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GRADUATION BY MOVING AVERAGES

by

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Summary

The stochastic variables $X_1, X_2, \ldots, X_N$ are observed values of a given quantity at N distinct points of time. A diagram of $\{X_t\}$, plotted against t, will have quite a ragged apperance. We assume, however, that the "real" quantity follows a "smooth" curve, and that any irregularities of the observed curve are due to accidential circumstances. Therefore, we want to graduate $\{X_t\}$ to get a "smoother" curve.

There exists quite a lot of granduation methods. In the present report we discuss graduation by moving averages. This is a more than hundred years old method which estimates the actual quantity by

$$\hat{\xi}_t = \sum_{\nu=\alpha}^{\beta} r_\nu X_{t-\tau+\nu}$$

for $t = \tau-\alpha+1, \tau-\alpha+2, \ldots, N+\tau-\beta$, where $\alpha \leq \tau \leq \beta$ and the weights $r_\alpha, r_{\alpha+1}, \ldots, r_\beta$ are known numbers. Two important problems in moving average graduation are how to choose the "best possible" weights $r_\alpha, r_{\alpha+1}, \ldots, r_\beta$ and the "best possible" centre τ for given range (α, β).

In this report we discuss these problems, and we give a criterion for judging the properties of moving averages. Furthermore, we derive moving averages which are optimal according to this criterion under general assumptions. When the X_t's are uncorrelated and have equal variance, our optimal moving averages generalize two well known optimal moving averages: The minimum-variance and the minimum-R_z moving averages. These are constructed to minimize $\sum_\nu r_\nu^2$ and $\sum_\nu (\Delta^z r_\nu)^2$, respectively, under certain constraints. Here Δ is the usual difference operator.

For the situation with uncorrelated observations with equal variance, we thoroughly discuss the optimal moving averages theoretically and by means of Monte Carlo experiments. These investigations indicate that our generalization of the well-known optimal moving averages is not only of theoretical, but also of practical interest.

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12	5 ↓	contant with the risk	content with the risk
13	1 ↑	$r_\beta z^{\beta-\alpha}+r_{\beta-1}+z^{\beta-\alpha-1}+\dots$	$r_\beta z^{\beta-\alpha}+r_{\beta-1}z^{\beta-\alpha-1}+\dots$
15	1 ↑	will make the form	will make the term
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1. INTRODUCTION

1.A. We have stochastic variables $X_1, X_2, \ldots, X_N$, which are observations of a given quantity at N different points of time. A diagram of $\{X_t\}$, plotted against t, will have quite a ragged appearance. We assume, however, that $\xi_t = EX_t$ follows a smooth curve as a function of t. Therfore, we want to graduate $\{X_t\}$ to get a smoother curve. Or in other words: We want to estimate the <u>sequence</u> $\{\xi_t\}$.

The situation described above arises e.g. in demography and actuarial science where one wants to graduate vital rates like mortality and fertility rates.

1.B. There exists quite a lot of graduation methods. Miller (1946, p.7) groups them into graphical methods, interpolation methods, difference-equation methods, analytic graduation and graduation by moving averages. Most of these methods have, at least originally, been developed by intuitive and heuristic reasoning and by experience with the methods in practical work. The statistical properties of some of the methods have later on been investigated. Hoem (1972) discusses analytic graduation and gives references to several other works on different graduation methods.

1.C. In this report we will discuss moving average as a graduation method *. This method estimates ξ_t by

$$(1.1) \qquad \hat{\xi}_t = \sum_{\nu=\alpha}^{\beta} r_\nu X_{t-\tau+\nu}$$

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* Moving averages are also used in other connections, e.g. to eliminate the periodic component in time series analysis. Such uses of moving averages are not the subject of this report.

for $t = \tau - \alpha+1,\ \tau - \alpha+2, \ldots, N+\tau - \beta$, where $\alpha \leq \tau \leq \beta$ and the weights $r_\alpha, r_{\alpha+1}, \ldots, r_\beta$ are known real numbers.

In connection with (1.1) we define: The moving average is centralized if $\tau-\alpha = \beta-\tau$. It is symmetric if we in addition have $r_{\tau-\nu} = r_{\tau+\nu}$ for all ν. (α,β) is called the range of the moving average and τ its centre. The length of the moving average is $l = \beta - \alpha+1$. Finally the moving average (1.1) is said to be exact for a function φ if

$$\varphi(t) = \sum_{\nu=\alpha}^{\beta} r_\nu \, \varphi(t - \tau+\nu) \tag{1.2}$$

for $t = \tau - \alpha+1,\ \tau - \alpha+2, \ldots, N+\tau - \beta$.

1.D. Moving averages have been used for more than a hundred years to graduate mortality tables. Historically we have three different types of moving averages: The summation formulas, the minimum-variance and the minimum-R_z moving averages.

The summation formulas were developed, chiefly by British actuaries, in the years 1870 to 1910. At that time the available calculating equipment was very primitive and, consequently, it was computationally very advantageous to employ more additions and fewer multiplications. Therefore, these moving averages are obtained by superimposing several unweighted summations, together with an adjustment to make the formulas exact for cubics. The probably most celebrated of these moving averages is the Spencer's 21-term formula. This may be given by:

$$\begin{aligned} \hat{\xi}_t = \frac{1}{350}(&- X_{t-10} - 3X_{t-9} - 5X_{t-8} - 5X_{t-7} - 2X_{t-6} \\ &+ 6X_{t-5} + 18X_{t-4} + 33X_{t-3} + 47X_{t-2} + 57X_{t-1} \\ &+ 60X_t + \ldots), \end{aligned} \tag{1.3}$$

where the dots indicate that the moving average is symmetric. For a thorough discussion of the Spencer's formula and the summation formulas in general we refer to Whittaker and Robinson (1924, pp.288-290).

The American E.L. De Forest did in the years 1870-80, independent of the British actuaries, a series of works on graduation. These works were published in obscure places and therefore were little noticed or used until attention was drawn to them by Wolfenden (1924). De Forest constructed moving averages, exact for cubics, which minimize

$$(1.4) \qquad R_z^2 = \binom{2z}{z}^{-1} \sum_{\nu=\alpha-z}^{\beta} (\Delta^z r_\nu)^2$$

for $z = 0$ and $z = 4$. Here Δ is the usual difference operator and we use the convention that $r_\nu = 0$ for $\nu < \alpha$ or $\nu > \beta$. (A motivation for minimizing R_z^2 is given by Miller, 1946, pp.31-33.) Independent of De Forest, Hardy later on suggested that one might construct moving averages which minimize R_0^2, R_2^2 or R_3^2 (see Wolfenden, 1924, p.108).

In a series of important works Sheppard (1913 , 1914a , 1914b , 1914c , 1915 , 1921) followed Hardy's suggestion. He found moving averages, exact for polynomials of given degree, that minimize R_0^2, and he showed the close connection between these moving averages and the method of least squares. (We shall have a closer look at this in chapter 3.) Furthermore, Sheppard found general expressions for moving averages, exact for polynomials of given degree, which minimize R_2^2 and R_3^2. But, as Hardy, he did not find these moving averages "convenient for practical purposes" (Sheppard, 1915, p.151) because of their complex form.

While De Forest, Hardy and Sheppard all discussed both moving averages that minimize R_0^2 and moving averages that minimize R_z^2 for $z>0$, this is not the case for many later authors. Most European authors after Sheppard only discuss moving averages that minimize R_0^2. This historical division between these moving averages makes it convenient to look upon them as two different types of moving averages. We will denote the moving averages which minimize R_0^2 and R_z^2, for $z>0$, for minimum-variance and minimum-R_z moving averages respectively. (In some connections, however, we will say minimum-R_0 moving averages for the minimum-variance moving averages.)

The minimum-variance moving averages have been given a thorough statistical treatment by many authors. Among important works from the last decades we may mention Kendall (1946), Jecklin and Strickler (1954), Weichselberger (1964), Sverdrup (1967) and Anderson (1971). To day the statistical theory of these moving averages are well developed. It is based on the method of least squares. Furthermore, the minimum-variance moving averages are constructed not only exact for polynomials, but also for other functions.

While the minimum-R_z moving averages are little used and discussed in Europe, they are much in use in North-America. These moving averages are, as far as the present author knows*,

* When the manuscript of this report was ready for typing, the authors attention was drawn to a paper by Gerber (1977). He shows how one may construct minimum-R_z moving averages exact for other functions than polynomials. For this purpose he minimizes a quadratic form analogous to what we do in chapter 3 in this report. For the minimization Gerber uses a result similar to our lemma 3.5. However, even if some of the results given in this report are similar to results given by Gerber, his work does not change the main ideas in this report. Therefore, we have not found it necessary to rewrite the manuscript.

up to now only constructed exact for polynomials, and their theory is in little extent expressed in statistical terms. Important contributions to the theory of the minimum-R_z moving averages are in the last decades given e.g. by Greville (1947, 1957, 1966, 1972, 1974) and Pollard (1971a, 1971b).

An interesting attempt to combine the philosophy behind the minimum-variance and the minimum-R_z moving averages is given by Michalup (1956). He finds the symmetric moving averages of length up to 9, exact for cubics, which minimize $AR_0^2 + BR_3^2$ for given A and B.

I.E. One main problem in graduation by moving average is the following: What is the "best" choise of the weights $r_\alpha, r_{\alpha+1}, \ldots, r_\beta$? Close related to this is the question how we "best" possible may choose the centre τ for given range (α,β).

To answer these questions we have to know what to mean by "best". Thus, we have to ask the fundamental question: What is the purpose of the graduation? We discuss this question in Chapter 2, and we try to answer it by defining a loss function for judging the graduation. The risk, i.e. the expected loss, consists of two components. One depends on the <u>trend</u> $\{\xi_t\}$ and the other on the <u>erratic component</u> $V_t = X_t - \xi_t$ *. These two quantities are studied separately. In connection with the first we also discuss for which functions a moving average is exact.

In Chapter 3 we show how we may find moving averages, exact

* The terms trend and erratic component are used according to Sverdrup (1967).

for a given function, which minimize the second component of the risk among moving averages with given range and centre. In the case where the observations are uncorrelated with equal variance, moving averages that are optimal in this sense generalize the minimum-variance and the minimum-R_z moving averages.

The important case where $X_1, X_2, \ldots, X_N$ are uncorrelated with equal variance is thoroughly discusses in Chapter 4. We give a short review of the known theory of the minimum-variance and the minimum-R_z moving averages and find some new results for these. It is worth mentioning that we construct minimum-R_z moving averages exact for other functions than polynomials*. Further, in this chapter we find moving averages that generalize Michalup's (1956) results. In connection with the actual moving averages we also discuss different choices of the centre. Previously, as far as the present author knows, such a discussion is only done for minimum-variance moving averages exact for polynomials (see Weichselberger, 1964, Kockelkorn and Rüger, 1974).

In Chapter 5 we compare different moving averages theoretically and by means of Monte Carlo experiments. These investigations seem to imply that our generalization of the well-known optimal moving averages not only is of theoretical, but also practical interest. In this chapter we also find the interesting result that Spencer's 21-terms formula (1.3) is approximately equal to the corresponding minimum-R_5 moving average.

By graduation with the moving average (1.1) we only get estimates $\hat{\xi}_t$ for the central values $t = \tau-\alpha+1, \tau-\alpha+2, \ldots, N+\tau-\beta$.

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* See footnote on page 4.

Therefore, an important problem is how we may find estimates for the end values $t = 1,2,\ldots,\tau-\alpha$ and $\tau = N+\tau-\beta+1, N+\tau-\beta+2,\ldots,N$. This problem is not considered in this report. In the last chapter we comment upon this and some other unsolved problems.

As seen in the historical review in the preceeding paragraph, there does not exist a unified statistical theory for moving average as a graduation method. In addition to the new results we give in this report, we hope this work also will give a contribution to such a theory.

2. THE GENERAL THEORY OF GRADUATION BY MOVING AVERAGES.

2.A. In this chapter we first clarify our statistical model. Then we discuss the fundamental question: What is the purpose of graduating the sequence $\{X_t\}$? This discussion justifies that the graduation of sequence of observations may be judged by the loss function (2.4). The risk, i.e. the expected loss, consists of two components, depending on the trend and the erratic component respectively. These two components are studied separately. In this connection we also discuss for which functions a moving average is exact.

2.B. We observe a given quantity at N different points of time. The observed values are $X_1, X_2, \ldots, X_N$. We suppose that the moments of first and second order exist, and we define

$$\underset{\sim}{X} = (X_1, X_2, \ldots, X_N)'$$

$$\underset{\sim}{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)'$$

$$\underset{\sim}{\Sigma} = \text{Covm}\ (\underset{\sim}{X}) = E(\underset{\sim}{X} - \underset{\sim}{\xi})(\underset{\sim}{X} - \underset{\sim}{\xi})'.$$

Here the prime denotes a transpose. We do not specify the model any further.

We let $\underset{\sim}{V} = \underset{\sim}{X} - \underset{\sim}{\xi}$ denote the erratic component. Then, of course, $E\underset{\sim}{V} = \underset{\sim}{0}$ and $\text{Covm}\ (\underset{\sim}{V}) = \underset{\sim}{\Sigma}$.

In connection with the moving average (1.1) we introduce the vector

$$\underset{\sim}{r} = (r_\alpha, r_{\alpha+1}, \ldots, r_\beta)' \tag{2.1}$$

and the $(N-\beta+\alpha) \times N$ matrix

$$(2.2) \qquad \underset{\sim}{R} = \begin{pmatrix} r_\alpha & r_{\alpha+1} & r_{\alpha+2} & \cdots & r_\beta & 0 & 0 & 0 & \cdots & 0 \\ 0 & r_\alpha & r_{\alpha+1} & \cdots & r_{\beta-1} & r_\beta & 0 & 0 & \cdots & 0 \\ 0 & 0 & r_\alpha & \cdots & r_{\beta-2} & r_{\beta-1} & r_\beta & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & r_\alpha & r_{\alpha+1} & r_{\alpha+2} & & r_\beta & 0 \\ 0 & 0 & \cdots & 0 & 0 & r_\alpha & r_{\alpha+1} & \cdots & r_{\beta-1} & r_\beta \end{pmatrix} .$$

In this report we will only discuss how to estimate ξ_t for the central values $t = \tau-\alpha+1,\ \tau-\alpha+2,\ldots,N+\tau-\beta$. Therefore if we define:

$$\hat{\underset{\sim}{\xi}} = (\hat{\xi}_{\tau-\alpha+1}, \hat{\xi}_{\tau-\alpha+2},\ldots,\hat{\xi}_{N+\tau-\beta})'$$

we have:

$$(2.3) \qquad \hat{\underset{\sim}{\xi}} = \underset{\sim}{R}\underset{\sim}{X} = \underset{\sim}{R}\underset{\sim}{\xi} + \underset{\sim}{R}\underset{\sim}{V} .$$

We will in the sequel feel free to also denote (2.1) and (2.2) a moving average.

2.C. As seen in Paragraph 1.D., there are in the literature different opinions of the purpose of graduating a sequence of observations by moving average. The two main points of view are reflected in the two most used criteria for optimal moving averages: Minimize R_0^2 or minimize R_z^2, usually with $z=3$, under certain constraints. These optimal moving averages are constructed to give as good "fit" and "smoothness", respectively, as possible. Michalup (1956) tries to balance these two requirements.

The present author believes that different requirements must be taken into account when graduating. We want $\hat{\xi}_t$ to be a good estimator for ξ_t for each t, that the sequences $\{\hat{\xi}_t\}$ and $\{\xi_t\}$ shall have nearly the same gradients and curvatures etc. Altogether we want the sequence $\{\hat{\xi}_t\}$ of estimates to agree with the sequence $\{\xi_t\}$ of means in a general way. How successfully this is obtained may be measured by the quantities $(\Delta^z\hat{\xi}_t - \Delta^z\xi_t)^2$; $t = \tau-\alpha+1, \tau-\alpha+2, \ldots, N+\tau-\beta-z$, $z = 0,1,2,\ldots$. Small values for $z = 0$ indicate good "fit", small values for $z = 1$ good reproduction of the gradient etc.

From this discussion it follows that we may judge the graduation of a sequence of observations by a loss function of the form

$$\sum_{z=0}^{K} \sum_{t=\tau-\alpha+1}^{N+\tau-\beta-z} a_{tz} (\Delta^z\hat{\xi}_t - \Delta^z\xi_t)^2 ,$$

where $K \leq N-\beta+\alpha-1$ and the weights a_{tz} are non-negative real numbers. In this report we will only consider the case where the a_{tz}'s are equal independent of t. Then, if we put $a_z = \sum_t a_{tz}$, the loss function may be written

$$(2.4) \qquad T_{\underset{\sim}{a}}(\underset{\sim}{\xi},\hat{\underset{\sim}{\xi}}) = \sum_{z=0}^{K} \frac{a_z}{N-\beta+\alpha-z} \sum_{t=\tau-\alpha+1}^{N+\tau-\beta-z} (\Delta^z\hat{\xi}_t - \Delta^z\xi_t)^2 ,$$

where $\underset{\sim}{a} = (a_0, a_1, \ldots, a_K)$ is a vector of non-negative real numbers. We will always let $\sum_{z=0}^{K} a_z = 1$. We note that the choice of $\underset{\sim}{a}$ depends on what we aim at by the graduation: a_0 is the weight for the "fit", a_1 for the gradient, a_2 for the curvature, a_3 for the "smoothness", etc.

It is of interest to note that in the case of analytic graduation a loss function as described above is of no use. In this case the requirements leading to (2.4) is apriori satisfied by the choice of graduation function.

We will give (2.4) in matrix notation. Introduce the $(N-\beta+\alpha)\times N$ matrix

$$\underset{\sim}{B} = (\underset{\sim}{O}^{(N-\beta+\alpha)\times(\tau-\alpha)}\ \underset{\sim}{I}^{(N-\beta+\alpha)}\ \underset{\sim}{O}^{(N-\beta+\alpha)\times(\beta-\tau)}),$$

where $\underset{\sim}{O}^{m\times n}$ is the $m\times n$ zero matrix and $\underset{\sim}{I}^{n}$ is the $n\times n$ identity matrix, and the $(N-\beta+\alpha-z)\times(N-\beta+\alpha)$ matrix

$$\underset{\sim}{D}_z = \begin{pmatrix} \binom{z}{0}(-1)^z & \binom{z}{1}(-1)^{z-1} & \cdots & \binom{z}{z}(-1)^0 & 0 & 0\dots0 \\ 0 & \binom{z}{0}(-1)^z & \cdots & \binom{z}{z-1}(-1)^1 & \binom{z}{z}(-1)^0 & 0\dots0 \\ \vdots & & \ddots & & \ddots & \vdots \\ & & & & & 0 \\ 0 & \cdots & 0 & \binom{z}{0}(-1)^z & \cdots & \binom{z}{z}(-1)^0 \end{pmatrix}.$$

Then we find

$$\underset{\sim}{D}_z\underset{\sim}{B}\underset{\sim}{\xi} = (\Delta^z\xi_{\tau-\alpha+1}, \Delta^z\xi_{\tau-\alpha+2}, \dots, \Delta^z\xi_{N+\tau-\beta-z})' ,$$

$$\underset{\sim}{D}_z\hat{\xi} = (\Delta^z\hat{\xi}_{\tau-\alpha+1}, \Delta^z\hat{\xi}_{\tau-\alpha+2}, \dots, \Delta^z\hat{\xi}_{N+\tau-\beta-z})' ,$$

and the loss function (2.4) may be written

$$(2.5)\qquad \underset{\sim}{T}_a(\underset{\sim}{\xi},\hat{\underset{\sim}{\xi}}) = (\hat{\underset{\sim}{\xi}}-\underset{\sim}{B}\underset{\sim}{\xi})'\,\underset{\sim}{T}_a(\hat{\underset{\sim}{\xi}}-\underset{\sim}{B}\underset{\sim}{\xi}) ,$$

where $\underset{\sim}{T}_a$ is the $(N-\beta+\alpha)\times(N-\beta+\alpha)$ matrix

$$(2.6)\qquad \underset{\sim}{T}_a = \sum_{z=0}^{K} \frac{a_z}{N-\beta+\alpha-z}\,\underset{\sim}{D}_z'\,\underset{\sim}{D}_z .$$

It is easy to prove that $\underset{\sim}{T}_a$ is non-negative definite and positive definite if and only if $a_0 > 0$.

If the trend $\{\xi_t\}$ was known, we could judge the graduation of a sequence of observations by the loss function (2.4) or (2.5). Since this is usually not the case we have to be contant with the risk, i.e. the expected loss. Thus, we want to graduate the sequence of observations such that the risk

$$(2.7) \qquad r(\underset{\sim}{\xi},\underset{\sim}{\Sigma},\underset{\sim}{R}) = ET_{\underset{\sim}{a}}(\underset{\sim}{\xi},\hat{\underset{\sim}{\xi}})$$

is as small as possible.

By (2.3) and $E\underset{\sim}{V} = \underset{\sim}{0}$ we find

$$(2.8) \qquad r(\underset{\sim}{\xi},\underset{\sim}{\Sigma},\underset{\sim}{R}) = (\underset{\sim}{R}\underset{\sim}{\xi}-\underset{\sim}{B}\underset{\sim}{\xi})'\underset{\sim}{T}_a(\underset{\sim}{R}\underset{\sim}{\xi}-\underset{\sim}{B}\underset{\sim}{\xi}) + E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W} ,$$

where $\underset{\sim}{W} = \underset{\sim}{R}\underset{\sim}{V}$. This shows that the risk consists of two components. One depends on how the moving average operates on the trend, the other on how it operates on the erratic component.

In the succeeding paragraphs we shall study these two components more in detail.

2.D. In this paragraph we will have a closer look at the first term in the risk function (2.8). Since $\underset{\sim}{T}_a$ is positive definite for $a_0 > 0$, a necessary and sufficient condition for $(\underset{\sim}{R}\underset{\sim}{\xi}-\underset{\sim}{B}\underset{\sim}{\xi})'\underset{\sim}{T}_a(\underset{\sim}{R}\underset{\sim}{\xi}-\underset{\sim}{B}\underset{\sim}{\xi})$ to vanish for all vectors $\underset{\sim}{a}$ is

$$\underset{\sim}{R}\underset{\sim}{\xi} = \underset{\sim}{B}\underset{\sim}{\xi} ,$$

or equivalently

$$\xi_t = \sum_{\nu=\alpha}^{\beta} r_\nu \, \xi_{t-\tau+\nu}$$

for $t = \tau-\alpha+1, \tau-\alpha+2, \ldots, N+\tau-\beta$. That is, the moving average has to be exact for the trend $\{\xi_t\}$. Thus, we want to study for which functions a given moving average is exact, and which moving averages that are exact for a given function (or rather class of functions). Since these questions are studied extensively, in the literature (see e.g. Elphinstone, 1951 , Jecklin and Strickler, 1954, and Sverdurp, 1967) we will here only give a brief review of the most important results.

By (1.2) the moving average (1.1) is exact for a function φ if and only if

$$r_\beta \varphi(t-\tau+\beta)+r_{\beta-1}\varphi(t-\tau+\beta-1)+\ldots+(r_\tau-1)\varphi(t)+\ldots+r_\alpha\varphi(t-\tau+\alpha) = 0 \tag{2.9}$$

for $t = \tau-\alpha+1, \tau-\alpha+2, \ldots, N+\tau-\beta$. This shows that a moving average is exact for φ if and only if $\{\varphi(t)\}$, for integer-valued t , is a solution of the homogeneous linear difference equation (2.9). Hence, by the theory of linear difference equations (see e.g. Henrici, 1964, pp. 137-140) we get the following well-known result:

Theorem 2.10: The moving average (1.1) is exact for a function φ if and only if φ for all integer-valued t may be written

$$\varphi(t) = \sum_{j=1}^{p} \sum_{m=0}^{h_j-1} \theta_{jm} t^m c_j^{\,t} , \tag{2.11}$$

where $c_1, c_2, \ldots, c_p$ are the (real or complex) distinct roots of the characteristic equation

$$r_\beta z^{\beta-\alpha}+r_{\beta-1}z^{\beta-\alpha-1}+\ldots+(r_\tau-1)z^{\tau-\alpha}+\ldots+r_\alpha = 0 \tag{2.12}$$

with multiplicity $h_1, h_2, \dots, h_p$ respectively and the θ_{jm}'s are unknown parameters.

This theorem characterizes the functions for which a given moving average is exact. We will now show that if φ is of the form (2.11) for arbitrary $c_1, c_2, \dots, c_p$, $h_1, h_2, \dots, h_p$ we can always find a moving average that is exact for φ, and we will find which conditions the moving average has to fulfil for this to be true. The following lemma is useful.

Lemma 2.13: A moving average $\underset{\sim}{r} = (r_\alpha, r_{\alpha+1}, \dots, r_\beta)'$ with range (α, β) and centre τ is exact for $t^m c^t$ if and only if

$$(2.14) \qquad \sum_{\nu=\alpha}^{\beta} r_\nu \nu^n c^\nu = \tau^n c^\tau$$

for $n = 0, 1, \dots, m$.

Proof: By (1.2) $\underset{\sim}{r}$ is exact for $t^m c^t$ if and only if

$$t^m c^t = \sum_{\nu=\alpha}^{\beta} r_\nu (t-\tau+\nu)^m c^{t-\tau+\nu} ,$$

or equivalently

$$c^\tau t^m = \sum_{\nu=\alpha}^{\beta} r_\nu (t-\tau+\nu)^m c^\nu = \sum_{j=0}^{m} \binom{m}{j} \left\{ \sum_{\nu=\alpha}^{\beta} r_\nu (\nu-\tau)^{m-j} c^\nu \right\} t^j$$

for $t = \tau-\alpha+1, \tau-\alpha+2, \dots, N+\tau-\beta$. If we compare the coefficients of t^j for $j = 0, 1, \dots, m$ this gives

$$\sum_{\nu=\alpha}^{\beta} r_\nu (\nu-\tau)^k c^\nu = \delta_{ko} c^\tau \text{ for } k = 0, 1, \dots, m ,$$

where δ_{ij} is the Kronecker delta symbol. This is again equivalent to

$$\sum_{\nu=\alpha}^{\beta} r_\nu \nu^n c^\nu = \tau^n c^\tau \quad \text{for } n = 0, 1, \dots, m$$

□

From this lemma we immediately find:

Theorem 2.15: Let $c_1, c_2, \ldots, c_p$ be distinct real or complex numbers and $h_1, h_2, \ldots, h_p$ integers. Suppose the real function φ for integer-valued t may be written

$$(2.16) \qquad \varphi(t) = \sum_{j=1}^{p} \sum_{m=0}^{h_j-1} \theta_{jm} t^m c_j^t ,$$

where the θ_{jm}'s are unknown parameters. Then a moving average $\underset{\sim}{r} = (r_\alpha, r_{\alpha+1}, \ldots, r_\beta)'$ with range (α, β) and centre τ is exact for φ if and only if

$$(2.17) \qquad \sum_{\nu=\alpha}^{\beta} r_\nu \nu^n c_j^{\ \nu} = \tau^n c_j^{\ \tau}$$

for $n = 0, 1, \ldots, h_j - 1$; $j = 1, 2, \ldots, p$.

Further we find:

Corollary 2.18: Suppose φ is of the form (2.16). Then there exist non-trivial moving averages of length $l \geq \sum_{j=1}^{p} h_j + 1$ which are exact for φ .

Proof: The trivial moving average given by $r_\tau = 1$ and $r_\nu = 0$ for $\nu \neq \tau$ is exact for all functions. If we exclude this, a moving average with range (α, β) and centre τ is exact for φ if and only if it satisfies the $\sum_{j=1}^{p} h_j$ linear independent equations (2.17). These may always be fulfilled for a moving average of length $l = \beta - \alpha + 1 \geq \sum_{j=1}^{p} h_j + 1$. □

By Theorem 2.10 and Corollary 2.18 there exists a moving average, exact for a function φ , if and only if φ may be given by (2.16) for suitable c_j's and h_j's . Now any trend $\{\xi_t\}$ may be given in this way if only $\sum_{j=1}^{p} h_j$ is large enough. However, this is of little practical interest because then, by Corollary 2.18, only very long moving averages are exact for φ . This in turn will make the form $E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W}$

in the risk function (2.8) very large. Hence, in general, the first term in the risk function (2.8) will not vanish completely. Suppose, however, that the trend for each interval (indexed by t) of length l may be given approximately by

$$\xi_{t+\mu} \simeq \sum_{j=1}^{p} \sum_{m=0}^{h_j-1} \theta_{jm}^{(t)} (t+\mu)^m c_j^{\,t+\mu} ,$$

where the coefficients are allowed to depend on the actual interval. If we then construct the moving average exact for the class of functions given by (2.16), the term $(\underset{\sim}{B}\underset{\sim}{\xi} - \underset{\sim}{B}\underset{\sim}{\xi})' \underset{\sim}{T}_a (\underset{\sim}{B}\underset{\sim}{\xi} - \underset{\sim}{B}\underset{\sim}{\xi})$ in the risk function will nearly vanish. We will then denote the class of functions (2.16) a basis for the moving average.

We note that more than one class of functions may be used to approximate a given trend $\{\xi_t\}$. We may, for example, approximate the trend with a linear function on each interval of length 5, with a polynomial of degree two on each interval of length 9 or with a polynomial of degree three on each interval of length 13. The best of these choices of basis is the one that minimizes the risk function (2.8).

2.E. We will now find an expression for the term $E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W}$ in the risk function (2.8). We have under the usual assumption $r_\nu = 0$ for $\nu < \alpha$ or $\nu > \beta$:

$$\Delta^z W_t = \sum_{j=0}^{z} \binom{z}{j} (-1)^{z-j} W_{t+j} = (-1)^z \sum_{j=0}^{z} \sum_{\nu} \binom{z}{j} (-1)^j r_{\nu-j} V_{t-\tau+\nu}$$

$$= (-1)^z \sum_{\nu=\alpha-z}^{\beta} (\Delta^z r_\nu) V_{t-\tau+\nu+z} .$$

Hence

$$(2.19) \qquad E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W} = \sum_{z=0}^{K} \frac{a_z}{N-\beta+\alpha-z} \sum_{t=\tau-\alpha+1}^{N+\tau-\beta-z} E(\Delta^z W_t)^2$$

$$= \sum_{z=0}^{K} \frac{a_z}{N-\beta+\alpha-z} \sum_{t=\tau-\alpha+1}^{N+\tau-\beta-z} \sum_{\nu}\sum_{\mu} (\Delta^z r_\nu)(\Delta^z r_\mu)\sigma_{t-z+\nu+z,\,t-z+\mu+z} \quad ,$$

where $\underset{\sim}{\Sigma} = \{\sigma_{ij}\}$. If we introduce

$$(2.20) \qquad \underset{\sim}{\Sigma}_{tz} = \{\sigma_{ij}\}_{i,j=t-\tau+\alpha}^{t-\tau+\beta+z}$$

and the $(1+z)\times 1$ matrix

$$(2.21) \qquad \underset{\sim}{A}_z = \begin{pmatrix} \binom{z}{z}(-1)^0 & 0 & 0 & \cdots & & & 0 \\ \binom{z}{z-1}(-1)^1 & \binom{z}{z}(-1)^0 & 0 & \cdots & & & 0 \\ \vdots & & \ddots & & & & \vdots \\ \binom{z}{0}(-1)^z & & & \binom{z}{z}(-1)^0 & 0 & \cdots & 0 \\ 0 & \ddots & & & & \ddots & 0 \\ \vdots & & \binom{z}{0}(-1)^z & \cdots & & & \binom{z}{z}(-1)^0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & \cdots & & & \binom{z}{0}(-1)^0 \end{pmatrix}$$

we find

$$\underset{\sim}{A}_z\underset{\sim}{r} = (\Delta^z r_{\alpha-z}, \Delta^z r_{\alpha+1-z}, \ldots, \Delta^z r_\beta)' ;$$

and by (2.19)

$$E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W} = \sum_{z=0}^{K} \frac{a_z}{N-\beta+\alpha-z} \sum_{t=\tau-\alpha+1}^{N+\tau-\beta-z} (\underset{\sim}{A}_z\underset{\sim}{r})' \, \underset{\sim}{\Sigma}_{tz}(\underset{\sim}{A}_z\underset{\sim}{r})$$

$$= \underset{\sim}{r}' \, \Big(\sum_{z=0}^{K} \frac{a_z}{N-\beta+\alpha-z} \sum_{t=\tau-\alpha+1}^{N-\tau-\beta-z} \underset{\sim}{A}_z' \, \underset{\sim}{\Sigma}_{tz} \, \underset{\sim}{A}_t \Big)\underset{\sim}{r} \, .$$

Thus we have proved:

<u>Theorem 2.22:</u> The term $E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W}$ in the risk function (2.9) is given by

$$(2.23) \qquad E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W} = \underset{\sim}{r}'\underset{\sim}{S}_{a,\Sigma}\underset{\sim}{r} ,$$

where $\underset{\sim}{S}_{a,\Sigma}$ is the 1×1 matrix

$$(2.24) \qquad \underset{\sim}{S}_{a,\Sigma} = \sum_{z=0}^{K} \frac{a_z}{N-\beta+\alpha-z} \sum_{t=\tau-\alpha+1}^{N+\tau-\beta-z} \underset{\sim}{A}_z{}' \underset{\sim}{\Sigma}_{tz} \underset{\sim}{A}_z .$$

We further have by (2.19) and the preceding theorem:

<u>Corollary 2.25</u>: If $\underset{\sim}{\Sigma} = \sigma^2\underset{\sim}{I}$, where $\underset{\sim}{I}$ is the identity matrix, we have

$$(2.26) \qquad E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W} = \sigma^2 \sum_{z=0}^{K} a_z\binom{2z}{z}R_z^2 = \sigma^2\underset{\sim}{r}'\underset{\sim}{S}_a\underset{\sim}{r} ,$$

where R_z^2 is given by (1.4) and $\underset{\sim}{S}_a$ is the 1×1 matrix

$$(2.27) \qquad \underset{\sim}{S}_a = \sum_{z=0}^{K} a_z \underset{\sim}{A}_z' \underset{\sim}{A}_z .$$

Thus, if the observations are uncorrelated and have equal variance σ^2 , the quantities R_z^2 enter in a simple way in (2.26). We will have a closer look at this in Chapter 3.

3. OPTIMAL MOVING AVERAGES.

3.A. In the preceding chapter we found that we will perform the graduation such that the risk (2.8) is minimized. Therefore, the question is whether we can find a moving average which does this. This problem is a very complicated one since it involves a simultaneous choice of length, centre, basis and weights for the moving average. Because of this we instead find moving averages which are optimal in the sense that they minimize $E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W}$ in the class of moving averages with given range, centre and basis. In this chapter we see how this can be done and how these optimal moving averages generalize the minimum-variance and minimum-R_z moving averages.

3.B. Let now $\underset{\sim}{\Sigma} = \sigma^2\underset{\sim}{C}$, where $\underset{\sim}{C}$ is a known positive definit matrix, and suppose the trend in each interval of length 1 may be given approximately by

$$\xi_{t+\mu} \simeq \sum_{j=1}^{p} \sum_{m=0}^{h_j-1} \theta_{jm}^{(t)}(t+\mu)^m c_j^{t+\mu} ,$$

where $c_1, c_2, \ldots, c_p$ are distinct real or complex numbers and $h_1, h_2, \ldots, h_p$ are integers. The coefficients $\theta_{jm}^{(t)}$ may depend on the actual interval (compare Paragraph 2.D.). Then, if we choose φ given by (2.16) as basis for the moving average and restrict ourselves to moving averages of length 1 , the term $(\underset{\sim}{R}\underset{\sim}{\xi} - \underset{\sim}{B}\underset{\sim}{\xi})' \underset{\sim}{T}_a (\underset{\sim}{R}\underset{\sim}{\xi} - \underset{\sim}{B}\underset{\sim}{\xi})$ in the risk will nearly vanish. Thus, we want to find a moving average that is optimal in the sense that it minimizes the term $E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W}$ in the risk (2.8) in the class of moving averages of given length 1 , given centre τ and basis φ .

We may now write for the basis

(3.1) $\varphi(t) = \sum_{i=1}^{m} \theta_i A_i(t)$,

where $m = \sum_{j=1}^{p} h_j$, the $A_i(t)$'s are of the form $t^m c_j^t$ and the θ_i's are unknown parameters. Then, according to Theorem 2.15, a moving average with range (α,β) and centre τ is exact for φ if and only if

(3.2) $\sum_{\nu=\alpha}^{\beta} r_\nu A_i(\nu) = A_i(\tau)$; $i = 1,2,\dots,m$.

We now introduce the vectors

$$\underset{\sim}{X}(t) = (X_{t-\tau+\alpha}, X_{t-\tau+\alpha+1},\dots,X_{t-\tau+\beta})'$$

$$\underset{\sim}{A}(\nu) = (A_1(\nu), A_2(\nu),\dots,A_m(\nu))'$$

and the $m\times l$ matrix

$$\underset{\sim}{A} = (\underset{\sim}{A}(\alpha), \underset{\sim}{A}(\alpha+1),\dots,\underset{\sim}{A}(\beta)) .$$

The constraints (3.2) may then be given equivalently by

(3.3) $\underset{\sim}{r}'\underset{\sim}{A}'\underset{\sim}{\theta} = \underset{\sim}{A}'(\tau)\underset{\sim}{\theta}$ for all $\underset{\sim}{\theta} \in R^m$.

Furthermore, by Theorem 2.22

(3.4) $E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W} = \sigma^2 \underset{\sim}{r}'\underset{\sim}{S}_{a,C}\underset{\sim}{r}$,

where $\underset{\sim}{S}_{a,C}$ is defined similarly to $\underset{\sim}{S}_{a,\Sigma}$ in (2.24).

Thus, we have to minimize $\underset{\sim}{r}'\underset{\sim}{S}_{a,C}\underset{\sim}{r}$ under the constraint (3.3). This problem is solved by the following lemmas.

<u>Lemma 3.5:</u> Let $\underset{\sim}{A}$ be a $m\times l$ matrix of rank m $(m < l)$ and let $\underset{\sim}{S}$ be a positive definite $l\times l$ matrix. Then there

exists a unique $\underset{\sim}{r}^* \in \mathbb{R}^l$ which mimimizes $\underset{\sim}{r}'\underset{\sim}{S}\underset{\sim}{r}$ among all $\underset{\sim}{r} \in \mathbb{R}^l$ under the constraint

$$\underset{\sim}{r}'\underset{\sim}{A}'\underset{\sim}{\theta} = \underset{\sim}{c}'\underset{\sim}{\theta} \quad \text{for all} \quad \underset{\sim}{\theta} \in \mathbb{R}^m ,$$

where $\underset{\sim}{c} \in \mathbb{R}^m$ is a known vector. $\underset{\sim}{r}^*$ is given by:

$$\underset{\sim}{r}^* = \underset{\sim}{S}^{-1}\underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{S}^{-1}\underset{\sim}{A}')^{-1}\underset{\sim}{c} .$$

A proof of this lemma is given in Appendix A. *

Lemma 3.6: The 1×1 matrix $\underset{\sim}{S}_{a,C}$ is positive definite.

Proof: Each $\underset{\sim}{C}_{t,z}$, defined similarly to $\underset{\sim}{\Sigma}_{t,z}$ in (2.20), is positive definite because $\underset{\sim}{C}$ is positive definite. Furthermore, each $\underset{\sim}{A}'_z \underset{\sim}{C}_{t,z}\underset{\sim}{A}_z$ is positive definite because $\underset{\sim}{x}'(\underset{\sim}{A}'_z\underset{\sim}{C}_{t,z}\underset{\sim}{A}_z)\underset{\sim}{x} = (\underset{\sim}{A}_z\underset{\sim}{x})'\underset{\sim}{C}_{t,z}(\underset{\sim}{A}_z\underset{\sim}{x}) \geq 0$ and equal to zero if and only if $\underset{\sim}{A}_z\underset{\sim}{x} = \underset{\sim}{0}$, which is equivalent to $\underset{\sim}{x} = \underset{\sim}{0}$. Thus we have

$$\underset{\sim}{x}'\underset{\sim}{S}_{a,C}\underset{\sim}{x} = \sum_{z=0}^{K} \frac{a_z}{N-\beta+\alpha-z} \sum_{t=\tau-\alpha+1}^{N+\tau-\beta-z} \underset{\sim}{x}'(\underset{\sim}{A}_z'\underset{\sim}{C}_{t,z}\underset{\sim}{A}_z)\underset{\sim}{x} \geq 0$$

and equal to zero if and only if $\underset{\sim}{x}'(\underset{\sim}{A}'_z\underset{\sim}{C}_{t,z}\underset{\sim}{A}_z)\underset{\sim}{x} = 0$ for all z such that $a_z > 0$ and all t. Therefore, $\underset{\sim}{x}'\underset{\sim}{S}_{a,C}\,\underset{\sim}{x} = 0$ if and only if $\underset{\sim}{x} = 0$, and the lemma is proved. □

If we combine Lemma 3.5 and 3.6 we get the following main result:

Theorem 3.7: Suppose $\underset{\sim}{\Sigma} = \sigma^2\underset{\sim}{C}$, where $\underset{\sim}{C}$ is a known positive definite matrix. Then there exists a unique moving average $\underset{\sim}{r}^* = (r^*_\alpha, r^*_{\alpha+1}, \ldots, r^*_\beta)'$ which minimizes $E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W}$ in the class of moving averages with range (α,β), centre τ

* By using a result of Gerber (1977), it is possible to give an easier proof of Lemma 3.5, compare footnote on page 4.

and basis φ given by (3.1). $\underset{\sim}{r}^*$ is given by

$$(3.8) \qquad \underset{\sim}{r}^* = \underset{\sim}{S}_{a,C}^{-1} \underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{S}_{a,C}^{-1} \underset{\sim}{A}')^{-1}\underset{\sim}{A}(\tau) .$$

3.C. We will investigate somewhat closer the case where $X_1, X_2, \ldots, X_N$ are uncorrelated observations with variances equal to σ^2, i.e. $\underset{\sim}{C} = \underset{\sim}{I}$ in the preceding paragraph. Then, by Corollary 2.25, $\underset{\sim}{r}^*$ given by (3.8) with $\underset{\sim}{S}_{a,C} = \underset{\sim}{S}_a$, where $\underset{\sim}{S}_a$ is given by (2.27), is the unique moving average which minimizes

$$\sum_{z=0}^{K} a_z \binom{2z}{z} R_z^2$$

in the class of moving averages with range (α,β), centre τ and basis φ as above. We find that $a_0 = 1$ gives the minimum-variance and $a_z = 1$, for $z > 0$, the minimum-R_z moving average. (Remember that $a_i \geq 0$ and $\sum_{i=1}^{K} a_i = 1$). Hence the optimal moving averages given in Theorem 3.7. generalize the well known optimal moving averages. The theorem also gives a generalization of Michalup's moving average, which we get for $a_0 + a_3 = 1$.

We will have a thorough discussion of the case with uncorrelated observations with equal variance in the next chapter.

3.D. Theorem 3.7 gives moving averages which are optimal in the sense that they minimize $E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W}$ in the class of moving averages with given range (α,β), centre τ and basis φ given by (3.1). It is now of interest to find the value(s) of τ, $\alpha \leq \tau \leq \beta$, which minimize $E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W}$. A value of τ with this property will be called an optimal centre for moving averages with range (α,β) and basis φ as above.

The following corollary to Theorem 3.7 is of interest when we will find optimal centres.

Corollary 3.9: The minimum value of $\frac{1}{\sigma^2} E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W} = \underset{\sim}{r}'\underset{\sim}{S}_{a,C}\underset{\sim}{r}$, where $\underset{\sim}{r}$ is a moving average with range (α,β) , centre τ and basis φ given by (3.1), is

$$\underset{\sim}{S}_{a,C}(\tau)\underset{\sim}{r}^* = \underset{\sim}{A}'(\tau)(\underset{\sim}{A}\underset{\sim}{S}_{a,C}^{-1}\underset{\sim}{A}')^{-1}\underset{\sim}{A}(\tau) ,$$

where $\underset{\sim}{S}_{a,C}(\tau)$ is the $(\tau-\alpha+1)$-th column- (or row-) vector in the 1×1 matrix $\underset{\sim}{S}_{a,C}$, and $\underset{\sim}{r}^*$ is the optimal moving average given in Theorem 3.7.

Proof: By Theorem 3.7. the minimum value of $\underset{\sim}{r}'\underset{\sim}{S}_{a,C}\underset{\sim}{r}$ is attained by $\underset{\sim}{r}^*$ given by (3.8). Now we have

$$\underset{\sim}{r}^{*\prime}\underset{\sim}{S}_{a,C}\underset{\sim}{r}^* = \underset{\sim}{A}'(\tau)(\underset{\sim}{A}\underset{\sim}{S}_{a,C}^{-1}\underset{\sim}{A}')^{-1}\underset{\sim}{A}(\tau) = \underset{\sim}{S}_{a,C}(\tau)\underset{\sim}{r}^* ,$$

and the corollary is proved. □

By this corollary $\tau^*, \alpha \le \tau^* \le \beta$, is an optimal centre for moving averages with range (α,β) and basis φ as above if and only if

$$\underset{\sim}{A}'(\tau^*)(\underset{\sim}{A}\underset{\sim}{S}_{a,C}^{-1}\underset{\sim}{A}')^{-1}\underset{\sim}{A}(\tau^*) = \min_{\tau=\alpha,\dots,\beta} \underset{\sim}{A}'(\tau)(\underset{\sim}{A}\underset{\sim}{S}_{a,C}^{-1}\underset{\sim}{A})^{-1}\underset{\sim}{A}(\tau) .$$

In the case $\underset{\sim}{C} = \underset{\sim}{I}$ Corollary 3.9 shows that the minimum value of $\sum_{z=0}^{K} a_z\binom{2z}{z}R_z^2$ is

$$(3.10) \qquad \underset{\sim}{S}_a(\tau)\underset{\sim}{r}^* = \sum_{z=0}^{K} a_z(-1)^z\,\delta^{2z}r_\tau^* .$$

This follows from (2.27) and the fact that

$$(3.11) \quad \underset{\sim}{A}_z' \underset{\sim}{A}_z = (-1)^z \begin{pmatrix} \binom{2z}{z}(-1)^z \ldots \binom{2z}{2z}(-1)^0 & 0 \ldots\ldots\ldots\ldots\ldots 0 \\ \vdots & \ddots \\ \binom{2z}{0}(-1)^{2z} \ldots \binom{2z}{z}(-1)^z \ldots \binom{2z}{2z}(-1)^0 & 0 \ldots\ldots 0 \\ 0 & \ddots \quad 0 \\ \vdots \quad \ddots & \\ 0 \quad 0 & \binom{2z}{0}(-1)^{2z} \ldots \binom{2z}{z}(-1)^z \ldots \binom{2z}{2z}(-1)^0 \\ \vdots \quad \ddots & \\ 0 \ldots\ldots\ldots\ldots 0 & \binom{2z}{0}(-1)^{2z} \ldots \binom{2z}{z}(-1)^z \end{pmatrix} .$$

(3.10) generalizes Sheppard's (1912) result that for the minimum-variance moving averages we have $r_T^* = R_0^2$, and Greville's (1947) result that for the minimum-R_z moving averages we have $(-1)^z \delta^{2z} r_T^* = \binom{2z}{z} R_z^2$.

We will have a closer look at the choice of optimal centres for the case $\underset{\sim}{C} = \underset{\sim}{I}$ in the next chapter.

3.E. It was known already by Sheppard (1912) that the smoothed values one gets by the minimum-variance moving averages are equivalent to the values one gets by the method of least squares (See e.g. Sverdrup, 1967, pp.347-351, for a thorough discussion of this.) Greville (1947, p.258) has generalized this result to the minimum-R_z moving averages.

Sheppard's result may be generalized also to the optimal moving averages given in Theorem 3.7.

<u>Theorem 3.12</u>: Let $\hat{\underset{\sim}{\theta}}$ be the $\underset{\sim}{\theta} \in \mathbb{R}^m$ which minimizes

$$(\underset{\sim}{X}(t) - \underset{\sim}{A}'\underset{\sim}{\theta})' \, \underset{\sim}{S}_{a,C}^{-1} (\underset{\sim}{X}(t) - \underset{\sim}{A}'\underset{\sim}{\theta}) .$$

Then we have

$$\hat{\xi}_t = \underset{\sim}{r}^{*'} \underset{\sim}{X}(t) = \underset{\sim}{A}'(\tau)\, \hat{\underset{\sim}{\theta}}$$

<u>Proof</u>: It is well known (see e.g. Scheffé, 1959, pp. 19 - 21) that

$$\hat{\underset{\sim}{\theta}} = (\underset{\sim}{A}\underset{\sim}{S}_{a,c}^{-1}\underset{\sim}{A}')^{-1}\underset{\sim}{A}\underset{\sim}{S}_{a,c}^{-1}\underset{\sim}{X}(t) .$$

Thus, by (3.8), we have

$$\underset{\sim}{A}'(\tau)\hat{\underset{\sim}{\theta}} = \underset{\sim}{A}'(\tau)(\underset{\sim}{A}\underset{\sim}{S}_{a,c}^{-1}\underset{\sim}{A}')^{-1}\underset{\sim}{A}\underset{\sim}{S}_{a,c}^{-1}\underset{\sim}{X}(t) = \underset{\sim}{r}^{*'}\underset{\sim}{X}(t) .$$

□

3.F. We complete this chapter of the general theory of optimal moving averages by to useful corollaries to Theorem 3.7.

<u>Corollary 3.13</u>: $\underset{\sim}{r} = (r_\alpha, r_{\alpha+1}, \dots, r_\beta)'$ is the unique optimal moving average given in Theorem 3.7. if and only if $\underset{\sim}{r}$ satisfies the following conditions:

i) There exists $\underset{\sim}{\theta}_1 \in \mathbb{R}^m$ such that

$$\underset{\sim}{S}_{a,c}\underset{\sim}{r} = \underset{\sim}{A}'\underset{\sim}{\theta}_1 .$$

ii) $\underset{\sim}{r}'\underset{\sim}{A}'\underset{\sim}{\theta} = \underset{\sim}{A}'(\tau)\underset{\sim}{\theta}$ for all $\underset{\sim}{\theta} \in \mathbb{R}^m$.

<u>Proof</u>: The necessity of the two conditions is trivial. We have to prove that they are sufficient. Suppose that i) and ii) are satisfied. Then there exists $\underset{\sim}{\theta}_1 \in \mathbb{R}^m$ such that

$$\underset{\sim}{r} = \underset{\sim}{S}_{a,c}^{-1}\underset{\sim}{A}'\underset{\sim}{\theta}_1 .$$

We will find $\underset{\sim}{\theta}_1$. From ii) we have

$$\underset{\sim}{\theta}_1'\underset{\sim}{A}\underset{\sim}{S}_{a,c}^{-1}\underset{\sim}{A}'\underset{\sim}{\theta} = \underset{\sim}{A}'(\tau)\underset{\sim}{\theta}$$

for all $\underset{\sim}{\theta} \in \mathbb{R}^m$, or equivalently

$$\underset{\sim}{\theta}_1' \underset{\sim}{A}\underset{\sim}{S}_{a,c}^{-1} \underset{\sim}{A}' = \underset{\sim}{A}'(\tau) .$$

Thus, we have

$$\underset{\sim}{\theta}_1' = \underset{\sim}{A}'(\tau)(\underset{\sim}{A}\underset{\sim}{S}_{a,c}^{-1} \underset{\sim}{A}')^{-1}$$

and

$$\underset{\sim}{r} = \underset{\sim}{S}_{a,c}^{-1} \underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{S}_{a,c}^{-1} \underset{\sim}{A}')^{-1}\underset{\sim}{A}(\tau) .$$

This proves that the conditions are sufficient. □

Corollary 3.14: Let $\mathcal{F}$ be the class of sequences $\{x_\nu\}$ which are solutions of the linear difference equation

$$(3.15) \qquad \sum_{z=0}^{K} a_z(-1)^z \delta^{2z} x_\nu = \sum_{i=1}^{m} \theta_i A_i(\nu)$$

for some $\theta_1, \theta_2, \dots, \theta_m$. Suppose there exists a sequence $\{r^*_\nu\} \in \mathcal{F}$ such that

$$r^*_\nu = 0 \quad \text{for} \quad \nu = \alpha-K,\ \alpha-K+1, \dots, \alpha-1,$$
$$\text{and} \quad \nu = \beta+1,\ \beta+2, \dots, \beta+K ,$$

and

$$\sum_{\nu=\alpha}^{\beta} r^*_\nu A_i(\nu) = A_i(\tau) \qquad \text{for } i=1,2,\dots,m .$$

Then $\underset{\sim}{r}^* = (r^*_\alpha, r^*_{\alpha+1}, \dots, r^*_\beta)'$ is the optimal moving average with range (α,β), centre τ and basis φ given by (3.1) for the case $\underset{\sim}{\Sigma} = \sigma^2 \underset{\sim}{I}$.

Proof: It is sufficient to prove that $\underset{\sim}{r}^*$ satisfies i) and ii) in Corollary 3.13. Condition ii) is satisfied by assumption. Further, by (3.11) and $r^*_\nu = 0$ for $\nu = \alpha-K, \dots, \alpha-1, \beta+1, \dots, \beta+K$, we find

$$\underset{\sim}{S}_a \underset{\sim}{r}^* = \sum_{z=0}^{K} a_z \underset{\sim}{A}_z' \underset{\sim}{A}_z \underset{\sim}{r}^*$$

$$= \sum_{z=0}^{K} a_z (-1)^z \begin{pmatrix} \delta^{2z} r_\alpha \\ \vdots \\ \delta^{2z} r_\beta \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{m} \theta_{1i} A_i(\alpha) \\ \vdots \\ \sum_{i=1}^{m} \theta_{1i} A_i(\beta) \end{pmatrix} = \underset{\sim}{A}' \underset{\sim}{\theta}_1$$

for some $\underset{\sim}{\theta}_1 = (\theta_{11}, \theta_{12}, \ldots, \theta_{1m})'$. Thus, i) is satisfied and the corollary is proved.

□

4. OPTIMAL MOVING AVERAGES WHEN THE OBSERVATIONS ARE UNCORRELATED AND HAVE EQUAL VARIANCES.

4.A. In this chapter we use the general theory from the preceding chapter to find optimal moving averages when the observations are uncorrelated and have equal variances.

First we give some general results for this case. Then we give a brief review of some well known results for minimum-variance and minimum-R_z moving averages. We also find some new results. Of special interest is perhaps the discussion of optimal centres for minimum-R_z moving averages and that we find minimum-R_2 moving averages exact for other functions than polynomials.* At last we discuss how we can construct optimal moving averages when more than one a_z in the loss function (2.4) is positive.

Throughout the chapter we assume $\underset{\sim}{\Sigma} = \sigma^2 \underset{\sim}{I}$.

4.B. Suppose we have given a moving average $\underset{\sim}{r}^* = (r^*_\alpha, r^*_{\alpha+1}, \ldots, r^*_\beta)'$ with range (α,β) and centre τ . Then we can define a new moving average $\underset{\sim}{\tilde{r}} = (\tilde{r}_\beta, \tilde{r}_{-\beta+1}, \ldots, \tilde{r}_{-\alpha})'$ with range $(-\beta,-\alpha)$ and centre $-\tau$ by

$$(4.1) \qquad \tilde{r}_{-\nu} = r^*_\nu \ ; \ \nu = \alpha, \alpha+1, \ldots, \beta \ .$$

We will investigate the connection between these two moving averages.

We first give two lemmas

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* See footnote on page 4.

Lemma 4.2: Let $\underset{\sim}{r}^*$ and $\tilde{\underset{\sim}{r}}$ be the moving averages given by (4.1). Then we have

$$\sum_{\nu=\alpha-z}^{\beta} (\Delta^z r^*_\nu)^2 = \sum_{\mu=-\beta-z}^{-\alpha} (\Delta^z \tilde{r}_\mu)^2$$

for $z = 0,1,2,\ldots,$ where as usual $r^*_\nu = 0$ for $\nu < \alpha$ or $\nu > \beta$ and $\tilde{r}_\mu = 0$ for $\mu < -\beta$ or $\mu > -\alpha$.

Lemma 4.3: Let $\underset{\sim}{r}^*$ and $\tilde{\underset{\sim}{r}}$ be the moving averages given by (4.1). Then $\underset{\sim}{r}^*$ is exact for $t^m c^t$ if and only if $\tilde{\underset{\sim}{r}}$ is exact for $t^m c^{-t}$.

The proofs of these lemmas are straightforward. (Compare, however, Lemma 2.13 for the proof of Lemma 4.3.) From these lemmas and Corollary 2.25 we immediately have:

Theorem 4.4: Let $\underset{\sim}{r}^* = (r^*_\alpha, r^*_{\alpha+1}, \ldots, r^*_\beta)'$ be the optimal moving average with range (α,β) , centre τ and basis

$$\varphi(t) = \sum_{j=1}^{p} \sum_{m=0}^{h_j-1} \theta_{jm} t^m c_j^{\,t} .$$

Then $\tilde{\underset{\sim}{r}} = (\tilde{r}_{-\beta}, \tilde{r}_{-\beta+1}, \ldots, \tilde{r}_{-\alpha})'$ given by (4.1) is the optimal moving average with range $(-\beta,-\alpha)$, centre $-\tau$ and basis

$$\Psi(t) = \sum_{j=1}^{p} \sum_{m=0}^{h_j-1} \rho_{jm} t^m c_j^{\,-t} ,$$

and the two moving averages have the same value of

$$\sum_{z=0}^{K} a_z \binom{2z}{z} R_z^2 .$$

Especially for optimal moving averages with range $(-k,k)$ and polynomials as basis, the theorem shows that we by (4.1) easily can find those with centre $\tau < 0$ from those with centre $\tau > 0$. Thus, we may in this case concentrate on finding optimal moving averages with centres $\tau \geq 0$. Furthermore, we see that if τ^* is an optimal centre, then $-\tau^*$ also is an optimal centre. At last, by letting $\tau = 0$, we see that an optimal centralized moving average, exact for polynomials, is symmetric.

4.C. The minimum-variance moving averages, exact for polynomials, have been well known for a long time and are studied extensively in the literature, compare Paragraph 1.D. Most authors, however, only discuss the centralized case. Exeptions are Greville (1947), Weichselberger (1964), Pollard (1971b) and Kockelkorn and Rüger (1974).

Weichselberger (1964) gives a thorough discussion of how one may find the minimum-variance moving averages of range $(-k,k)$, centre τ and polynomials of degree m as basis, and he gives explicit formulas for the weights $r_{\nu\tau}^{(m)}$ for m up to five. (The formula Weichselberger gives for $m = 5$ is somewhat wrong. The correct formula is given by Borgan, 1976, p. 84.) Weichselberger also discusses how one may find optimal centres. His result is that for $m = 0$ all centres are equally good, for $m = 1$ $\tau^* = 0$ is the optimal centre and for $m = 2$ the optimal centres are τ^* and $-\tau^*$, where $\tau^* = [\sqrt{\frac{(2k+1)^2+1}{20}}]$ or $\tau^* = [\sqrt{\frac{(2k+1)^2+1}{20}}]+1$ depending on which gives the smallest value of $r_{\tau^*\tau^*}^{(2)}$ (see Weichselberger, 1964, p. 223) . From this and numeric computations he conjectures that $\tau^* = 0$ is the optimal centre when m is odd, while this is not the case for m even. Later on Kockelkorn and Rüger (1974, p. 326) have proved that $\tau^* = 0$ is the optimal centre for $m = 3$. Numeric computations by the present author for $m = 4,5,\ldots,9$ also indicates that Weichselberger's conjecture is true (see Borgan, 1976, pp. 84 - 87) . We have, however, not succeeded in proving any general results in this direction.

In Table 4.20 we have given the optimal centres $\tau^* \geq 0$ for $m = 2,4$ and different values of k . In Appendix C we give the weights of the centralized minimum-variance moving averages, exact for cubics, for $k = 3,4,\ldots,10$. We also give the values of R_z^2 for $z = 0,1,2,3,4$ for the actual moving averages. More tables for the minimum-variance moving averages, exact for polynomials are given in Borgan (1976, Appendix B).

4.D. We will in this report also see how we may find minimum-variance moving averages when the basis has exponential terms, i.e. at least one $c_j \neq 1$ in (2.16). This problem was first discussed thoroughly by Jecklin and Strickler (1954). They, however, only discuss the centralized case. We will also see how we may find the optimal centres. We illustrate the technique by two examples:

The most used function for analytic graduation of mortality rates for adult ages, is the Gompertz-Makeham's formula (see e.g. Hoem, 1972, p. 569) :

$$\alpha + \beta c^t ,$$

where $\beta > 0$, $c > 1$ and $\alpha > -\beta c^{t_{min}}$. For mortality rates c is usually close to 1.1. We will see how we may construct minimum-variance moving averages with this function as basis.

It is convenient here to use range $(1,l)$ and centre τ , $1 \leq \tau \leq l$. By Corollary 3.13 the weights $r_{\nu\tau}$ for the minimum-variance moving average are given uniquely by:

$$r_{\nu\tau} = a_\tau + b_\tau c^\nu \; ; \; \nu = 1,2,\ldots,l \, ,$$

where a_τ and b_τ are found from the equations:

$$\sum_{\nu=1}^{l} r_{\nu\tau} = 1 \quad , \quad \sum_{\nu=1}^{l} r_{\nu\tau} c^\nu = c^\tau$$

or equivalently

$$(4.5) \qquad \begin{aligned} a_\tau l + b_\tau \sum_{\nu=1}^{l} c^\nu &= 1 \\ a_\tau \sum_{\nu=1}^{l} c^\nu + b_\tau \sum_{\nu=1}^{l} c^{2\nu} &= c^\tau \, . \end{aligned}$$

If we solve these equations for a_τ and b_τ and substitute the expression for $r_{\nu\tau}$, we find:

$$(4.6) \qquad r_{\nu\tau} = \frac{(1-c)[1-c^{2l}-(c^{\tau-1}+c^{\nu-1})(1-c^l)(1+c)+lc^{\tau+\nu-2}(1-c^2)]}{l(1-c^{2l})(1-c)-(1-c^l)^2(1+c)} \, .$$

By the remark succeeding Corollary 3.9 the optimal centre τ^* is the value of τ which minimizes $r_{\tau\tau}$. If we consider τ as a continuous variable we may differentiate $r_{\tau\tau}$ with respect to τ . We find:

$$\frac{dr_{\tau\tau}}{d\tau} = \frac{2(1-c^2)c^{\tau-1} \log c \, [lc^{\tau-1}(1-c)-(1-c^l)]}{l(1-c^{2l})(1-c)-(1-c^l)^2(1+c)} \, ,$$

and hence $\frac{dr_{\tau\tau}}{d\tau} = 0$ for

$$(4.7) \qquad \tau^{\#} = 1 + \frac{\log \frac{1-c^l}{l(1-c)}}{\log c} \, .$$

Discussion of the sign of $\frac{dr_{\tau\tau}}{d\tau}$ shows that $\tau^{\#}$ is the value of τ that minimizes $r_{\tau\tau}$. Thus the optimal centre is

$\tau^* = [\tau^{\#}]$ or $\tau^* = [\tau^{\#}]+1$ depending on which gives the smallest value or $r_{\tau^*\tau^*}$.

c \ l	5	7	9	11	13	15	17	19	21
1.06	3.1	4.1	5.2	6.3	7.4	8.5	9.7	10.9	12.1
1.08	3.1	4.2	5.3	6.4	7.5	8.7	9.9	11.1	12.4
1.10	3.1	4.2	5.3	6.5	7.7	8.9	10.1	11.4	12.7
1.12	3.1	4.2	5.4	6.6	7.8	9.0	10.3	11.6	13.0
1.14	3.1	4.3	5.4	6.6	7.9	9.2	10.5	11.9	13.3

Table 4.8: The value of $\tau^{\#}$ given in (4.7) for different values of c and l .

Table 4.8 gives $\tau^{\#}$ for different values of c and l . We see that for small values of c and l the centralized moving averages are optimal. If c or l (or both) increases the optimal centres also increase.

Jecklin and Strickler (1954, pp. 144 and 155) have given the centralized minimum-variance moving averages of length 5,7,9,11 and 13 when $c = 1.10$. The optimal centres in this case are given in Table 4.23.(In the table we use range $(-k,k)$. Hence, to get the optimal centres corresponding to range $(1,l)$ we have to add $k+1$ to the values given in the table.) We find that the centralized moving averages are best for $l = 5,7,9,11$. For $l = 13$, however, the optimal centre is $\tau^* = 8$, and the corresponding moving average is

(.06728, .06837, .06957, .07088, .07233, .07392, .07567, .07760, .07972, .08205, .08462, .08744, .09054), while the centralized minimum-variance moving average of length 13 is

$$(.09467, .09267, .09047, .08804, .08538, .08245, \underline{.07922}, .07567, .07177, .06743, .06276, .05756, .05185).$$

We notice that the weights for $\tau=8$ is strictly increasing and for $\tau=7$ (i.e. the centralized case) strictly decreasing. This is due to the following:

The weights are generally given by $r_{\nu\tau} = a_\tau + b_\tau c^\nu$, where a_τ and b_τ are the solutions of (4.5). Some computations now give

$$b_\tau \begin{cases} < 0 & \text{for} \quad \tau < \tau^{\#} \\ = 0 & \text{for} \quad \tau = \tau^{\#} \\ > 0 & \text{for} \quad \tau > \tau^{\#} , \end{cases}$$

where $\tau^{\#}$ is given by (4.7). Thus, the weights $r_{\nu\tau}$ are strictly increasing if $\tau > \tau^{\#}$ and strictly decreasing if $\tau < \tau^{\#}$. In the case considered above we find from Table 4.8 $\tau^{\#} = 7.7$.

In Appendix C we have given the weights $r_{\nu\tau^*}$ corresponding to the optimal centre τ^* for $l = 7,9,\ldots,21$ when $c = 1.10$ (Also in the appendix we use range $(-k,k)$). We also give τ^* and the values of R_z^2 for $z = 0,1,2,3,4$ for the actual moving averages.

Jecklin and Strickler (1954, p.151) also give the centralized minimum-variance moving average of length 7 with basis $\alpha+\beta c^t+\delta t$ when $c = 1.10$. We will have a closer look at this case. Then consider moving averages with range $(1,l)$, centre τ and basis as given above. By Corollary 3.13 the weights $r_{\nu\tau}$ of the minimum-variance moving average are given uniquely by

$$(4.9)\qquad r_{\nu\tau} = a_\tau + b_\tau c^\nu + d_\tau \nu \;;\quad \nu = 1,2,\ldots,l \;,$$

where a_τ, b_τ and d_τ are the solutions of the equations

$$(4.10)\qquad \begin{aligned} a_\tau l + b_\tau \sum_\nu c^\nu + d_\tau \sum_\nu \nu &= 1 \\ a_\tau \sum_\nu c^\nu + b_\tau \sum_\nu c^{2\nu} + d_\tau \sum_\nu \nu c^\nu &= c^\tau \\ a_\tau \sum_\nu \nu + b_\tau \sum_\nu \nu c^\nu + d_\tau \sum_\nu \nu^2 &= \tau \;. \end{aligned}$$

From (4.9) and (4.10) we easily find the minimum-variance moving average for different values of c, l and τ. From this we may also find the optimal centre τ^*. For example for $l = 7$ and $c = 1.10$, which is the case Jecklin and Strickler (1954) consider, we find the optimal centre $\tau^* = 3$ and the corresponding moving average

(.0895, .2082, <u>.2722</u>, .2760, .2135, .0781, -.1375),

while the centralized minimum-variance moving average $(\tau=4)$ is

(-.0853, .1343, .2760, <u>.3321</u>, .2940, .1523, -.1034).

We note that the technique used above also will give minimum-variance moving averages with basis for example $\alpha+\beta c_1{}^t + \gamma c_2{}^t$ or $\alpha+\beta c^t + \gamma t + \delta t^2$. In each case we only have to solve a system of linear equations similar to (4.10). This is easily done with modern computing equipment.

4.E. <u>The minimum-R_z moving averages</u>, exact for polynomials, are discussed e.g. by Greville (1947, 1972) and Pollard (1971a, 1971b). Like most authors Greville (1972) and Pollard

(1971a) only discuss the centralized case. A treatment of the non-centralized case is given by Greville (1947) and Pollard (1971b). By different techniques they find non-centralized minimum-R_z moving averages exact for polynomials. Greville (1947, p. 257) find a general expression for the minimum-R_z moving average with range $(0,n-1)$, centre τ and polynomial of degree m as basis. In Greville (1948, pp. 13-30) he uses this to give tables of the minimum-R_3 and -R_4 moving averages of different lengths and centres when $m = 3$. Pollard (1971b) uses a technique similar to that we will give below.

Neither Greville (1947) nor Pollard (1971b), however, discuss optimal centres in connection with the non-centralized moving averages. As far as the present author knows, the only example in the literature of a non-centralized minimum-R_z moving average that is better than the corresponding centralized one is given by Greville (1974, pp. 395-396). (In this connection we exclude the minimum-R_0 or minimum-variance moving averages, compare Paragraph 4.C.) We will see below that a non-centralized moving average usually is best when we use polynomials of even degree as basis.

We now proceed to see how we may find the minimum-R_z moving averages, exact for polynomials, from the theory given in Chapter 3. The following theorem, which is a generalization of a theorem of Greville (1972 , p.9) and a result of Pollard (1971b, p.8), is an easy consequence of Corollary 3.14 and the theory of linear difference equations (see e.g. Henrici, 1964, pp. 137-140) .

Theorem 4.11: The minimum-R_z moving average of range (α,β) , centre τ and polynomials of degree m as basis is given uniquely by

$$(4.12) \qquad r_{\nu\tau}^{(m)} = c_{o\tau} + c_{1\tau}\nu + \ldots + c_{2z+m,\tau}\nu^{2z+m} ;$$

$\nu = \alpha,\ \alpha+1,\ldots,\beta$, where the $c_{i\tau}$'s are given by

$$(4.13) \qquad r_{\nu\tau}^{(m)} = 0 \quad \text{for} \quad \nu = \alpha-z,\ldots,\alpha-1 \quad \text{and} \quad \nu = \beta+1,\ldots,\beta+z$$

and

$$(4.14) \qquad \sum_{\nu=\alpha}^{\beta} r_{\nu\tau}^{(m)} \nu^p = \tau^p ; \; p = 0,1,\ldots,m .$$

This theorem gives us a procedure for computing the minimum-R_z moving average of range (α,β) , centre τ and polynomials of degree m as basis. We first find the $c_{i\tau}$'s as solutions of the $2z + m+1$ linear equations (4.13) and (4.14) and thereafter the weights $r_{\nu\tau}^{(m)}$ by (4.12).

By this method we have computed the minimum-R_z moving averages of range $(-k,k)$ for $z = 1,2,3,4;\ m = 1,2,3,4;\ k = 3,4,\ldots,10$ and all centres $\tau \geq 0$. Those with centres $\tau < 0$ are found by the remark succeeding Theorem 4.4. Also in this case we want to find the optimal centre (or centres) τ^*, i.e. the value of τ which minimizes $R_z^2 = (-1)^z \delta^{2z} r_{\tau\tau}^{(m)}$ (compare (3.10)). The Figures 4.15 - 4.19 show how R_3^2 varies with τ for the minimum-R_3 moving averages for $m = 1,2,3,4,5$ when $k = 10$. The figures also give the weights $r_{\nu\tau^*}^{(m)}$ corresponding to the optimal centre $\tau^* \geq 0$. (Similar figures for $z = 1,2,4$ are given in Borgan, 1976, Appendix A.)

As for the minimum-variance moving averages the optimal centre is $\tau^* = 0$ for $m = 1,3,5$ (for the values of k and z we have considered). However, for $m = 2,4$ the centralized moving averages are not the best. Table 4.20 gives the optimal centres $\tau^* \geq 0$ in these cases for different values of z and k.

In Appendix C we have given the weights of the centralized minimum-R_z moving averages, exact for cubics, for $k = 3,4,\ldots,10$ and $z = 1,2,3,4$. We also given the values of R_z^2 for $z = 0,1,2,3,4$ for the actual moving averages. More tables for the minimum-R_z moving averages, exact for polynomials, are given in Borgan (1976, Appendix B).

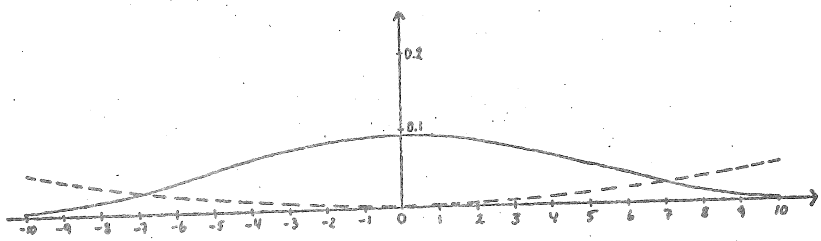

Figure 4.15: $10^3 \cdot R_3^2$ as a function of τ for the minimum-R_3 moving averages with range $(-10,10)$ and polynomials of degree one as basis (dotted line) and the weights $r_{\nu 0}^{(1)}$ corresponding to the optimal centre (drawn line).

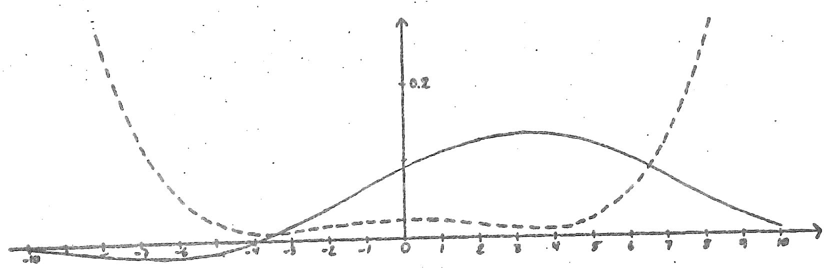

Figure 4.16: $10^3 \cdot R_3^2$ as a function of τ for the minimum-R_3 moving averages with range $(-10,10)$ and polynomials of degree two as basis (dotted line) and the weights $r_{\nu 4}^{(2)}$ corresponding to the optimal centre (drawn line).

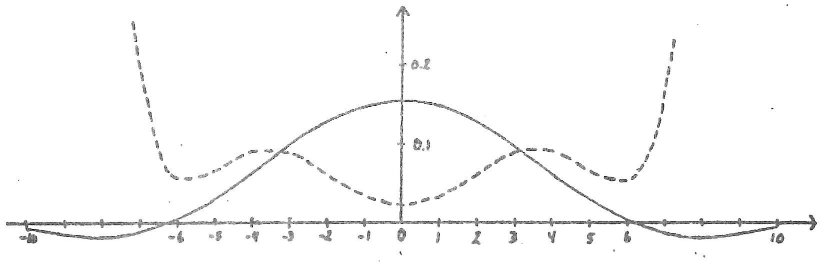

Figure 4.17: $10^3 \cdot R_3^2$ as a function of τ for the minimum-R_3 moving averages with range (-10,10) and polynomials of degree three as basis (dotted line) and the weights $r_{\nu 0}^{(3)}$ corresponding to the optimal centre (drawn line).

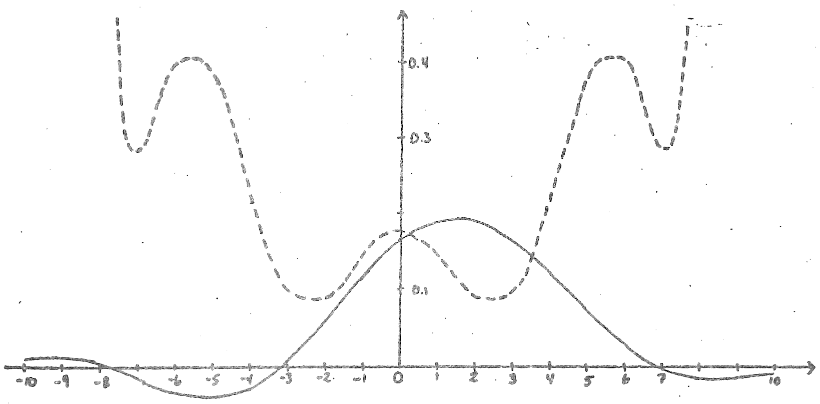

Figure 4.18: $10^3 \cdot R_3^2$ as a function of τ for the minimum-R_3 moving averages with range (-10,10) and polynomials of degree four as basis (dotted line) and the weights $r_{\nu 2}^{(4)}$ corresponding to the optimal centre (drawn line).

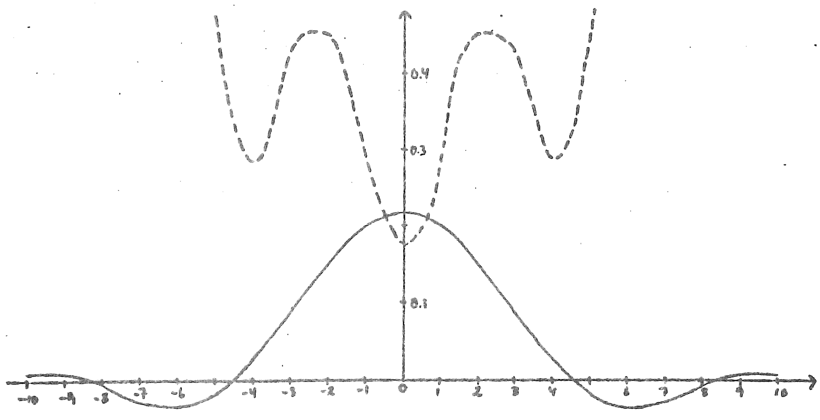

Figure 4.19: $10^3 \cdot R_3^2$ as a function of τ for the minimum-R_3

moving averages with range $(-10,10)$ and polynomials of degree five as basis (dotted line) and the weights $r_{\nu o}^{(5)}$ corresponding to the optimal centre (drawn line).

z	0		1		2		3		4	
k \ m	2	4	2	4	2	4	2	4	2	4
3	1	1	1	1	1	1	1	1	1	1
4	2	1	2	1	2	1	2	1	2	1
5	2	2	2	1	2	1	2	1	2	1
6	3	2	3	2	2	2	2	2	2	1
7	3	2	3	2	3	2	3	2	3	2
8	4	2	3	2	3	2	3	2	3	2
9	4	3	4	2	3	2	3	2	3	2
10	5	3	4	3	4	3	4	2	3	2

Table 4.20: The optimal centres $\tau^* \geq 0$ for the minimum-R_z moving averages of range $(-k,k)$ and polynomial of degree m as basis.

4.F. In this paragraph we will see how we may find minimum-R_z moving averages with exponential basis. As an example we shall find the minimum-R_z moving averages exact for Gompertz-Makeham's function. Other exponential bases may be treated in a similar way.

We will use the same technique as in the preceding paragraph. From Corollary 3.14 and the theory of linear difference equations we find that the minimum-R_z moving average with range $(-k,k)$, centre τ and Gompertz-Makeham's formula, $\alpha+\beta c^t$, as basis is given by

$$(4.21)\qquad r_{\nu\tau} = c_{0\tau}+c_{1\tau}\nu+\ldots+c_{2z,\tau}\nu^{2z}+c_{2z+1,\tau}c^{\nu} ,$$

where the $c_{i\tau}$'s are the unique solutions of the equations

$$\sum_{\nu=-k}^{k} r_{\nu\tau} = 1$$

$$(4.22)\qquad \sum_{\nu=-k}^{k} r_{\nu\tau}c^{\nu} = c^{\tau}$$

$$r_{\nu\tau} = 0 \; ; \; \nu = \pm (k+1), \pm (k+2),\ldots,\pm(k+z) .$$

Thus, we may find the actual minimum-R_z moving averages from (4.21) and (4.22) for different values of c,k and τ .

We have computed the minimum-R_z moving averages for $z = 1,2,3,4$ and $k = 3,4,\ldots,10$ when $c = 1.10$ for all centres. The optimal centres in these cases are given in Table 4.23. We note that the centralized moving averages not are the best in general. Figure 4.24 shows how R_z^2 varies with τ for $z = 1$ and $k = 5$. The figure also gives the weights $r_{\nu o}$ corresponding to the optimal centre $\tau^* = 0$. (Similar figures for $z = 2,3,4$ are given in Borgan, 1976, pp. 113-114). In Appendix C we have given the weights $r_{\nu\tau}{}^*$ corresponding to the optimal centre τ^* for $z = 1,2,3,4$ and $k = 3,4,\ldots,10$ when $c = 1.10$. We also give τ^* and the values of R_z^2 for $z = 0,1,2,3,4$ for the actual moving averages.

k \ z	0	1	2	3	[illegible]
3	0	0	0	0	0
4	0	0	0	0	0
5	0	0	0	0	0
6	1	0	0	0	0
7	1	1	0	0	0
8	1	1	1	0	0
9	1	1	1	1	1
10	2	1	1	1	1

Table 4.23: Optimal centres for minimum-R_z moving averages with range $(-k,k)$ and basis $\alpha+\beta(1.1)^t$.

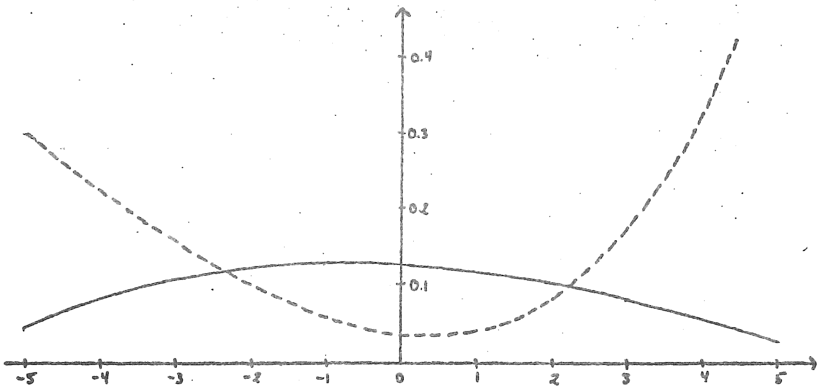

Figure 4.24: $10 \cdot R_1^2$ as a function of τ for the minimum-R_1 moving averages with range $(-5,5)$ and basis $\alpha+\beta(1.1)^t$ (dotted line) and the weights $r_{\nu o}$ corresponding to the optimal centre (drawn line).

4.G. The only example in the literature on moving averages which minimize $\sum_{z=0}^{K} a_z \binom{2z}{z} R_z^2$ when more than one a_z is positive is given by Michalup (1956). He finds the symmetric moving averages of length 5,7 and 9, exact for cubics, which minimize $AR_0^2 + BR_3^2$. In this paragraph we will discuss how we in general may find the optimal moving averages when more than one a_z is positive. It is convenient to discuss the cases $a_0 + a_1 = 1$ and $a_z > 0$ for some $z \geq 2$ separately.

When $a_0 + a_1 = 1$ and $a_0, a_1 > 0$ we may find the optimal moving averages by Corollary 3.14. The linear difference equation (3.15) may now be written

$$(4.25) \quad (a_0 - a_1 \delta^2) x_\nu = \sum_{i=1}^{m} \theta_i A_i(\nu) .$$

Some simple computation shows that the corresponding homogenous equation has the general solution

$$c_1 z_1^{\nu} + c_2 z_2^{\nu} ,$$

where

$$(4.26) \quad z_{1,2} = \tfrac{1}{2}\left(1 + \frac{1}{a_1} \pm \sqrt{\frac{2}{a_1} + \frac{1}{a_1^2} - 3}\right) .$$

Thus, the optimal moving average with range $(-k,k)$, centre τ and polynomials of degree m as basis is given by

$$(4.27) \quad r_{\nu\tau}^{(m)} = c_{0\tau} + c_{1\tau}\nu + \dots + c_{m\tau}\nu^m + c_{m+1,\tau} z_1^{\nu} + c_{m+2,\tau} z_2^{\nu} ,$$

where the $c_{i\tau}$'s are given by:

$$\sum_{\nu=-k}^{k} r_{\nu\tau}^{(m)} \nu^p = \tau^p \; ; \; p = 0,1,\ldots,m$$

$$r_{\nu\tau}^{(m)} = 0 \; ; \; \nu = \pm (k+1) \; .$$

Further, the optimal moving average with range $(-k,k)$, centre τ and $\alpha+\beta c^t$ as basis is given by $(a_1 \neq c/(c^2-c+1))$

$$(4.28) \quad r_{\nu\tau} = c_{0\tau} + c_{1\tau}c^{\nu} + c_{2\tau}z_1^{\nu} + c_{3\tau}z_2^{\nu} \; ,$$

where the $c_{i\tau}$'s now are given by

$$\sum_{\nu=-k}^{k} r_{\nu\tau} = 1$$

$$\sum_{\nu=-k}^{k} r_{\nu\tau}c^{\nu} = c^{\tau}$$

$$r_{\nu\tau} = 0 \; ; \; \nu = \pm (k+1) \; .$$

We have computed the moving averages given by (4.27) for different values of a_1, k and m. As for the minimum-variance and minimum-R_z moving averages the optimal centre is $\tau^* = 0$ for $m = 1,3,5$ (for the values of k and a_1 we have considered) while the centralized moving averages not are the best for $m = 2,4$. In Appendix C we have given the weights corresponding to the optimal centre $\tau^* = 0$ for $k = 3,4,\ldots,10$ when $m = 3$ and $a_1 = 0.75$. We have also given the values of R_z^2 for $z = 0,1,2,3,4$ for the actual moving averages. Similar tables for $m = 1,2,4,5$ are given in Borgan (1976, Appendix B).

We have also computed the moving averages given by (4.28) for different values of a_1 and k when $c = 1.0$. In Appendix C we have given the weights $r_{\nu\tau}{}^*$ corresponding to the optimal centre τ^* for $k = 3,4,\ldots,10$ when $a_1 = 0.75$. We have also given τ^* and the values of R_z^2 for $z = 0,1,2,3,4$ for the moving averages.

The technique used above is not useable in general when $a_z > 0$ for some $z \geq 2$. This is due to the fact that we usually not are able to find the general solution of the linear homogenous difference equation

$$\sum_{z=0}^{K} a_z(-1)^z \delta^{2z} x_\nu = 0$$

corresponding to (3.15).

Therefore, in this case, we have to find the optimal moving averages directly by Theorem 3.7. According to this theorem the optimal moving average with range (α,β), centre τ and basis $\sum_{i=1}^{m} \theta_i A_i(t)$ is given uniquely by

$$(4.29) \quad \underset{\sim}{x}^* = \underset{\sim}{S}_a^{-1} \underset{\sim}{A}'(\underset{\sim}{A}\, \underset{\sim}{S}_a^{-1} \underset{\sim}{A}')^{-1}\underset{\sim}{A}(\tau),$$

where $\underset{\sim}{S}_a$ is given by (2.27) and $\underset{\sim}{A}$ and $\underset{\sim}{A}(\tau)$ are given in paragraph 3.B.

By means of modern electronic computors the optimal moving averages may be computed directly from (4.29) with high accuracy*.

We have found the moving averages with range $(-k,k)$, centre τ and polynomials of degree m as basis which minimize $\sum_{z=0}^{K} a_z \binom{2z}{z} R_z^2$ for different values of k, τ, m and

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* When CDC CYBER 74, University of Oslo, was used minimum-R_z moving averages computed as described in the paragraphs 4.E. and 4.F. and directly by (4.29) agreed in all 6 decimals given.

$a_0, a_1, \ldots, a_K$. Also in these cases we find that $\tau^* = 0$ is the optimal centre when $m = 1,3,5$, while the centralized moving averages not are the best for $m = 2,4$. In Appendix C we have given the weights corresponding to the optimal centre $\tau^* = 0$ for $k = 3,4,\ldots,10$ when $m = 3$ and $(a_0, a_3) = (0.10, 0.90)$. We have also given the values of R_z^2 for $z = 0,1,2,3,4$ for the actual moving averages. Similar tables for $m = 1,2,4,5$ and for other values of the a_z's are given by Borgan (1976, Appendix B).

We have also computed the optimal moving averages with range $(-k,k)$, centre τ and **Gompertz-Makeham's** formula, $\alpha+\beta c^t$, with $c = 1.10$, as basis for different values of k,τ and $a_0, a_1, \ldots, a_K$. In Appendix C we have given the weights $r_{\nu\tau^*}$ corresponding to the optimal centre τ^* for $k = 3,4,\ldots,10$ when $(a_0, a_3) = (0.25, 0.75)$. We have also given τ^* and the values of R_z^2 for $z = 0,1,2,3,4$ for the actual moving averages. Tables for other values of the a_z's are given by Borgan (1976, Appendix B).

5. COMPARISON OF DIFFERENT MOVING AVERAGES.

5.A. In this chapter we discuss a sort of "robustness" of different moving averages when the observations are uncorrelated and have equal variance. That is, we discuss how given moving averages perform according to different criteria of optimality (i.e. different values of the a_z's in (2.4)). For this purpose we introduce the R_z-efficiency defined in Paragraph 5.B, and we use this and Monte-Carlo experiments to compare different optimal moving averages, exact for cubics, of length 21 and different optimal moving averages of length 11 with Gompertz-Makcham's function as basis. The first moving averages are also compared with the Spencer 21-term formula, which is found to be approximately equal to the corresponding minimum-R_5 moving average.

5.B. In Paragraph 2.C. we found that a_0 in the loss function (2.4) is a weight for the "fit", a_1 a weight for the "gradient", a_2 a weight for the "curvature", etc. Furthermore, when $\underset{\sim}{\Sigma} = \sigma^2 \underset{\sim}{I}$ we have by (2.26):

$$E\underset{\sim}{W}'\underset{\sim}{T}_a\underset{\sim}{W} = \sigma^2 \sum_{z=0}^{k} a_z \binom{2z}{z} R_z^2 \; .$$

Thus, if we have a reasonable choice of the basis, the quantities R_z^2 , $z = 0,1,2,\ldots$ measure how a moving average reproduces different properties of the trend $\{\xi_t\}$. A measure of the relative size of R_z^2 for a given moving average is now the R_z-efficiency defined by:

<u>Definition 5.1</u>: Let $\underset{\sim}{r} = (r_\alpha, r_{\alpha+1}, \ldots, r_\beta)'$ be a moving average with range (α,β) , centre τ and given basis, and let

$\underset{\sim}{r}^* = (r_\alpha^*, r_{\alpha+1}^*, \ldots, r_\beta^*)'$ be the minimum-R_z moving average with the same range and basis and optimal centre τ^*. Then the R_z-efficiency of $\underset{\sim}{r}$ is given by

$$(5.2) \qquad e_z(\underset{\sim}{r}) = \frac{\sum_{\nu=\alpha-z}^{\beta} (\Delta^z r_\nu^*)^2}{\sum_{\nu=\alpha-z}^{\beta} (\Delta^z r_\nu)^2} .$$

The R_z-efficiency defined by (5.2) generalizes the "error-reducing efficiency" and the "smoothing efficiency" defined by Pollard (1971a, p. 22). These correspond to $\sqrt{e_0(\underset{\sim}{r})}$ and $\sqrt{e_3(\underset{\sim}{r})}$ respectively.

5.C. We will compare different centralized moving averages, exact for cubics, of length 21. The actual moving averages are the minimum-variance moving average, the minimum-R_z moving averages for $z = 1,2,3,4$, the optimal moving averages for $(a_0,a_1) = (0.25, 0.75)$ and $(a_0,a_3) = (0.1, 0.9)$, and the Spencer's 21-term formula (1.3). The weights r_ν of these moving averages are given in Figure 5.3.

The figure shows that the curve for the weights of the Spencer's 21-term formula has the same shape as the curve for the minimum-R_z moving averages for $z = 1,2,3,4$, but it is more peaked. Because of this we have also compared the Spencer's 21-term formula with the centralized

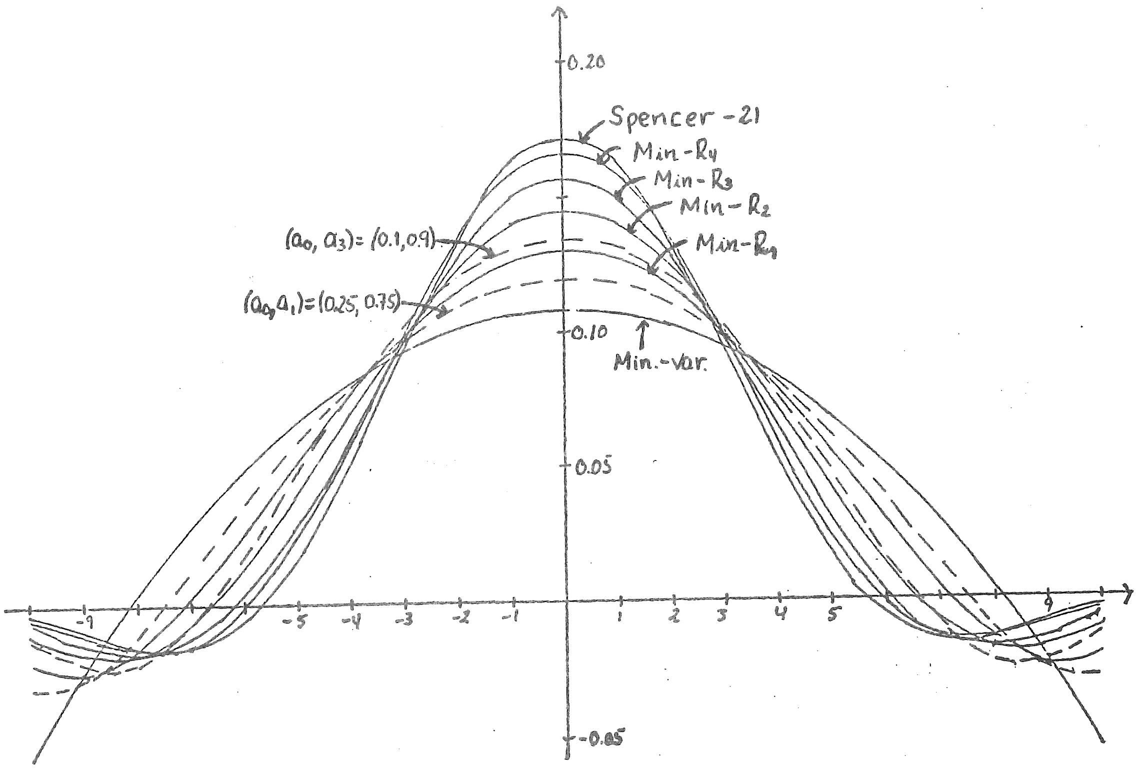

Figure 5.3: The weights of different centralized moving averages, exact for cubics, of length 21.

minimum-R_5 moving average, exact for cubics, of length 21. The weights of these two moving averages are given in Table 5.4. We find that the two moving averages are approximately equal.

We will now have a closer look at the R_z-efficiencies of the different moving averages. These are given in Table 5.5. First we note that the R_z-efficiencies for the minimum-variance moving average are very low for $z > 0$. This indicates that this moving

ν	0	±1	±2	±3	±4	±5	±6	±7	±8	±9	±10
Spencer	.1714	.1629	.1343	.0943	.0514	.0171	-.0057	-.0143	-.0143	-.0086	-.0029
min-R_5	.1745	.1638	.1346	.0939	.0153	.0158	-.0069	-.0157	-.0140	-.0077	-.0022

Table 5.4: The weights of Spencer's 21-term formula and the corresponding minimum-R_5 moving average.

moving average \ z	0	1	2	3
Min-var	1.000	0.533	0.065	0.007
Min-R_1	0.937	1.000	0.587	0.112
Min-R_2	0.866	0.877	1.000	0.635
Min-R_3	0.813	0.736	0.856	1.000
Min-R_4	0.774	0.633	0.681	0.850
$(a_0,a_1)=(0.25, 0.70)$	0.971	0.933	0.306	0.048
$(a_0,a_3)=(0.10, 0.90)$	0.891	0.913	0.898	0.500
Spencer	0.751	0.577	0.576	0.591

Table 5.5: R_z-efficiencies for some centralized moving averages, exact for cubics, of length 21.

average will perform unsatisfactorily if we are interested in estimating the trend $\{\xi_t\}$ as a sequence, and not only in getting a good estimate of ξ_t for each t. We get significantly higher R_1- and R_2-efficiencies, and only slightly lower R_0-efficiencies, if we use the optimal moving average with $(a_0, a_1) = (0.25, 0.75)$ instead of the minimum-variance moving averages. Thus, we will expect the optimal moving average with $(a_0, a_1) = (0.25, 0.75)$ to give nearly as good "fit" as the minimum-variance moving average, and much better reproduction of the gradient and the curvature of the trend $\{\xi_t\}$. The same conclusions are valid for the minimum-R_1 moving average, which has somewhat lower R_0-efficiency and higher R_1- and R_2-efficiencies. Summed up, the minimum-variance moving average is little "robust" to other criteria of optimality, while the other two moving averages are more "robust".

Greville (1972) and Pollard (1971a, p.22) claim that the minimum-R_3 moving average is the best for "smoothing" purposes. Table 5.5 shows that this moving average has rather low values for the R_0- and R_1-efficiencies. Thus, even if this moving average will produce a "smooth" curve, we will expect it to give a rather bad "fit" and especially an unsatisfactory reproduction of the gradient. If we instead use the optimal moving average with $(a_0, a_3) = (0.1, 0.9)$ we get a reasonable high R_3-efficiency and much better values of the R_0- and R_1-efficiencies.

Further, we see from Table 5.5 that our objections against the minimum-R_3 moving average are even more valid for the minimum-R_4 moving average and Spencer's 21-term formula.

To see how the actual moving averages perform in practical work, and to test our reasoning above, we have done some Monte-Carlo experiments. On an electronic computer (CDC 3300, University of Oslo) we have generated independent and normally distributed "observations" $X_0, X_1, \ldots, X_{100}$ with equal variance σ^2 and

$$\xi_t = EX_t = 0.0001\, t^3 - 0.02\, t^2 + t \,,$$

$t = 0.1, \ldots, 100$. Then we have graduated these "observations" by the moving averages given above. Results for a randomly drawn experiment when $\sigma = 1.5$ are given in Figure 5.6 a-h.

The Monte Carlo experiments confirm the conclusions we gave from the discussion of the R_z-efficiencies. Typically the minimum-variance moving average produces a somewhat ragged curve, while the minimum-R_3 and R_4 moving averages and the Spencer's 21-term formula give "smooth" curves that badly follow the trend $\{\xi_t\}$. By eyeball-inspection the minimum-R_1 moving average and the optimal moving averages with $(a_0, a_1) = (0.25, 0.75)$ and $(a_0, a_3) = (0.1, 0.9)$ give the best results if the purpose of the graduation is more than estimating ξ_t for each t .

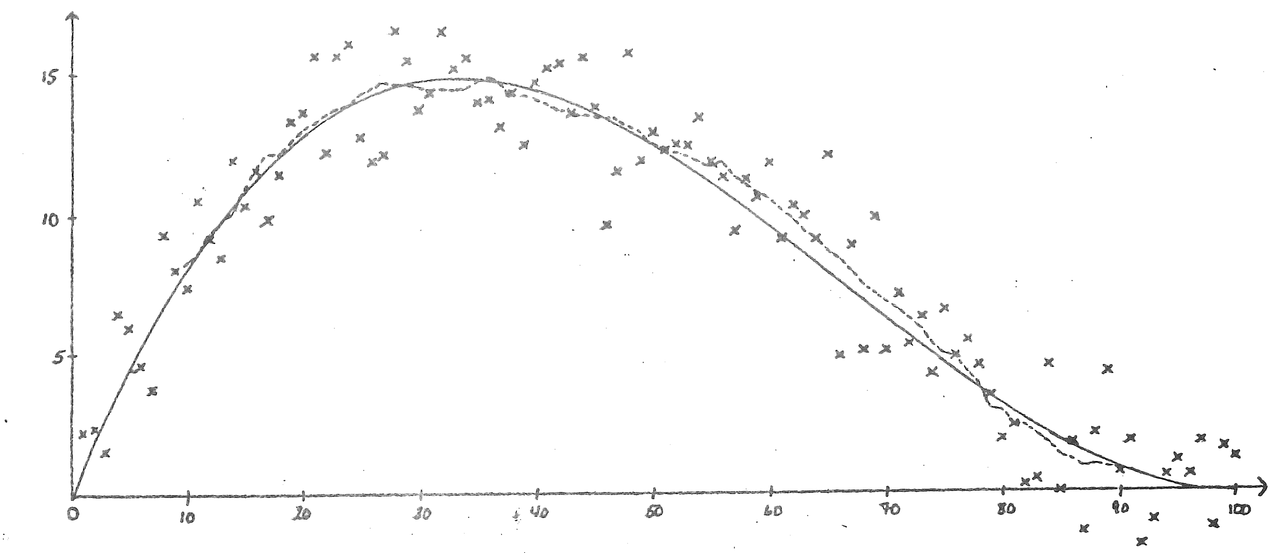

Figure 5.6.a: Independent normally distributed "observations" with expected value $\xi_t = 0.0001\ t^3 - 0.02\ t + t$ (drawn line) and standard deviation 1.5 graduated by means of the centralized minimum-variance moving average, exact for cubics, of length 21 (dotted line).

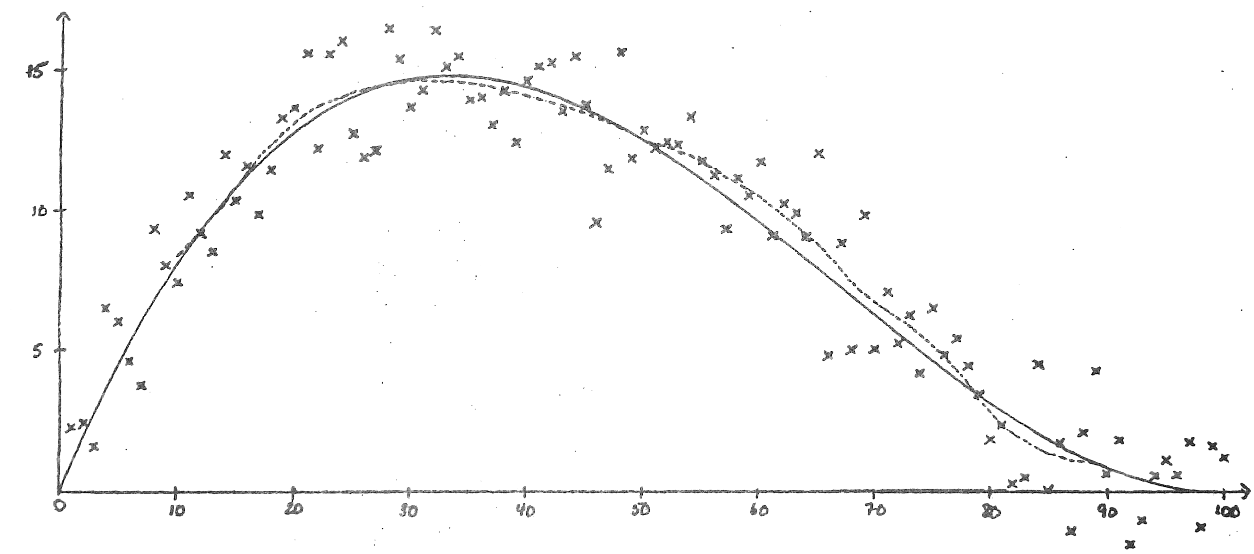

Figure 5.6.b: Graduated by means of the centralized minimum-R_1 moving average, exact for cubics, of length 21.

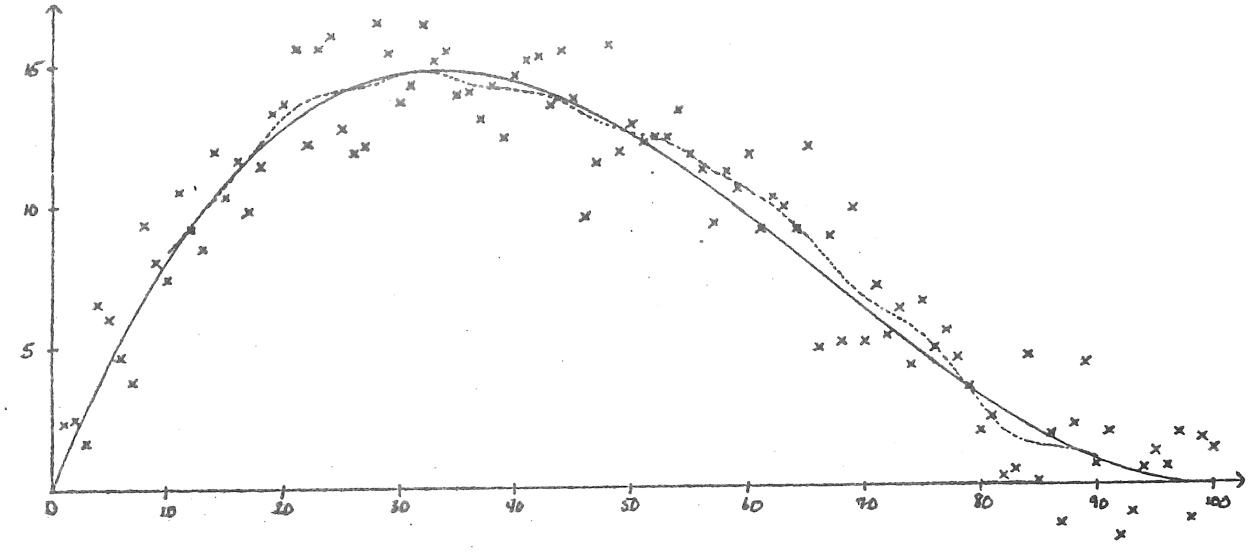

Figure 5.6.c: Graduated by means of the centralized minimum-R_2 moving average, exact for cubics, of length 21.

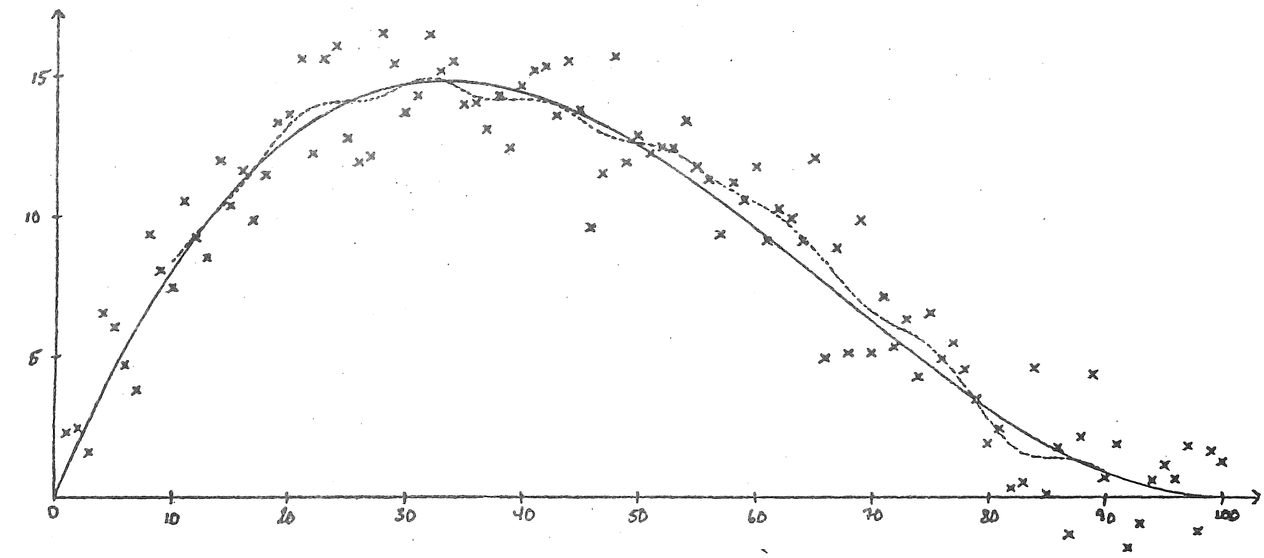

Figure 5.6.d: Graduated by means of the centralized minimum-R_3 moving average, exact for cubics, of length 21.

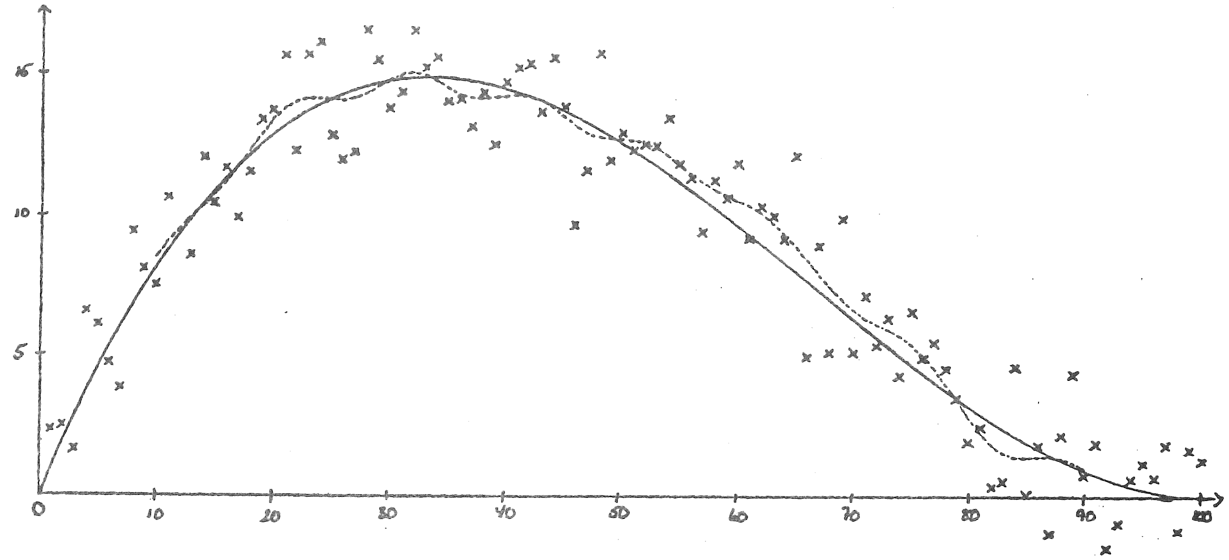

Figure 5.6.e: Graduated by means of the centralized minimum-R_4 moving average, exact for cubics, of lenght 21.

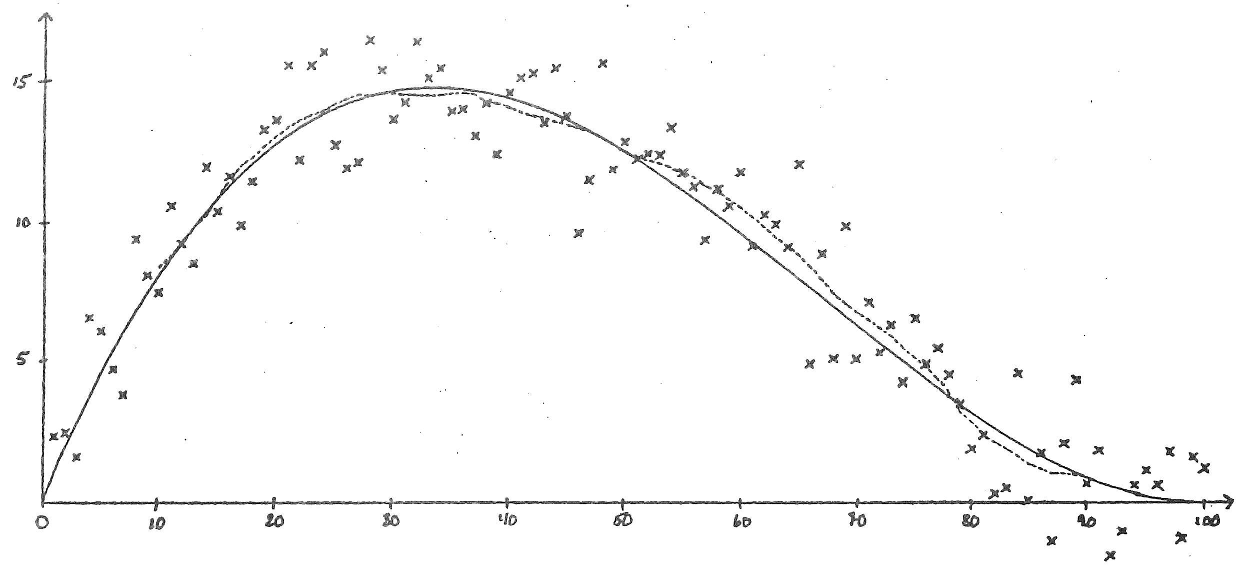

Figure 5.6.f: Graduated by means of the centralized optimal moving average with $(a_0, a_1) = (0.25, 0.75)$, exact for cubics, of length 21.

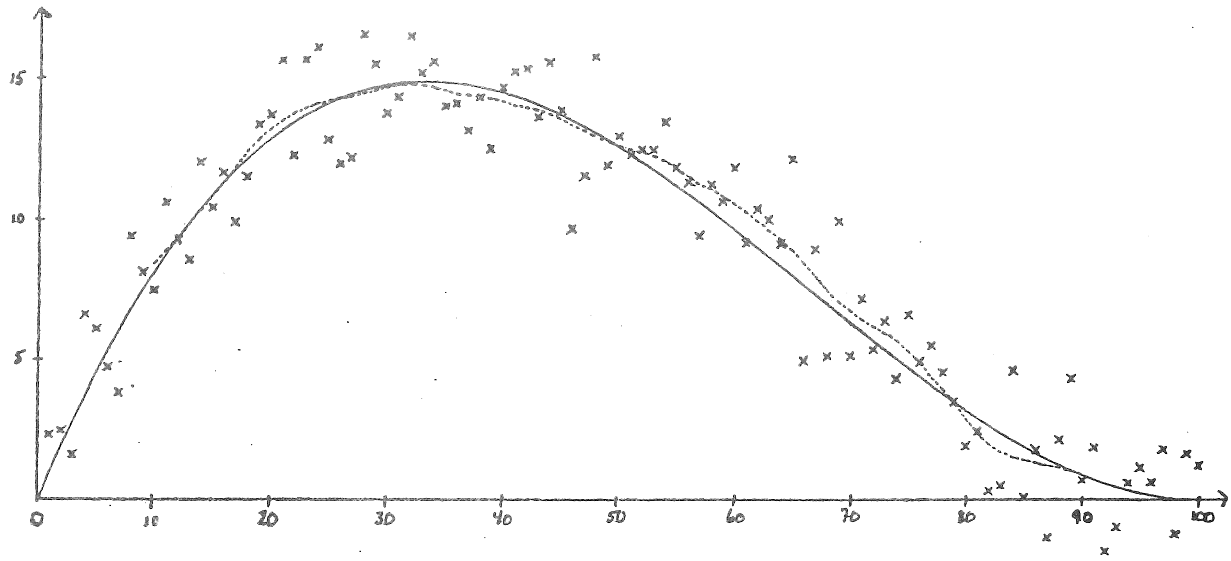

Figure 5.6.g: Graduated by means of the centralized optimal moving average with $(a_0, a_3) = (0.1, 0.9)$, exact for cubics, of length 21.

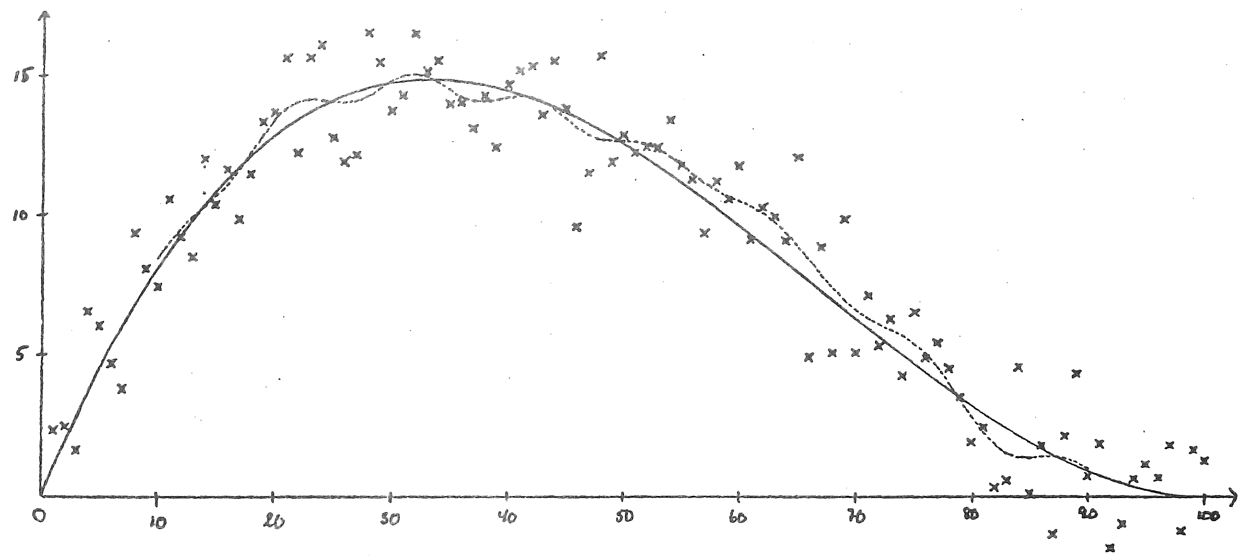

Figure 5.6.h: Graduated by means of Spencer's 21-term formula.

5.D. We will also compare different centralized moving averages of length 11 with basis $\alpha+\beta(1.1)^t$. The actual moving averages are the minimum-variance moving average, the minimum-R_z moving averages for $z = 1,2,3,4$ and the optimal moving averages for $(a_0, a_1) = (0.25, 0.75)$ and $(a_0, a_3) = (0.25, 0.75)$. The weights of these moving averages are given in Figure 5.7.

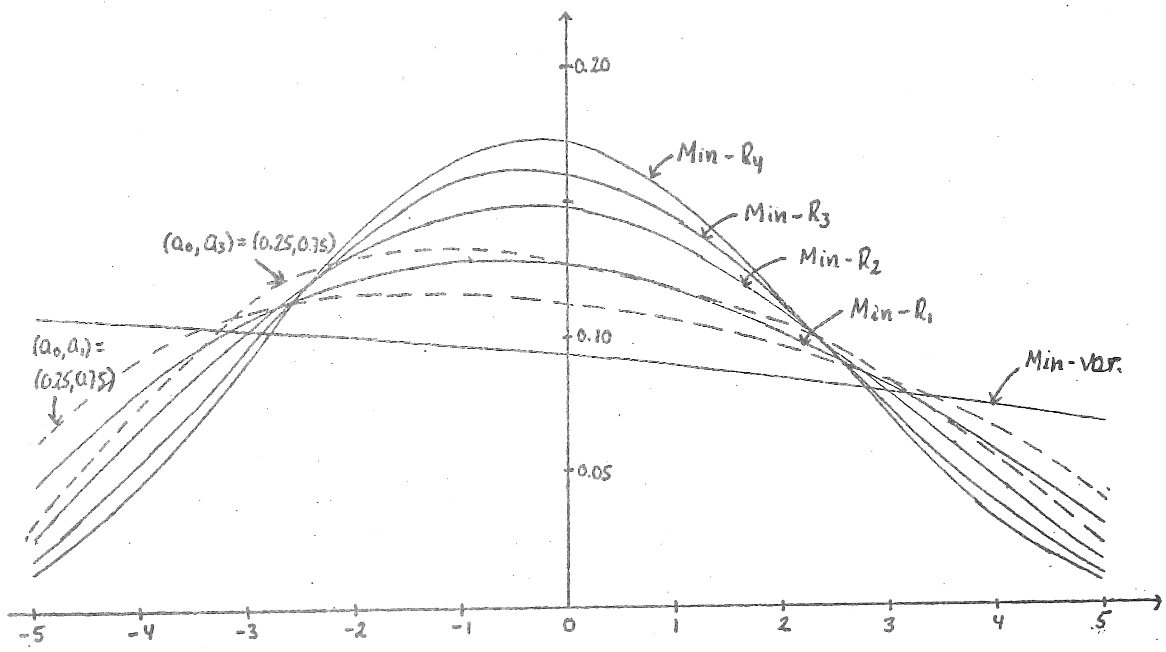

Figure 5.7: The weights of different centralized moving averages of length 11 with basis $\alpha+\beta(1.1)^t$.

Moving average \ z	0	1	2	3
Min-var.	1.000	0.442	0.065	0.011
Min-R_1	0.902	1.000	0.585	0.153
Min-R_2	0.816	0.852	1.000	0.680
Min-R_3	0.760	0.700	0.851	1.000
Min-R_4	0.722	0.600	0.677	0.858
$(a_0,a_1)=(0.25,0.75)$	0.951	0.913	0.309	0.067
$(a_0,a_3)=(0.25,0.75)$	0.870	0.933	0.785	0.374

Table 5.8: R_z-efficiencies for some centralized moving averages of length 11 with basis $\alpha+\beta(1.1)^t$.

As in Paragraph 5.C we will study the R_z-efficiencies for the actual moving averages. These are given in Table 5.8. We find a high resemblance between the R_z-efficiencies of Table 5.5 and Table 5.8. In this case as well the R_z-efficiencies for the minimum-variance moving average are very low for $z > 0$, and for the minimum-R_3 moving average rather low for $z = 0$ and $z = 1$. The optimal moving averages with more than one $a_z > 0$ give intermediate values for the R_z-efficiencies.

Also in this case we have made some Monte Carlo experiments. On an electronic computer we have generated independent and normally distributed "observations" $X_{30}, X_{31}, \ldots, X_{70}$ with equal variance σ^2 and expected values

$$\xi_t = 0.9 + 0.044(1.1)^t .$$

Then we have graduated these "observations" by the moving averages given above. Results for a randomly drawn experiment when $\sigma = 2.0$ are given in Appendix B.

The Monte Carlo experiments give results similar to those given in Paragraph 5.C. In this case as well the minimum-variance moving average gives a somewhat ragged curve, while the minimum-R_3 and -R_4 moving averages produce "smooth" curves that do not follow the trend $\{\xi_t\}$ too well. Only in this situation the rapidly increasing trend to some extent eliminates the difference between the actual moving averages. By eyeball-inspection the minimum-R_1 moving average and the optimal moving averages for $(a_0, a_1) = (0.25, 0.75)$ and $(a_0, a_3) = (0.25, 0.75)$ give the best result if the purpose of the graduation is more than estimating ξ_t for each t.

6. CONCLUDING REMARKS.

6.A. In the present report we have discussed how to choose the weights r_ν for a moving average, and we have found optimal moving averages which generalize the well-known minimum-variance and minimum-R_z moving averages. Furthermore, the comparison of the different moving averages in Chapter 5 indicates that our generalization of the usual criteria of optimality is not only of theoretical, but also of practical interest.

However, many problems remain to be solved. We have only discussed in detail the situation with uncorrelated observations with equal variance. A more thorough discussion of other co-variance-structures would be of interest e.g. in connection with graduation of mortality tables. Even when the observations are uncorrelated and have equal variance, several problems are unsolved. The most important of these are:

i) By numeric methods we have found the optimal centre(s) for all the situations we have considered. Analytically we have only been able to do this in some situations for the minimum-variance moving averages. The numeric computations, however, indicate that Weichselberger's conjecture (see page 30) is true, not only for the minimum-variance, but for the optimal moving averages in general. Hence, our conjecture is: When the observations are uncorrelated and have equal variance the optimal centre is $\tau^* = 0$ for all optimal moving averages of range $(-k,k)$ and with polynomials of odd degree as basis.

ii) For graduation of mortality tables it seems natural to use the Gompertz-Maheham's function as basis. For most other situations there does not exist a natural basis. Therefore, one problem is to find a rule for choosing between different bases.

iii) A given trend may be approximated by polynomials of different degrees over intervals of different lengths. Hence, polynomials of different degrees may be used as basis when we graduate by moving averages. It is of interest to find which degree of the polynomials that give the best result. Kockelkorn and Rüger (1974, pp.327-328) discuss this problem for the minimum-variance moving averages. They compute which lengths of moving averages with optimal centre, exact for polynomials of different degrees, that give the same value of $E\underset{\sim}{W}'\underset{\sim a}{T}\underset{\sim}{W} = r_{\tau\tau}^{(m)}$ (compare Paragraph 4.C). From this they conclude that polynomials of first and third degree as basis should be preferred to polynomials of second degree. For example, the minimum-variance moving averages, exact for polynomials of first, second and third degree, of lengths 7, 13 and 15-17 respectively, have approximately the same value of $r_{\tau\tau}^{(m)}$. The corresponding values for the minimum-R_1 moving averages are 7, 11-13 and 15 and for the minimum-R_2 and $-R_3$ moving averages 7, 11 and 13. These, and similar computations, indicate that it is convenient to use polynomials of third degree as basis. This is, in fact, also usually done in practical work. There are, however, necessary to investigate this problem more closely.

iv) By the moving average (1.1) we do not get estimates of ξ_t for the end values of t, i.e. for $t = 1,2,\ldots,\tau-\alpha$

and $t = N+\tau-\beta+1,\ldots,N$. Quite a lot of methods to solve this problem have been proposed in the literature. References to such works are given e.g. by Weichselberger (1964, pp.208-214). By all the proposed methods one estimates ξ_t for the end values by a linear combination of the X_t's , but the weights are now allowed to depend on t . Thus, the problem is to find which weights to use for the different end values. According to the discussion in Paragraph 2.C. it is in this case natural to judge the graduation by a loss function of the form

$$(6.1) \qquad T_{\underset{\sim}{a}}(\underset{\sim}{\xi},\hat{\underset{\sim}{\xi}}) = \sum_{z=0}^{K} \frac{a_z}{N-z} \sum_{t=1}^{N-z} (\Delta^z\hat{\xi}_t - \Delta^z\xi_t)^2 ,$$

where now $\hat{\underset{\sim}{\xi}} = (\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N)'$ and $\underset{\sim}{a} = (a_0,a_1,\ldots,a_K)$ is a vector of non-negative real numbers such that $\sum_{z=0}^{K} a_z = 1$. The criterion of optimality given in Paragraph 3.B is easily generalized to the present situation. For $a_0 = 1$ the method proposed by Weichselberger (1964) is optimal. By this method one uses the usual minimum-variance moving average with optimal centre for the central values of t , and for the end values one uses the minimum-variance moving average with the one of the possible centres τ that minimize $r_{\tau\tau}$. For $a_3 = 1$ Greville (1947) proposed, analogous to the method given above, to estimate ξ_t for the end values of t by non-centralized minimum-R_3 moving averages. As Greville (1972, pp. 17-18) himself points out, this does not give a correct solution of the end value problem. When $a_z > 0$ for some $z > 0$ in (6.1) it is not easy to find the optimal method for estimating

ξ_t for the end values. The difficulty is that we do not get a result analogous to (2.19) when the weights in the moving averages may depend on t. Therefore, we do not know how to get an optimal solution of the end value problem according to (6.1) when $a_z > 0$ for some $z > 0$.

6.B. To conclude the report we will point out that there exists quite a lot of graduation methods in addition to graduation by moving averages (compare Paragraph 1.B). In many situations some of these are probably better graduation methods than moving average graduation. The last method, however, has the advantage that it is easy to use, and that it requires only little apriori knowledge of the trend and the distribution of the X_t's. In this respect graduation by moving averages is a robust method similar to the non-parametric methods.

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APPENDIX A - PROOF OF LEMMA 3.5.

We will prove:

Lemma 3.5: Let A be a $m\times l$ matrix of rank m $(m < l)$ and let S be a positive definite $l\times l$ matrix. Then there exists a unique $r^*\in \mathbb{R}^l$ which minimizes $r'Sr$ among all $r\in\mathbb{R}^l$ under the constraint

$$\text{(A.1)} \qquad r'A'\theta = c'\theta \quad \text{for all} \quad \theta\in\mathbb{R}^m ,$$

where $c\in\mathbb{R}^m$ is a known vector. r^* is given by

$$\text{(A.2)} \qquad r^* = S^{-1}A'(AS^{-1}A')^{-1}c .$$

Proof: Suppose first that S is the identity matrix. Since A' is a $l\times m$ matrix of rank m, there exists a $l\times(l-m)$ matrix B' such that (A',B') is non-singular $l\times l$ matrix. Hence, any $r\in\mathbb{R}^l$ may be given by $r = (A',B')\tau$ for a $\tau\in\mathbb{R}^l$. Write now $\tau = (\tau'_{(1)},\tau'_{(2)})'$, where $\tau_{(1)}\in\mathbb{R}^m$ and $\tau_{(2)}\in\mathbb{R}^{l-m}$. If $r = (A',B')\tau$ is to satisfy the constraint (A.1), we have

$$r'A'\theta = (\tau'_{(1)},\tau'_{(2)})\binom{A}{B}A'\theta = c'\theta \quad \text{for all} \quad \theta\in\mathbb{R}^m ,$$

or

$$\tau'_{(1)}AA'\theta + \tau'_{(2)}BA'\theta = c'\theta \quad \text{for all} \quad \theta\in\mathbb{R}^m .$$

Especially for $\theta = \tau_{(1)}$ this gives:

$$\text{(A.3)} \qquad \tau'_{(1)}AA'\tau_{(1)} + \tau'_{(2)}BA'\tau_{(1)} = c'\tau_{(1)} .$$

We now concentrate on finding the $r\in\mathbb{R}^l$ which minimizes $r'r$ under the constraint (A.3). We find using (A.3):

$$\underset{\sim}{r}'\underset{\sim}{r} = (\underset{\sim}{\tau}'_{(1)},\underset{\sim}{\tau}'_{(2)})\binom{\underset{\sim}{A}}{\underset{\sim}{B}}(\underset{\sim}{A}',\underset{\sim}{B}')\binom{\underset{\sim}{\tau}_{(1)}}{\underset{\sim}{\tau}_{(2)}}$$

$$= 2\underset{\sim}{c}'\underset{\sim}{\tau}_{(1)} + \underset{\sim}{\tau}_{(2)}\underset{\sim}{B}\underset{\sim}{B}'\underset{\sim}{\tau}_{(2)} - \underset{\sim}{\tau}'_{(1)}\underset{\sim}{A}\underset{\sim}{A}'\underset{\sim}{\tau}_{(1)} \ .$$

We will minimize this expression with respect to $\underset{\sim}{\tau}$. Now $\underset{\sim}{B}\underset{\sim}{B}'$ is positive definit since $\underset{\sim}{B}$ is of full rank. Thus, the minimizing value of $\underset{\sim}{\tau}_{(2)}$ is $\underset{\sim}{\tau}^*_{(2)} = \underset{\sim}{0}$. Further, we have to minimize

$$Q(\underset{\sim}{\tau}_{(1)}) = 2\underset{\sim}{c}'\underset{\sim}{\tau}_{(1)} - \underset{\sim}{\tau}'_{(1)}\underset{\sim}{A}\underset{\sim}{A}'\underset{\sim}{\tau}_{(1)}$$

$$= 2\sum_{i=1}^{m} c_i\tau_i - \sum_{i,j} \tau_i\tau_j(\underset{\sim}{A}\underset{\sim}{A}')_{ij}$$

with respect to $\underset{\sim}{\tau}_{(1)} = (\tau_1,\tau_2,\ldots,\tau_m)'$. Here $(\underset{\sim}{A}\underset{\sim}{A}')_{ij}$ is the (i,j)-th element of $\underset{\sim}{A}\underset{\sim}{A}'$. Partial differentiation of $Q(\underset{\sim}{\tau}_{(1)})$ with respect to τ_j ; $j = 1,2,\ldots,m$, gives:

$$\frac{\partial Q(\underset{\sim}{\tau}_{(1)})}{\partial \tau_j} = 2c_j - 2\sum_{i=1}^{m} \tau_i(\underset{\sim}{A}\underset{\sim}{A}')_{ij} \ .$$

This gives the following equations for the minimizing values $\tau^*_1,\tau^*_2,\ldots,\tau^*_m$:

$$\sum_{i=1}^{m} \tau^*_i \, (\underset{\sim}{A}\underset{\sim}{A}')_{ij} = c_j \ ; \ j = 1,2,\ldots,m \ .$$

In matrix notation this may be written

$$AA'\tau^*_{(1)} = \underset{\sim}{c} \ .$$

Since $\underset{\sim}{A}$ is of full rank $\underset{\sim}{A}\underset{\sim}{A}'$ is non-singular and we have

$$\underset{\sim}{\tau}^*_{(1)} = (\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{c} \ .$$

This shows that

$$r^* = (A', B')\tau^* = A'\tau^*_{(1)} = A'(AA')^{-1}c$$

is the unique vector $r \in \mathbb{R}^l$ which minimizes $r'r$ under the constraint (A.3). Now r^* satisfyes the constraint (A.1) and the lemma is proved for the case $S = I$.

Let now S be any positive definite matrix. Then there exists a non-singular matrix P such that $P'SP = I$ (see e.g. Anderson, 1958, p. 339). We have $S = (PP')^{-1}$. Introduce for each $r \in \mathbb{R}^l$ $\bar{r} = P^{-1}r$. We then find

$$r'Sr = r'(PP')^{-1}r = (P^{-1}r)'(P^{-1}r) = \bar{r}'\bar{r} .$$

The constraint (A.1) may now be written

$$\text{(A.4)} \qquad c'\theta = r'A'\theta = \bar{r}'P'A'\theta = \bar{r}'(AP)'\theta$$

for all $\theta \in \mathbb{R}^m$. Since AP is a $m \times l$ matrix of full rank we have by what we have just proved, that there exists a unique $\bar{r}^* \in \mathbb{R}^l$ which minimizes $\bar{r}'\bar{r}$ under the constraint (A.4). $\bar{r}^*$ is given by

$$\bar{r}^* = (AP)'(APP'A')^{-1}c = P'A'(AS^{-1}A')^{-1}c .$$

Thus, there exists a unique $r^* \in \mathbb{R}^l$ which minimizes $r'Sr$ under the constraint (A.1), and this r^* is given by

$$r^* = P\bar{r}^* = S^{-1}A'(AS^{-1}A')^{-1}c . \qquad \square$$

APPENDIX B - FIGURES

This appendix contains results for one of the Monte Carlo experiments described in Paragraph 5.D.

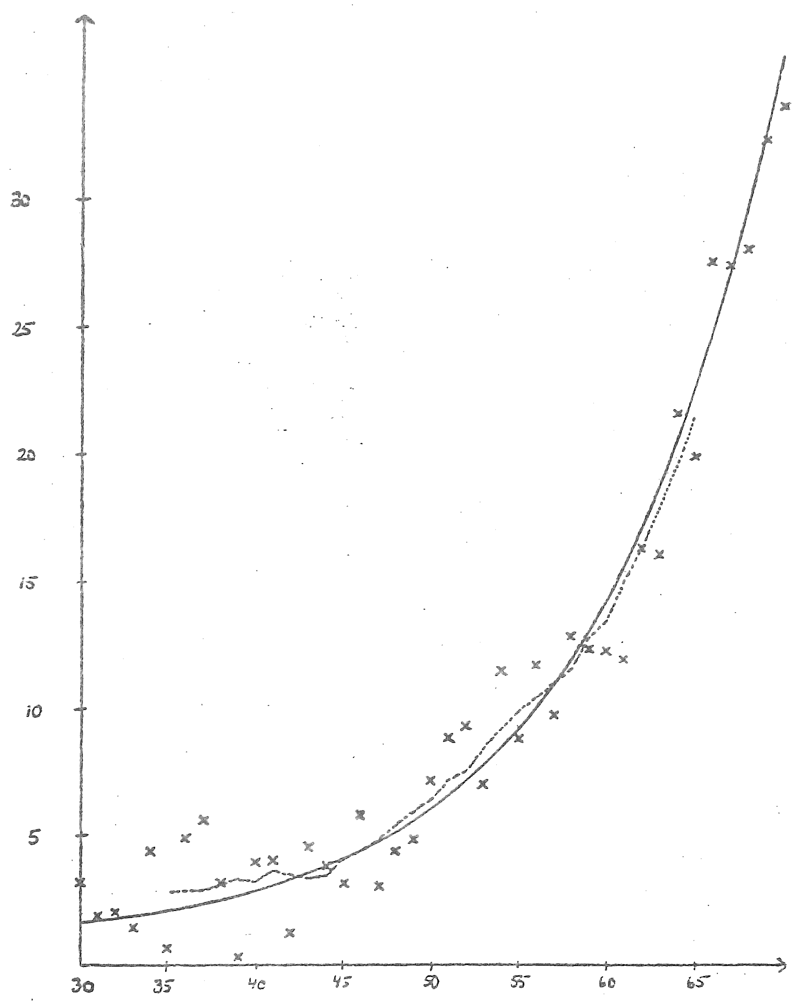

Figure B.1: Independent normally distributed "observations" with expected values $\xi_t = 0.9 + 0.044 \cdot (1.1)^t$ (drawn line) and standard deviation 2.0 graduated by the centralized minimum-variance moving average of length 11 and with basis $\alpha+\beta(1.1)^t$.

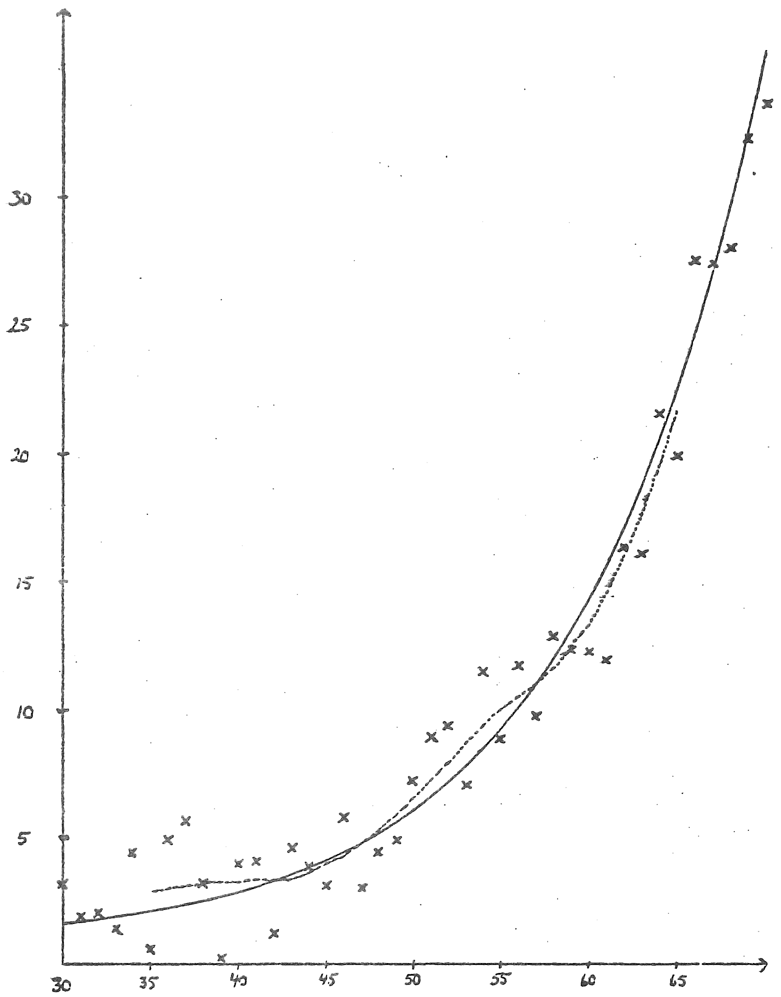

Figure B.2: Graduated by the centralized minimum-R_1 moving average of length 11 and with basis $\alpha+\beta(1.1)^t$.

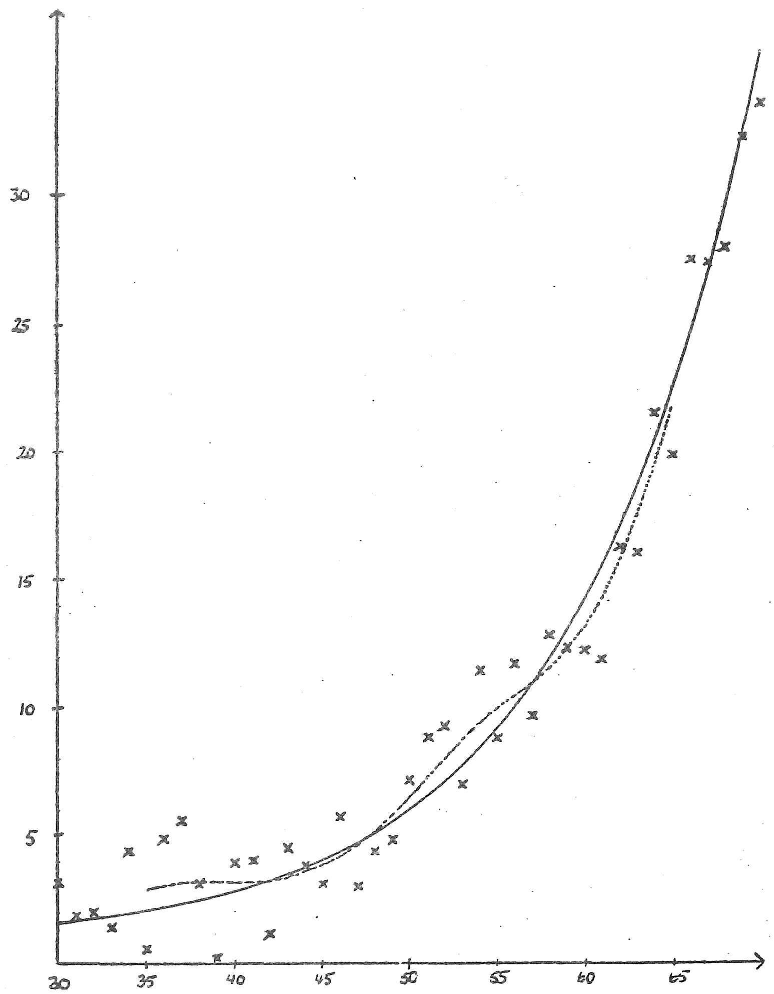

Figure B.3: Graduated by the centralized minimum-R_2 moving average of length 11 and with basis $\alpha+\beta(1.1)^t$.

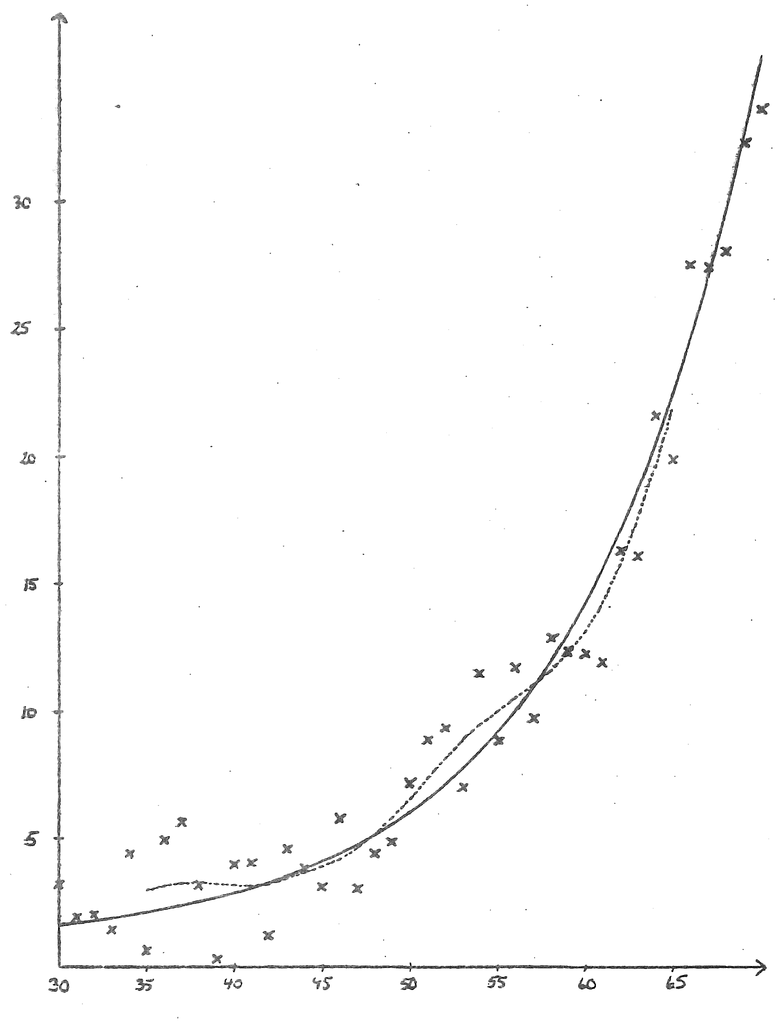

Figure B.4: Graduated by the centralized minimum-R_3 moving average of length 11 and with basis $\alpha+\beta(1.1)^t$.

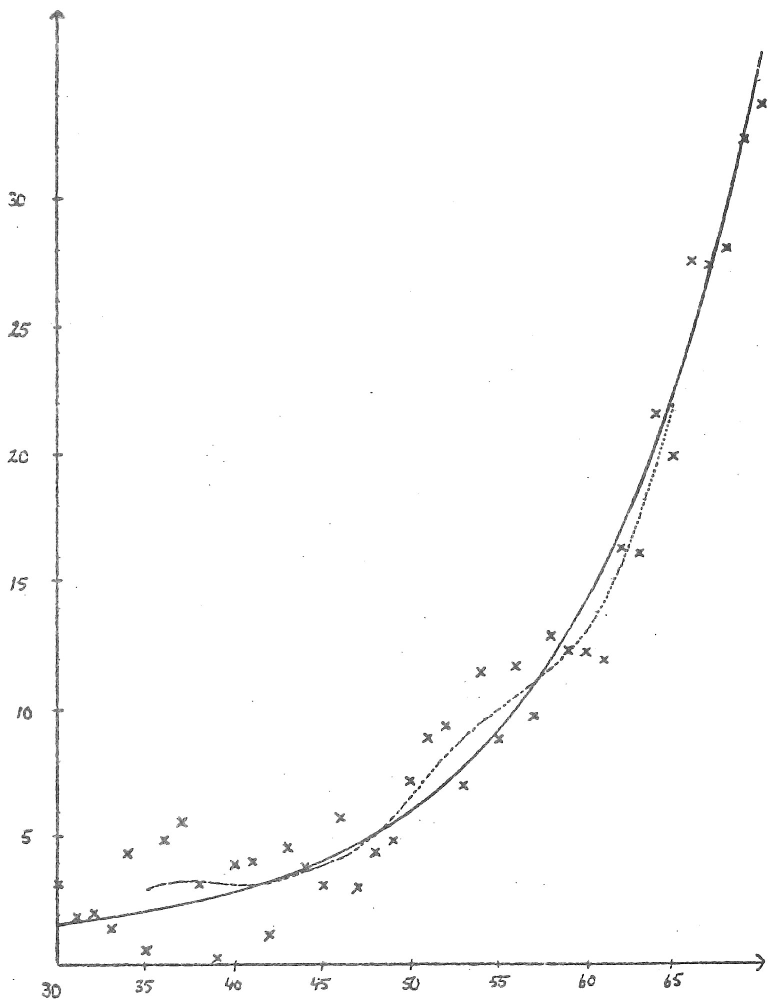

Figure B.5: Graduated by the centralized minimum-R_4 moving average of length 11 and with basis $\alpha+\beta(1.1)^t$.

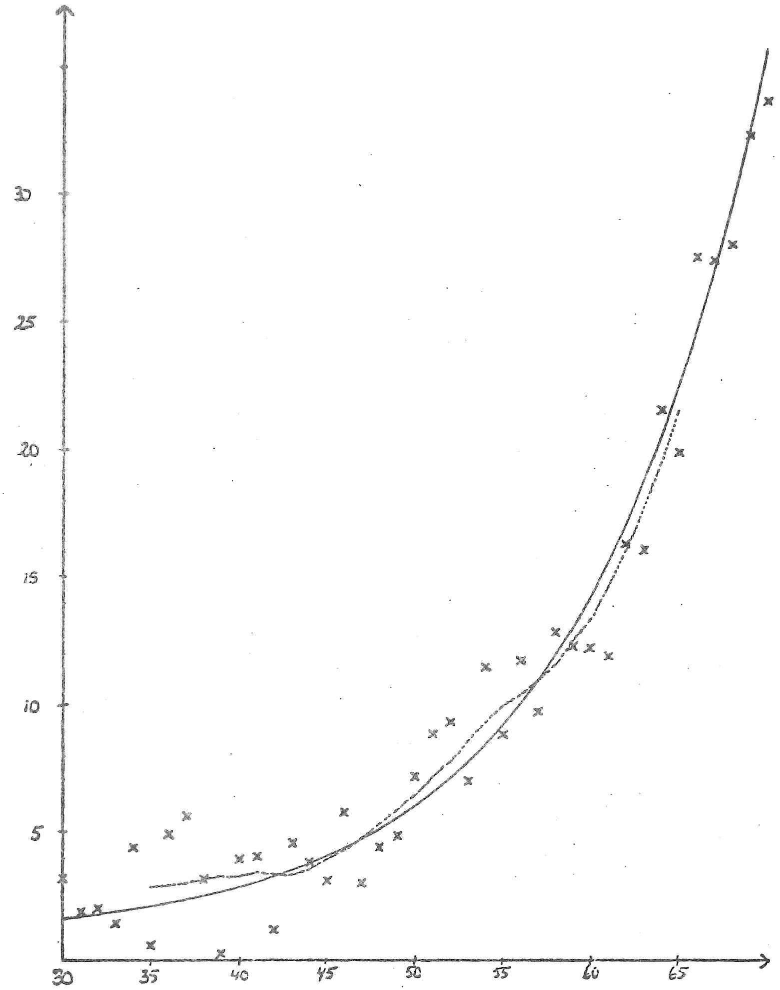

Figure B.6: Graduated by the centralized optimal moving average with $(a_0, a_1) = (0.25, 0.75)$ of length 11 and with basis $\alpha+\beta(1.1)^t$.

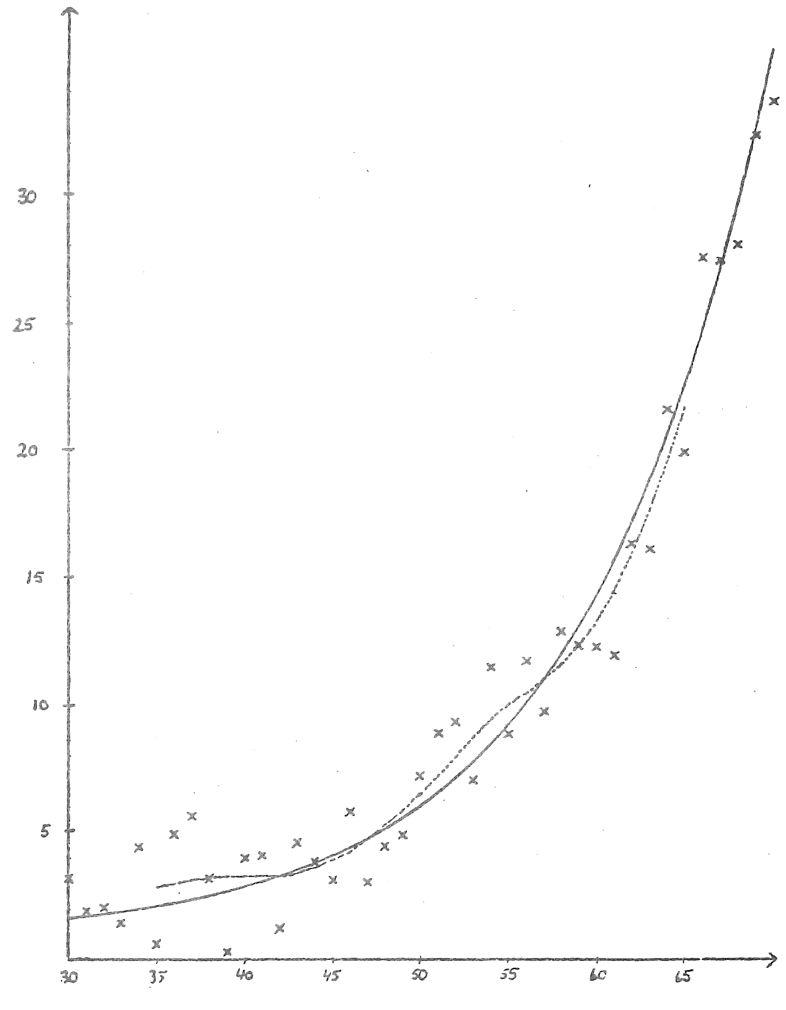

Figure B.7: Graduated by the centralized optimal moving average with $(a_0, a_3) = (0.25,\ 0.75)$ of length 11 and with basis $\alpha+\beta(1.1)^t$.

APPENDIX C - TABLES OF MOVING AVERAGES.

In this appendix we give tables of moving averages which are optimal when the observations are uncorrelated and have equal variance (compare Chapter 4).

For different values of $a_0, a_1, \ldots, a_K$ in the loss function (2.4) and for polynomials of third degree and for Gompertz-Makeham's function, $\alpha+\beta(1.1)^t$, as basis we give optimal moving averages of range $(-k,k)$; $k = 3,4,\ldots,10$; and with optimal centre τ^*. For the actual moving averages we also give the length, $l = 2k+1$, the optimal centre τ^* and the values of R_z^2 for $z = 0,1,2,3,4$ (R0, R1, R2, R3 and R4 in the tables). The values of the a_z's are $a_0 = 1$ (minimum-variance), $a_z = 1$ for $z = 1,2,3,4$ (minimum-R_z), $(a_0,a_1) = (0.25, 0.75)$, $(a_0,a_3) = (0.1, 0.9)$ (for polynomials of third degree as basis) and $(a_0,a_3) = (0.25, 0.75)$ (for Gompertz-Makeham's function as basis).

More tables of optimal moving averages are given by Borgan (1976, Appendix B).

Table 1: Minimum-variance moving averages, exact for cubics, of range (-k,k) and optimal centre τ^*.

l \ ν	7	9	11	13	15	17	19	21
-10								-.055901
-9							-.060150	-.024845
-8						-.065015	-.022556	.002942
-7					-.070588	-.018576	.010615	.027460
-6				-.076923	-.011765	.021672	.039363	.048709
-5			-.083916	-.000000	.038009	.055728	.063689	.066688
-4		-.090909	.020979	.062937	.078733	.083591	.083591	.081399
-3	-.095238	.060606	.102564	.111888	.110407	.105263	.099071	.092841
-2	.142857	.168831	.160839	.146853	.133032	.120743	.110128	.101013
-1	.285714	.233766	.195804	.167832	.146606	.130031	.116762	.105917
0	.333333	.255411	.207459	.174825	.151131	.133127	.118974	.107551
1	.285714	.233766	.195804	.167832	.146606	.130031	.116762	.105917
2	.142857	.168831	.160839	.146853	.133032	.120743	.110128	.101013
3	-.095238	.060606	.102564	.111888	.110407	.105263	.099071	.092841
4		-.090909	.020979	.062937	.078733	.083591	.083591	.081399
5			-.083916	-.000000	.038009	.055728	.063689	.066688
6				-.076923	-.011765	.021672	.039363	.048709
7					-.070588	-.018576	.010615	.027460
8						-.065015	-.022556	.002942
9							-.060150	-.024845
10								-.055901
τ^*	0	0	0	0	0	0	0	0
R0	.3333E+00	.2554E+00	.2075E+00	.1748E+00	.1511E+00	.1331E+00	.1190E+00	.1076E+00
R1	.8844E-01	.4762E-01	.2946E-01	.1990E-01	.1430E-01	.1074E-01	.8357E-02	.6678E-02
R2	.4762E-01	.2453E-01	.1505E-01	.1022E-01	.7421E-02	.5646E-02	.4446E-02	.3596E-02
R3	.3764E-01	.2010E-01	.1264E-01	.8734E-02	.6415E-02	.4921E-02	.3899E-02	.3167E-02
R4	.3434E-01	.1866E-01	.1184E-01	.8218E-02	.6055E-02	.4654E-02	.3692E-02	.3003E-02

Table 2: Minimum-R_1 moving averages, exact for cubics, of range (-k,k) and optimal centre τ^*.

l / ν	7	9	11	13	15	17	19	21
-10								-.021344
-9							-.024845	-.024845
-8						-.029240	-.026479	-.015246
-7					-.034830	-.027520	-.011769	.003210
-6				-.042017	-.027090	-.005160	.013230	.026776
-5			-.051282	-.023271	.006966	.028896	.043267	.052210
-4		-.062937	-.011655	.029089	.053584	.067079	.073900	.076763
-3	-.075758	.018648	.069930	.092848	.101512	.103199	.101494	.098190
-2	.097403	.146853	.155400	.150732	.141998	.132439	.123224	.114743
-1	.292208	.251748	.217560	.190398	.168790	.151359	.137069	.125175
0	.372294	.291375	.240093	.204443	.178138	.157895	.141819	.128735
1	.292208	.251748	.217560	.190398	.168790	.151359	.137069	.125175
2	.097403	.146853	.155400	.150732	.141998	.132439	.123224	.114743
3	-.075758	.018648	.069930	.092848	.101512	.103199	.101494	.098190
4		-.062937	-.011655	.029089	.053584	.067079	.073900	.076763
5			-.051282	-.023271	.006966	.028896	.043267	.052210
6				-.042017	-.027090	-.005160	.013230	.026776
7					-.034830	-.027520	-.011769	.003210
8						-.029240	-.026479	-.015246
9							-.024845	-.024845
10								-.021344
τ^*	0	0	0	0	0	0	0	0
R0	.3398E+00	.2634E+00	.2159E+00	.1833E+00	.1594E+00	.1411E+00	.1266E+00	.1148E+00
R1	.8009E-01	.3963E-01	.2253E-01	.1404E-01	.9347E-02	.6536E-02	.4750E-02	.3560E-02
R2	.3139E-01	.1166E-01	.5266E-02	.2712E-02	.1533E-02	.9301E-03	.5961E-03	.3993E-03
R3	.1835E-01	.6343E-02	.2773E-02	.1407E-02	.7915E-03	.4801E-03	.3085E-03	.2074E-03
R4	.1404E-01	.4945E-02	.2204E-02	.1136E-02	.6463E-03	.3955E-03	.2558E-03	.1729E-03

Table 3: Minimum=R_2 moving averages, exact for cubics, of range $(-k,k)$ and optimal centre τ^*.

ν \ l	7	9	11	13	15	17	19	21
-10								-.010145
-9							-.012648	-.018445
-8						-.016018	-.021344	-.018445
-7					-.020640	-.024409	-.017974	-.007905
-6				-.027090	-.027090	-.015076	-.000666	.012378
-5			-.036199	-.027864	-.007224	.012563	.028194	.039609
-4		-.048951	-.022624	.010717	.036120	.053080	.063679	.069898
-3	-.065268	.000000	.049362	.077399	.091688	.097994	.099776	.099022
-2	.073427	.130536	.148087	.149792	.145173	.138197	.130574	.123050
-1	.293706	.261072	.230358	.204594	.183377	.165837	.151191	.138825
0	.396270	.314685	.262032	.224902	.197190	.175663	.158433	.144317
1	.293706	.261072	.230358	.204594	.183377	.165837	.151191	.138825
2	.073427	.130536	.148087	.149792	.145173	.138197	.130574	.123050
3	-.065268	.000000	.049362	.077399	.091688	.097994	.099776	.099022
4		-.048951	-.022624	.010717	.036120	.053080	.063679	.069898
5			-.036199	-.027864	-.007224	.012563	.028194	.039609
6				-.027090	-.027090	-.015076	-.000666	.012378
7					-.020640	-.024409	-.017974	-.007905
8						-.016018	-.021344	-.018445
9							-.012648	-.018445
10								-.010145
τ^*	0	0	0	0	0	0	0	0
R0	.3489E+00	.2742E+00	.2272E+00	.1944E+00	.1701E+00	.1514E+00	.1364E+00	.1242E+00
R1	.8254E-01	.4175E-01	.2420E-01	.1533E-01	.1034E-01	.7314E-02	.5368E-02	.4058E-02
R2	.2914E-01	.1010E-01	.4251E-02	.2041E-02	.1078E-02	.6132E-03	.3696E-03	.2335E-03
R3	.1391E-01	.3772E-02	.1297E-02	.5250E-03	.2393E-03	.1195E-03	.6412E-04	.3648E-04
R4	.8664E-02	.2161E-02	.7097E-03	.2797E-03	.1256E-03	.6219E-04	.3323E-04	.1887E-04

Table 4: Minimum-R_3 moving averages, exact for cubics, of range (-k,k) and optimal centre τ^* .

ν \ l	7	9	11	13	15	17	19	21
-10								-.005570
-9							-.007378	-.013455
-8						-.009960	-.016601	-.017614
-7					-.013730	-.020370	-.018972	-.012896
-6				-.019350	-.024499	-.018639	-.008155	.003119
-5			-.027864	-.027864	-.014134	.002467	.017475	.029628
-4		-.040724	-.026792	-.000000	.024027	.042093	.054685	.063038
-3	-.058741	-.009872	.035723	.065492	.082918	.092293	.096658	.097956
-2	.058741	.118470	.141267	.147357	.145904	.141112	.134965	.128423
-1	.293706	.266557	.238693	.214337	.193742	.176390	.161691	.149136
0	.412587	.331139	.277945	.240057	.211541	.189231	.171266	.156469
1	.293706	.266557	.238693	.214337	.193742	.176390	.161691	.149136
2	.058741	.118470	.141267	.147357	.145904	.141112	.134965	.128423
3	-.058741	-.009872	.035723	.065492	.082918	.092293	.096658	.097956
4		-.040724	-.026792	-.000000	.024027	.042093	.054685	.063038
5			-.027864	-.027864	-.014134	.002467	.017475	.029628
6				-.019350	-.024499	-.018639	-.008155	.003119
7					-.013730	-.020370	-.018972	-.012896
8						-.009960	-.016601	-.017614
9							-.007378	-.013455
10								-.005570
τ^*	0	0	0	0	0	0	0	0
R0	.3566E+00	.2833E+00	.2367E+00	.2038E+00	.1793E+00	.1602E+00	.1448E+00	.1322E+00
R1	.8659E-01	.4518E-01	.2686E-01	.1736E-01	.1191E-01	.8539E-02	.6339E-02	.4838E-02
R2	.3002E-01	.1066E-01	.4591E-02	.2249E-02	.1209E-02	.6985E-03	.4267E-03	.2727E-03
R3	.1315E-01	.3373E-02	.1093E-02	.4168E-03	.1791E-03	.8448E-04	.4291E-04	.2316E-04
R4	.7105E-02	.1469E-02	.3974E-03	.1297E-03	.4867E-04	.2034E-04	.9270E-05	.4534E-05

Table 5: Minimum-R_4 moving averages, exact for cubics, of range (-k,k) and optimal centre τ^*.

ν \ l	7	9	11	13	15	17	19	21
-10								-.003388
-9							-.004736	-.010040
-8						-.006763	-.013078	-.015837
-7					-.009886	-.017005	-.018268	-.015000
-6				-.014807	-.021785	-.019649	-.012271	-.002909
-5			-.022704	-.026652	-.017639	-.003987	.009789	.021857
-4		-.035366	-.028380	-.006663	.015434	.033536	.047112	.056800
-3	-.054299	-.015718	.026197	.056266	.075456	.086888	.093160	.096077
-2	.048869	.109312	.135350	.144538	.145522	.142527	.137673	.132046
-1	.293213	.270064	.244503	.221419	.201491	.184447	.169840	.157246
0	.424434	.343415	.290069	.251799	.222813	.200011	.181559	.166297
1	.293213	.270064	.244503	.221419	.201491	.184447	.169840	.157246
2	.048869	.109312	.135350	.144538	.145522	.142527	.137673	.132046
3	-.054299	-.015718	.026197	.056266	.075456	.086888	.093160	.096077
4		-.035366	-.028380	-.006663	.015434	.033536	.047112	.056800
5			-.022704	-.026652	-.017639	-.003987	.009789	.021857
6				-.014807	-.021785	-.019649	-.012271	-.002909
7					-.009886	-.017005	-.018268	-.015000
8						-.006763	-.013078	-.015837
9							-.004736	-.010040
10								-.003388
τ^*	0	0	0	0	0	0	0	0
R0	.3628E+00	.2907E+00	.2444E+00	.2115E+00	.1868E+00	.1675E+00	.1519E+00	.1390E+00
R1	.9051E-01	.4849E-01	.2943E-01	.1934E-01	.1345E-01	.9753E-02	.7308E-02	.5624E-02
R2	.3164E-01	.1169E-01	.5203E-02	.2622E-02	.1444E-02	.8506E-03	.5284E-03	.3427E-03
R3	.1350E-01	.3548E-02	.1178E-02	.4587E-03	.2010E-03	.9649E-04	.4978E-04	.2724E-04
R4	.6800E-02	.1344E-02	.3459E-03	.1072E-03	.3823E-04	.1520E-04	.6594E-05	.3075E-05

Table 6: Optimal moving averages, exact for cubics, with range (-k,k) and optimal centre τ^* when $(a_0,a_1) = (0.25, 0.75)$.

ν \ l	7	9	11	13	15	17	19	21
-10								-.030100
-9							-.033314	-.028754
-8						-.037277	-.029124	-.011858
-7					-.042249	-.028676	-.006914	.011595
-6				-.048581	-.026546	.001100	.022292	.036517
-5			-.056706	-.020893	.014361	.038079	.052263	.060031
-4		-.066902	-.007516	.036979	.061970	.074367	.079476	.080511
-3	-.077990	.024007	.077092	.099068	.105855	.105516	.101946	.097043
-2	.102514	.151319	.157752	.150869	.140193	.129041	.118566	.109115
-1	.291853	.249105	.213087	.184506	.161822	.143594	.128733	.116448
0	.367245	.284943	.232583	.196106	.169188	.148512	.132152	.118906
1	.291853	.249105	.213087	.184506	.161822	.143594	.128733	.116448
2	.102514	.151319	.157752	.150869	.140193	.129041	.118566	.109115
3	-.077990	.024007	.077092	.099068	.105855	.105516	.101946	.097043
4		-.066902	-.007516	.036979	.061970	.074367	.079476	.080511
5			-.056706	-.020893	.014361	.038079	.052263	.060031
6				-.048581	-.026546	.001100	.022292	.036517
7					-.042249	-.028676	-.006914	.011595
8						-.037277	-.029124	-.011858
9							-.033314	-.028754
10								-.030100
τ^*	0	0	0	0	0	0	0	0
R0	.3384E+00	.2612E+00	.2131E+00	.1800E+00	.1558E+00	.1373E+00	.1226E+00	.1108E+00
R1	.8020E-01	.3980E-01	.2274E-01	.1428E-01	.9599E-02	.6794E-02	.5009E-02	.3816E-02
R2	.3244E-01	.1266E-01	.6129E-02	.3432E-02	.2134E-02	.1434E-02	.1022E-02	.7623E-03
R3	.1981E-01	.7520E-02	.3675E-02	.2103E-02	.1339E-02	.9201E-03	.6683E-03	.5066E-03
R4	.1565E-01	.6121E-02	.3061E-02	.1779E-02	.1144E-02	.7914E-03	.5775E-03	.4392E-03

Table 7: Optimal moving averages, exact for cubics, with range (-k,k) and optimal centre τ^* when $(a_o,a_3) = (0.10, 0.90)$.

l / ν	7	9	11	13	15	17	19	21
-10								-.010835
-9							-.012025	-.021633
-8						-.013777	-.022899	-.021704
-7					-.016601	-.024564	-.020309	-.007610
-6				-.021295	-.026697	-.017593	-.000857	.017614
-5			-.029019	-.028505	-.011599	.010320	.031202	.047920
-4		-.041293	-.026501	.002869	.030703	.052755	.068228	.077402
-3	-.058942	-.009273	.037920	.069689	.088914	.099163	.102909	.101995
-2	.059185	.119489	.142717	.148744	.146349	.139463	.130131	.119753
-1	.293737	.266189	.237352	.211153	.187641	.166358	.147160	.130250
0	.412039	.329776	.275065	.234689	.202579	.175751	.152919	.133697
1	.293737	.266189	.237352	.211153	.187641	.166358	.147160	.130250
2	.059185	.119489	.142717	.148744	.146349	.139463	.130131	.119753
3	-.058942	-.009273	.037920	.069689	.088914	.099163	.102909	.101995
4		-.041293	-.026501	.002869	.030703	.052755	.068228	.077402
5			-.029019	-.028505	-.011599	.010320	.031202	.047920
6				-.021295	-.026697	-.017593	-.000857	.017614
7					-.016601	-.024564	-.020309	-.007610
8						-.013777	-.022899	-.021704
9							-.012025	-.021633
10								-.010835
τ^*	0	0	0	0	0	0	0	0
R0	.3563E+00	.2826E+00	.2350E+00	.2008E+00	.1742E+00	.1528E+00	.1352E+00	.1207E+00
R1	.8644E-01	.4487E-01	.2636E-01	.1665E-01	.1101E-01	.7524E-02	.5314E-02	.3899E-02
R2	.2996E-01	.1059E-01	.4496E-02	.2148E-02	.1116E-02	.6264E-03	.3845E-03	.2601E-03
R3	.1315E-01	.3375E-02	.1098E-02	.4249E-03	.1922E-03	.1025E-03	.6462E-04	.4629E-04
R4	.7133E-02	.1497E-02	.4231E-03	.1531E-03	.7011E-04	.3999E-04	.2698E-04	.2008E-04

Table 8: Minimum-variance moving averages of range $(-k,k)$, optimal centre τ^* and basis $\alpha+\beta(1,1)^t$.

ν \ l	7	9	11	13	15	17	19	21
-10								.044572
-9							.057330	.044721
-8						.060678	.057053	.044885
-7					.064090	.060544	.056747	.045065
-6				.067282	.064320	.060396	.056412	.045263
-5			.109150	.068370	.064574	.060234	.056042	.045481
-4		.129695	.106486	.069566	.064852	.060056	.055636	.045720
-3	.161498	.126042	.103555	.070882	.065159	.059860	.055188	.045984
-2	.156252	.122025	.100332	.072330	.065497	.059645	.054697	.046274
-1	.150481	.117606	.096785	.073923	.065868	.059408	.054156	.046593
0	.144133	.112745	.092884	.075674	.066276	.059147	.053561	.046944
1	.137150	.107398	.088594	.077601	.066725	.058860	.052906	.047329
2	.129468	.101516	.083874	.079721	.067219	.058544	.052186	.047754
3	.121019	.095046	.078682	.082053	.067763	.058197	.051394	.048221
4		.087928	.072970	.084617	.068361	.057815	.050523	.048735
5			.066688	.087439	.069018	.057395	.049564	.049300
6				.090542	.069741	.056933	.048510	.049921
7					.070537	.056424	.047350	.050605
8						.055865	.046075	.051357
9							.044671	.052184
10								.053094
τ^*	0	0	0	1	1	1	1	2
R0	.1441E+00	.1127E+00	.9288E-01	.7760E-01	.6673E-01	.5886E-01	.5291E-01	.4775E-01
R1	.2050E-01	.1239E-01	.8277E-02	.6387E-02	.4543E-02	.3402E-02	.2647E-02	.2405E-02
R2	.1353E-01	.8144E-02	.5419E-02	.4313E-02	.3042E-02	.2260E-02	.1746E-02	.1616E-02
R3	.1216E-01	.7316E-02	.4868E-02	.3884E-02	.2738E-02	.2034E-02	.1570E-02	.1455E-02
R4	.1157E-01	.6965E-02	.4635E-02	.3700E-02	.2608E-02	.1937E-02	.1495E-02	.1386E-02

Table 9: Minimum-R_1 moving averages of range $(-k,k)$, optimal centre τ^* and basis $\alpha+\beta(1.1)^t$.

ν \ l	7	9	11	13	15	17	19	21
-10								.012380
-9							.013904	.023505
-8						.015645	.026407	.033383
-7					.017574	.029808	.037503	.042019
-6				.035748	.033796	.042446	.047184	.049423
-5			.046720	.064114	.048514	.053509	.055443	.055603
-4		.064150	.082311	.085450	.061566	.062943	.062271	.060569
-3	.094603	.110189	.107397	.100142	.072770	.070689	.067659	.064332
-2	.156189	.139324	.122666	.108615	.081927	.076680	.071595	.066902
-1	.187393	.152883	.128872	.111338	.088819	.080845	.074069	.068295
0	.191118	.152326	.126848	.108823	.093205	.083105	.075067	.068522
1	.170552	.139259	.117507	.101637	.094819	.083372	.074576	.067601
2	.129206	.115451	.101855	.090403	.093371	.081551	.072580	.065548
3	.070939	.082844	.081000	.075804	.088540	.077536	.069063	.062383
4		.043576	.056157	.058593	.079974	.071211	.064004	.058126
5			.028666	.039600	.067285	.062449	.057385	.052800
6				.019735	.050047	.051109	.049182	.046430
7					.027791	.037037	.039371	.039044
8						.020063	.027924	.030673
9							.014811	.021350
10								.011112
τ^*	0	0	0	0	1	1	1	1
R0	.1558E+00	.1237E+00	.1030E+00	.8839E-01	.7593E-01	.6705E-01	.6026E-01	.5489E-01
R1	.1214E-01	.6255E-02	.3658E-02	.2336E-02	.1531E-02	.1044E-02	.7524E-03	.5658E-03
R2	.3037E-02	.1247E-02	.6050E-03	.3291E-03	.2072E-03	.1211E-03	.7604E-04	.5055E-04
R3	.1867E-02	.7549E-03	.3626E-03	.1959E-03	.1276E-03	.7384E-04	.4598E-04	.3032E-04
R4	.1579E-02	.6408E-03	.3084E-03	.1668E-03	.1092E-03	.6322E-04	.3935E-04	.2594E-04

Table 10: Minimum-R_2 moving averages of range $(-k,k)$, optimal centre τ^* and basis $\alpha+\beta(1.1)^t$.

l \ ν	7	9	11	13	15	17	19	21
-10								.004091
-9							.004838	.011217
-8						.005729	.013272	.020407
-7					.013394	.015798	.024133	.030780
-6				.018533	.034134	.028826	.036337	.041544
-5			.026834	.046172	.057336	.043461	.048891	.052001
-4		.041256	.064847	.075548	.079214	.058406	.060899	.061542
-3	.068927	.095474	.102340	.101221	.097015	.072446	.071571	.069658
-2	.149124	.142834	.131264	.119534	.108943	.084477	.080236	.075936
-1	.204248	.171217	.146896	.128458	.114074	.093543	.086352	.080062
0	.216868	.175347	.147475	.127416	.112269	.098868	.089512	.081829
1	.185822	.155855	.133802	.117094	.104069	.099899	.089467	.081132
2	.123311	.118239	.108802	.099228	.090588	.096348	.086131	.077980
3	.051700	.071732	.077043	.076376	.073391	.088246	.079602	.072491
4		.028046	.044204	.051661	.054359	.075988	.070179	.064905
5			.016492	.028494	.035546	.060399	.058380	.055580
6				.010264	.019014	.042796	.044960	.045004
7					.006655	.025059	.030942	.033796
8						.009711	.017634	.022712
9							.006663	.012655
10								.004679
τ^*	0	0	0	0	0	1	1	1
R0	.1681E+00	.1355E+00	.1139E+00	.9843E-01	.8685E-01	.7635E-01	.6866E-01	.6253E-01
R1	.1353E-01	.7180E-02	.4296E-02	.2789E-02	.1923E-02	.1310E-02	.9499E-03	.7166E-03
R2	.2227E-02	.8117E-03	.3541E-03	.1753E-03	.9534E-04	.5252E-04	.3045E-04	.1890E-04
R3	.7438E-03	.2214E-03	.8150E-04	.3484E-04	.1665E-04	.8934E-05	.4553E-05	.2512E-05
R4	.4547E-03	.1317E-03	.4765E-04	.2012E-04	.9529E-05	.5294E-05	.2670E-05	.1460E-05

Table 11: Minimum-R_3 moving averages of range $(-k,k)$, optimal centre τ^* and basis $\alpha+\beta(1.1)^t$.

ν \ l	7	9	11	13	15	17	19	21
-10								.001683
-9							.002080	.005951
-8						.005309	.007365	.013060
-7					.007632	.017365	.016159	.022758
-6				.011427	.024369	.035074	.028092	.034415
-5			.018006	.035376	.047897	.055911	.042276	.047144
-4		.030310	.053490	.067099	.074015	.076796	.057467	.059021
-3	.055728	.084933	.096622	.099469	.098083	.094759	.072235	.071688
-2	.142781	.142827	.135075	.125501	.116073	.107411	.085125	.081452
-1	.213775	.182510	.158612	.140019	.125233	.113231	.094807	.088370
0	.233407	.190745	.161701	.140552	.124423	.111700	.100222	.091820
1	.194457	.166086	.144408	.127550	.114151	.103281	.100695	.091453
2	.118046	.118181	.111875	.104061	.096363	.089294	.096029	.087235
3	.041806	.063795	.072683	.074951	.074047	.071688	.086556	.079457
4		.020613	.036452	.045832	.050687	.052742	.073147	.068734
5			.011077	.021827	.029652	.034741	.057168	.055967
6				.006339	.013575	.019629	.040363	.042283
7					.003803	.008702	.024673	.028939
8						.002364	.011952	.017188
9							.003588	.008107
10								.002374
τ^*	0	0	0	0	0	0	1	1
R0	.1772E+00	.1443E+00	.1222E+00	.1062E+00	.9413E-01	.8463E-01	.7556E-01	.6887E-01
R1	.1551E-01	.8523E-02	.5229E-02	.3460E-02	.2419E-02	.1765E-02	.1256E-02	.9506E-03
R2	.2459E-02	.9275E-03	.4161E-03	.2107E-03	.1167E-03	.6924E-04	.3942E-04	.2469E-04
R3	.6042E-03	.1641E-03	.5542E-04	.2189E-04	.9726E-05	.4739E-05	.2254E-05	.1163E-05
R4	.2424E-03	.5479E-04	.1584E-04	.5460E-05	.2150E-05	.9395E-06	.4377E-06	.2021E-06

Table 12: Minimum-R_4 moving averages of range $(-k,k)$, optimal centre τ^* and basis $\alpha+\beta(1.1)^t$.

ν \ l	7	9	11	13	15	17	19	21
-10								.000803
-9							.001028	.003442
-8						.003200	.004427	.008776
-7					.004904	.012416	.011300	.017234
-6				.007856	.018468	.028481	.022136	.028693
-5			.013292	.028477	.040961	.049979	.036630	.042476
-4		.024098	.045797	.060448	.069173	.073777	.053674	.057441
-3	.047798	.077325	.091575	.097008	.097706	.095938	.071515	.072138
-2	.137704	.141781	.136835	.129047	.120714	.112699	.088014	.085005
-1	.219844	.190163	.166872	.148410	.133515	.121286	.100995	.094581
0	.244996	.201860	.172209	.150437	.133712	.120435	.108595	.099710
1	.199960	.173022	.151890	.135145	.121642	.110561	.109594	.099708
2	.113838	.117288	.113286	.106934	.100129	.093584	.103652	.094473
3	.035860	.058072	.068857	.073043	.073681	.072469	.091426	.084525
4		.016393	.031205	.041264	.047319	.050584	.074516	.070961
5			.008181	.017569	.025339	.031010	.055252	.055329
6				.004362	.010289	.015927	.036302	.039413
7					.002448	.006225	.020163	.024967
8						.001429	.008603	.013411
9							.002178	.005549
10								.001365
τ^*	0	0	0	0	0	0	1	1
R0	.1838E+00	.1509E+00	.1286E+00	.1123E+00	.9990E-01	.9008E-01	.8124E-01	.7413E-01
R1	.1728E-01	.9748E-02	.6100E-02	.4097E-02	.2899E-02	.2134E-02	.1568E-02	.1192E-02
R2	.2851E-02	.1126E-02	.5230E-03	.2721E-03	.1539E-03	.9290E-04	.5582E-04	.3529E-04
R3	.6576E-03	.1848E-03	.6428E-04	.2602E-04	.1181E-04	.5853E-05	.2906E-05	.1521E-05
R4	.2086E-03	.4385E-04	.1181E-04	.3809E-05	.1408E-05	.5802E-06	.2502E-06	.1085E-06

Table 13: Optimal moving averages of range $(-k,k)$, optimal centre τ^* and basis $\alpha+\beta(1.1)^t$ when $(a_o,a_1) = (0.25, 0.75)$.

ν \ l	7	9	11	13	15	17	19	21
-10								.025977
-9							.027652	.040524
-8						.029431	.043205	.048589
-7					.031235	.046150	.051904	.052970
-6				.032840	.049361	.055683	.056715	.055249
-5			.061858	.052802	.060109	.061154	.059313	.056319
-4		.078277	.094524	.065586	.066720	.064330	.060645	.056680
-3	.106785	.118680	.110401	.074404	.071023	.066207	.061244	.056616
-2	.159387	.137084	.116448	.081041	.074037	.067342	.061405	.056281
-1	.179677	.142275	.116513	.086440	.076314	.068036	.061290	.055758
0	.179185	.138898	.112633	.091004	.078112	.068439	.060978	.055087
1	.162995	.129032	.105734	.094720	.079482	.068590	.060493	.054280
2	.131482	.112917	.095953	.097135	.080273	.068440	.059816	.053325
3	.080488	.089060	.082713	.097197	.080085	.067827	.058878	.052191
4		.053777	.064554	.092881	.078122	.066424	.057537	.050811
5			.038669	.080499	.072926	.063629	.055536	.049071
6				.053451	.061878	.058362	.052416	.046769
7					.040322	.048703	.047366	.043555
8						.031254	.038954	.038817
9							.024653	.031479
10								.019651
τ*	0	0	0	1	1	1	1	1
R0	.1515E+00	.1188E+00	.9766E-01	.8184E-01	.6998E-01	.6141E-01	.5495E-01	.4991E-01
R1	.1246E-01	.6600E-02	.4005E-02	.2796E-02	.1873E-02	.1347E-02	.1020E-02	.8019E-03
R2	.4004E-02	.1967E-02	.1147E-02	.8633E-03	.5608E-03	.3934E-03	.2912E-03	.2243E-03
R3	.2832E-02	.1408E-02	.8259E-03	.6325E-03	.4104E-03	.2874E-03	.2124E-03	.1634E-03
R4	.2484E-02	.1236E-02	.7253E-03	.5566E-03	.3610E-03	.2527E-03	.1867E-03	.1436E-03

Table 14: Optimal moving averages of range (-k,k) , optimal centre τ^* and basis $\alpha+\beta(1.1)^t$ when (a_o,a_3) = (0.25, 0.75) .

l \ ν	7	9	11	13	15	17	19	21
-10								.014311
-9							.015147	.034457
-8						.016008	.036514	.050756
-7					.016843	.038705	.053886	.059326
-6				.027638	.040934	.057347	.063152	.061294
-5			.032798	.065894	.061020	.067546	.065482	.059751
-4		.041694	.078107	.095733	.072383	.070442	.064106	.057388
-3	.062498	.099785	.113570	.109806	.076136	.069352	.061818	.055594
-2	.147698	.146016	.130882	.110789	.075892	.067227	.060049	.054611
-1	.208850	.168674	.133308	.105239	.075170	.065740	.059073	.054096
0	.222824	.168888	.127728	.098698	.076287	.065539	.058664	.053633
1	.188961	.151284	.118411	.093457	.080133	.066809	.058654	.053028
2	.121728	.118918	.104923	.088410	.085991	.069498	.059153	.052391
3	.047442	.075168	.084029	.080143	.091133	.072985	.060349	.052034
4		.029574	.054414	.065026	.090750	.075415	.061978	.052218
5			.021829	.042254	.079405	.073407	.062716	.052738
6				.016914	.054840	.063085	.059987	.052511
7					.023082	.042985	.050843	.049494
8						.017910	.034270	.041423
9							.014157	.027630
10								.011316
τ^*	0	0	0	0	1	1	1	1
R0	.1718E+00	.1335E+00	.1068E+00	.8869E-01	.7415E-01	.6436E-01	.5713E-01	.5157E-01
R1	.1427E-01	.6994E-02	.3922E-02	.2593E-02	.1881E-02	.1365E-02	.1033E-02	.8131E-03
R2	.2275E-02	.8345E-03	.4513E-03	.3154E-03	.2505E-03	.1747E-03	.1287E-03	.9962E-04
R3	.6468E-03	.2413E-03	.1483E-03	.1044E-03	.8258E-04	.5706E-04	.4217E-04	.3259E-04
R4	.3281E-03	.1364E-03	.8518E-04	.5797E-04	.4570E-04	.3179E-04	.2354E-04	.1815E-04