Navier-Stokes equations perturbed by a rough transport noise

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Abstract

We consider a Navier-Stokes equation in two and three space dimensions subject to periodic boundary conditions and perturbed by a transport type noise. The perturbation is sufficiently smooth in space, but rough in time. The system is studied within the framework of rough path theory and, in particular, the recently developed theory of unbounded rough drivers. We introduce an intrinsic notion of weak solution to the Navier-Stokes system, establish suitable a priori estimates and prove existence. In two dimensions, we also present uniqueness and stability results with respect to the driving signal.

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1 Introduction

The theory of rough paths was introduced by Terry Lyons in his seminal work [Lyo98]. It can be briefly described as an extension of the classical theory of controlled differential equations which is robust enough to allow for a deterministic treatment of stochastic differential equations. Since its introduction, the rough path theory has found a large number of applications and tremendous progress has been made in application of rough path ideas to ordinary as well as partial differential equations driven by rough signals. We refer the reader for instance to the works by Friz et al. [CF09, CFO11], Gubinelli et al. [GT10, DGT12, GLT06], Gubinelli–Imkeller–Perkowski [GIP15], Hairer [Hai14] for a tiny sample of the exponentially growing literature on the subject. However, in view of these exciting developments, it is remarkable that many basic PDE methods have not yet found their rough path analogues. For instance, until recently it was an open problem how to construct (weak) solutions to rough partial differential equations (RPDEs) using energy methods.

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The first results in this direction were given in [BG15, DGHT16b, HH17]. In [BG15], the basics of the so-called theory of unbounded rough drivers were laid down and applied in order to show well-posedness for a linear rough path driven transport equation. In a similar spirit, the paper [DGHT16b] was concerned with nonlinear scalar conservation laws with rough flux and well-posedness was proved. The work [HH17] was then concerned with linear parabolic PDEs with discontinuous coefficients driven by rough paths and existence, uniqueness and stability was shown. Within the framework of unbounded rough drivers it was possible to give an intrinsic notion of weak solution to rough path driven PDEs as well as to derive a priori estimates similar to the classical PDE theory. These (and related) problems remained long open in the literature due to the lack of a suitable Gronwall Lemma applicable in the rough path setting. In addition, the same difficulty applied to uniqueness for reflected rough differential equations which was established by similar techniques in [DGHT16a].

The aim of the present paper is to continue in this direction and study one of the most prominent equations in fluid dynamics, namely, the Navier-Stokes system, subject to rough transport noise. We study the following equation that governs the time evolution of the velocity field $u : \mathbf{R}_+ \times \mathbf{T}^d \to \mathbf{R}^d$ and the pressure $p : \mathbf{R}_+ \times \mathbf{T}^d \to \mathbf{R}$ of an incompressible viscous fluid on the d-dimensional torus $\mathbf{T}^d$ subject to a transport type noise

$$\begin{aligned}\partial_t u + ((u - \dot{a}) \cdot \nabla)u + \nabla p &= \nu \Delta u, \\ \nabla \cdot u &= 0, \\ u(0) &= u_0 \in L^2(\mathbf{T}^d; \mathbf{R}^d),\end{aligned} \tag{1.1}$$

where $\nu > 0$ is the viscosity and $\dot{a}$ is the formal time-derivative of a function $a : \mathbf{R}_+ \times \mathbf{T}^d \to \mathbf{R}^d$ that is divergence-free in space and has finite p-variation for $p \in [2, 3)$ in time. As an example, $\dot{a}$ may stand for a white in time, colored in space noise, which is formally a time derivative of a Wiener process. Nevertheless, one of the main advantages of the rough path theory is that it allows to go much further as far as the driving (stochastic) process is concerned. For instance, it allows to consider drivers beyond the martingale world, in contrast to the classical Itô stochastic integration theory. Consequently, $\dot{a}$ may represent for instance time derivative of a more general Gaussian or Markov process, such as fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2}]$.

Due to the low regularity, a classical interpretation of the transport noise term is out of reach. Instead, we consider the equation integrated in time as an evolution type equation with values in some spatial distribution space. To this end, it is necessary make sense of the time integral $\int_0^t (\dot{a}_s \cdot \nabla) u_s \, ds$ as a spatial distribution; that is, tested against some smooth $\phi : \mathbf{T}^d \to \mathbf{R}^d$:

$$\int_0^t (\dot{a}_s \cdot \nabla) u_s ds(\phi) = - \int_0^t u_s((\dot{a}_s \cdot \nabla)\phi \,) ds, \tag{1.2}$$

where we used the assumption $\nabla \cdot \dot{a} = 0$. Nevertheless, this time integral is not a priori well defined. Indeed, we expect a solution to inherit the same time regularity as the noise a. In other words, we are confronted with a multiplication of a distribution $\dot{a}$ with a non-smooth function u. The theory of integration introduced L.C. Young in [You36] cannot be applied unless $p \in [1, 2)$.

Lyons' rough paths theory [Lyo98], however, enables us to give meaning to this integral provided we possess additional information about the driving path, namely its iterated integral.

In the case of transport noise, this iteration also leads to an iteration of the spatial derivative. This is easier to see if we consider a pure transport equation

$$\partial_t u = (\dot{a} \cdot \nabla) u. \tag{1.3}$$

Integrating the equation in time and testing against ϕ gives us

$$\begin{aligned} u_t(\phi) &= u_s(\phi) - \int_s^t u_r((\dot{a}_r \cdot \nabla)\phi) dr \\ &= u_s(\phi) - u_s\left(\int_s^t (\dot{a}_r \cdot \nabla)\phi dr\right) + \int_s^t \int_s^{r_1} u_{r_2}\left((\dot{a}_{r_2} \cdot \nabla)(\dot{a}_{r_1} \cdot \nabla)\phi\right) \, dr_2 dr_1 \\ &= u_s(\phi) - u_s\left(\int_s^t (\dot{a}_r \cdot \nabla)\phi dr\right) + u_s\left(\int_s^t \int_s^{r_1} (\dot{a}_{r_2} \cdot \nabla)(\dot{a}_{r_1} \cdot \nabla)\phi \, dr_2 dr_1\right) \\ &\quad - \int_s^t \int_s^{r_1} \int_s^{r_2} u_{r_3}\left((\dot{a}_{r_3} \cdot \nabla)(\dot{a}_{r_2} \cdot \nabla)(\dot{a}_{r_1} \cdot \nabla)\phi\right) dr_3 dr_2 dr_1, \end{aligned} \tag{1.4}$$

where we have iterated the equation into itself and used the fact that $\dot{a}$ is divergence free in space. Let us now define the operators

$$A^1_{st}\phi = \int_s^t (\dot{a}_r \cdot \nabla) dr \phi,$$

and

$$A^2_{st}\phi = \int_s^t \int_s^{r_1} (\dot{a}_{r_2} \cdot \nabla)(\dot{a}_{r_1} \cdot \nabla) dr_2 dr_1 \phi$$

and write $\delta u_{st} = u_t - u_s$. Solving the transport equation (1.3) then corresponds to finding a map $t \mapsto u_t$ such that $u^\natural$ defined by

$$\delta u_{st}(\phi) - u_s\left([A^{1,*}_{st} + A^{2,*}_{st}]\phi\right) =: u^\natural_{st}(\phi) \tag{1.5}$$

is a negligible remainder, that is, it is of order $o(|t-s|)$. As a consequence, tested against ϕ, the germ $[A^1_{st} + A^2_{st}]u_s$ provides a good local approximation of the time integral (1.2), which is then uniquely defined in view of the Sewing Lemma (see Lemma B.1). We point out that in the smooth setting, (1.5) gives an equivalent formulation of the transport equation (1.3). Furthermore, (1.5) does not contain any time derivatives and is therefore well-suited for irregular drivers.

In order to guarantee the time regularity of the remainder $u^\natural$, it has to be regarded as a distribution of third order with respect to the space variable; it is third order since three derivatives are taken in (1.4). This is yet another example of the trade-off between time and space regularity pertinent to many PDE problems, also in the classical setting. More precisely, if a is α-Hölder continuous with respect to time and the solution u has the same regularity, the first two terms on the right hand side of (1.5) are proportional to $|t-s|^\alpha$, whereas the last term can be bound by $|t-s|^{2\alpha}$. Hence, in the case of $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ there has to be a cancellation between these terms to guarantee that $u^\natural$ is indeed of order $o(|t-s|)$. On the other hand, the right hand side of (1.5) is a distribution of second order with respect to the space variable. Accordingly, the necessary improvement of time regularity can be obtained at the cost of loss of space regularity, i.e. considering $u^\natural$ rather as a distribution of third order.

In this paper, we assume that the noise can be factorized as follows:

$$a_t(x) = \sigma_k(x) z_t^k = \sum_{k=1}^{K} \sigma_k(x) z_t^k, \tag{1.6}$$

where we employ the summation over repeated indices $k \in \{1, \dots, k\}$. The vector fields $\sigma_k : \mathbf{T}^d \to \mathbf{R}^d$ are bounded, twice differentiable with bounded derivatives and divergence free. The driving signal z is a $\mathbf{R}^K$-valued path of finite p-variation for some $p \in [2,3)$ that can be lifted to a geometric rough path $(Z, \mathbb{Z})$. The first component is given by the increments of z, i.e. $Z_{st} = z_t - z_s$, whereas the second one is the so-called Lévy's area which plays the role of the iterated integral $\mathbb{Z}_{st} =: \int_s^t \int_s^r dz_{r_1} \otimes dz_r$. In the smooth setting, this iterated integral is well-defined, whereas in the rough setting, it has to be given as an input datum. For instance, if z is a Wiener process then a suitable iterated integral can be constructed using the Stratonovich stochastic integration, nevertheless, many other important stochastic processes give rise to a (2-step) rough paths. For more details we refer to Section 2.3 and the literature mentioned therein.

The motivation for a perturbation of this form comes from modeling of a turbulent flow of a viscous fluid. Namely, it was observed e.g. in [BCF91, BCF92, MR04, MR$^+$05] that an equation of the form (1.1) with (1.6) (modulo certain zero order terms) stems from the dynamics of fluid particles given by the stochastic flow map

$$\dot{\eta}_t(x) = u_t(\eta_t(x)) - \sigma_k(\eta_t(x)) \circ \dot{w}_t^k, \qquad \eta_0(x) = x \in \mathbf{T}^d,$$

where w^k is a sequence of independent Wiener process and $\circ$ denotes the Stratonovich product. In this formulation, the velocity of the fluid splits into a regular (slow oscillating, deterministic) component u and a turbulent (fast oscillating, stochastic) component $\sigma_k \circ \dot{w}^k$. Accordingly, it is natural to assume that the vector fields σ_k are divergence free. In other words, the random fluctuations are superimposed to the velocity field at the Lagrangian level and are energy neutral, i.e. they do not contribute to the energy balance. Such a splitting idea goes back to Krachnan's turbulence theory [Kra68] and was then further developed in [GK96, GV00] and other works. The solvability of (1.1) with transport noise driven by a Wiener process, was presented in [BCF92, FG95, MR$^+$05]. In general, k may range over $\mathbf{N}$, but for simplicity, in this paper, we consider only on $\{1, \dots, K\}$.

The present paper puts forward a (rough) pathwise approach to (1.1), (1.6), which, in particular, yields pathwise energy estimates for a wide class of driving signals. We establish the existence of weak solutions in two and three space dimensions, see Theorem 2.10, including the pressure recovery, see Section 4.1.2. The main tool is a Galerkin approximation combined with a suitable mollification of the driving signal, uniform estimates and a compactness argument. In addition, in two space dimensions and for constant vector fields we prove uniqueness as well as pathwise stability with respect to the given driver and initial datum (also called a Wong-Zakai result), see Theorem 2.11 and Corollary 2.12. To the best of our knowledge, this is the first Wong-Zakai type result for the Navier-Stokes system perturbed by such a general transport noise. It is an immediate consequence of the construction of solutions within rough path theory, which allows to overcome the lack of continuity of solutions as functions of the noise pertinent to the classical stochastic integration theory. There is a substantial number of Wong-Zakai results for infinite dimensional stochastic evolution equations in various settings. We mention only the work [CM11] of A. Millet and I. Chueshov in which the authors derive a Wong-Zakai result and support theorem for a general class of stochastic 2D hydrodynamical systems, including 2D stochastic Navier-Stokes. However, the diffusion coefficients in [CM11] are assumed to have linear growth on H, and hence do not cover transport-type noise. We note, however, that in [CM10], A. Millet and I. Chueshov establish a large deviations result for stochastic 2D hydrodynamical systems that does hold for transport-type noise.

Our approach relies on a suitable formulation of the problem (1.1), (1.6) in the spirit of (1.5). Due to the delicate structure of (1.1) and the fact that a solution is a couple of velocity and pressure (u, p), we observe an interesting phenomenon. To be more precise, in Section 2.5, we derive two equivalent (rough) formulations of (1.1), (1.6).

Let P be the Helmholtz-Leray projection and $Q = I - P$. Applying P and Q separately to (1.1), we get the system of coupled equations

$$\begin{aligned} &\partial_t u + P[(u \cdot \nabla)u] = \nu \Delta u + P[(\dot{a} \cdot \nabla)u] \\ &Q[(u \cdot \nabla)u] + \nabla p = Q[(\dot{a} \cdot \nabla)u] \end{aligned}$$

We take this formulation as the starting point as it makes clear the dependence of p on $\dot{a} = \sigma_k \dot{z}^k$. We then perform an iteration similar to that of the transport equation (1.3) for the terms $P[\dot{a} \cdot \nabla u]$ and $Q[\dot{a} \cdot \nabla u]$. In doing so, we arrive at a coupled system of equations for the velocity field and pressure in which the associated unbounded rough drivers are intertwined and a particular version of the so-called Chen's relation holds true (cf. (2.15) and Definition 2.5). We obtain the second formulation by summing the coupled equations in the first formulation. The second formulation is a single equation for the velocity field where the Chen's relation also has to be modified (cf. (2.19) and Definition 2.7). An alternative way to arrive at the second formulation is by iterating (1.1) and using that $\nabla p = Q[(\dot{a} \cdot \nabla)u] - Q[(u \cdot \nabla)u]$.

The presentation is organized as follows. In Section 2 we introduce the basic set up and state our main results. Section 3 is devoted to a priori estimates for various quantities. This is then used in Section 4 where the main results are proved. Several auxiliary results needed in the main body of the paper are collected in the appendix.

2 Mathematical framework and main results

2.1 Notation and definitions

To begin with, let us fix the notation we use throughout the paper. For a given Banach space V with norm $|\cdot|_V$, we denote by V^* its continuous dual and by $\mathcal{B}(V)$ the Borel sigma-algebra. For given Banach spaces V_1 and V_2, we denote by $\mathcal{L}(V_1, V_2)$ the space of continuous linear operators from V_1 to V_2 with the operator norm denoted by $|\cdot|_{\mathcal{L}(V_1,V_2)}$. For a given $d \in \mathbf{N}$, let $\mathbf{T}^d = \mathbf{R}^d/(2\pi\mathbf{Z})^d$ be the d-dimensional flat torus and denote by dx the unormalized Lebesgue measure on $\mathbf{T}^d$. For a given sigma-finite measured space $(X, \mathcal{X}, \mu)$, separable Banach space V with norm $|\cdot|_V$, and $p \in [1, \infty]$, we denote by $L^p(X; V)$ the Banach space of all μ-equivalence-classes of strongly-measurable functions $f : X \to V$ such that

$$|f|_{L^p(X;V)} := \left(\int_X |f|_V^p \, d\mu\right)^{\frac{1}{p}} < \infty,$$

equipped with the norm $|\cdot|_{L^p(X;V)}$. We denote by $L^\infty(X; V)$ the Banach space of all μ-equivalence-classes of strongly-measurable functions $f : X \to V$ such that

$$|f|_{L^\infty(X;V)} := \operatorname{esssup}_X |f|_V := \inf\{a \in \mathbf{R} : \mu(|f|_V^{-1}((a, \infty)) = 0)\} < \infty,$$

where $|f|_V^{-1}((a, \infty))$ denotes the preimage of the set (a, ∞) under the map $|f|_V : X \to \mathbf{R}$, equipped with the norm $|\cdot|_{L^\infty(X;V)}$. It is well-known that if $V = H$ is a Hilbert space with inner product $(\cdot, \cdot)_H$, then $L^2(X; H)$ is a Hilbert space equipped with the inner product

$$(f, g)_{L^2(X;H)} = \int_X (f, g)_H \, d\mu, \quad f, g \in L^2(X; H).$$

For a given Hilbert space H, we let $L^2_T H = L^2([0, T]; H)$, $L^\infty_T H = L^\infty([0, T]; H)$. For a given Hilbert space V, and real number $T > 0$ we let $C_T H$ denote the Banach space $C([0, T]; H)$ consisting of continuous functions from $[0, T]$ to G endowed with the supremum norm in time.

For a given $n \in \mathbf{Z}^d$, let $e_n : \mathbf{T}^d \to \mathbf{C}$ be defined by $e_n(x) = (2\pi)^{-\frac{d}{2}} e^{in\cdot x}$. It is well-known that $\{e_n\}_{n\in\mathbf{Z}^d}$ constitutes an orthonormal system of $L^2(\mathbf{T}^d; \mathbf{C})$, and hence for all $f, g \in \mathbf{L}^2 := L^2(\mathbf{T}^d; \mathbf{R}^d)$,

$$f = \sum_{n\in\mathbf{Z}^d} \hat{f}_n e_n, \quad (f, g)_{(L^2)^d} = \sum_{n\in\mathbf{Z}^d} \hat{f}_n \cdot \overline{\hat{g}_n},$$

where for each $n \in \mathbf{Z}^d$,

$$\hat{f}^i_n = \int_{\mathbf{T}^d} f^i(x) e_{-n}(x) \, dx, \quad i \in \{1, \ldots, d\}.$$

Let $\mathcal{S}$ be the Fréchet space of infinitely differentiable periodic complex-valued functions with the usual set of seminorms. Let $\mathcal{S}'$ be the continuous dual space of $\mathcal{S}$ endowed with the weak-star topology. We denote by $\Lambda(\phi)$ the value of a distribution $\Lambda \in \mathcal{S}'$ at a test function $\phi \in \mathcal{S}$. Since $e_n \in \mathcal{S}$, for a given $f \in \mathcal{S}'$ and $n \in \mathbf{Z}^d$, we define $\hat{f}_n = f(e_n)$. It is well-known that $f = \sum_{n \in \mathbf{Z}^d} \hat{f}_n e_n$, where convergence holds in $\mathcal{S}$ if $f \in \mathcal{S}$ and in $\mathcal{S}'$ if $f \in \mathcal{S}'$. This extends trivially to the set $\mathbf{S}' = (\mathcal{S}')^d$ of continuous linear functions from $\mathbf{S} = (\mathcal{S})^d$ to $\mathbf{C}$ endowed with the weak-star topology.

For a given $m \in \mathbf{N} \cup \{0\}$, we denote by $\mathbf{W}^{m,2}$ the Hilbert space

$$\mathbf{W}^{m,2} = (I - \Delta)^{-\frac{m}{2}} \mathbf{L}^2 = \{f \in \mathbf{S}' : (I - \Delta)^{\frac{m}{2}} f \in \mathbf{L}^2\}$$

with inner product

$$(f, g)_m = ((I - \Delta)^{\frac{m}{2}} f, (I - \Delta)^{\frac{m}{2}} g)_{\mathbf{L}^2} = \sum_{n \in \mathbf{Z}^d} (1 + |n|^2)^m \hat{f}_n \cdot \overline{\hat{g}_n}, \qquad f, g \in \mathbf{W}^{m,2}$$

and induced norm $|\cdot|_m$. For notational simplicity, when $m = 0$ we omit the index in the inner product, i.e. $(\cdot,\cdot) := (\cdot,\cdot)_0$. It is easy to see that $\mathbf{W}^{m_1,2} \subset \mathbf{W}^{m_2,2}$ for $m_1, m_2 \in \mathbf{Z}$ with $m_1 > m_2$ and that $\mathbf{S}$ is dense in $\mathbf{W}^{m,2}$ for all $m \in \mathbf{Z}$. It can be shown that for all $m, k \in \mathbf{Z}$, the map $i_{k-m,k+m}$: $\mathbf{W}^{k-m,2} \to (\mathbf{W}^{k+m,2})^*$ defined by

$$i_{k-m,k+m}(g)(f) = \langle g, f \rangle_{k-m,k+m} := ((I - \Delta)^{\frac{-m}{2}} g, (I - \Delta)^{\frac{m}{2}} f)_k,$$

for all $f \in \mathbf{W}^{k+m,2}$, and $g \in \mathbf{W}^{k-m,2}$, is an isometric isomorphism.

Let

$$\mathbf{H}^0 = \left\{ f \in \mathbf{W}^{0,2} : \nabla \cdot f = 0 \right\} = \left\{ f \in \mathbf{W}^{0,2} : \hat{f}_n \cdot n = 0, \ \forall n \in \mathbf{Z}^d \right\},$$

define $P : \mathbf{S}' \to \mathbf{S}'$ by

$$Pf = \sum_{n \in \mathbf{Z}^d} \left(\hat{f}_n - \frac{n \cdot \hat{f}_n}{|n|^2} n \right) e_n, \qquad f \in \mathbf{L}^2,$$

and let $Q : I - P$. It follows that P is a projection of $\mathbf{L}^2$ onto $\mathbf{H}^0 = P\mathbf{L}^2$ and that $\mathbf{L}^2$ possesses the orthogonal decomposition

$$\mathbf{L}^2 = P\mathbf{L}^2 \oplus Q\mathbf{L}^2.$$

It follows that $P, Q \in \mathcal{L}(\mathbf{W}^{m,2}, \mathbf{W}^{m,2})$ and that P and Q have operator norm less than or equal to one for all $m \in \mathbf{Z}$. We set $\mathbf{H}^m = P\mathbf{W}^{m,2}$ and $\mathbf{H}^m_\perp = Q\mathbf{W}^{m,2}$. As in Lemma 3.7 in [Mik02], it can be shown that the following direct sum decomposition holds for all $m \in \mathbf{Z}$,

$$\mathbf{W}^{m,2} = \mathbf{H}^m \oplus \mathbf{H}^m_\perp,$$

where

$$\langle u, v \rangle_{-m,m} = 0, \quad \forall v \in \mathbf{H}^m_\perp, \quad \forall u \in \mathbf{H}^{-m}, \tag{2.1}$$

and

$$\mathbf{H}^m = \left\{ f \in \mathbf{W}^{m,2} : \nabla \cdot f = 0 \right\},$$

$$\mathbf{H}_\perp^m = \{u \in \mathbf{W}^{m,2} : \langle v, u\rangle_{-m,m} = 0, \quad \forall v \in \mathbf{H}^{-m}\}.$$

Using (2.1), one can check that $i_{-m,m} : \mathbf{H}^{-m} \to (\mathbf{H}^m)^*$ and $i_{-m,m} : \mathbf{H}_\perp^{-m} \to (\mathbf{H}_\perp^m)^*$ are isometric isomorphisms for all $m \in \mathbf{Z}$.

For each vector $n \in \mathbf{Z}^d$, $n \neq 0$, we can find a $d-1$-vectors $\{m_1(n), \cdots, m_{d-1}(n)\} \subseteq \mathbf{R}^d$ that are of unit length and orthogonal to both $-n$ and n. It follows that $m_j(n) = m_j(-n)$ for all $j \in \{1, \ldots, d-1\}$ and $n \in \mathbf{Z}^d$ such that $n \neq 0$, and that $\mathbf{f}_{j,n} = m_j(n)e_n$, $n \in \mathbf{Z}^d$, $n \neq 0$, $j \in \{1, \ldots, d-1\}$, is an orthonormal family of $\{u \in L^2(\mathbf{T}^d; \mathbf{C}^d) : \nabla \cdot u = 0\}$. In particular, for $n = [n_1, n_2]^T \in \mathbf{Z}^2$, $n \neq 0$, the vector $\frac{n^\perp}{|n|} = [\frac{n_2}{|n|}, -\frac{n_1}{|n|}]^T$ is orthogonal to n and $-n$ and has length one. Thus, if $d = 2$, $\mathbf{f}_n = \frac{n^\perp}{|n|}e_n$, $n \in \mathbf{Z}^2$, $n \neq 0$ is an orthonormal family $\mathbf{H}_\mathbf{C}^0$. Notice that in this case

$$\nabla^T \mathbf{f}_n = \frac{n^\perp}{|n|} i n^T e_n = -\frac{n^\perp}{|n|}(-in^T)e_n = -\nabla^T \mathbf{e}_{-n} = -\nabla^T \overline{\mathbf{f}_n}. \tag{2.2}$$

One can then check that $\mathbf{w}_{j,n}^{sin}(x) := \frac{\sqrt{2}}{(2\pi)^{\frac{d}{2}}} m_j(n)\sin(n\cdot x)$ and $\mathbf{w}_{j,n}^{cos}(x) := \frac{\sqrt{2}}{(2\pi)^{\frac{d}{2}}} m_j(n)\cos(n\cdot x)$, where $j \in \{1, \ldots, d-1\}$ and $n \in \mathbf{Z}^d$ such that $n_1 \geq 0$ form an orthonormal basis of $\mathbf{H}^0$, an orthogonal basis of $\mathbf{H}^1$. Moreover, they are the eigenfunctions of the Stokes operator $P\Delta$ on $\mathbf{H}^0$ with corresponding eigenvalues $\lambda_{j,n} = |n|^2$. We re-index the sequence of eigenfunctions and eigenvalues by $\{h_n\}_{n\in\mathbf{N}}$ and $\{\lambda_n\}_{n\in\mathbf{N}}$, respectively, where $\{\lambda_n\}_{n\in\mathbf{N}}$ is a strictly positive increasing sequence tending to infinity.

The following considerations shall enlighten the construction of unbounded rough drivers in Section 2.5. Let $\sigma : \mathbf{T}^d \to \mathbf{R}^d$ be 2-times differentiable and divergence-free. Moreover, assume that all of its derivatives up to order 2 are bounded uniformly by a constant N_0. Let $\mathcal{A}^1 = \sigma\cdot\nabla = \sum_{i=1}^d \sigma^i D_i$ and note that $\mathcal{A}^2 = (\sigma\cdot\nabla)(\sigma\cdot\nabla)$. Then we find that

$$|\mathcal{A}^1|_{\mathcal{L}(\mathbf{W}^{m+1,2},\mathbf{W}^{m,2})} \leq N,\ \forall m \in \{0,2\}, \quad |\mathcal{A}^2 f|_{\mathcal{L}(\mathbf{W}^{m+2,2},\mathbf{W}^{m,2})} \leq N,\ \forall m \in \{0,1\},$$

for some constant N depending only on d and N_0.

Since $P \in \mathcal{L}(\mathbf{W}^{m,2}, \mathbf{H}^m)$ and $Q \in \mathcal{L}(\mathbf{W}^{m,2}, \mathbf{H}_\perp^m)$ for all $m \in \mathbf{Z}$, both of which have operator norm bounded by 1, we have

$$|P\mathcal{A}^1|_{\mathcal{L}(\mathbf{H}^{m+1},\mathbf{H}^m)} \leq N, \quad |Q\mathcal{A}^1|_{\mathcal{L}(\mathbf{H}_\perp^{m+1},\mathbf{H}_\perp^m)} \leq N,\ \forall m \in \{0,2\}, \tag{2.3}$$

and

$$|P\mathcal{A}^2|_{\mathcal{L}(\mathbf{H}^{m+2},\mathbf{H}^m)} \leq N, \quad |Q\mathcal{A}^2|_{\mathcal{L}(\mathbf{H}_\perp^{m+2},\mathbf{H}_\perp^m)} \leq N,\ \forall m \in \{0,1\}, \tag{2.4}$$

and hence $(P\mathcal{A}^1)^* \in \mathcal{L}((\mathbf{H}^m)^*, (\mathbf{H}^{m+1})^*)$ and $(Q\mathcal{A}^1)^* \in \mathcal{L}((\mathbf{H}_\perp^m)^*, (\mathbf{H}_\perp^{m+1})^*)$ for $m \in \{0,2\}$ and $(P\mathcal{A}^2)^* \in \mathcal{L}((\mathbf{H}^m)^*, (\mathbf{H}^{m+2})^*)$ and $(Q\mathcal{A}^2)^* \in \mathcal{L}((\mathbf{H}_\perp^m)^*, (\mathbf{H}_\perp^{m+2})^*)$ for $m \in \{0,1\}$. A simple calculation shows

$$((-P\mathcal{A}^1)f, g) = (f, P\mathcal{A}^1 g), \quad \forall f, g \in \mathbf{S}\cap\mathbf{H}^0,$$

$$((-Q\mathcal{A}^1)f, g) = (f, Q\mathcal{A}^1 g), \quad \forall f, g \in \mathbf{S}\cap\mathbf{H}_\perp^0,$$

which implies that $(-P\mathcal{A}^1)^* = P\mathcal{A}^1$, $(-Q\mathcal{A}^1)^* = Q\mathcal{A}^1$. Thus, owing to the characterization of the duality between $\mathbf{W}^{m,2}$ and $\mathbf{W}^{-m,2}$ through the $\mathbf{L}^2$ inner product, we have that $P\mathcal{A}^1 \in \mathcal{L}(\mathbf{H}^{-m}, \mathbf{H}^{-(m+1)})$, $Q\mathcal{A}^1 \in \mathcal{L}(\mathbf{H}_\perp^{-m}, \mathbf{H}_\perp^{-(m+1)})$, $P\mathcal{A}^2 \in \mathcal{L}(\mathbf{H}^{-m}, \mathbf{H}^{-(m+2)})$ and $Q\mathcal{A}^2 \in \mathcal{L}(\mathbf{H}_\perp^{-m}, \mathbf{H}_\perp^{-(m+2)})$.

In order to describe the convective term, we employ the classical notation and bounds. Owing to Lemma 2.1 in [Tem83], the trilinear form

$$b(u,v,w) = \int_{\mathbf{T}^d} ((u \cdot \nabla)v) \cdot w\, b dx = \sum_{i,j=1}^{d} \int_{\mathbf{T}^d} u^i D_i v^j w^j \, dx$$

is continuous on $\mathbf{W}^{m_1,2} \times \mathbf{W}^{m_2+1,2} \times \mathbf{W}^{m_3,2}$ if $m_1, m_2, m_3 \in \mathbf{N}_0$ satisfy

$$m_1 + m_2 + m_3 \geq \frac{d}{2}, \quad \text{if} \quad m_i \neq \frac{d}{2} \text{ for all } i \in \{1,2,3\}, \tag{2.5}$$

$$m_1 + m_2 + m_3 > \frac{d}{2}, \quad \text{if } m_i = \frac{d}{2} \text{ for some } i \in \{1,2,3\}. \tag{2.6}$$

Moreover, for all $u, v, w \in \mathbf{H}^1$,

$$b(u,v,w) = -b(u,w,v) \quad \text{and} \quad b(u,v,v) = 0. \tag{2.7}$$

For m_1, m_2, and m_3 that satisfy (2.6) or (2.5) and any given $(u,v) \in \mathbf{H}^{m_1} \times \mathbf{H}^{m_2+1}$, we define $B(u,v) \in \mathbf{H}^{-m_3}$ by

$$\langle B(u,v), w \rangle_{-m_3,m_3} = b(u,v,w), \quad \forall w \in \mathbf{H}^{m_3}.$$

At times we may simply write $(u \cdot \nabla)v(w)$ instead of $B(u,v)(w)$, and $B(u)(w) := B(u,u)(w)$.

2.2 Smoothing operators

As in [BG15], we shall need a family of smoothing operators $(J^\eta)_{\eta \in (0,1)}$ acting on the scale of spaces $(\mathbf{W}^{m,2})_{m \in \mathbf{Z}}$, i.e.

$$|(I - J^\eta) f|_m \lesssim \eta^k |f|_{m+k} \qquad \text{and} \qquad |J^\eta f|_{m+k} \lesssim \eta^{-k} |f|_m \tag{2.8}$$

for all $m \in \mathbf{Z}$ and $k \in \mathbf{N}$. To construct these operators, denote by $\tilde{J}^N : \mathbf{S}' \to \mathbf{S}$ the frequency cut-off operator

$$\tilde{J}^N f = \sum_{|n|<N} \hat{f}_n e_n.$$

It follows that for all $m \in \mathbf{Z}$ and $k \in \mathbf{N}$,

$$|f - \tilde{J}^N f|_m^2 = \sum_{|n| \geq N} (1 + |n|^2)^m |\hat{f}_n|^2 \leq N^{-2k} \sum_{|n| \geq N} (1 + |n|^2)^{m+k} |\hat{f}_n|^2 \leq N^{-2k} |f|_{m+k}^2$$

and

$$|\tilde{J}^N f|_{m+k}^2 = \sum_{|n|<N} (1 + |n|^2)^{m+k} |\hat{f}_n|^2 \leq (1 + N^2)^k \sum_{|n| \geq N} (1 + |n|^2)^m |\hat{f}_n|^2 \lesssim N^{2k} |f|_m^2.$$

We define $J^\eta := \tilde{J}^{\lfloor \eta^{-1} \rfloor}$ and J^η is a smoothing operator on $\mathbf{W}^{m,2}$. A pleasant feature of the frequency cut-off smoothing operator is that it leaves the the subspaces $\mathbf{H}^m$ and $\mathbf{H}_\perp^m$ invariant.

2.3 Rough paths

For an interval I we define $\Delta_I := \{(s,t) \in I^2 : s \le t\}$, and for $T > 0$ we let $\Delta_T := \Delta_{[0,T]}$. We also set $\Delta_I^{(2)} := \{(s,\theta,t) \in I^3 : s \le \theta \le t\}$. Let $\mathcal{P}(I)$ denote the set of all partitions of an interval I and let E be a Banach space with norm $|\cdot|$. A function $g : \Delta_I \to E$ is said to have finite p-variation for some $p > 0$ on I if

$$|g|_{p-\text{var};I} := \sup_{(t_i)\in\mathcal{P}(I)} \left(\sum_i |g_{t_i t_{i+1}}|^p \right)^{\frac{1}{p}} < \infty.$$

and we denote by $C_2^{p-var}(I;E)$ the set of all continuous functions with finite p-variation on I equipped with the seminorm $|\cdot|_{p-var;I}$. For notational simplicity, we do not specify explicitly the space E in the norm as it will be always clear from the context. We denote by $C^{p-var}(I;E)$ the set of all paths $z : I \to E$ such that $\delta z \in C_2^{p-var}(I;E)$, where $\delta z_{st} := z_t - z_s$.

A two-index map $\omega : \Delta_I \to [0,\infty)$ is called superadditive if for all $s \le \theta \le t \in I$

$$\omega(s,\theta) + \omega(\theta,t) \le \omega(s,t),$$

and ω is called a control if it is superadditive, continuous on Δ_I and for all $s \in I$, $\omega(s,s) = 0$.

If for a given $p > 0$, $g \in C_2^{p-\text{var}}(I;E)$, then it can be shown that the 2-index map $\omega_g : \Delta_I \to [0,\infty)$ defined by $\omega_g(s,t) = |g|^p_{p-var;[s,t]}$ is a control (see, e.g., Proposition 5.8 in [FV10]) and we obviously have $|g_{st}| \le \omega_g(s,t)^{\frac{1}{p}}$. Conversely, if ω is a control such that $|g_{st}| \le \omega(s,t)^{\frac{1}{p}}$ then for any partition (t_i) of $[s,t]$ we have using the superadditivity,

$$\sum_i |g_{t_i t_{i+1}}|^p \le \sum_i \omega(t_i, t_{i+1}) \le \omega(s,t).$$

Taking supremum over all partitions gives $\omega_g(s,t) \le \omega(s,t)$ and we could equivalently define the semi-norm

$$|g|_{p-var;[s,t]} = \inf\{\omega(s,t)^{\frac{1}{p}} : |g_{uv}| \le \omega(u,v)^{\frac{1}{p}} \text{ for all } (u,v) \in \Delta_{[s,t]}\}.$$

For our analysis we shall need a local version of the p-variation spaces where we restrict the mesh size of the partitions. This restriction shall be given by a control as follows.

Definition 2.1. Given a control ϖ and a number $L > 0$ we define the space $C_{2,\varpi,L}^{p-\text{var}}(I;E)$ of continuous functions $g : \Delta_I \to E$ such that $(s,t) \in \Delta_I$ with $\varpi(s,t) \le L$ implies

$$|g_{st}| \le \omega(s,t)^{\frac{1}{p}}$$

for some control ω. We define a semi-norm on this space by

$$|g|_{p-var,\varpi,L;[s,t]} := \inf\left\{\omega(s,t)^{\frac{1}{p}} : |g_{uv}| \le \omega(u,v)^{\frac{1}{p}} \text{ for all } (u,v) \in \Delta_{[s,t]} \text{ such that } \varpi(s,t) \le L\right\}.$$

From the above analysis it is clear that we could equivalently define the semi-norm as

$$|g|_{p-var,\varpi,L;I} = \sup_{(t_i)\in\mathcal{P}_{\varpi,L}(I)} \left(\sum_i |g_{t_it_{i+1}}|^p\right)^{\frac{1}{p}},$$

where $\mathcal{P}_{\varpi,L}(I)$ denotes the family of all partitions of an interval I such that $\varpi(t_i,t_{i+1}) \leq L$ for all neighboring partition points t_i and t_{i+1}. Clearly we have

$$C^{p-\mathrm{var}}_{2,\varpi_1,L_1}(I;E) \subset C^{p-\mathrm{var}}_{2,\varpi_2,L_2}(I;E) \tag{2.9}$$

for $\varpi_1 \leq \varpi_2$ and $L_2 \leq L_1$.

Remark 2.2. We could define the corresponding local p-variation space for 1-index maps $C^{p-\mathrm{var}}_{\varpi,L}(I;E)$ as above, but in this case there is no difference between the local and global spaces, i.e.

$$C^{p-\mathrm{var}}_{\varpi,L}(I;E) \subset C^{p-\mathrm{var}}(I;E) \subset C^{p-\mathrm{var}}_{\varpi,L}(I;E)$$

under our assumptions on ϖ. In fact, if ϖ is such that we can choose a partition $(s_j)_{j=1}^J$ of I such that $\varpi(s_j,s_{j+1}) \leq L$, then for any partition (t_i) of I we may choose a refinement $(\tilde{t}_k)$ of (t_i) containing (s_j). We then have for any $g \in C^{p-\mathrm{var}}(I;E)$,

$$\sum_{(t_i)} |\delta g_{t_it_{i+1}}|^p \leq J^{p-1} \sum_{(\tilde{t}_i)} |\delta g_{\tilde{t}_i\tilde{t}_{i+1}}|^p$$

where we have used

$$|\delta g_{st}|^p \leq J^{p-1} \sum_{1\leq j\leq J: s_j\in[s,t]} |\delta g_{s_js_{j+1}}|^p$$

for all $(s,t) \in \Delta_I$.

We now introduce the notion of a rough path. For a thorough introduction to the theory of rough paths we refer the reader to the monographs [LCL07, FV10, FH14].

Definition 2.3. Let $K \in \mathbf{N}$, $p \in [2,3)$. A continuous p-rough path is a pair

$$\mathbf{Z} = (Z,\mathbb{Z}) \in C_2^{p-\mathrm{var}}([0,T];\mathbf{R}^K) \times C_2^{\frac{p}{2}-\mathrm{var}}([0,T];\mathbf{R}^{K\times K}) \tag{2.10}$$

that satisfies the Chen's relation

$$\delta\mathbb{Z}_{s\theta t} = Z_{s\theta} \otimes Z_{\theta t}\,, \qquad (s,\theta,t) \in \Delta^{(2)}_{[0,T]}.$$

A rough path $(Z,\mathbb{Z})$ is said to be geometric if it can be obtained as the limit in the p-variation topology given in (2.10) of a sequence of rough paths $(Z^\varepsilon,\mathbb{Z}^\varepsilon)$ explicitly defined as

$$Z^\varepsilon_{st} := \delta z^\varepsilon_{st}, \qquad \mathbb{Z}^\varepsilon_{st} := \int_s^t \delta z^\varepsilon_{s\theta} \otimes \mathrm{d}z^\varepsilon_\theta,$$

for some smooth path $z^\varepsilon : [0,T] \to \mathbf{R}^K$. Denote by $C_g^{p-\mathrm{var}}([0,T];\mathbf{R}^K)$ the set of geometric p-rough paths.

Throughout this paper, we are only concerned with geometric rough paths. An advantage of working with geometric rough paths is a first order chain rule formula similar to the one known for smooth paths. Recall that this is not true within the Itô stochastic integration theory, where only a (second order) Itô formula is available. However, for the Stratonovich stochastic integral, the first order chain rule holds true. Thus in case of a Brownian motion we employ Stratonovich integration for the construction of the iterated integrals of a geometric rough path, whereas the Itô integral would lead to a non-geometric setting.

2.4 Unbounded rough drivers

Since the rough perturbation considered in (1.1) is actually (unbounded) operator valued, it is necessary to generalize the notion of a rough path accordingly. To this end, we define unbounded rough drivers, which can be regarded as operator valued rough paths with values in a suitable space of unbounded operators. In what follows, we call a scale any sequence $(E^n, |\cdot|_n)_{n\in\mathbf{N}_0}$ of Banach spaces such that E^{n+1} is continuously embedded into E^n. Besides, for $n \in \mathbf{N}_0$ we denote by E^{-n} the topological dual of E^n and note that in general $E^{-0} \neq E^0$.

Definition 2.4. Let $p \in [2,3)$ be given. A continuous unbounded p-rough driver with respect to the scale $(E^n, |\cdot|_n)_{n\in\mathbf{N}_0}$, is a pair $\mathbf{A} = (A^1, A^2)$ of 2-index maps such that

$$A^1_{st} \in \mathcal{L}(E^{-n}, E^{-(n+1)}) \ \text{ for } \ n \in \{0,2\}, \qquad A^2_{st} \in \mathcal{L}(E^{-n}, E^{-(n+2)}) \ \text{ for } \ n \in \{0,1\},$$

and there exists a continuous control ω_A on $[0,T]$ such that for every $s,t \in [0,T]$,

$$\begin{aligned} |A^1_{st}|^p_{\mathcal{L}(E^{-n},E^{-(n+1)})} &\le \omega_A(s,t) \qquad \text{for } \ n \in \{0,2\}, \\ |A^2_{st}|^{\frac{p}{2}}_{\mathcal{L}(E^{-n},E^{-(n+2)})} &\le \omega_A(s,t) \qquad \text{for } \ n \in \{0,1\}, \end{aligned}$$

and, in addition, the Chen's relation holds true, that is,

$$\delta A^1_{s\theta t} = 0, \qquad \delta A^2_{s\theta t} = A^1_{\theta t} A^1_{s\theta}, \qquad \text{for all} \quad (s,\theta,t) \in \Delta^{(2)}_T. \tag{2.11}$$

Similarly to the introduction, it will become immediately clear from the sequel, that Definition 2.4 allows a formulation of (1.1), (1.6) which is well-suited for our rough path analysis.

2.5 Formulation of the equation

In this section we derive a rough path formulation of (1.1), (1.6), which will be satisfied by solutions constructed by our main result below, Theorem 2.10. The main ideas of this step were already discussed in Section 1 in the simpler setting of the transport equation (1.3).

We study the system of Navier-Stokes equations

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u + \nabla p &= \nu\Delta u + (\sigma_k\cdot\nabla)u\,\dot z^k_t, \\ \nabla\cdot u &= 0, \\ u(0) &= u_0, \end{aligned} \tag{2.12}$$

where $(t,x) \in [0,T] \times \mathbf{T}^d$ and the velocity field $u : [0,T] \times \mathbf{T}^d \to \mathbf{R}^d$ together with the pressure $p : [0,T] \times \mathbf{T}^d \to \mathbf{R}$ are the unknown functions, and $z \in C^{p-\mathrm{var}}([0,T]; \mathbf{R}^K)$ can be lifted to a continuous p-rough path $\mathbf{Z} = (Z, \mathbb{Z})$. For simplicity we will let $\nu = 1$ for the remainder of the paper. We read the term

$$(u \cdot \nabla)u = \sum_{j=1}^{d} u^j \frac{\partial u}{\partial x_j}$$

and similarly

$$(\sigma_k \cdot \nabla)u\, \dot{z}_t^k = \sum_{k=1}^{K} (\sigma_k \cdot \nabla)u\, \dot{z}_t^k = \sum_{k=1}^{K} \sum_{j=1}^{d} \sigma_k^j \frac{\partial u}{\partial x_j}.$$

The Laplacian Δ is taken component-wise. The initial condition u_0 is assumed to be in $\mathbf{H}^0$ and we assume that the vector fields $\sigma_k : \mathbf{T}^d \to \mathbf{R}^d$ are bounded, twice continuously differentiable with bounded derivatives and divergence-free, i.e. $\nabla \cdot \sigma_k = 0$ for all $k = 1, \ldots, K$.

The classical way of studying the Navier-Stokes equation in a variational framework is to decouple the velocity field and the pressure into two equations using the Leray projection P defined in Section 2.1. Starting from (2.12) this leads to

$$\begin{aligned} \partial_t u + P[(u \cdot \nabla)u] &= \Delta u + P[(\sigma_k \cdot \nabla)u]\dot{z}_t^k, \\ \nabla p + Q[(u \cdot \nabla)u] &= Q[(\sigma_k \cdot \nabla)u]\dot{z}_t^k, \end{aligned} \tag{2.13}$$

where we recall that $P : \mathbf{W}^{m,2} \to \mathbf{H}^m$ and $Q : \mathbf{W}^{m,2} \to \mathbf{H}^m_\perp$ are solenoidal and gradient projection, respectively. As in the introduction, we integrate the equations from $[s,t]$ and iterate them into themselves and arrive at

$$\begin{aligned} \delta u_{st} + \int_s^t P[(u_r \cdot \nabla)u_r]dr &= \int_s^t \Delta u_r dr + [A_{st}^{P,1} + A_{st}^{P,2}]u_s + u_{st}^{P,\natural}, \\ \delta \pi_{st} + \int_s^t Q[(u_r \cdot \nabla)u_r)]dr &= [A_{st}^{Q,1} + A_{st}^{Q,2}]u_s + u_{st}^{Q,\natural}, \end{aligned} \tag{2.14}$$

where we denoted $\pi = \int_0^\cdot \nabla p_r\, dr$ with $\pi_0 = 0$ and

$$A_{st}^{P,1}\varphi := P[(\sigma_k \cdot \nabla)\varphi]\, Z_{st}^k, \qquad A_{st}^{P,2}\varphi := P[(\sigma_k \cdot \nabla)P[(\sigma_i \cdot \nabla)\varphi]]\mathbb{Z}_{st}^{i,k},$$

$$A_{st}^{Q,1}\varphi := Q[(\sigma_k \cdot \nabla)\varphi]\, Z_{st}^k, \qquad A_{st}^{Q,2}\varphi := Q[(\sigma_k \cdot \nabla)P[(\sigma_i \cdot \nabla)\varphi]]\, \mathbb{Z}_{st}^{i,k}.$$

To do this derivation, let us assume that we have a solution u only with energy bounds, i.e. $u \in L_T^2\mathbf{H}^1 \cap L_T^\infty \mathbf{H}^0$. If we denote by μ the drift term $\int_0^\cdot \Delta u_r - u_r \cdot \nabla u_r dr$, then from the energy bound $\mu \in C^{1-\mathrm{var}}([0,T]; \mathbf{W}^{-1,2})$.

Iterating the first equation iterated into itself gives

$$\begin{aligned}
\delta u_{st} &= P\delta\mu_{st} + \int_s^t P(\sigma_k \cdot \nabla)(u_s + \int_s^r P(\sigma_i \cdot \nabla) u_{r_1} dz^i_{r_1}) dz^k_r \\
&= P\delta\mu_{st} + P(\sigma_k \cdot \nabla) u_s Z^k_{st} + \int_s^t P(\sigma_k \cdot \nabla)\delta\mu_{sr} dz^k_r + \int_s^t P(\sigma_k \cdot \nabla) \int_s^r P(\sigma_i \cdot \nabla) u_{r_1} dz^i_{r_1} dz^k_r \\
&= P\delta\mu_{st} + P(\sigma \cdot \nabla_k) u_s Z^k_{st} + \int_s^t P(\sigma_k \cdot \nabla)\delta\mu_{sr} dz^k_r + \\
&+ \int_s^t P(\sigma_k \cdot \nabla) \int_s^r P(\sigma_i \cdot \nabla) \left(u_s + P\delta\mu_{sr_1} + P \int_s^{r_1} (\sigma_j \cdot \nabla) u_{r_2} dz^j_{r_2} dz^i_{r_1} \right) dz^k_r \\
&= P\delta\mu_{st} + P(\sigma_k \cdot \nabla) u_s Z^k_{st} + P(\sigma_k \cdot \nabla) P(\sigma_i \cdot \nabla) u_s \mathbb{Z}^{i,k}_{st} + \int_s^t P(\sigma_k \cdot \nabla)\delta\mu_{sr} dz^k_r + \\
&+ \int_s^t P(\sigma_k \cdot \nabla) \int_s^r P(\sigma_i \cdot \nabla) \left(P\delta\mu_{sr_1} + P \int_s^{r_1} (\sigma_j \cdot \nabla) u_{r_2} dz^j_{r_2} dz^i_{r_1} \right) dz^k_r \\
&= P\delta\mu_{st} + P(\sigma_k \cdot \nabla) u_s Z^k_{st} + P(\sigma_k \cdot \nabla) P(\sigma_i \cdot \nabla) u_s \mathbb{Z}^{i,k}_{st} + u^{P,\natural}_{st}
\end{aligned}$$

where

$$u^{P,\natural}_{st} := \int_s^t P(\sigma_k \cdot \nabla)\delta\mu_{sr} dz^k_r + \int_s^t P(\sigma_k \cdot \nabla) \int_s^r P(\sigma_i \cdot \nabla) \left(P\delta\mu_{sr_1} + P \int_s^{r_1} (\sigma_j \cdot \nabla) u_{r_2} dz^j_{r_2} dz^i_{r_1} \right) dz^k_r$$

which is expected to be in $C_2^{\frac{p}{3}-var}([0,T];\mathbf{H}^{-3})$ using $\mu \in C^{1-\text{var}}([0,T];\mathbf{W}^{-1,2})$ and $u \in L^\infty_T \mathbf{H}^0$.

For the second equation we put in the first equation

$$\begin{aligned}
\delta\pi_{st} &= -Q\delta\mu_{st} - Q \int_s^t (\sigma_k \cdot \nabla) u_r dz^k_r \\
&= -Q\delta\mu_{st} - Q \int_s^t (\sigma_k \cdot \nabla) \left(u_s + P\delta\mu_{sr} + P \int_s^r (\sigma_i \cdot \nabla) u_{r_1} dz^i_{r_1} \right) dz^k_r \\
&= -Q\delta\mu_{st} - Q(\sigma_k \cdot \nabla) u_s Z^k_{st} - Q \int_s^t (\sigma_k \cdot \nabla) P\delta\mu_{sr} dz^k_r \\
&- Q \int_s^t (\sigma_k \cdot \nabla) P \int_s^r (\sigma_i \cdot \nabla) \left(u_s + P\delta\mu_{sr_1} + P \int_s^{r_1} (\sigma_j \cdot \nabla) u_{r_2} dz^j_{r_2}) dz^i_{r_1} \right) dz^k_r \\
&= -Q\delta\mu_{st} - Q(\sigma_k \cdot \nabla) u_s Z^k_{st} - Q(\sigma_k \cdot \nabla) P(\sigma_i \cdot \nabla) u_s \mathbb{Z}^{i,k}_{st} - u^{Q,\natural}_{st}
\end{aligned}$$

where

$$u^{Q,\natural}_{st} = Q \int_s^t (\sigma_k \cdot \nabla) P\delta\mu_{sr} dz^k_r + Q \int_s^t (\sigma_k \cdot \nabla) P \int_s^r (\sigma_i \cdot \nabla) \left(P\delta\mu_{sr_1} + P \int_s^{r_1} (\sigma_j \cdot \nabla) u_{r_2} dz^j_{r_2}) dz^i_{r_1} \right) dz^k_r$$

which is expected to be in $C_2^{\frac{p}{3}-\text{var}}([0,T];\mathbf{H}^{-3}_\perp)$ by similar reasoning.

Recall that the equations (2.14) are to be understood in the sense that we *define* the remainder terms $u^{P,\natural}$ and $u^{Q,\natural}$ from the solution u and π and have to verify that they are indeed negligible remainders, namely, they are of order $o(|t-s|)$. This will be made precise in Definition 2.5 below.

We point out that the couple $(A^{P,1}, A^{P,2})$ is an unbounded rough driver on the scale $(\mathbf{H}^n)_n$ according to the Definition 2.4. Indeed, the analytical bounds follow from the discussion in Section 2.1 and the fact that $(Z, \mathbb{Z})$ is a continuous p-rough path according to Definition 2.3, which also implies the Chen's relation. The pair $(A^{Q,1}, A^{Q,2})$ is, somewhat surprisingly, not quite an unbounded rough driver on $(\mathbf{H}^n_\perp)_n$, because it fails to satisfy Chen's relation (2.11). Newertheless, it satisfies

$$\delta A^{Q,2}_{s\theta t} = A^{Q,1}_{\theta t} A^{P,1}_{s\theta}, \qquad \text{for all} \quad (s,\theta,t) \in \Delta^{(2)}_T, \tag{2.15}$$

which is the correct Chen's relation for the system of equations (2.13) needed for establishing the necessary time regularity of the remainders $u^{P,\natural}, u^{Q,\natural}$. This in turn justifies the choice of the second order component $A^{Q,2}$.

Our first notion of solution to (2.12) reads as follows.

Definition 2.5. A pair of weakly continuous functions $(u, \pi) : [0,T] \to \mathbf{H}^0 \times \mathbf{H}^{-3}_\perp$ is called a solution of (2.12) if

$$\sup_{t\in[0,T]} |u_t|_0^2 + \int_0^T |\nabla u_r|_0^2 dr \lesssim |u_0|_0^2$$

and if $u^{P,\natural} : \Delta_T \to \mathbf{H}^{-3}$ and $u^{Q,\natural} : \Delta_T \to \mathbf{H}^{-3}_\perp$ defined by

$$u^{P,\natural}_{st}(\phi) := \delta u_{st}(\phi) + \int_s^t [(\nabla u_r, \nabla\phi) + (u_r\cdot\nabla u_r, \phi)]\, dr - u_s([A^{P,1,*}_{st} + A^{P,2,*}_{st}]\phi), \tag{2.16}$$

$$u^{Q,\natural}_{st}(\psi) := \delta \pi_{st}(\psi) + \int_s^t (u_r\cdot\nabla u_r, \psi) dr - u_s([A^{Q,1,*}_{st} + A^{Q,2,*}_{st}]\psi), \tag{2.17}$$

for all $\phi \in \mathbf{H}^3$, $\psi \in \mathbf{H}^3_\perp$ and $(s,t) \in \Delta_T$, satisfy

$$u^{P,\natural} \in C^{\frac{p}{3}-\mathrm{var}}_{2,\varpi,L}([0,T]; \mathbf{H}^{-3}), \qquad u^{Q,\natural} \in C^{\frac{p}{3}-\mathrm{var}}_{2,\varpi,L}([0,T]; \mathbf{H}^{-3}_\perp), \tag{2.18}$$

for some control ϖ and $L > 0$.

Remark 2.6. From (2.9) there is no restriction in taking the same ϖ and $L > 0$ for both local variation spaces in (2.18).

Another possible way of formulating the equation is by performing the iteration directly on the equation, i.e. we start from the original equation (2.12) and iterate it into itself. Again, we only want to assume $u \in L^2_T\mathbf{H}^1 \cap L^\infty_T\mathbf{H}^0$. We obtain

$$\begin{aligned}
\delta u_{st} &= \delta\mu_{st} - \delta\pi_{st} + \int_s^t (\sigma_k\cdot\nabla)u_r dz^k_r \\
&= \delta\mu_{st} - \delta\pi_{st} + \int_s^t (\sigma_k\cdot\nabla)\left(u_s + \delta\mu_{sr} - \delta\pi_{sr} + \int_s^r (\sigma_i\cdot\nabla)u_{r_1} dz^i_{r_1}\right) dz^k_r \\
&= \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)u_s Z^k_{st} + \int_s^t (\sigma_k\cdot\nabla)\delta\mu_{sr} dz^k_r - \int_s^t (\sigma_k\cdot\nabla)\delta\pi_{sr} dz^k_r \\
&\quad + \int_s^t \int_s^r (\sigma_k\cdot\nabla)(\sigma_i\cdot\nabla)u_{r_1} dz^i_{r_1} dz^k_r \\
&= \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)u_s Z^k_{st} + \int_s^t (\sigma_k\cdot\nabla)\delta\mu_{sr} dz^k_r - \int_s^t (\sigma_k\cdot\nabla)\delta\pi_{sr} dz^k_r \\
&\quad + \int_s^t (\sigma_k\cdot\nabla) \int_s^r (\sigma_i\cdot\nabla)\left(u_s + \delta\mu_{sr_2} - \delta\pi_{sr_2} + \int_s^{r_1} (\sigma_j\cdot\nabla)u_{r_2} dz^j_{r_2}\right) dz^i_{r_1} dz^k_r \\
&= \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)u_s Z^k_{st} + (\sigma_k\cdot\nabla)((\sigma_i\cdot\nabla)u_s)\mathbb{Z}^{i,k}_{st} + \int_s^t (\sigma_k\cdot\nabla)\delta\mu_{sr} dz^k_r - \int_s^t (\sigma_k\cdot\nabla)\delta\pi_{sr} dz^k_r \\
&\quad + \int_s^t (\sigma_k\cdot\nabla) \int_s^r (\sigma_i\cdot\nabla)\left(\delta\mu_{sr_1} - \delta\pi_{sr_1} + \int_s^{r_1} (\sigma_j\cdot\nabla)u_{r_2} dz^j_{r_2}\right) dz^i_{r_1} dz^k_r
\end{aligned}$$

Notice that $\int_s^t (\sigma_k\cdot\nabla)\delta\pi_{sr} dz^k_r$ is not regular enough in time to assume it is a negligible remainder. Indeed, we expect π to have finite p-variation, so $\int_0^\cdot (\sigma_k\cdot\nabla)\delta\pi_{sr} dz^k_r$ should have finite $\frac{p}{2}$-variation only. If we define

$$\bar{u}^\natural_{st} = \int_s^t (\sigma_k\cdot\nabla)\delta\mu_{sr} dz^k_r + \int_s^t (\sigma_k\cdot\nabla) \int_s^r (\sigma_i\cdot\nabla)\left(\delta\mu_{sr_2} - \delta\pi_{sr_2} + \int_s^{r_1} (\sigma_j\cdot\nabla)u_{r_2} dz^j_{r_2}\right) dz^i_{r_1} dz^k_r$$

then we expect $\bar{u}^\natural$ to be in $C_2^{\frac{p}{3}-var}([0,T];\mathbf{W}^{-3,2})$, so it holds

$$\delta u_{st} = \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)u_s Z^k_{st} + (\sigma_k\cdot\nabla)(\sigma_i\cdot\nabla)u_s \mathbb{Z}^{i,k}_{st} + \bar{u}^\natural_{st} - \int_s^t (\sigma_k\cdot\nabla)\delta\pi_{sr} dz^k_r.$$

In order to close the argument, we use the equation (2.13) for π to deduce (note that that $Q\mu_t = -\int_0^t Q[u_r\cdot\nabla)u_r]dr$)

$$\begin{aligned}
-\int_s^t (\sigma_k\cdot\nabla)\delta\pi_{sr} dz^k_r &= \int_s^t (\sigma_k\cdot\nabla)Q\delta\mu_{sr} dz^k_r - \int_s^t (\sigma_k\cdot\nabla) \int_s^r Q(\sigma_i\cdot\nabla)u_{r_1} dz^i_{r_1} dz^k_r \\
&= \int_s^t (\sigma_k\cdot\nabla)Q\delta\mu_{sr} dz^k_r \\
&\quad - \int_s^t (\sigma_k\cdot\nabla) \int_s^r Q(\sigma_i\cdot\nabla)\left(u_s + \delta\mu_{sr_1} - \delta\pi_{sr_1} + \int_s^{r_1} (\sigma_j\nabla)u_{r_2} dz^j_{r_2}\right) dz^i_{r_1} dz^k_r
\end{aligned}$$

$$
\begin{aligned}
&= -(\sigma_k\cdot\nabla)Q((\sigma_i\cdot\nabla)u_s)\mathbb{Z}^{i,k}_{st} + \int_s^t (\sigma_k\cdot\nabla)Q\delta\mu_{sr}dz^k_r \\
&\quad - \int_s^t (\sigma_k\cdot\nabla)\int_s^r Q(\sigma_i\cdot\nabla)\left(\delta\mu_{sr_1} - \delta\pi_{sr_1} + \int_s^{r_1}(\sigma_j\cdot\nabla)u_{r_1}dz^j_{r_1}\right)dz^i_{r_1}dz^k_r.
\end{aligned}
$$

All the terms above except for $(\sigma_k\cdot\nabla)Q((\sigma_i\cdot\nabla)u_s)\mathbb{Z}^{i,k}_{st}$ belong to $C_2^{\frac{p}{3}-var}([0,T];\mathbf{W}^{-3,2})$ and hence we may include them in the new remainder

$$
u^\natural_{st} := \bar{u}^\natural_{st} - \int_s^t (\sigma_k\cdot\nabla)Q\delta\mu_{sr}dz^k_r - \int_s^t(\sigma_k\cdot\nabla)\int_s^r Q(\sigma_i\cdot\nabla)\left(\delta\mu_{sr_1} + \delta\pi_{sr_1} + \int_s^{r_1}(\sigma_j\cdot\nabla)u_{r_1}dz^j_{r_2}\right)dz^i_{r_1}dz^k_r.
$$

Finally, we obtain

$$
\begin{aligned}
\delta u_{st} &= \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)u_s Z^k_{st} + (\sigma_k\cdot\nabla)((\sigma_i\cdot\nabla)u_s)\mathbb{Z}^{i,k}_{st} - (\sigma_k\cdot\nabla)(Q(\sigma_i\cdot\nabla)u_s)\mathbb{Z}^{i,k}_{st} + u^\natural_{st} \\
&= \delta\mu_{st} - \delta\pi_{st} + (\sigma_k\cdot\nabla)u_s Z^k_{st} + (\sigma_k\cdot\nabla)(P(\sigma_i\cdot\nabla)u_s)\mathbb{Z}^{i,k}_{st} + u^\natural_{st},
\end{aligned}
$$

thus the corresponding unbounded rough driver should be defined as $A^1_{st}\varphi = (\sigma_k\cdot\nabla)\varphi Z^k_{st}$ and $A^2_{st}\varphi = (\sigma_k\cdot\nabla)(P(\sigma_k\cdot\nabla)\varphi)\mathbb{Z}^{i,k}_{st}$. This pair no longer satisfies the Chen's relation (2.11), but

$$
\delta A^2_{s\theta t} = A^1_{\theta t}PA^1_{s\theta}, \qquad \text{for all} \quad (s,\theta,t)\in\Delta^{(2)}_T. \tag{2.19}
$$

holds true.

Alternatively, we may therefore formulate a solution to (2.12) as follows.

Definition 2.7. A pair of weakly continuous functions $(u,\pi) : [0,T] \to \mathbf{H}^0\times\mathbf{H}^{-3}_\perp$ is called a solution of (2.12) if

$$
\sup_{t\in[0,T]}|u_t|_0^2 + \int_0^T |\nabla u_r|_0^2 dr \lesssim |u_0|_0^2
$$

and if $u^\natural : \Delta_T \to \mathbf{W}^{-3,2}$ defined by

$$
u^\natural_{st}(\phi) = \delta u_{st}(\phi) + \int_s^t [(\nabla u_r,\nabla\phi) + (u_r\cdot\nabla u_r,\phi)]\,dr - u_s([A^{1,*}_{st} + A^{2,*}_{st}]\phi) + \delta\pi_{st}(\phi), \tag{2.20}
$$

for all $\phi\in\mathbf{W}^{3,2}$ and $(s,t)\in\Delta_T$, satisfies $u^\natural \in C^{\frac{p}{3}-\text{var}}_{2,\varpi,L}([0,T];\mathbf{W}^{-3,2})$ for some control ϖ and $L>0$.

Remark 2.8. Notice that since we require $\phi\in\mathbf{W}^{3,2}$ and $u\in L^2_T\mathbf{H}^1$ all the terms above are well-defined. Indeed, let $\Gamma\subset[0,T]$ be a set of full Lebesgue measure such that $u_r\in\mathbf{H}^1$ for all $r\in\Gamma$. Then

$$
|(u_r\cdot\nabla u_r,\phi)| = |(u_r\cdot\nabla\phi,u_r)| \le |\nabla\phi|_{L^\infty}|u_r|_0^2 \lesssim |\phi|_3|u_r|_0^2
$$

where we have used that $(u_r\cdot\nabla u_r,\phi) = -(u_r\cdot\nabla\phi,u_r)$ since u is divergence free and the Sobolev embedding $\mathbf{W}^{2,2}\subset(L^\infty(\mathbf{T}^d))^d$ valid for $d=2,3$. Integrating w.r.t. r shows that the drift term is well-defined.

The following result justifies that both formulations were derived in a consistent way and are equivalent.

Lemma 2.9. *Definition 2.5 and Definition 2.7 are equivalent.*

Proof. Start by noticing that $PA^i_{st} = A^{P,i}_{st}$ and $QA^i_{st} = A^{Q,i}_{st}$ for $i = 1,2$. Moreover, the mapping

$$C^{\frac{p}{3}-var}_{2,\varpi,L}([0,T];\mathbf{W}^{-3,2}) \to C^{\frac{p}{3}-var}_{2,\varpi,L}([0,T];\mathbf{H}^{-3}) \times C^{\frac{p}{3}-var}_{2,\varpi,L}([0,T];\mathbf{H}^{-3}_{\perp})$$
$$u^{\natural} \mapsto (u^{P,\natural}, u^{Q,\natural}) := (Pu^{\natural}, Qu^{\natural})$$

is continuous and invertible with inverse

$$C^{\frac{p}{3}-var}_{2,\varpi_1,L_1}([0,T];\mathbf{H}^{-3}) \times C^{\frac{p}{3}-var}_{2,\varpi_2,L_2}([0,T];\mathbf{H}^{-3}_{\perp}) \to C^{\frac{p}{3}-var}_{2,\varpi,L}([0,T];\mathbf{W}^{-3,2})$$
$$(u^{P,\natural}, u^{Q,\natural}) \mapsto u^{P,\natural} + u^{Q,\natural}$$

where $\varpi := \varpi_1 + \varpi_2$ and $L := L_1 \wedge L_2$. The proof is straightforward. □

In the remainder of the paper we will work with Definition 2.5.

2.6 Main results

Let us now formulate our main results.

Theorem 2.10. *Let $d = 2,3$. Assume the vector fields $\sigma_k : \mathbf{T}^d \to \mathbf{R}^d$ are divergence free, twice differentiable with bounded derivatives up to order 2. Then there exists a solution of* (2.12) *in the sense of Definition 2.5.*

The proof of this result is presented in Section 4.1 as a consequence of the stronger statement in Theorem 4.1. It proceeds in two steps: First the velocity field is constructed using compactness as a limit of suitable Galerkin approximations combined with an approximation of the driving signal z by smooth paths, see Section 4.1.1. Second, the pressure is recovered in Section 4.1.2.

In two space dimensions and for constant vector fields, also uniqueness holds true.

Theorem 2.11. *Let $d = 2$ and assume $\sigma_k(x) = \sigma_k$ are constant for all $k = 1,\dots,K$. Then uniqueness holds for solutions to* (2.12) *in the sense of Definition 2.5.*

The proof of uniqueness can be found in Section 4.2 as a consequence of the stronger statement in Theorem 4.3. Considering the evolution of a difference of two arbitrary solutions, the key is a suitable tensorization argument which allows to estimate the difference of these solutions by the difference of their initial conditions.

In addition, in two space dimensions we obtain the following stability result which is proved in Section 4.3.

Corollary 2.12. *Let $d = 2$ and $p \in [2,3)$. Let $C_g^{p-\mathrm{var}}([0,T];\mathbf{R}^K)$ denote the space of continuous geometric p-rough paths according to Definition 2.3 equipped with the topology given by* (2.10)*. Let Γ denote the solution map to* (3.1) *corresponding to an initial condition u_0, a family of constant vector fields σ and a continuous geometric p-rough path $\mathbf{Z} = (Z, \mathbb{Z})$, that is, $\Gamma(u_0, \sigma, \mathbf{Z}) := (u, \pi)$ where the couple (u, π) is the unique solution constructed in Theorem 4.1 and Theorem 4.3. Then*

$$\Gamma : \mathbf{H}^0 \times \mathbf{R}^{2\times K} \times C_g^{p-\mathrm{var}}([0,T];\mathbf{R}^K) \to L_T^2\mathbf{H}^0 \cap C_T\mathbf{H}_w^0 \times C^{1-var}([0,T];\mathbf{H}_\perp^{-2})$$
$$(u_0, \sigma, \mathbf{Z}) \mapsto (u, \pi)$$

is continuous.

Remark 2.13. Following ideas in e.g. [FO14, CFO11] and [DFO14] it is tempting to try to rewrite (1.1) using a flow-transformation. More specifically, suppose that there is sufficiently regular invertible map $\varphi : [0,T] \times \mathbf{T}^d \to \mathbf{T}^d$ such that

$$\dot{\varphi}_t(x) = \dot{a}_t(\varphi_t(x)), \quad \varphi_0(x) = 0,$$

and let us define $v_t(x) := u_t(\varphi_t(x))$. Then we get

$$\begin{aligned} \partial_t v_t(x) &= \partial_t u_t(\varphi_t(x)) + \dot{a}_t(\varphi_t(x)) \cdot \nabla u_t(\varphi_t(x)) \\ &= \Delta u_t(\varphi_t(x)) - u_t(\varphi_t(x)) \cdot \nabla u_t(\varphi_t(x)) - \nabla p_t(\varphi_t(x)) \end{aligned}$$

which could be rewritten in terms of v using $\nabla v_t(x) = \nabla u_t(\varphi_t(x))\nabla\varphi_t(x)$ provided $\varphi_t(\cdot)$ is a diffeomorphism. If we assume all the driving vector fields are divergence free we have $\det(\nabla\varphi_t(x)) = 1$ so that the equation for v is a Navier-Stokes type equation including coefficients from a unimodular matrix depending on t and x. This could account for further diffulties although it seems plausible that one can solve such an equation. The added value of our construction is that it allows for an intrinsic notion of solution to (1.1) and the necessary estimates for the corresponding rough integral.

3 A priori estimates

As the next step, we derive the basic a priori estimates for the system (2.13). Let (u, π) be a solution of (2.13) in the sense of Definition 2.5. For $t \in [0,T]$, let

$$\mu_t(\phi) = -\int_0^t [(\nabla u_r, \nabla\phi) + (u_r \cdot \nabla u_r, \phi)]\, dr, \quad \phi \in \mathbf{H}^1,$$

and

$$\omega_\mu(s,t) := \int_s^t \left[|u_r|_1 + |u_r|_1^2\right] dr.$$

Since $u \in L^2_T\mathbf{H}^1$, ω_μ is a control. Using (2.6) with $m_1 = m_3 = 1$ and $m_2 = 0$ we have $|B(u_r, u_r)|_{-1} \lesssim |u_r|_1^2$ which gives

$$|\delta\mu_{st}|_{-1} \lesssim \omega_\mu(s,t).$$

Note that for all $(s,t) \in \Delta_T$,

$$\delta u_{st} = \delta\mu_{st} + A^{P,1}_{st} u_s + A^{P,2}_{st} u_s + u^{P,\natural}_{st} \tag{3.1}$$

on $(\mathbf{H}^k)_k$. Implicit in the definition of the solution of (3.1) is time-regularity of the solution as a spatial distribution. In order to establish this result, we explore the trade-off between time and space regularity, which will be an important technique in the remaining proofs of the paper.

We begin with an important lemma that enables us to estimate $u^{P,\natural}$ in terms of the given data. The following Lemma is a special case of [DGHT16b, Theorem 2.5], but we include a proof in this easier setting.

Lemma 3.1. *Assume (u,π) solves (2.14), in particular $u^{P,\natural}$ defined by (2.16) is in $C^{\frac{p}{3}-var}_{2,\varpi,L}([0,T];\mathbf{H}^{-3})$ for some control ϖ and $L > 0$. Let $\omega_{P,\natural}(s,t) := |u^{P,\natural}|^{\frac{p}{3}}_{\frac{p}{3}-var;[s,t]}$. Then provided L is small enough depending only on p and $\omega_A(s,t) \le L$ we have*

$$\omega_{P,\natural}(s,t) \lesssim |u|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_A(s,t) + \omega_\mu(s,t)^{\frac{p}{3}}(\omega_A(s,t)^{1/3} + \omega_A(s,t)^{2/3}), \tag{3.2}$$

and

$$\omega_{P,\natural}(s,t) \lesssim |u|^{\frac{p}{3}}_{L^\infty_T\mathbf{H}^0}\omega_A(s,t) + (t-s)^{\frac{p}{3}}(|u|_{L^\infty_T\mathbf{H}^0} + |u|^2_{L^\infty_T\mathbf{H}^0})^{\frac{p}{3}}(\omega_A(s,t)^{1/3} + \omega_A(s,t)^{2/3}), \tag{3.3}$$

where the proportionality constants only depends p.

Proof. We apply δ to (2.16) and recalling $\delta\delta = 0$ we get for any $\phi \in \mathbf{H}^3$ that

$$\delta u^{P,\natural}_{s\theta t}(\phi) = \delta u_{s\theta}(A^{P,2,*}_{\theta t}\phi) + (\delta u_{s\theta} - A^{P,1}_{s\theta}u_s)(A^{P,1,*}_{\theta t}\phi).$$

As in the proof of Lemma 3.2 we decompose $\delta u^{P,\natural}_{s\theta t}(\phi)$ into a smooth and non-smooth part

$$\delta u^{P,\natural}_{s\theta t}(\phi) = \delta u^{P,\natural}_{s\theta t}(J^\eta\phi) + \delta u^{P,\natural}_{s\theta t}((I - J^\eta)\phi),$$

for some $\eta \in (0,1]$ that will be specified later and then analyse term by term. For the non-smooth part, we obtain

$$\begin{aligned}\left|\delta u^{P,\natural}_{s\theta t}((I-J^\eta)\phi)\right| &\lesssim |u|_{L^\infty_T\mathbf{H}^0}\left(\left|A^{P,1*}_{\theta t}((I-J^\eta)\phi)\right|_0 + \left|A^{P,1,*}_{s\theta}A^{P,1*}_{\theta t}((I-J^\eta)\phi)\right|_0 + \left|A^{2*}_{\theta t}((I-J^\eta)\phi)\right|_0\right)\\ &\lesssim |u|_{L^\infty_T\mathbf{H}^0}\left(\omega_A(s,t)^{\frac{1}{p}}|(I-J^\eta)\phi|_1 + \omega_A(s,t)^{\frac{2}{p}}|(I-J^\eta)\phi|_2\right)\\ &\lesssim |u|_{L^\infty_T\mathbf{H}^0}\left(\omega_A(s,t)^{\frac{1}{p}}\eta^2 + \omega_A(s,t)^{\frac{2}{p}}\eta\right)|\phi|_3\end{aligned}$$

For the smooth part, we notice that we can write $\delta u_{s\theta} - A^{P,1}_{s\theta} u_s = \delta \mu_{s\theta} + A^{P,2}_{s\theta} u_s + u^{P,\natural}_{s\theta}$, we have

$$\begin{aligned}\delta u^{P,\natural}_{s\theta t}(J^\eta \phi) &= \delta\mu_{s\theta}(A^{P,1,*}_{\theta t} J^\eta\phi) + u_s(A^{P,2,*}_{s\theta} A^{P,1,*}_{\theta t} J^\eta\phi) + u^{P,\natural}_{s\theta}(A^{P,1,*}_{\theta t} J^\eta \phi)\\ &\quad + \delta\mu_{s\theta}(A^{P,2,*}_{\theta t} J^\eta\phi) + u_s(A^{P,1,*}_{s\theta} A^{P,2,*}_{\theta t} J^\eta\phi) + u_s(A^{P,2,*}_{s\theta} A^{P,2,*}_{\theta t} J^\eta\phi) + u^{P,\natural}_{s\theta}(A^{P,2,*}_{\theta t} J^\eta\phi),\end{aligned}$$

Estimating each term we obtain

$$\begin{aligned}|\delta u^{P,\natural}_{s\theta t}(J^\eta\phi)| &\le \omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}|J^\eta\phi|_2 + |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}|J^\eta\phi|_3 + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{1}{p}}|J^\eta\phi|_4\\ &\quad + \omega_\mu(s,t)\omega_A(s,t)^{\frac{2}{p}}|J^\eta\phi|_3 + |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}|J^\eta\phi|_3 + |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{4/p}|J^\eta\phi|_4\\ &\quad + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{2}{p}}|J^\eta\phi|_5\\ &\le \Big(\omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}} + |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}} + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{1}{p}}\eta^{-1}\\ &\quad + \omega_\mu(s,t)\omega_A(s,t)^{\frac{2}{p}} + |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}} + |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{4/p}\eta^{-1}\\ &\quad + \omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{2}{p}}\eta^{-2}\Big)|\phi|_3.\end{aligned}$$

Setting $\eta = \omega_A(s,t)^{\frac{1}{p}}\lambda$ for some constant $\lambda > 0$ to be determined later, we find

$$\begin{aligned}|\delta u^{P,\natural}_{s\theta t}|_{-3} &\lesssim |u|_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}(\lambda^{-1} + 1 + \lambda + \lambda^2) + \omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}\\ &\quad + \omega_\mu(s,t)\omega_A(s,t)^{\frac{2}{p}} + \omega_{P,\natural}(s,t)^{\frac{3}{p}}(\lambda^{-1} + \lambda^{-2})\\ &\lesssim \Big(|u|^{\frac{p}{3}}_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)(\lambda^{-1} + 1 + \lambda + \lambda^2)^{\frac{p}{3}} + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{1/3}\\ &\quad + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{2/3} + \omega_{P,\natural}(s,t)(\lambda^{-1} + \lambda^{-2})^{\frac{p}{3}}\Big)^{\frac{3}{p}}\end{aligned}$$

By Lemma B.1, we get

$$\begin{aligned}|u^{P,\natural}_{st}|_{-3} &\le C\Big(|u|^{\frac{p}{3}}_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)(\lambda^{-1} + 1 + \lambda + \lambda^2)^{\frac{p}{3}} + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{1/3}\\ &\quad + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{2/3} + \omega_{P,\natural}(s,t)(\lambda^{-1} + \lambda^{-2})^{\frac{p}{3}}\Big)^{\frac{3}{p}}\end{aligned}$$

for a constant C that only depends on p. Since $\omega_{P,\natural}$ was chosen as the infimum over all controls satisfying $|u^{P,\natural}_{st}|_{-3} \le \omega_{P,\natural}(s,t)^{\frac{3}{p}}$ we must have

$$\begin{aligned}\omega_{P,\natural}(s,t) &\le C^{\frac{p}{3}}\Big(|u|^{\frac{p}{3}}_{L^\infty_T \mathbf{H}^0}\omega_A(s,t)(\lambda^{-1} + 1 + \lambda + \lambda^2)^{\frac{p}{3}} + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{1/3}\\ &\quad + \omega_\mu(s,t)^{\frac{p}{3}}\omega_A(s,t)^{2/3} + \omega_{P,\natural}(s,t)(\lambda^{-1} + \lambda^{-2})^{\frac{p}{3}}\Big).\end{aligned}$$

Choosing λ such that $C^{\frac{p}{3}}(\lambda^{-1} + \lambda^{-2})^{\frac{p}{3}} \le \frac{1}{2}$, and L such that $\omega_A(s,t)^{\frac{1}{p}}\lambda \le 1$ we obtain (3.2).

The proof of (3.3) is done in a similar way replacing $\omega_\mu(s,t)$ by the estimate

$$
\begin{aligned}
|\delta\mu_{st}(\phi)| &\leq \int_s^t |(u_r, \Delta\phi)| + |B(u_r,u_r)(\phi)| dr \\
&\lesssim \int_s^t |u_r|_0|\phi|_2 + |u_r|_0^2|\phi|_3 dr \\
&\leq (t-s)(|u|_{L_T^\infty \mathbf{H}^0} + |u|^2_{L_T^\infty \mathbf{H}^0})|\phi|_3.
\end{aligned}
$$

Above we have used the antisymmetric nature of B as well as (2.5) with $m_1 = m_3 = 0$ and $m_2 = 2$. Notice that this is only possible when $d \leq 3$. The rest of the proof follows exactly the same lines substituting this bound instead of ω_μ. □

Lemma 3.2. *Assume (u,π) is a solution to* (2.14). *There exists a constant $L > 0$ such that whenever $\omega_{P,\natural}(s,t), \omega_A(s,t) \leq L$ we have*

$$
\omega_u(s,t) \lesssim (1 + |u|_{L_T^\infty \mathbf{H}^0})^p(\omega_{P,\natural}(s,t) + \omega_\mu(s,t)^p + \omega_A(s,t)),
$$

where $\omega_u(s,t) := |u|^p_{p-var;[s,t]}$.

Proof. For any $\eta \in (0,1]$ which will be specified later, we can decompose any $\phi \in \mathbf{H}^1$ into smooth and non-smooth part using J^η as follows: $\phi = J^\eta\phi + (I - J^\eta)\phi$. Thus, for all $(s,t) \in \Delta_T$ we have

$$
\delta u_{st}(\phi) = \delta u_{st}(J^\eta\phi) + \delta u_{st}((I - J^\eta)\phi).
$$

Applying (2.8) we find

$$
|\delta u_{st}((I - J^\eta)\phi)| \leq 2|u|_{L_T^\infty \mathbf{H}^0}|(I - J^\eta)\phi|_0 \lesssim \eta|u|_{L_T^\infty \mathbf{H}^0}|\phi|_1.
$$

For the smooth part we expand δu_{st} using (3.1) and then apply (2.8) to get

$$
\begin{aligned}
|\delta u_{st}(J^\eta\phi)| &\leq |u^{P,\natural}_{st}(J^\eta\phi)| + |\delta\mu_{st}(J^\eta\phi)| + |u_s(A^{P,1,*}_{st}J^\eta\phi)| + |u_s(A^{P,2,*}_{st}J^\eta\phi)| \\
&\lesssim \omega_{P,\natural}(s,t)^{\frac{3}{p}}|J^\eta\phi|_3 + \omega_\mu(s,t)|J^\eta\phi|_1 + |u|_{L_T^\infty \mathbf{H}^0}\omega_A(s,t)^{\frac{1}{p}}|J^\eta\phi|_1 + |u|_{L_T^\infty \mathbf{H}^0}\omega_A(s,t)^{\frac{2}{p}}|J^\eta\phi|_2 \\
&\leq \left(\omega_{P,\natural}(s,t)^{\frac{3}{p}}\eta^{-2} + \omega_\mu(s,t) + |u|_{L_T^\infty \mathbf{H}^0}\omega_A(s,t)^{\frac{1}{p}} + |u|_{L_T^\infty \mathbf{H}^0}\omega_A(s,t)^{\frac{2}{p}}\eta^{-1}\right)|\phi|_1.
\end{aligned}
$$

Put $\eta = \omega_{P,\natural}(s,t)^{\frac{1}{p}} + \omega_A(s,t)^{\frac{1}{p}}$ and choose L such that $\eta \in (0,1]$.Then

$$
\begin{aligned}
|\delta u_{st}|_{-1} &\lesssim (1 + |u|_{L_T^\infty \mathbf{H}^0})\omega_{P,\natural}(s,t)^{\frac{1}{p}} + \omega_\mu(s,t) + |u|_{L_T^\infty \mathbf{H}^0}\omega_A(s,t)^{\frac{1}{p}} \\
&\lesssim (1 + |u|_{L_T^\infty \mathbf{H}^0})\left(\omega_{P,\natural}(s,t) + \omega_\mu(s,t)^p + \omega_A(s,t)\right)^{\frac{1}{p}},
\end{aligned}
$$

which proves the claim. □

We now go on to prove an intermediate step, which in the language of controlled rough paths shows that the solution u is controlled by $A^{P,1}$. We define the map

$$u^{\sharp}_{st} = \delta u_{st} - A^{P,1}_{st} u_s.$$

Lemma 3.3. *Assume* (u,π) *is a solution to* (2.14). *There exists constant* $L > 0$ *such that whenever* $\omega_{P,\natural}(s,t), \omega_A(s,t) \le L$ *we have*

$$\omega_{\sharp}(s,t) \lesssim (1 + |u|_{L^\infty_T \mathbf{H}^0})^p (\omega_{P,\natural}(s,t) + \omega_\mu(s,t)^{\frac{p}{2}} + \omega_A(s,t)),$$

where $\omega_{\sharp}(s,t) := |u^{\sharp}|^{\frac{p}{2}}_{\frac{p}{2}-var;[s,t]}$.

Proof. For any $\eta \in (0,1]$ which will be specified later, we can decompose any $\phi \in \mathbf{H}^2$ into smooth and non-smooth part using J^η as follows: $\phi = J^\eta \phi + (I - J^\eta)\phi$. Thus, for all $(s,t) \in \Delta_T$ we have

$$u^{\sharp}_{st}(\phi) = u^{\sharp}_{st}(J^\eta \phi) + u^{\sharp}_{st}((I - J^\eta)\phi).$$

Since by definition we have two formulaes for $u^\sharp$, namely,

$$u^{\sharp}_{st} = \delta u_{st} - A^{P,1}_{st} u_s = \delta \mu_{st} + A^{P,2}_{st} u_s + u^{P,\natural}_{st},$$

we employ the first formulae to the non-smooth part of ϕ whereas the second one to the smooth part of ϕ. Applying (2.8) we find

$$\begin{aligned} |u^{\sharp}_{st}((I - J^\eta)\phi)| &\le |\delta u_{st}((I - J^\eta)\phi)| + |u_s(A^{P,1,*}_{st}(I - J^\eta)\phi)| \\ &\lesssim |u|_{L^\infty_T \mathbf{H}^0} |(I - J^\eta)\phi|_0 + |u|_{L^\infty_T \mathbf{H}^0} \omega_A(s,t)^{\frac{1}{p}} |(I - J^\eta)\phi|_1 \\ &\lesssim \left(\eta^2 |u|_{L^\infty_T \mathbf{H}^0} + \eta |u|_{L^\infty_T \mathbf{H}^0} \omega_A(s,t)^{\frac{1}{p}}\right) |\phi|_2. \end{aligned}$$

For the smooth part we apply (2.8) to get

$$\begin{aligned} |u^{\sharp}_{st}(J^\eta \phi)| &\le |u^{P,\natural}_{st}(J^\eta \phi)| + |\delta \mu_{st}(J^\eta \phi)| + |u_s(A^{P,2,*}_{st} J^\eta \phi)| \\ &\lesssim \omega_{P,\natural}(s,t)^{\frac{3}{p}} |J^\eta \phi|_3 + \omega_\mu(s,t) |J^\eta \phi|_1 + |u|_{L^\infty_T \mathbf{H}^0} \omega_A(s,t)^{\frac{2}{p}} |J^\eta \phi|_2 \\ &\le \left(\omega_{P,\natural}(s,t)^{\frac{3}{p}} \eta^{-1} + \omega_\mu(s,t) + |u|_{L^\infty_T \mathbf{H}^0} \omega_A(s,t)^{\frac{2}{p}}\right) |\phi|_2. \end{aligned}$$

Put $\eta = \omega_{P,\natural}(s,t)^{\frac{1}{p}} + \omega_A(s,t)^{\frac{1}{p}}$ and choose L such that $\eta \in (0,1]$.Then

$$\begin{aligned} |u^{\sharp}_{st}|_{-2} &\lesssim (1 + |u|_{L^\infty_T \mathbf{H}^0}) \omega_{P,\natural}(s,t)^{\frac{2}{p}} + \omega_\mu(s,t) + |u|_{L^\infty_T \mathbf{H}^0} \omega_A(s,t)^{\frac{2}{p}} \\ &\lesssim (1 + |u|_{L^\infty_T \mathbf{H}^0}) \left(\omega_{P,\natural}(s,t) + \omega_\mu(s,t)^{\frac{p}{2}} + \omega_A(s,t)\right)^{\frac{2}{p}}, \end{aligned}$$

which proves the claim. □

Similary we can find estimates for the rough integral of the pressure term (2.17). The computations below show why (2.15) is the correct Chen's relation for this system.

Lemma 3.4. *Suppose* (u,π) *solves* (2.14)*, in particular* $u^{Q,\natural}$ *defined by* (2.17) *is in* $C^{\frac{p}{3}-var}_{2,\varpi,L}([0,T];\mathbf{H}^{-3})$ *for some control* ϖ *and* $L>0$*. Let* $\omega_{Q,\natural}(s,t)=|u^{Q,\natural}|^{\frac{p}{3}}_{\frac{p}{3}-var;[s,t]}$*. Then there exists* $L>0$ *such that* $\omega_A(s,t)\le L$ *implies*

$$\omega_{Q,\natural}(s,t)\lesssim |u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}+\omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}+\omega_{P,\natural}(s,t)^{\frac{3}{p}}+\omega_u(s,t)^{\frac{1}{p}}\omega_A(s,t)^{\frac{2}{p}}, \qquad (3.4)$$

and

$$\omega_{Q,\natural}(s,t)\lesssim |u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{1}{p}}(\omega_A(s,t)^{\frac{2}{p}}+(t-s))+\omega_{P,\natural}(s,t)^{\frac{3}{p}}+\omega_u(s,t)^{\frac{1}{p}}\omega_A(s,t)^{\frac{2}{p}}. \qquad (3.5)$$

In particular, using Lemma 3.1 and Lemma 3.2, $u^{Q,\natural}\in C^{\frac{p}{3}-var}_{2,\varpi,L}([0,T];\mathbf{H}^{-3}_\perp)$ *where* ϖ *depends only on* $|u|_{L^\infty_T\mathbf{H}^0}$*,* ω_A *and* ω_μ*, and* L *depends only on* p*.*

Proof. We apply δ to (2.17) and recalling $\delta\delta=0$ we get for any $\psi\in\mathbf{H}^3_\perp$

$$\begin{aligned}\delta u^{Q,\natural}_{s\theta t}(\psi)&=u^{Q,\natural}_{st}(\psi)-u^{Q,\natural}_{s\theta}(\psi)-u^{Q,\natural}_{\theta t}(\psi)\\&=\delta u_{s\theta}(A^{Q,1,*}_{\theta t}\psi)+\delta u_{s\theta}(A^{Q,2,*}_{\theta t}\psi)-u_s(A^{P,1,*}_{\theta t}A^{Q,1,*}_{\theta t}\psi)\\&=(\delta u_{s\theta}-A^{P,1}_{s\theta}u_s)(A^{Q,1,*}_{\theta t}\psi)+\delta u_{s\theta}(A^{Q,2,*}_{\theta t}\psi)\end{aligned}$$

where we have used (2.15) in the second equality. Using Lemma 3.2 the last term obviously satisfies (3.4) and (3.5), so we focus on the first term.

We split up the equality into smooth and non-smooth parts $\psi=J^\eta\psi+(I-J^\eta)\psi$ for η to be determined later. For the non-smooth part we have

$$\begin{aligned}(\delta u_{s\theta}-A^{P,1}_{s\theta}u_s)((I-J^\eta)\psi)&=\delta u_{s\theta}(A^{Q,1,*}_{\theta t}(I-J^\eta)\psi)-u_s(A^{P,1,*}_{s\theta}A^{Q,1,*}_{\theta t}(I-J^\eta)\psi)\\&\le 2|u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{1}{p}}|(I-J^\eta)\psi|_1+|u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{2}{p}}|(I-J^\eta)\psi|_2\\&\lesssim |u|_{L^\infty_T\mathbf{H}^0}(\omega_A(s,t)^{\frac{1}{p}}\eta^2+\omega_A(s,t)^{\frac{2}{p}}\eta)|\psi|_3\end{aligned}$$

For the smooth part we use (2.16) to write

$$\begin{aligned}(\delta u_{s\theta}-A^{P,1}_{s\theta}u_s)(J^\eta\psi)&=\delta\mu_{s\theta}(A^{Q,1,*}_{\theta t}J^\eta\psi)+u_s(A^{P,2,*}_{s\theta}A^{Q,1,*}_{\theta t}J^\eta\psi)+u^{P,\natural}_{s\theta}(A^{Q,1,*}_{\theta t}J^\eta\psi)\\&\le\omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}|J^\eta\psi|_2+|u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}|J^\eta\psi|_3+\omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{1}{p}}|J^\eta\psi|_4\\&\le(\omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}+|u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}+\omega_{P,\natural}(s,t)^{\frac{3}{p}}\omega_A(s,t)^{\frac{1}{p}}\eta^{-1})|\psi|_3.\end{aligned}$$

Choosing $\eta=\omega_A(s,t)^{\frac{1}{p}}$ and L such that $\eta\in(0,1]$ we get

$$|\delta u^{Q,\natural}_{s\theta t}|_{-3}\lesssim|u|_{L^\infty_T\mathbf{H}^0}\omega_A(s,t)^{\frac{3}{p}}+\omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}+\omega_{P,\natural}(s,t)^{\frac{3}{p}}+\omega_u(s,t)^{\frac{1}{p}}\omega_A(s,t)^{\frac{2}{p}}.$$

Using Lemma B.1 gives the first inequality. The second one is similar. □

From Lemma 3.4 and (2.17) we see immediately that $\pi\in C^{p-var}([0,T];\mathbf{H}^{-3}_\perp)$, although we conjecture that there is much better spatial reqularity.

4 Proofs of the main results

4.1 Existence, proof of Theorem 2.10

4.1.1 Galerkin approximation

Existence of a solution is proved using a standard Galerkin approximation as follows. Let $\{h_n\}_{n\in\mathbf{N}}$ be the eigenfunctions of the Stokes operator with corresponding eigenvalues $\{\lambda_n\}_{n\in\mathbf{N}}$. As mentioned in Section 2.1, the collection $\{h_n\}_{n\in\mathbf{N}}$ is an orthonormal basis of $\mathbf{H}^0$ and an orthogonal basis of $\mathbf{H}^1$. For a given $N\in\mathbf{N}$, let $\mathbf{H}_N=\operatorname{span}(\{h_1,\dots,h_N\})$ and $P_N:\mathbf{H}^{-1}\to\mathbf{H}_N$ be defined by

$$P_N v := \sum_{n=1}^{N} v(h_n)h_n, \quad v\in\mathbf{H}_{-1}.$$

Besides, let $\{z^N\}_{N\in\mathbf{N}}$ be a sequence of smooth paths such that their canonical lifts $\mathbf{Z}^N=(Z^N,\mathbb{Z}^N)$ converge to $\mathbf{Z}$ in the rough path topology, and the uniform bounds

$$|Z^N_{st}|\le\omega_Z(s,t)^{\frac{1}{p}} \quad \text{and} \quad |\mathbb{Z}^N_{st}|\le\omega_Z(s,t)^{\frac{2}{p}} \tag{4.1}$$

are satisfied.

We consider the following N-th order Galerkin approximations

$$\partial_t u^N + P_N B(u^N,u^N) = P_N\Delta u^N + \sum_{k=1}^{K} P_N\left(P(\sigma_k\cdot\nabla u^N)\right)\dot z_t^{N,k} \tag{4.2}$$

where $u^N(0)=P_N u_0$. Assume u^N takes the form $u^N(t,x)=\sum_{i=1}^N c_i^N(t)h_i(x)$. Plugging u^N into (4.2) and then testing against h_i, we find

$$\dot c_i^N(t)+\sum_{j,l=1}^{N} c_j^N(t)c_l^N(t)P_N B(h_j,h_l)(h_i)=\lambda_i c_i^N(t)+\sum_{k=1}^{K}\sum_{j=1}^{N} c_j^N(t)(\sigma_k\cdot\nabla h_j,h_i)\dot z_t^{N,k}.$$

Set $B_{j,l,i}=P_N B(h_j,h_l)(h_i)$ and $A_{k,j,i}=(\sigma_k\cdot\nabla h_j)(h_i)$ then we have $|B_{j,l,i}|\le|h_j|_1|h_l|_1|h_i|_1$ and $|A_{k,j,i}|\le|\sigma_k|_{\mathbf{L}^\infty}|h_j|_1|h_i|_0$. The system of N-ODEs

$$\dot c_i^N(t)=\lambda_i c_i^N(t)-\sum_{j,l=1}^{N} B_{j,l,i}c_j^N(t)c_l^N(t)+\sum_{k=1}^{K}\sum_{j=1}^{N} A_{k,j,i}c_j^N(t)\dot z_t^{N,k}, \quad c_i^N(0)=(u_0,h_i), \ \ i\in\{1,\dots,N\}, \tag{4.3}$$

has locally Lipschitz coefficients, and hence there exists a unique solution $(c_i)_{i=1}^N$ of (4.3) on a time interval $[0,T_N)$, where $T_N>0$. Thus, $u^N(t,x)=\sum_{i=1}^N c_i^N(t)h_i(x)\in C^1([0,T_N);\mathbf{H}_N)$ is a solution of (4.2) on the time interval $[0,T_N)$.

Testing (4.2) against u^N and using (2.7), the divergence theorem, and that the σ_k, $k \in \{1, \ldots, K\}$, are divergence free, we get

$$
\begin{aligned}
|u_t^N|_0^2 + 2\int_0^t |\nabla u_s^N|_0^2 ds &= |P_N u_0|_0^2 - 2\int_0^t P_N B(u_s^N, u_s^N)(u_s^N) ds \\
&\quad + \sum_{k=1}^K \int_0^t (\sigma_k \cdot \nabla u_s^N, u_s^N) \dot{z}_s^{N,k} ds \\
&= |P_N u_0|_0^2 \leq |u_0|_0^2, \quad \forall t \in [0, T_N).
\end{aligned}
$$

It follows that the $\mathbf{L}^2$-norm of u^N is non-increasing in time, and hence that $(c_i)_{i=1}^N$ do not blow-up in finite time. Therefore, for all $T > 0$, u^N solves (4.2) and $u^N \in C_T \mathbf{H}^0 \cap L_T^2 \mathbf{H}^1$.

Define the operators

$$
\begin{aligned}
A_{st}^{N,1}\phi &= \tilde{P}_N \sigma_k \cdot \nabla \phi Z_{st}^{N,k}, \\
A_{st}^{N,2}\phi &= \tilde{P}_N(\sigma_k \cdot \nabla \tilde{P}_N[\sigma_j \cdot \nabla \phi]) \mathbb{Z}_{st}^{N,j,k},
\end{aligned}
$$

where $\tilde{P}_N := P_N P$. Notice that by (4.1) it follows that $(A^{N,1}, A^{N,2})$ is uniformly bounded in N as a family of unbounded rough drivers on the scale $(\mathbf{H}^n)_n$. In addition, we have for $\phi \in \mathbf{H}^3$

$$
\begin{aligned}
|(A_{st}^{N,1} - A_{st}^{P,1})\phi|_0 &\leq |P_N P(\sigma_k \cdot \nabla \phi) Z_{st}^{N,k} - P(\sigma_k \cdot \nabla \phi) Z_{st}^k|_0 \\
&\leq |P_N P(\sigma_k \cdot \nabla \phi)(Z_{st}^{N,k} - Z_{st}^k)|_0 + |(I - P_N) P(\sigma_k \cdot \nabla \phi) Z_{st}^k|_0
\end{aligned}
$$

which converges to 0 as $N \to \infty$. Indeed, from (2.3) the first term above is bounded by

$$
|P\sigma_k \cdot \nabla \phi|_0 |Z_{st}^{N,k} - Z_{st}^k| \lesssim |\phi|_1 |Z_{st}^{N,k} - Z_{st}^k|.
$$

Similary for the second order term,

$$
\begin{aligned}
|(A_{st}^{N,2} - A_{st}^{P,2})\phi|_0 &\leq |\tilde{P}_N(\sigma_k \cdot \nabla \tilde{P}_N[\sigma_j \cdot \nabla \phi])(\mathbb{Z}_{st}^{N,j,k} - \mathbb{Z}_{st}^{j,k})|_0 + |(I - P_N) P(\sigma_k \cdot \nabla P[\sigma_j \cdot \nabla \phi]) \mathbb{Z}_{st}^{j,k}|_0 \\
&\quad + |\tilde{P}_N(\sigma_k \cdot \nabla (I - P_N) P[\sigma_j \cdot \nabla \phi]) \mathbb{Z}_{st}^{j,k}|_0.
\end{aligned}
$$

Now, by (2.4) for fixed $\phi \in \mathbf{H}^3$, $\left(P(\sigma_k \cdot \nabla \tilde{P}_N[\sigma_j \cdot \nabla \phi])\right)_N$ is a bounded set in $\mathbf{H}^0$, and the first term above is bounded by

$$
|P(\sigma_k \cdot \nabla \tilde{P}_N[\sigma_j \cdot \nabla \phi])|_0 |\mathbb{Z}_{st}^{N,j,k} - \mathbb{Z}_{st}^{j,k}|
$$

which converges to 0 by assumption on $\mathbb{Z}^N$. The second term also converges to 0 since $P_N \psi \to \psi$ uniformly in ψ from a bounded set in $\mathbf{H}^0$. The last term is bounded by

$$
|\sigma_k \cdot \nabla (I - P_N) P[\sigma_j \cdot \nabla \phi]) \mathbb{Z}_{st}^{j,k}|_0 \lesssim |(I - P_N) P[\sigma_j \cdot \nabla \phi])|_1 |\mathbb{Z}_{st}^{j,k}|
$$

which converges to 0 since $P[\sigma_j \cdot \nabla \phi] \in \mathbf{H}^1$.

Moreover, we may equivalently rewrite (4.2) as

$$\delta u^N_{st} = \int_s^t \Big(P_N \Delta u^N_r - P_N B(u^N_r, u^N_r) \Big) dr + A^{N,1}_{st} u^N_s + A^{N,2}_{st} u^N_s + u^{N,\natural}_{st}, \tag{4.4}$$

where

$$\begin{aligned} u^{N,\natural}_{st} = & \int_s^t \tilde{P}_N(\sigma_k \cdot \nabla \delta \mu^N_{sr}) \dot{z}^{N,k}_r dr + \tilde{P}_N \int_s^t \int_s^r (\sigma_k \cdot \nabla \tilde{P}_N(\sigma_i \cdot \nabla \delta \mu^N_{sr_1})) \dot{z}^{N,i}_{r_1} \dot{z}^{N,k}_r dr_1 dr \\ & + \int_s^t \int_s^r \int_s^{r_1} \tilde{P}_N(\sigma_i \cdot \nabla \tilde{P}_N(\sigma_i \cdot \nabla \tilde{P}_N(\sigma_j \cdot \nabla u^N_{r_2}))) \dot{z}^{N,j}_{r_2} \dot{z}^{N,i}_{r_1} \dot{z}^{N,k}_r dr_2 dr_1 dr \end{aligned}$$

where we have defined $\mu^N_t := P_N \int_0^t \Big(\Delta u^N_r - B(u^N_r, u^N_r) \Big) dr$. Clearly we have $u^{N,\natural} \in C_2^{\zeta-var}([0,T]; H_N)$ for any $0 < \zeta < 1$. Similary as in Lemma 3.1 we get

$$\begin{aligned} \omega_{N,\natural}(s,t) &\lesssim |u^N|_{L^\infty_T H_N} \omega_{A^N}(s,t) + (t-s)^{\frac{p}{3}} (|u^N|_{L^\infty_T H_N} + |u^N|^2_{L^\infty_T H_N})(\omega_{A^N}(s,t)^{\frac{1}{3}} + \omega_{A^N}(s,t)^{\frac{2}{3}}) \\ &\le |u_0|_0 \omega_Z(s,t) + (t-s)^{\frac{p}{3}} (|u_0|_0 + |u_0|_0^2)(\omega_Z(s,t)^{\frac{1}{3}} + \omega_Z(s,t)^{\frac{2}{3}}), \end{aligned} \tag{4.5}$$

where the proportionality constant is independent of N.

Theorem 4.1. *The sequence $\{u^N\}$ is relatively compact in $L^2_T \mathbf{H}^0 \cap C_T \mathbf{H}^{-1}$ and there exists a subsequence converging to the solution of* (2.16).

Proof. The proof is similar to the proof of Lemma 3.2, except we need a slight different bound on the drift term. This bound does in particular not give p-variation of the solution, which is why we include it here.

Let $\phi \in \mathbf{H}^1$. Decomposing into smooth and non-smooth parts we get

$$\begin{aligned} |\delta u^N_{st}(\phi)| &\le |\delta u^N_{st}(J^\eta \phi)| + |\delta u^N_{st}((I - J^\eta)\phi)| \\ &\lesssim \omega_{N,\natural}(s,t)^{\frac{3}{p}} |J^\eta \phi|_3 + (t-s)(|u^N|_{L^\infty_T H_N} + |u^N|^2_{L^\infty_T H_N}) |J^\eta \phi|_3 \\ &\quad + |u^N|_{L^\infty_T H_N} (\omega_{A^N}(s,t)^{\frac{1}{p}} |\phi|_1 + \omega_{A^N}(s,t)^{\frac{2}{p}} |J^\eta \phi|_2) + |u^N|_{L^\infty_T H_N} |(I - J^\eta)\phi|_0 \\ &\lesssim \omega_{N,\natural}(s,t)^{\frac{3}{p}} \eta^{-2} |\phi|_1 + (t-s)\eta^{-2} |\phi|_1 \\ &\quad + (\omega_{A^N}(s,t)^{\frac{1}{p}} + \omega_{A^N}(s,t)^{\frac{2}{p}} \eta^{-1}) |\phi|_1 + \eta |\phi|_1. \end{aligned}$$

Using (4.5) together with $\eta = \omega_A(s,t)^{\frac{1}{p}} + (t-s)^{\frac{1}{p}}$ and $L \ge 0$ chosen such that $\eta \in (0,1]$ we see that

$$\begin{aligned} |\delta u^N_{st}|_{-1} &\lesssim \Big(\omega_{A^N}(s,t)^{\frac{3}{p}} + (t-s)\omega_{A^N}(s,t)^{\frac{1}{p}} \Big) \eta^{-2} \\ &\quad + (t-s)\eta^{-2} + (\omega_{A^N}(s,t)^{\frac{1}{p}} + \omega_{A^N}(s,t)^{\frac{2}{p}} \eta^{-1}) + \eta \\ &\lesssim \omega_A(s,t)^{\frac{1}{p}} + (t-s)^{1-\frac{2}{p}}. \end{aligned} \tag{4.6}$$

Using Lemma A.2 there is a subsequence u^N converging to an element u in $C_T\mathbf{H}^{-1} \cap L^2_T\mathbf{H}^0$. From Lemma A.3 it is also clear that u is continuous with values in $\mathbf{H}^0$ equipped with the weak topology. Fix $\phi \in \mathbf{H}^3$, then

$$A^{N,i,*}_{st}\phi \to A^{P,i,*}_{st}\phi$$

in $\mathbf{H}^0$ for $i = 1, 2$ as $N \to \infty$. Then

$$\begin{aligned}|(u^N_s, A^{N,i,*}_{st}\phi) - (u_s, A^{P,i,*}_{st}\phi)| &\leq (u^N_s - u_s, A^{N,i,*}_{st}\phi) - (u_s, (A^{P,i,*}_{st} - A^{N,i,*}_{st})\phi)| \\ &\leq |u^N_s - u_s|_{-1}|A^{N,i,*}_{st}\phi|_1 + |u_s|_0|(A^{P,i,*}_{st} - PA^{N,i,*}_{st})\phi|_0 \to 0\end{aligned}$$

as $N \to \infty$. Finally, using strong convergence in $L^2_T\mathbf{H}^0$ we have

$$\begin{aligned}\left|\int_s^t B(u_r, u_r)(\phi) - B(u^N_r, u^N_r)(\phi)dr\right| &\leq \left|\int_s^t B(u_r - u^N_r, u_r)(\phi)dr\right| \quad + \left|\int_s^t B(u^N_r, u_r - u^N_r)(\phi)dr\right| \\ &\lesssim \int_s^t |u_r - u^N_r|_0|u_r|_0 dr|\phi|_3 \quad + \int_s^t |u_r - u^N_r|_0|u^N_r|_0 dr|\phi|_3 \to 0\end{aligned}$$

as $N \to \infty$.

Since all the terms in equation (4.4) converge when applied to ϕ, also the remainder $u^{N,\natural}_{st}(\phi)$ converges to some limit $u^{P,\natural}_{st}(\phi)$. By the uniform bounds (4.5) we have $u^{P,\natural} \in C^{\frac{p}{3}-\mathrm{var}}_{2,\varpi,L}([0,T]; \mathbf{H}^{-3})$ for some control ϖ depending only on ω_Z and $L > 0$ depending only on p which proves that u is a solution of (2.16). □

4.1.2 Pressure recovery

To finalize the proof of existence we need to prove that also the pressure term π as in (2.17) exists. To this end, we first observe that according to the a priori estimates from Lemma 3.2 and Lemma 3.3, we can construct the rough integral $I_t = Q\int_0^t (\sigma_k \cdot \nabla)u_r\, dZ^k_r$, $I_0 = 0$, using the sewing lemma, Lemma B.1. Indeed, let us define

$$h_{st} = A^{Q,1}_{st}u_s + A^{Q,2}_{st}u_s.$$

We apply the δ-operator to the above and obtain

$$\begin{aligned}\delta h_{s\theta t} &= (\delta A^{Q,2}_{s\theta t})u_s - A^{Q,1}_{\theta t}\delta u_{s\theta} - A^{Q,2}_{\theta t}\delta u_{s\theta} \\ &= A^{Q,1}_{\theta t}A^{P,1}_{s\theta}u_s - A^{Q,1}_{\theta t}\delta u_{s\theta} - A^{Q,2}_{\theta t}\delta u_{s\theta} \\ &= -A^{Q,1}_{\theta t}u^{\sharp}_{s\theta} - A^{Q,2}_{\theta t}\delta u_{s\theta}\end{aligned}$$

where we have used (2.15). In view of the regularity of δu and $u^{\sharp}$ and the bounds on $(A^{Q,1}, A^{Q,2})$ there exists $(I, I^{\natural})$ such that $\|I^{\natural}_{s\theta t}\|_{-3} \lesssim \omega(s,t)^{\frac{3}{p}}$,

$$\delta I_{st} = A^{Q,1}_{st}u_s + A^{Q,2}_{st}u_s + I^{\natural}_{st}.$$

As the next step, we define

$$\pi_t := -Q \int_0^t u_r \cdot \nabla u_r dr + I_t,$$

or alternatively using the local approximation

$$\delta\pi_{st} = -Q \int_s^t u_r \cdot \nabla u_r dr + A_{st}^{Q,1} u_s + A_{st}^{Q,2} u_s + u_{st}^{Q,\natural},$$

where $u_{st}^{Q,\natural} := I_{st}^{\natural}$. As a consequence of Lemma 3.4 we therefore obtain that $\pi \in C^{p-\mathrm{var}}([0,T]; \mathbf{H}_{\perp}^{-3})$.

4.2 Uniqueness in two spatial dimensions, proof of Theorem 2.11

The objective of this section is to prove uniqueness of solutions to the Navier-Stokes equation when $d = 2$. For technical reasons we are forced to consider the case where σ_k is a constant for all $k = 1, \dots, K$. Assume for a moment that all our objects are smooth, and we have two solutions of our equation i.e.

$$\partial_t u_t^i = \Delta u_t^i - B(u_t^i) + P\sigma_k \cdot \nabla u_t^i \dot{z}_t^k.$$

Then $v := u^1 - u^2$ satisfies

$$\partial_t v_t = \Delta v_t - \left(B(u_t^1) - B(u_t^2)\right) + P\sigma_k \cdot \nabla v_t \dot{z}_t^k,$$

and the usual calculus rules give

$$\frac{1}{2}\partial_t |v_t(x)|^2 = v_t(x)^T \Delta v_t(x) - v_t(x)^T \left(B(u_t^1(x)) - B(u_t^2(x))\right) + v_t(x)^T \sigma_k \cdot \nabla v_t(x) \dot{z}_t^k.$$

Then one could proceed by integration w.r.t. x to obtain uniqueness and energy estimates.

However, in the rough case many of our objects are distributions, and so the action of integrating w.r.t. x is actually applying a distribution to a test function. Unfortunately, we do not expect our solution to be regular enough to perform this operation but instead we shall employ a usual doubling of the variables trick, i.e. we consider $t \mapsto v_t^{\otimes 2}(x,y) := v_t(x) v_t(y)^T$ where T denotes the transpose. This is a well defined operation for any distribution and we get the formula for the square by testing this distribution against an approximation of the Dirac-delta in $x = y$. We remark that one cannot use directly the techniques from [DGHT16b], since this way of approximating the Dirac-delta violates the divergence-free condition.

Let u^1 and u^2 be solutions of (3.1). We have that for all $\phi \in \mathbf{H}^3$ and $(s,t) \in \Delta_T$,

$$\delta u_{st}^i(\phi) = \delta \mu_{st}^i(\phi) + u_s^i([A_{st}^{P,1,*} + A_{st}^{P,2,*}]\phi) + u_{st}^{i;P,\natural}(\phi),$$

where

$$\mu_t^i(\phi) = -\int_0^t \left[(\nabla u_r^i, \nabla \phi) + B(u_r^i)(\phi) \right] dr.$$

Setting $v = u^1 - u^2$ and $v^\natural = u^{1;P,\natural} - u^{2;P,\natural}$ and $\mu_t(\phi) = -\int_0^t [(\nabla v_r, \nabla\phi) + (B(u_r^1) - B(u_r^2))(\phi)]dr$, we have

$$\delta v_{st}(\phi) = \delta\mu_{st}(\phi) + v_s([A_{st}^{P,1,*} + A_{st}^{P,2,*}]\phi) + v_{st}^\natural.$$

Define

$$\omega_\mu(s,t) = \omega_{\mu^1}(s,t) + \omega_{\mu^2}(s,t),$$

and notice that

$$\left|\delta\mu_{st}(\phi)\right|_{-1} \le \omega_\mu(s,t).$$

We denote by $a \hat{\otimes} b$ the symmetrisation of the tensor product of two functions $a, b : \mathbf{T}^2 \to \mathbf{R}^2$, i.e.

$$a \hat{\otimes} b(x,y) := \frac{1}{2}(a\otimes b + b\otimes a)(x,y) = \frac{1}{2}\Big(a(x)b(y)^T + b(x)a(y)^T\Big).$$

Lemma 4.2. *The tensorised mapping $t \mapsto v_t^{\otimes 2}$ satisfies the equation*

$$\delta v_{st}^{\otimes 2} - 2\int_s^t \Big(v_r \hat{\otimes} \Delta v_r - v_r \hat{\otimes} (B(u_r^1) - B(u_r^2))\Big)dr = \Big(\Gamma_{st}^1 + \Gamma_{st}^2\Big)v_s^{\otimes 2} + v_{st}^{\otimes 2,\natural} \tag{4.7}$$

where

$$\Gamma^1 = A^{P,1}\otimes I + I \otimes A^{P,1}, \quad \Gamma^2 = A^{P,2}\otimes I + I\otimes A^{P,2} + A^{P,1}\otimes A^{P,1}$$

where $v^{\otimes 2,\natural} \in C_{2,\varpi,L}^{\gamma-\mathrm{var}}([0,T]; \mathbf{H}_x^{-3}\otimes\mathbf{H}_y^{-3})$ for some $\gamma < 1$, a control ϖ and $L > 0$.

Proof. Elementary algebraic manipulations give

$$\begin{aligned}\delta v_{st}^{\otimes 2} &= 2v_s \hat{\otimes} \delta v_{st} + \delta v_{st}\otimes\delta v_{st} = 2v_s\hat{\otimes} v_{st}^\natural + 2v_s \hat{\otimes}\delta\mu_{st} + 2v_s\hat{\otimes} A_{st}^1 v_s + 2v_s\hat{\otimes} A_{st}^{P,2}v_s \\ &\quad + (v_{st}^\natural + \delta\mu_{st} + A_{st}^{P,2}v_s)^{\otimes 2} + 2(v_{st}^\natural + \delta\mu_{st} + A_{st}^{P,2}v_s)\hat{\otimes} A_{st}^{P,1}v_s + A_{st}^{P,1}v_s \otimes A_{st}^{P,1}v_s.\end{aligned}$$

Thus,

$$\delta v_{st}^{\otimes 2} - 2\int_s^t \Big[v_r\hat{\otimes}\Delta v_r - v_r\hat{\otimes}(B(u_r^1) - B(u_r^2))\Big]dr = \Big(\Gamma_{st}^1 + \Gamma_{st}^2\Big)v_s^{\otimes 2} + v_{st}^{\otimes 2,\natural}, \tag{4.8}$$

where

$$\begin{aligned} v_{st}^{\otimes 2,\natural} &:= -2\int_s^t \delta v_{sr}\hat{\otimes}[\Delta v_r + (B(u_r^1) - B(u_r^2))]dr + 2v_{st}^\natural \hat{\otimes} v_s \\ &\quad + (v_{st}^\natural + \delta\mu_{st} + A_{st}^{P,2}v_s)^{\otimes 2} + 2(v_{st}^\natural + \delta\mu_{st} + A_{st}^{P,2}v_s)\hat{\otimes} A_{st}^{P,1}v_s \\ &= -2\int_s^t \delta v_{sr}\hat{\otimes}\Delta v_r dr + 2\int_s^t \delta v_{sr}\hat{\otimes}\Big[B(u_r^1) - B(u_r^2)\Big]dr + 2v_{st}^\natural\hat{\otimes} v_s \\ &\quad + v_{st}^\natural\otimes v_{st}^\natural + v_{st}^\natural\hat{\otimes}\delta\mu_{st} + v_{st}^\natural\hat{\otimes} A_{st}^{P,2}v_s + \delta\mu_{st}\otimes\delta\mu_{st} + \delta\mu_{st}\hat{\otimes} A_{st}^{P,2}v_s + A_{st}^{P,2}v_s\otimes A_{st}^{P,2}v_s \\ &\quad + 2v_{st}^\natural\hat{\otimes} A_{st}^{P,1}v_s + \delta\mu_{st}\hat{\otimes} A_{st}^{P,1}v_s + A_{st}^{P,2}v_s\hat{\otimes} A_{st}^{P,1}v_s, \end{aligned} \tag{4.9}$$

where the second equality is simply expanding the tensor products. The result follows. □

Let $\mathbf{f}_n$ be the orthonormal basis of $\mathbf{H}^0$ as described in Section 2.1, and define $F_N(x,y) := \sum_{|n|\le N} \mathbf{f}_n(x) \otimes \overline{\mathbf{f}_n(y)}$. From (2.2) we get that $\nabla_x F_N + \nabla_y F_N = 0$. Moreover, we have for $f, g \in \mathbf{H}^0$

$$(f \otimes g, F_N) = \sum_{|n|\le N} (f, \mathbf{f}_n)\overline{(g, \mathbf{f}_n)} \to (f, g)$$

as $N \to \infty$. Motivatied by this, we would like to test our equation (4.8) for $v^{\otimes 2}$ against F_N, which converges to $|v|_0^2$ as $N \to \infty$.

Notice that

$$\Gamma_{st}^{1,*} F_N = (\sigma_k \cdot \nabla_x F_N + \sigma_k \cdot \nabla_y F_N) Z_{st}^k = 0$$

when we assume σ_k is constant. Moreover we have

$$\begin{aligned}
\Gamma_{st}^{2,*} F_N &= \sigma_k \cdot \nabla_x (\sigma_j \cdot \nabla_x F_N) \mathbb{Z}_{st}^{j,k} + \sigma_k \cdot \nabla_y (\sigma_j \cdot \nabla_y F_N) \mathbb{Z}_{st}^{j,k} + \sigma_k \cdot \nabla_x (\sigma_j \cdot \nabla_y F_N) Z_{st}^j Z_{st}^k \\
&= \sigma_k \cdot \nabla_x (\sigma_j \cdot \nabla_x F_N) \mathbb{Z}_{st}^{j,k} + \sigma_k \cdot \nabla_x (\sigma_j \cdot \nabla_x F_N) \mathbb{Z}_{st}^{k,j} - \sigma_k \cdot \nabla_x (\sigma_j \cdot \nabla_x F_N) Z_{st}^j Z_{st}^k \\
&= 0,
\end{aligned}$$

where we have used $\sigma_k \cdot \nabla(\sigma_j \cdot \nabla) = \sigma_j \cdot \nabla(\sigma_k \cdot \nabla)$ and $\mathbb{Z}_{st}^{j,k} + \mathbb{Z}_{st}^{k,j} = Z_{st}^j Z_{st}^k$.

For the drift terms it holds

$$\begin{aligned}
\int_s^t v_r \otimes \Delta v_r(F_N) dr = - \int_s^t v_r \otimes \nabla v_r (\nabla_y F_N) dr = \int_s^t v_r \otimes \nabla v_r (\nabla_x F_N) dr \\
= - \int_s^t \nabla v_r \otimes \nabla v_r (F_N) dr
\end{aligned}$$

so that

$$2 \int_s^t v_r \hat{\otimes} \Delta v_r (F_N) dr = -2 \int_s^t \nabla v_r \otimes \nabla v_r (F_N) dr.$$

Since $v \in L_T^2 \mathbf{H}^1$ we have $\nabla v_r \otimes \nabla v_r(F_N) \to |\nabla v_r|_0^2$ for almost all $r \in [s,t]$, and $|\nabla v_r \otimes \nabla v_r(F_N)| \le |\nabla v_r|_0^2$ if follows by dominated convergence that

$$\lim_{N\to\infty} 2 \int_s^t v_r \hat{\otimes} \Delta v_r(F_N) dr = -2 \int_s^t |\nabla v_r|_0^2 dr.$$

For the convective term we write

$$\int_s^t v_r \otimes u_r^i \cdot \nabla u_r^i (F_N) dr = - \int_s^t v_r \otimes (u_r^i)^T u_r^i (\nabla_y F_N) dr = - \int_s^t \nabla v_r \otimes (u_r^i)^T u_r^i (F_N) dr.$$

By Sobolev embedding we have $(u^i)^T u^i \in L_T^2 \mathbf{H}^0$. In fact, we have $|(u_r^i)^T u_r^i|_0 \lesssim |u_r^i|_0^{1/2} |\nabla u_r^i|_0^{1/2}$. This also shows we may use dominated convergence to get

$$\lim_{N\to\infty} 2 \int_s^t v_r \hat{\otimes} B(u_r^i)(F_N) dr = 2 \int_s^t B(u_r^i)(v_r) dr$$

We are now ready to finish the proof of uniqueness.

Theorem 4.3. *Suppose $d = 2$ and assume $\sigma_k(x) = \sigma_k$ are constant for all $k = 1, \dots, K$. Then $v = u^1 - u^2$ satisfies the following energy equality*

$$|v_t|_0^2 + 2\int_0^t (B(u_r^1) - B(u_r^2))(v_r)dr + 2\int_0^t |\nabla v_r|_0^2 dr = |v_0|_0^2$$

and we have

$$|v_t|_0^2 + \int_0^t |\nabla v_r|_0^2 dr \lesssim |v_0|_0^2 \exp\left\{c\int_0^t |u_r^1|_0^2 |\nabla u_r^1|_0^2 dr\right\} \tag{4.10}$$

for some universal constant c. Consequently, uniqueness holds.

Remark 4.4. The right hand side of (4.10) is finite. Indeed, we have

$$\int_0^t |u_r^1|_0^2 |\nabla u_r^1|_0^2 dr \leq \sup_{t\in[0,T]} |u_t^1|_0^2 \int_0^T |\nabla u_r^1|_0^2 dr$$

which is finite by assumption.

Proof of Theorem 4.3. We test equation (4.8) for $v^{\otimes 2}$ against F_N, and use that $\Gamma_{st}^{i,*} F_N = 0$ to see that

$$\delta v_{st}^{\otimes 2}(F_N) - 2\int_s^t v_r \hat{\otimes} \Delta v_r + v_r \hat{\otimes} (B(u_r^1) - B(u_r^2))dr(F_N) = v_{st}^{\otimes 2,\natural}(F_N).$$

Since the above left hand side is an increment of a function from s to t, so is the right hand side $(s,t) \mapsto v_{st}^{\otimes 2,\natural}(F_N)$. From Lemma 4.2 we see that $v^{\otimes 2,\natural}(F_N)$ has finite $\frac{p}{3}$-variation which is only possible if $v_{st}^{\otimes 2,\natural}(F_N) = 0$, so

$$\delta v_{st}^{\otimes 2}(F_N) - 2\int_s^t v_r \hat{\otimes} \Delta v_r - v_r \hat{\otimes} (B(u_r^1) - B(u_r^2))dr(F_N) = 0$$

for every N. Letting $N \to \infty$ we get from the above discussion

$$\delta(|v|_0^2)_{st} + 2\int_s^t (B(u_r^1) - B(u_r^2))(v_r)dr + 2\int_s^t |\nabla v_r|_0^2 dr = 0.$$

Using (2.7), (2.6) and the interpolation inequality $|\phi|_{L^4(\mathbf{T}^2)} \lesssim |\phi|_0^{1/2} |\nabla\phi|_0^{1/2}$ yields

$$\begin{aligned}
(B(u_r^1) - B(u_r^2))(v_r) &= -B(v_r, v_r)(u_r^1) \leq |v_r u_r^1|_0 |\nabla v_r|_0 \\
&\leq |v_r|_{L^4(\mathbf{T}^2)} |u_r^1|_{L^4(\mathbf{T}^2)} |\nabla v_r|_0 \lesssim |v_r|_0^{1/2} |\nabla v_r|_0^{3/2} |u_r^1|_{L^4(\mathbf{T}^2)} \\
&\leq \epsilon |\nabla v_r|_0^2 + c_\epsilon |v_r|_0^2 |u_r^1|_{L^4(\mathbf{T}^2)}^4
\end{aligned}$$

for any $\epsilon > 0$ where we have used Young's inequality, $ab \leq \epsilon a^4 + c_\epsilon b^{4/3}$ in the last step. This gives

$$|v_t|_0^2 + 2\int_0^t |\nabla v_r|_0^2 dr \lesssim |v_0|_0^2 + \epsilon\int_0^t |\nabla v_r|_0^2 dr + c_\epsilon \int_0^t |v_r|_0^2 |u_r^1|_0^2 |\nabla u_r^1|_0^2 dr$$

Choose ϵ small enough to get

$$|v_t|_0^2 + \int_0^t |\nabla v_r|_0^2 dr \lesssim |v_0|_0^2 + c_\epsilon \int_0^t |v_r|_0^2 |u_r^1|_0^2 |\nabla u_r^1|_0^2 dr$$

which gives the result by Gronwall's lemma.

Note that from the uniqueness of the velocity and the pressure recovery in Section 4.1.2 we immediately obtain the uniqueness of the associated pressure π. □

4.2.1 Energy equality and continuity

Taking $u = u^1$ and $u^2 = 0$ in (4.10) we get the following.

Corollary 4.5. *The solution to the 2-d Navier-Stokes equation satisfies the energy equality*

$$|u_t|_0^2 + 2\int_0^t |\nabla u_r|_0^2 dr = |u_0|_0^2. \tag{4.11}$$

Moreover, u is continuous as a mapping with values in $\mathbf{H}^0$.

Proof. We start by showing that u is continuous as a mapping with values in $\mathbf{H}^0$ equipped with the weak topology. It is immediate from (3.1) that $\lim_{s\to t} u_s(\phi) = u_t(\phi)$ for any $\phi \in \mathbf{H}^3$. Moreover, since $\{|u_s|_0\}_{s\in[0,T]}$ is bounded there exists a subsequence $\{u_{s_n}\}_n \subset \{u_s\}_{s\to t}$ such that $u_{s_n}(\phi)$ has a limit for all $\phi \in \mathbf{H}^3$. Since $\mathbf{H}^3$ is dense in $\mathbf{H}^0$ and weak limits are unique we see that we must have convergence $\lim_{s\to t} u_s(\phi) = u_t(\phi)$ for all $\phi \in \mathbf{H}^0$.

From the energy equality, (4.11) we see that $\lim_{s\to t} |u_s|_0 = |u_t|_0$ which implies strong convergence. □

Remark 4.6. The reader should notice that in the simpler case of σ_k being constant, the unbounded rough driver $(A^{P,1}, A^{P,2})$ leaves the decomposition $\mathbf{W}^{m,2} = \mathbf{H}^m \oplus \mathbf{H}^m_\perp$ invariant. In particular we have $A^{Q,i}_{st} u_s = 0$ for $i = 1, 2$ and so (2.17) reduces to the standard case, i.e.

$$\pi_t = -Q \int_0^t (u_r \cdot \nabla) u_r dr,$$

from which it is easy to see that π is of bounded variation with values in $\mathbf{H}^1_\perp$.

4.3 Stability in two spatial dimension, proof of Corollary 2.12

Proof of Corollary 2.12. For $n \in \mathbf{N}$ consider an initial condition $u_0^n \in \mathbf{H}^0$, a family of constant vector fields σ^n and a continuous geometric p-rough path $\mathbf{Z}^n = (Z^n, \mathbb{Z}^n)$. According to Theorem 4.1, there exists (u^n, π^n) which is a solution to (2.13) corresponding to the datum $(u_0^n, \sigma^n, \mathbf{Z}^n)$. Moreover, due to (4.11) it holds true

$$|u_t^n|_0^2 + 2\int_0^t |\nabla u_r^n|^2 dr = |u_0^n|^2$$

and in view of Lemma 3.2 and Lemma 3.1 it follows

$$|u^n|_{p-\text{var}} \leq c\Big(|u_0^n|_0, |\sigma^n|, |Z^n|_{p-\text{var};[0,\mathrm{T}]}, |\mathbb{Z}^n|_{\frac{p}{2}-\text{var};[0,\mathrm{T}]}\Big).$$

Assume now that $u_0^n \to u_0$ in $\mathbf{H}^0$, $\sigma^n \to \sigma$ in $\mathbf{R}^{2\times K}$ and $\mathbf{Z}^n \to \mathbf{Z} = (Z, \mathbb{Z})$ in the topology given by (2.10), namely, $Z^n \to Z$ in $C_2^{p-\text{var}}([0,T];\mathbf{R}^K)$ and $\mathbb{Z}^n \to \mathbb{Z}$ in $C_2^{p-\text{var}}([0,T];\mathbf{R}^{K\times K})$. Then the above estimates yield a uniform (in n) bound for the sequence $(u^n)_{n\in\mathbf{N}}$ in $L_T^\infty \mathbf{H}^0 \cap L_T^2 \mathbf{H}^1 \cap C^{p-\text{var}}([0,T];\mathbf{H}^{-1})$. Hence due to Lemma A.3 there exists $u \in L_T^\infty \mathbf{H}^0 \cap L_T^2 \mathbf{H}^1 \cap C^{p-\text{var}}([0,T];\mathbf{H}^{-1})$ such that (up to a subsequence)

$$u^n \to u \quad \text{in} \quad L_T^2 \mathbf{H}^0 \cap C_T \mathbf{H}_w^0.$$

Similarly to the proof of Theorem 4.1 we may pass to the limit in the equation and verify that u solves (2.13) with the datum $(u_0, \sigma, \mathbf{Z})$. Since uniqueness holds true for (2.13) in 2-d with constant vector fields, we deduce that the whole sequence u^n converges to u in $L_T^2 \mathbf{H}^0 \cap C_T \mathbf{H}_w^0$.

To see the convergence of π^n, we simply note that since the vector fields are constant we have $A_{st}^{Q,i} u_s = 0$ for $i = 1, 2$ and so

$$\pi_t^n = -Q \int_0^t u_r^n \cdot \nabla u_r^n dr.$$

The convergence $\pi^n \to \pi$ in $C^{1-var}([0,T];\mathbf{H}_\perp^{-2})$ follows since u^n converges to u in $L_T^2 \mathbf{H}^0$ as follows

$$\begin{aligned}\left|\int_s^t B(u_r,u_r)(\psi) - B(u_r^n,u_r^n)(\psi)dr\right| &\leq \left|\int_s^t B(u_r - u_r^n, u_r)(\psi)dr\right| + \left|\int_s^t B(u_r^n, u_r - u_r^n)(\psi)dr\right| \\ &\lesssim \int_s^t |u_r - u_r^n|_0 |u_r|_1 |\psi|_2 dr + \int_s^t |u_r^n|_1 |\psi|_2 |u_r - u_r^n|_0 dr\end{aligned}$$

for every $\psi \in \mathbf{H}_\perp^3$ where we have used (2.6) with $m_1 = m_2 = 0$ and $m_3 = 2$ as well as (2.7) and (2.6) with $m_1 = 1$, $m_2 = 1$ and $m_3 = 0$ □

4.4 Final remark

In 3 dimensions, it is known that the Stratonovich Navier-Stokes equation

$$du(t,x) + u(t,x)\cdot\nabla u(t,x)dt + \nabla p(t,x) = \Delta u(t,x)dt + \nabla u(t,x)\circ dW$$

has a probabilistically weak solution (see e.g. [BCF92, FG95, MR+05]). Nevertheless, the question of whether it is probabilistically strong is still an open problem. In other words, it is not known whether the solution to the above equation is adapted to the filtration generated by the Brownian motion W. Even though a prime example of a driving rough path in our equation is a Brownian motion with its Stratonovich lift and solving rough PDEs corresponds to a non-probabilistic (pathwise) contruction of solutions, we still can not answer this question at this point. The reader should notice that using the compactness criterion Lemma A.2 we obtain a a subsequence of the approximate solutions that a priori depends the randomness variable ω. The question whether the full sequence converges is very difficult to answer as it is intimately related to the issue of uniqueness. Indeed, if uniqueness held true, then every subsequence of $\{u^N\}_{N\geq 1}$ would converge to the same limit, hence the full sequence would converge. As a consequence, the proof of stability in Corollary 2.12 would imply that the solution (u,π) depends continuously on the given data $(u_0,\sigma,\mathbf{Z})$ and is thus adapted to the filtration generated by the Brownian motion.

A Compact embedding results

The following compact embedding result is comparable to the fractional version of the Aubin-Lions compactness result (see e.g. [FG95, Theorem 2.1]). We include a proof for the sake of being self-contained. Before we come to the embedding itself, we need to prove a simple lemma.

Lemma A.1. *If ω is a continuous control, then*

$$\lim_{a\to 0}\sup_{s\in[0,T]}\sup_{t\in[s,s+a]}\omega(s,t)=0.$$

Proof. Owing to superadditivity, for any $t\in[s,s+a]$, we have $\omega(s,t)\leq\omega(s,s+a)$, and hence the claim follows once we show that

$$\lim_{a\to 0}\sup_{s\in[0,T]}\omega(s,s+a)=0.$$

Suppose, by contradiction, there exists an $\epsilon>0$ and a sequence $\{(s_n,a_n)\}_{n\in\mathbf{N}}\subset[0,T]\times[0,1]$ such that $\lim_{n\to\infty}a_n=0$ and

$$\omega(s_n,s_n+a_n)>\epsilon,\quad\forall n\in\mathbf{N}.$$

Since $[0,T]$ is compact, there exists an $s\in[0,T]$ and a subsequence $\{(s_{n_k},a_{n_k})\}_{k\in\mathbf{N}}\subset\{(s_n,a_n)\}_{n\in\mathbf{N}}$ converging to $(s,0)$. By continuity of the control, we find

$$\epsilon\leq\lim_{k\to\infty}\omega(s_{n_k},s_{n_k}+a_{n_k})=\omega(s,s)=0,$$

which is a contradiction.

□

Lemma A.2. *Let ω be a control $L>0$, $\kappa>0$, and*

$$X=L^2_T\mathbf{H}^1\cap\left\{g\in C_T\mathbf{H}^{-1}:\omega(s,t)\leq L\text{ implies }|\delta g_{st}|_{-1}\leq\omega(s,t)^\kappa\right\}.$$

Then X is compactly embedded into $C_T\mathbf{H}^{-1}$ and $L^2_T\mathbf{H}^0$.

Proof. For each $a \in [0, L]$, define the operator J_a on $L^2_T\mathbf{H}^1$ by

$$J_a g_s = \frac{1}{a}\int_s^{s+a} g_t dt = \frac{1}{a}\int_0^a g_{s+t} dt,$$

where we extend g to $\mathbf{R}_+$ by letting $g = g_T$ outside $[0, T]$. Clearly, $s \mapsto (J_a g)(s)$ is continuous from $[0, T]$ into $\mathbf{H}^{-1}$. Moreover,

$$|J_a g_s|_{-1} \leq \frac{1}{a}\int_0^a |g_{s+t}|_{-1} dt \leq \frac{1}{\sqrt{a}}\left(\int_0^a |g_{s+t}|^2_{-1} dt\right)^{1/2},$$

which implies

$$\int_0^T |J_a g_s|^2_{-1} ds \leq \frac{1}{a}\int_0^T \int_0^a |g_{s+t}|^2_{-1} dt ds = \int_0^T |g_t|^2_{-1} dt,$$

and hence $|J_a g|_{L^2([0,T];\mathbf{H}^{-1})} \leq |g|_{L^2([0,T];\mathbf{H}^{-1})}$. Let us first show that $J_a g \to g$ in $C_T\mathbf{H}^{-1}$ as $a \to 0$. To see this consider

$$\begin{aligned} |J_a g_s - g_s|_{-1} &= \left|\frac{1}{a}\left(\int_s^{s+a} g_t dt - \int_s^{s+a} g_s dt\right)\right|_{-1} \leq \frac{1}{a}\int_s^{s+a} |g_t - g_s|_{-1}\, dt \\ &\leq \frac{1}{a}\int_s^{s+a} \omega(s,t)^\kappa dt \leq \sup_{t\in[s,s+a]} \omega(s,t)^\kappa \end{aligned}$$

which converges uniformly in s to 0 by Lemma A.1 as $a \to 0$.

On the other hand, notice that for fixed $a \in [0, L]$, the set $J_a L^2_T\mathbf{H}^1$ is relatively compact in $C_T\mathbf{H}^{-1}$. In fact, for $s, \bar{s} \in [0, T]$ we have

$$|J_a g_s - J_a g_{\bar{s}}|_1 = \frac{1}{a}\left|\int_{\bar{s}+a}^{s+a} g_t dt - \int_{\bar{s}}^{s} g_t dt\right|_1 \leq \frac{2}{a}\sqrt{|s-\bar{s}|}|g|_{L^2_T\mathbf{H}^1} \tag{A.1}$$

where we have used Hölder's inequality in the last step. Since $\mathbf{H}^1$ is compactly embedded into $\mathbf{H}^{-1}$ we can use Arzelà–Ascoli to see that $J_a L^2_T\mathbf{H}^1$ is indeed relatively compact.

To conclude the proof, assume g^n is a sequence of functions bounded in $L^2_T\mathbf{H}^1$ and $|\delta g^n_{st}|_{-1} \leq \omega(s,t)^\kappa$. In particular there exists a $g \in L^2_T\mathbf{H}^1$ such that g^n converges weak* to g (we omit the subsequence for simplicity). We may assume $g = 0$ and the result is proved if we can prove that $|g^n|_{C_T\mathbf{H}^{-1}}$ converges to 0 as $n \to \infty$. For any $a \in [0, L]$, $J_a g^n$ has a converging subsequence $\{J_a g^{n^a_k}\}_k \subset \{J_a g^n\}_n$ in $C_T\mathbf{H}^{-1}$ by the above Arzelà–Ascoli theorem. Notice that this subsequence may depend on a. Combining with weak* compactness we see that for any $f \otimes \phi \in C_T \otimes \mathbf{H}^1$ we have

$$\lim_{k\to\infty}\int_0^T J_a g_r^{n^a_k}(\phi) f_r dr = \lim_{k\to\infty}\int_0^T g_r^{n^a_k}(\phi) J^*_a f_r dr = 0,$$

so that $\lim_{k\to\infty} J_a g^{n^a_k} = 0$ in $C_T\mathbf{H}^{-1}$. Since all subsequences converges to the same limit, this means the full sequence converges.

For any $0 < a \le L$

$$|g^n|_{C_T\mathbf{H}^{-1}} \le |J_a g^n|_{C_T\mathbf{H}^{-1}} + |J_a g^n - g^n|_{C_T\mathbf{H}^{-1}} \le |J_a g^n|_{C_T\mathbf{H}^{-1}} + \sup_{s\in[0,T]} \sup_{t\in[s,s+a]} \omega(s,t)^\kappa.$$

Letting first $n \to \infty$ and then $a \to 0$ we get the result.

To show that the set is also relatively compact in $L^2_T\mathbf{H}^0$ we use Young's inequality for $h \in \mathbf{H}^1$ and any $\epsilon > 0$

$$|h|_0^2 = h(h) \le |h|_{-1}|h|_1 \le C_\epsilon |h|_{-1}^2 + \epsilon |h|_1^2$$

for some appropriate constant C_ϵ. Consequently,

$$|g^n|^2_{L^2_T\mathbf{H}^0} \le C_\epsilon |g^n|^2_{L^2_T\mathbf{H}^{-1}} + \epsilon |g^n|^2_{L^2_T\mathbf{H}^1} \le C_\epsilon |g^n|^2_{C_T\mathbf{H}^{-1}} + \epsilon \sup_k |g^k|^2_{L^2_T\mathbf{H}^1}.$$

Letting first $n \to \infty$ we have by the above result for $C_T\mathbf{H}^{-1}$ that

$$\lim_{n\to\infty} |g^n|^2_{L^2_T\mathbf{H}^0} \le \epsilon \sup_k |g^k|^2_{L^2_T\mathbf{H}^1}.$$

Let $\epsilon \to 0$ to conclude the proof. ◻

Similarly, we obtain weak continuity of trajectories.

Lemma A.3. *Let ω be a control $L > 0$, $\kappa > 0$, and*

$$Y = L^\infty_T\mathbf{H}^0 \cap \left\{ g \in C_T\mathbf{H}^{-1} : \omega(s,t) \le L \text{ implies } |\delta g_{st}|_{-1} \le \omega(s,t)^\kappa \right\}.$$

Then Y is compactly embedded into $C_T\mathbf{H}^0_w$, the space of weakly continuous functions with values in $\mathbf{H}^0$.

Proof. Let $g \in Y$. First, we will show that for all $\varphi \in \mathbf{H}^0$ the mapping

$$t \mapsto \langle g_t, \varphi\rangle \in C\big([0,T]\big). \tag{A.2}$$

To this end, we observe that since $g \in L^\infty_T\mathbf{H}^0$ it follows that there exists $R > 0$ such that $g_t \in B_R$ for all $t \in [0,T]$, where $B_R \subset \mathbf{H}^0$ is a ball of radius R. Take any family $(h_n)_{n\in\mathbf{N}}$ that belong to $\mathbf{H}^1$ and their finite linear combinations are dense in $\mathbf{H}^0$. Then

$$\begin{aligned}
&|\langle g_t, \varphi\rangle - \langle g_s, \varphi\rangle| \\
&\le \left|\left\langle g_t - g_s, \sum_{n\le M} \beta_n h_n \right\rangle\right| + \left|\left\langle g_t - g_s, \varphi - \sum_{n\le M} \beta_n h_n, \right\rangle\right| \\
&\le \left|\left\langle g_t - g_s, \sum_{n\le M} \beta_n h_n, \right\rangle\right| + R\left|\varphi - \sum_{n\le M} \beta_n h_n\right|_0 \\
&\le c(M)\omega(s,t)^\kappa + R\left|\varphi - \sum_{n\le M} \beta_n h_n\right|_0,
\end{aligned} \tag{A.3}$$

where the last term can be made small uniformly for all $s,t \in [0,T]$ by taking suitable β_m and M large enough. Hence (A.2) follows. The compactness of the embedding follows from the abstract Arzelà–Ascoli theorem. Indeed the ball B_R is relatively weakly compact, and the desired equicontinuity follows from (A.3). □

B Sewing Lemma

Lemma B.1. *Fix a subinterval I of $[0,T]$, a Banach space E and a parameter $\zeta < 1$. Consider a two-index map $h : \Delta_I \to E$ such that for all $(s,u,t) \in \Delta_I^{(2)}$ with $\varpi(s,t) \leq L$ implies*

$$|\delta h_{sut}| \leq \omega(s,t)^{\frac{1}{\zeta}},$$

for some controls ω, ϖ on I and $L > 0$. Then there exists a unique path $\mathcal{I}h : I \to E$ with $\mathcal{I}h_0 = 0$ such that $\Lambda h := h - \delta\mathcal{I}h \in C^{\zeta-var}_{2,\varpi,L}(I;E)$. If $h \in C^{p-var}_{2,\varpi,L}(I;E)$, then $\mathcal{I}h \in C^{p-var}(I;E)$. Moreover, there exists a universal constant $C_\zeta > 0$ such that for all $(s,t) \in \Delta_I$ with $\varpi(s,t) \leq L$, we have

$$|(\Lambda h)_{st}| \leq C_\zeta \omega(s,t)^{\frac{1}{\zeta}}.$$

The following corollary is immediate since $\mathcal{I}h$ is a path.

Corollary B.2. *Assume the hypothesis of Lemma B.1. If $h \in C^{p-var}_{2,\varpi,L}(I;E)$ for some $p > 1$, then for all $(s,t) \in \Delta_I$,*

$$|h_{st}| \leq C_\zeta \omega(s,t)^{\frac{1}{\zeta}}.$$

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