} + \frac{(1 + r_{t+1})A_{o,t+1}}{p_{t+1}} $$

(2)

The bequest cannot be negative; if the household leave negative financial assets, this must be compensated for by the housing capital. The financial assets are divided by the prices for the next period to show how much housing the bequest receivers can purchase for it.

The budget constraint for generation $i$ at time $t$ is defined as

$$c_{i,t} + p_t h^o_{i,t} + q_t h^l_{i,t} + (p_t - q_t) h^l_{i,t} + A_{i,t+1} = y_{i,t} $$

(3)

where the disposable resources of each generation are given by

$$y_{y,t} = w_y$$

$$y_{m,t} = w_m + (1 + r_t) A_{y,t} + p_t (h^o_{y,t-1} + h^l_{y,t-1})$$

$$y_{o,t} = w_o + (1 + r_t) A_{m,t} + p_t (h^o_{m,t-1} + h^l_{m,t-1} + b_{o,t})$$

The following no arbitrage condition ensures that the agent is indifferent between holding financial assets and investing in buy-to-let housing

$$1 + r_{t+1} = \frac{p_{t+1}}{p_t - q_t} $$

(4)
One unit of housing corresponds to one square meter, and housing is perfectly divisible. The total amount of housing capital that a household holds is the sum of owner-occupied housing and buy-to-let housing

\[ h^l_{i,t} = h^l_{i,t} + h^o_{i,t} \]

Housing capital is of fixed supply every period. For market clearing on the housing market, the total supply of housing capital, \( H \), must be equal to the total amount of housing capital held by each generation for all \( t \geq 0 \).

\[ H = h^l_{y,t} + h^l_{m,t} + h^l_{o,t} \]

4 The benchmark model

We can define a new variable

\[ a_{i,t+1} = \frac{A_{i,t+j+1}}{p_{t+j} - q_{t+j}} \] (5)

Interpreted as financial assets in terms of buy-to-let housing.

We can use this equation and insert it into the budget constraint, and use the fact that \( h^l_{i,t} = h^l_{i,t} - h^o_{i,t} \) to rewrite the constraint

\[ c_{i,t} + q_t(h^o_{i,t} + h^r_{i,t}) + (p_t - q_t)(h^l_{i,t} + a_{i,t+1}) = y_{i,t} \] (6)

Then we can use \( 1 + r_{t+1} = \frac{p_{t+1}}{p_t - q_t} \) and rewrite the bequest as

\[ b_{o,t+1} = h^l_{o,t} + h^o_{o,t} + \frac{(1 + r_{t+1})A_{o,t+1}}{p_{t+1}} = h^l_{o,t} + a_{o,t+1} \] (7)

From this, we get the period by period budget constraints for an agent through the life cycle:

\[ c_{y,t} + q_t(h^o_{y,t} + h^r_{y,t}) + (p_t - q_t)(h^l_{y,t} + a_{y,t+1}) = w_y \]
\[ c_{m,t+1} + q_{t+1}(h^o_{m,t+1} + h^r_{m,t+1}) + (p_{t+1} - q_{t+1})(h^l_{m,t+1} + a_{m,t+2}) = w_m + p_{t+1}(h^l_{y,t} + a_{y,t+1}) \]
\[ c_{o,t+2} + q_{t+2}(h^o_{o,t+1} + h^r_{o,t+2}) + (p_{t+2} - q_{t+2})b_{o,t+3} = w_o + p_{t+2}(h^l_{m,t+1} + a_{m,t+2} + b_{o,t+2}) \]
As there are no borrowing constraints for the households in this setup, there are no limitations of how much credit they can borrow, neither for housing or other consumption goods. Combined with the preference for ownership, nobody will rent their home and everybody becomes homeowners, so the rental market drops out of the model. Proof for this is provided in Appendix A.1. Thus, we get $h_{i,t}^t = h_{i,t}^o$.

We can use no arbitrage condition again, $1 + r_{t+1} = \frac{p_{t+j+1}}{p_{t+j} - q_{t+j}}$, divide the third constraint by $(1 + r_{t+1})(1 + r_{t+2})$ and the second constraint by $(1 + r_{t+1})$, and rearrange to get the lifetime budget constraint when nobody rents

$$c_{y,t} + q_t h_{y,t}^o + \frac{c_{m,t+1} + q_{t+1} h_{m,t+1}^o}{1 + r_{t+1}} + \frac{c_{o,t+2} + q_{t+2} h_{o,t+2}^o + (p_{t+2} - q_{t+2}) b_{o,t+3}}{(1 + r_{t+1})(1 + r_{t+2})} = W_t$$

Where lifetime wealth for an agent born in period $t$ is

$$W_t = w_y + \frac{w_m}{1 + r_{t+1}} + \frac{w_o + p_{t+2} b_{o,t+2}}{(1 + r_{t+1})(1 + r_{t+2})}$$

Households born in period $t$ maximize utility function (1) subject to the lifetime budget constraint and the additional constraints

$$h_{i,t}^o \geq 0$$
$$h_{r,t}^r \geq 0$$

And the non-negativity constraint on bequests, which is always satisfied when using log-utility.

When $h_{i,t}^r = 0$, the optimality conditions are

$$\gamma_i h_{i,t+j}^o = \frac{q_{t+j}}{c_{i,t+j}}$$
$$c_{k,t+j+1} = \beta(1 + r_{t+j+1})c_{i,t+j}$$
$$b_{o,t+j+3} = \frac{\alpha c_{i,t+j+2}}{p_{t+j+2} - q_{t+j+2}}$$

Housing demand is a fraction of consumption of other goods in the same period, and the age dependent utility weight on housing. Consumption increases according to the
standard Euler equation, where the intertemporal elasticity of substitution is equal to one, and the income- and substitution effects cancel each other out in the case of a change in the interest rate.

Inserting this into the lifetime budget constraint yields the solution for consumption of the young in period t

\[ c_{y,t} = \frac{W_t}{1 + \gamma_y + \beta(1 + \gamma_m) + \beta^2(1 + \gamma_o + \alpha)} \]  

(11)

Use the solution for \( c_{y,t} \) to solve the rest of the demand functions.

\[
\begin{align*}
    c_{m,t+1} &= \beta(1 + r_{t+1})c_{y,t}, \quad c_{o,t+2} = \beta^2(1 + r_{t+1})(1 + r_{t+2})c_{y,t} \\
    h_{y,t} &= \frac{\gamma_y}{q_t}c_{y,t}, \quad h_{m,t+1} = \frac{\gamma_m}{q_{t+1}}c_{m,t+1}, \quad h_{o,t+2} = \frac{\gamma_o}{q_{t+2}}c_{o,t+2} \\
    b_{o,t+3} &= \frac{\alpha}{p_{t+2} - q_{t+2}}c_{o,t+2}
\end{align*}
\]

4.1 Equilibrium

An equilibrium when nobody rents is characterized by a set of prices \( \{p_t, q_t\} \) and a set of optimal allocations \( \{c_{i,t}, h_{i,t}, b_{o,t}\} \) for \( i = y, m, o \) such that agents maximize utility function (1) over all three periods subject to

\[
\begin{align*}
    c_{y,t} + q_t h_{o,t}^o + \frac{c_{m,t+1} + q_t h_{m,t+1}^o}{(1 + r_{t+1})} + \frac{c_{o,t+2} + q_{t+2} h_{o,t+2}^o + (p_{t+2} - q_{t+2})}{(1 + r_{t+1})(1 + r_{t+2})} &\leq W_t \\
    h_{o,t} &\geq 0
\end{align*}
\]

The generations born before time \( t \) maximize utility, and the housing market clears for all \( t \geq 0 \)

\[
H = h_{y,t}^o + h_{m,t}^o + h_{o,t}^o
\]

\[
= \frac{1}{q_t} \frac{\gamma_y W_t + \gamma_m \beta(1 + r_t)W_{t-1} + \beta(1 + r_{t-1})(1 + r_t)\gamma_o W_{t-2}}{1 + \gamma_y + \beta(1 + \gamma_m + \beta^2(1 + \gamma_o + \alpha)}
\]

where housing demand for the young, middle-aged and old depends on the lifetime wealth \( W_t, W_{t-1} \) and \( W_{t-2} \) respectively, the real interest rate \( r_{t+j} \) and the endowment level \( w_i \) for each generation is given exogenously, and the no-arbitrage condition is satisfied.
\[ 1 + r_{t+1} = \frac{p_{t+1}}{p_t - q_t} \] (12)

### 4.2 Steady state

I will now proceed to the steady state solution of the model, where consumption for each generation is constant, as well as the prices \( q_t, p_t \) and the interest rate \( r_t \) are constant.

Starting with consumption of other consumption goods for the young in steady state

\[ c_y = \frac{w_y + \frac{w_m}{(1+r)} + \frac{w_o + pb_o}{(1+r)^2}}{(1 + \gamma_y) + \beta(1 + \gamma_m) + \beta^2(1 + \gamma_o + \alpha)} \]

The bequest the old agent leaves is the same as the bequest the old agent receives in the beginning of the same period

\[ b_o = \frac{\alpha}{p - q} \beta^2(1 + r)^2 c_y \]

From the no arbitrage condition, we see that

\[ \frac{p}{1 + r} = p - q \]

Which we can use to solve for \( \frac{pb_o}{(1+r)^2} \) and get

\[ \frac{pb_o}{(1+r)^2} = \frac{(p - q) \frac{\alpha}{p - q} \beta^2(1 + r)^2 c_y}{(1 + r)} = \alpha \beta^2(1 + r)c_y \]

By rearranging, we can rewrite the solution for steady state consumption of the young to depend on exogenous variables and parameters only

\[ c_y = \frac{w_y + \frac{w_m}{(1+r)} + \frac{w_o}{(1+r)^2}}{(1 + \gamma_y) + \beta(1 + \gamma_m) + \beta^2(1 + \gamma_o - r\alpha)} \] (13)

Steady state consumption for the middle-aged and old are defined by the Euler equations with a constant real interest rate.
In order to solve for prices, we use the market clearing condition on the housing market when nobody rents

\[ H = h_y^o + h_m^o + h_o^o \]

\[ = \frac{1}{q}(\gamma_y c_y + \gamma_m c_m + \gamma_o c_o) \]

Which we can use to solve for the rental price

\[ q = \frac{1}{H}(\gamma_y c_y + \gamma_m c_m + \gamma_o c_o) \]

(14)

And we insert for \( q \) in the no arbitrage condition to solve for the purchase price.

5 Model where the agents are borrowing constrained

Because the initial net financial position of the households is important for the wealth change when the interest rate falls, it is interesting to see how the outcome might be modified when the households are borrowing constrained. I therefore extend the benchmark model with the following borrowing constraint

\[ A_{t+j+1} \geq -(1 - \psi)p_{t+j}(h_{y,t+j}^o + h_{l,t+j}^l) \]

(15)

meaning that the agent can borrow up to a \((1 - \psi)\) fraction of the total house value, and they cannot borrow to finance other consumption goods. The constraint can be interpreted as down payment the household has to do in the same period. This is a typical demand from the banks. Typically, the borrowing constraint is binding only for the youngest groups, as they have low income and have not accumulated any wealth yet. Older agents who have already accumulated some housing wealth are not borrowing constrained. The approach to solve the constrained model, including the calculation of the endowment level for the young in the Appendix, is taken from the master thesis of Stiansen (2016).
The rental market dropped out of the benchmark model due to the relative preference for ownership. When the household is borrowing constrained, the same happens as long as
\[ q_t > \psi p_t \]  
\[ (16) \]
Meaning that the price of renting housing in a period is higher than the down payment they must pay in the same period. With the calibrations I will make, this condition holds in steady state.

5.1 The borrowing constrained households

In the optimality conditions presented below, the borrowing constraint holds with equality for the young group only, and not for the middle-aged and old. The first order conditions and the condition providing that everybody become homeowners are presented in the Appendix A.2.

When the borrowing constraint holds with equality for the young household, the intratemporal optimality condition is modified to
\[ \frac{\gamma_y}{h_{y,t}} = \frac{q_t}{c_{y,t}} - \theta^y_t(q_t - \psi p_t) \]  
\[ (17) \]
As long as condition (16) holds, the last part of the expression will be negative.

The intertemporal optimality condition is found by inserting from the first order condition for consumption into the first order condition for financial asset holdings.
\[ \frac{1}{c_{y,t}} = \frac{\beta(1 + r_{t+1})}{c_{m,t+1}} + \theta^y_t \]  
\[ (18) \]

5.2 Equilibrium

An equilibrium when the economy faces a borrowing constraint is characterized by a set of prices \( \{p_t, q_t\} \) and a set of optimal allocations \( \{c_{i,t}, h_{i,t}, b_{o,t}\} \) for \( i = y, m, o \) such
that households maximize utility function (1) over all three periods subject to the period budget constraints, the borrowing constraint, and the non-negativity constraint of housing

\[
\begin{align*}
c_{y,t} + q_t(h_{y,t} + h_{y,t}^r) + (p_t - q_t)(h_{h,y,t}^t + a_{y,t+1}) &= w_y \\
c_{m,t+1} + q_{t+1}(h_{y,m,t+1}^o + h_{m,t+1}^r) + (p_{t+1} - q_{t+1})(h_{m,t+1}^t + a_{m,t+2}) &= w_m + p_{t+1}(h_{h,y,t}^t + a_{y,t+1}) \\
c_{o,t+2} + q_{t+2}(h_{o,t+2}^o + h_{o,t+2}^r) + (p_{t+2} - q_{t+2})b_{o,t+3} &= w_o + p_{t+2}(h_{m,t+1}^t + a_{m,t+2} + b_{o,t+2}) \\
h_{o,t}^o &\geq 0 \\
h_{r,t}^o &\geq 0 \\
h_{l,t}^l &\geq 0 \\
A_{i,t+1} &\geq -(1 - \psi)p_t(h_{o,t}^o + h_{l,t}^l)
\end{align*}
\]

The generations born before period \( t \) maximize their utility, and the housing market clears for all \( t \geq 0 \)

\[
H = h_{y,t}^o + h_{m,t}^o + h_{o,t}^o
\]

The real interest rate \( r_{t+j} \) and the endowment level \( w_i \) for each generation is given exogenously, and the no-arbitrage condition is satisfied

\[
1 + r_{t+1} = \frac{p_{t+1}}{p_t - q_t}
\] (19)

With the assumption that the borrowing constraint is not binding for the initial middle-aged and old, they will not rent just as in the benchmark model. We can therefore solve the maximization problem for these two groups by using the lifetime budget constraint from period \( t+1 \) and forward and get

\[
c_{m,t+1} = \frac{w_m + (p_{t+1} - (1 - \psi)p_t(1 + r_{t+1}))h_{y,t} + \frac{w_o + p_{t+2}b_{o,t+2}}{1 + r_{t+2}}}{1 + \gamma_m + \beta(1 + \gamma_o + \alpha)}
\] (20)

and

\[
c_{o,t+2} = \beta(1 + r_{t+1})c_{m,t+1}
\]
When the borrowing constraint is binding, consumption of other goods for the young can be found by inserting the constraint into the period budget constraint of the young. We can insert this into the optimality conditions of the young households and get

\[ c_{y,t} = w_y - \psi_p h_{y,t} \]  \hspace{1cm} (21)

\[ \frac{\gamma_y}{h_{y,t}^o} = \frac{q_t}{w_y - \psi_p h_{y,t}^o} - \theta_t^p (q_t - \psi_p t) \]  \hspace{1cm} (22)

\[ \frac{1}{w_y - \psi_p h_{y,t}^o} = \beta (1 + r_{t+1}) c_{m,t+1} + \theta_t^p \]  \hspace{1cm} (23)

Thus, the equilibrium condition on the housing market can be written as

\[ H = h_{y,t}^o + \frac{1}{q_t} (\gamma_m c_{m,t+1} + \gamma_o c_{o,t+2}) \]  \hspace{1cm} (24)

### 5.3 Steady state

In steady state, equations (20)-(25) are given by

\[ 0 = c_m - \frac{w_m + p (1 - (1 - \psi)(1 + r)) h_{y}^o + \frac{w_o}{1 + r}}{1 + \gamma_m + \beta (1 + \gamma_o - r \alpha)} \]

\[ 0 = \frac{\gamma_y}{h_{y}^o} - \frac{q}{w_y - \psi_p h_{y}^o} - \theta^o (q - \psi_p) \]

\[ 0 = \frac{1}{w_y - \psi_p h_{y}^o} - \beta (1 + r) c_m + \theta^o \]

\[ 0 = H - h_{y}^o + \frac{1}{q} (\gamma_m c_m + \gamma_o c_o) \]

\[ 0 = p - q \frac{1 + r}{r} \]

We have five equations and five unknowns, \( \{h_{y}^o, c_m, p, q, \theta^o\} \), which we can solve numerically.
6 A fall in the real interest rate

Before I show the simulation results, it is useful to study the mechanisms of the model that cause redistribution when the interest rate falls. I base this chapter on a one-period fall in period t. These are the effects I address in chapter 7. Most of the intuitions below are based on the benchmark model.

Due to consumption smoothing, the sign of $A_{i,t+j}$ changes over the life cycle. When the agents expect their income to increase in the future, as the young agent does, they may borrow credit and consume more today. They therefore tend to be net borrowers when they become middle-aged. When they are middle-aged they also have higher income, and expect their income to decrease when they become old. They may therefore save enough during the period in order to live off their wealth when they become old, given that they have a sufficiently high income when they are middle-aged compared to when they become old.

Because I assume that the economy has been in steady state for the generations born in t-1 and t-2, their holdings of financial assets $A_{i,t+j+1}$ and housing capital $h_{i,t-1}$ matches the steady state solutions.

The immediate effect is a wealth redistribution between lenders and borrowers, as the absolute value of $(1 + r_t)A_{i,t+1}$ falls. Households who start the period as net borrowers, $A_{t+1} < 0$, experience a wealth increase and households starting as net lenders, $A_{t+1} > 0$ experience a wealth loss.

As long as the old agents are lenders and the middle-aged group are borrowers, the first-round effect is therefore a wealth redistribution from generation t-2 to t-1. The young agents, generation t, are born without initial assets. Given that the shock occur in the beginning of the period and the interest costs incur when the interest rate is back at the initial level, there are no first-round effect for the young agents.
The initial wealth redistribution effect can also be seen in the lifetime wealth of the agents. At time $t$, the lifetime wealth of the young, the middle-aged and the old, respectively, is defined as

$$W_t = w_y + \frac{w_m}{(1 + r_{t+1})} + \frac{w_o + p_{t+2}b_{o,t+2}}{(1 + r_{t+1})(1 + r_{t+2})}$$

$$W_{t-1} = w_y + \frac{w_m}{(1 + r_t)} + \frac{w_o + p_{t+1}b_{o,t+1}}{(1 + r_t)(1 + r_{t+1})}$$

$$W_{t-2} = w_y + \frac{w_m}{(1 + r_{t-1})} + \frac{w_o + p_tb_{o,t}}{(1 + r_{t-1})(1 + r_t)}$$

Thus, lifetime wealth of generation $t$ do not depend on the interest rate in period $t$, as the interest costs they may have on any debt incurs in period $t+1$ and $t+2$. Their only source to increased lifetime wealth, and thereby increased consumption is an increase in $p_{t+2}b_{o,t+2}$ given by generation $t-1$. Further on, it can be seen that the interest rate in period $t$ incurs in more periods for generation $t-1$ than $t-2$, reflecting how the number of remaining periods matters for how much they responds to a shock and how this may induce generations to respond asymmetrically to changes in the interest rate.

As long as demand for other goods changes on the aggregate level due to the wealth gain, the prices on the housing market might be affected. How much the prices reacts to the shock depends both on aggregate consumption and on the utility weights $\gamma_i$. In other words, generations do not affect prices equally. Even if aggregate consumption fall, the prices can still increase given that consumption of the groups with the highest weight increase sufficiently.

Define net accumulated wealth of the middle-aged and old agents at time $t$ as the sum of financial wealth and housing wealth

$$d_m = (1 + r_t)A_{y,t} + p_th_{y,t-1}$$

$$d_o = (1 + r_t)A_{m,t} + p_t(h_{m,t-1} + b_{o,t})$$

How the net wealth is affected by the real interest rate fall thus depends on how much financial- and housing wealth changes. If the value change of the two goes in opposite directions, as it will do with for the old lenders, the question is which of these dominates.
Finally, as mentioned previously, the bequest motive have implications for the total wealth effect of the generations. The old generation alive will receive their bequest from a generation that died prior to the shock, and will thus get the steady state value of bequest. Given higher prices, the bequest is also a dampening effect to their initial loss. The bequest they then give to the middle-aged at time $t$ is a function of wealth and house prices. A lower bequest from the old can therefore dampen some of the gains to the middle-aged.

The last implication for welfare of the generations is how the house price change compared increase in lifetime wealth for the different generations. If their lifetime wealth declines, or increases by less than the prices, they will consume less housing when the interest rate falls. Because the young barely receive any change in lifetime wealth, their housing demand might decline when the price increases. It must be noted that housing demand depends on the rental price $q_t$ and not the purchase price $p_t$, but they move in the same direction in the simulations in chapter 7.

The intuition in the constrained model is quite similar to the unconstrained. The most notable difference is that we do not have an explicit solution for consumption of the young, it is presented as a linear function of endowment and the total house expenditure in the same period.

$$c_{y,t} = w_y - \psi p_t h_{y,t}^o$$

Since I use log-utility, the income- and substitution effect will cancel each other out when we look at savings behaviour. In the simulation results, I will focus both on how welfare for the different age groups are affected in the period of the shock, and how their lifetime consumption will be compared to the steady state.

7 Numerical exercise

In the following section, I will simulate a temporary fall in the real interest rate. The shock is caused by factors outside of the model. For instance, as mentioned by Doepke,
Schneider, and Selezneva (2015), there can be a change in the world interest rate driven by the global savings glut. I assume that the steady state level of the real interest rate is equivalent to the "natural level", defined as the level where monetary policy is neither expansive nor contractive. According to a publication by Bernhardsen et al. (2006) in Norges Bank, the natural level lies between 2.5 and 3.5 percent annually. I assume that the real interest rate has been 2.5 percent annually prior to the shock. In period 1, the interest rate drops to 1.5 percent annually for the whole period. I extend the analysis in chapter 7.4 by one extra shock period where the interest rate reach 2 percent in the second period before it is back to the initial level in period 3. I do the simulation by using the Dynare-application Matlab.

7.1 Calibration

In the simulations, I extend the model with an additional group, making them four in total. The reason for the extension is that the borrowing constraint limits the amount of debt the young are able to accumulate and we will not get any substantial amounts of debt for the middle aged compared to the amount of financial wealth for the old. Such a solution would not match the financial positions for Norwegian households. When I include one extra cohort, the second group has housing wealth and is therefore not borrowing constrained and can borrow more. The only change from the three-period model presented above is an extra demand function on the housing market and more parameters in the consumption function. I provide the four period solution for consumption in Appendix A.3. The groups are now denoted \(i = 1, 2, 3, 4\) instead of young, middle-aged and old.

The following table shows the distribution of financial wealth and estimated market value of owner-occupied housing capital from Statistics Norway \(^3\).

On average, households aged 25-24, 35-44 and 45-54 are net debtors, while the rest of the age groups are lenders. Debt levels are highest for the two youngest groups and the accumulated savings highest for the two oldest groups. Thus, the middle-aged pay

\(^3\)https://www.ssb.no/statistikkbanken/selectvarval/saveselections.asp
Table 1: Data on financial wealth and housing wealth for Norwegian households for 2015

<table>
<thead>
<tr>
<th>Age groups</th>
<th>Net financial wealth</th>
<th>Housing wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>25-34</td>
<td>-1,117,100</td>
<td>1,1495,700</td>
</tr>
<tr>
<td>34-44</td>
<td>-1,190,200</td>
<td>2,318,700</td>
</tr>
<tr>
<td>45-54</td>
<td>-163,500</td>
<td>2,544,100</td>
</tr>
<tr>
<td>55-66</td>
<td>350,500</td>
<td>2,555,200</td>
</tr>
<tr>
<td>67-70</td>
<td>944,300</td>
<td>2,409,900</td>
</tr>
<tr>
<td>80-</td>
<td>865,700</td>
<td>1,901,100</td>
</tr>
</tbody>
</table>

down their debts and start saving such that they are lenders when they become 55. On the aggregate level, the households are net borrowers. Housing wealth follow a similar pattern, where the youngest have the smallest values, the middle-aged the highest, before it declines some when they become old, but still at a higher level than for the youngest.

The calibration done in the simulations are therefore chosen to somehow match these patterns, in both the constrained and the unconstrained model. Because I have four in stead of six groups as in table 1, the values of $A_{i,t+1}$ that an agent accumulated in the previous period will be negative for group 2 and 3 and positive for group 4.

As stated earlier, the old can leave bequest with negative financial assets as long as the housing capital compensate for it. As a result, the net financial wealth of group 4 consists of the wealth they accumulated in previous periods, in addition to the financial wealth they receive from the previous old generation. Due to this, the calibration is made to ensure that they have a substantial level of financial wealth even if they inherit debt.

The endowment levels are chosen to give us the correct values of $A_{i,t+1}$ and do not match empirical values. First of all, group 1 must receive an endowment that is sufficiently small for them to borrow up to the borrowing constraint in the constrained model. Second, group 3 must have an endowment level that is sufficiently large given the endowment level of group 4 for them to save enough and become lenders in the last period in the unconstrained model. Hence, the endowment level for group 3 is very large compared to that of the other groups, and the endowment level of group 4 is smaller than that of group
1. When the households expect such a large difference in income between period 3 and 4, they save enough to compensate for the debt they may inherit. How the endowment levels for group 1 and group 3 are calculated given the endowment levels of group 2 and 4 can be seen in Appendix A.4 and A.5.

The utility weights on housing are chosen to ensure a similar pattern of housing wealth as shown in the table. The utility weight on housing consumption for group 2 and 3 is therefore larger than for group 1 and 4. An argument for this is that middle-aged agents who tend to have children may put a larger weight on housing than young and unestablished agents, and old agents who might want to downsize their home.

The discount factor is associated with patient households. The down payment requirement is chosen to match the requirement in Norwegian banks, which is 15 percent.

One model period can be assumed to last for 15 years. I interpret group one as those between 20 and 34 years, group two as those between 35 and 49 years, group three as those between 50 and 64 years, and group four as those from 65 and above. Because the age groups are quite wide, I call group 1 the young agents, group 2 between young and middle-aged, group 3 middle-aged and group 4 old.

<table>
<thead>
<tr>
<th>Table 2: Values for the simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1 = 0.4$</td>
</tr>
<tr>
<td>$T = 15$</td>
</tr>
<tr>
<td>$\gamma_2 = 0.2$</td>
</tr>
<tr>
<td>$H = 1$</td>
</tr>
</tbody>
</table>

This calibration also ensures that the condition

$$q > \psi p$$
$$q > \psi q \frac{1 + r}{r}$$
$$1 > \psi \frac{1 + r}{r}$$

is satisfied in steady state, and we can be sure that the borrowing constrained agents
become homeowners.

In the graphs below, the shock occurs in the beginning of period 1 and lasts for the whole period. This means that the values for period 0 in the graphs are equal to the steady state values. Because I compare the results from chapter 7.2 and 7.3, the numerical results from these are added in Appendix A.6. The graphs in the following sections show levels of the variables of interest for the different age groups. Hence, when I look at consumption for generation 1 through the life cycle, this corresponds to group 1 in period 1, group 2 in period 2 etc. Period 0 corresponds to the steady state level.

7.2 The benchmark model

In steady state, consumption of other goods rises substantially with age due to the high discount factor. Consumption of housing is lowest for group 1 and highest for group 3. It declines for group 4, but to a higher level than for group 1. Group 2 and 3 are net borrowers, group 4 is net lender, and the net financial position of the economy is negative.

The debt levels for group 2 and 3 are higher than their housing wealth meaning negative net wealth. Hence, group 4 is the only group with positive net wealth. Group 4 borrow credit; the bequest they leave to their offspring therefore includes debt.

Figure 1 shows the first-round effect of the fall in the interest rate, improving the financial wealth for generations 2 and 3 while declining it for generation 4. Generations 2 and 3 have higher financial wealth for the remaining periods of their lives as well, implying that they save more in the period of the shock.

As predicted in chapter 6, generation 1 barely experience any increase in their consumption of other goods through the life cycle, meaning that they receive a slightly increased bequest value in period 4. The indebted generations 2 and 3, on the other hand, have a much higher increase in consumption, both in the period of the shock and in the remaining periods of the life cycle. Generation 4 lose in terms of other consumption goods. Because they are in their last period of the life cycle, this causes their lifetime consumption to be
Figure 1: Financial wealth after a one-period fall in the real interest rate in the benchmark model

lower than if the shock had not occurred.

Because the increase in wealth for generations 2 and 3 are higher than the loss for generation 4, aggregate consumption rises in the period of the shock as shown in figure 3. The increase reflects the fact that the initial net financial position of the economy was negative, as there were two groups of borrowers and only one group of lenders, and also because the consumption increase of generation 2 and 3 individually is higher than the consumption reduction of generation 4 due to longer remaining planning horizon. Aggregate consumption increases even more the next period when the only generation who had a disadvantage of the shock is dead. The house prices increase, and the effect is amplified because the winners also have the highest utility weight on housing. The differences in utility weights can also explain why the prices decline in period 2 even though aggregate consumption is even higher. Generation 1 do not have a significant higher consumption level in period 2 but a higher utility weight and thereby a higher influence of the house price, while generation 3 with the highest consumption increase
Figure 2: Consumption of other goods after a one-period fall in the real interest rate in the benchmark model

now have a lower weight and a lower influence.

Higher house prices adjust the initial wealth change of the generations. Generations 2 and 3 receive even higher wealth gains, and their net wealth is above steady state for the rest of their life cycles, indicating that their wealth gain induce them to accumulate more housing capital. The net wealth for group 3 also rises from a small negative value to a small positive value. Increased housing wealth compensate for some of the initial loss for generation 4, but not enough to avoid an overall decline. Generation 4 also responds by giving a lower bequest to generation 3, which dampens the gain for them slightly. Generation 2 and 3 give a higher bequest than in steady state to their offspring, explaining why consumption for generation 1 and those who are not yet born experience some consumption increase, together with the still higher prices than in steady state.

As indicated from the effects on net wealth, generations 2 and 3 have higher housing consumption both in the period of the shock and over the life cycle. Generation 1 have
Figure 3: Aggregate consumption and purchase price for housing after a one-period fall in the real interest rate in the benchmark model.

(a) Aggregate consumption

![Aggregate consumption graph]

(b) Purchase price of housing

![Purchase price of housing graph]

A lower housing consumption in all periods of life because their small wealth increase is lower than the effect on the prices. Generation 4 reduce their housing demand as well, making housing consumption over the life cycle smaller than in steady state.

Figure 6 and 7 show how every generation increase their savings in the period of the shock, such that aggregate savings rise.
Figure 4: Net wealth and bequest after a one-period fall in the real interest rate in the benchmark model

The results show how an initial redistribution of wealth from old lenders to middle-aged borrowers affects macroeconomic aggregates, which in turn both amplify and modify the initial redistribution effect between the old and the middle-aged agents. The gains to those who are born in the period of the shock are very limited, while they consume less housing overall. Because the prices remain high for several periods, the generation born in period 2 consume less housing over the life cycle as well, meaning that low interest rates have effects also on generations who are not yet born. The large gains to the winners contribute to higher house prices, and the effect is amplified as they have a higher utility weight on housing. The effects are persistent, and the house price is not back to the initial level before generation 1 is dead. Based on this small experiment, we can conclude that the age of a generation matters for how their welfare is affected by the real interest rate fall.
Figure 5: Consumption of owner-occupied housing after a one-period fall in the real interest rate in the benchmark model

Figure 6: Savings after a one-period fall in the real interest rate in the benchmark model
Figure 7: Aggregate savings after a one-period fall in the real interest rate in the benchmark model
7.3 The constrained economy

The initial position of the constrained economy differs some from the unconstrained. As expected, the aggregate debt level in the economy is smaller when the agents are borrowing constrained. The youngest agents consume less housing and other goods in the first period of life than in the unconstrained case, but more in the remaining periods.

The direction of the effects is similar to the unconstrained model, but smaller in magnitude. The exception is generation 4 who reduce their consumption of other goods by more as their initial financial wealth is higher. I will here show the graphs for consumption of housing and other goods for all generations, aggregate consumption, and prices.

As seen in figure 8, aggregate consumption increase only slightly in the period of the shock when the overall gain in financial wealth is smaller. This cause a lower increase of house prices. The price declines in period 2 just as in the benchmark case.

Figure 9 presents the effects on housing and other consumption goods. Consumption of other goods for generation 1 increases even less in the constrained model. Housing demand falls due to higher prices, and the decrease is high enough for their total housing expenditures to decline some such that consumption of other goods do not fall. Their housing consumption is lower for all periods except the last. Finally, aggregate savings behave in the same manner as in the benchmark model.

The results for generation 2 and 3 follow the same pattern as in the benchmark case, but to a lower extent. Due to the borrowing constraint, generation 2 had a much lower level of debt than in the benchmark case and hence a lower gain. Generation 4 reduce their consumption of other goods by more than in the benchmark model, but reduce housing by less. The effect on savings, bequest and net wealth is similar to the benchmark case, all generations save more when the interest rate fall, causing the aggregate debt level to decline.
Figure 8: Aggregate consumption and purchase price for housing after a one-period fall in the real interest rate in the constrained model.
Figure 9: Consumption of other goods and housing after a one-period fall in the real interest rate in the constrained model
Figure 10: Aggregate savings after a one-period fall in the real interest rate in the constrained model
7.4 A two-period shock

In the previous simulations, the young did not get any benefits initially from the low interest rates because they did not have any accumulated wealth. None of the generations borrowed more when the interest rate fell, because it would be back to the initial level in the period of down payment. I therefore provide a short analysis of how the outcome might change if the interest rate remains low for one extra period, based on the borrowing constrained model. The interest rate falls down to 1.5 percent annually in period 1, rises to 2 percent in period 2 and is back at the initial level in period 3.

Figure 11 shows that the effects on aggregate consumption and prices differs from the previous case. They both increase in period 1, and to a much larger extent, as the effect on lifetime wealth of the agents is bigger. On the other hand, they both fall below the steady state level in period 2.

From figure 12, we can see how consumption of housing and other goods of the households react to the two-period interest rate fall. The effect on welfare for generation 1 is still quite small. The difference is that they reduce their consumption both of housing and other goods in the first period of life, but they have a higher level of consumption of both goods over the life cycle as a whole. Thus, the two-period fall causes a welfare loss in the first period but a gain overall. They borrow more today than in steady state and still have a financial wealth gain in the next period compared because of the low interest rates. Their consumption in period 2 is dampened by lower housing wealth, because their housing wealth is declines.

Generation 2 still have higher lifetime consumption both of housing and other goods when the interest rate is lower for two periods. However, their consumption of other goods is only higher in period 1 and lower in the remaining periods. They borrow so much in period 2 that they have higher debt costs to pay in the next period even though the interest rate is lower, such that they have reduced financial wealth in period 2 compared to the steady state. Their net wealth in period 2 is reduced also because their housing wealth is lower than in steady state. They save less in period 2 than in steady state,
Figure 11: Aggregate consumption and house prices after a two-period fall in the real interest rate

implying lower net wealth in the last period of the life cycle.

Generation 3 experience an immediate wealth gain, and increase their consumption both of housing and other goods. However, the gain lasts only in period 1 and not in period 2. They save more because of higher wealth, but the returns are smaller in period 2 and their financial wealth declines because of that. Combined with the low house prices and lower
Figure 12: Consumption of housing other goods after a two-period fall in the real interest rate

bequest from generation 4, their net wealth is lower in the last period of life. Their wealth in period 2 declines more than the prices, which causes them to consume less housing.
Lifetime consumption of other goods is still higher than in steady state, but consumption of housing is lower. They also give lower bequest to generation 2.

Generation 4 represents the most notable difference from the one-period fall. Despite an initial wealth loss, the house prices increase to such a large extent that their net wealth rises. Because of this, they consume more of other goods in their last period of life. Housing demand falls when the house price increase. They borrow more, and the bequest they leave to generation 3 is therefore lower.

**Figure 13:** Net wealth after a two-period fall in the real interest rate

Figures 13 and 14 contain the effects on savings from the two-period fall. The effect the net financial position of the economy is the opposite of the two-period case, namely more borrowing in the first period of the shock despite the borrowing constraint, but converging back to the initial level in the period after. Aggregate debt thus increases when the interest rates are low also in the next period.

The implications for welfare for the different generations is not as clear when the interest rates are low for two periods. All of the generations except from the oldest
experience fluctuations in consumption of both goods over the life cycle, both above and below the steady state levels. Over the life cycle, there is a small gain from being born into a period of low interest rates when they can enjoy some of the wealth benefits in the next period, but they are worse off in the first period of life. We can no longer say that the old lenders are the losers when their housing wealth rises to such a large extent that their net wealth rises, and they are dead in the next period when the prices fall. Generation 2 increase consumption both of housing and other goods in the period of the shock, but consumes lower levels for the remaining periods. The same goes from generation 3 - their lifetime consumption is higher than in steady state, and their housing consumption is lower. The persistence of the effects are of a similar length as in the one-period fall.

8 Conclusion

I have used a simple overlapping generations model to explain the mechanisms that cause redistribution between generations when the real interest rate falls temporarily. When the initial net financial position of the economy is negative, the fall leads to increased
aggregate consumption and house prices because the households experience an overall gain. In addition, households respond asymmetrically according to how many periods they have left of their lives. Consumption smoothing due to different levels of income over the life cycle is an explanation to why old agents tend to be lenders and middle-aged agents tend to be borrowers, which is why the fall have an intergenerational distributive effect.

The model predicts that when a one-period fall in the real interest rate occurs, the middle-aged borrowers are unambiguously better off for all remaining periods of the life cycle on the expense the old lenders. Whether it is advantageous to be born into a period of low interest rates is ambiguous; they have no direct benefit from the low interest rate, while their housing consumption falls compared to the situation of a constant interest rate due to higher prices. When the interest rates are low from the beginning of the life cycle, accumulation of housing capital becomes more challenging when the house prices are so high that they must reduce their house purchases.

Further on, the outcome for the generations is more diversified when the shock lasts for two periods, and the effects on welfare of a generation in the first period of the shock is not necessarily with the same sign as the those for the rest of the life cycle. The two-period shock enables the young to enjoy some of the direct benefits from lower interest rates. The effect on aggregate demand and prices is higher in the period of the shock, which might lead to a net wealth increase for the initial old despite their initial loss.

The total effect on net wealth of the households depends on the initial financial wealth change and the change in house wealth that arise due to increased prices. For the middle-aged borrowers, both types of wealth changes goes in the same, positive direction. For the old lenders, however, their total effect depends on how much the house prices can offset the loss in their financial wealth. Furthermore, the old agents influence the outcome by the bequest they give their offspring, which might dampen some of the wealth increase to the middle-aged. In all three experiments I have conducted, the initial old have responded by giving a lower bequest to the initial middle-aged. In the one-period shocks, the bequest have been the only source of wealth gain for the youngest. A possible
extension of the analysis could be to study how much different values for the strength of
the bequest motive could matter for the total outcome. The analysis highlights that some
age groups have a higher influence on the house prices, both due to their wealth gain and
higher weight on housing.

Several more experiments could have been performed within this framework. For instance,
we could add more periods to the model and study a period of low interest rates followed
by an increase above the initial level. How the initial fall caused changes in the net wealth
of the agents could then lead to more serious outcomes if the house price would suddenly
fall while their debts would make a larger share of their expenditures.

A major limitation of the explanation power of the model is the exclusion of the production
side of the economy, especially regarding the housing market. The fixed housing supply
can to a certain extent capture the scarcity of housing that cause high price effects from
income shocks. However, a more flexible supply would affect how prices behaved in the
long run.

Another extension that would have been valuable would be to add a certain degree
of indivisibility to housing. By assuming a minimum size, higher prices due to the
interest rate fall could cause young households to become renters, affecting their wealth
accumulation over the life cycle.

Despite the simplicity of the model, I believe that it can provide some useful insights
into the mechanisms that arise after an initial wealth redistribution that in turn can have
further implications for the welfare for different generations.
References

Auclert, Adrien et al. (2015). “Monetary policy and the redistribution channel”. In: *Unpublished manuscript*.


Gary-Bobo, Robert J and Jamil Nur (2015). “Housing, Capital Taxation and Bequests in a Simple OLG Model”. In:


A Appendix

A.1 Proof that nobody rents in the absence of borrowing constraints

We get the following first order constraints:

\[ c_{y,t} : \frac{1}{c_{y,t}} - \lambda = 0 \]

\[ h_{r,t}^r = \frac{\gamma_y}{h_{r,t}^r + \kappa h_{g,t}^o} - q_t \lambda + \delta_t = 0 \]

\[ h_{g,t}^o = \frac{\kappa \gamma_y}{h_{r,t}^r + \kappa h_{g,t}^o} - q_t \lambda + \mu_t = 0 \]

\[ c_{m,t+1} : \beta \frac{1}{c_{m,t+1}} - \frac{\lambda}{1 + r_{t+1}} = 0 \]

\[ h_{r,m,t+1} = \beta \frac{\gamma_m}{h_{r,m,t+1} + \kappa h_{m,t+1}^o} - \frac{q_{t+1} \lambda}{(1 + r_{t+1})} + \delta_{t+1} = 0 \]

\[ h_{m,t+1}^o = \beta \frac{\kappa \gamma_m}{h_{r,m,t+1} + \kappa h_{m,t+1}^o} - \frac{q_{t+1} \lambda}{(1 + r_{t+1})} + \mu_{t+1} = 0 \]

\[ c_{o,t+2} : \beta^2 \frac{1}{c_{o,t+2}} - \frac{\lambda}{(1 + r_{t+1})(1 + r_{t+2})} = 0 \]

\[ h_{r,o,t+2} = \beta^2 \frac{\gamma_o}{h_{r,o,t+2} + \kappa h_{o,t+2}^o} - \frac{q_{t+2} \lambda}{(1 + r_{t+1})(1 + r_{t+2})} + \delta_{t+2} = 0 \]

\[ h_{o,t+2}^o = \beta^2 \frac{\kappa \gamma_o}{h_{r,o,t+2} + \kappa h_{o,t+2}^o} + \beta^2 \frac{\alpha}{h_{o,t+2} + a_{o,t+3}} - \frac{p_{t+2} \lambda}{(1 + r_{t+1})(1 + r_{t+2})} + \mu_{t+2} = 0 \]

\[ a_{o,t+3} = \beta^2 \frac{\alpha}{h_{r,o,t+2} + a_{o,t+3}} - \frac{(p_{t+2} - q_{t+2}) \lambda}{(1 + r_{t+1})(1 + r_{t+2})} = 0 \]

Complementary slackness conditions:

\[ \mu_{t,t} \geq 0, = 0 \text{ if } h_{o,t}^o > 0 \]

\[ \delta_{t,t} \geq 0, = 0 \text{ if } h_{r,t}^r > 0 \]

Combine first order conditions in time \( t \) with respect to housing rented and owned and get:

\[ \frac{\gamma_y (\kappa - 1)}{h_{r,y}^r + \kappa h_{g,t}^o} - \mu_t + \delta_t = 0 \]
Both complementary slackness conditions cannot be binding, that would result in no consumption of housing. Hence, at least one of the housing types must be positive.

What if $h_{i,t}^o > 0$ meaning $\mu_{i,t} = 0$:

$$\delta_{y,t} = \frac{\gamma_y(\kappa - 1)}{h_{y,t}^r + \kappa h_{y,t}^o} > 0$$ \hspace{1cm} (26)

$\delta$ is granted to be positive, because of the assumption that $\kappa > 0$. Thus, households do not rent.

What if $h_{y,t}^r > 0$, meaning $\delta = 0$:

$$\mu_{y,t} = -\frac{\gamma_y(\kappa - 1)}{h_{y,t}^r + \kappa h_{y,t}^o} < 0$$ \hspace{1cm} (27)

Thus, we get a contradiction for the complementary slackness conditions. The same intuition goes for the middle-aged, thus neither young or middle-aged rents in this unconstrained model.

For the old, combine first the first-order conditions for $a_{a,t+3}$ and $h_{o,t+2}^o$, and then combine this with the first-order condition for $h_{o,t+2}^r$ and get the same result as for the young.

A.2 First order conditions with borrowing constraints

Budget constraints

First order conditions for the young household
\[ c_{y,t} : \frac{1}{c_{y,t}} - \lambda_t^y = 0 \]
\[ h_{y,t}^o : \frac{k \gamma y}{h_{y,t}^r + k h_{y,t}^o} - \lambda_t^y p_t + \beta \lambda_{t+1}^o p_{t+1} + \theta_t^y (1 - \psi) p_t + \mu_t^y = 0 \]
\[ h_{y,t}^r : \frac{\gamma}{h_{y,t}^r + k h_{y,t}^o} - q_t \lambda_t^y + \delta_t^y \]
\[ h_{y,t}^l : -(p_t - q_t) \lambda_t^y + \beta \lambda_{t+1}^m p_{t+1} + \theta_t^y (1 - \psi) p_t + \varepsilon_t^y = 0 \]
\[ A_{y,t+1} : -\lambda_t^y + \beta \lambda_{t+1}^m (1 + r_{t+1}) + \theta_t^y = 0 \]

Complementary slackness conditions

\[ \theta \geq 0 \quad (=0 \text{ if } A_{t+1} > -(1 - \psi) p_t (h_{y,t}^o + h_{y,t}^l)) \]
\[ \delta \geq 0 \quad (=0 \text{ if } h^r > 0) \]
\[ \mu \geq 0 \quad (=0 \text{ if } h^o > 0) \]
\[ \varepsilon \geq 0 \quad (=0 \text{ if } h^l > 0) \]

1. Solve for \( \beta \lambda_{t+1} \) in the FOC for financial assets, insert into FOC for owner-occupied housing, and use the no arbitrage condition

\[ \beta \lambda_{t+1} = \frac{\lambda_t^y - \theta_t^y}{1 + r_{t+1}} \]
\[ \frac{\gamma y}{h_{y,t}^r + k h_{y,t}^o} - \lambda_t^y q_t + \theta_t^y (q_t - \psi p_t) + \mu_t^y = 0 \]

Add the first order conditions for consumption of rented and owner-occupied housing

\[ \frac{\gamma (\kappa - 1)}{h_{y,t}^r + k h_{y,t}^o} + \theta_t^y (q_t - \psi p_t) + \mu_t^y - \delta_t^y = 0 \]

As long as \( q_t > \psi p_t \), the second term is positive.

1. Assume that \( h^r > 0 \) so that \( \delta = 0 \)

\[ -\mu_t = \frac{\omega (\kappa - 1)}{h_{y,t}^r + k h_{y,t}^o} + \theta_t^y (q_t - \psi p_t) \]

43
If $q_t > \psi p_t$, that would yield a negative $\mu_t$, and in that case we would get a contradiction.

How do we know that $q_t > \psi p_t$? From the steady state solutions, this is clear, because purchase prices are a function of rental prices and the interest rate, whereas $1 + \frac{r}{r} > 1$.

2. Assume that $h^o > 0$ so that $\mu = 0$

$$\delta_t = \frac{\gamma y (\kappa - 1)}{h_{y,t} + \kappa h^o_{y,t}} + \theta^y_t (q_t - \psi p_t)$$

As long as $q_t > \psi p_t$, this would not be a contradiction.

A.3 Solutions with four groups

I will state the demand functions when I have included one extra cohort, denoting $i = 1, 2, 3, 4$. The budget constraints are similar to the three-period model, except with one extra group.

In the unconstrained model, consumption of group 1 is

$$c_{1,t} = \frac{w_1 + \frac{w_2}{(1+r_{t+1})} + \frac{w_3}{(1+r_{t+1})(1+r_{t+2})} + \frac{w_4 + p_{t+1} b_{o,t+1}}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})}}{1 + \omega_1 + \beta(1 + \omega_2) + \beta^2(1 + \omega_3) + \beta^3(1 + \omega_4 + \alpha)}$$

The Euler equations and demand for housing is defined in the same way as previously, and the market clearing condition is modified to

$$H = h^o_{1,t} + h^o_{2,t} + h^o_{3,t} + h^o_{4,t}$$

In the constrained model, consumption of the middle-aged when group 1 is borrowing constrained is defined as

$$c_{2,t+1} = \frac{w_2 + (p_{t+1} - (1 - \psi)p_t(1 + r_{t+1})) h^o_{1,t} + \frac{w_3}{(1+r_{t+2})} + \frac{w_4 + p_{t+1} b_{o,t+3}}{(1+r_{t+2})(1+r_{t+3})}}{1 + \gamma_2 + \beta(1 + \gamma_3) + \beta^2(1 + \gamma_4 + \alpha)}$$

And the rest of the equations is solved in the same way as in the three-period case.
A.4 Endowment lever ensuring that the youngest borrow up to the borrowing constraint

I use the model with four periods as in the simulations when calculating the endowment of group 1. the same as in the simulations. Look at the optimality condition for housing of the youngest, use steady state solutions

\[
\frac{\gamma_1}{h_1^o} - \frac{q}{w_1 - \psi ph_1^o} + \theta^1(q - \psi p) = 0
\]

Using the no arbitrage condition in steady state and insert for p

\[
q\theta^1(1 - \psi \frac{1 + r}{r}) = \frac{q}{w_1 - \psi ph_1^o} - \frac{\gamma_1}{h_1^o} \geq 0
\]

\[
h_1^o \leq \frac{\gamma_1 w_1}{q(1 + \gamma_1 \psi \frac{1 + r}{r})}
\]

Then use the Euler equation

\[
\frac{1}{c_1} = \frac{\beta(1 + r)}{c_2} + \theta^1
\]

\[
\theta^1 = \frac{1}{c_1} - \frac{\beta(1 + r)}{c_2} \geq 0
\]

\[
\frac{c_2}{c_1} \geq \beta(1 + r)
\]

Then: inserting the steady state solution for consumption of group 1 and 2 in the last expression

\[
\frac{c_2}{c_1} = \frac{\frac{w_2 + p(1 - (1 - \psi)(1 + r))h_1^o + \frac{w_3}{(1 + r)} + \frac{w_4}{(1 + r)^2}}{1 + \gamma_2 + \beta(1 + \gamma_3) + \beta^2(1 + \gamma_4 - r\alpha)}}{w_1 - \psi ph_1^o} \geq \beta(1 + r)
\]

Define \(\epsilon = 1 + \gamma_2 + \beta(1 + \gamma_3) + \beta^2(1 + \gamma_4 - r\alpha)\)

And solve for \(w_1\), so that the expression is reduced to

\[
w_1 \leq \frac{w_2 + \frac{w_3}{(1 + r)} + \frac{w_4}{(1 + r)^2}}{\epsilon \beta(1 + r) - \frac{\gamma_1 h_1^o}{1 + \gamma_1 \psi \frac{1 + r}{r}}[\epsilon \beta(1 + r)\psi \frac{1 + r}{r} + \frac{1 + r}{r}(1 - (1 - \psi)(1 + r))]} (28)
\]
A.5 Endowment ensuring that the middle-aged become savers in the unconstrained model

Given their demand function and accumulated wealth, I calculate a value for \( w_3 \) given the rest of the endowments to ensure that group 3 save enough to be lenders when they enter their last period of life. I base the calculations on the steady state of the model.

Use the period budget constraint for period three, then insert for \( A_2 \) and \( A_1 \).

\[
A_3 = w_3 + (1 + r)A_2 + ph_2 - c_3 - ph_3 \geq 0
\]
\[
w_3 + (1 + r)(w_2 + (1 + r)A_1 + ph_1 - c_2 - ph_2) + ph_2 - c_3 - ph_3 \geq 0
\]

Inserting for \( A_1 \) and the demand functions

\[
w_3 + (1 + r)w_2 + (1 + r)^2w_1 - c_1(1 + r)^2[1 + \gamma_1 + \beta(1 + \gamma_2) + \beta^2(1 + \gamma_3\frac{1 + r}{r})] \geq 0
\]

This gives a messy expression, so I define two sums of coefficients

\[
\omega = 1 + \gamma_1 + \beta(1 + \gamma_2) + \beta^2(1 + \gamma_3\frac{1 + r}{r})
\]
\[
\eta = \beta^2\frac{\gamma_3}{r} - \beta^3(1 + \gamma_4 - r\alpha)
\]
\[
w_3 \geq \frac{w_3}{(1+r)}\omega + \frac{[w_1(1 + r)^2 + w_2(1 + r)]\eta}{\beta^3(1 + \gamma_4 - r\alpha) - \beta^2\frac{2\gamma_3}{r}}
\]

Hence, for given values of the three other endowments, this will secure positive financial wealth in the last period. In the calibration I have chosen endowment levels ensuring that the conditions in 9.3 and 9.4 holds.
A.6 Numerical results from the one-period shock

The following tables shows the values for all variables of interest from the simulations.

\[
\begin{align*}
  s_{2,t} &= (1 + r_{t+1})A_{1,t} \\
  s_{3,t} &= (1 + r_{t+1})A_{2,t} \\
  s_{4,t} &= (1 + r_{t+1})(A_{3,t} + A_{4,t}) \\
  d_{2,t} &= s_{2,t} + p_t h_{1,t-1} \\
  d_{3,t} &= s_{3,t} + p_t h_{2,t-1} \\
  d_{4,t} &= s_{4,t} + p_t (h_{3,t-1} + b_{4,t})
\end{align*}
\]

**Table 3:** The unconstrained model where \( t=0 \) denoted steady state values and \( t=1 \) denotes the period of the shock

<table>
<thead>
<tr>
<th>Period</th>
<th>( c_{1,t} )</th>
<th>( c_{2,t} )</th>
<th>( c_{3,t} )</th>
<th>( c_{4,t} )</th>
<th>( h_{1,t} )</th>
<th>( h_{2,t} )</th>
<th>( h_{3,t} )</th>
<th>( h_{4,t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t=0 )</td>
<td>0.4478</td>
<td>0.5578</td>
<td>0.6948</td>
<td>0.8654</td>
<td>0.1173</td>
<td>0.2922</td>
<td>0.3639</td>
<td>0.2267</td>
</tr>
<tr>
<td>( t=1 )</td>
<td>0.4480</td>
<td>0.5710</td>
<td>0.7177</td>
<td>0.8468</td>
<td>0.1157</td>
<td>0.2949</td>
<td>0.3707</td>
<td>0.2187</td>
</tr>
<tr>
<td>( t=2 )</td>
<td>0.4478</td>
<td>0.5581</td>
<td>0.7113</td>
<td>0.8940</td>
<td>0.1154</td>
<td>0.2876</td>
<td>0.3666</td>
<td>0.2304</td>
</tr>
<tr>
<td>( t=3 )</td>
<td>–</td>
<td>0.5578</td>
<td>0.6952</td>
<td>0.8860</td>
<td>0.1166</td>
<td>0.2905</td>
<td>0.3621</td>
<td>0.2307</td>
</tr>
<tr>
<td>( t=4 )</td>
<td>–</td>
<td>–</td>
<td>0.6948</td>
<td>0.8659</td>
<td>0.1173</td>
<td>0.2921</td>
<td>0.3639</td>
<td>0.2268</td>
</tr>
<tr>
<td>( t=5 )</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.8654</td>
<td>–</td>
<td>0.2922</td>
<td>–</td>
<td>0.2267</td>
</tr>
</tbody>
</table>

Define \( M_{t+1} \) as aggregate savings in the economy, which is the sum of \( A_{i,t+1} \). The values are determined in period \( t \).
Table 4: Unconstrained model continued.

<table>
<thead>
<tr>
<th>Period</th>
<th>$A_{1,t+1}$</th>
<th>$A_{2,t+1}$</th>
<th>$A_{3,t+1}$</th>
<th>$A_{4,t+1}$</th>
<th>$M_{t+1}$</th>
<th>$b_{4,t+1}$</th>
<th>$c_t$</th>
<th>$q_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>t=0</td>
<td>-0.1924</td>
<td>-0.2522</td>
<td>0.2515</td>
<td>-0.1065</td>
<td>-0.2996</td>
<td>0.1016</td>
<td>2.5657</td>
<td>0.3818</td>
</tr>
<tr>
<td>t=1</td>
<td>-0.1920</td>
<td>-0.2326</td>
<td>0.2639</td>
<td>-0.1027</td>
<td>-0.2581</td>
<td>0.0988</td>
<td>2.5835</td>
<td>0.3872</td>
</tr>
<tr>
<td>t=2</td>
<td>-0.1910</td>
<td>-0.2496</td>
<td>0.2629</td>
<td>-0.1072</td>
<td>-0.2849</td>
<td>0.1048</td>
<td>2.6111</td>
<td>0.3880</td>
</tr>
<tr>
<td>t=3</td>
<td>-0.1919</td>
<td>-0.2508</td>
<td>0.2514</td>
<td>-0.1079</td>
<td>-0.2993</td>
<td>0.1040</td>
<td>2.5867</td>
<td>0.3840</td>
</tr>
<tr>
<td>t=4</td>
<td>-0.1924</td>
<td>-0.2521</td>
<td>0.2515</td>
<td>-0.1065</td>
<td>-0.2996</td>
<td>0.1017</td>
<td>2.5662</td>
<td>0.3819</td>
</tr>
<tr>
<td>t=5</td>
<td>–</td>
<td>-0.2522</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.1016</td>
<td>2.5657</td>
<td>0.3818</td>
</tr>
<tr>
<td>t=6</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 5: Unconstrained model continued.

<table>
<thead>
<tr>
<th>Period</th>
<th>$s_{2,t}$</th>
<th>$s_{3,t}$</th>
<th>$s_{4,t}$</th>
<th>$d_{2,t}$</th>
<th>$d_{3,t}$</th>
<th>$d_{4,t}$</th>
<th>$p_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>t=0</td>
<td>-0.2787</td>
<td>-0.3652</td>
<td>0.2100</td>
<td>-0.1340</td>
<td>-0.0048</td>
<td>0.9385</td>
<td>1.2335</td>
</tr>
<tr>
<td>t=1</td>
<td>-0.2406</td>
<td>-0.3153</td>
<td>0.1813</td>
<td>-0.0947</td>
<td>0.0482</td>
<td>0.9161</td>
<td>1.2443</td>
</tr>
<tr>
<td>t=2</td>
<td>-0.2781</td>
<td>-0.3369</td>
<td>0.2412</td>
<td>-0.1345</td>
<td>0.0291</td>
<td>0.9728</td>
<td>1.2413</td>
</tr>
<tr>
<td>t=3</td>
<td>-0.2766</td>
<td>-0.3614</td>
<td>0.2255</td>
<td>-0.1341</td>
<td>-0.0060</td>
<td>0.9632</td>
<td>1.2357</td>
</tr>
<tr>
<td>t=4</td>
<td>-0.2779</td>
<td>-0.3633</td>
<td>0.2078</td>
<td>-0.1340</td>
<td>-0.0049</td>
<td>0.9391</td>
<td>1.2336</td>
</tr>
<tr>
<td>t=5</td>
<td>-0.2787</td>
<td>-0.3652</td>
<td>0.2099</td>
<td>–</td>
<td>-0.0049</td>
<td>0.9385</td>
<td>1.2335</td>
</tr>
<tr>
<td>t=6</td>
<td>–</td>
<td>–</td>
<td>0.2100</td>
<td>–</td>
<td>-0.0048</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
Table 6: Constrained model model where $t=0$ denoted steady state values and $t=1$ denotes the period of the shock

<table>
<thead>
<tr>
<th>Period</th>
<th>$c_{1,t}$</th>
<th>$c_{2,t}$</th>
<th>$c_{3,t}$</th>
<th>$c_{4,t}$</th>
<th>$h_{1,t}$</th>
<th>$h_{2,t}$</th>
<th>$h_{3,t}$</th>
<th>$h_{4,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t=0$</td>
<td>0.3795</td>
<td>0.5918</td>
<td>0.7371</td>
<td>0.9182</td>
<td>0.1059</td>
<td>0.2959</td>
<td>0.3686</td>
<td>0.2296</td>
</tr>
<tr>
<td>$t=1$</td>
<td>0.3795</td>
<td>0.5999</td>
<td>0.7552</td>
<td>0.8954</td>
<td>0.1051</td>
<td>0.2978</td>
<td>0.3749</td>
<td>0.2222</td>
</tr>
<tr>
<td>$t=2$</td>
<td>0.3796</td>
<td>0.5920</td>
<td>0.7472</td>
<td>0.9407</td>
<td>0.1048</td>
<td>0.2929</td>
<td>0.3696</td>
<td>0.2327</td>
</tr>
<tr>
<td>$t=3$</td>
<td>0.3795</td>
<td>0.5917</td>
<td>0.7374</td>
<td>0.9307</td>
<td>0.1055</td>
<td>0.2949</td>
<td>0.3676</td>
<td>0.2320</td>
</tr>
<tr>
<td>$t=4$</td>
<td>–</td>
<td>0.5917</td>
<td>0.7370</td>
<td>0.9185</td>
<td>0.1059</td>
<td>0.2959</td>
<td>0.3686</td>
<td>0.2297</td>
</tr>
<tr>
<td>$t=5$</td>
<td>–</td>
<td>0.5918</td>
<td>0.7371</td>
<td>0.9181</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.2296</td>
</tr>
<tr>
<td>$t=6$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.9181</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 7: Constrained model continued.

<table>
<thead>
<tr>
<th>Period</th>
<th>$A_{1,t+1}$</th>
<th>$A_{2,t+1}$</th>
<th>$A_{3,t+1}$</th>
<th>$A_{4,t+1}$</th>
<th>$M_{t+1}$</th>
<th>$h_{4,t+1}$</th>
<th>$c_t$</th>
<th>$q_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t=0$</td>
<td>-0.1163</td>
<td>-0.2057</td>
<td>0.2710</td>
<td>-0.1130</td>
<td>-0.1639</td>
<td>0.1029</td>
<td>2.6265</td>
<td>0.3999</td>
</tr>
<tr>
<td>$t=1$</td>
<td>-0.1161</td>
<td>-0.1944</td>
<td>0.2851</td>
<td>-0.1095</td>
<td>-0.1349</td>
<td>0.1000</td>
<td>2.6300</td>
<td>0.4029</td>
</tr>
<tr>
<td>$t=2$</td>
<td>-0.1156</td>
<td>-0.2036</td>
<td>0.2780</td>
<td>-0.1137</td>
<td>-0.1550</td>
<td>0.1053</td>
<td>2.6595</td>
<td>0.4043</td>
</tr>
<tr>
<td>$t=3$</td>
<td>-0.1160</td>
<td>-0.2050</td>
<td>0.2710</td>
<td>-0.1139</td>
<td>-0.1638</td>
<td>0.1043</td>
<td>2.6394</td>
<td>0.4012</td>
</tr>
<tr>
<td>$t=4$</td>
<td>-0.1163</td>
<td>-0.2057</td>
<td>–</td>
<td>-0.1130</td>
<td>-0.1641</td>
<td>0.1030</td>
<td>2.6268</td>
<td>0.3999</td>
</tr>
<tr>
<td>$t=5$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>-0.1640</td>
<td>0.1029</td>
<td>2.6264</td>
<td>–</td>
</tr>
<tr>
<td>$t=6$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>-0.1639</td>
<td>–</td>
<td>2.6265</td>
<td>–</td>
</tr>
</tbody>
</table>
Table 8: Constrained model continued.

<table>
<thead>
<tr>
<th>Period</th>
<th>$s_{2,t}$</th>
<th>$s_{3,t}$</th>
<th>$s_{4,t}$</th>
<th>$d_{2,t}$</th>
<th>$d_{3,t}$</th>
<th>$d_{4,t}$</th>
<th>$p_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>t=0</td>
<td>-0.1684</td>
<td>-0.2979</td>
<td>0.2289</td>
<td>-0.0316</td>
<td>0.0844</td>
<td>1.0018</td>
<td>1.2920</td>
</tr>
<tr>
<td>t=1</td>
<td>-0.1454</td>
<td>-0.2572</td>
<td>0.1976</td>
<td>-0.0079</td>
<td>0.1271</td>
<td>0.9745</td>
<td>1.2987</td>
</tr>
<tr>
<td>t=2</td>
<td>-0.1681</td>
<td>-0.2816</td>
<td>0.2542</td>
<td>-0.0317</td>
<td>0.1047</td>
<td>1.0289</td>
<td>1.2973</td>
</tr>
<tr>
<td>t=3</td>
<td>-0.1674</td>
<td>-0.2949</td>
<td>0.2379</td>
<td>-0.0318</td>
<td>0.0839</td>
<td>1.0169</td>
<td>1.2934</td>
</tr>
<tr>
<td>t=4</td>
<td>-0.1681</td>
<td>-0.2969</td>
<td>0.2276</td>
<td>-0.0317</td>
<td>0.0842</td>
<td>1.0023</td>
<td>1.2920</td>
</tr>
<tr>
<td>t=5</td>
<td>-0.1684</td>
<td>-0.2980</td>
<td>0.2287</td>
<td>-0.0316</td>
<td>0.0843</td>
<td>1.0017</td>
<td>–</td>
</tr>
<tr>
<td>t=6</td>
<td>–</td>
<td>-0.2980</td>
<td>0.2289</td>
<td>–</td>
<td>0.0844</td>
<td>1.0017</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 9: Percentage change from steady state in unconstrained and constrained model

<table>
<thead>
<tr>
<th>Model</th>
<th>$\Delta c_1$</th>
<th>$\Delta c_2$</th>
<th>$\Delta c_3$</th>
<th>$\Delta c_4$</th>
<th>$\Delta c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconstrained</td>
<td>0.0578</td>
<td>2.3762</td>
<td>3.3030</td>
<td>-2.1549</td>
<td>0.6942</td>
</tr>
<tr>
<td>Constrained</td>
<td>0.0098</td>
<td>1.3698</td>
<td>2.4573</td>
<td>-2.4812</td>
<td>0.1323</td>
</tr>
</tbody>
</table>

Table 10: Percentage change from steady state in unconstrained and constrained model

<table>
<thead>
<tr>
<th>Model</th>
<th>$\Delta h_1$</th>
<th>$\Delta h_2$</th>
<th>$\Delta h_3$</th>
<th>$\Delta h_4$</th>
<th>$\Delta p$</th>
<th>$\Delta q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconstrained</td>
<td>-1.3378</td>
<td>0.9482</td>
<td>1.8620</td>
<td>-3.5197</td>
<td>0.8704</td>
<td>1.4146</td>
</tr>
<tr>
<td>Constrained</td>
<td>-0.6916</td>
<td>0.6183</td>
<td>1.6978</td>
<td>-3.2041</td>
<td>0.5131</td>
<td>0.7468</td>
</tr>
</tbody>
</table>