OPTIMAL MANAGEMENT OF GREEN CERTIFICATES IN THE SWEDISH-NORWEGIAN MARKET

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ABSTRACT. We propose and investigate a valuation model for the income of selling tradeable green certificates (TGCs) in the Swedish-Norwegian market, formulated as a singular stochastic control problem. Our model takes into account the production rate of renewable energy from a "typical" plant, the dynamics market price of TGCs and the cumulative amount of certificates sold. We assume that the production rate has a dynamics given by an exponential Ornstein-Uhlenbeck process and the TGC logrpice a Levy process. For this class of dynamics for the state variables we find optimal decision rules and a closed form solution to the control problem.

Furthermore, we perform an empirical analysis of the TGC logreturns based on data between November 2009 until May 2013. The empirical analysis strongly indicates that the TGC logprice is a normal inverse Gaussian distributed Lévy process. For this case the valuation model is explicitly calculated.

1. INTRODUCTION

The need for renewable energy sources has increased a lot during the last decade as the awareness of climate change has increased. The renewable energy technologies are often very expensive and requires external financial support to be realized. An alternative to direct government funding is the development of a market for green certificates, also called tradeable green certificates (TGCs). The certificates are pure financial objects used to reach a desirable production capacity of electricity from renewable resources. The idea is that the end consumers finance the renewable energy technologies by purchasing TGCs.

The producer of "green" electricity has the right to sell one certificate per unit "green" electricity produced, while the consumers are obliged to cover their share of electricity consumption from renewable sources. This is done by purchasing green certificates. The producer is given an amount of certificates from the government based on production to sell in the market. In that way government subsidies can be removed and a direct link between power consumption and "green energy" generation is created.

The Swedish electricity certificate market was established in 2003, and in 2012 Norway enrolled in the market. Before the common market between Sweden and Norway 13.3 TWh was financed via the Swedish certificate market. The common goal between 2012-2020 is to increase renewable electricity production by 26.4 TWh, shared equally between Sweden and Norway. The market will continue until 2035, with a target of totally financing 198 TWh of new renewable electricity production.[1]

In this paper, the objective is to find the value of an income stream obtained by selling certificates in the particular case of the Swedish-Norwegian green certificate market. From observing the market price of the certificates, the holder can maximize the income stream by finding the best time and amount to sell. We formulate the optimization problem as a finite horizon singular stochastic control problem with state variable being the certificate spot price, the production rate of power from renewable sources, and the amount of certificates sold. We allow certificates to be sold in gulps or all at once, which includes singular stochastic controls in our optimization problem.

Since one of the purposes with TGCs is to cover the producer from losses due to high costs of the installation of renewable energy technologies, the consumer pays a higher price for electricity, namely the sum of the market prices for electricity and TGCs. Both the electricity price and the TGC price will be driven by the demand for power which makes the two strongly dependent.

Date: Tuesday 26th August, 2014.

Key words and phrases. Green certificates, Optimal decision rule, Empirical analysis , NIG distribution.

We thank Montel for provding us with data.

[1] *The Swedish-Norwegian Electricity Certificate Market*, Annual Report 2012.
http://www.nve.no/Global/Elsertifikater/Elcertifikat2013_Eng_TA%20%282%29.pdf

This, and the fact that the TGC market is very immature, motivate us to consider an exponential Lévy process to model the TGC price. This gives a great flexibility and a price model able to incorporate features as, heavy tails, skewness and kurtosis in the TGC price distribution, all which are typically observed in electricity prices [1].

The production rate is highly influenced by weather factors such as rain, solar and wind, having in mind renewable power generation plants like hydro power, photovoltaic and wind mills. It is reasonable to assume that these weather factors are stationary varying around some seasonal mean level. To describe statistical features of wind speed dynamics in discrete time, commonly used models are the ARMA (Autoregressive moving-average) time series models. These models has also been used for time series of temperature. The analogue to continuous time is the CARMA process. Applications of these, and other related weather models, in the weather market can be found in [2] and references therein. In particular they suggest a CARMA model for temperature and an exponential CARMA model for wind speed. As we are interested in the production rate of electricity, this motivates us to assume that the dynamics of the production rate follows an exponential Ornstein-Uhlenbeck (OU) process.

Previous work relating to green certificates markets is [6]. This term paper gives an overview of the market and its implications from a political and economical point of view. Furthermore they discuss the equilibrium price and the Swedish-Norwegian market. A paper on green certificates related to solar renewable energy certificates (SREC), in the American market has been investigated by [4]. They focus on understanding the price dynamics and propose a structural model for renewable energy certificates, able to incorporate important features. However, the existing academic literature is still sparse on this subject.

As the TGC market is very new there are, to the best of our knowledge, no papers discussing the price dynamics of TGC's in the European market or how a producer should sell them optimally. We provide a general framwork for a valuation model for selling TGC optimally, as well as a general model for the underlying TGC price dynamics. A closed form solution to the valuation model is provided and explicitly calculated when the TGC logprice process is normal inverse Gaussian (NIG) distributed. We also conduct an empirical analysis that support that our proposed model for the price dynamics work and that the NIG-distribution is very adequate. In this case, we also calculate the numerical value of the contract based on the empirical data.

We also want to emphisize the flexibility of the valuation model as it can be used for any exponential Lévy process with finite moments to model the TGC price dynamic, yet being analytically tractable.

The paper is organized as follows. In section 2 we give the framework for our valuation model, and introduce the dynamics of the production rate. In section 3 we introduce the exponential Lévy process for the TGC price dynamics and derive a HJB equation for the valuation problem. We then derive criterions for the optimal strategy, and show the optimality via a verification theorem. The main result of the paper gives a closed form solution to the valuation model. In section 4 we make a case study followed by an explicit numerical calculation of the optimal value. For the numerical part we make an empirical analysis of TGC spot price data. Proofs and intermediate results can be found in Appendix.

2. THE SINGULAR STOCHASTIC CONTROL PROBLEM

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, be a complete filtered probability space satisfying the usual conditions. We define $\mathcal{F}_t := \{\mathbf{X}(u), u \leq t\}$ to be the σ-algebra generated by the state process $\mathbf{X}(u)$. Also, we assume that the state process has the Markov property.

We formulate the singular stochastic control problem for optimal management of the green certificates held by the producer as follows. Let $X(t)$ be the price at time t of a green certificate, and denote $P(t)$ as the accumulated production of "clean power" from a producer entitled to receive certificates. We introduce $R(t)$ as the *production rate* at time t, and thus

$$dP(t) = R(t)\, dt\,.$$

Furthermore, we denote by $A(t)$ the cumulative amount of certificates *sold* up to time t.

The market is organized such that the producer of "clean power" is granted certificates proportional to the production on a regular basis. We approximate this as a continuous income of certificates proportional to the production rate $R(t)$. Hence, at time t we have $cP(t)$ accumulated certificates obtained from production, with $c > 0$ being the proportionality constant. These certificates can, once received, be sold at any later time. Our selling strategy is modeled by the control $A(t)$. We assume the following conditions to hold on the set of controls:

Condition: A is a positive, non-decreasing and adapted stochastic process with paths being right-continuous with left-limits (RCLL). We let $A(0-) = 0$. In addition, $A(t) \leq cP(t)$ for all $t \leq T$. We call such controls *admissible*, and denote the set of admissible A's by $\mathcal{A}$.

Remark that T is a finite trading horizon, typically being the total duration of the certificate market. The condition $A(t) \leq cP(t)$ is to prohibit short-selling of certificates.

Introduce the process $Z(t)$ measuring the amount of certificates held at time t, i.e.,

$$Z(t) = cP(t) - A(t)\,.$$

We observe that for any $A \in \mathcal{A}$, it holds that $Z(t) \geq 0$ for all $t \leq T$.

With these notations, we have the state variable

$$\mathbf{X}(t) = (X(t), R(t), Z(t))$$

being controlled by $A \in \mathcal{A}$. The expected value of the income flow from selling certificates becomes,

$$J(t, x, \varrho, z : A) = \mathbb{E}\left[\int_t^T \mathrm{e}^{-r(s-t)} X(s)\, dA(s) \,|\, \mathbf{X}(t-) = (x, \varrho, z)\right]$$

for any $A \in \mathcal{A}(t)$, where $\mathcal{A}(t)$ is the set of admissible controls where time starts at t. We have denoted by $r > 0$ the constant discount rate. Note that as A is monotonely non-decreasing, it is of finite variation on the interval $[t, T]$. Hence, the integral with respect to A inside the expectation operator above is interpreted in the Lebesgue-Stieltjes sense. Our stochastic control problem is now to find an optimal $\widehat{A} \in \mathcal{A}(t)$ such that

$$V(t, x, \varrho, z) := \sup_{A \in \mathcal{A}(t)} J(t, x, \varrho, z : A) = J(t, x, \varrho, z : \widehat{A})\,. \tag{2.1}$$

We analyze this singular stochastic control problem by the method of dynamic programming.

We observe that if $t = T$, the optimal control is to sell all the certificates that the producer hold. Hence, if $Z(T-) = z$, the optimal control is $\Delta\widehat{A}(T) = z$. The value for selling these certificates is then given by

$$V(T, x, \varrho, z) = \mathbb{E}\left[X(T)\Delta\widehat{A}(T) \,|\, \mathbf{X}(T-) = (x, \varrho, z)\right] = xz\,. \tag{2.2}$$

This provides us with a terminal condition for the value function.

We will focus our optimal control problem on some particular classes of state processes X and R of practical relevance and interest. As the production rate R is highly influenced by weather factors, which are stationary varying around some seasonal mean level, a simple, yet natural model is to assume that the dynamics of R follows an exponential Ornstein-Uhlenbeck (OU) process,

$$R(s) = \mathrm{e}^{U(s)}\,, \tag{2.3}$$

where $U(s)$ is a mean-reverting OU-process driven by a Brownian motion,

$$dU(s) = (\mu - \alpha U(s))ds + \sigma_u dB^u(s), \quad U(t) = \ln(R(t))\,.$$

Then the dynamics of $R(s)$ reads as

$$dR(s) = a_R(R(s))R(s)dt + \sigma_u R(s)dB^u(s), \quad R(t) = \varrho,$$

where

$$a_R(R(s)) := \mu - \alpha \ln(R(s)) + \frac{1}{2}\sigma_u^2. \tag{2.4}$$

The constants μ, α and σ_u represents the mean-reversion level, rate of mean-reversion and the volatility of the process $U(s)$. B^u is a Brownian motion, where the superindex u indicate that it is related to the process $U(s)$. The explicit solution to $R(s)$, starting at time t is given by

$$R(s) = \exp\left[\ln(R(t))\mathrm{e}^{-\alpha(s-t)} + \frac{\mu}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-v)} dB^u(v)\right]. \tag{2.5}$$

In next section we specify the price process X.

3. A PRICE MODEL FOR TGC AND DYNAMIC PROGRAMMING

In this section we find a closed form solution for the optimal value (2.1). We assume the logprice, denoted by Y, to be a Lévy process with finite moments. The production rate $R(s)$ is assumed to be an exponential OU-process given by (2.5). As a biproduct we also obtain the optimal strategy, i.e. the optimal control. The result is concluded in Theorem 3.8. We start to specify the price model for X.

3.1. **The TGC price model.** Let

$$X(s) = x\exp(Y(s)),$$

where the dynamics of the Lévy process Y is given by

$$dY(v) = \gamma dv + \sigma_Y dB^Y(v) + \int_{|\xi|<1} \xi\widetilde{N}(dv,d\xi) + \int_{|\xi|\geq 1}\xi N(dv,d\xi), \tag{3.1}$$

and B^Y is a Brownian motion correlated with B^u with correlation coefficient ρ. N is a Poisson random measure with Lévy measure $\nu(d\xi)$ as compensator. Furthermore, assume that X has finite moments, i.e., we suppose that the condition

$$\int_{|\xi|\geq 1} e^{k|\xi|}\nu(d\xi) < \infty \tag{3.2}$$

holds for some $k > 2$. As a consequence we have

$$\int_{\mathbb{R}\backslash\{0\}} |\mathrm{e}^{\xi} - 1 - \xi|\nu(d\xi) < \infty. \tag{3.3}$$

and

$$\int_{|\xi|\geq 1} |\xi|\nu(d\xi) < \infty. \tag{3.4}$$

We can write

$$\int_{|\xi|<1} \xi\widetilde{N}(dv,d\xi) + \int_{|\xi|\geq 1}\xi N(dv,d\xi) = \int_{|\xi|<1} \xi\widetilde{N}(dv,d\xi) + \int_{|\xi|\geq 1}\xi \widetilde{N}(dv,d\xi) + \int_{|\xi|\geq 1}\xi\nu(d\xi).$$

Hence,

$$dY(v) = \tilde{\gamma} dv + \sigma_Y dB^Y(v) + \int_{\mathbb{R}\backslash\{0\}} \xi\widetilde{N}(dv,d\xi),$$

where

$$\tilde{\gamma} := \gamma + \int_{|\xi|\geq 1}\xi\nu(d\xi).$$

By Itô formula for semimartingales we obtain the dynamics for $X(v)$:

$$dX(v) = a_X X(v)dv + \sigma_Y X(v) dB^Y(v) + \int_{\mathbb{R}\backslash\{0\}} X(v-)(\mathrm{e}^{\xi}-1)\widetilde{N}(dv,d\xi),$$

where

$$a_X := \tilde{\gamma} + \frac{1}{2}\sigma_Y^2 + \int_{\mathbb{R}\backslash\{0\}} (\mathrm{e}^{\xi} - 1 - \xi)\nu(d\xi). \tag{3.5}$$

Lemma 3.1. *Let $Y(1)$ have Lévy triplet $(\gamma, \sigma_Y^2, \nu(d\xi))$ and characteristic function $\phi(u)$. Suppose that* (3.3) *holds. Then*

$$a_X = \ln\phi(-i),$$

where i is the imaginary unit.

Proof. We have that

$$a_X := \gamma + \int_{|\xi|\geq 1}\xi\nu(d\xi) + \int_{\mathbb{R}\backslash\{0\}} (\mathrm{e}^{\xi} - 1 - \xi)\nu(d\xi) + \frac{1}{2}\sigma_Y^2.$$

By Lévy-Khintchine formula for the logarithm of the characteristic function we have

$$\ln\phi(u) = i\gamma u - \frac{1}{2}\sigma_Y^2 u^2 + \int_{\mathbb{R}\backslash\{0\}} (\mathrm{e}^{iu\xi} - 1 - iu\xi\mathbf{1}(|\xi|<1))\nu(d\xi)$$

$$= i\gamma u - \frac{1}{2}\sigma_Y^2 u^2 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{iu\xi} - 1 - iu\xi\mathbf{1}(|\xi| < 1) + iu\xi\mathbf{1}(|\xi| > 1) - iu\xi\mathbf{1}(|\xi| > 1))\nu(d\xi)$$
$$= i\tilde{\gamma} u - \frac{1}{2}\sigma_Y^2 u^2 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{iu\xi} - 1 - iu\xi)\nu(d\xi)$$

for $u \in \mathbb{R}$. Here we used (3.4) in the second line. By condition (3.3), u can be extended to the complex plane. Taking $u = -i$ yields

$$\ln \phi(-i) = \tilde{\gamma} + \frac{1}{2}\sigma_Y^2 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{\xi} - 1 - \xi)\nu(d\xi).$$

Hence, the result follows. □

3.2. **The valuation model.** We will now solve the control problem defined in (2.1). First we define the space $\mathcal{M}(t, T, B_1, B_2)$ as all $F(t, x, \varrho, z) \in C^{1,2,2,1}$ such that for any admissible control $A \in \mathcal{A}(t)$ the processes

$$\theta \mapsto \int_t^\theta \mathrm{e}^{-rs} F_x(s, X(s), R(s), Z(s))\sigma_1 X(s) dB_1(s)$$
$$\theta \mapsto \int_t^\theta \mathrm{e}^{-rs} F_\varrho(s, X(s), R(s), Z(s))\sigma_2 R(s) dB_2(s)$$

are martingales, for $t \le \theta \le T$. The Hamilton-Jacobi-Bellman (HJB) equation is now derived via Bellman's principle of optimality.

Proposition 3.2. *Suppose that $V(t, x, \varrho, z) \in \mathcal{M}(t, T, B^Y, B^u)$, and that the process*

$$\theta \mapsto \int_t^\theta \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{-rs} \left[V(s, X(s)\mathrm{e}^{\xi}, R(s), Z(s)) - V(s, X(s), R(s), Z(s))\right] \widetilde{N}(ds, d\xi)$$

is a martingale. Then, for all $t \in [0, T]$, the corresponding HJB-equation associated to the value function V is

$$\max\left(V_t + \mathcal{L}V - rV, -V_z + x\right) = 0, \tag{3.6}$$

where the operator $\mathcal{L}$ acting on functions $F \equiv F(t, x, \varrho, z) \in C^{1,2,2,1}$, is defined as

$$\begin{aligned}\mathcal{L}F :=& a_X x F_x + a_R(\varrho)\varrho F_\varrho + c\varrho F_z + \frac{1}{2}\sigma_Y^2 x^2 F_{xx} + \frac{1}{2}\sigma_u^2 \varrho^2 F_{\varrho\varrho} + \rho\sigma_Y\sigma_u x\varrho F_{x\varrho} \\ &+ \int_{\mathbb{R}\setminus\{0\}} \left[F(t, x\mathrm{e}^{\xi}, \varrho, z) - F(t, x, \varrho, z) - x(\mathrm{e}^{\xi} - 1)F_x(t, x, \varrho, z)\right] \nu(d\xi)\end{aligned} \tag{3.7}$$

with $a_R(\varrho)$ given by (2.4) *and a_X by* (3.5).

Proof. See Appendix. □

To proceed, the following lemma will be useful for further calculations.

Lemma 3.3. *Let $\phi(u)$ be the characteristic function of $Y(1)$, defined in* (3.1)*, and suppose the conditions of Lemma* 3.1 *holds. Then*

(i)

$$\begin{aligned}\mathbb{E}[X(s)R(s)\,|\,\mathbf{X}(t) = (x, \varrho, z)] = x \exp\Big[&\ln(\varrho)\mathrm{e}^{-\alpha(s-t)} + \frac{\mu}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}) \\ &+ \frac{\sigma_u^2}{4\alpha}(1 - \mathrm{e}^{-2\alpha(s-t)})(2 - \rho^2) + \frac{\sigma_u\sigma_Y}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}) + a_X(s - t)\Big]\end{aligned} \tag{3.8}$$

(ii)

$$\mathbb{E}[X(T)\,|\,\mathbf{X}(t) = (x, \varrho, z)] = x \exp\left[a_X(T - t)\right] \tag{3.9}$$

(iii)

$$\mathbb{E}[X(T)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)] = x\exp\Big[\ln(\varrho)\mathrm{e}^{-\alpha(s-t)}+\frac{\mu}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)})$$
$$+\frac{\sigma_u^2}{4\alpha}(1-\mathrm{e}^{-2\alpha(s-t)})(2-\rho_u^2)+\frac{\sigma_u\sigma_Y}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)})+a_X(T-t)\Big] \tag{3.10}$$

Proof. See Appendix. □

Define

$$h\equiv h(t,\varrho,s):=\frac{1}{x}\mathbb{E}[X(s)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)],$$
$$\tilde{h}\equiv\tilde{h}(t,\varrho,s):=\frac{1}{x}\mathbb{E}[X(T)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)],$$

and

$$H\equiv H(t,\varrho,T):=\int_t^T \mathrm{e}^{-r(s-t)}h(t,\varrho,s)ds, \tag{3.11}$$

$$\tilde{H}\equiv\tilde{H}(t,\varrho,T):=\int_t^T \mathrm{e}^{-r(s-t)}\tilde{h}(t,\varrho,s)ds. \tag{3.12}$$

Consider the admissible control $\widetilde{A}_1$, defined as $\Delta\widetilde{A}_1(t)=z$ and $d\widetilde{A}_1(s)=cR(s)ds$ for $s>t$. Define

$$\Phi(t,x,\varrho,z):=J(t,x,\varrho,z:\widetilde{A}_1)=\mathbb{E}\left[\int_t^T \mathrm{e}^{-r(s-t)}X(s)\,d\widetilde{A}_1(s)\,|\,\mathbf{X}(t-)=(x,\varrho,z)\right]. \tag{3.13}$$

Then,

$$\begin{aligned}\Phi(t,x,\varrho,z)&=\mathbb{E}\left[xz+\int_t^T \mathrm{e}^{-r(s-t)}X(s)cR(s)ds\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\\&=xz+c\int_t^T \mathrm{e}^{-r(s-t)}\mathbb{E}\left[X(s)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]ds\\&=xz+cxH(t,\varrho,T).\end{aligned}$$

Proposition 3.4. *If*

$$a_X\le r, \tag{3.14}$$

then $\Phi(t,x,\varrho,z)$ *defined in* (3.13) *solves the HJB-equation* (3.6)*. Furthermore,* $\Phi(t,x,\varrho,z)$ *is dominated by the value function. I.e.,*

$$\Phi(t,x,\varrho,z)\le V(t,x,\varrho,z),$$

and $\Phi(T,x,\varrho,z)=V(T,x,\varrho,z)=xz$*.*

Proof. From the relation

$$\Phi(t,x,\varrho,z)==xz+cxH(t,\varrho,T),$$

it is clear that $\Phi(T,x,\varrho,z)=xz$. Also we see that $x-\Phi_z=0$. It remains to show that $\Phi_t+\mathcal{L}\Phi-r\Phi\le 0$. Using the definition of $H(t,\varrho,T)$ in (3.11), we calculate

$$\begin{aligned}&\Phi_t+\mathcal{L}\Phi-r\Phi\\&=cx\int_t^T \mathrm{e}^{-r(s-t)}(rh+h_t)ds-cx\varrho+a_Xx\left(z+c\int_t^T \mathrm{e}^{-r(s-t)}hds\right)\\&\quad+cxa_R(\varrho)\varrho\int_t^T \mathrm{e}^{-r(s-t)}h_\varrho ds+cx\sigma_Y\sigma_u\rho\varrho\int_t^T \mathrm{e}^{-r(s-t)}h_\varrho ds\\&\quad+cx\varrho+cx\frac{1}{2}\sigma_u^2\varrho^2\int_t^T \mathrm{e}^{-r(s-t)}h_{\varrho\varrho}ds\end{aligned}$$

$$
\begin{aligned}
&- r\left(xz + cx\int_t^T \mathrm{e}^{-r(s-t)}h ds\right)\\
&+ \int_{\mathbb{R}\setminus\{0\}} \left[x\mathrm{e}^{\xi}z + cx\mathrm{e}^{\xi}H(t,\varrho,T) - (xz + cxH(t,\varrho,T)) - x(\mathrm{e}^{\xi}-1)(z + cH(t,\varrho,T))\right]\nu(d\xi)\\
&= (a_X - r)xz + cx\int_t^T \mathrm{e}^{-r(s-t)}\left[h_t + a_X h + a_R(\varrho)\varrho h_\varrho + \rho\sigma_Y\sigma_u\varrho h_\varrho + \frac{1}{2}\sigma_u^2\varrho^2 h_{\varrho\varrho}\right]ds
\end{aligned}
$$

Inserting the derivatives of h, given in Appendix, yields

$$
\begin{aligned}
&\Phi_t + \mathcal{L}\Phi - r\Phi = (a_X - r)xz\\
&+ cx\int_t^T \mathrm{e}^{-r(s-t)}h\left[M(s,t) + a_X + \mathrm{e}^{-\alpha(s-t)}\left(a_R(\varrho) + \rho\sigma_Y\sigma_u - \frac{1}{2}\sigma_u^2\right) + \frac{1}{2}\sigma_u^2\mathrm{e}^{-2\alpha(s-t)}\right]ds
\end{aligned}
$$

The first term is clearly non-positive due to (3.14), since $xz > 0$. For the integrand we have, from the expressions for $a_R(\varrho) = \mu - \alpha\ln(\varrho) + \frac{1}{2}\sigma_u^2$ and $M(s,t)$ in (6.8)

$$
\begin{aligned}
&M(s,t) + a_X + \mathrm{e}^{-\alpha(s-t)}\left(a_R(\varrho) + \rho\sigma_Y\sigma_u - \frac{1}{2}\sigma_u^2\right) + \frac{1}{2}\sigma_u^2\mathrm{e}^{-2\alpha(s-t)}\\
&= \mathrm{e}^{-\alpha(s-t)}\left[\alpha\ln(\varrho) - \mu - \sigma_u\sigma_Y\right] + \frac{1}{2}\sigma_u^2(\rho^2-2)\mathrm{e}^{-2\alpha(s-t)} - \frac{1}{2}\sigma_Y^2\\
&\quad - \ln\phi(-i) + a_X\\
&\quad + \mathrm{e}^{-\alpha(s-t)}\left(\mu - \alpha\ln(\varrho) + \frac{1}{2}\sigma_u^2 + \rho\sigma_Y\sigma_u - \frac{1}{2}\sigma_u^2\right) + \frac{1}{2}\sigma_u^2\mathrm{e}^{-2\alpha(s-t)}\\
&= \mathrm{e}^{-\alpha(s-t)}\left[\sigma_u\sigma_Y(\rho-1) + \frac{1}{2}\sigma_u^2(\rho^2-2)\mathrm{e}^{-\alpha(s-t)}\right]\\
&\quad + a_X - a_X \le 0.
\end{aligned}
$$

The inequality follows since $\sigma_u\sigma_Y(\rho-1) + \frac{1}{2}\sigma_u^2(\rho^2-2)\mathrm{e}^{-\alpha(s-t)} \le 0$. Hence, $\Phi_t + \mathcal{L}\Phi - r\Phi \le 0$ since $cx \ge 0$. The domination follows since $\widetilde{A}_1$ is admissible. □

If $a_X > r$ the control $\widetilde{A}_1$ would violate the HJB-equation. Consider instead the admissible control $\tilde{A}_2$ defined as $\tilde{A}_2(s) = 0$ for $s \in [t,T)$ and $\Delta\tilde{A}_2(T) = Z(T-) = z + \int_t^T cR(s)ds$. Define

$$
\Phi(t,x,\varrho,z) = J(t,x,\varrho,z : \tilde{A}_2). \tag{3.15}
$$

Then,

$$
\begin{aligned}
\Phi(t,x,\varrho,z) &= \mathbb{E}\left[\mathrm{e}^{-r(T-t)}X(T)\Delta\widetilde{A}(T)\,|\,\mathbf{X}(t-) = (x,\varrho,z)\right]\\
&= \mathrm{e}^{-r(T-t)}\mathbb{E}\left[X(T)\left(z + \int_t^T cR(s)\,ds\right)\,|\,\mathbf{X}(t) = (x,\varrho,z)\right]\\
&= \mathrm{e}^{-r(T-t)}z\mathbb{E}[X(T)\,|\,\mathbf{X}(t) = (x,\varrho,z)] + \mathrm{e}^{-r(T-t)}c\int_t^T \mathbb{E}\left[X(T)R(s)\,|\,\mathbf{X}(t) = (x,\varrho,z)\right]ds.
\end{aligned}
$$

Thus, by Lemma 3.3 (ii) and (3.12) we obtain,

$$
\Phi(t,x,\varrho,z) = xz\mathrm{e}^{(a_X - r)(T-t)} + cx\tilde{H}(t,\varrho,T). \tag{3.16}
$$

Proposition 3.5. *If*

$$
r \le a_X \tag{3.17}
$$

then $\Phi(t,x,\varrho,z)$ *defined in* (3.15) *solves the HJB-equation* (3.6). *Furthermore,* $\Phi(t,x,\varrho,z)$ *is dominated by the value function. I.e.,*

$$
\Phi(t,x,\varrho,z) \le V(t,x,\varrho,z),
$$

and $\Phi(T,x,\varrho,z) = V(T,x,\varrho,z) = xz.$

Proof. By (3.16) it is clear that the terminal condition holds. Turning to the HJB-equation, we have

$$x - \Phi_z = x\left(1 - \mathrm{e}^{(a_X - r)(T-t)}\right) \leq 0, \tag{3.18}$$

by condition (3.17). Similar to the proof of Proposition 3.4 we get

$$\begin{aligned}\Phi_t + \mathcal{L}\Phi - r\Phi &= \left[\mathrm{e}^{(a_X-r)(T-t)}(r - a_X)\right] xz \\ &+ cx\int_t^T \mathrm{e}^{-r(T-t)}\left[\tilde{h}_t + a_X\tilde{h} + a_R(\varrho)\varrho\tilde{h}_\varrho + \rho\sigma_Y\sigma_u\varrho\tilde{h}_\varrho + \frac{1}{2}\sigma_u^2\varrho^2\tilde{h}_{\varrho\varrho}\right] ds.\end{aligned}$$

By (3.17) the first term is non-positive as well as the second by similiar calculations as in the proof of Proposition 3.4. The derivatives of Φ and $\tilde{h}$ can be found in Appendix. The domination follows since $\widetilde{A}_2$ is an admissible control. □

Theorem 3.6. *Suppose that $\Phi(t, x, \varrho, z) \in \mathcal{M}(t, T, B^Y, B^u)$, and that the process*

$$\theta \mapsto \int_t^\theta \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{-rs}\left[\Phi(s, X(s)\mathrm{e}^\xi, R(s), Z(s)) - \Phi(s, X(s), R(s), Z(s))\right] \widetilde{N}(ds, d\xi) \tag{3.19}$$

is a martingale. If Φ solves the HJB-equation (3.6), with $\Phi(T, X(T), R(T), Z(T)) = V(T, X(T), R(T), Z(T))$, then $\Phi \geq V$ for all $(t, x, \varrho, z) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times [0, M]$.

Proof. For $t \leq \theta \leq T$ we obtain from Itô formula

$$\begin{aligned}&\int_t^\theta d(\mathrm{e}^{-rs}\Phi(s, X(s), R(s), Z(s))) \\ &= \int_t^\theta \mathrm{e}^{-rs}\Phi_t(s, X(s), R(s), Z(s)) - r\mathrm{e}^{-rs}\Phi_t(s, X(s), R(s), Z(s))ds \\ &\quad + \int_t^\theta \mathrm{e}^{-rs}\Phi_x(s, X(s), R(s), Z(s))a_X X(s)ds + \int_t^\theta \mathrm{e}^{-rs}\Phi_\varrho(s, X(s), R(s), Z(s))a_R(R(s))R(s)ds \\ &\quad + \int_t^\theta \mathrm{e}^{-rs}\Phi_z(s, X(s), R(s), Z(s))cR(s)ds \\ &\quad + \int_t^\theta \mathrm{e}^{-rs}\Phi_x(s, X(s), R(s), Z(s))\sigma_Y X(s)dB^Y(s) + \int_t^\theta \mathrm{e}^{-rs}\Phi_\varrho(s, X(s), R(s), Z(s))\sigma_u R(s)dB^u(s) \\ &\quad - \int_t^\theta \mathrm{e}^{-rs}\Phi_z(s, X(s), R(s), Z(s))dA(s) \\ &\quad + \int_t^\theta \mathrm{e}^{-rs}\Phi_{xx}(s, X(s), R(s), Z(s))\frac{1}{2}\sigma_Y^2 X^2(s)ds \\ &\quad + \int_t^\theta \mathrm{e}^{-rs}\Phi_{\varrho\varrho}(s, X(s), R(s), Z(s))\frac{1}{2}\sigma_u^2 R^2(s)ds \\ &\quad + \int_t^\theta \mathrm{e}^{-rs}\Phi_{x\varrho}(s, X(s), R(s), Z(s))\sigma_Y\sigma_u\rho X(s)R(s)ds \\ &\quad + \int_t^\theta \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{-rs}\left[\Phi(s, X(s-)\mathrm{e}^\xi, R(s-), Z(s-)) - \Phi(s, X(s-), R(s-), Z(s-))\right]\widetilde{N}(ds, d\xi) \\ &\quad + \int_t^\theta \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{-rs}\left[\Phi(s, X(s) + X(s-)(\mathrm{e}^\xi - 1), R(s), Z(s)) - \Phi(s, X(s), R(s), Z(s)) - X(s)(\mathrm{e}^\xi - 1)\right]\nu(d\xi)ds\end{aligned}$$

Taking expectation and using the conditions in the Theorem yields,

$$\Phi(t, x, \varrho, z) - \mathrm{e}^{-r(\theta-t)}\mathbb{E}[\Phi(\theta, X(\theta), R(\theta), Z(\theta)) \,|\, \mathbf{X}(t) = (x, \varrho, z)]$$

$$= \mathbb{E}\left[\int_t^\theta \mathrm{e}^{-r(s-t)}\Phi_z(s,X(s),R(s),Z(s))\,dA(s) + \int_t^\theta \mathrm{e}^{-r(s-t)}[-(\Phi_t+\mathcal{L}\Phi-r\Phi)(s,X(s),R(s),Z(s))]\,ds \,|\, \mathbf{X}(t)=(x,\varrho,z)\right].$$

Take $\theta = T$, then

$$\Phi(t,x,\varrho,z) = \mathbb{E}\left[\int_t^T \mathrm{e}^{-r(s-t)}\Phi_z(s,X(s),R(s),Z(s))\,dA(s) + \int_t^T \mathrm{e}^{-r(s-t)}[-(\Phi_t+\mathcal{L}\Phi-r\Phi)(s,X(s),R(s),Z(s))]\,ds + \mathrm{e}^{-r(T-t)}\Phi(T,X(T),R(T),Z(T)) \,|\, \mathbf{X}(t)=(x,\varrho,z)\right].$$

Since Φ satisfies the HJB-equation we have at time s

$$\Phi_z(s,X(s),R(s),Z(s)) \geq X(s), \quad -(\Phi_t+\mathcal{L}\Phi-r\Phi)(s,X(s),R(s),Z(s)) \geq 0,$$

and by assumption we have $\Phi(T,X(T),R(T),Z(T)) = V(T,X(T),R(T),Z(T)) \geq 0$. It follows that

$$\Phi(t,x,\varrho,z) \geq \mathbb{E}\left[\int_t^T \mathrm{e}^{-r(s-t)}X(s)\,dA(s) \,|\, \mathbf{X}(t)=(x,\varrho,z)\right].$$

Since this inequality holds for any admissible control, it also holds for the supremum over all such controls. Hence,

$$\Phi(t,x,\varrho,z) \geq V(t,x,\varrho,z).$$

□

Lemma 3.7. *For the functions $\Phi(t,x,\varrho,z)$ defined in* (3.13) *and* (3.15)*. Condition* (3.19) *holds and $\Phi(t,x,\varrho,z) \in \mathcal{M}(t,T,B^Y,B^u)$.*

Proof. See Appendix. □

From the Verification Theorem 3.6 we have the following conclusion.

Theorem 3.8. *For $a_X \leq r$*

$$V(t,x,\varrho,z) = xz + cxH(t,\varrho,T) \tag{3.20}$$

and the optimal control is $\widetilde{A}_1$.
For $r \leq a_X$,

$$V(t,x,\varrho,z) = xz\mathrm{e}^{(a_X-r)(T-t)} + cx\tilde{H}(t,\varrho,T) \tag{3.21}$$

and the optimal control is $\widetilde{A}_2$.

Proof. Since $\widetilde{A}_1$ and $\widetilde{A}_2$ are admissible the conclusion follows directly from Proposition 3.4 and Proposition 3.5 together with Lemma 3.7 and the Verification Theorem 3.6. □

The optimal value of the income from selling certificates is thus given by Theorem 3.8, and as a biproduct we get the optimal strategy. Furthermore, Theorem 3.8 also provide us with conditions for what strategy to use, depending on the sign of $a_X - r$, that is $\widetilde{A}_1$ or $\widetilde{A}_2$. Hence, in practice we only need to determine the sign of $a_X - r$, then by Theorem 3.8 a closed form solution for the optimal value is given by (3.20) or (3.21).

Furthermore, note the interpretation of the optimal stratgies $\widetilde{A}_1$ and $\widetilde{A}_2$: when the expected rate of return is smaller than r the holder of TGCs should sell all certificates held at time t and continue selling at the same rate as certificates is obtained from production of renewable energy. On the other hand, when the expected rate of return is bigger than r the holder of TGCs should wait and then sell all certificates at maturity. Here r is the risk free return.

In next section we will make a profound case study, where we also explicitly calculate the optimal value. First, just to illustrate the simplicity of what control to choose, consider the following simple example.

Let the logprice follow a Brownian diffusion process,

$$dY(t) = \gamma dt + \sigma_Y dB^Y(t).$$

This results in a price X having geometric Brownian motion dynamics

$$dX(v) = a_X X(v)dv + \sigma_Y X(v)dB^Y(v),$$

where $a_X = \gamma + \frac{1}{2}\sigma_Y^2$. Consider the time series for the certificate spot price in Figure 1. The data[2] is collected daily (five days a week) between November 2009 until May 2013. The downward trend until 2011 (from Day 0 to around Day 500 in the Figure 1) is naturally modelled as a negative trend a_X in the GBM dynamics. Focusing on the data following, we will have a positive trend, for a_X. This results in a negative a_X in the GBM model, an thereafter positive. Without any deeper empirical analysis, in the case

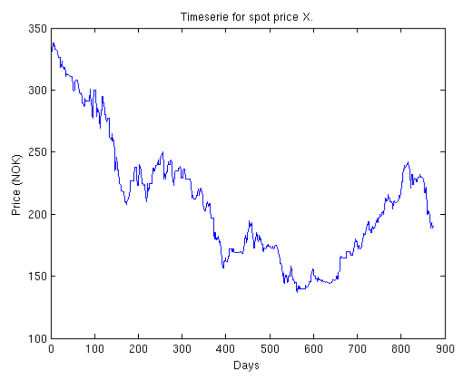


FIGURE 1. Time series for the spot price X. Collected data from 3 November 2009 until 10 May 2013.

of a GBM, the sign of $a_X - r$ is easily determined once the size of r is known. Then the optimal strategy is provided by Theorem 3.8.

4. A CASE STUDY

The time series for the logreturns in Figure 2 and the empirical density plots fitted with the normal distribution in Figure 3, clearly shows that the logreturns are not normally distributed. Clearly, we need to consider more sophisticated models than e.g. the GBM model in the example above. Motivated from an empirical analysis we consider a normal inverse Gaussian (NIG) distributed logprice process.

4.1. Logprice with normal inverse Gaussian distribution.

The normal inverse Gaussian distribution is a four parameter distribution, $NIG(\mu_{NIG}, \alpha_{NIG}, \beta_{NIG}, \delta_{NIG})$, with characteristic function (see e.g. [8])

$$\phi(u) = \exp\left[iu\mu_{NIG} - \delta_{NIG}(\sqrt{\alpha_{NIG}^2 - (\beta_{NIG} + iu)^2} - \sqrt{\alpha_{NIG}^2 - \beta_{NIG}^2})\right].$$

Figure 4 shows the empirical density plot, fitted with the NIG-distribution. The autocorrelation function of the logreturns and squared logreturns in shown in Figure 5. The autocorrelation are close to zero for both the logreturns and the squared logreturns for all lags, indicating that we have a weak relationship between todays logreturns and one or more days ahead. This suggest that a NIG distributed Lévy process is an adequate model for the logprice. To obtain this price model in our theoretical framework in the previous section, we let Y have the Lévy triplet $(\gamma, 0, \nu(d\xi))$, where

$$\gamma = \frac{2\delta_{NIG}\alpha_{NIG}}{\pi}\int_0^1 \sinh(\beta_{NIG}x)K_1(\alpha_{NIG}x)dx + \mu_{NIG},$$

[2]The data is provided from Montel

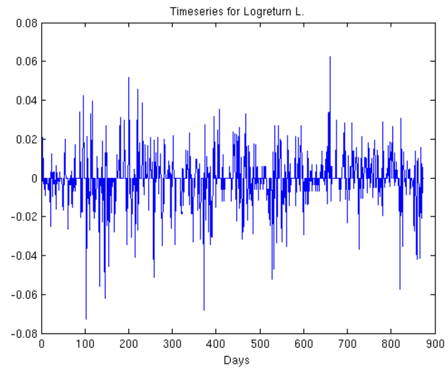
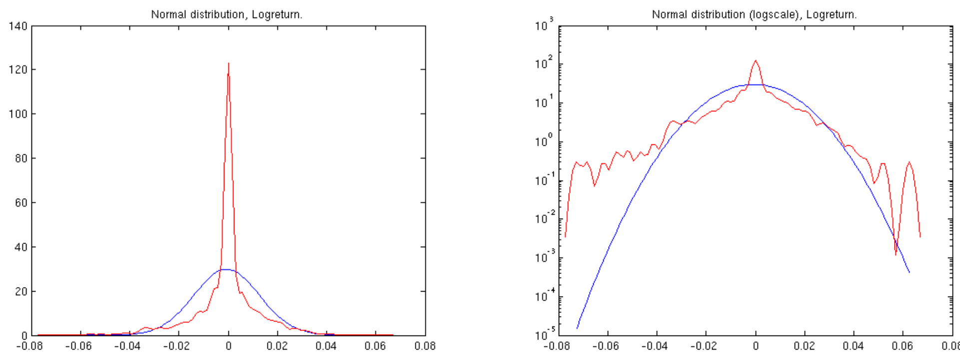


FIGURE 2. Time series for Logreturn L.



(a) (b)

FIGURE 3. (a) Normal distribution for logreturn L, (b) in logarithmic scale. The red curve is the sample distribution.

and

$$\nu(d\xi) = \frac{\delta_{NIG}\alpha_{NIG}}{\pi}\frac{\exp(\beta_{NIG}\xi)K_1(\alpha_{NIG}|\xi|)}{|\xi|}d\xi,$$

where the modified Bessel function $K_v(z)$, of third kind with index v, is given by

$$K_v(z) = \frac{1}{2}\int_0^\infty u^{v-1}\exp\left(-\frac{1}{2}z(u+u^{-1})\right)du, \quad z>0.$$

Since the NIG-distribution has finite moments, the conditions in Lemma 3.1 holds and we obtain

$$a_X = \ln\phi(-i) = \mu_{NIG} - \delta_{NIG}(\sqrt{\alpha_{NIG}^2-(\beta_{NIG}-1)^2} - \sqrt{\alpha_{NIG}^2-\beta_{NIG}^2}). \tag{4.1}$$

The estimation of the NIG parameters is done in the numerical example below. Note that (3.2) holds if

$$k+\beta_{NIG}-\frac{1}{2}\alpha_{NIG}<0. \tag{4.2}$$

To proceed, we investigate the production rate a bit closer in order to find a qualitative way to approximate the parameters from descriptive statistics. The logarithmic production rate, $\ln(R(s)) := U(s)$ has solution, for $s\geq t$ with $U(t)=\ln(\varrho)$, given by

$$\ln(R(s)) := U(s) = \ln(\varrho)\mathrm{e}^{-\alpha(s-t)} + \frac{\mu}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)}) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-v)}dB^u(v). \tag{4.3}$$

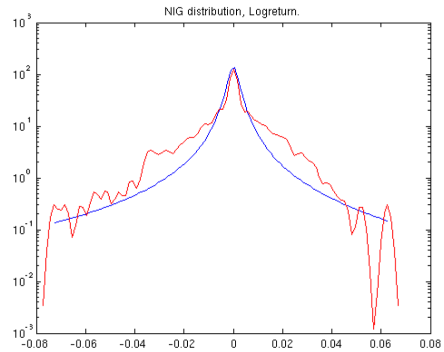
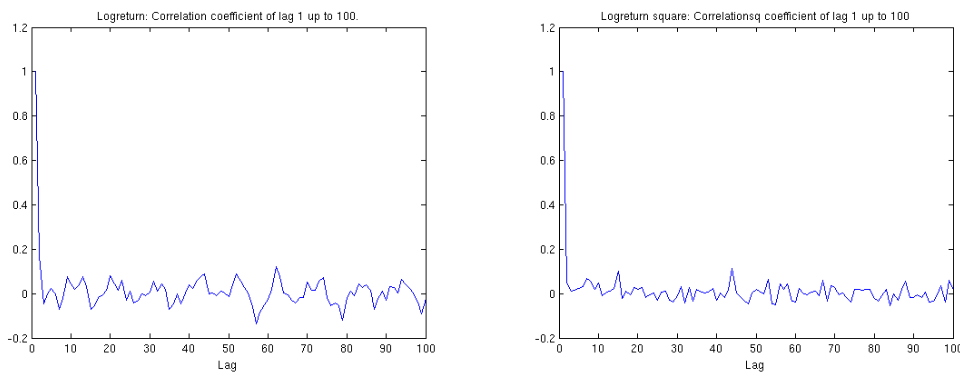


FIGURE 4. Fitted NIG distribution. Parameters obtained via MLE.



(a) (b)

FIGURE 5. The autocorrelation for logreturns (a) and squared logreturns (b).

The long term mean is easily calculated to be

$$\mathbb{E}[\ln(R)] = \frac{\mu}{\alpha}. \tag{4.4}$$

The variance is calculated as $Var(\ln(R)) := \mathbb{E}[(\ln(R))^2] - (\mathbb{E}[\ln(R)])^2$ and is found, via Itô isometry to be

$$Var(\ln(R)) = \frac{\sigma_u^2}{2\alpha} \tag{4.5}$$

in stationary. Clearly, the logarithmic production rate is normally distributed, $\mathcal{N}(\frac{\mu}{\alpha}, \frac{\sigma_u^2}{2\alpha})$. Thus R is lognormally distributed. In practice, it is reasonal to assume that we, at least in a qualitative manner, can estimate the mean and the variance of the process R without a deeper empirical analysis. Furthermore, we shall also assume that we can estimate a value on the rate at which $\ln(R)$ reverts towards the mean.

Denote the mean and variance of R by m and V. Since R is lognormally distributed the relation between the descriptive statistics of R and $\ln(R)$ is given by [3]

$$\mathbb{E}[\ln(R)] = \ln(\frac{m^2}{\sqrt{V + m^2}}),$$
$$Var[\ln(R)] = \ln(1 + \frac{V}{m^2}).$$

[3]This is independent on the choice of logarithmic base.

We thus obtain the parameters μ and σ_u in the process for R, via (4.4) and (4.5) as

$$\mu = \alpha \ln(\frac{m^2}{\sqrt{V+m^2}}), \tag{4.6}$$

$$\sigma_u^2 = 2\alpha \ln(1+\frac{V}{m^2}). \tag{4.7}$$

To find an approximation for α we proceed as follows. Let $\tau > 0$ denote the decay time, i.e. the time it takes for the process $\ln(R)$ to revert towards its mean. Since α is finite we will never hit the actual mean level. Define a strip $[\mathbb{E}[\ln(R)] - \mu^+, \mathbb{E}[\ln(R)] + \mu^+]$ for all t, $\mu^+ > 0$. We say that $\mathbb{E}[\ln(R)] \pm \mu^+$ are the acceptable mean reversion levels, i.e., we say that the process has mean revert once it enters this strip after mean reverting from a Brownian shock of size S. Under the assumption that we do not have any stochastic influence during the mean reversion we have the relation

$$\mathrm{e}^{-\alpha\tau} = \frac{\mu^+}{|S - \mathbb{E}[\ln(R)]|}.$$

Solving for α yields,

$$\alpha = -\frac{1}{\tau} \ln \frac{\mu^+}{|S - \mathbb{E}[\ln(R)]|}. \tag{4.8}$$

Depending on the practical situation we choose appropiate values on the size of the shock S and the acceptable diversion μ^+ from the mean. It is also natural to express these quantities in proportion to the mean level $\mathbb{E}[\ln(R)]|$. Hence, suppose m and V are known or are reasonably estimated. Then the remaning parameters μ, σ_u and α are found via (4.6), (4.7) and (4.8) provided that the decay time is known.

In the example below we illustrate how to estimate the parameters in the price process. For the production rate, the parameters is calculated by using the approxiamation (4.8) and reasonable approximations for the mean and variance of the production rate. This results in an "example-model" for R that aims to mimic the production rate. This enable us to calculate a_X in (4.1) and hence, to calculate the explicit value function.

4.2. **Numerical example.** The aim of this example is to illustrate the full application of Theorem 3.8. We start to investigate the price X. The logreturns L displayed in the time series in Figure 2 are calculated as $L(t) = Y(t+1) - Y(t)$. The parameters in the density plot in Figure 4 are obtained via maximum likelihood [4] estimation (MLE) in the software R[5]. The result is given in Table 1. With these parameters we obtain from (4.1) that $a_X = -4.5 \times 10^{-4}$. Since the discount rate $r > 0$ we have that $a_X - r < 0$. Also, note that (4.2) is clearly satisfied for some $k > 2$. Hence, by Theorem 3.8, the optimal strategy is to use the control $\tilde{A}_1$ and the optimal value is given by

$$V(t,x,\varrho,z) = xz + cxH(t,\varrho,T).$$

From the definition of H in (3.11), and Lemma 3.3 (i) we obtain

$$\begin{aligned} V(t,x,\varrho,z) = xz + cx \int_t^T \exp\Big[& -r(s-t) + U(\varrho)\mathrm{e}^{-\alpha(s-t)} + \frac{\mu}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)}) \\ & + \frac{\sigma_u^2}{2\alpha}(1-\mathrm{e}^{-2\alpha(s-t)}) \\ & + (\mu_{NIG} - \delta_{NIG}(\sqrt{\alpha_{NIG}^2 - (\beta_{NIG}+1)^2} - \sqrt{\alpha_{NIG}^2 - \beta_{NIG}^2}))(s-t)\Big] \, ds \end{aligned}$$

To proceed we need to calculate the parameters in the production rate R. In this example we consider the production of electricity from wind farms in Norway, and the background information for this case is based on the production during 2012 [6]. The mean m and variance V is calculated as follows. Let PP be

[4]The parameters was also calculated via moment estimation according to theorem 2.2 in [5]. However, MLE gave a better peak behavior, therefore we choosed to use these values for this illustration.

[5]We used the package 'ghyp'.

[6]The collection of data is published by Norges vassdrags -og energidirektorat and can be found at $http://webby.nve.no/publikasjoner/rapport/2013/rapport2013_13.pdf$

TABLE 1. NIG parameters

Parameter	MLE
α_{NIG}	6.618456
β_{NIG}	-1.736201
μ_{NIG}	-1.7×10^{-5}
δ_{NIG}	2.288627×10^{-3}

the total power production for 2012, and FL the number of hours when the wind turbins operate at full power, then we calculate the total power P produced in one year as

$$P = \frac{PP}{FL}.$$

Hence, the average production rate per day is

$$m = \frac{P}{365}.$$

The standard deviation sd is estimated as $sd = \pm Nm$, where N is a possitive number, and $V = (Nm)^2$. The data from the 2012 production is listed in Table 2. In order to consider different standard deviations and decay rates (units 'per day') we let N and τ vary.

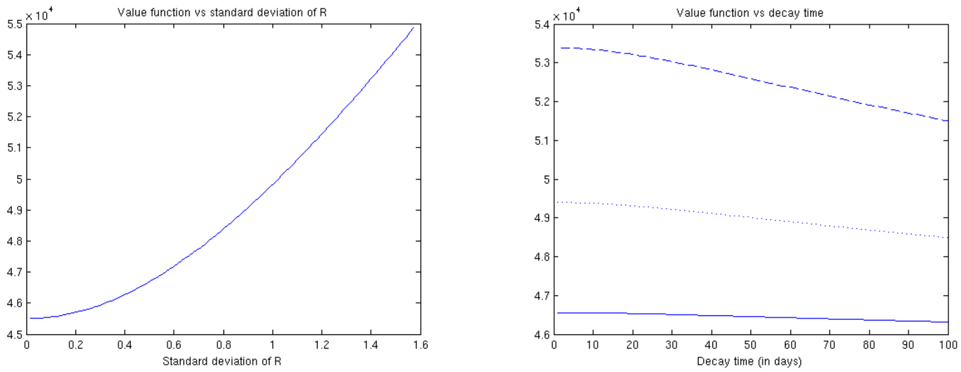
TABLE 2. Production parameters for wind power

PP [TWh]	1.57
FL [h]	2734
P [MW]	574
m	1.57
N	$0.01 - 1$
sd	$1.57N$
V	$(1.57N)^2$
τ	$1 - 100$
α	$-\frac{1}{\tau}\ln(0.1)$
U	$\ln(m)$

To calculate α we have used $\mu^+ = 0.1\mathbb{E}[\ln(R)]$ and $S = 2\mathbb{E}[\ln(R)]$. Using the initial values $x = 300, z = 100$ and a discount rate $r = 0.03$. In the Swedish-Norwegian market the government issue one certificate to the producer for each MWh of renewable electricity produced, i.e., here the proportionality constant is $c = 1$. The parameters in the production rate R is obtained from (4.6) – (4.8) and Table 2. The value of the contract is shown in Figure 6 when N and τ has been continuously varied. That is, for different standard deviations and decay rates. It shows that the value is stable with respect to the decay time. Also, we see that the value increases with increased volatility in the production rate.

5. CONCLUSIONS

We provide a valuation model for the income of selling TGCs, formulated as a singular stochastic control problem. Our model takes into account the production rate of renewable energy from a "typical" plant, the dynamics market price of TGCs and the cumulative amount of certificates sold. We assume that the production rate has a dynamics given by an exponential Ornstein-Uhlenbeck process and the TGC logprice is a Lévy process. The price model is able to incorporate spiky behavior and stylized distributional features such as heavy tails, skewness and exessive kurtosis. In spite of this flexibility we find a closed form solution to the control problem. As a consequence we also provide the optimal selling strategy. It should also be noted that the optimal value is easily calculated by Theorem 3.8 via Lemma 3.3 as soon as the characteristic function of the logprice is known.



(a) (b)

FIGURE 6. Figure $6(a)$ shows how the value function changes with the standard deviation sd for $\tau = 4$. Figure $6(b)$ shows how the value function changes with the decay time τ for $N = 0.3, 0.6, 0.9$, i.e for different values of sd.

Furthermore, we conduct an empirical analysis on data collected between November 2009 until May 2012. It shows that a NIG distributed Lévy process appears to be appropriate as a logprice model.

Finally, the numerical example illustrates the performance of the valuation model. In this example the parameters in the price process are estimated via MLE. For the production rate we relate the parameters with the moments and the decay time after a spike. The moments are reasonable approximated based on production parameters. The value is plotted as a function of the decay time and the standard deviation respectively. The plots strongly indicates that the optimal value is stable with respect to the decay time in the production rate, and that the value increases with increased volatility.

6. APPENDIX

6.1. **Proof of Proposition** 3.2**.** The value function is defined in (2.1) to be

$$V(t,x,\varrho,z) := \sup_{A\in\mathcal{A}(t)} \mathbb{E}\left[\int_t^T \mathrm{e}^{-r(s-t)} X(s)\,dA(s)\,|\,\mathbf{X}(t) = (x,\varrho,z)\right].$$

By Bellman's principle of optimality, we have, for $t \le \theta \le T$,

$$0 = \sup_{A\in\mathcal{A}(t)} \mathbb{E}\Big[\int_t^\theta \mathrm{e}^{-rs} X(s)\,dA(s) + \mathrm{e}^{-r\theta} V(\theta, X(\theta), R(\theta), Z(\theta)) - \mathrm{e}^{-rt} V(t,x,\varrho,z)\,|\,\mathbf{X}(t) = (x,\varrho,z)\Big].$$

By Itô formula we obtain

$$\begin{aligned}
&\int_t^\theta d(\mathrm{e}^{-rs} V(s, X(s), R(s), Z(s))) \\
&= \int_t^\theta \mathrm{e}^{-rs} V_t(s, X(s), R(s), Z(s)) - r\mathrm{e}^{-rs} V_t(s, X(s), R(s), Z(s))ds \\
&\quad + \int_t^\theta \mathrm{e}^{-rs} V_x(s, X(s), R(s), Z(s)) a_X X(s) ds + \int_t^\theta \mathrm{e}^{-rs} V_\varrho(s, X(s), R(s), Z(s)) a_R(R(s)) R(s) ds \\
&\quad + \int_t^\theta \mathrm{e}^{-rs} V_z(s, X(s), R(s), Z(s)) c R(s) ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^\theta \mathrm{e}^{-rs} V_x(s,X(s),R(s),Z(s))\sigma_Y X(s)dB^Y(s)\\
&+\int_t^\theta \mathrm{e}^{-rs} V_\varrho(s,X(s),R(s),Z(s))\sigma_u R(s)dB^u(s)\\
&-\int_t^\theta \mathrm{e}^{-rs} V_z(s,X(s),R(s),Z(s))dA(s)\\
&+\int_t^\theta \mathrm{e}^{-rs} V_{xx}(s,X(s),R(s),Z(s))\frac{1}{2}\sigma_Y^2 X^2(s)ds\\
&+\int_t^\theta \mathrm{e}^{-rs} V_{\varrho\varrho}(s,X(s),R(s),Z(s))\frac{1}{2}\sigma_u^2 R^2(s)ds\\
&+\int_t^\theta \mathrm{e}^{-rs} V_{x\varrho}(s,X(s),R(s),Z(s))\sigma_Y\sigma_u\rho X(s)R(s)ds\\
&+\int_t^\theta \mathrm{e}^{-rs}\left[V(s,X(s-)\mathrm{e}^\xi,R(s-),Z(s-))-V(s,X(s-),R(s-),Z(s-))\right]\widetilde{N}(ds,d\xi)\\
&+\int_t^\theta \mathrm{e}^{-rs}\Big[V(s,X(s)+X(s-)(\mathrm{e}^\xi-1),R(s),Z(s))\\
&\qquad\qquad -V(s,X(s),R(s),Z(s))-X(s)(\mathrm{e}^\xi-1)\Big]\nu(d\xi)ds
\end{aligned}
$$

By the conditions in the proposition we get by Bellman's principle,

$$
\sup_{A\in\mathcal{A}(t)}\mathbb{E}\left[\int_t^\theta \mathrm{e}^{-rs}(X(s)-V_z)\,dA(s)+\int_t^\theta \mathrm{e}^{-rs}[V_t+\mathcal{L}V-rV]\,ds\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]=0, \tag{6.1}
$$

where the operator L is given in (3.7). Clearly, (6.1) is satisfied by the HJB-equation

$$
\max\left(V_t+\mathcal{L}V-rV,-V_z+x\right)=0.
$$

6.2. **Proof of Lemma** 3.3**.** We have that

$$
X(s)R(s)=x\mathrm{e}^{Y(s)}\mathrm{e}^{U(s)}.
$$

We note that we can write the dynamics of Y as

$$
dY(v)=dY_0(v)+\sigma_Y dB^Y(v),
$$

where

$$
dY_0(v):=\gamma dv+\int_{|\xi|<1}\xi\widetilde{N}(dv,d\xi)+\int_{|\xi|\geq 1}\xi N(dv,d\xi).
$$

Using the representation

$$
Y(s)=Y_0(s)+\sigma_Y\int_t^s dB^Y(v),
$$

where Y_0 has characteristic triplet $(\gamma,0,\nu(d\xi))$. Furthermore, due to the correlation between B^Y and B^u we have

$$
B^u(s)=\rho B^Y(s)+\sqrt{1-\rho^2}W(s), \tag{6.2}
$$

where $W(s)$ is another Brownian motion, independent of $B^Y(s)$. It follows that

$$
\begin{aligned}
\mathbb{E}[X(s)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)] &= x\exp(\ln(\varrho)\mathrm{e}^{-\alpha(s-t)}+\frac{\mu}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)}))\\
&\times\mathbb{E}\left[\exp\left(\int_t^s\sigma_Y dB^Y(v)+\int_t^s\sigma_u\mathrm{e}^{-\alpha(s-t)}dB^u(v)\right)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\\
&\times\mathbb{E}\left[\exp(Y_0(s))\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]
\end{aligned} \tag{6.3}
$$

For the first expectation we have, by using (6.2)

$$\int_t^s \sigma_Y dB^Y(v) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-t)} dB^u(v)$$

$$= \int_t^s (\sigma_u \mathrm{e}^{-\alpha(s-v)}\rho + \sigma_Y) dB^Y(v) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-v)} \sqrt{1-\rho^2} dW(v). \tag{6.4}$$

Both integrals in (6.4) have zero expectation and

$$Var(\int_t^s (\sigma_u \mathrm{e}^{-\alpha(s-v)}\rho + \sigma_Y) dB^Y(v)) = \int_t^s (\sigma_u \mathrm{e}^{-\alpha(s-v)}\rho + \sigma_Y)^2 dv$$

$$= \frac{\sigma_u^2}{2\alpha}(1-\mathrm{e}^{-2\alpha(s-t)}) + \frac{2\sigma_u\sigma_Y}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)}) + \sigma_Y^2(s-t), \tag{6.5}$$

$$Var(\int_t^s \sigma_u \mathrm{e}^{-\alpha(s-v)} \sqrt{1-\rho^2} dW(v)) = \int_t^s \sigma_u^2 \mathrm{e}^{-2\alpha(s-v)}(1-\rho^2) dv$$

$$= \frac{\sigma_u^2(1-\rho^2)}{2\alpha}(1-\mathrm{e}^{-2\alpha(s-t)}) \tag{6.6}$$

where we used the Itô isometry. Consequently, (6.4) is normally distributed with mean zero and variance being the sum of (6.5) and (6.6). It follows that

$$\mathbb{E}\left[\exp\left(\sigma_Y \int_t^s dB^Y(v) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-v)} dB^u(v)\right) \mid \mathbf{X}(t) = (x,\varrho,z)\right]$$

$$= \exp\left[\frac{\sigma_u^2}{4\alpha}(1-\mathrm{e}^{-2\alpha(s-t)})(2-\rho^2) + \frac{\sigma_u\sigma_Y}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)}) + \frac{1}{2}\sigma_Y^2(s-t)\right].$$

We now turn to the second expectation in (6.3). By the Lévy-Kinchtine formula we obtain

$$\mathbb{E}[\mathrm{e}^{iuY_0(s)} \mid \mathbf{X}(t) = (x,\varrho,z)] = \phi_{Y_0}^{(s-t)}(u),$$

Where ϕ_{Y_0} is the characteristic function for $Y_0(1)$. From the assumption on ϕ, which is inherited to ϕ_{Y_0}, it follows that, with $u = -i$

$$\mathbb{E}[\mathrm{e}^{Y_0(s)} \mid \mathbf{X}(t) = (x,\varrho,z)] = \phi_{Y_0}^{(s-t)}(-i).$$

Hence,

$$\begin{aligned}
\mathbb{E}[X(s)R(s) \mid \mathbf{X}(t) = (x,\varrho,z)] &= x\exp\Big[\ln(\varrho)\mathrm{e}^{-\alpha(s-t)} + \frac{\mu}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)}) \\
&\quad + \frac{\sigma_u^2}{4\alpha}(1-\mathrm{e}^{-2\alpha(s-t)})(2-\rho^2) + \frac{\sigma_u\sigma_Y}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)}) + \frac{1}{2}\sigma_Y^2(s-t) \\
&\quad + (\ln\phi_{Y_0}(-i))(s-t)\Big] \\
&= x\exp\Big[\ln(\varrho)\mathrm{e}^{-\alpha(s-t)} + \frac{\mu}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)}) \\
&\quad + \frac{\sigma_u^2}{4\alpha}(1-\mathrm{e}^{-2\alpha(s-t)})(2-\rho^2) + \frac{\sigma_u\sigma_Y}{\alpha}(1-\mathrm{e}^{-\alpha(s-t)}) \\
&\quad + (\ln\phi(-i))(s-t)\Big].
\end{aligned}$$

By Lemma 3.1 the first claim, (i) follows. For (ii) we obtain

$$\begin{aligned}
&\mathbb{E}[X(T) \mid \mathbf{X}(t) \\
&= (x,\varrho,z)] = x\mathbb{E}\left[\mathrm{e}^{(Y_0(T))} \mid \mathbf{X}(t) = (x,\varrho,z)\right]\mathbb{E}\left[\exp\left(\int_t^T \sigma_Y dB^Y(v)\right) \mid \mathbf{X}(t) = (x,\varrho,z)\right] \\
&= x\phi^{(T-t)}(-i)\exp\left(\frac{1}{2}\sigma_Y^2(T-t)\right) \\
&= x\exp\left(\ln\phi_{Y_0}(-i) + \frac{1}{2}\sigma_Y^2\right)(T-t) \\
&= x\exp(\ln\phi(-i))(T-t) = x\exp(a_X(T-t)).
\end{aligned}$$

For (iii) we have

$$\begin{aligned}
&\mathbb{E}[X(T)R(s) \,|\, \mathbf{X}(t) = (x, \varrho, z)] \\
&= x \exp(\ln(\varrho) \mathrm{e}^{-\alpha(s-t)} + \frac{\mu}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)})) \\
&\quad \times \mathbb{E}\left[\exp\left(\int_t^T \sigma_Y dB^Y(v) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-t)} dB^u(v)\right) \,|\, \mathbf{X}(t) = (x, \varrho, z)\right] \\
&\quad \times \mathbb{E}\left[\exp(Y_0(T)) \,|\, \mathbf{X}(t) = (x, \varrho, z)\right].
\end{aligned}$$

Similarly as in (6.4) we get

$$\begin{aligned}
&\int_t^T \sigma_Y dB^Y(v) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-t)} dB^u(v) \\
&= \int_s^T \sigma_Y dB^Y(v) + \int_t^s \sigma_Y dB^Y(v) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-t)} dB^u(v) \\
&= \int_s^T \sigma_Y dB^Y(v) + \int_t^s (\sigma_u \mathrm{e}^{-\alpha(s-v)} \rho + \sigma_Y) dB^Y(v) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-v)} \sqrt{1-\rho^2} dW(v).
\end{aligned}$$

The first integral has expectation zero and variance $\sigma_Y^2(T-s)$. Thus,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\int_t^T \sigma_Y dB^Y(v) + \int_t^s \sigma_u \mathrm{e}^{-\alpha(s-t)} dB^u(v)\right) \,|\, \mathbf{X}(t) = (x, \varrho, z)\right] \\
&= \exp\left[\frac{\sigma_u^2}{4\alpha}(1 - \mathrm{e}^{-2\alpha(s-t)})(2 - \rho_u^2) + \frac{\sigma_u \sigma_Y}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}) + \frac{1}{2}\sigma_Y^2(T-t)\right].
\end{aligned}$$

as in the proof of (i). Hence,

$$\begin{aligned}
&\mathbb{E}[X(T)R(s) \,|\, \mathbf{X}(t) = (x, \varrho, z)] = x \exp\Big[\ln(\varrho) \mathrm{e}^{-\alpha(s-t)} + \frac{\mu}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}) \\
&\quad + \frac{\sigma_u^2}{4\alpha}(1 - \mathrm{e}^{-2\alpha(s-t)})(2 - \rho_u^2) + \frac{\sigma_u \sigma_Y}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}) + \frac{1}{2}\sigma_Y^2(T-t) \\
&\quad \ln \phi(-i)(T-t)] \\
&= x \exp\Big[\ln(\varrho) \mathrm{e}^{-\alpha(s-t)} + \frac{\mu}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}) \\
&\quad + \frac{\sigma_u^2}{4\alpha}(1 - \mathrm{e}^{-2\alpha(s-t)})(2 - \rho_u^2) + \frac{\sigma_u \sigma_Y}{\alpha}(1 - \mathrm{e}^{-\alpha(s-t)}) + \ln \phi(-i)(T-t)\Big]
\end{aligned}$$

The result follows.

6.3. **Calculation of the derivatives of ϕ in terms of h with the control $\widetilde{A}_1$.**

$$h := h(t, \varrho, s) := \frac{1}{x}\mathbb{E}\left[X(s)R(s) \,|\, \mathbf{X}(t) = (x, \varrho, z)\right],$$

$$H(t, \varrho, T) := \int_t^T \mathrm{e}^{-r(s-t)} h(t, \varrho, s) ds$$

For the control $\widetilde{A}_1$ we have

$$\Phi(t, x, \varrho, z) = xz + cxH(t, \varrho, T). \tag{6.7}$$

By elemetary differentiation we get

$$\begin{aligned}
\Phi_x &= z + c\int_t^T \mathrm{e}^{-r(s-t)} h ds \\
\Phi_{xx} &= 0 \\
\Phi_\varrho &= cxH_\varrho = cx\int_t^T \mathrm{e}^{-r(s-t)} h_\varrho ds
\end{aligned}$$

$$\Phi_{x\varrho} = cH_\varrho = c\int_t^T \mathrm{e}^{-r(s-t)} h_\varrho ds$$
$$\Phi_{\varrho\varrho} = cxH_{\varrho\varrho} = cx\int_t^T \mathrm{e}^{-r(s-t)} h_{\varrho\varrho} ds$$
$$\Phi_t = cxH_t = cx\int_t^T \mathrm{e}^{-r(s-t)}(h_t + rh)ds - cxh(t,\varrho,t)$$
$$\Phi_z = x.$$

The partial derivatives of $h(t,\varrho,s)$ are given by

$$h_\varrho = \frac{1}{\varrho} h\mathrm{e}^{-\alpha(s-t)}$$
$$h_{\varrho\varrho} = \frac{1}{\varrho^2} h\left(\mathrm{e}^{-2\alpha(s-t)} - \mathrm{e}^{-\alpha(s-t)}\right)$$
$$h_t = M(s,t)h$$

where

$$M(s,t) = \mathrm{e}^{-\alpha(s-t)}\left[\alpha\ln(\varrho) - \mu - \sigma_u\sigma_Y\right] + \frac{1}{2}\sigma_u^2(\rho^2-2)\mathrm{e}^{-2\alpha(s-t)} - a_X \tag{6.8}$$

6.4. Calculation of the derivatives of ϕ and $\tilde{h}$ with the control $\widetilde{A}_2$.

$$\tilde{h} := \tilde{h}(t,\varrho,s) := \frac{1}{x}\mathbb{E}\left[X(T)R(s)\,|\,\mathbf{X}(t) = (x,\varrho,z)\right],$$
$$\tilde{H}(t,\varrho,T) := \int_t^T \mathrm{e}^{-r(s-t)}\tilde{h}(t,\varrho,s)ds$$

For the control $\widetilde{A}_2$ we have

$$\Phi(t,x,\varrho,z) = xz\mathrm{e}^{(\mathcal{N}-r)(T-t)} + cx\tilde{H}(t,\varrho,T). \tag{6.9}$$

By elementary differentiation we get

$$\Phi_x = z\mathrm{e}^{a_X-r)(T-t)} + c\int_t^T \mathrm{e}^{-r(T-t)}\tilde{h}ds$$
$$\Phi_{xx} = 0$$
$$\Phi_\varrho = cx\tilde{H}_\varrho = cx\int_t^T \mathrm{e}^{-r(T-t)}\tilde{h}_\varrho ds$$
$$\Phi_{x\varrho} = c\tilde{H}_\varrho = c\int_t^T \mathrm{e}^{-r(T-t)}\tilde{h}_\varrho ds$$
$$\Phi_{\varrho\varrho} = cx\tilde{H}_{\varrho\varrho} = cx\int_t^T \mathrm{e}^{-r(s-t)}\tilde{h}_{\varrho\varrho} ds$$
$$\Phi_t = xz\mathrm{e}^{(a_X-r)(T-t)}(r-a_X) + cx\tilde{H}_t = xz\mathrm{e}^{(a_X-r)(T-t)}(r-a_X)$$
$$+ cx\int_t^T \mathrm{e}^{-r(T-t)}(\tilde{h}_t + r\tilde{h})ds - cx\varrho\mathrm{e}^{(a_X-r)(T-t)}$$
$$\Phi_z = x\mathrm{e}^{(a_X-r)(T-t)}.$$

The partial derivatives of $\tilde{h}(t,\varrho,s)$ are given by

$$\tilde{h}_\varrho = \frac{1}{\varrho}\tilde{h}\mathrm{e}^{-\alpha(s-t)}$$
$$\tilde{h}_{\varrho\varrho} = \frac{1}{\varrho^2}\tilde{h}\left(\mathrm{e}^{-2\alpha(s-t)} - \mathrm{e}^{-\alpha(s-t)}\right)$$
$$\tilde{h}_t = M(s,t)\tilde{h}$$

where

$$M(s,t) = \mathrm{e}^{-\alpha(s-t)}\left[\alpha\ln(\varrho) - \mu - \sigma_u\sigma_Y\right] + \frac{1}{2}\sigma_u^2(\rho^2-2)\mathrm{e}^{-2\alpha(s-t)} - a_X$$

6.5. **Proof of Lemma** 3.7**.** We start with a result, stated in e.g. [3]. Let ψ_i^n, for $i=1,2$ be any simple predictable function. Then the process

$$\theta \mapsto \int_t^\theta \int_{\mathbb{R}\setminus\{0\}} \psi_i^n(s,\xi)d\tilde{N}(ds,d\xi) \tag{6.10}$$

is a square integrable martingale, that verifies the isometry formula.

Denote by

$$\Phi_1(s,X(s),R(s),Z(s)) := X(s)Z(s) + cX(s)H(s,R(s),T), \tag{6.11}$$

and

$$\Phi_2(s,X(s),R(s),Z(s)) := X(s)Z(s)\mathrm{e}^{(a_X^{(2)}-r)(T-s)} + cX(s)\tilde{H}(s,R(s),T), \tag{6.12}$$

where $a_X^{(i)}$ is associated with Φ_i. Define

$$\begin{aligned}\psi_i(s,\xi) &:= \mathrm{e}^{-rs}\left[\Phi_i(s,X(s)\mathrm{e}^{\xi},R(s),Z(s)) - \Phi_i(s,X(s),R(s),Z(s))\right]\\ &= e^{-rs}\left(\mathrm{e}^{\xi}-1\right)\Phi_i(s,X(s),R(s),Z(s)).\end{aligned}$$

If

$$\mathbb{E}\left[\int_t^\theta \int_{\mathbb{R}\setminus\{0\}} |\psi_i(s,\xi)|^2\nu(d\xi)ds \,|\, \mathbf{X}(t) = (x,\varrho,z)\right] < \infty, \tag{6.13}$$

holds for $i=1,2$. Then there exist a sequence (ψ_i^n) of simple predicitable functions such that (6.10) converge, in $L^2(\mathbf{P})$, to a process

$$\theta \mapsto \int_t^\theta \int_{\mathbb{R}\setminus\{0\}} \psi_i(s,\xi)d\tilde{N}(ds,d\xi). \tag{6.14}$$

The limiting process (6.14) is also a square integrable martingale, that verifies the isometry formula. See e.g. [3] or [7]. Thus, the martingale property of (3.19) follows if (6.13) holds. By Fubini's Theorem and since $r>0$, we have

$$\begin{aligned}&\mathbb{E}\left[\int_t^\theta \int_{\mathbb{R}\setminus\{0\}} |\psi_i(s,\xi)|^2\nu(d\xi)ds \,|\, \mathbf{X}(t) = (x,\varrho,z)\right]\\ &\leq \int_{\mathbb{R}\setminus\{0\}} |\mathrm{e}^{\xi}-1|\nu(d\xi)\mathbb{E}\left[\int_t^\theta |\Phi_i(s,X(s),R(s),Z(s))|^2 ds \,|\, \mathbf{X}(t) = (x,\varrho,z)\right]\end{aligned} \tag{6.15}$$

The first integral is finite by condition (3.2). For notational convenience, define

$$D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s) := \frac{\mu}{\alpha}(1-\mathrm{e}^{-\alpha(\tau-s)}) + \frac{\sigma_u^2}{4\alpha}(1-\mathrm{e}^{-2\alpha(\tau-s)})(2-\rho^2) + \frac{\sigma_u\sigma_Y}{\alpha}(1-\mathrm{e}^{-\alpha(\tau-s)}).$$

Then

$$\begin{aligned}\Phi_1(s,X(s),R(s),Z(s)) &:= X(s)Z(s)\\ &+ c\int_s^T X(s)R(s)^{\mathrm{e}^{-\alpha(\tau-s)}}\exp((a_X^{(1)}-r)(\tau-s))D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)d\tau,\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}\Phi_2(s,X(s),R(s),Z(s)) &:= \exp((a_X^{(2)}-r)(T-s))\\ &\times\left[X(s)Z(s) + c\int_s^T X(s)R(s)^{\mathrm{e}^{-\alpha(\tau-s)}}D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)d\tau\right].\end{aligned} \tag{6.17}$$

Recall that by assumption, $a_X^{(1)}-r<0$ and $a_X^{(2)}-r>0$. Hence,

$$|\Phi_1(s,X(s),R(s),Z(s))| \leq X(s)Z(s)$$

$$+ c\int_s^T X(s)R(s)^{\mathrm{e}^{-\alpha(\tau-s)}} D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)d\tau$$
$$\leq |\Phi_2(s,X(s),R(s),Z(s))|. \tag{6.18}$$

It follows from (6.15) that (6.13) holds for $i = 1, 2$ if

$$\mathbb{E}\left[\int_t^\theta |\Phi_2(s,X(s),R(s),Z(s))|^2 ds \,|\, \mathbf{X}(t) = (x,\varrho,z)\right] \tag{6.19}$$

is finite. We obtain,

$$\begin{aligned}
&|\Phi_2(s,X(s),R(s),Z(s))|^2 \\
&= \mathrm{e}^{2(a_X^{(2)}-r)(T-s)}\left[X(s)Z(s) + c\int_s^T X(s)R(s)^{\mathrm{e}^{-\alpha(\tau-s)}} D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)d\tau\right]^2 \\
&= \mathrm{e}^{2(a_X^{(2)}-r)(T-s)}\Big[X^2(s)Z^2(s) \\
&\quad + 2c\int_s^T X^2(s)Z(s)R(s)^{\mathrm{e}^{-\alpha(\tau-s)}} D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)d\tau \\
&\quad + c^2\int_s^T X(s)R(s)^{\mathrm{e}^{-\alpha(\tau_1-s)}} D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_1-s)d\tau_1 \\
&\quad\quad \times \int_s^T X(s)R(s)^{\mathrm{e}^{-\alpha(\tau_2-s)}} D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_2-s)d\tau_2\Big] \\
&= \mathrm{e}^{2(a_X^{(2)}-r)(T-s)}\Big[X^2(s)Z^2(s) \\
&\quad + 2c\int_s^T X^2(s)Z(s)R(s)^{\mathrm{e}^{-\alpha(\tau-s)}}\mathbf{1}(R(s)\geq 1) D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)d\tau \\
&\quad + 2c\int_s^T X^2(s)Z(s)R(s)^{\mathrm{e}^{-\alpha(\tau-s)}}\mathbf{1}(R(s)< 1) D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)d\tau \\
&\quad + c^2\int_s^T\int_s^T \Big[X^2(s)R(s)^{\mathrm{e}^{-\alpha(\tau_1+\tau_2-2s)}}\mathbf{1}(R(s)\geq 1) \\
&\quad\quad\quad \times D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_1-s)D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_2-s)\Big]d\tau_1 d\tau_2 \\
&\quad + c^2\int_s^T\int_s^T \Big[X^2(s)R(s)^{\mathrm{e}^{-\alpha(\tau_1+\tau_2-2s)}}\mathbf{1}(R(s)< 1) \\
&\quad\quad\quad \times D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_1-s)D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_2-s)\Big]d\tau_1 d\tau_2\Big].
\end{aligned}$$

Taking conditional expectation and using Fubini's Theorem and that $\mathrm{e}^{-\alpha(\cdot)} \in (0,1)$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[|\Phi_2(s,X(s),R(s),Z(s))|^2 \,|\, \mathbf{X}(t) = (x,\varrho,z)\right] \\
&\leq \mathrm{e}^{2(a_X^{(2)}-r)(T-s)}\Big[\mathbb{E}\left[X^2(s)Z^2(s) \,|\, \mathbf{X}(t) = (x,\varrho,z)\right] \\
&\quad + 2c\int_s^T \mathbb{E}\left[X^2(s)Z(s)R(s) \,|\, \mathbf{X}(t) = (x,\varrho,z)\right] D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)d\tau \\
&\quad + 2c\int_s^T \mathbb{E}\left[X^2(s)Z(s) \,|\, \mathbf{X}(t) = (x,\varrho,z)\right] D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)d\tau \\
&\quad + c^2\int_s^T\int_s^T \Big(\mathbb{E}\left[X^2(s)R(s) \,|\, \mathbf{X}(t) = (x,\varrho,z)\right]
\end{aligned}$$

$$\times D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_1-s)D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_2-s)\Big)d\tau_1 d\tau_2$$
$$+c^2\int_s^T\int_s^T\Big(\mathbb{E}\left[X^2(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]$$
$$\times D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_1-s)D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau_2-s)\Big)d\tau_1 d\tau_2\Big].$$

Note that

$$D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)\leq\exp\left(\frac{\mu}{\alpha}+\frac{\sigma_u^2}{4\alpha}+\frac{\sigma_u\sigma_Y}{\alpha}\right)\quad\text{if}\quad\mu\geq 0,$$
$$D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)\leq\exp\left(\frac{\sigma_u^2}{4\alpha}+\frac{\sigma_u\sigma_Y}{\alpha}\right)\quad\text{if}\quad\mu<0,$$

Define

$$\bar{D}[\mu,\alpha,\sigma_u,\sigma_Y,\rho]:=\max\left[\exp\left(\frac{\mu}{\alpha}+\frac{\sigma_u^2}{4\alpha}+\frac{\sigma_u\sigma_Y}{\alpha}\right),\exp\left(\frac{\sigma_u^2}{4\alpha}+\frac{\sigma_u\sigma_Y}{\alpha}\right)\right].$$

Then

$$D[\mu,\alpha,\sigma_u,\sigma_Y,\rho](\tau-s)\leq\bar{D}[\mu,\alpha,\sigma_u,\sigma_Y,\rho].$$

We obtain

$$\begin{aligned}&\mathbb{E}\left[|\Phi_2(s,X(s),R(s),Z(s))|^2\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\\&\leq \mathrm{e}^{2(a_X^{(2)}-r)(T-s)}\Big[\mathbb{E}\left[X^2(s)Z^2(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\\&\quad+2c(T-s)\bar{D}[\mu,\alpha,\sigma_u,\sigma_Y,\rho]\mathbb{E}\left[X^2(s)Z(s)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\\&\quad+2c(T-s)\bar{D}[\mu,\alpha,\sigma_u,\sigma_Y,\rho]\mathbb{E}\left[X^2(s)Z(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\\&\quad+c^2(T-s)^2\bar{D}^2[\mu,\alpha,\sigma_u,\sigma_Y,\rho]\mathbb{E}\left[X^2(s)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\\&\quad+c^2(T-s)^2\bar{D}^2[\mu,\alpha,\sigma_u,\sigma_Y,\rho]\mathbb{E}\left[X^2(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\Big].\end{aligned}\tag{6.20}$$

Recall that

$$Z(s)=cP(s)-A(s),$$

where $P(s)=\int_0^s R(v)dv$. Since A is non-decreasing and $A(s)\leq cP(s)$ we have

$$Z(s)\leq c\int_t^s R(v)dv+Z(t),$$
$$Z^2(s)\leq\left[c\int_t^s R(v)dv+Z(t)\right]^2=\frac{c}{2}\int_t^s R(u)Z(u)du+2cZ(t)\int_t^s R(v)dv+Z^2(t).\tag{6.21}$$

We now calculate the conditional expectations in (6.20). In the calculations below we will use

$$\mathbb{E}[\cdot\,|\,\mathbf{X}(t)]=\mathbb{E}[\cdot\,|\,\mathcal{F}_t].$$

For $\mathbb{E}\left[X^2(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]$:
Similar as in the proof of (ii) in Lemma 3.3 we get

$$\begin{aligned}\mathbb{E}\left[X^2(s)|\mathcal{F}_t\right]&=X^2(t)\mathbb{E}\left[\mathrm{e}^{2Y_0(s)}|\mathcal{F}_t\right]\mathbb{E}\left[\exp(\int_t^s 2\sigma_Y dB^Y(v))|\mathcal{F}_t\right]\\&=X^2(t)\exp(\ln(\phi(-2i))(s-t)).\end{aligned}\tag{6.22}$$

Since $X(t)=x$ we have

$$\mathbb{E}\left[X^2(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]=x^2\exp(\ln(\phi(-2i))(s-t)).\tag{6.23}$$

This is clearly positive and finite for $s\in[t,T]$.

For $\mathbb{E}\left[X^2(s)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]$:
Similar to the proof of (i) in Lemma 3.3 we obtain

$$\begin{aligned}
&\mathbb{E}\left[X^2(s)R(s)|\mathcal{F}_t\right]=\mathbb{E}\left[X^2(t)\mathrm{e}^{2Y(s)}\mathrm{e}^{U(s)}|\mathcal{F}_t\right]\\
&=X^2(t)R(t)^{\mathrm{e}^{-\alpha(s-t)}}\exp\left(\frac{\mu}{\alpha}\left(1-\mathrm{e}^{-\alpha(s-t)}\right)\right)\\
&\quad\times\mathbb{E}\left[\exp\left(\int_t^s 2\sigma_Y dB^Y(v)+\int_t^s\sigma_u\mathrm{e}^{-\alpha(s-t)}dB^u(v)\right)|\mathcal{F}_t\right]\\
&\quad\times\mathbb{E}\left[\mathrm{e}^{2Y_0(s)}|\mathcal{F}_t\right]\\
&=X^2(t)R(t)^{\mathrm{e}^{-\alpha(s-t)}}D[\mu,\alpha,\sigma_u,2\sigma_Y,\rho](s-t)\exp\left[\ln(\phi(-2i))(s-t)\right]\\
&=X^2(t)R(t)^{\mathrm{e}^{-\alpha(s-t)}}\mathbf{1}(R(t)\geq 1)D[\mu,\alpha,\sigma_u,2\sigma_Y,\rho](s-t)\exp\left[\ln(\phi(-2i))(s-t)\right]\\
&\quad+X^2(t)R(t)^{\mathrm{e}^{-\alpha(s-t)}}\mathbf{1}(R(t)<1)D[\mu,\alpha,\sigma_u,2\sigma_Y,\rho](s-t)\exp\left[\ln(\phi(-2i))(s-t)\right]\\
&\leq X^2(t)(R(t)+1)\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-t)\right] \qquad (6.24)
\end{aligned}$$

Since $X(t)=x, R(t)=\varrho$ we have

$$\mathbb{E}\left[X^2(s)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\leq x^2(\varrho+1)\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-t)\right] \qquad (6.25)$$

This is clearly positive and finite for $s\in[t,T]$.

For $\mathbb{E}\left[X^2(s)Z(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]$:
By (6.21) we obtain

$$\mathbb{E}\left[X^2(s)Z(s)|\mathcal{F}_t\right]\leq\mathbb{E}\left[X^2(s)c\int_t^s R(v)dv|\mathcal{F}_t\right]+Z(t)\mathbb{E}\left[X^2(s)|\mathcal{F}_t\right]. \qquad (6.26)$$

The last term is positive and finite by (6.23). Thus, by Fubini's Theorem and the tower property we obtain

$$\begin{aligned}
&\mathbb{E}\left[X^2(s)Z(s)|\mathcal{F}_t\right]\leq c\int_t^s\mathbb{E}\left[X^2(s)R(v)|\mathcal{F}_t\right]dv=c\int_t^s\mathbb{E}\left[R(v)\mathbb{E}\left[X^2(s)|\mathcal{F}_v\right]|\mathcal{F}_t\right]dv\\
&=c\int_t^s\mathbb{E}\left[X^2(v)R(v)|\mathcal{F}_t\right]\exp(\ln(\phi(-2i))(s-v))dv, \qquad (6.27)
\end{aligned}$$

where we used (6.22) with t replaced by v. By (6.24) we have

$$\mathbb{E}\left[X^2(v)R(v)|\mathcal{F}_t\right]\leq X^2(t)(R(t)+1)\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(v-t)\right]. \qquad (6.28)$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left[X^2(s)Z(s)|\mathcal{F}_t\right]\leq c\int_t^s X^2(t)(R(t)+1)\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp(\ln(\phi(-2i))(s-t))dv\\
&=cX^2(t)(R(t)+1)(s-t)\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp(\ln(\phi(-2i))(s-t)) \qquad (6.29)
\end{aligned}$$

Since $X(t)=x, R(t)=\varrho$ we have

(6.30)
$$\mathbb{E}\left[X^2(s)Z(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\leq cx^2(\varrho+1)(s-t)\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp(\ln(\phi(-2i))(s-t))$$

This is clearly positive and finite for $s\in[t,T]$.

For $\mathbb{E}\left[X^2(s)Z(s)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]$:
By (6.21), (6.25), Fubini's Theorem and the tower property we have

$$\begin{aligned}
&\mathbb{E}\left[X^2(s)Z(s)R(s)|\mathcal{F}_t\right]\leq\mathbb{E}\left[X^2(s)R(s)\int_t^s R(v)dv|\mathcal{F}_t\right]+Z(t)\mathbb{E}\left[X^2(s)R(s)|\mathcal{F}_t\right]\\
&\leq\int_t^s\mathbb{E}\left[R(v)\mathbb{E}\left[X^2(s)R(s)|\mathcal{F}_v\right]|\mathcal{F}_t\right]dv. \qquad (6.31)
\end{aligned}$$

As in (6.24) we have

$$\begin{aligned}&\mathbb{E}\left[X^2(s)R(s)|\mathcal{F}_v\right]\\&= X^2(v)R(v)^{\mathrm{e}^{-\alpha(s-v)}}D[\mu,\alpha,\sigma_u,2\sigma_Y,\rho](s-v)\exp\left[\ln(\phi(-2i))(s-v)\right].\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{E}\left[X^2(s)Z(s)R(s)|\mathcal{F}_t\right]\\&\leq \int_t^s \mathbb{E}\left[X^2(v)R(v)^{(1+\mathrm{e}^{-\alpha(s-v)})}|\mathcal{F}_t\right]D[\mu,\alpha,\sigma_u,2\sigma_Y,\rho](s-v)\exp\left[\ln(\phi(-2i))(s-v)\right]dv.\\&= \int_t^s \Bigg(\mathbb{E}\left[X^2(v)R(v)^{(1+\mathrm{e}^{-\alpha(s-v)})}\mathbf{1}(R(v)\geq 1)|\mathcal{F}_t\right]\\&\qquad\qquad \times D[\mu,\alpha,\sigma_u,2\sigma_Y,\rho](s-v)\exp\left[\ln(\phi(-2i))(s-v)\right]\Bigg)dv.\\&\quad+\int_t^s \Bigg(\mathbb{E}\left[X^2(v)R(v)^{(1+\mathrm{e}^{-\alpha(s-v)})}\mathbf{1}(R(v)< 1)|\mathcal{F}_t\right]\\&\qquad\qquad \times D[\mu,\alpha,\sigma_u,2\sigma_Y,\rho](s-v)\exp\left[\ln(\phi(-2i))(s-v)\right]\Bigg)dv.\\&\leq \int_t^s \mathbb{E}\left[X^2(v)R(v)^2|\mathcal{F}_t\right]\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-v)\right]dv\\&\quad+\int_t^s \mathbb{E}\left[X^2(v)|\mathcal{F}_t\right]\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-v)\right]dv.\end{aligned}$$

The last term is positive and finite by (6.23). For $\mathbb{E}\left[X^2(v)R(v)^2|\mathcal{F}_t\right]$ we obtain similar to the derivation of (6.24) that

$$\begin{aligned}&\mathbb{E}\left[X^2(v)R(v)^2|\mathcal{F}_t\right]\\&= X^2(t)R(t)^{2\mathrm{e}^{-\alpha(v-t)}}D[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho](v-t)\exp\left[\ln(\phi(-2i))(v-t)\right]\\&\leq X^2(t)R^2(t)\mathbf{1}(R(v)\geq 1)D[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho](v-t)\exp\left[\ln(\phi(-2i))(v-t)\right]\\&\quad+X^2(t)\mathbf{1}(R(v)<1)D[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho](v-t)\exp\left[\ln(\phi(-2i))(v-t)\right]\\&\leq X^2(t)(R^2(t)+1)\bar{D}[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(v-t)\right].\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{E}\left[X^2(s)Z(s)R(s)|\mathcal{F}_t\right]\\&\leq \int_t^s X^2(t)(R^2(t)+1)\bar{D}[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho]\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-t)\right]\\&= X^2(t)(R^2(t)+1)(s-t)\bar{D}[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho]\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-t)\right]\end{aligned}$$

Since $X(t)=x, R(t)=\varrho$ we have

$$\begin{aligned}&\mathbb{E}\left[X^2(s)Z(s)R(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right]\\&\leq x^2(t)(\varrho^2(t)+1)(s-t)\bar{D}[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho]\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-t)\right]\end{aligned}\tag{6.32}$$

This is clearly positive and finite for $s\in[t,T]$.

For $\mathbb{E}\left[X^2(s)Z^2(s)|\mathcal{F}_t\right]$ **:**
Again, by (6.21) and Fubini's Theorem we have

$$\begin{aligned}\mathbb{E}\left[X^2(s)Z^2(s)|\mathcal{F}_t\right]\leq \frac{c}{2}\int_t^s \mathbb{E}\left[X^2(s)R(v)Z(v)|\mathcal{F}_t\right]dv\\+2cZ(t)\int_t^s \mathbb{E}\left[X^2(s)R(v)|\mathcal{F}_t\right]dv\\+Z^2(t)\mathbb{E}\left[X^2(s)|\mathcal{F}_t\right]dv.\end{aligned}$$

The last two terms is obtained to be positive and finite via the tower property and (6.25) and (6.23). Hence,

$$\begin{aligned}
&\mathbb{E}\left[X^2(s)Z^2(s)|\mathcal{F}_t\right] \leq \frac{c}{2}\int_t^s \mathbb{E}\left[X^2(s)R(v)Z(v)|\mathcal{F}_t\right]dv \\
&= \frac{c}{2}\int_t^s \mathbb{E}\left[R(v)Z(v)\mathbb{E}\left[X^2|\mathcal{F}_v\right]|\mathcal{F}_t\right]dv \\
&= \frac{c}{2}\int_t^s \exp\left[\ln(\phi(-2i))(s-v)\right]\mathbb{E}\left[X^2(v)R(v)Z(v)|\mathcal{F}_t\right]dv \\
&\leq \frac{c}{2}X^2(t)(R^2(t)+1)\bar{D}[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho]\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-t)\right]\int_t^s(v-t)dv \\
&= \frac{c}{4}X^2(t)(R^2(t)+1)(s-t)^2\bar{D}[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho]\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-t)\right].
\end{aligned}$$

We get,

$$\begin{aligned}
&\mathbb{E}\left[X^2(s)Z^2(s)\,|\,\mathbf{X}(t)=(x,\varrho,z)\right] \\
&\leq \frac{c}{4}x^2(\varrho^2+1)(s-t)^2\bar{D}[2\mu,\alpha,2\sigma_u,2\sigma_Y,\rho]\bar{D}[\mu,\alpha,\sigma_u,2\sigma_Y,\rho]\exp\left[\ln(\phi(-2i))(s-t)\right]
\end{aligned} \tag{6.33}$$

This is clearly positive and finite for $s \in [t,T]$. All the terms in (6.20) are finite and continuous in s. The square integrability of (6.15) follows, and (6.13) is indeed finite. It follows that the process

$$\theta \mapsto \int_t^\theta \int_{\mathbb{R}} \mathrm{e}^{-rs}\left[\Phi(s,X(s)\mathrm{e}^{\xi},R(s),Z(s)) - \Phi(s,X(s),R(s),Z(s))\right]\widetilde{N}(ds,d\xi)$$

is a martingale. To see that the processes

$$\theta \mapsto \int_t^\theta \mathrm{e}^{-rs}\Phi_x(s,X(s),R(s),Z(s))\sigma_Y X(s)dB^Y(s)$$

$$\theta \mapsto \int_t^\theta \mathrm{e}^{-rs}\Phi_\varrho(s,X(s),R(s),Z(s))\sigma_u R(s)dB^u(s)$$

are martingales, note that

$$\begin{aligned}
&\int_t^\theta \mathrm{e}^{-rs}\frac{\partial\Phi_i}{\partial x}(s,X(s),R(s),Z(s))\sigma_Y X(s)dB^Y(s) \\
&= \int_t^\theta \mathrm{e}^{-rs}\Phi_i(s,X(s),R(s),Z(s))\sigma_Y X(s)dB^Y(s).
\end{aligned}$$

and that

$$R(s)\frac{\partial\Phi_i}{\partial\varrho}(s,X(s),R(s),Z(s)) \leq \Phi_i(s,X(s),R(s),Z(s)). \tag{6.34}$$

Hence, the martingale property follows, as for (3.19), directly from the finitness of (6.19). The result of the Lemma follows.

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