# THE BOUNDARY QUOTIENT FOR ALGEBRAIC DYNAMICAL SYSTEMS

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Abstract. We introduce the notion of accurate foundation sets and the accurate refinement property for right LCM semigroups. For right LCM semigroups with this property, we derive a more explicit presentation of the boundary quotient. In the context of algebraic dynamical systems, we also analyse finiteness properties of foundation sets which lead us to a very concrete presentation. Based on Starling's recent work, we provide sharp conditions on certain algebraic dynamical systems for pure infiniteness and simplicity of their boundary quotient.

## Introduction

All semigroups in this paper are assumed to be countable, discrete and left cancellative. Recall from [BRRW14] that a semigroup is *right LCM* if the intersection of two principal right ideals is either empty or another principal right ideal. Examples of right LCM semigroups come from *algebraic dynamical systems* $(G, P, \theta)$, which consist of an action $\theta$ of a right LCM semigroup $P$ with identity by injective endomorphisms of a group $G$, subject to the condition that $pP \cap qP = rP$ implies $\theta_p(G) \cap \theta_q(G) = \theta_r(G)$ for all $p, q, r \in P$, see [BLS] for details and examples. It has been observed that the $C^*$-algebra $\mathcal{A}[G, P, \theta]$ associated to $(G, P, \theta)$ in [BLS] is isomorphic to the full semigroup $C^*$-algebra of the right LCM semigroup $G \rtimes_\theta P$, see [BLS, Theorem 4.4]. It is also know to be isomorphic to a Nica-Toeplitz algebra for a product system of right-Hilbert bimodules over the right LCM semigroup $P$, see [BLS, Theorem 7.9]. These two ways of viewing $\mathcal{A}[G, P, \theta]$ both indicate that this $C^*$-algebra tends to have proper ideals. Therefore, it is natural to search for a notion of a minimal quotient that is simple under reasonable assumptions on $(G, P, \theta)$.

With regards to $C^*$-algebras of product systems of right-Hilbert bimodules, this quotient ought to be a Cuntz-Nica-Pimsner algebra. But so far only Nica covariance has been defined for product systems over right LCM semigroups, see [BLS, Definition 6.4]. Even worse, it does not seem to be clear what the general notion of Cuntz-Pimsner covariance for product systems over quasi-lattice ordered pairs should be, compare [Fow02] and [SY10]. Recently, definitions for Cuntz-Pimsner covariance for product systems over

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Ore semigroups have been proposed in [KS] and [AM] which might lead to substantial progress in this direction. However, we remark that a right LCM semigroup can be far from satisfying the Ore condition.

There has been a successful attempt to identify the analogous quotient, called the *boundary quotient*, for full semigroup $C^*$-algebras of right LCM semigroups with identity, see [BRRW14]. In fact, the authors also indicate how one could define this object for general semigroups, see [BRRW14, Remark 5.5]. Let us briefly review the idea behind this quotient, which goes back to [CL07]: Firstly, recall from [BLS15, Lemma 3.3] that the family of constructible right ideals $\mathcal{J}(S)$ for a right LCM semigroup with identity $S$ consists only of $\emptyset$ and the principal right ideals in $S$. A finite subset $F$ of $S$ is called a *foundation set* if for every $s \in S$ there is $f \in F$ such that $sS \cap fS \neq \emptyset$. The boundary quotient $\mathcal{Q}(S)$ of $C^*(S)$ is then obtained by imposing the additional relation $\prod_{s\in F}(1 - e_{sS}) = 0$ for every foundation set $F$. It was shown in [BRRW14] that $\mathcal{Q}(S)$ recovers classical objects such as $\mathcal{O}_n$, provides an appealing perspective on Toeplitz and Cuntz-Pimsner algebras associated to self-similar actions, see [BRRW14, Subsection 6.4], and may yield plenty of interesting new $C^*$-algebras related to Zappa-Szép products of monoids which had not been considered before.

As we know that $G \rtimes_\theta P$ is right LCM for each algebraic dynamical system $(G, P, \theta)$, the boundary quotient $\mathcal{Q}(G \rtimes_\theta P)$ deserves a closer examination. As it turns out, for most standard examples of such dynamics, the resulting right LCM semigroup $S = G \rtimes_\theta P$ has two additional features: There are plenty of foundation sets $F$ such that $f_1S$ and $f_2S$ are disjoint for all distinct $f_1, f_2 \in F$. Such finite subsets $F$ will be called *accurate foundation sets*. More importantly, every foundation set $F$ can be refined to an accurate foundation set $F_a$ in the sense that for every $f_a \in F_a$ there is $f \in F$ such that $f_a \in fS$. This feature will be named the *accurate refinement property*, or property (AR) for short. If a right LCM semigroup $S$ has property (AR), then the defining relation

$$\prod_{f\in F}(1 - e_{fS}) = 0 \qquad \text{for every foundation set } F$$

can be replaced by the more familiar-looking relation

$$\sum_{f\in F_a} e_{fS} = 1 \qquad \text{for every accurate foundation set } F_a,$$

see Proposition 2.4. We show that property (AR) is enjoyed by various types of known right LCM semigroups.

If we are given additional information on $S$ in the sense that $S = G \rtimes_\theta P$ for a (nontrivial) algebraic dynamical system $(G, P, \theta)$, then we can say more about the structure of (accurate) foundation sets and hence about property (AR). This is the aim of Section 3, where we present a useful sufficient criterion on $(G, P, \theta)$ for $G \rtimes_\theta P$ to have property (AR), see Proposition 3.9. As an application, we show that $G \rtimes_\theta P$ has property (AR) provided that $P$ is directed or that incomparable elements in $P$ have disjoint principal right ideals, where we use $p \geq q :\Leftrightarrow p \in qP$, see Corollary 3.11. We note that these two options include the cases where $P$ is a group, an abelian semigroup, a free semigroup, or a Zappa-Szép product $X^* \bowtie G$ for some self-similar action $(G, X)$ as in [BRRW14]. In particular, the semigroups $G \rtimes_\theta P$ arising from irreversible algebraic dynamical systems as defined in [Sta15] have property (AR). To achieve Proposition 3.9 and hence the aforementioned results, we use a celebrated lemma of B. H. Neumann from [Neu54] about finite covers of groups by cosets of subgroups to conclude that it suffices to consider (accurate) foundation sets $F$ for $G \rtimes_\theta P$ such that the index of $\theta_p(G)$ of $G$ is finite for all $(g, p) \in F$, see Proposition 3.5.

Let $(G, P, \theta)$ satisfy the assumptions of Proposition 3.9, so that $G \rtimes_\theta P$ has property (AR). If we combine the alternative presentation for $\mathcal{Q}(G \rtimes_\theta P)$ obtained in Proposition 2.4 with the dynamic description $\mathcal{A}[G, P, \theta]$ of $C^*(G \rtimes_\theta P)$, we arrive at a presentation of $\mathcal{Q}(G \rtimes_\theta P)$ which emphasises that it originates from a dynamical system, see Corollary 4.1. However, we observe that $\mathcal{Q}(G \rtimes_\theta P)$ may fail to admit a natural representation on $\ell^2(G)$: The representation exists if and only if $P$ is directed, see Proposition 4.3. This is somewhat surprising as $\ell^2(G)$ is arguably a very natural state space for a dynamical system given by injective group endomorphisms of a group $G$. Nevertheless, we immediately get that the boundary quotient $\mathcal{Q}(G \rtimes_\theta P)$ is canonically isomorphic to the $C^*$-algebra $\mathcal{O}[G, P, \theta]$ studied in [Sta15] for irreversible algebraic dynamical systems $(G, P, \theta)$, see Corollary 4.2. Thus, one can regard the present paper as a continuation, and a vast generalisation of essential parts from [Sta15], though the employed techniques are quite different.

The topic we have not addressed so far is simplicity and pure infiniteness of $\mathcal{Q}(G \rtimes_\theta P)$. In [Sta15], the author showed that $\mathcal{O}[G, P, \theta]$ is purely infinite and simple provided a certain amenability condition and $\bigcap_{p \in P} \theta_p(G) = \{1\}$ hold, see [Sta15, Theorem 3.26]. But it remained unclear whether these sufficient conditions where also necessary for irreversible algebraic dynamical systems. They were known to be sharp for the case where $G$ is abelian and $G/\theta_p(G)$ is finite for all $p \in P$ by [Sta, Corollary 5.10].

Fortunately, Starling has recently applied deep results from [EP] and [BOCFS14] precisely to boundary quotients of right LCM semigroups to obtain a characterisation of simplicity, see [Star15, Theorem 4.12]. We analyse his conditions in the context of algebraic dynamical systems in order to express them directly in terms of $(G, P, \theta)$. This leads to much more explicit conditions in important special cases, see Corollary 4.13. Mostly, we restrict our attention to the case where $P$ is right cancellative, simply because we lack examples for algebraic dynamical systems with a right LCM $P$ that is not right cancellative. Regarding simplicity of $\mathcal{O}[G, P, \theta]$ for irreversible algebraic dynamical systems, we now achieve a proper characterisation, see Corollary 4.14, and the conditions turn out to be slightly milder than in [Sta15]. Finally, we address classifiability of $\mathcal{Q}(G \rtimes_\theta P)$ in Theorem 4.17.

The paper is organised as follows: In Section 1 we recall the notions of the boundary quotient and the inverse semigroup of a right LCM semigroup as well as the key result from [Star15] concerning simplicity. Accurate foundation sets and property (AR) are introduced and studied for certain right LCM semigroups in Section 2. In Section 3 we focus on establishing property (AR) for right LCM semigroups constructed from algebraic dynamical systems and analyse finiteness properties of (accurate) foundation sets. In the final Section 4, we start off with some observations concerning basic structural properties of the boundary quotient for algebraic dynamical systems $(G,P,\theta)$, before we discuss simplicity and pure infiniteness.

## 1. Background

In this section we give the necessary background on semigroups and their $C^*$-algebras, including the full semigroup $C^*$-algebra $C^*(S)$, and its boundary quotient $\mathcal{Q}(S)$. In the second subsection we discuss Starling's results from [Star15], where he studied the boundary quotient of right LCM semigroups using an inverse semigroup (and groupoid) approach.

### 1.1. **The boundary quotient for right LCM semigroups.**

Within this section, we briefly recall the construction of $C^*(S)$ from [Li12] and the notion of the boundary quotient $\mathcal{Q}(S)$ of $C^*(S)$ for right LCM semigroups from [BRRW14, Definition 5.1].

In [Li12], the full semigroup $C^*$-algebra $C^*(S)$ of a discrete and left cancellative semigroup $S$ is defined using additional relations for projections $e_X$ arising from right ideals $X$ in $S$ that are part of the *family of constructible right ideals* $\mathcal{J}(S)$. This is the smallest family of right ideals of $S$ satisfying

(a) $S, \emptyset \in \mathcal{J}(S)$ and
(b) $X \in \mathcal{J}(S)$ and $s \in S$ implies $sX, s^{-1}X \in \mathcal{J}(S)$.

The general form of a constructible right ideal is given in [Li12, Equation (5)]. We note that $\mathcal{J}(S)$ is also closed under finite intersections, a fact that can be derived from (a) and (b) using $sS \cap tS = s(s^{-1}(tS))$.

**Definition 1.1.** Let $S$ be a discrete left cancellative semigroup. The *full semigroup $C^*$-algebra* $C^*(S)$ is the universal $C^*$-algebra generated by isometries $(v_s)_{s\in S}$ and projections $(e_X)_{X\in\mathcal{J}(S)}$ satisfying

$$\text{(L1)} \quad v_s v_t = v_{st}, \qquad\qquad \text{(L2)} \quad v_s e_X v_s^* = e_{sX},$$
$$\text{(L3)} \quad e_S = 1, e_\emptyset = 0, \quad \text{and} \quad \text{(L4)} \quad e_X e_Y = e_{X\cap Y},$$

for all $s,t \in S$, $X,Y \in \mathcal{J}(S)$.

Note that (L2) and (L3) give $v_s v_s^* = e_{sS}$ for all $p \in S$. If $S$ is a right LCM semigroup with identity, then $\mathcal{J}(S) = \{sS \mid s \in S\} \cup \{\emptyset\}$, see [BLS15, Lemma 3.3]. From now on, let $S$ be a right LCM semigroup with identity.

**Definition 1.2.** A finite subset $F \subset S$ is called a *foundation set* for $S$ if, for every $s \in S$, there exists $t \in F$ satisfying $sS \cap tS \neq \emptyset$. The collection of foundation sets for $S$ is denoted by $\mathfrak{F}(S)$.

**Remark 1.3.** We note the following simple observations:

(a) If $S$ is directed, then every finite subset of $S$ is a foundation set.
(b) $F \subset S$ is a foundation set if and only if it is finite and $sS \cap \bigcup_{t \in F} tS \neq \emptyset$ for all $s \in S$. Since $S$ is right LCM, this means that for each principal right ideal $sS$, there is $s' \in S$ such that $ss' \in \bigcup_{t \in F} tS$. So this union can be thought of as a cofinal subset of $S$ with respect to the partial order on $S$ induced by reverse inclusion of associated principal right ideals.

**Definition 1.4.** The *boundary quotient* $\mathcal{Q}(S)$ is the quotient of $C^*(S)$ by

$$\prod_{s \in F}(1 - e_{sS}) = 0 \qquad \text{for every foundation set } F. \tag{Q}$$

We shall denote the images of the isometries $v_s$ and the projections $e_{sS}$ for $s \in S$ under the quotient map by $\bar{v}_s$ and $\bar{e}_{sS}$, respectively.

We point out that (Q) has the flavour of the summation relation used for $\mathcal{O}_n, 2 \leq n < \infty$. This is the essence of Proposition 2.4.

1.2. **The inverse semigroup approach.**

In [Star15] Starling uses techniques and machinery from inverse semigroups and groupoids to study the boundary quotient $\mathcal{Q}(S)$ of a right LCM semigroup $S$. In particular, he applies the machinery from [EP] and the results of [BOCFS14]. In this section we recall the construction of an inverse semigroup $I(S)$ for a discrete, left cancellative semigroup $S$, and then some of the terminology, notation and results from [Star15].

**Definition 1.5.** For a discrete, left cancellative semigroup $S$ let $I(S)$ be the multiplicative subsemigroup of $C^*(S)$ generated by 0 and $v_s, v_s^*$ for $s \in S$. The set of idempotents in $I(S)$ is denoted by $E(S)$.

**Lemma 1.6.** *$I(S)$ is an inverse semigroup with identity and zero. $E(S)$ is given by $\{e_X \mid X \in \mathcal{J}(S)\}$, where $\mathcal{J}(S)$ denotes the family of constructible right ideals in $S$. If $S$ is right LCM, then $I(S)$ equals $\{0\} \cup \{v_s v_t^* \mid s, t \in S\}$ and $E(S) = \{0\} \cup \{e_{sS} \mid s \in S\}$.*

*Proof.* The first claim is straightforward. If we consider an arbitrary finite product $v_{s_1}^* v_{s_2} \cdots v_{s_{n-1}}^* v_{s_n}$, then its range projection

$$v_{s_1}^* v_{s_2} \cdots v_{s_{n-1}}^* v_{s_n} (v_{s_1}^* v_{s_2} \cdots v_{s_{n-1}}^* v_{s_n})^* = e_{s_1^{-1}(s_2(\dots(s_{n-1}^{-1}(s_nS)\dots)}$$

is the projection corresponding to the constructible right ideal

$$s_1^{-1}(s_2(\dots(s_{n-1}^{-1}(s_nS)\dots) \in \mathcal{J}(S).$$

Hence we get $E(S) = \{e_X \mid X \in \mathcal{J}(S)\}$. Now suppose $S$ is right LCM. Then $v_s^* v_t = v_s^* e_{sS \cap tS} v_t$ vanishes unless $sS \cap tS = rS, r = ss' = tt'$ for some $r, s', t' \in S$, in which case we get $v_s^* v_t = v_{s'} v_{t'}^*$. Finally, we know that $\mathcal{J}(S) = \{\emptyset\} \cup \{sS \mid s \in S\}$ for right LCM semigroups. $\square$

$I(S)$ can also be defined via partial bijections $\Lambda_s : S \to sS$ and their partial inverses $\Lambda_s^* : sS \to S$ since $S$ is left cancellative.

To every inverse semigroup $I$, we can associate the *tight groupoid* $\mathcal{G}_{\text{tight}}(I)$, which is a groupoid of germs for a certain action of the inverse semigroup on a particular spectrum, see [Exe08] for details. For $I(S)$ of a right LCM semigroup $S$, the boundary quotient $\mathcal{Q}(S)$ is isomorphic to $C^*_{\text{tight}}(I(S)) \cong C^*(\mathcal{G}_{\text{tight}}(I(S)))$, see [Star15, Theorem 3.7 and Subsection 4.1].

We need two more concepts before we can state Starling's main theorem on simplicity of $\mathcal{Q}(S)$ for right LCM $S$. Following the convention of [Star15], we denote

$$[s,t] := v_s v_t^*. \tag{1.1}$$

Note that $[s,t] = [s',t']$ holds if and only if we have $s' = sx$ and $t' = tx$ for some $x \in S^*$.

The following is [Star15, Definition 4.6], which is inspired by the work of Crisp and Laca, see [CL07, Definition 5.4].

**Definition 1.7.** For a right LCM semigroup $S$ with identity, the *core* of $S$ is the set $S_0 := \{s \in S \mid sS \cap tS \neq \emptyset \text{ for all } t \in S\}$.

It is immediate that the group of invertible elements $S^*$ in $S$ is a subset of the core $S_0$.

We now state Starling's [Star15, Theorem 4.12], although we have not yet written down conditions (H) and (EP); we will do so after the statement.

**Theorem 1.8.** *Let $S$ be a right LCM semigroup with identity which satisfies* (H)*. Then $\mathcal{Q}(S)$ is simple if and only if*

(1) *$\mathcal{Q}(S) \cong C_r^*(\mathcal{G}_{tight}(I(S)))$, and*
(2) *for all $s, t \in S_0$, the element $[s,t]$ satisfies* (EP)*.*

Let us explain the conditions (H) and (EP). Condition (H) characterises Hausdorffness of the tight groupoid of $I(S)$, see [Star15, Proposition 4.1]:

(H) For all $s, t \in S$ with $sS \cap tS \neq \emptyset$, there is a finite subset $F \subset S$ with $sf = tf$ for all $f \in F$ such that the following holds: If $r \in S$ satisfies $sr = tr$, then there exists $f \in F$ with $rS \cap fS \neq \emptyset$.

**Remark 1.9.** If we have $s = t$, then we can simply choose $F = \{1\}$ . Now if $S$ is right cancellative, then $sr = tr$ implies $s = t$. Hence (H) holds whenever $S$ is right cancellative.

To present condition (EP), we need to define the notion of weakly fixed idempotents in inverse semigroups, see [Star15, Definition 4.8].

**Definition 1.10.** Let $I$ be an inverse semigroup, $a \in I$ and $e \in E(I)$ such that $a^*a \geq e$. The idempotent $e$ is said to be *weakly fixed* by $a$ if $afa^*f \neq 0$ for all $f \in E(I) \setminus \{0\}$ with $f \leq e$.

Since we are interested in inverse semigroups built from right LCM semigroups with identity, let us recall the conclusion of [Star15, Lemma 4.9]:

**Lemma 1.11.** *Let $S$ be a right LCM semigroup with identity and $I(S)$ the associated inverse semigroup. $[s,t]$ fixes $[tt',tt']$ weakly if and only if $st'rS \cap tt'rS \neq \emptyset$ for all $r \in S$.*

As stated in [Star15, Lemma 4.11], condition (EP) for $[s,t] \in I(S)$ is given by:

(EP) Whenever $[s,t]$ fixes $[tt',tt']$ weakly, there is a foundation set $F$ such that $st'f = tt'f$ for all $f \in F$.

**Remark 1.12.** There are some special cases:

(a) If there is no $t' \in S$ such that $[s,t]$ fixes $[tt',tt']$ weakly, then (EP) holds for $[s,t]$.
(b) For $s = t$ the foundation set $F = \{1\}$ gives (EP) for $[s,t]$.
(c) Suppose $S$ is right cancellative and $[s,t] \in I(S)$ fixes some $[tt',tt']$ weakly. If $[s,t]$ satisfies (EP), then $s$ and $t$ have to be the same.

## 2. Foundation sets made accurate

Throughout this section let $S$ be a right LCM semigroup with identity. We will now introduce accurate foundation sets and accurate refinements of foundation sets. These lead to a clearer picture of the boundary quotient $\mathcal{Q}(S)$ provided that accurate refinements are always possible. This feature of $S$ is called the accurate refinement property, or property (AR) for short. We show that many known right LCM semigroups have property (AR). In fact, we are not aware of an example of a right LCM semigroup that does not have property (AR).

**Definition 2.1.** $F \in \mathfrak{F}(S)$ is called an *accurate foundation set* for $S$ if $sS \cap tS = \emptyset$ holds for all $s,t \in F, s \neq t$. The collection of accurate foundation sets is denoted by $\mathfrak{F}_a(S)$.

**Remark 2.2.** If $S$ is directed, then $\mathfrak{F}(S)$ consists of all finite subsets of $S$, see Remark 1.3 (a), and it is apparent that $\mathfrak{F}_a(S)$ is equal to $S$ as a set. For general right LCM $S$, accurate foundation sets consisting of a single point correspond to elements of the core $S_0$ of $S$.

**Definition 2.3.** $S$ is said to have the *accurate refinement property*, or property (AR) for short, if, for all $F \in \mathfrak{F}(S)$, there exists $F_a \in \mathfrak{F}_a(S)$ such that $F_a \subset FS$. This means that for every $f_a \in F_a$, there is a $f \in F$ with $f_a \in fS$.

**Proposition 2.4.** *$\sum_{s \in F_a} \bar{e}_{sS} = 1$ holds for all accurate foundation sets $F_a$ for $S$. If $S$ has property (AR), then $\mathcal{Q}(S)$ is the quotient of $C^*(S)$ by the relation*

$$(\mathrm{Q}_a) \qquad \sum_{s \in F} e_{sS} = 1 \qquad \text{for every accurate foundation set } F.$$

*In other words, $\mathcal{Q}(S)$ is the universal $C^*$-algebra generated by a representation of $S$ by isometries $\bar{v}_s$ and projections $(\bar{e}_X)_{X\in\mathcal{J}(S)}$ subject to the relations* (L1) – (L4)*, and* $(\mathrm{Q}_a)$*.*

*Proof.* Let $F_a$ be an accurate foundation set for $S$, that is

1) $F_a$ is a foundation set and
2) $sS \cap tS = \emptyset$ for all distinct $s,t \in F_a$.

1) implies $\prod_{s\in F_a}(1-\bar{e}_{sS}) = 0$, see Definition 1.4. Now (L4) and 2) yield $\prod_{s\in F_a}(1-\bar{e}_{sS}) = 1-\sum_{s\in F_a}\bar{e}_{sS}$, so we get $(\mathrm{Q}_a)$.

Now let $F \in \mathfrak{F}(S)$. We need to show that (Q) holds for $F$ by using the structure of $C^*(S)$ and $(\mathrm{Q}_a)$. If $S$ has property (AR), then there is $F_a \in \mathfrak{F}_a(S)$ which refines $F$, that is, for each $s \in F_a$, there exists $t \in F$ with $s \in tS$. On the level of $C^*(S)$, this implies $1-e_{tS} \leq 1-e_{sS}$. Since all these projections commute, we get

$$0 \leq \prod_{t\in F}(1-e_{tS}) \leq \prod_{s\in F_a}(1-e_{sS}) = 1-\sum_{s\in F_a} e_{sS}.$$

But the right hand side vanishes once we impose relation $(\mathrm{Q}_a)$ and hence (Q) holds for $F$. This shows that (Q) and $(\mathrm{Q}_a)$ are equivalent relations provided that $S$ has property (AR). □

Similar presentations of $\mathcal{Q}(S)$ for right LCM semigroups with property (AR) have been obtained in special cases, see for instance [LR10, Corollary 6.2] or [BRRW14, Subsection 6.4]. We will now show, that these examples and many more right LCM semigroups have property (AR).

Recall that reverse inclusion of principal right ideal defines a partial order on $S$, i.e. $s \leq t$ if $t \in sS$ for $s,t \in S$. If $s \leq t$ or $s \geq t$ holds, then $s,t \in S$ are said to be *comparable*.

**Remark 2.5.** Every left cancellative semigroup with the property that incomparable elements have disjoint principal right ideals is right LCM.

**Proposition 2.6.** *Suppose that*

(1) *$S$ is directed with respect to $\leq$, or*
(2) *incomparable elements have disjoint principal right ideals.*

*Then every $F \in \mathfrak{F}(S)$ has an accurate refinement $F_a \in \mathfrak{F}_a(S)$ satisfying $F_a \subset F$. In particular, $S$ has property* (AR)*.*

*Proof.* Suppose first that $S$ is directed and let $F \in \mathfrak{F}(S)$. Then $F \neq \emptyset$ and every $p \in F$ yields an accurate refinement $F_a := \{p\} \in \mathfrak{F}_a(S)$ for $F$. Now let $S$ satisfy (2) and $F \in \mathfrak{F}(S)$. If there are $p,q \in F$ with $p \neq q$ and $pS \cap qS \neq \emptyset$, (2) implies that $p \in qP$ or $q \in pP$. If $p \in qP$, then $F' := F \setminus \{p\} \in \mathfrak{F}(S)$, and otherwise we get $F' := F \setminus \{q\} \in \mathfrak{F}(S)$. Hence we can remove redundant elements from $F$ until there are only those left that correspond to mutually disjoint right ideals and the output is an accurate refinement of $F$. □

The class of right LCM semigroups to which Proposition 2.6 applies is large and we list a number special cases to demonstrate this.

**Corollary 2.7.** *If $S$ is*

(3) *a group,*
(4) *abelian,*
(5) *isomorphic to $\mathbb{F}_n^+$ for some $1 \leq n < \infty$, or*
(6) *given by $X^* \bowtie G$ for a self-similar action $(G, X)$,*

*then either* (1) *or* (2) *holds. In particular, $S$ has property* (AR)*.*

*Proof.* (3) and (4) both imply (1), so $S$ has property (AR) by Proposition 2.6. (5) is a special case of (6). Due to [BRRW14, Theorem 3.8], (6) forces (2) and hence Proposition 2.6 shows property (AR). $\square$

**Remark 2.8.** $S = \mathbb{F}_\infty^+$ also has property (AR), but for trivial reasons since any foundation set $F$ for $S$ has to contain the identity of $S$. For completeness, we note that $\mathfrak{F}_a(S) = \big\{\{1\}\big\}$.

**Remark 2.9.** In [Law08], Lawson considered so-called *left Rees monoids*, which are left cancellative semigroups with identity that satisfy condition (2) from Proposition 2.6 and the ascending chain condition for principal right ideals. The last condition means that every principal right ideal is properly contained in only a finite number of principal right ideals. By Proposition 2.6, all left Rees monoids have property (AR).

According to [Law08, Theorem 3.7], attributed to Perrot [Per72], left Rees monoids can be characterised as Zappa-Szép products of free monoids by groups. Moreover, new examples of left Rees monoids can be constructed out of known ones, see [Law08, Section 4] for details.

Finally, let us mention that [Law08, Examples 2.8] provides a number of interesting examples of left Rees monoids. [Law08, Examples 2.8 (iv)] might be particularly interesting because a left Rees monoid is constructed from an arbitrary left cancellative semigroup using Rhodes expansions.

From what we have gathered so far, it seems feasible to explore property (AR) for other kinds of Zappa-Szép products $U \bowtie A$. Indeed, [BRRW14, Lemma 3.3] provides a sufficient criterion for $U \bowtie A$ to be a right LCM semigroup. More importantly, [BRRW14, Remark 3.4] explains that, given the requirements of [BRRW14, Lemma 3.3], the structure of $\mathcal{J}(U \bowtie A)$ is governed by $\mathcal{J}(U)$, i.e.,

$$(u,a)U \bowtie A \cap (v,b)U \bowtie A \neq \emptyset \Longleftrightarrow uU \cap vU \neq \emptyset \tag{2.1}$$

for all $(u,a),(v,b) \in U \bowtie A$. The proof of [BRRW14, Lemma 3.3] actually shows that if $w$ is a right LCM for $u$ and $v$ in $U$, then there is $c \in A$ such that

$$(u,a)U \bowtie A \cap (v,b)U \bowtie A = (w,c)U \bowtie A. \tag{2.2}$$

**Proposition 2.10.** *Suppose $S = U \bowtie A$ is such that $U$ is right LCM, $\mathcal{J}(A)$ is totally ordered by inclusion and $U \to U, u \mapsto a \cdot u$ is bijective for every $a \in A$. Then $S$ is a right LCM semigroup with identity and $S$ has property (AR) if and only if $U$ has property (AR).*

*Proof.* $S$ is right LCM by [BRRW14, Lemma 3.3]. For each $E \subset U$ and every family $(a_u)_{u\in E}$ we let $F(E,(a_u)) := \{(u,a_u) \mid u \in E\}$. Similarly, given $F \subset S$, we set $E(F) := \{u \mid (u,a) \in F\}$. By (2.1), we have $E \in \mathfrak{F}(U)$ if and only if $F(E,(a_u)) \in \mathfrak{F}(S)$, and moreover, $E$ is accurate if and only if $F(E,(a_u))$ is accurate (for every family $(a_u)_{u\in E}$). Likewise, (2.1) implies that $F \in \mathfrak{F}(S)$ holds if and only if $E(F) \in \mathfrak{F}(U)$. In addition, accuracy of $F$ is equivalent to $E(F)$ being accurate.

Now suppose $U$ has property (AR). Starting with $F \in \mathfrak{F}(S)$, we can refine $E(F) \in \mathfrak{F}(U)$ to some accurate foundation set $E(F)_a$. Take $u \in E(F)_a$. Since $E(F)_a$ is an accurate refinement for $E(F)$, there is $(v,b) \in F$ such that $u \in vU$. By (2.2), there is $a_u \in A$ satisfying $(u,a_u) \in (v,b)S$. It follows that $F(E(F)_a,(a_u))$ is an accurate refinement of $F$ because $E(F)_a$ is accurate.

Conversely, assume that $S$ has property (AR). If $E \in \mathfrak{F}(U)$, then we know that $F(E,(a_u)) \in \mathfrak{F}(S)$ (for every family $(a_u)_{u\in E}$) and we can refine this foundation set by some $F_a \in \mathfrak{F}_a(S)$. By construction, $E(F_a)$ is an accurate refinement of $E$. $\square$

**Examples 2.11.** We have already seen that the Zappa-Szép product $X^* \bowtie G$ associated to a self-similar action $(G,X)$ has property (AR). In fact, we can use Proposition 2.10 to see that all of the examples of right LCM Zappa-Szép products in [BRRW14, Section 3] have property (AR).

(a) For $m$ and $n$ positive integers the positive cone $\mathrm{BS}(m,n)^+$ of the Baumslag-Solitar group $\mathrm{BS}(m,n) = \langle a,b | ab^m = b^n a\rangle$ is a Zappa-Szép product of the form $\mathbb{F}_n^+ \bowtie \mathbb{N}$. Since we know from Corollary 2.7 that $\mathbb{F}_n^+$ has property (AR), Proposition 2.10 says that each $\mathrm{BS}(m,n)^+$ has property (AR).
(b) The semigroups $\mathbb{N} \rtimes \mathbb{N}^\times$ and $\mathbb{Z} \rtimes \mathbb{Z}^\times$ can be described as Zappa-Szép products $U \bowtie A$ with $U = \{(r,x) : x \geq 1,\, 0 \leq r < x\}$. To see that $U$ has property (AR), suppose $F = \{(r_1,x_1),\dots,(r_n,x_n)\}$ is a foundation set. Then for $x$ the least common multiple of $x_1,\dots,x_n$ the set $F_a = \{(0,x),\dots,(x-1,x)\}$ is an accurate foundation set. Morover, because of the structure of the principal right ideals and since $F$ is a foundation set, for each $(r,x) \in F_a$ the ideal $(r,x)U$ must be contained in one of $(r_i,x_i)U$. So $F_a$ is an accurate refinement of $F$, and hence $U$ has property (AR). Proposition 2.10 now says that $\mathbb{N} \rtimes \mathbb{N}^\times$ and $\mathbb{Z} \rtimes \mathbb{Z}^\times$ both have property (AR).
(c) If $G$ acts self-similarly on two alphabets $X$ and $Y$, and there is a bijection $\theta : Y \times X \to X \times Y$ such that the conditions given in [BRRW14, Proposition 3.10] hold, then there is a natural Zappa-Szép product $\mathbb{F}_\theta^+ \bowtie G$, where the semigroup $\mathbb{F}_\theta^+$ is a 2-graph with a single vertex. In general $\mathbb{F}_\theta^+$ is not right LCM, but, for instance, it is right LCM when the sizes of $X$ and $Y$ are coprime, and $G = \mathbb{Z}$ acts as an odometer on both $X$ and $Y$. In this case, and if $X$ has size $m$ and $Y$ has size $n$, then $\mathbb{F}_\theta^+$ is isomorphic to the subsemigroup

of $U$ from (ii) generated by $(0,m),\ldots,(m-1,m),(0,n),\ldots,(n-1,n)$. The arguments above in (b) apply, and hence $\mathbb{F}_\theta^+$ has property (AR). Proposition 2.10 now says that the product of two (coprime) odometer actions $\mathbb{F}_\theta^+ \bowtie \mathbb{Z}$ has property (AR).

Example 2.11 (b) can also be viewed as an elementary example of a semigroup built from an algebraic dynamical system $(G,P,\theta)$ as $S = G \rtimes_\theta P$. The natural question whether property (AR) passes from $P$ to $S$ under suitable conditions requires some preparation and will be examined in the first part of the next section.

## 3. Foundation sets for algebraic dynamical systems

Recall from [BLS, Definition 2.1] that an *algebraic dynamical system* $(G,P,\theta)$ is an action $\theta$ of a right LCM semigroup with identity $P$ by injective endomorphisms of a group $G$, subject to the condition that $pP \cap qP = rP$ implies $\theta_p(g) \cap \theta_q(G) = \theta_r(G)$ for all $p,q,r \in P$. In this section we aim to establish property (AR) for a large class of right LCM semigroups $S$ built from algebraic dynamical systems $(G,P,\theta)$.

From now on let $(G,P,\theta)$ denote an algebraic dynamical system. In addition, let $P^{(\mathrm{fin})}$ denote the subsemigroup of $P$ consisting of those $p \in P$ for which $G/\theta_p(G)$ is finite. For convenience, we shall usually denote $G \rtimes_\theta P$ by $S$ within this section.

Let us remind ourselves of the structure of $\mathcal{J}(S)$ as described in [BLS, Proposition 4.2] since this will be essential.

**Lemma 3.1.** *For all $(g,p),(h,q) \in S$, we have*

$$(g,p)S \cap (h,q)S = \begin{cases} (g\theta_p(k),r)S & \text{if there are } r \in P \text{ and } k \in G \text{ with} \\ & pP \cap qP = rP \text{ and } g\theta_p(k) \in h\theta_q(G), \\ \emptyset & \text{otherwise.} \end{cases}$$

Recall that $p \geq q$ is the same as $p \in qP$.

**Lemma 3.2.** *Given a finite subset $F \subset S$, there exists a finite set $P_F \subset P$ with the following properties:*

(i) *Whenever $p \in P$ and $(h,q) \in F$ satisfy $pP \cap qP \neq \emptyset$, there is $q' \in P_F$ such that $pP \cap q'P \neq \emptyset$.*
(ii) *For each $q \in P_F$ there exists $p \in P$ such that $pP \cap qP \neq \emptyset$ and $pP \cap q'P = \emptyset$ for all $q' \in P_F$ with $q' \not\leq q$.*
(iii) *For each $q \in P_F$ there exists $(h',q') \in F$ such that $q \geq q'$.*

*Proof.* Let $F_1 \subset P$ be a complete set of representatives for

$$\Big\{ \bigcap_{(h,q)\in F'} qP \mid F' \subset F \Big\} \setminus \{\emptyset\} \subset \mathcal{J}(P).$$

Pick $q_1 \in F_1$ which is minimal in the sense that $q_1 \geq q$ implies $q_1P = qP$ for all $q \in F_1$. Let $F_1' := \{q \in F_1 \mid qP = q_1P\}$ and $E_0 := \emptyset$. If there is $p \in P$ such that $pP \cap q_1P \neq \emptyset$ whereas $pP \cap qP = \emptyset$ for all $q \in (F_1 \setminus F_1') \cup \{q' \in$

$E_0 \mid q' \not\leq q_1\}$, then we say that $q_1$ is indispensable and set $E_1 := E_0 \cup \{q_1\}$. If $q_1$ is dispensable, we choose $E_1 = E_0$ and note that by construction of $F_1$ we have: Whenever $q_1P \cap pP \neq \emptyset$ for some $p \in P$, there exists $q \in F_1 \setminus F_1'$ with $q \geq q_1$ such that $qP \cap pP \neq \emptyset$.

Next, we define $F_2 := F_1 \setminus F_1'$ and repeat the procedure for some $q_2 \in F_2$ which is minimal in the sense that $q_2 \geq q$ implies $q_2P = qP$ for all $q \in F_2$. Let $F_2' := \{q \in F_2 \mid qP = q_2P\}$. If there is $p \in P$ such that $pP \cap q_2P \neq \emptyset$ whereas $pP \cap qP = \emptyset$ for all $(h,q) \in (F_2 \setminus F_2') \cup \{q' \in E_1 \mid q' \not\leq q_2\}$, then we set $E_2 := E_1 \cup \{q_2\}$. Otherwise we take $E_2 := E_1$. Finally, setting $F_3 := F_2 \setminus F_2'$ allows us to iterate this procedure. After finitely many steps, we get a finite set $E_n =: P_F$ which satisfies (i)–(iii) because it is a minimal subset of indispensable elements of $F_1$. □

It is clear from the construction that $P_F$ is non-empty if and only if $F$ is. If $P$ is directed, it is easy to see that $P_F$ consists of a single element $p_F$ with $\bigcap_{(h,q)\in F} qP = p_FP$.

**Lemma 3.3.** *A finite subset $F$ of $S$ is a foundation set for $S$ if and only if there exists a foundation set $P_F$ for $P$ such that*

$$\bigcup_{\substack{(h',q')\in F: \\ q'\leq q}} h'\theta_{q'}(G) = G \text{ holds for all } q \in P_F. \tag{3.1}$$

*Proof.* Suppose $F$ is a foundation set and $P_F \subset P$ is obtained via Lemma 3.2. So for every $(g,p) \in S$, there exists $(h,q) \in F$ such that $(g,p)S \cap (h,q)S \neq \emptyset$. According to Lemma 3.1, this implies $pP \cap qP \neq \emptyset$. By condition (i) for $P_F$ from Lemma 3.2, there is $q' \in P_F$ satisfying $pP \cap q'P \neq \emptyset$, so we get $P_F \in \mathfrak{F}(P)$. Concerning (3.1), we note that it suffices to prove this for all minimal elements of $P_F$. But if $q \in P_F$ is minimal among the elements of $P_F$, then (ii) implies that there exists $p \in P$ such that $pP \cap qP \neq \emptyset$ whereas $pP \cap q'P = \emptyset$ for all $q' \in P_F, q' \neq q$. Without loss of generality, we can assume that $p$ belongs to $qP$ since we may replace it with $p' \in qP$ satisfying $pP \cap qP = p'P$. Let $g \in G$. Since $F$ is a foundation set for $S$ there is $(h',q') \in F$ such that $(g,p)S \cap (h',q')S \neq \emptyset$. In particular, we get $(g,q)S \cap (h',q')S \neq \emptyset$. We remark that $(g,p)S \cap (h'',q'')S = \emptyset$ for all $h'' \in G$ and $q'' \in P_F \setminus \{q\}$. This forces $q \in q'P$, so Lemma 3.1 implies $g \in h'\theta_{q'}(G)\theta_q(G) = h'\theta_{q'}(G)$. Since $g$ was arbitrary, we get (3.1) for every minimal $q \in P_F$ and hence for all $q \in P_F$.

The converse direction is straightforward. For each $(g,p) \in S$, there is $q \in P_F$ such that $pP \cap qP \neq \emptyset$. This means $(g,p)S \cap (g,q)S \neq \emptyset$, see Lemma 3.1. By (3.1), there exists $(h',q') \in F$ satisfying $(h',q') \leq (g,q)$. In particular, this implies $(g,p)S \cap (h',q')S \supset (g,p)S \cap (g,q)S \neq \emptyset$, so $F$ is a foundation set for $S$. □

Note that if $F \subset S$ and $P_F \subset P$ satisfy (3.1), then we have $P_F \subset \bigcup_{(h,q)\in F} qP$.

For the next step, we will need a celebrated lemma of B.H. Neumann on finiteness properties for covers of groups, see [Neu54, Lemma 4.1].

**Lemma 3.4.** *Let $G$ be a group and $G_1, \ldots, G_n$ subgroups of $G$. If there are $g_1, \ldots, g_n \in G$ such that $G = \bigcup_{1 \leq i \leq n} g_i G_i$, then there is $1 \leq i \leq n$ such that the index $[G : G_i] < \infty$ and $G = \bigcup_{\substack{1 \leq i \leq n: \\ [G:G_i] < \infty}} g_i G_i$.*

**Proposition 3.5.** *Let $F$ be a finite subset of $S$. Then $F$ is a foundation set for $S$ if and only if $F \cap G \rtimes_\theta P^{(fin)}$ is a foundation set for $S$.*

*Proof.* If $F$ is a foundation set, then Lemma 3.3 states that there exists $P_F \in \mathfrak{F}(P)$ satisfying (3.1) for $F$. Now if we let $F^{(\mathrm{fin})} := F \cap G \rtimes_\theta P^{(\mathrm{fin})}$, then Lemma 3.4 shows that $P_F$ also satisfies (3.1) for $F^{(\mathrm{fin})}$. Hence $F^{(\mathrm{fin})}$ is a foundation set for $S$ by Lemma 3.3. The reverse implication is obvious. □

**Corollary 3.6.** *If $F$ is an accurate foundation set, then $F \subset G \rtimes_\theta P^{(fin)}$.*

*Proof.* Let $F \in \mathfrak{F}_a(S)$. By Proposition 3.5 we know that $F^{(\mathrm{fin})} := F \cap G \rtimes_\theta P^{(\mathrm{fin})}$ is also a foundation set for $S$. So if there was $(g,p) \in F$ with $p \in P^{(\mathrm{inf})}$, then there would be $(h,q) \in F^{(\mathrm{fin})}$ satisfying $(g,p)S \cap (h,q)S \neq \emptyset$. But then $F$ would not be accurate and hence we conclude $F = F^{(\mathrm{fin})}$. □

**Definition 3.7.** If

$$F = \{(g_1^{(1)}, p_1), \ldots, (g_1^{(n_1)}, p_1), (g_2^{(1)}, p_2), \ldots, (g_m^{(n_m)}, p_m)\} \subset S$$

is such that

(1) $\{p_1, \ldots, p_m\}$ is contained in $P^{(\mathrm{fin})}$ and an element of $\mathfrak{F}_a(P)$, and
(2) $(g_\ell^{(k)})_{1 \leq k \leq n_\ell}$ is a transversal for $G/\theta_{p_\ell}(G)$ for each $1 \leq \ell \leq m$,

then $F$ is called an *elementary foundation set*. The collection of all elementary foundation sets is denoted by $\mathfrak{F}_e(G, P, \theta)$.

Every elementary foundation set is an accurate foundation set.

**Example 3.8.** Let us consider $\mathbb{Z} \rtimes |2\rangle \subset \mathbb{Z} \rtimes \mathbb{Z}^\times$ built from the irreversible algebraic dynamical system $(\mathbb{Z}, |2\rangle, \cdot)$. The set $\{(0,2),(1,2)\}$ forms an elementary foundation set whereas $\{(0,2),(1,4),/3,4)\}$ is an accurate foundation set, which is non-elementary.

**Proposition 3.9.** *Suppose that for every $F \in \mathfrak{F}(P)$ with $F \subset P^{(fin)}$ there exists an accurate refinement $F_a \in \mathfrak{F}_a(P)$ with $F_a \subset P^{(fin)}$. Then every foundation set for $S$ can be refined accurately by an elementary foundation set for $S$. In particular, $S$ has property* (AR).

*Proof.* Let $F' \in \mathfrak{F}(S)$. Using Proposition 3.5, we may assume $F' \subset G \rtimes_\theta P^{(\mathrm{fin})}$. In particular, $F := \{p \in P \mid (g,p) \in F \text{ for some } g \in G\} \subset P^{(\mathrm{fin})}$ forms a foundation set for $P$. By our assumption, there is $F_a \in \mathfrak{F}_a(P)$ with $F_a \subset P^{(\mathrm{fin})}$ which refines $F$. Next, pick a transversal $T_p$ for $G/\theta_p(G)$ for every $p \in F_a$. Then $F'_e := \{(g,p) \mid p \in F_a, g \in T_p\}$ yields an elementary foundation set that refines $F'$. Since elementary foundation set are accurate, $S$ has property (AR). □

**Remark 3.10.** The converse of the first statement in Proposition 3.9 might be true in some cases, but there is a subtlety we would like to point out: Suppose $S$ has property (AR) and let $F \in \mathfrak{F}(P)$ with $F \subset P^{(\text{fin})}$. Choose a transversal $T_p$ for $G/\theta_p(G)$ for every $p \in F$. As $F \subset P^{(\text{fin})}$, the set $F' := \{(g,p) \mid p \in F, g \in T_p\}$ is a foundation set for $S$. Thus there exists $F'_a \in \mathfrak{F}_a(S)$ which refines $F'$. By Proposition 3.5, we know that we can assume $F'_a \subset G \rtimes_\theta P^{(\text{fin})}$. It follows that $F_a := \{p \in P \mid (g,p) \in F'_a \text{ for some } g \in G\}$ is a foundation set for $P$. However, this need not imply that $F_a$ is accurate. In fact, this depends on the choice of a suitable $F'_a$.

We note the following consequence of Proposition 3.9:

**Corollary 3.11.** *$S$ has property* (AR) *provided that*

(1) *$P$ is directed with respect to $\leq$, or*
(2) *incomparable elements in $P$ have disjoint principal right ideals.*

*In particular, $S$ has property* (AR) *if $P$ satisfies one of the conditions* (3)–(6) *from Corollary 2.7.*

*Proof.* By Proposition 2.6, the prerequisites for Proposition 3.9 are fulfilled and hence $S$ has property (AR). Since each of the conditions (3)–(6) implies (1) or (2), the additional claim is clear, see Corollary 2.7. □

## 4. The boundary quotient $\mathcal{Q}(G \rtimes_\theta P)$

Recall from [BLS15] that the authors associated a $C^*$-algebra $\mathcal{A}[G,P,\theta]$ to every algebraic dynamical system $(G,P,\theta)$, and showed that it is canonically isomorphic to the full semigroup $C^*$-algebra $C^*(G \rtimes_\theta P)$. In this section we use the insights gained in Section 3 to give an alternative presentation of the boundary quotient $\mathcal{Q}(G \rtimes_\theta P)$ provided that $G \rtimes_\theta P$ has property (AR). For irreversible algebraic dynamical systems, we conclude that $\mathcal{Q}(G \rtimes_\theta P)$ is canonically isomorphic to the algebra $\mathcal{O}[G,P,\theta]$ from [Sta15]. We also indicate that $\mathcal{Q}(G \rtimes_\theta P)$ can be represented on $\ell^2(G)$ in the obvious way if and only if $P$ is directed, which raises the question of a natural state space for $\mathcal{Q}(G \rtimes_\theta P)$ for the case where $P$ is not directed, see Proposition 4.3 and Remark 4.4. The majority of this Section appears in Subsection 4.2, in which we address the issues of pure infiniteness and simplicity for $\mathcal{Q}(G \rtimes_\theta P)$.

We will again denote $G \rtimes_\theta P$ by $S$ within this section.

### 4.1. **Basic structure.**

In this short subsection we obtain a dynamic description of $\mathcal{Q}(G \rtimes_\theta P)$ for algebraic dynamical systems $(G,P,\theta)$ with the property that for every $F \in \mathfrak{F}(P)$ with $F \subset P^{(\text{fin})}$ there exists an accurate refinement $F_a \in \mathfrak{F}_a(P)$ with $F_a \subset P^{(\text{fin})}$. This allows us to identify $\mathcal{Q}(G \rtimes_\theta P)$ as the $C^*$-algbra $\mathcal{O}[G,P,\theta]$ from [Sta15] for irreversible algebraic dynamical systems. Moreover, we discuss representability of $\mathcal{Q}(G \rtimes_\theta P)$ on $\ell^2(G)$

**Proposition 4.1.** *If $(G,P,\theta)$ is an algebraic dynamical system such that for every $F \in \mathfrak{F}(P)$ with $F \subset P^{(fin)}$ there exists an accurate refinement $F_a \in$*

*$\mathfrak{F}_a(P)$ with $F_a \subset P^{(fin)}$, then $\mathcal{Q}(S)$ is the universal $C^*$-algebra generated by a unitary representation $\bar{u}$ of the group $G$ and a representation $\bar{s}$ of the semigroup $P$ by isometries subject to the relations:*

$$\text{(A1)} \qquad \bar{s}_p\bar{u}_g = \bar{u}_{\theta_p(g)}\bar{s}_p \qquad \textit{for all } p \in P, g \in G.$$

$$\text{(A2)} \qquad \bar{s}_p^*\bar{u}_g\bar{s}_q = \begin{cases} \bar{u}_k\bar{s}_{p'}\bar{s}_{q'}^*\bar{u}_\ell^* & \textit{if } pP \cap qP = rP,\ pp' = qq' = r \textit{ and} \\ & g = \theta_p(k)\theta_q(\ell^{-1}) \textit{ for some } k, \ell \in G, \\ 0 & \textit{otherwise.} \end{cases}$$

$$\text{(O)} \qquad 1 = \sum_{(g,p)\in F} \bar{e}_{g,p} \qquad \textit{for every } F \in \mathfrak{F}_e(G,P,\theta),$$

*where $\bar{e}_{g,p} = \bar{u}_g\bar{s}_p\bar{s}_p^*\bar{u}_g^*$.*

*Proof.* By [BLS, Theorem 4.4], $C^*(S)$ is isomorphic to $\mathcal{A}[G,P,\theta]$. (A1) and (A2) represent the defining relations for $\mathcal{A}[G,P,\theta]$, see [BLS, Definition 2.2]. Hence we need to argue that (Q) and (O) are equivalent. Since $S$ has property (AR), relation (Q) is equivalent to (Q$_a$), see Proposition 2.4. Clearly, this implies (O) as $F \in \mathfrak{F}_e(G,P,\theta)$ is always an accurate foundation set. Now suppose (O) holds and we have $F \in \mathfrak{F}_a(S)$. By Corollary 3.6, we now that $F \subset G \rtimes_\theta P^{(\mathrm{fin})}$. Hence there exists $F_e \in \mathfrak{F}_e(G,P,\theta)$ refining $F$, see Proposition 3.9. This leads to

$$1 \geq \sum_{(g,p)\in F} e_{g,p} \geq \sum_{(g,p)\in F_e} e_{g,p} = 1,$$

which establishes (Q$_a$) using (O). □

**Corollary 4.2.** *Suppose $(G,P,\theta)$ is an irreversible algebraic dynamical system. Then $\mathcal{O}[G,P,\theta]$ is canonically isomorphic to $\mathcal{Q}(S)$.*

*Proof.* $P$ is a countably generated, free abelian monoid, hence directed, so Corollary 3.11 applies and we arrive at the description of $\mathcal{Q}(S)$ from Corollary 4.1. A comparison of this presentation with [Sta15, Definition 3.1] shows that the two $C^*$-algebras are canonically isomorphic since (CNP1) and (CNP 2) correspond to (A1) and (A2), respectively, and (CNP 3) corresponds to (O) because $P$ is directed, see Remark 2.2. □

The algebra $\mathcal{O}[G,P,\theta]$ was constructed from the natural representation of $(G,P,\theta)$ on $\ell^2(G)$. Therefore, we would like to discuss this approach for $\mathcal{Q}(S)$ for algebraic dynamical systems:

Let $(\xi_g)_{g\in G}$ denote the standard orthonormal basis for $\ell^2(G)$ and $(U_g)_{g\in G}$ the unitary representation of $G$ on $\ell^2(G)$ given by $U_g\xi_h = \xi_{gh}$. Moreover, the map $\xi_h \mapsto \xi_{\theta_p(h)}$ defines an isometry $S_p \in \mathcal{L}(\ell^2(G))$ for every $p \in P$.

**Proposition 4.3.** *For an algebraic dynamical system $(G,P,\theta)$ and $S = G \rtimes_\theta P$, the assignment $\bar{u}_g\bar{s}_p \mapsto U_gS_p$ defines a representation $\lambda$ of $\mathcal{Q}(S)$ on $\ell^2(G)$ if and only if $P$ is directed.*

*Proof.* If $P$ is directed, then $S$ has property (AR) by Corollary 3.11 and hence $\mathcal{Q}(S)$ can be described as in Corollary 4.1. So we need to show that $(U_g)_{g\in G}$ and $(S_p)_{p\in P}$ satisfy (A1), (A2) and (O). (A1) is obvious and (O) is also easy once we observe that $\mathfrak{F}_a(P)\cap P^{(\text{fin})}$ is given by $P^{(\text{fin})}$, see Remark 2.2. This means that the families in $\mathfrak{F}_a^{(\text{fin})}(G,P,\theta)$ consist of one element $p\in P^{(\text{fin})}$ together with a transversal $T_p$ for $G/\theta_p(G)$. The verification of (A2) is a straightforward calculation that is omitted here. Thus we get a $*$-homomorphism $\lambda:\mathcal{Q}(S)\to\mathcal{L}(\ell^2(G))$ with $\bar{u}_g\bar{s}_p\mapsto U_gS_p$.

Now suppose $P$ is not directed, that is, there are $p,q\in P$ with disjoint principal right ideals. Then (A2) implies $s_p^*s_q=0$. But $\theta_p(G)\cap\theta_q(G)$ is a subgroup of $G$ and hence $S_p^*S_q\xi_1=\xi_1$. In particular, we get $S_p^*S_q\neq 0$, so (A2) does not hold for $S_p,U_1$ and $S_q$. $\square$

**Remark 4.4.** The $C^*$-algebra $\mathcal{A}[G,P,\theta]\cong C^*(S)$ introduced in [BLS15] is a $C^*$-algebraic model for the dynamical system $(G,P,\theta)$ based on the state space $\ell^2(S)$ and $\mathcal{Q}(S)$ is derived from this construction as a quotient. Although $\ell^2(G)$ is arguably a natural state space, we lose this representation for $\mathcal{Q}(S)$ once we leave the realm of actions of directed semigroups $P$. It seems that $\ell^2(G)$ can be too small to accommodate a representation of a $C^*$-algebraic model for $(G,P,\theta)$ that incorporates relations on the ideal structure of $P$. This raises the question whether there is a natural Hilbert space associated to $(G,P,\theta)$ on which we can represent $\mathcal{Q}(S)$.

### 4.2. **Simplicity and pure infiniteness à la Starling.**

The remainder of this section is devoted to applying the results of [Star15] which we recalled in Section 1.2 to right LCM semigroups $S=G\rtimes_\theta P$. This yields necessary and sufficient conditions on $(G,P,\theta)$ for $\mathcal{Q}(S)$ to be purely infinite and simple. We show that these conditions look quite familiar in the case where $P$ is right cancellative, an extra assumption which is satisfied by many interesting examples.

We first address the issue of simplicity, and then discuss pure infiniteness starting after Remark 4.15. Before we can state any results, though, we have to do some work on translating conditions (H) and (EP) from Theorem 1.8 into the setting of algebraic dynamical systems.

Recall from Definition 1.7 that the group of units in a semigroup is always contained in the core. While this inclusion is proper in many cases, we will show that we have equality for algebraic dynamical systems $(G,P,\theta)$ provided that $\theta_p\in\operatorname{Aut}(G)$ implies $p\in P^*$ for all $p\in P$.

**Standing Assumption 4.5.** For the rest of this section we will assume that for $(G,P,\theta)$ an algebraic dynamical system we have $\theta_p\in\operatorname{Aut}(G)\Longrightarrow p\in P^*$ for all $p\in P$.

This is a very reasonable assumption since the original $P$ can always be replaced by the right LCM semigroup $\{\theta_p\mid p\in P\}$.

**Proposition 4.6.** *For an algebraic dynamical system $(G,P,\theta)$ we have $S_0=S^*$.*

Recall from [BLS15, Lemma 2.4] that $S^* = G \rtimes_\theta P^*$.

*Proof of Proposition 4.6.* Let $(g,p),(h,q) \in S$. According to Lemma 3.1, $(g,p)S \cap (h,q)S \neq \emptyset$ holds if and only if $pP \cap qP \neq \emptyset$ and $h \in g\theta_p(G)\theta_q(G)$. Thus, $(g,p) \in S_0$ if and only if $p \in P_0$ and $h \in g\theta_p(G)\theta_q(G)$ for all $h \in G$ and $q \in P$. If we choose $q = p$, then this implies $G = g\theta_p(G)$ as $\theta_p(G)$ is a subgroup of $G$. Hence we get $\theta_p \in \mathrm{Aut}(G)$ as a necessary condition. But this is also sufficient as $h \in g\theta_p(G)\theta_q(G) = G\theta_q(G) = G$. Thus we see that $S_0 = S^*$. □

**Remark 4.7.** Recall from Remark 1.9 that condition (H) always holds for a right cancellative right LCM semigroups. We note that for algebraic dynamical systems $(G,P,\theta)$ with $S = G \rtimes_\theta P$, this is equivalent to $P$ having right cancellation. So the non-trivial case for (H) is the one where $S$ is not right cancellative. The set $S_{s,t} := \{r \in S \mid sr = tr\}$ for $s,t \in S$ is a proper right ideal in $S$ unless $s = t$, in which case $S_{s,t} = S$. We note that $S_{s,t}$ is a left cancellative semigroup that may also be empty. From this perspective, (H) is equivalent to

(H') The semigroup $S_{s,t}$ has a foundation set for all $s,t \in S$.

Here, the term foundation set is meant in the sense of Definition 1.2, even though $S_{s,t}$ need not be right LCM.

Due to a lack of examples of algebraic dynamical systems with a right LCM semigroup $P$ that is not right cancellative, we stop the discussion of condition (H) and commence on (EP).

Recall from Proposition 4.6 that $S_0 = G \rtimes_\theta P^*$ and from (1.1) the notation $[s,t]$ for an element in the inverse semigroup $I(S)$ corresponding to $v_s v_t^* \in C^*(S)$. In particular, we have $[s,s] = 1$ for all $s \in S_0$ and hence $[s,s] \geq [s',s']$ for all $s' \in S$. Since $S_0 = G \rtimes_\theta P^*$ is a group, it suffices to consider the case $t = 1$ for (EP) because $[s,t] = [st^{-1},1]$. Our aim is to find a precise dynamic condition on $(G,P,\theta)$ which guarantees that $s = t$ holds as soon as there exists some $[tt',tt']$ that is weakly fixed by $[s,t]$ with $s,t \in S_0$.

**Notation 4.8.** *For $p \in P^*$ and $(h,q) \in S$, we let*

$$G_{p,h,q} := \bigcap_{(k,r)\in S} h\theta_q(k)\theta_{qr}(G)\theta_{pqr}(G)\theta_p(h\theta_q(k))^{-1}.$$

We can now state the first of our simplicity results.

**Theorem 4.9.** *Suppose $(G,P,\theta)$ is an algebraic dynamical system with right cancellative $P$. Let $S = G \rtimes_\theta P$. The boundary quotient $\mathcal{Q}(S)$ is simple if and only if $\mathcal{Q}(S) \cong C_r^*(\mathcal{G}_{tight}(I(S)))$, and*

(EP') *For $(h,q) \in S$ and all $p \in P^*$ with $pqrP \cap qrP \neq \emptyset$ for all $r \in P$, the set $G_{p,h,q}$ is empty unless $p = 1$, in which case $G_{1,h,q} = \{1\}$.*

*Proof.* We want to show that $[s,t]$ satisfies (EP) for all $s,t \in S_0$ so that Theorem 1.8 applies. Recall from Lemma 4.6 that $S_0 = G \rtimes_\theta P^*$. Moreover, as remarked before Notation 4.8, it suffices to consider $[(g,p),(1,1)]$ with $(g,p) \in S_0$. So let $(g,p) \in S^* = G \rtimes_\theta P^*$. We start by observing that $[(g,p),(1,1)]$ fixes $[(h,q),(h,q)]$ weakly if and only if

$$qrP \cap pqrP \neq \emptyset \text{ for all } r \in P \text{ and } g \in G_{p,h,q}. \tag{4.1}$$

Indeed, $[(g,p),(1,1)]$ fixes $[(h,q),(h,q)]$ weakly if and only if

$$(h,q)(k,r)S \cap (g,p)(h,q)(k,r)S \neq \emptyset \text{ for all } (k,r) \in S.$$

Using Lemma 3.1, we translate this to $qrP \cap pqrP \neq \emptyset$ and

$$\big(h\theta_q(k)\big)^{-1} g\theta_p(h)\theta_{pq}(k) \in \theta_{qr}(G)\theta_{pqr}(G) \text{ for all } (k,r) \in S.$$

The second equation can be reformulated as $g \in G_{p,h,q}$. Let us note that $G_{p,h,q}$ may be empty in which case $[(g,p),(1,1)]$ cannot fix $[(h,q),(h,q)]$ weakly irrespective of the choice of $g$.

Since $P$ is right cancellative, so is $S$. In view of Remark 1.12 (c), we want to use (4.1) to show that (EP') is equivalent to:

(4.2) If $(g,p) \in G \rtimes_\theta P^*$ has the property that $[(g,p),(1,1)]$ fixes some $[(h,q),(h,q)]$ weakly, then $(g,p) = (1,1)$.

If (EP') holds, then the only $(g,p)$, for which $[(g,p),(1,1)]$ may fix some $[(h,q),(h,q)]$ weakly, is $(g,p) = (1,1)$. Thus (4.2) is valid. Conversely, suppose there is $(h,q) \in S$ and $p \in \{p' \in P^* \mid p'qrP \cap qrP \neq \emptyset \text{ for all } r \in P\}$ for which either $p \neq 1$ and $G_{p,h,q} \neq \emptyset$ or $p = 1$ and there exists $g \in G_{1,h,q} \setminus \{1\}$. In both cases, we get a $(g,p) \in S_0$ such that $[(g,p),(1,1)]$ fixes $[(h,q),(h,q)]$ weakly by (4.1), but $(g,p) \neq (1,1)$. So we arrive at a contradiction to (4.2) and the proof is complete. □

The condition (EP') is technical and lacks an immediate interpretation. But we will see that it takes a simpler form in special cases.

**Corollary 4.10.** *Suppose $(G,P,\theta)$ is an algebraic dynamical system with right cancellative $P$ and $P^*p \subset pP^*$ for all $p \in P$. Let $S = G \rtimes_\theta P$. The boundary quotient $\mathcal{Q}(S)$ is simple if and only if*

1. $\mathcal{Q}(S) \cong C_r^*(\mathcal{G}_{tight}(I(S)))$, *and*
2. $\bigcap_{(k,r)\in S} k\theta_r(G)k^{-1} = \{1\}$, *and*
3. $\bigcap_{(k,r)\in S} k\theta_r(G)\theta_{\tilde{p}}(k)^{-1} = \emptyset$ *for all $\tilde{p} \in P^*$ arising from $pq = q\tilde{p}$ for some $p \in P^* \setminus \{1\}$ and $q \in P$.*

*Proof.* We claim that (EP') holds if and only if (2) and (3) are true. Let $p \in P^*$ and $(h,q) \in S$. We start by observing that (4.1) holds for all $p \in P^*$ as $pqrP = qrP$ for all $q,r \in P$. Moreover, writing $pq = q\tilde{p}$ with $\tilde{p} \in P^*$, the set $G_{p,h,q}$ becomes

$$\begin{aligned} G_{p,h,q} &= \bigcap_{(k,r)\in S} h\theta_q(k)\theta_{qr}(G)\theta_{pqr}(G)\theta_p(h\theta_q(k))^{-1} \\ &\overset{a)}{=} \bigcap_{(k,r)\in S} h\theta_q(k)\theta_{qr}(G)\theta_{q\tilde{p}}(k)^{-1}\theta_p(h)^{-1} \\ &\overset{b)}{=} h\theta_q\big(\bigcap_{(k,r)\in S} k\theta_r(G)\theta_{\tilde{p}}(k)^{-1}\big)\theta_p(h)^{-1}, \end{aligned}$$

where we used that:

a) $\theta_{pqr}(G) = \theta_{qr}(G)$ for $pqr = qrp'$ and $\theta_{p'} \in \mathrm{Aut}(G)$.
b) $G$ is a group and $\theta_q$ is injective.

This proves the claim and hence we can apply Theorem 4.9. □

**Remark 4.11.** The existence of $p \in P^*$ and $q \in P$ such that $p \neq 1$, but $pq = q$, i.e. $\tilde{p} = 1$, immediately leads to a violation of (EP') as $\bigcap_{(k,r)\in S} k\theta_r(G)k^{-1}$ is a subgroup of $G$. Note that this phenomenon can only occur in the case where $P$ is not right cancellative.

**Remark 4.12.** Suppose that $P$ is right cancellative. If $\theta$ separates the points of $G$, i.e. $\bigcap_{p\in P}\theta_p(G) = \{1\}$, and $\theta : P^* \curvearrowright G$ is faithful, that is, for each $p \in P^* \setminus \{1\}$ there exists $g \in G$ with $\theta_p(g) \neq g$, then conditions (2) and (3) from Corollary 4.10 are satisfied. Indeed, (2) is obvious. If we take $p \in P^* \setminus \{1\}$ and $q \in P$ to get $\tilde{p} \in P^*$ with $pq = q\tilde{p}$, right cancellation for $P$ implies $\tilde{p} \neq 1$. Since $\theta : P^* \curvearrowright G$ is faithful, there is $g \in G$ with $\theta_{\tilde{p}}(g) \neq g$. If $\theta$ separates the points in $G$, we can choose $r' \in P$ large enough such that $g^{-1}\theta_{\tilde{p}}(g) \notin \theta_{r'}(G)$. Therefore,

$$\bigcap_{(k,r)\in S} k\theta_r(G)\theta_{\tilde{p}}(k)^{-1} \subset \bigcap_{r\in P}\theta_r(G) \cap g\theta_{r'}(G)\theta_{\tilde{p}}(g)^{-1} = \emptyset,$$

which shows (3).

For $P^* = \{1\}$, we recover a condition that has already appeared in [BLS15]. Recall that an action $H \curvearrowright J$ of a group $H$ on a set $J$ is said to be *effective* if for every $h \neq 1$ there is $X \in J$ such that $h.X \neq X$.

**Corollary 4.13.** *Suppose $(G, P, \theta)$ is an algebraic dynamical system with $P^* = \{1\}$ and right cancellative $P$. Let $S = G \rtimes_\theta P$. The boundary quotient $\mathcal{Q}(S)$ is simple if and only if*

(1) $\mathcal{Q}(S) \cong C_r^*(\mathcal{G}_{tight}(I(S)))$, *and*
(2) $S^* \curvearrowright \mathcal{J}(S)$ *is effective.*

*Proof.* In the case of $P^* = \{1\}$, [BLS15, Lemma 8.5 and Lemma 8.6] states that the action $S^* \curvearrowright \mathcal{J}(S)$ for $S = G \rtimes_\theta P$ is effective if and only if $\bigcap_{(k,r)\in S} k\theta_r(G)k^{-1} = \{1\}$. Now Corollary 4.10 applies because condition (3) is void due to $P^* = \{1\}$. □

Corollary 4.13 yields a sophisticated answer to the question of a characterisation of simplicity of $\mathcal{O}[G, P, \theta]$ for irreversible algebraic dynamical systems

$(G,P,\theta)$ considered in [Sta15], where sufficient conditions were discussed, see [Sta15, Theorem 3.26]. Due to [Sta15, Definition 1.5 (C)], $\theta_p \in \mathrm{Aut}(G)$ implies $p = 1 \in P^*$. Moreover, $P$ is right cancellative and $P^* = \{1\}$ since $P$ is a countably generated, free abelian monoid.

**Corollary 4.14.** *Let $(G,P,\theta)$ be an irreversible algebraic dynamical system. Then $\mathcal{O}[G,P,\theta]$ is simple if and only if*

(1) $\mathcal{O}[G,P,\theta] \cong C_r^*(\mathcal{G}_{tight}(I(G \rtimes_\theta P)))$, *and*
(2) $\bigcap_{(g,p)\in G\rtimes_\theta P} g\theta_p(G)g^{-1} = \{1\}$.

*Proof.* By Corollary 4.2, we have $\mathcal{Q}(G \rtimes_\theta P) \cong \mathcal{O}[G,P,\theta]$. As $P$ is right cancellative and $P^* = \{1\}$, the claim follows from Corollary 4.13. $\square$

**Remark 4.15.** In [Sta15, Theorem 3.26], the author proved that $\mathcal{O}[G,P,\theta]$ is simple (and purely infinite) given that the canonical action $\hat{\tau}$ of $S^* \cong G$ on the spectrum $G_\theta$ of the diagonal of $\mathcal{O}[G,P,\theta]$ is amenable and that the action $\theta$ is *minimal* in the sense that $\bigcap_{p\in P}\theta_p(G) = \{1\}$. It is not hard to see that amenability of $\hat{\tau}$ yields amenability of $\mathcal{G}_{\text{tight}}(I(G \rtimes_\theta P))$ and hence $\mathcal{O}[G,P,\theta] \cong C^*(\mathcal{G}_{\text{tight}}(I(G \rtimes_\theta P))) \cong C_r^*(\mathcal{G}_{\text{tight}}(I(G \rtimes_\theta P)))$. In addition, minimality of $(G,P,\theta)$ clearly implies (2) from Corollary 4.14. So we see that, in general, the conditions on $(G,P,\theta)$ are slightly milder than the ones obtained in [Sta15]. Note however, that minimality of $(G,P,\theta)$ is in fact necessary and sufficient for simplicity of $\mathcal{O}[G,P,\theta]$ in the case where $G$ is abelian, as assumed in [CV13].

Let us now briefly discuss pure infiniteness of $\mathcal{Q}(S)$ for $S = G \rtimes_\theta P$. It was proven in [Star15, Theorem 4.15] that, for general right LCM $T$, the boundary quotient $\mathcal{Q}(T)$ is purely infinite if and only if $\mathcal{G}_{\text{tight}}(T)$ is not a single point, provided that $\mathcal{Q}(T)$ is simple and $T$ satisfies condition (H). Hence, pure infiniteness is almost automatic in this case. Indeed, $\mathcal{G}_{\text{tight}}(T)$, as a set, is given by the equivalence classes of

$$\begin{aligned}&\mathcal{G}'_{\text{tight}}(T)\\ &\quad := \{\big([s,t],\xi\big) \mid s,t \in T, \xi \subset \mathcal{J}(T) \text{ tight filter with } r \in tT \text{ for all } rT \in \xi\}\end{aligned}$$

with respect to $\big([s,t],\xi\big) \sim \big([s',t'],\xi'\big)$ defined as

$$\xi = \xi' \text{ and there exists } rT \in \xi \text{ such that } [s,t].rT = [s',t'].rT,$$

where $[s,t].rT := s(t^{-1}(rT))$, see [Star15, Subsection 4.1] for details.

**Corollary 4.16.** *Suppose that $(G,P,\theta)$ is an algebraic dynamical system such that* (H) *holds for $S$ and $\mathcal{Q}(S)$ is simple. Then $\mathcal{Q}(S)$ is purely infinite if and only if $P$ is not a group.*

*Proof.* We start by observing that $\mathcal{J}(S) = \{\emptyset, S\}$ holds if and only if $S$ is a group (as $sS = S$ implies $s \in S^*$ for all $s \in S$). In this case, the only tight filter on $\mathcal{J}(S)$ is $\{S\}$ and every $[s,t] \in I(S)$ fixes $S$, so $\mathcal{G}_{\text{tight}}(I(S))$

is just a singleton. So if $S$ is a group, which is equivalent to $P$ being a group, then $\mathcal{Q}(S) \cong \mathbb{C}$. If $P$ is not a group, then there is $p \in P$ such that $[G : \theta_p(G)] \geq 2$. So there are $g_1, g_2 \in G$ with $g_1^{-1} g_2 \notin \theta_p(G)$. This amounts to $(g_1, p)S \cap (g_2, p)S = \emptyset$ using Lemma 3.1. There is at least one ultrafilter $\xi_i$ of $\mathcal{J}(S)$ containing $(g_i, p)S$ for $i = 1, 2$. We clearly have $(g_2, p)S \notin \xi_1$ as $(g_1, p)S \cap (g_2, p)S = \emptyset$, so $\xi_1 \neq \xi_2$. Therefore, $\mathcal{G}_{\text{tight}}(S)$ contains at least two distinct points and, with the help of [Star15, Theorem 4.15], we conclude that $\mathcal{Q}(S)$ is purely infinite. □

The essential ingredient in here is that $S$ fails to be left reversible as soon as $P$ is not a group. It was pointed out to us by Xin Li that there is a deeper connection between pure infiniteness of $C^*$-algebras associated to left cancellative semigroups (without assuming simplicity) and failure with respect to left reversibility.

As a final result, we collect a number of cases in which we now know that $\mathcal{Q}(G \rtimes_\theta P)$ belongs to a well-understood class of $C^*$-algebras classifiable by K-theory, see [Kir, Phi00].

**Theorem 4.17.** *Suppose $(G, P, \theta)$ is an algebraic dynamical system such that $P$ is not a group and $\mathcal{G}_{tight}(I(G \rtimes_\theta P))$ is amenable. Then $\mathcal{Q}(G \rtimes_\theta P)$ is a unital UCT Kirchberg algebra provided that one of the following conditions holds:*

1. *$G \rtimes_\theta P$ satisfies* (H) *and for all $s, t \in (G \rtimes_\theta P)_0$, the element $[s, t]$ satisfies* (EP).
2. *$P$ is right cancellative, $P^* p \subset p P^*$ for all $p \in P$, and $G \rtimes_\theta P$ satisfies* (2) *and* (3) *from Corollary 4.10.*
3. *$P$ is right cancellative, $P^* = \{1\}$ and $\bigcap_{(g,p) \in G \rtimes_\theta P} g\theta_p(G)g^{-1} = \{1\}$.*

*Proof.* In each case, $\mathcal{Q}(G \rtimes_\theta P)$ is simple, see Theorem 1.8, Theorem 4.9, Corollary 4.10, and Corollary 4.13, respectively. Since $P$ is not a group, Corollary 4.16 shows that $\mathcal{Q}(G \rtimes_\theta P)$ is also purely infinite. $\mathcal{Q}(G \rtimes_\theta P)$ is separable since $G \rtimes_\theta P$ is countable. Finally, nuclearity and the UCT follow from amenability of $\mathcal{G}_{\text{tight}}(I(G \rtimes_\theta P))$, see for instance [BO08, Theorem 5.6.18] and [Tu99]. □

We remark that Theorem 4.17 generalises [Sta15, Corollary 3.28].

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