ON ORIENTABLE MATROID SYSTEMS AND RELIABILITY EQUIVALENCE

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Abstract

A binary monotone system is an ordered pair \((E, \phi)\) where \(E\) is the component set, and \(\phi\), the structure function of the system, is a binary function defined for all subsets \(A \subseteq E\) which is non-decreasing with respect to set inclusion. If all components have the same probability \(p\) of functioning, the system reliability can be expressed in terms of the reliability polynomial \(h(p)\). We say that two binary systems are equivalent if their reliability polynomials are equal. An important class of binary monotone systems is the class of undirected network systems. A generalization of this class is the class of matroid systems. A matroid is an ordered pair \((F, M)\) where \(M\) is a family of incomparable subsets of \(F\), called circuits, satisfying certain axioms. A matroid system is a binary monotone system, \((E, \phi)\) which can be associated with a matroid \((E \cup \{x\}, M)\) in such a way that the minimal path sets of \((E, \phi)\) can be recovered by extracting all the circuits \(M \in M\) containing the element \(x\), and then deleting \(x\) from these circuits. A subclass of such systems is the class of orientable matroid systems, i.e., matroid system where the associated matroid is orientable. In particular, if \((E, \phi)\) is a 2-terminal undirected network system, it is an orientable matroid systems because we may assign directions to all the edges in the system. By using the properties of orientable matroid systems we obtain a way of constructing equivalent systems. We show that this construction can be carried out for all orientable matroid systems except for series-parallel systems.

1 Introduction

The context of the present paper is the theory of binary monotone systems as introduced in the book by Barlow and Proschan [1]. While this approach provides a very general framework for analysing a wide range of systems, it is difficult to find smart solutions to various problems related to reliability calculations. Thus, many papers within the reliability literature focuses on certain subclasses of systems with additional structure which can be utilized in order to obtain stronger results and faster algorithms for calculating the system reliability. One such subclass is the class of network systems, where the system can be represented as a directed or undirected graph. Classical references on this topic are Satyanarayana and Prabhakar [11], Satyanarayana [12] and Satyanarayana and Chang [13]. Network systems also fit within more general classes of systems associated with matroids. Huseby [5] introduced the class of matroid systems, then referred to as regular systems, and used this to obtain generalisation of results on undirected network systems. This work was extended in Huseby [6] and Huseby [7]. Other work in this area includes Rodriguez and Traldi [10]. In Huseby [8] the class of oriented matroid systems was introduced as a
generalization of directed network systems. In the present paper we introduce a related
class of systems, called **orientable matroid systems** and explore how this class can be used
in relation to the study of **equivalent systems**.

# 2 Basic concepts and results

In the present paper we define a **binary monotone system** as an ordered pair \((E, \phi)\) where
\(E\) is the component set, and \(\phi\) is a binary function defined for all subsets \(A \subseteq E\) which
is non-decreasing with respect to set inclusion. The elements of set set \(E\) are interpreted
as components of some technological system. Each component can be either functioning
or failed. The function \(\phi\) is called the **structure function** of the system. If \(A\) is the set of
functioning components of the system, then \(\phi(A)\) represents the resulting system state. If
\(\phi(A) = 1\), the system is functioning, while if \(\phi(A) = 0\), the system is failed. The **order**
of the system is the number of components in the system, i.e., \(|E|\). A component \(e \in E\) is
said to be **relevant** if there exists at least one set \(A \subseteq E \setminus \{e\}\) such that \(\phi(A) = 0\) while
\(\phi(A \cup \{e\}) = 1\). A binary monotone system is **coherent** if all components are relevant.

A set \(P \subseteq E\) is said to be a **minimal path set** of a binary monotone system \((E, \phi)\) if
\(\phi(P) = 1\) and \(\phi(A) = 0\) for all \(A \subset P\). We denote the family of minimal path sets of
\((E, \phi)\) by \(\mathcal{P}\). By the definition of a minimal path set and the monotonicity of the structure
function it follows that \(\phi(A) = 1\) if and only if there exists a minimal path set \(P \in \mathcal{P}\)
such that \(P \subseteq A\). Thus, the family of minimal path sets uniquely determines the binary
monotone system.

If \((E, \phi)\) is a binary monotone system, and \(e \in E\), we introduce two lower order systems
\((E \setminus \{e\}, \phi_+e)\) and \((E \setminus \{e\}, \phi_-e)\) called respectively the **contraction** and **restriction** of \((E, \phi)\)
with respect to \(e\), where:

\[
\phi_+e(A) = \phi(A \cup \{e\}), \quad \text{for all } A \subseteq (E \setminus \{e\}),
\]

\[
\phi_-e(A) = \phi(A), \quad \text{for all } A \subseteq (E \setminus \{e\}).
\]

It is easy to see that we have the following **pivotal decomposition**:

\[
\phi(A) = I(e \in A)\phi_+e(A \setminus \{e\}) + I(e \notin A)\phi_-e(A \setminus \{e\}). \tag{2.1}
\]

Contractions and restrictions are referred to as **minor operations**. If \((D, \psi)\) can be obtained
from \((E, \phi)\) by performing a sequence of minor operations, we say that \((D, \psi)\) is a **minor**
of \((E, \phi)\). In particular, if \(D \subset E\), and \(\psi(A) = \phi(A)\) for all \(A \subseteq D\), \((D, \psi)\) is the restriction
of \((E, \phi)\) to the set \(D\), and hence by definition also a minor of \((E, \phi)\).

Contraction and restriction can also be expressed in terms of the family of minimal path
sets. Thus, let \(\mathcal{P}\) be the family of minimal path sets of a binary monotone system \((E, \phi)\),
let \(e \in E\), and introduce the following families:

\[
\mathcal{P}_+e = \text{Min}\{P \setminus \{e\} : P \in \mathcal{P}\}, \tag{2.2}
\]

\[
\mathcal{P}_-e = \{P \in \mathcal{P} : e \notin P\}. \tag{2.3}
\]

Then \(\mathcal{P}_+e\) is the family of minimal path sets of \((E \setminus e, \phi_+e)\), while \(\mathcal{P}_-e\) is the family of
minimal path sets of \((E \setminus e, \phi_-e)\).
It is well-known (see e.g., Huseby [5]) that there exists a unique integer valued function \( \delta \) defined for all subsets of \( E \), called the domination function of the system, such that the structure function can be written in terms of \( \delta \) as:

\[
\phi(A) = \sum_{B \subseteq A} \delta(B), \quad \text{for all } A \subseteq E. \tag{2.4}
\]

Using Möbius inversion Huseby [5] showed the converse result that the domination function can be expressed in terms of the structure function as:

\[
\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B), \quad \text{for all } A \subseteq E. \tag{2.5}
\]

The signed domination of a binary monotone system \( (E, \phi) \) with domination function \( \delta \), denoted \( d(\phi) \), is defined as:

\[
d(\phi) = \delta(E).
\]

Moreover, using (2.5) it is easy to show the following decomposition formula:

**Proposition 2.2** Let \( (E, \phi) \) be a binary monotone system and let \( e \in E \). Then

\[
d(\phi) = d(\phi_+e) - d(\phi_-e), \quad \text{for all } e \in E. \tag{2.6}
\]

**Proof:** See Huseby [5] ■

If \( E = \{1, \ldots, n\} \), we introduce the component state vector \( X = (X_1, \ldots, X_n) \), where \( X_e = 1 \) if the component \( e \) is functioning, and zero otherwise, \( e \in E \). In this case the set of functioning components, \( A \), is given by \( A = \{e \in E : X_e = 1\} \). Hence, the structure function \( \phi \), expressed as a function of \( X \), can be written as \( \phi(X) = \phi(A(X)) \). We observe that if \( B \subseteq E \), then:

\[
\prod_{e \in B} X_e = I(B \subseteq A)
\]

Hence, we can write \( \phi(X) \) in terms of \( \delta \) as a multilinear function:

\[
\phi(X) = \sum_{B \subseteq E} \delta(B) \prod_{e \in B} X_e. \tag{2.7}
\]

By (2.7) it follows that \( d(\phi) \) is the coefficient of the highest order term in \( \phi(X) \), the term associated with \( \prod_{e \in E} X_e \).

If \( A \subseteq E \), we denote the subvector of \( X \) corresponding to the components in \( A \) by \( X_A \). Using familiar notation introduced in Barlow and Proschan [1] the structure functions of the contraction and restriction of the system with respect to \( e \) can then be expressed as:

\[
\phi_+(X_{E \setminus \{e\}}) = \phi(1_e, X),
\]

\[
\phi_-(X_{E \setminus \{e\}}) = \phi(0_e, X)
\]
Moreover, the pivotal decomposition (2.1) can be expressed in terms of the component state vector as:

$$\phi(X) = X_e \phi_e(X_{E \setminus \{e\}}) + (1 - X_e) \phi_{-e}(X_{E \setminus \{e\}}).$$  \hfill (2.8)

Throughout this paper we assume that the component state variables are independent. If the component reliabilities are $P(X_e = 1) = p_e$ for $e \in E$, the reliability of the system can be expressed as a function of the vector $p$ of the component reliabilities as:

$$h(p) = P(\phi(X) = 1) = \sum_{B \subseteq E} \delta(B) E[\prod_{e \in B} X_e] = \sum_{B \subseteq E} \delta(B) \prod_{e \in B} p_e.$$  \hfill (2.9)

Of special interest is the case where all the component state variables are identically distributed, i.e., we have $p_e = q$, for all $e \in E$ for some $q \in [0,1]$. In this case the system reliability $h = P(\phi = 1)$, can be expressed as a polynomial function of the common component reliability $q$, i.e., $h = h(q)$. The function $h$ is referred to as the reliability polynomial of the system, and by (2.9) it is given by:

$$h(q) = \sum_{B \subseteq E} \delta(B) q^{\lvert B \rvert} = \sum_{k=0}^{n} \left( \sum_{B \subseteq E : \lvert B \rvert = k} \delta(B) \right) q^k. \quad \hfill (2.10)$$

From this it follows that the degree of the polynomial $h$, denoted $\deg(h)$ is less than or equal to the order of the system, i.e. $\deg(h) \leq n$, where $n = \lvert E \rvert$. Moreover, the coefficient associated with the highest order term $q^{\lvert E \rvert}$ in $h$ is the signed domination of the system, $d(\phi)$. Thus, $\deg(h) = n$ if and only if $d(\phi) \neq 0$.

If $(E, \phi)$ is a binary monotone system, and $e, f \in E$, we say that $e$ and $f$ are in series in $(E, \phi)$ if the structure function $\phi$ depends on $X_e$ and $X_f$ only through $X_e \cdot X_f$. Similarly, we say that $e$ and $f$ are in parallel in $(E, \phi)$ if the structure function $\phi$ depends on $X_e$ and $X_f$ only through $X_e \bigoplus X_f$. If $e$ and $f$ are in series, we may replace $e$ and $f$ by a single component $g$ with reliability $p_g = p_e \cdot p_f$ without changing the reliability of the system. This operation is called a series reduction. If $e$ and $f$ are in parallel, we may replace $e$ and $f$ by a single component $g$ with reliability $p_g = p_e \bigoplus p_f$ without changing the reliability of the system. This operation is called a parallel reduction. We refer to series and parallel reductions by the common notion of $s$-$p$-reductions. We observe that whenever an $s$-$p$-reduction is performed, the number of components in the system is reduced by 1. An $s$-$p$-system is a binary monotone system which can be reduced to a single component by performing a sequence of $s$-$p$-reductions.

According to Navarro et al. [9] two systems are said to be equivalent if their reliability polynomials are equal as functions of $q$, for $q \in [0,1]$. Two systems may be equivalent even if they do not have the same order. See Lindqvist et al. [3] and Lindqvist et al. [4].

More generally, if $(E, \phi)$ is a binary monotone system, we let $q = (q_1, \ldots, q_m) \in [0,1]^m$ where $m \leq n$, and let $\{S_j\}_{j=1}^m$ be a partition of the component set $E$. We then assume that $P(X_e = 1) = q_j$ for $e \in S_j$, $j = 1, \ldots, m$, and write $h$ as $h(q)$. We say that two systems are equivalent if their respective reliability functions can be expressed as identical functions of a common vector $q$ for all $q \in [0,1]^m$. We note that the definition of equivalence considered by Navarro et al. [9] corresponds to the case where $m = 1$. As before two systems may be equivalent even if they do not have the same order. Still, from the above discussion the following proposition is immediate:

\[ \text{Proposition:} \]
Proposition 2.3 Let \((E, \phi)\) be a binary monotone system of order \(n\). Then a necessary condition for the existence of an equivalent lower order system is that \(d(\phi) = 0\).

It is well-known that if \((E, \phi)\) is not coherent, then \(d(\phi) = 0\). For such systems lower order equivalent systems can be constructed simply by deleting irrelevant components. See Lindqvist et al. [4]. However, there are many binary monotone systems where \(d(\phi) = 0\) even though all components are relevant. See e.g., Satyanarayana and Prabhakar [11], Satyanarayana [13], and Huseby [8]. This opens up the possibility for constructing non-trivial equivalent systems of different order.

As an example motivated by the results of Satyanarayana and Prabhakar [11], we consider the 2-terminal undirected network system \((E, \phi)\) shown in Figure 2.1. The component set \(E\) consists of the undirected edges of the system labelled 1, \ldots, 7, i.e., \(E = \{1, \ldots, 7\}\), and the system is functioning if the terminals \(S\) and \(T\) can communicate through the network. We let \(q = (q_1, \ldots, q_7) \in [0, 1]^m\), and assume that \(P(X_e = 1) = q_e\), for all \(e \in E\).

![Figure 2.1: A 2-terminal undirected network system.](image)

We then compare \((E, \phi)\) to the 2-terminal directed network system \((E', \phi')\) shown in Figure 2.2. The component set \(E'\) consists of the undirected edges of the system, and the system is functioning if the terminal \(S\) could send signals to terminal \(T\) through the network. We observe that \(E'\) is obtained from \(E\) by replacing the undirected edges 1, 2, 4, 6, 7 in \((E, \phi)\) by directed edges, and by replacing the undirected edges 3 and 5 in \((E, \phi)\) respectively by the two directed anti-parallel edges 3' and 3'', and 5' and 5''. Note that we could in fact have replaced all the undirected edges in \((E, \phi)\) by directed anti-parallel edges in \((E', \phi')\). However, it is easy to see that any additional directed edge would be irrelevant. Thus, since we want \((E', \phi')\) to be coherent, we skipped this.

![Figure 2.2: A 2-terminal directed network system.](image)
For the directed network we assume that \( P(X_e = 1) = q_e \) for \( i \in \{1, 2, 4, 6, 7\} \). Moreover, \( P(X_{3'} = 1) = P(X_{3''} = 1) = q_3 \), while \( P(X_{5'} = 1) = P(X_{5''} = 1) = q_5 \). By letting \( h = P(\phi = 1) \) and \( h' = P(\phi' = 1) \), we see that both reliabilities can be expressed as functions of \( q \). We claim that \( h(q) = h'(q) \) for all for all \( q \in [0,1]^7 \), i.e., that the two systems are equivalent. In order to show this, we use an argument consisting of two steps. In the first step we assume that the correlations between the states of the anti-parallel components are 1, so that one component functions if and only if the other functions. That is, we have:

\[
E[X_{3'} \cdot X_{3''}] = q_3, \\
E[X_{5'} \cdot X_{5''}] = q_5.
\]

Under this assumption the anti-parallel components behave like single undirected components, and thus, the undirected and directed systems must have equal reliability. Denoting the domination function of \((E', \phi')\) by \( \delta' \), this implies that we must have:

\[
h(q) = \sum_{B \subseteq E'} \delta'(B) E[\prod_{e \in B} X_e]
\]

In the second step we apply a result of Satyanarayana and Prabhakar [11] which states that \( \delta'(B) = 0 \) if \( B \) contains at least one directed cycle. This implies that if \( \{3', 3''\} \subseteq B \) or \( \{5', 5''\} \subseteq B \) then \( \delta'(B) = 0 \). Conversely, if \( \delta'(B) \neq 0 \), then \( B \) contains at most one of the components \( 3' \) and \( 3'' \) and at most one of the components \( 5' \) and \( 5'' \). Thus, we have:

\[
E[\prod_{e \in B} X_e] = \prod_{e \in B} P(X_e = 1), \text{ for all } B \subseteq E' \text{ such that } \delta'(B) \neq 0.
\]

Combining all this, we see that we may write:

\[
h(q) = \sum_{B \subseteq E'} \delta'(B) \prod_{e \in B} P(X_e = 1) = h'(q),
\]

where, as before, \( h'(q) \) is the reliability function of \((E', \phi')\) assuming that all component states are independent.

The above equivalence result is clearly just a special case of the more general result that any undirected network system can be converted to an equivalent directed network system by replacing undirected edges by directed edges in anti-parallel. Thus, any reliability calculation method for directed network systems can easily be applied to undirected network systems as well. Hence, the study of equivalent systems actually leads to a better understanding of the relationship between various computational methods.

3 Matroid systems

Rather than studying the class of network systems further, we now use a more general approach using the framework of matroid theory. In this framework the results for network systems follow as special cases. We start out by reviewing some basic definitions and results.

An ordered pair \((F, \mathcal{M})\), where \( F \) is a finite set and \( \mathcal{M} \) is a family of subsets of \( F \) is said to be a matroid if \( \mathcal{M} \) satifies the following properties:
Axiom M1. The sets in $\mathcal{M}$ are distinct, incomparable subsets of $F$, i.e., no set is a proper subset of another.

**Axiom M2.** If $M_1, M_2 \in \mathcal{M}$ and $x \in M_1 \cap M_2$, then there exists a third set $M_3 \in \mathcal{M}$ such that $M_3 \subseteq (M_1 \cup M_2) \setminus \{x\}$.

The sets in $\mathcal{M}$ is called the circuits of the matroid. This concept is motivated by graph theory. In fact, if $F$ is the edge set of an undirected graph $G$ and $\mathcal{M}$ is the family of circuits of $G$ (i.e., the minimal cycles in $G$), $(F, \mathcal{M})$ is a matroid. This particular type of matroid is called a graphic matroid. While graphic matroids represent an important class of matroids, matroids are indeed a much more general concept. See Welsh [14].

A connection between matroids and binary monotone systems was first introduced in Huseby [5]. In the special case where the binary monotone system $(E, \phi)$ is a 2-terminal undirected network system, the corresponding matroid is found by adding an extra artificial component $x$ between the two terminal nodes and then letting $F = E \cup \{x\}$ and $\mathcal{M}$ be the family of circuits of the extended graph. See Figure 3.1. The artificial component $x$ is referred to as the extension component.

![Figure 3.1: Undirected graph with an extra edge $x$ between the terminal nodes $S$ and $T$.](image)

More generally, if $(F, \mathcal{M})$ is a matroid, and $x \in F$ is the extension component, a binary monotone system can be constructed as follows: We start out by identifying the subfamily $\mathcal{M}_x$ of all circuits in $\mathcal{M}$ containing the element $x$. We then construct $(E, \phi)$ by letting $E = F \setminus \{x\}$, and defining $\phi(A) = 1$ if and only if there exists a set $P \subseteq A$ such that $P \cup \{x\} \in \mathcal{M}_x$. A binary monotone system constructed this way, is called a matroid system. Note that if $\mathcal{P}$ is the family of the minimal path sets of $(E, \phi)$, then $\mathcal{P}$ can be derived from the corresponding matroid $(F, \mathcal{M})$ as $\mathcal{P} = \{M \setminus \{x\} : M \in \mathcal{M}_x\}$. As mentioned earlier the family of minimal path sets, $\mathcal{P}$, uniquely determines the binary monotone system $(E, \phi)$.

If $(F, \mathcal{M})$ is a matroid and $(E, \phi)$ is the matroid system derived from $(F, \mathcal{M})$, we write the mapping from the matroid $(F, \mathcal{M})$ to the corresponding matroid system $(E, \phi)$ as:

$$(F, \mathcal{M}) \rightarrow (E, \phi).$$

The class of matroid systems contains all 2-terminal undirected network systems. In fact Huseby [7] shows more generally that the class of matroid systems also includes $k$-terminal undirected network systems and many other types of undirected network systems. In Huseby [5] it was shown that all $k$-out-of-$n$-systems are matroid systems as well. Moreover, the following proposition was proved:
Proposition 3.1 If \((E, \phi)\) is a matroid system, then \(|d(\phi)| > 0\) if and only if \((E, \phi)\) is coherent. Moreover, \(|d(\phi)| = 1\) if and only if \((E, \phi)\) is a coherent s-p-system.

By combining Proposition 2.3 and the first part of Proposition 3.1 we get the following result:

Proposition 3.2 If \((E, \phi)\) is a coherent matroid system, then no lower order equivalent system exists.

While Proposition 3.2 excludes the possibility of finding lower order equivalent systems for coherent matroid systems, there may still be possible to find coherent higher order equivalent systems. In the remaining part of the present paper we will show how this can be done.

### 3.1 Orientable matroid systems

As the name indicates, orientable matroid systems are matroid systems where the underlying matroid can be oriented. In order to introduce the concept of an oriented matroid, we need some notation. A signed set is a set \(M\) along with a mapping \(\sigma_M : M \to \{+, -\}\), called the sign mapping of the set. With a slight abuse of notation, \(M\) refers both to the signed set itself as well as the underlying unsigned set of elements. The sign mapping \(\sigma_M\) of a signed set \(M\) defines a partition of \(M\) into two subsets, \(M^+\) and \(M^-\), where \(M^+ = \{e \in M : \sigma_M(e) = +\}\) and \(M^- = \{e \in M : \sigma_M(e) = -\}\). The subsets \(M^+\) and \(M^-\) are referred to as respectively the positive and negative parts of \(M\). If \(M = M^+\), we say that \(M\) is a positive signed circuit, while if \(M = M^-\), we say that \(M\) is a negative signed circuit. For any signed set \(M\), the signed set \(-M\) is obtained from \(M\) by reversing all signs. An oriented matroid is defined as follows:

**Definition 3.3** An oriented matroid is an ordered pair \((F, \bar{M})\) where \(F\) is a non-empty finite set, and \(\bar{M}\) is a family of non-empty signed subsets of \(F\), called signed circuits. The signed circuits satisfy the following properties:

**Axiom O1.** If \(M\) is a signed circuit, then so is \(-M\).

**Axiom O2.** For all \(M_1, M_2 \in \bar{M}\) such that \(M_1 \subseteq M_2\), we either have \(M_1 = M_2\) or \(M_1 = -M_2\).

**Axiom O3.** If \(M_1\) and \(M_2\) are signed circuits such that \(M_1 \neq -M_2\), and \(e \in M_1^+ \cap M_2^-\), then there exists a third signed circuit \(M_3\) with \(M_3^+ \subseteq (M_1^+ \cup M_2^+)\setminus \{e\}\) and \(M_3^- \subseteq (M_1^- \cup M_2^-)\setminus \{e\}\).

If \((F, \bar{M})\) is an oriented matroid, and \(\mathcal{M}\) is obtained from \(\bar{M}\) by replacing signed sets by unsigned sets and deleting redundant sets, then \((F, \mathcal{M})\) is a matroid. Thus, for any oriented matroid \((F, \bar{M})\) there exists a unique underlying matroid \((F, \mathcal{M})\). Since in general there may be many different ways of assigning signs to a family of sets, several different oriented matroids may have the same underlying matroid. On the other hand, for a given matroid \((F, \mathcal{M})\) it may not be possible to find any oriented matroid having \((F, \bar{M})\) as its underlying matroid. In cases where such an oriented matroid can be found, the matroid is said to be orientable. As indicated in the beginning of this section, a matroid system where the corresponding matroid is orientable is called an orientable matroid system. The class of orientable matroid systems include all undirected network systems as well as all \(k\)-out-of-\(n\)-systems.
If \((F, \bar{M})\) is an oriented matroid, and \((F, M)\) is the corresponding underlying matroid, for each circuit \(M \in \mathcal{M}\) there are exactly two signed circuits \(M_1, M_2 \in \mathcal{M}\) containing the same elements as \(M\), and such that \(M_1 = -M_2\). In order to determine the corresponding oriented matroid system \((E, \phi)\), where \(E = F \setminus \{x\}\), we need the family \(\mathcal{M}_x\) of sets in \(\mathcal{M}\) containing the extension component \(x\). This can now be determined by considering signed circuits in \(\bar{M}\) containing the extension component \(x\). As a convention, we eliminate redundant sets at this stage by defining \(\mathcal{M}_x = \{M \in \mathcal{M} : x \in M^+\}\). From this family we get the following family of signed minimal path sets, \(\bar{P} = \{M \setminus \{x\} : M \in \mathcal{M}_x\}\). The unsigned minimal path sets determining \((E, \phi)\), are then derived simply by replacing the signed sets in \(\bar{P}\) by unsigned sets.

Just as for matroid system, we write the mapping from the oriented matroid \((F, \bar{M})\) to the corresponding orientable matroid system \((E, \phi)\) as:

\[
(F, \bar{M}) \rightarrow (E, \phi).
\]

Note that even though \((E, \phi)\) is derived from an oriented matroid, we ignore all the signs in the last step of the construction. Thus, the resulting system is exactly the same system as the one we derived from the original matroid \((F, \mathcal{M})\). Still the signed minimal path sets will play an important role in the construction of higher order equivalent systems. In relation to this we say that a component \(e \in E\) is two-way relevant in \((E, \phi)\) with respect to \(\bar{P}\) if there exists signed minimal path sets \(P_1, P_2 \in \bar{P}\) such that \(e \in P_1^+\) and \(e \in P_2^-\). A relevant component which is not two-way relevant, is said to be one-way relevant. In particular, if \(e \in P_1^+\) for all \(P \in \bar{P}\) such that \(e \in P\), we say that \(e\) is forward relevant, while if \(e \in P_1^-\) for all \(P \in \bar{P}\) such that \(e \in P\), we say that \(e\) is backward relevant.

If \((F, \bar{M})\) is an oriented matroid and \(e \in F\), it is possible to modify \((F, \bar{M})\) as follows: For any signed circuit \(M \in \mathcal{M}\) such that \(e \in M\) we replace \(M\) by \(\bar{M}\) containing the same elements as \(M\), but where \(\bar{M}\) is obtained from \(M\) by reversing the sign of \(e\). That is, if \(e \in M^+\), then \(\bar{M}^+ = M^+ \setminus \{e\}\), while \(\bar{M}^- = M^- \cup \{e\}\). Similarly, if \(e \in M^-\), then \(\bar{M}^+ = M^+ \cup \{e\}\), while \(\bar{M}^- = M^- \setminus \{e\}\). The resulting family of signed sets is denoted \(\bar{M}\), and it can be shown that \((F, \bar{M})\) is an oriented matroid as well. See Björner et al. [2]. Reversing the signs of an element \(e\) is referred to as reorientation. Obviously, it is possible to reorientate more than a single element. Thus, if we reorientate all elements in some set \(A \subseteq F\), we denote the resulting oriented matroid by \((F_{\rightarrow A}, \bar{M})\).

Note that the underlying matroid is not changed when elements are reoriented. Thus, the corresponding orientable matroid system is not changed either as a result of this. Thus, if \((F, \bar{M}) \rightarrow (E, \phi)\) and \((F_{\rightarrow A}, \bar{M}) \rightarrow (E_{\rightarrow A}, \phi)\), then \(\phi(B) = _A \phi(B)\) for all \(B \subseteq E\). However, if \((F, \bar{M}) \rightarrow (E, \phi)\), and \(e\) is a forward relevant component, then \(e\) becomes a backward relevant component if it is reoriented. Similarly, if \(e\) is a backward relevant component, then \(e\) becomes a forward relevant component if it is reoriented.

While the class of orientable matroid systems is nothing more than a subclass of matroid systems, it is possible to use this class as a basis for defining oriented or partially oriented matroid systems as well. Oriented matroid systems were studied in Huseby [8]. In this context we introduce the more general concept of partially oriented matroid systems. This can be done as follows: If \((E, \phi)\) is an orientable matroid system with \(\mathcal{P}\) as its family of signed minimal path sets, and \(A \subseteq E\), we introduce:

\[
\mathcal{P}_{[A]} = \{P \in \mathcal{P} : A \cap P^- = \emptyset\}
\]
Thus, the family $\overline{P}_{[A]}$ where components in the set $A$ only are allowed to belong to the positive parts of the signed minimal path sets. The resulting binary monotone system is denoted by $(E, \phi_{[A]})$. We refer to the components in $A$ as the oriented components of the system. If $A = E$, all components are oriented, and the resulting system is an oriented matroid system. On the other hand, if $A \subset E$, we say that the system is a partially oriented matroid system.

We observe that if $(F, \mathcal{M}) \rightarrow (E, \phi)$ and $e \in A \subseteq E$, then $e$ is relevant in $(E, \phi_{[A]})$ if and only if $e$ is either two-way relevant or forward relevant in $(E, \phi)$. Moreover, if $(E, \phi)$ is coherent, and $B \subseteq E$ is the set of backward relevant components, then $(E_{(-A_\cap B)} \phi_{[A]})$ is coherent as well. This follows since in $(E_{(-A_\cap B)} \phi_{[A]})$ all the backward relevant components in $A$ have been reoriented.

### 3.2 Minors of matroids

The minor operations introduced for binary monotone systems correspond to similar operations on matroid. If $(F, \mathcal{M})$ is a matroid and $e \in F$, we introduce the following families of subsets of $F \setminus \{e\}$:

$$\mathcal{M}_{+e} = \text{Min}\{ (M \setminus \{e\}) : M \in \mathcal{M}, (M \setminus \{e\}) \neq \emptyset \}, \quad (3.1)$$
$$\mathcal{M}_{-e} = \{ M \in \mathcal{M} : e \notin M \}. \quad (3.2)$$

It can then be shown that $(F \setminus \{e\}, \mathcal{M}_{+e})$ and $(F \setminus \{e\}, \mathcal{M}_{-e})$, called respectively the contraction and restriction of $(F, \mathcal{M})$ with respect to $e$, are matroids as well. See Welsh [14]. If a matroid $(G, \mathcal{N})$ can be obtained from $(F, \mathcal{M})$ by performing a sequence of minor operations, we say that $(G, \mathcal{N})$ is a minor of $(F, \mathcal{M})$.

Note that while the definition of $\mathcal{M}_{-e}$ in (3.2) is completely similar to the definition of $\mathcal{P}_{-e}$ in (2.3), there is a slight difference in how the family $\mathcal{M}_{+e}$ is defined in (3.1) compared to how the family $\mathcal{P}_{+e}$ is defined in (2.2). When constructing the contraction of a matroid, it may happen that some of the circuits contain only a single element. Such circuits are called loops. If $\{e\}$ is such a loop, then by (3.1) $\mathcal{M}_{+e}$ is simply the family of all the other circuits. Thus, in this case we in fact have $\mathcal{M}_{+e} = \mathcal{M}_{-e}$. Moreover, if $\{e\}$ is a loop, then $e$ does not belong to any other circuit in the matroid. Hence, in particular $e$ cannot belong to any minimal path set either. Thus, it follows by (2.2) that $\mathcal{P}_{+e}$ is simply the family of all the minimal path sets. Thus, in this case we have $\mathcal{P}_{+e} = \mathcal{P}_{-e}$ as well.

More generally, the following result was proved in Huseby [6]:

**Proposition 3.4** Let $(E, \phi)$ be a matroid system derived from the matroid $(F, \mathcal{M})$. Moreover, let $e \in E$. Then both $(E \setminus \{e\}, \phi_{+e})$ and $(E \setminus \{e\}, \phi_{-e})$ are matroid systems as well, and we have:

$$(F \setminus \{e\}, \mathcal{M}_{+e}) \rightarrow (E \setminus \{e\}, \phi_{+e}), \quad (3.3)$$
$$(F \setminus \{e\}, \mathcal{M}_{-e}) \rightarrow (E \setminus \{e\}, \phi_{-e}). \quad (3.4)$$

Minor operations on oriented matroids are defined in exactly the same way. Moreover, orientability of matroids is preserved under minor operations. See Björner et al. [2]. It is easy to see that the same holds for orientability of matroid systems. Thus, the following proposition is immediate:

**Proposition 3.5** If $(E, \phi)$ is an orientable matroid system, and $(D, \psi)$ is a minor of $(E, \phi)$, then $(D, \psi)$ is an orientable matroid system as well.
The class of oriented or partially oriented matroid systems is not closed under minor operations. This issue was addressed in Huseby [8]. Still, for our purpose the following result is sufficient:

**Proposition 3.6** Let \((E, \phi)\) be an orientable matroid system derived from the oriented matroid \((\mathcal{F}, \mathcal{M})\). We then let \(A \subseteq E\) and consider the partially oriented matroid system \((E, \phi|_A)\).

If \(e \in E \setminus A\), then \((E \setminus \{e\}, \phi|_A + e)\) is a partially oriented matroid system, and we have:

\[
(F \setminus \{e\}, \bar{\mathcal{M}} + e) \rightarrow (E \setminus \{e\}, \phi|_A + e).
\]

If \(e \in E\), then \((E \setminus \{e\}, \phi|_{A \setminus \{e\}} - e)\) is a partially oriented matroid system, and we have:

\[
(F \setminus \{e\}, \bar{\mathcal{M}} - e) \rightarrow (E \setminus \{e\}, \phi|_{A \setminus \{e\}} - e).
\]

**Proof:** The result is a consequence of Proposition 3.4.

If \((E, \phi|_A)\) is a partially oriented matroid system derived from the oriented matroid \((\mathcal{F}, \mathcal{M})\), we say that \((E, \phi|_A)\) cyclic if there exists a positive signed circuit \(M \in \mathcal{M}\) such that \(M \subseteq A\). A partially oriented matroid system which is not cyclic is said to be acyclic.

In Huseby [8] the following result was proved:

**Proposition 3.7** If an oriented matroid system \((E, \phi|_E)\) is cyclic, then \(d(\phi|_E) = 0\).

A generalization of this result is the following:

**Theorem 3.8** If for \(A \subseteq E\) the partially oriented matroid system \((E, \phi|_A)\) is cyclic, then \(d(\phi|_A) = 0\).

**Proof:** The proof is by induction on \(|E \setminus A|\). If \(|E \setminus A| = 0\), we have \(A = E\). Thus, in this case the result holds by Proposition 3.7. We then assume that the result is proved for all partially oriented matroid systems where \(|E \setminus A| < n\), and consider a partially oriented matroid system \((E, \phi|_A)\) where \(|E \setminus A| = n\). We then choose a component \(e \in E \subseteq A\).

By Proposition 3.6 we know that both \((E \setminus \{e\}, \phi|_A + e)\) and \((E \setminus \{e\}, \phi|_A - e)\) are partially oriented matroid systems. Moreover, both these minors must be cyclic as well since by the assumption \(A\) contains a positive signed circuit. Hence, using Proposition 2.2 and the induction hypothesis we get:

\[
d(\phi|_A) = d(\phi|_A + e) - d(\phi|_A - e) = 0 - 0 = 0.
\]

Hence, the result is proved by induction.

**Corollary 3.9** Assume that for \(A \subseteq E\), the partially oriented matroid system \((E, \phi|_A)\) is cyclic, and that \(\delta\) is the domination function of \((E, \phi|_A)\). Then \(\delta(B) = 0\) for all \(B \subseteq E\) such that \(B \cap A\) contains a positive signed circuit.

**Proof:** Let \(B \subseteq E\) and \(C = E \setminus B\), and let \((B, \psi|_{A \setminus C})\) denote the restriction of \((E, \phi|_A)\) to the set \(B \subseteq E\). By repeated application of Proposition 3.6 it follows that \((B, \psi|_{A \setminus C})\) is a partially oriented matroid system. Moreover, if \(B \cap A\) contains a positive signed circuit, then \((B, \psi|_{A \setminus C})\) is cyclic. Hence, by Theorem 3.8 we have that \(d(\psi|_{A \setminus C}) = 0\). Finally, by Proposition 2.1 we also have that \(d(\psi|_{A \setminus C}) = \delta(B)\), and thus the result follows.

Huseby [8] also proved the following result:
Proposition 3.10 If an oriented matroid system \((E, \phi_{|E|})\) is coherent and acyclic, then \(|d(\phi_{|E|})| = 1\).

This result has a consequence that is relevant for the present paper:

Theorem 3.11 Let \((E, \phi)\) be a coherent orientable matroid system. If all components are one-way relevant, then \((E, \phi)\) is an s-p-system.

Proof: Assume that all the components in \((E, \phi)\) are one-way relevant, and let \(B \subseteq E\) be the set of backward relevant components. By reorienting the components in \(B\), we obtain a matroid system \((E, -B \phi)\) where all the components are forward relevant. Since the underlying matroid is not changed by reorientation, it follows that \(\phi(A) = -B \phi(A)\) for all \(A \subseteq E\). We now consider the oriented matroid system \((E, -B \phi_{|E|})\). Since all the components in \((E, -B \phi)\) are forward relevant, it follows that we also have \(\phi(A) = -B \phi(A)_{|E|}\) for all \(A \subseteq E\). Thus, in particular \(d(\phi) = d(-B \phi(A)_{|E|})\). Since \((E, \phi)\) is assumed to be coherent, it follows by the first part of Proposition 3.1 that \(|d(\phi)| > 0\). On the other hand, by combining Proposition 3.7 and Proposition 3.10 it follows that \(|d(-B \phi(A)_{|E|})| \leq 1\). Hence, the only possibility is that \(|d(\phi)| = |d(-B \phi(A)_{|E|})| = 1\). Hence, by the second part of Proposition 3.1, it follows that \((E, \phi)\) is an s-p-system.

Using matroid theory it is possible to show the converse result as well:

Theorem 3.12 Let \((E, \phi)\) be a coherent orientable matroid system. If \((E, \phi)\) is an s-p-system, then all components are one-way relevant.

The proof of this result, however, is beyond the scope of the present paper. We will return to this result in an upcoming paper.

3.3 Anti-parallel extensions of oriented matroids

If \((F, \bar{M})\) is an oriented matroid where \(f \in F\) and \(f' \notin F\), we can extend \((F, \bar{M})\) by adding the component \(f'\) in such a way that \(f\) and \(f'\) forms a new signed circuit. Thus, we add an extra signed circuit \(N_0\) such that \(N_0 = N_0^+ = \{f, f'\}\) to the family \(\bar{M}\). In order to preserve the oriented matroid axioms after adding \(N_0\), however, the family \(\bar{M}\) needs to be extended further by adding more signed circuits containing the new component \(f'\). In order to explain this in further detail, we partition the family \(\bar{M}\) into:

\[
\bar{M}_0 = \{M \in \bar{M} : f \notin M\}, \\
\bar{M}_1 = \{M \in \bar{M} : f \in M^+\}, \\
\bar{M}_2 = \{M \in \bar{M} : f \in M^-\}.
\]

We then introduce the following families of subsets of the extended ground set \((F \cup \{f'\})\):

\[
\bar{E}_1 = \{N : N^+ = M^+ \setminus \{f\}, \ N^- = M^- \cup \{f'\}, \ M \in \bar{M}_1\}, \\
\bar{E}_2 = \{N : N^+ = M^+ \cup \{f'\}, \ N^- = M^- \setminus \{f\}, \ M \in \bar{M}_2\}.
\]

We then define:

\[
\bar{M}^* = \bar{M}_0 \cup \bar{M}_1 \cup \bar{M}_2 \cup \bar{E}_1 \cup \bar{E}_2 \cup \{N_0, -N_0\}.
\]
It turns out that \((F \cup \{f\}, \mathcal{M}^*)\) indeed is an oriented matroid, and we call this the \textit{anti-parallel extension} of \((F, \mathcal{M})\) with respect to \(f\). The validity of this construction is well established. See e.g., Björner et al. \cite{2}.

In order to explain this in more detail we consider the directed network shown in Figure 3.2.

![Figure 3.2: A simple directed network.](image)

The corresponding oriented matroid \((F, \mathcal{M})\) is obtained by letting \(F = \{1, \ldots, 7, x\}\) and \(\mathcal{M}\) be the family of signed circuits in the network.

We then add an extra component \(3'\) in anti-parallel with the component \(3\). The resulting network is shown in Figure 3.3. The extended network obviously has a new signed circuit \(N\), where \(N = N^+ = \{3, 3'\}\). However, there are of course lots of other signed circuits also containing the new component \(3'\). One such signed circuit is e.g., \(\{2, 3', 4, 6, x\}\). Moreover, for each new signed circuit we also need to include its reversed version in the extended family of signed circuits. In particular, we need to include the signed circuit \(-N\), where \(-N = (-N)^- = \{3, 3'\}\). The family \(\mathcal{M}^*\) is constructed so that it contains all these new signed circuits as well as all the signed circuits in \(\mathcal{M}\).

By interpreting the nodes labelled \(S\) and \(T\) as terminals in a network system, and the edge \(x\) between \(S\) and \(T\) as the extension component, we can derive the two corresponding 2-terminal network systems. That is, we identify the families \(\mathcal{M}_x\) and \(\mathcal{M}^*_x\) of signed circuits containing \(x\) in their positive parts. The resulting families of signed minimal path sets, denoted respectively by \(\mathcal{P}\) and \(\mathcal{P}^*\), are obtained by deleting the extension component \(x\)
from all the identified signed circuits. Since $\mathcal{P}^*$ is derived from a network with the extra component $3'$, we clearly have that $\mathcal{P} \subseteq \mathcal{P}^*$. The family $\mathcal{P}^*$ also includes new signed minimal path sets obtained by replacing the component 3 by 3' wherever possible. One such new signed minimal path set is $\{2, 3', 4, 6\}$. More specifically, if $P \in \mathcal{P}$ and $3 \in P^+$, then there exists a set $P' \in \mathcal{P}^*$ such that $P'^+ = P^+ \setminus \{3\}$ and $P'^- = P^- \cup \{3'\}$. Similarly, if $P \in \mathcal{P}$ and $3 \in P^-$, then there exists a set $P' \in \mathcal{P}^*$ such that $P'^+ = P^+ \cup \{3'\}$ and $P'^- = P^- \setminus \{3\}$.

Motivated by the above example, we now define the anti-parallel extension of an orientable matroid system as follows. Let $(E, \phi)$ be an orientable matroid system and let $(E \cup \{x\}, \mathcal{M})$ denote the corresponding oriented matroid from which $(E, \phi)$ is derived. We also let $\mathcal{P}$ denote the family of signed minimal path sets of $(E, \phi)$. Furthermore, let $f \in E$ and let $f' \notin E$, and denote by $(F \cup \{f'\}, \mathcal{M}')$ the anti-parallel extension of $(F, \mathcal{M})$ with respect to $f$. We proceed by identifying the family $\mathcal{M}'^* = \{M \in \mathcal{M}^* : x \in M^+\}$, and letting $\mathcal{P}^* = \{M \setminus \{x\} : M \in \mathcal{M}'^*\}$. We then define the anti-parallel extension of $(E, \phi)$ with respect to $f$ as the binary monotone system $(E \cup \{f'\}, \phi^*)$ with $\mathcal{P}^*$ as its family of signed minimal path sets.

In order to study the family $\mathcal{P}^*$ further, we partition $\mathcal{P}$ into:

$$
\mathcal{P}_0 = \{P \in \mathcal{P} : f \notin P\},
$$

$$
\mathcal{P}_1 = \{P \in \mathcal{P} : f \in P^+\},
$$

$$
\mathcal{P}_2 = \{P \in \mathcal{P} : f \in P^-\}.
$$

We then introduce the following families of subsets of the extended ground set $(E \cup \{f'\})$:

$$
\mathcal{Q}_1 = \{Q : Q^+ = P^+ \setminus \{f\}, \quad Q^- = P^- \cup \{f\}, \quad P \in \mathcal{P}_1\}, \quad (3.9)
$$

$$
\mathcal{Q}_2 = \{Q : Q^+ = P^+ \cup \{f'\}, \quad Q^- = P^- \setminus \{f\}, \quad P \in \mathcal{P}_2\}. \quad (3.10)
$$

By (3.7) and (3.8) it is easy to see that we have:

$$
\mathcal{P}^* = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2.
$$

We note that the signed circuits $N_0$ and $-N_0$ do not contain the extension component $x$, so these sets are not included in the family $\mathcal{M}'^*$.

Having arrived at the anti-parallel extension $(E \cup \{f'\}, \phi^*)$ we consider the partially oriented system $(E \cup \{f'\}, \phi^*_{[f,f']})$. In this system the components $f$ and $f'$ are only allowed to belong to the positive parts of the minimal path sets. Using the above notation this implies that the minimal path sets in the families $\mathcal{P}_0$, $\mathcal{P}_1$, and $\mathcal{Q}_2$ are still valid minimal path sets, while the minimal path sets in the families $\mathcal{P}_2$, and $\mathcal{Q}_1$ are excluded from the system. That is, we have:

$$
\mathcal{P}^*_{[f,f']} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{Q}_2.
$$

We now have the following results:

**Theorem 3.13** Let $(E, \phi)$ be an orientable matroid system, and assume that $f \in E$ and $f' \notin E$. We also let $(E \cup \{f'\}, \phi^*)$ be the anti-parallel extension of $(E, \phi)$ with respect to $f$.

(a) If $f$ is forward relevant in $(E, \phi)$, then $f$ is relevant while $f'$ is irrelevant with respect to $(E \cup \{f'\}, \phi^*_{[f,f']})$. 

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(b) If \( f \) is backward relevant in \((E, \phi)\), then \( f \) is irrelevant while \( f' \) is relevant with respect to \((E \cup \{f'\}, \phi^*_{\{f,f'\}})\).

(c) If \( f \) is two-way relevant in \((E, \phi)\), then both \( f \) and \( f' \) are relevant with respect to \((E \cup \{f'\}, \phi^*_{\{f,f'\}})\).

**Proof:** If \( f \) is forward relevant in \((E, \phi)\), this implies that \( \overline{P}_1 \) is non-empty, while \( \overline{Q}_2 \) is empty. Hence, \( f \) is relevant while \( f' \) is irrelevant with respect to \((E \cup \{f'\}, \phi^*_{\{f,f'\}})\).

Similarly, if \( f \) is backward relevant in \((E, \phi)\), this implies that \( \overline{P}_1 \) is empty, while \( \overline{Q}_2 \) is non-empty. Hence, \( f \) is irrelevant while \( f' \) is relevant with respect to \((E \cup \{f'\}, \phi^*_{\{f,f'\}})\).

Finally, if \( f \) is two-way relevant then both \( \overline{P}_1 \) and \( \overline{Q}_2 \) are non-empty. Hence, in this case both \( f \) and \( f' \) are relevant with respect to \((E \cup \{f'\}, \phi^*_{\{f,f'\}})\). \( \blacksquare \)

Using Theorem 3.13 the following corollary is immediate:

**Corollary 3.14** Let \((E, \phi)\) be a coherent orientable matroid system. Then there exists a coherent, partially oriented anti-parallel extension of \((E, \phi)\) if and only if \((E, \phi)\) contains at least one two-way relevant component.

By combining Corollary 3.14, Theorem 3.11 and Theorem 3.12, we also get:

**Corollary 3.15** Let \((E, \phi)\) be a coherent orientable matroid system. Then there exists a coherent, partially oriented anti-parallel extension of \((E, \phi)\) if and only if \((E, \phi)\) is not an s-p-system.

The following results deal with anti-parallel extensions and domination:

**Theorem 3.16** Let \((E, \phi)\) be an orientable matroid system, and assume that \( f \in E \) and \( f' \notin E \). We also let \((E \cup \{f'\}, \phi^*)\) be the anti-parallel extension of \((E, \phi)\) with respect to \( f \). Then we have:

\[
d(\phi^*_{\{f,f'\}}) = 0.
\]

(3.11)

**Proof:** The result can easily be proved by using Proposition 2.2 and induction. However, by realizing that \((E \cup \{f'\}, \phi^*_{\{f,f'\}})\) is a cyclic, partially oriented matroid system, the result is in fact just a special case of Theorem 3.8 \( \blacksquare \)

**Corollary 3.17** Let \((E, \phi)\) be an orientable matroid system, and assume that \( f \in E \) and \( f' \notin E \). We also let \((E \cup \{f'\}, \phi^*)\) be the anti-parallel extension of \((E, \phi)\) with respect to \( f \). Finally, let \( \delta \) be the domination function of \((E \cup \{f'\}, \phi^*_{\{f,f'\}})\). Then \( \delta(B) = 0 \) for all \( B \subseteq E \) such that \( \{f, f'\} \subseteq B \).

**Proof:** The result is a special case of Corollary 3.9.

We close this section by pointing out that the concept of anti-parallel extensions can be extended further in the obvious way by considering anti-parallel extensions of all the components in the original system. As long as the original system is coherent, and the anti-parallel extensions are constructed with respect to two-way relevant components, the resulting extended system is coherent as well. Moreover, by the same argument we used to prove Theorem 3.16 and Corollary 3.17 the domination function, \( \delta \), of the extended system is zero for all sets \( B \) containing at least one pair of anti-parallel components.
4 Constructing equivalent systems

Having established the results we need on orientable matroid systems, we can proceed by applying this to the construction of equivalent systems. The idea is to use the same technique as we did in the example considered in Section 2. Thus, we consider a coherent orientable matroid system \((E, \phi)\), and let \(F \subseteq E\) be the set of two-way relevant components in this system. We assume that \(F \neq \emptyset\). For all \(f \in F\) we extend the system by adding a new component \(f'\) in anti-parallel as described in Subsection 3.3. We let \(F'\) denote the set of added components, and consider the orientable system \((E \cup F', \phi^*\mid_{E \cup F'})\), where \(\phi^*\) denotes the structure function of the system after all the components are added. Finally, we arrive at the partially oriented system \((E \cup F', \phi^*\mid_{E \cup F'})\) where all the components in \(F\) and \(F'\) are oriented. By Theorem 3.13 it follows that \((E \cup F', \phi^*\mid_{E \cup F'})\) is coherent.

We then assume that component \(e \in E\) has reliability \(q_e\), and let \(q\) be the vector of component reliabilities of the original orientable matroid system \((E, \phi)\). We denote the reliability of \((E, \phi)\) by \(h\). Thus, assuming as before that all the components are independent, we may write \(h = h(q)\).

If \(f \in F\), and \(f'\) is the corresponding anti-parallel component, we assume that \(f'\) has the same reliability as \(f\). The reliability of \((E \cup F', \phi^*\mid_{E \cup F'})\) is denoted by \(h'\). Since no new component reliabilities are introduced, it follows that we may write \(h' = h'(q)\), once again assuming independence of the components. We can now state our main result:

**Theorem 4.1** Let \((E, \phi)\) be a coherent orientable matroid system, and let \((E \cup F', \phi^*\mid_{E \cup F'})\) be obtained from \((E, \phi)\) as described above. Moreover, let the component and system reliabilities be as described above. We then have:

\[
h(q) = h'(q), \quad \text{for all } q \in [0, 1]^{|E|}.
\]

That is, \((E, \phi)\) and \((E \cup F', \phi^*\mid_{E \cup F'})\) are equivalent systems.

**Proof:** As in the example considered in Section 2, the proof consists of two steps. In the first step we assume that the correlations between the states of the anti-parallel components are 1, so that one component functions if and only if the other functions. That is, if \(X_f\) and \(X_{f'}\) are the state variables of two components in anti-parallel, we have:

\[
E[X_f \cdot X_{f'}] = q_f.
\]

Under this assumption the anti-parallel components behave like single undirected components, and thus, the two systems must have the same system reliability. Denoting the domination function of \((E \cup F', \phi^*\mid_{E \cup F'})\) by \(\delta'\), this implies that we have:

\[
h(q) = \sum_{B \subseteq E \cup F'} \delta'(B) E[\prod_{e \in B} X_e]. \quad (4.1)
\]

In the second step we apply Corollary 3.17. By this corollary \(\delta'(B) = 0\) for all \(B \subseteq E\) containing at least one pair of anti-parallel components, and from this it follows that:

\[
E[\prod_{e \in B} X_e] = \prod_{e \in B} P(X_e = 1), \quad \text{for all } B \subseteq E \cup F' \text{ such that } \delta'(B) \neq 0. \quad (4.2)
\]

Hence, if we drop the assumption that the correlations between the states of the anti-parallel components are 1, and instead assume that all the components in the extended
system \((E \cup F', \phi'_{c_{\cup F'}})\) are independent, this does not change the reliability of the system. This follows since all the terms in the reliability function where a possible dependence between pairs of anti-parallel components could affect the system reliability, vanish. Thus, by combining (4.1) and (4.2) we get that:
\[
h(q) = \sum_{B \subseteq E \cup F'} \delta'(B) \prod_{e \in B} P(X_e = 1) = h'(q),
\]
and this concludes the proof. □

Since undirected network systems are special cases of orientable matroid systems, Theorem 4.1 allows us to construct equivalent network systems by adding anti-parallel edges to all two-way relevant edges. In particular the example considered in Section 2 is obtained this way. In order to demonstrate that Theorem 4.1 is not restricted to network systems, we present a class of orientable matroid systems which can be derived from a set of vectors.

Thus, we let \((E, \phi)\) be a coherent orientable matroid system which is defined as follows:

For each component \(e \in E\) we associate a vector \(v_e \in \mathbb{R}^k\) where the dimension \(k\) is suitably chosen. We also add a target vector \(v_x \in \mathbb{R}^k\), where \(x \notin E\) denotes the extension component. For all subsets \(B \subseteq E\), we define \(\text{Span}(B)\) as the subspace of \(\mathbb{R}^k\) spanned by the vectors in the set \(\{v_e : e \in B\}\). We then define \(\phi(B)\) for all \(B \subseteq E\) as:
\[
\phi(B) = \text{I}(v_x \in \text{Span}(B)).
\]

Then it can be shown that \((E, \phi)\) is an orientable matroid system. If \((E \cup \{x\}, \mathcal{M})\) denotes the corresponding matroid, the circuits of this matroid are the minimal sets \(M \in \mathcal{M}\) such that the corresponding set of vectors \(\{v_e : e \in M\}\) are linearly dependent. See Björner et al. [2]. Thus, if \(M \in \mathcal{M}\), there exists real numbers \(\{\lambda_e(M) \neq 0 : e \in M\}\) satisfying:
\[
\sum_{e \in M} \lambda_e(M)v_e = 0.
\]

In particular, if \(M \in \mathcal{M}_x\), we have:
\[
\lambda_x(M)v_x + \sum_{e \in M \setminus \{x\}} \lambda_e(M)v_e = 0.
\]

Since \(\lambda_x(M) \neq 0\), we may divide both sides of this equation by \(\lambda_x(M)\), and obtain:
\[
v_x = \sum_{e \in M \setminus \{x\}} \tilde{\lambda}_e(M)v_e,
\]
where we have introduced \(\tilde{\lambda}_e(M) = -\lambda_e(M)/\lambda_x(M)\), for all \(e \in M \setminus \{x\}\). This implies that \(v_x \in \text{Span}(M \setminus \{x\})\), and by the definition of \(\phi\) we conclude that \(M \setminus \{x\}\) is a minimal path set of \((E, \phi)\). Since all the minimal path sets of \((E, \phi)\) can be derived this way, we indeed have that \((E \cup \{x\}, \mathcal{M}) \to (E, \phi)\).

If \(M \in \mathcal{M}\), the corresponding signed circuit is obtained, following Björner et al. [2], by letting:
\[
M^+ = \{e \in M : \lambda_e(M) > 0\},
\]
\[
M^- = \{e \in M : \lambda_e(M) < 0\}.
\]

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From the signed circuits we also derive the family of signed minimal path sets in the usual way by considering the signed circuits in $\mathcal{M}_x$ and defining $\mathcal{P} = \{ M \setminus \{ x \} : M \in \mathcal{M}_x \}$.

It is easy to see that this type of orientable matroid systems in fact includes all 2-terminal undirected network systems. This follows by letting the vectors $v_e, e \in E$ be the columns of the signed node-edge incidence matrix, and by letting $v_x$ be an incidence vector corresponding to an additional edge between the two terminals in the network. However, the class of orientable matroid systems derived from vectors contains many other types of systems as well.

Assuming that $(E, \phi)$ contains at least one two-way relevant component, we may construct a coherent equivalent system by using the procedure described above. In order to explain this in further detail, we consider a specific example where $E = \{1, 2, 3, 4\}$, and where:

$$v_1 = (5, 0), \ v_2 = (4, 3), \ v_3 = (3, 4), \ v_4 = (0, 5).$$

Moreover, we let $v_5 = (-5, -5)$ be the target vector. Since any pair of the vectors $v_1, \ldots, v_4$ span $\mathbb{R}^2$, it follows that the minimal path sets of the system are:

$$P_1 = \{1, 2\}, \ P_2 = \{1, 3\}, \ P_3 = \{1, 4\}, \ P_4 = \{2, 3\}, \ P_5 = \{2, 4\}, \ P_6 = \{3, 4\}.$$

Thus, it follows that $(E, \phi)$ is in fact a 2-out-of-4 system.

By considering the signs of the coefficients in the different linear combinations representing the target vector $v_x$, we obtain the following signed minimal path sets:

$$P_1 = \{1, 2\}, \ P_2 = \{1, 3\}, \ P_3 = \{1, 4\}, \ P_4 = \{2, 3\}, \ P_5 = \{2, 4\}, \ P_6 = \{3, 4\},$$

where the components in the negative parts of the signed sets are marked by a bar. Thus, $1 \in P_1^-$ and $4 \in P_6^-$. We observe that the components 1 and 4 are two-way relevant, while the other components are forward relevant.

The structure function of $(E, \phi)$ are given by:

$$\phi = X_1X_2 + X_1X_3 + X_1X_4 + X_2X_3 + X_2X_4 + X_3X_4 - 2X_1X_2X_3 - 2X_1X_2X_4 - 2X_1X_3X_4 - 2X_2X_3X_4 + 3X_1X_2X_3X_4.$$

Moreover, assuming independence and that $P(X_e = 1) = q_e, e \in E$, the reliability of the system is given by:

$$h = q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4 - 2q_1q_2q_3 - 2q_1q_2q_4 - 2q_1q_3q_4 - 2q_2q_3q_4 + 3q_1q_2q_3q_4.$$

We can now construct an equivalent system by adding components in anti-parallel to the two-way relevant components 1 and 4. We denote the new components by $1'$ and $4'$ respectively. The corresponding vectors are $v_{1'} = -v_1 = (-5, 0)$ and $v_{4'} = -v_4 = (0, -5)$. We simplify the notation by denoting the resulting system by $(E', \phi')$, where $E' = E \cup \{1', 4'\}$, and where $\phi'(B) = 1$ if and only if there exists a set of positive real numbers $\{\lambda_e(B) : e \in B\}$ such that:

$$v_x + \sum_{e \in B} \lambda_e(B)v_e = 0.$$
Thus, in \((E', \phi')\) we only include positive signed minimal path sets. This means that \((E', \phi')\) is an oriented matroid system. Moreover, the signed minimal path sets of the \((E', \phi')\) are:

\[
P_1' = \{1', 2\}, P_2 = \{1, 3\}, P_3 = \{1, 4\}, P_4 = \{2, 3\}, P_5 = \{2, 4\}, P_6' = \{3, 4'\}.
\]

Given these signed minimal path sets it is easy to verify that the structure function \(\phi'\) is given by:

\[
\phi' = X_1'X_2 + X_1X_3 + X_1X_4 + X_2X_3 + X_2X_4 + X_3X_4'
- X_1'X_2X_3 - X_1X_2X_3 - X_1'X_2X_4 - X_1X_2X_4
- X_1X_3X_4 - X_1X_3X_4' - X_2X_3X_4 - X_2X_3X_4'
+ X_1X_2X_3X_4 + X_1X_2X_3X_4 + X_1X_2X_3X_4'.
\]

We observe that all the coefficients in this expression are either 1 or \(-1\). This is a consequence of a result proved in Huseby [8]. In fact, by using the results of Huseby [8], the expression for \(\phi'\) can be derived very easily. Moreover, we observe that in the above expression there are no terms including both \(X_1\) and \(X_1'\), and no terms including both \(X_4\) and \(X_4'\). This is exactly as expected according to Corollary 3.17. Finally, there is no term containing the product \(X_1'X_2X_3X_4\). This is a consequence of Corollary 3.9 since the set \(B = \{1', 2, 3, 4'\}\) happens to be a positive signed circuit of the corresponding oriented matroid. To see this, we must verify that we can find a set of real numbers \(\{\lambda_e(B) > 0 : e \in B\}\) such that

\[
\sum_{e \in B} \lambda_e(B)v_e = 0.
\]

This holds true if we e.g., let \(\lambda_{1'}(B) = \lambda_{4'}(B) = \frac{1}{2}\) and \(\lambda_2(B) = \lambda_3(B) = 1\).

We complete the construction by letting \(P(X_1' = 1) = q_1\), while \(P(X_4' = 1) = q_4\), and obtain the reliability of \((E', \phi')\), denoted \(h'\), as:

\[
h' = q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4
- q_1q_2q_3 - q_1q_2q_4 - q_1q_3q_4
- q_1q_3q_4 - q_1q_3q_4 - q_2q_3q_4
+ q_1q_2q_3q_4 + q_1q_2q_3q_4 + q_1q_2q_3q_4
= q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4
- 2q_1q_2q_3 - 2q_1q_2q_4 - 2q_1q_3q_4 - 2q_2q_3q_4 + 3q_1q_2q_3q_4
= h.
\]

Hence, we conclude that \((E, \phi)\) and \((E', \phi')\) are equivalent.

We note that the structure function of the original system \((E, \phi)\), is symmetric with respect to the component state variables. However, this symmetry is not present in the equivalent system \((E', \phi')\). It is easy to see that the loss of symmetry is introduced as soon as we have chosen the vectors \(v_1, \ldots, v_4\) as well as the target vector \(v_x\). Other choices of vectors would result in different equivalent systems. Still all these systems will lack the symmetry of the original system.

The construction we have presented here, can be extended to any \(k\text{-out-of-}n\) system. The key idea is to choose the vectors \(v_1, \ldots, v_n \in \mathbb{R}^k\) so that any subset of \(k\) vectors
span $\mathbb{R}^k$. Moreover, the target vector must be chosen so that it cannot be represented by fewer than $k$ of the chosen vectors. This is accomplished by arranging all the vectors in so-called general position. If $1 < k < n$, there will always be at least one two-way relevant component, allowing the construction of an equivalent system. If $k = 1$ the system is a parallel system, while if $k = n$ the system is a series system. Parallel systems and series systems are special cases of s-p-systems. For such systems we know by Theorem 3.12 that none of the components can be two-way relevant. Hence no coherent equivalent system can be constructed by adding components in anti-parallel. This point was also mentioned in Lindqvist et al. [4].

5 Conclusions

In this paper we have developed new results regarding the construction of equivalent systems of unequal order. This study is facilitated by introducing the class of orientable matroid systems. Using this class as a framework we have shown how to construct higher order equivalent systems using anti-parallel extensions. Our construction immediately applies to undirected network systems and $k$-out-of-$n$-systems, and many other types of systems.

Our results are proved using a combination of domination theory and matroid theory, and provides new insight into well-known algorithms for reliability computation. It is worth pointing out that, although many of our results are similar to those obtained in Lindqvist et al. [4], our proofs are very different from the previous proofs. Thus, we feel that our approach represents a new and very promising way of investigating the problem.

There are still many remaining questions regarding equivalent systems. One problem mentioned in Lindqvist et al. [4] is determining the maximal order of a coherent equivalent system. Through the use of anti-parallel extensions, we have shown that the number of two-way relevant components plays an important role here. The author plans to investigate this further using matroid theory as a framework. Within this framework, it is also of interest to establish a better understanding of the relation between the original system and the different possible associated oriented matroids.

The class of systems which can be derived form a set of vectors also needs to explored further, both in order to construct a wider range of examples of systems, and to better understand how the choice of vector representation influences the resulting equivalent system.

Finally, it should also be pointed out that orientable matroid systems is most likely just one class of systems allowing equivalent systems to be constructed. By investigating other classes of systems, it may be possible to obtain new and non-overlapping results.

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