RUSSIAN ROULETTE, INSURANCE,
AND OTHER HAZARDOUS GAMES

by

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Can risk theory be of any use in such practical matters?\(^{(1)}\)

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As our society of Norwegian actuarial students completes its first half hundred years it feels natural to review and celebrate our past. However, we must also look ahead and question what our future will be. To try to give an answer, we should all agree on some basics: What is an actuary, and what does he study?

A marked trend in modern economic life is the growing professional specialization that accompanies scientific and technical progress. Gradually the universal scholar of the classics has been replaced by teams of highly skilled specialists. Clearly, the actuary cannot survive as an academic jack-of-all-trades in insurance, coping with specialists in accounting, law, economics of the firm, public finance, automatic data-processing, and so on, and at the same time be an expert in calculation of premiums and funds. He can hold his own only by developing the characteristics of his profession, which is mathematical risk theory (in a broad sense), and applying it in practical insurance.

But then do we have a future as actuaries? Aren't theory and practice after all separate matters? In recent time the dichotomy of theory and practice has been referred to on repeated occasions in insurance circuits, in fact by actuaries, what is alarming.

\(^{(1)}\) Based on an address delivered on occasion of the 50th anniversary of the Actuarial Students' Society in Oslo, 12th of March 1982.
If they were right, risk theory would be little more than a stylish jargon. It would play the same role for us as Hippocrates and Latin for the medical doctors, useful mainly as a justification of our academic title (and, not to forget, a permit to the boards of directors and splendid receptional dinners).

Questions of vital importance are:

(i) Can risk theory do more than paraphrase in formal terms results that are obvious to practical people in possession of sound intuition?

(ii) Can it give deep, non-trivial insight into the nature of the risk business?

And even if it can, looking at the simple models and the complex real world, one may still ask:

(iii) Can it be developed into practical instruments?

My own opinion is that the answer to all these questions is affirmative. For what is theory?

Theory combines practical experience and a priori insight, presents this knowledge in an organized and condensed form, and derives from it general methods for solution of special problems.

Rational behaviour implies theoretical work. We must restore our faith in the practical relevance of risk theory, which is the raison d'être of the actuarial profession.

A suitable first reference, which marks the bounds of the scope of risk theory, is the short story "A fatalist" by the great Russian author Lermontov (1841). Here we meet Lieutenant Vulich who wants to demonstrate the predestination of fate. He seizes a colleague's gun - they are at the front and it is impossible to
know whether the gun is loaded or not - aims at his forehead and pulls the trigger: Nothing happens. Then he aims at the wall and pulls the trigger anew: The shot thunders and the bullet blows a hole in the paling. All who have witnessed the event get convinced that destiny has chosen another day for Vulich's last hour, and that this predestination cannot be changed by human whims or other accidental circumstances.

If this conclusion were correct, there would be no need for risk theory: there is no room for probabilistic statements in a deterministic world. From the Russians we have, however, also a game, Russian roulette, in which probability considerations obviously make sense. The ingredients of this game are an empty sixshooter, a sharp ball-cartridge, and two rivals (not necessarily so sharp and not necessarily Russians both of them, see Figure 1). The revolver is charged with the one cartridge, the barrel rotated vigorously and the game starts. The gamblers alternate pointing the gun at their heads and pulling the trigger, without rotating the barrel anew. The one who shoots himself through the head has lost the game.

In Vulich's case it was impossible to assess in an objective way the probability that he would be killed. In Russian roulette, however, we can easily agree on ground of the symmetry in the situation that both gamblers have the same chance 50% of being shot. This game appears also in another version, attributed to the Mexicans, by which the participants are seated on powder-barrels as in Figure 2. This is one of the rare situations in which a slow ignition is a good thing; the fuses are lighted simultaneously by a referee, and the one whose barrel blows up first loses. The barrels are assigned by coin-tossing, and again it is a priori clear that each gambler has an equal chance of losing.
In principle - that is in the theory - this game is played only once. In practice, however, large scale preparations are made to play it repeatedly. (And here, please, is actually an example where theory and practice seem to disagree. In this case, interestingly enough, it is the theory that is sound whereas practice is insane.) Let us now, just for the sake of illustration, allow for the absurd and imagine that the game can be repeated. Then theory tells us that the game is fair in the sense that the two gamblers will be destroyed equally often in the long run. In formal terms, if \( S_n \) is the number of times the first gambler (say) is blown up in \( n \) rounds, we have that

\[
\Pr\left\{ \lim_{n \to \infty} \frac{S_n}{n} = \frac{1}{2} \right\} = 1.
\]

Thus on theoretical grounds the rivals can rejoice at having achieved the constantly requested strategic balance.

Perhaps this result is judged as intuitively obvious. But that may be due to a shift in observational outlook caused by the development of the theory itself. The result was not at all obvious, neither to mathematicians nor to laymen, until its theoretical foundation was laid by the great Russian mathematician Kolmogorov (1933). Only afterwards has it gradually become commonplace.

The theory also says something about how quickly balance will be reached in actual repetitions. The very useful Central limit theorem, proved in its general form by Khinchin (yet another of those Russians, see Feller 1966), implies that for \( n \) not too small

\[
\Pr\left\{ \left| \frac{S_n}{n} - \frac{1}{2} \right| < \epsilon \right\} = 2\Phi(2\epsilon \sqrt{n}) - 1,
\]

where \( \Phi \) is the cumulative standard normal distribution function.

From this we obtain almost without any computations that in 100
Figure 1  Russian roulette.

Figure 2  Russian roulette, modern version.
rounds a gambler can be about 95% sure of losing between 40 and 60 times. So if he is blown up 55 out of 100 times, say, he should consider that as a quite normal result of the game he has accepted to play. On the other hand, if he is blown up 70 out of 100 times, he would be in his right to be displeased and to question the fairness of the game. The choice of the limits 40 and 60 in 100 rounds is not trivial, and without knowledge of this theoretical result the gambler might make a fuss even after having been destroyed 51 times as compared to 49 for his adversary.

Theory provides even more sophisticated results. The Arcsine law (see Feller, 1966) tells us that most likely one of the two gamblers will have a lead over the other through a majority of repeated trials. In fact, the least probable event is that they are in the lead equally long. This is one example of a theoretical result that on the face of it seems to contradict sound practical intuition, but represents deep insight into the nature of the phenomena. Furthermore, the practical consequences are significant, namely that we should not give much heed to a gambler who claims that there is a permanent strategic bias in favour of the rival; that is a typical result of pure chance and need not be intended.

At this point in our development we are already in a position to answer affirmatively to questions (i) and (ii) above. Now, Russian roulette is a simple game which can be analyzed by elementary mathematical tools. To test the potential of risk theory, we should address ourselves also to more complex problems. We do this with question (iii) above in mind. Thus let us consider the uncontested queen amongst games of chance, the roulette. Figure 3 shows the design of a roulette wheel. The wheel is spun and a tiny ball is thrown round the inside rim until it finally settles in one of the 37 equally large sectoral positions. The players place their
bets on any single or combination of positions while the ball is still in orbit. For example, a player may bet on a single number, the first four numbers, all black numbers, etc. A player who has placed an amount of \( l \) on \( m \) positions wins \( 36/m \) if the ball lands in one of his \( m \) numbers, otherwise he loses. Thus his net gain in one round is

\[
X = \begin{cases} 
\frac{36}{m} - 1 = a & \text{with probability } p = \frac{m}{37} \\
-1 & \text{otherwise} 
\end{cases} \quad (1)
\]

Figure 4 exhibits the cumulative distribution function

\[
F(x) = P\{X \leq x\}
\]

for some different values of \( m \).

If one is prepared to examine the game by means of such theoretical concepts, one can make statements about its chance aspects, some of which are quite obvious, some not so obvious, and some entirely out of range of direct reasoning.

To begin, we can calculate the expected value

\[
E(X) = -\frac{1}{37},
\]

which shows that "on the average" the game is favourable to the bank. The player has an expected loss, which is independent of the number of positions, \( m \).

But averages do not tell the whole story. An essential aspect is the uncertainty in the outcomes which makes the casino a risky business. The casino boss (managing director?) could try to use his experience and intuition to assess this risk and bring it under control. It would, however, be more practical to employ risk theory, which comprises the insight that we already have in the notion of risk.
Figure 3  The roulette wheel (hatching represents red).

Figure 4  The distribution function (1) for some different m.

Figure 5  A player who cannot bear losing.

Figure 6  Risk theoretic advice.
A simple measure of the amount of risk is the standard deviation,

$$\text{SD}(X) = \frac{36}{37} \left( \frac{37}{m} - 1 \right)^{\frac{1}{4}}.$$ 

It is a decreasing function of $m$, which is reasonable.

The standard deviation is related only to the outcome of one single round. A more refined concept is the probability that the accumulated sum of net gains of the players reaches a given level, $u$. Interpreting $u$ as the initial fortune of the bank, actuaries speak of the "probability of ruin". It seems to be in fashion to assert that this concept hampers the communication between risk theoreticians and practitioners; managing directors dislike the word ruin; ruin in the above sense does not necessarily imply bankruptcy in practice; etc. etc. A real practical mind will, of course, see through this and take the probability of ruin for what it is; a workable criterion for assessing the stability of a risk business. It provides an estimate of the financial capacity that is required to run a given business safely. In particular the event of ruin may refer to the end of a certain accounting period. Then the ruin probability regarded as a function of $u$ simply determines the probability distribution of the net result in the period, which is quite useful in practice! It may also refer to the entire period, not only the result at the end, and the length of the period may be short or long, or even infinite.

Let us now determine what this risk theoretic device can tell the casino manager. Assume initially that there is only one player with an infinite capital who places the same bet in each round, a unit amount on $m$ positions. (Those who at this stage chuckle, "Totally unrealistic", are, of course, right. As in most complex studies one begins with a very simple, and most likely unrealistic, model. When this is fully understood, complexity may be added
until the final model approaches reality.) Denote by \( X_i \) the net gain of the one player in the \( i \)-th round. Then \( X_1, X_2, \ldots \) are independently and identically distributed (i.i.d.) as \( X \) in (1). Introduce also the accumulated net gain of the player in the \( n \) first rounds,

\[
S_n = \sum_{i=1}^{n} X_i.
\]  

The probability of eventual ruin of the bank by initial fortune \( u \) is

\[
\psi(u) = P\{ \bigcup_{n=1}^{\infty} S_n > u \}. 
\]

Consider first a player who cannot bear losing and, therefore, almost safeguards by placing his bet on \( m = 36 \) positions, e.g. the black and the red as in Figure 5. This is a quite safe business for the bank. And a most unfavourable game on the part of the player; in 36 out of 37 cases his net gain is 0 and in the last case he loses his stake. Now the player may have counsellors with some background in theory who can explain this (Figure 6). Suppose he is persuaded to change his strategy and instead places the whole amount on the blacks, leaving nothing to the Reds! Then \( m = 18 \), which is far more in accordance with the idea of the game; the player still has an expected loss of \( 1/37 \) in each single round, but in return he now has a gambling chance of winning. And the bank, consequently runs a risk of losing. How large should \( u \) be to make \( \psi(u) \) less than 1%? Those who rely on their instincts can write their guess here for later reference (take this opportunity to have your results appear in a research report):

\[
u = \ldots 
\]

Now to theory. First note that when \( m = 18 \) the \( X_i \) can assume only the values \(-1\) and \(+1\). Figure 7 shows a typical sequence
of partial sums $S_n$ (this particular one being unfavourable to the bank). It is realized that losing the fortune $u$ is equivalent to losing a unit repeatedly $u$ times, which leads to the equation

$$\psi(u) = \psi(1)^u.$$  \hspace{1cm} (5)

Also, by conditioning on the value of $X_1$, we obtain the general difference equation

$$\psi(u) = p\psi(u-a) + (1-p)\psi(u+1), \hspace{1cm} u = 1, 2, \ldots,$$  \hspace{1cm} (6)

with $p$ and $a$ defined as in (1). Known initial conditions are

$$\psi(u) = 1, \hspace{1cm} u = -a+1, \ldots, -1, 0.$$  \hspace{1cm} (7)

Upon inserting $m = 18$, which gives $p = 18/37$ and $a = 1$, and (5) into (6) and cancelling $\psi(1)^{u-1}$, we obtain a quadratic equation in $\psi(1)$ which has roots $18/19$ and $1$. Since $\psi(u)$ must tend to 0 as $u$ increases, only the first root is admissible, so that

$$\psi(u) = \left(\frac{18}{19}\right)^u = e^{-0.054u}.$$  

We calculate $\psi(1) = 0.95$, $\psi(10) = 0.58$, $\psi(50) = 0.07$, and in particular $\psi(84) = 0.01$. Thus, with an initial reserve of $u = 84$, the bank has an acceptable small risk (1%) of running short of financial reserves. Compare with your guess in (4)!

Now a person who places his bet on $m = 18$ positions is not really playing high. Dostoyevsky (1866) says:

"In my opinion the game roulette was invented especially for the Russians... They are tempted by hazardous games, but they play carelessly..."

and are likely to place their entire stake on one single number as in Figure 8. It is felt that this fellow represents a real threat against the security (of the bank); he has a small probability of winning in a single round, but when he wins the consequences are
serious. How large a risk does he represent, e.g. as compared to the one who plays at 18 numbers? That we can assess by calculating the probability of ruin and in particular by finding the \( u \)-value corresponding to 1\%. But first you should make your own guess:

\[
u = \ldots \quad (8)
\]

When \( m = 1 \) the cumulative net gain of the player may evolve as shown in Figure 9. The difference equation (6) is now of order 37. It can be shown that \( \psi(1) = 1 - (37 - m)^{-1} \), which together with the side conditions (7) is sufficient to calculate all \( \psi(u) \)-values recursively using relation (6). A computer-program carried out this procedure, and it turned out that \( \psi(2900) = 1\% \). Compare with your guess in (8)!

The same procedure was used to calculate \( \psi(u) \) also for other values of \( m \). In Figure 10 the graph of \( \psi(u) \) is depicted for those \( m \) for which the distribution of \( X \) is shown in Figure 4.

So far we have dealt only with the simplest type of playing strategies, but even for these the calculation of ruin probabilities is cumbersome. Before turning to more complex strategies it is worthwhile looking for simple methods for the (possibly approximate) calculation of ruin probabilities.

As early as in 1903 the famous Russian probabilist Markov made a thorough study of the equation (6) and established close lower and upper bounds to the solution \( \psi \) (see Seal, 1966). A very handy general result is that \( \psi(u) \) can be approximated by a function of the exponential form. In fact we have

\[
\psi(u) < e^{-Ru} \quad ,
\]

where \( R \) is determined as follows (for a simple proof, see von Bahr, 1974).
Figure 7  The random walk of the cumulative net gain when $m = 18$. Large dots indicate first passages of levels 1, 2, ...

\[ S_n = \sum_{i=1}^{n} X_i \]

Figure 8  A real gambler.
Figure 9  The random walk of the cumulative net gain when $m=1$.

Figure 10  The probability of ruin $\psi(u)$ (unbroken line) and its approximation $e^{-Ru}$ (dotted) for some different $m$. 

$$S_n = \sum_{i=1}^{n} X_i$$
The moment-generating function of $X$, 

$$\chi(r) = \int_{-\infty}^{\infty} e^{rx} dF(x),$$  \hspace{0.5cm} (10) 

has a graph as shown in Figure 11 below. It is strictly convex, with $\chi(0) = 1$ and $\chi'(0) < 0$ (remember that $\chi'(0) = EX = -\frac{1}{37}$). Hence there exists a unique $R > 0$ for which

$$\chi(R) = 1,$$  \hspace{0.5cm} (11) 

and this is the $R$ occurring in (9).

The quality of the approximation $\psi(u) \approx e^{-Ru}$ is illustrated in Figure 10 where the graph of $e^{-Ru}$ can be compared to that of $\psi(u)$. No comment is needed.

We turn now to the study of more complex playing strategies. Let the pair $(b,m)$ represent the strategy that consists in always placing the same amount $b$ on $m$ positions, and let $\psi(u|b,m)$, $\chi(r|b,m)$, and $R(b,m)$ denote the corresponding ruin probability, moment generating function, and solution of (11), respectively. The above results for $b = 1$ are readily carried over to general $b$ by a simple change of scale (or monetary unit). In fact, we have for all $u$ that are multiples of $b$,

$$\psi(u|b,m) = \psi\left(\frac{u}{b}\right|1,m)$$

and, since $\chi(r|b,m) = \chi(br|1,m)$,

$$R(b,m) = \frac{1}{b}R(1,m).$$

In practice there is, of course, more than one player. What can we do if there are $H$ players who play independently of one another with fixed strategies $\theta = \{(b_h,m_h) : h = 1, \ldots, H\}$? If each player $h$ picks his $m_h$ positions in a purely random manner in each round, the individual net gains are mutually independent.
Then the total net gain in one round, which is the sum of the individual net gains, has a moment generating function given by

\[ \chi(x|\theta) = \prod_{h=1}^{H} \chi(x|b_h,m_h) . \] (12)

By numerical methods we can determine the positive root \( R(\theta) \) of the equation \( \chi(x|\theta) = 1 \), which gives the approximation \( e^{-R(\theta)u} \) to the probability of eventual ruin, \( \psi(u|\theta) \). A straightforward analysis of (12) yields the inequalities

\[ \min_h R(b_h,m_h) < R(\theta) < \max_h R(b_h,m_h) . \]

Thus \( \psi(u) \) may be approximated from above and below by ruin probabilities associated with simple strategies of the kind examined above. (This, by the way, demonstrates the usefulness of studying special "unrealistic" cases.)

It can still be argued that our model assumptions are not completely realistic since each player may change his strategy, and the number and mix of players may vary from one evening to another. Moreover, only the total net gain is observed for each round, whereas it is impossible to observe the underlying strategies of the individual players. To take some of these points into account we can introduce for the \( i \)-th evening, \( i = 1,2,... \), a "risk parameter" \( \theta_i \) representing the risk of the casino on that evening. It is a complex quantity comprising all risk characteristics such as the number of players, their wealth, style of playing, and so on. Such risk characteristics may change from one evening to another, and the \( \theta_i \)'s are, therefore, conceived as independent and identically distributed random elements. Let \( X_{ij} \) be the total net gain of the players in round \( j \) on evening \( i, j = 1,...,J; i = 1,2,... \). Although the individual players may vary their bets \((b,m)\) from one round to another, it is not unreasonable to assume
that the "resultant" strategy of the clientèle as a whole remains roughly constant throughout the evening. In mathematical terms this can be made precise by assuming that for fixed \( \theta_i \) the net gains \( X_{ij}, \ldots, X_{ij} \) are conditionally independent and identically distributed.

How can we assess the risk of the casino under these rather weak assumptions? Let us try and find (an approximation to) the probability of ruin in the course of the \( J \) rounds on evening \( i \). This is not greater than the probability of eventual ruin in the course of infinitely many rounds with the same clientèle, which is

\[
\psi(u) = E\psi(u|\theta_i) - R(\theta_i)u, \quad u < E e^{-\theta_i}. \tag{13}
\]

The value of the expression in (13) cannot be set up immediately since the probability distribution of the \( R(\theta_i) \)'s is unknown. A further complication is that this distribution cannot be estimated directly since the \( R(\theta_i) \)'s are not observable. Nonetheless we can try to estimate the expression in (13) from the observations \( X_{ij}, j = 1, \ldots, J; i = 1, \ldots, I \). We sketch below one possible procedure.

The empirical analogue of the conditional momentgenerating function \( \chi(r|\theta_i) \) is

\[
\chi_i(r) = \frac{1}{J} \sum_{j=1}^{J} e^{rX_{ij}}.
\]

A "natural" estimator of \( R(\theta_i) \) is the largest solution \( R_i \) of the equation \( \chi_i(r) = 1 \). By the law of large numbers we have \( \chi_i(r) + \chi(r|\theta_i) \) (almost surely) and

\[
E(R_i|\theta_i) + R(\theta_i) \tag{14}
\]

as \( J \to \infty \) (the stakes are bounded since all private estates are). Using first the law of large numbers, then Jensen's inequality \( (Eg(X) \geq g(EX) \text{ for convex } g) \), and finally (14), we obtain
as $I$ and $J$ tend to $\infty$. On combining (13) and (15) we conclude that for $I$ and $J$ not too small the left hand side expression in (15) is a conservative estimate of the probability of ruin.

The above models for roulette and estimates of risk are not exhaustive. We could continue to refine our models and methods to take into account more complex or less specified strategies, but we shall not pursue this task here. Our purpose was not to provide a complete theory, but rather to give an indication of the applicability of risk theory. Instead we comment briefly on certain practical implications. The motive for defining and calculating measures of riskiness such as the probability of ruin is the need for a means of comparing and choosing between different risks, that is, controlling the risk business. Our previous discussions suggest $b$ and $u$ as the obvious instruments on the hand of the casino manager. In fact, upper limits to the stakes are usually defined in roulette, and the need for cash is certainly currently valued. The following result by Ohlin (1969) gives an indication of how the risk could be further reduced if the casino could impose a wider set of restrictions on the strategies of the players.

Let $F_1$ and $F_2$ be distribution functions with equal mean values and assume that there exists an $x_0$ such that
\[ F_1(x) > F_2(x) \text{ for } x < x_0 \]
\[ F_1(x) < F_2(x) \quad \text{ if } x > x_0. \quad (16) \]

Then for each convex function \( g \) we have
\[ \int g(x)dF_1(x) > \int g(x)dF_2(x). \quad (17) \]

If \( g \) is strictly convex and \( F_1 \neq F_2 \), the inequality \( (17) \) is strict.

An example can be picked already from Figure 4, from which we readily see that \( (16) \) is satisfied with \( F_1 \neq F_2 \) if \( F_1 \) and \( F_2 \) correspond to strategies \((b,m_1)\) and \((b,m_2)\), respectively, with \( m_1 < m_2 \). Putting \( g(x) = e^{rx} \) in \( (17) \), we conclude that \( \chi(r|b,m_1) > \chi(r|b,m_2) \) for all \( r \neq 0 \). Then, by a glance at Figure 11, it is realized that \( R(b,m_1) < R(b,m_2) \), hence \((b,m_1)\) is a more dangerous strategy than \((b,m_2)\) as measured by the approximation \( (9) \) to the probability of ruin. Thus a lower limit to the number of positions \( m \) represents a potential risk preventing measure.

In principle the problem of measuring and controlling risk is much the same for an insurance company as it is for a roulette casino. To see the parallelism clearly, turn to Figure 12 which depicts the risk process of an insurance portfolio. The \( Y_i \)'s are the individual claim amounts recorded in chronological order and the \( V_i \)'s are the interoccurrence times, \( i = 1,2,\ldots \). Premiums are earned at a (positive) rate of \( B \) per time unit. Suppose an initial cash reserve \( u \) is set aside at time \( t = 0 \) to defray possible extreme losses on the business. Clearly the probability that the accumulated net losses will ever exceed the initial reserve is given by \( (2) \) and \( (3) \) with
\[ X_i = Y_i - BV_i, \quad i = 1,2,\ldots \]
work of Guus Wolf who carried out the computer work. It was he who conjectured that \( \phi(1) = 1-(37-m)^{-1} \) by simple roulette strategies, and Luciano Molinari gave a formal proof.

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