SOME ACTUARIAL CALCULATIONS
ON ELECTRONIC DESK CALCULATOR

by

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1. Introduction.

It seems clear that the small (pocket) desk calculators with programming facilities will have an important impact on the work of actuaries, since actuarial values will be available on any standard (interest rate, mortality table) on a few seconds notice.

This is shown by the programs for the Hewlett-Packard 25 electronic desk calculator below. The Gompertz-Makeham's law of mortality has been assumed, but it will be clear that programs could be worked out for any mortality representable by a small set of parameters.

2. Notations.

The force of mortality at age \( x \) is given by

\[
\mu_x = \alpha + \beta e^x, \quad \alpha > 0, \beta > 0, \ c > 1.
\]

The rate of interest is \( i \), the force of interest \( \delta = \ln(1+i) \).

Furthermore the survival function and discounted survival function are given by

\[
l_x = e^{-\int^x \mu_s \, ds}, \quad D_x = l_x \varepsilon^x = e^{-\int^x (\mu_s + \delta) \, ds}
\]

where \( \varepsilon = (1+i)^{-1} \). The one year survival probability at age \( x \) is

\[
P_x = \frac{l_{x+1}}{l_x}
\]

and the present expected value of a \( n \) year temporary annuity-due to a person of age \( x \) is given by

\[
\ddot{a}_{x \, \overline{n}} = \sum_{t=0}^{n-1} \frac{D_{x+t}}{D_x}.
\]

The corresponding joint life annuity-due to a group of \( m \) persons of age \( x_1, \ldots, x_m \) is
3. Program I.

A. Single values of \( \bar{x}_{n+1} \):

The following program I can be used

\[
\bar{x}_{n+1} = \frac{n-1}{m} \sum_{t=0}^{m} \prod_{i=1}^{t} \frac{1}{x_i}
\]

\[
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\]

Set: \( n \), \( STO 1 \), \( x+n \), \( STO 2 \), \( c \), \( STO 3 \), \( \beta \), enter, \( \alpha \), enter, 1+i, R/S.

\( \bar{x}_{n+1} \) is shown in the window.

Example 1: The Norwegian life companies standard

\( N \) 1955: \( i=0.03, \alpha=0.0025, \beta=0.00002, \log_{10} c=0.046 \), is used in all the examples presented here.

With \( x=40, n=10 \), we get

\[ \bar{x}_{10} = 8.62112 \]
B. Other values of $a_{x \overline{n}}$ on the same standard $(i, x, z, c)$.

After one value $a_{x \overline{n}}$ has been found as explained above, values of $a_{x \overline{n}}$ for other $(x,n)$, but for the same standard, can be found by setting

- $n$, STO 1,
- $x+n$, STO 2,
- GTO 13,
- R/S.

Example 2: For $x=20$, $n=80$, we get

$$a_{x \overline{n}} = 26.00100.$$  

C. Single values of $D_{x+n}/D_x$.

Make the following change in program I:

1. NOP

Set:

- 1, STO 7,
- $n$, STO 1,
- $x+n$, STO 2,
- $c$, STO 3,
- $\alpha$, enter,
- $\beta$, enter,

1+i, R/S.

$$D_{x+n}/D_x$$ is shown in the window.

D. Other values of $D_{x+n}/D_x$ on the same standard.

Having found one value $D_{x+n}/D_x$ as explained above, values of $D_{x+n}/D_x$ can be found for other $(x,n)$, but the same standard, by setting:

- 1, STO 7,
- $n$, STO 1,
- $x+n$, STO 2,
- GTO 13,
- R/S.

E. Tabulation of $a_{x \overline{w-x}}$ with respect to $x$.

Make the following changes in program I above:

1. f pause
2. RCL 7
3. R/S
4. GTO 21
Set
\( w(=x+n) \), STO 1, STO 2, c, STO 3, \( \beta \), enter, \( \alpha \), enter 1+i, R/S.

By repeatedly pressing \( \text{R/S} \) we get \( x \) in the window when pausing and

\[
\begin{align*}
\text{SS} \ x \ &= \ x - 1, \ x - 2, \ x - 3, \\
\text{when stopping.}
\end{align*}
\]

**Example 3.** \( w(=x+n) = 70 \), standard = N 1955.

We get

<table>
<thead>
<tr>
<th>( x )</th>
<th>( n = w - x )</th>
<th>( \text{SS} \ x \ \overline{n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>69</td>
<td>1</td>
<td>1.00000</td>
</tr>
<tr>
<td>68</td>
<td>2</td>
<td>1.94140</td>
</tr>
<tr>
<td>67</td>
<td>3</td>
<td>2.83284</td>
</tr>
<tr>
<td>66</td>
<td>4</td>
<td>3.68130</td>
</tr>
<tr>
<td>65</td>
<td>5</td>
<td>4.49240</td>
</tr>
<tr>
<td>64</td>
<td>6</td>
<td>5.27072</td>
</tr>
</tbody>
</table>

etc.

**F. Joint life annuities** \( \text{SS} \ x_1, x_2, \ldots, x_m \overline{n} \)

It is known that

\[
\text{SS} \ x_1, \ldots, x_m \overline{n} = a^{(m)}
\]

where the sum age \( w \) is given by \( c^w = \sum_{i=1}^{m} c_i \) and \( a^{(m)} \) is computed on the standard \((i, \alpha, \beta, c)\). Hence the program presented in 3 A can be used.

Set
\( n, \ \text{STO} \ 1, \ c, \ \text{STO} \ 3, \ x_1 + n, \ \text{f} \ y^x, \ \text{RCL} \ 3, \ x_2 + n, \ \text{f} \ y^x, +, \ \text{RCL} \ 3, \ x_3 + n, \ \text{f} \ y^x, +, \ldots, +, \ \text{f} \ \text{log}, \ \text{RCL} \ 3, \ \text{f} \ \text{log}, +, \ \text{STO} \ 2, \ \beta, \ \text{enter,} \ \alpha, \ \text{enter 1+i, R/S.} \)

\( \text{SS} \ x_1, \ldots, x_m \overline{n} \) is shown in the window.
Example 3. \( x_1 = 32, x_2 = 32, n = 10 \), standard N 1955.

\[ \ddot{a}_{32 \overset{2}{\varepsilon} 10} = 8.54101. \]

Example 4. \( x_1 = 32, x_2 = 37, x_3 = 40, n = 10 \), standard N 1955,

\[ \ddot{a}_{32 \overset{37}{\varepsilon} 40 \overset{10}{\varepsilon}} = 8.36240. \]

(Sum age \( x_1 + n, x_2 + n, x_3 + n \) is 57.25467).

G. The register in Program I.

The register has been used as follows:

<table>
<thead>
<tr>
<th>STO 0</th>
<th>STO 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c+n )</td>
<td>(-c^{x+n-1-t} e^{-t} (c-1)/\ln(c))</td>
</tr>
</tbody>
</table>

1 \( n-t \) 
2 \( x+n-1 \) 
3 \( c \) 
5 not used 
6 \(-e^{-t} (c-1)/\ln(c)\) 
7 \( \ddot{a} \)

4. Program II.

A. Tabulation of \( \ddot{a}_{x \overset{\varepsilon}{\varepsilon} t} \) with respect to \( t \).

Sometimes it is desired to tabulate \( \ddot{a} \) w.r.t. duration for given initial age, not w.r.t. initial age for given terminal age as in I. D.

Then the following Program II can be used

| 1 \( \ln \) | 15 \( \text{RCL} \) 1 | 29 - |
| 2 + | 16 \( e^x \) | 30 \( e^x \) |
| 3 \( \text{STO} \) 0 | 17 \( x \) | 31 \( \text{STO} \times 5 \) |
| 4 \( \text{R} \ \downarrow \) | 18 \( \text{STO} \) 4 | 32 \( \text{RCL} \) 5 |
| 5 \( \text{RCL} \) 3 | 19 \( 1 \) | 33 \( \text{STO} \) +7 |
| 6 \( \ln \) | 20 \( \text{STO} \) 2 | 34 \( 1 \) |
| 7 \( + \) | 21 \( \text{STO} \) 5 | 35 \( \text{STO} \) +1 |
| 8 \( 1 \) | 22 \( \text{STO} \) 7 | 36 \( \text{RCL} \) 3 |
| 9 \( \text{RCL} \) 3 | 23 \( 1 \) | 37 \( \text{STO} \) \times 4 |
| 10 - | 24 \( \text{STO} \) +2 | 38 \( \text{RCL} \) 7 |
| 11 \( \times \) | 25 \( \text{RCL} \) 2 | 39 \( \text{R/S} \) |
| 12 \( \text{STO} \) 6 | 26 \( \text{pause} \) | 40 \( \text{GTO} \) 23 |
| 13 \( \text{RCL} \) 6 | 27 \( \text{RCL} \) 4 |
| 14 \( \text{RCL} \) 3 | 28 \( \text{RCL} \) 0 |
Set
x, STO 1, c STO 3, \( \beta \), enter, \( \alpha \), enter, 1+i, R/S.
Press R/S repeatedly, \( t \) is shown in the window when pausing
and
\[
\text{PDF} \quad ; \quad t=2,3,\ldots
\]
when stopping.

Example 5: \( x = 47 \), standard = N 1955.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \text{PDF} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.96549</td>
</tr>
<tr>
<td>3</td>
<td>2.89733</td>
</tr>
<tr>
<td>4</td>
<td>3.79637</td>
</tr>
<tr>
<td>5</td>
<td>4.66338</td>
</tr>
<tr>
<td>6</td>
<td>5.49912</td>
</tr>
<tr>
<td>7</td>
<td>6.30429</td>
</tr>
</tbody>
</table>

E. Tabulation of \( \text{PDF} \) for other \( x \) on the same standard.
Having tabulated \( \text{PDF} \) for one value of \( x \) as explained above
in 4. A, tabulations for other \( x \), but with the same standard,
is performed by setting

x, STO 1, GTO 13, R/S.

C. Single values of \( \text{PDF} \), by program II.

Make the following alterations in program II in 4. A above:

```
20  STO -2  26  GTO 39
23  RCL 2   36  GTO 23
24  RCL 1   39  RCL 7
25  f x=y   40  GTO 00
```

Set
x, STO 1, \( x+n \), STO 2, c, STO 3, \( \beta \), enter, \( \alpha \), enter, 1+i, R/S.

Then \( \text{PDF} \) is shown in the window.
D. Other values of $a_x^N$ by Program II on the same standard.

After one value $a_x^N$ has been run off as explained in II.C, $a_x^N$ can be found for other $(x,n)$, but on the same standard, by setting

\[ x, \text{STO} 1, x+n, \text{STO} 2, \text{GTO} 13, \text{R/S}. \]

E. Single values of $D_{x+n}/D_x$ on Program II.

Make the setting as for $a_x^{n+1}$ in 4. C, i.e. set

\[ x, \text{STO} 1, x+n+1, \text{STO} 2, c, \text{STO} 3, \beta, \text{enter}, a, \text{enter}, 1+i, \text{R/S}. \]

By RCL 5 $D_{x+n}/D_x$ is shown in the window.

Other $D_{x+n}/D_x$ on the same standard can be found as explained in 4. D. Use $n+1$ in place of $n$ and RCL 5.

F. The register in Program II.

The register has been used as follows,

- **STO 0**: $\alpha + 6$
- **STO 1**: $x + t$
- **STO 2**: $t (or x + n - 1)$
- **STO 3**: $c$
- **STO 4**: $-c^{x+t} \beta (c-1)/\text{lnc}$
- **STO 5**: $D_{x+t-1}/D_{x+t}$
- **STO 6**: $-\beta (c-1)/\text{lnc}$
- **STO 7**: $a$

5. Comments on the programs.

**Comment (i):** The two programs are based respectively on the recursions

(I) \[ a_{x+t}^{n-t} = 1 + v P_{x+t} \frac{a_{x+t+1, n-t-1}}{a_{x+n, c+1}} ; a_{x+n, c} = 0 \]

(II) \[ a_{x-t} = a_{x, t-1} + D_{x+t-1}/D_x ; a_{x, t} = 1. \]

**Comment (ii):** The factor $c^{x+y}$ in STO 4 in both programs was found by recursive divisions or multiplications. Compared to using $f y^x$ it shortens the running time considerably. On the other hand using $f y^x$ would have required fewer program lines and fewer stores.
Comment (iii): 12 STO 6, 13 RCL 6 are found in both programs. They facilitates finding many \( x \) on the same standard. They can be left out if such facilities are not needed.

Comment (iv): If values of \( x \) only on one standard are needed, then program lines 1-12 can be left out and the programs can start with program line 13 as program line 1. This would require permanent storage of \( a + \delta \), \( c \) and \( -\beta(c-1)/\ln c \) in STO 0, 3 and 6 respectively: The setting would be as in 3.B or 4.B (without GTO 13).

6. Acknowledgement.

I am indebted to Bjørn R. Sundt for having suggested interesting improvements in my original programs.