Local risk minimizing strategy in a market driven by time-changed Lévy noises

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The front page depicts a section of the root system of the exceptional Lie group $E_8$, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.
Preface

Abstract

The purpose of this thesis is to study the hedging of financial derivatives, using the so-called local risk-minimizing strategy, which is a popular quadratic hedging strategy in incomplete markets. The local risk minimization aims at perfectly replicating the derivative. However such strategies cannot be self-financing in general, and therefore allowing for a cost. Then a good strategy should have minimal cost. The problem of finding the local risk minimizing strategy is in this thesis tackled by two methods, via a change of measure using the minimal martingale measure and via backward stochastic differential equations.

The financial market model studied is driven by a time-changed Lévy noise, where the time change is independent of the Lévy process.

In this thesis two different information flows are considered. Both filtrations are naturally linked to the noise. Information flow $\mathcal{G}$ is the large filtration containing information about the future, which captures all statistical properties of the noise. While the smaller information flow $\mathcal{F}$ is a more realistic information flow from a financial modeling perspective.
Acknowledgement

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Chapter 1

Introduction

In an incomplete market not all claims are perfectly replicable. In such incomplete markets one can apply quadratic hedging. The strategy in quadratic hedging is to minimize the hedging error in a mean square sense. One of the two main approaches in quadratic hedging is local risk minimizing. The main feature of the local risk minimizing approach is that one works with strategies which are not self-financing. So one is looking for a strategy that insists on perfectly replicating the contingent claim, however can not be self-financing. Then a good strategy should have minimal cost.

Time-changed Lévy noises are in the financial market considered as a standard method for constructing financial models. The time-changed Lévy process introduces a new stochastic process through a random change of time. Models driven by these time-changed Lévy processes are based on the fact that the volatility is stochastic and increasing with the intensity of trades. So these models are a good way of describing the business activities. The new stochastic process, also called the time change, can be considered as the new random time, the so-called business time clock, and the original clock is considered as the calendar time. There are several well-known applications to financial models driven by time-changed Lévy noises. Models driven by time-changed Brownian motions appear in financial price modeling in the class of stochastic volatility. See the papers [BNS02], [Hes93], [HW87] and [SS91].

This thesis aims at studying the following question: Given a contingent claim F how can we characterize the local risk minimizing strategy in such time-changed markets? We tackle the problem by considering two methods. The first method, is to study the relationship between the Föllmer-Schweizer decomposition and the
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Galtchouk-Kunita-Watanabe decomposition. For this we will need a change of measure that preserves orthogonality of local martingales, with the so-called minimal martingale measure. The necessary background on local risk minimizing strategies in the setting of a general semimartingale is found in [Sch99] and [FS91]. We extend this theory to the time-change setting. The second method that we consider uses backward stochastic differential equations driven by the time-changed Lévy noises, whose terminal condition is the contingent claim $F$. Then we will look at the relation between the backward stochastic differential equations and the Föllmer-Schweizer decomposition. This method is based on the work in [DKV15].

In this thesis two information flows are considered. The first filtration is generated by the natural filtration of the time-changed Brownian motion. The second, is generated by the time-changed Brownian motion and in addition has knowledge of the entire future of the time-change process. These information flows are denoted by $\mathcal{F}$ and $\mathcal{G}$, respectively. From a financial point of view it is more realistic to consider the information flow $\mathcal{F}$. While the information flow $\mathcal{G}$, captures all the statistical properties of the noise.

In the setting of the two different filtrations, one can expect that the associated portfolio value of the local risk-minimizing strategy, will lead to different solutions depending on the filtrations measurability properties.

The structure for the thesis is as follows. In Chapter 2, we start by introducing the preliminary theory for the noise, and present two popular time change models, subordinator and absolutely continuous time change. Then we will narrow the theory down to time-changed Brownian motions. The market model is also presented. In Appendix A there are more theoretical results and definitions to support the theory in Chapter 2.

In Section 3.1, we discuss the local risk minimizing problem, under a measure change, with respect to the two filtrations $\mathcal{G}$ and $\mathcal{F}$.

In Section 3.2, we discuss the local risk minimizing problem with backward stochastic differential equations, with respect to the two filtrations $\mathcal{G}$ and $\mathcal{F}$.

For both methods, we first consider the information flow $\mathcal{G}$, here we give important results to show that the theory on local risk minimizing in the setting of a general semimartingale can be extended to the case of a semimartingale driven by a time-changed Brownian motion. When considering the information flow $\mathcal{F}$, we will need to apply some other techniques to find the solution to our problem.
My work for this thesis was structured as follows. I started by studying the necessary background on time change techniques, where the focus was kept on time-changed Brownian motion. This included background on Lévy processes, subordinators, and absolutely continuous time change. Then I defined the market model driven by a time-changed Brownian motion, and the necessary properties for time-changed Brownian motion.

Next, I continued with acquiring basic knowledge on the local risk minimizing (LRM) strategy and the minimal martingale measure (MMM).

Then, started tackling the LRM via MMM in the setting of the time-changed Brownian motion with respect to the filtration $G$.

I also acquire the basic knowledge on backward stochastic differential equations (BSDEs), both in the setting of general martingales and BSDEs driven by càdlàg martingales.

I continued by tackling the LRM problem via BSDEs with respect to filtration $G$.

Then I extended the LRM strategy via MMM and BSDEs to the case of filtration $F$, using a partial information approach.

My main contribution to this thesis is the work in Section 3.1.2 and Section 3.2.2, when we consider the LRM problem under the information flow $F$, generated by the time-changed Brownian motion. In addition, I contributed with the following:

- Chapter 2:
  - Proposition 2.3.5 (The Lévy characterization of a time-changed Brownian motion) and its proof.
  - Proof (i) of Proposition 2.3.4 (The proof of the time-changed Brownian motion being a martingale with respect to filtration $G$).

- Chapter 3:
  - Theorem 3.1.7 (Girsanov’s Theorem in the setting of a time-changed Brownian motion) and its proof.
  - The Explicit solution of the LRM strategy under filtration $G$.
  - Proposition 3.1.21 (The representation of the predictable quadratic variation of $B$ with respect to $F$, as a projection of the time change process) and its proof.

My original contributions are marked with $\clubsuit$. 
Chapter 2

The Market model and the noise

2.1 Lévy processes

In this section we formally introduce Lévy processes. The theory is from the book of Applebaum (2009) [App09].

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space.

**Definition 2.1.1.** Let \(X = \{X_t, t \geq 0\}\) be a stochastic process on \((\Omega, \mathcal{F}, P)\).

Then \(X\) is a Lévy process if

(i) \(X_0 = 0\) a.s.

(ii) \(X\) has independent and stationary increments

(iii) \(X\) is stochastically continuous, i.e. for all \(k > 0\) and for all \(s \geq 0\)

\[
\lim_{t \to s} P(|X_t - X_s| > k) = 0
\]

Let \(X = \{X_t, t \geq 0\}\) be a Lévy process on \(\mathbb{R}^d\). Then the random variable \(X_t\) is characterized by the triplet \((\beta, A, \nu)\), for all \(t \geq 0\), called Lévy triplet and given by the Lévy-Itô decomposition. Moreover the whole process \(X\) is described by the Lévy triplet of \(X_1\), that is \((\beta, A, \nu)\).

Denote the Borel \(\sigma\)-algebra by \(\mathcal{B}\). The measure \(\nu\) on \(\mathbb{R}^d\) is defined by

\[
\nu(B) = E[\#t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in B], \quad B \in \mathcal{B}(\mathbb{R}^d)
\]

called the Lévy measure.

The characteristic function of the random variable \(X_t\), is defined by

\[
\Psi_{X_t}(u) := E[exp(iu^tX_t)] = exp(\psi_{X_t}(u)), \quad u \in \mathbb{R}^d
\]

for all \(t \geq 0\), where \(\psi_X(u) = tu^t\psi(u)\) and \(\psi_X : \mathbb{R}^d \to \mathbb{R}\) is a continuous function called the characteristic exponent of \(X\).

\[^1\text{See Appendix A.2}\]
2.2. TIME CHANGED PROCESS

The Lévy-Khinchine representation for a d-dimensional Lévy process \( X = \{X_t, t \geq 0\} \) gives an expression of the characteristic function of \( X \) in terms of its Lévy triplet \( (\beta, A, \nu) \). That is, the characteristic exponent is given by

\[
\psi_X(u) = i\beta' u - \frac{1}{2} u' Au + \int_{\mathbb{R}^d} \left( e^{iu'y} - 1 - iu'y1_{|y| \leq 1}\right) \nu(dy) \tag{2.1}
\]

(where \( ' \) is the transposed).

Example 2.1.2. The (standard) Brownian motion \( W = \{W_t, t \geq 0\} \) on \( \mathbb{R}^d \) is a Lévy process, with Lévy triplet \( (0, I_d, 0) \).

2.2 Time changed process

Time changing is a transformation of a stochastic process into a new stochastic process through random change of time. In finance, applications of price dynamics driven by time-changed noises are based on the fact that volatility is increasing with intensity of trades. So the time change process represents the transition between the real-time clock to the trading clock.

Let \( X = \{X_t, t \geq 0\} \) be a stochastic process on \( \mathbb{R}^d \) and \( \Lambda = \{\Lambda_t, t \geq 0\} \) be a nonnegative, non-decreasing stochastic process on \( \mathbb{R} \), not necessarily independent of \( X \). Then the time-changed process define by \( Y = \{Y_t, t \geq 0\} \) with respect to the filtration \( \mathcal{G}_t \) is defined as

\[
Y_t = X_{\Lambda_t}
\]

Where \( \mathcal{G}_t = \sigma\{Y_s, \Lambda_s, s \leq t\} \vee \mathcal{N} \) and \( \mathcal{N} \) is the collection of all P-measure zero sets in \( \mathcal{F} \). The process \( \Lambda \) is called the time change process.

Later on we will only work with time-changed Brownian motions. First we will introduce some useful results (see e.g. [VW10]).

Theorem 2.2.1 (Dubins-Schwarz). Every continuous local martingale \( M \) can be written as a time-changed Brownian motion \( B_{\langle M \rangle} \). Here \( \langle M \rangle = \{\langle M \rangle_t, t \geq 0\} \) is the predictable quadratic variation of \( M \) (see Appendix A.3).

Example 2.2.2. Let \( M_t = \int_0^t \sigma_s dW_s \), where \( W \) is a Brownian motion with respect to it natural filtration \( \mathcal{F}^W \), for all \( t \geq 0 \) and \( \sigma \) a independent non-negative càdlàg stochastic process, such that \( M \) is a local martingale. In fact for a sequence of stopping times \( T_n, n \in \mathbb{N} \), with \( \lim_{n \to \infty} T_n = \infty \), we have
that $M_{1\wedge T_n}1_{T_n>0}$ is a square integrable martingale. Let $\mathcal{F}^W_t = \mathcal{G}_{T_t}$. Then by Optional Sampling,

$$E[M_t|\mathcal{G}_{T_s}] = \lim_{n\to\infty} E[M_{1\wedge T_n}|\mathcal{G}_{T_s}] = \lim_{n\to\infty} M_{s\wedge T_n} = M_s$$

So we have $\langle M \rangle_t = \int_0^t \sigma^2_s ds =: \Lambda_t$, and by the Dubins-Schwarz Theorem, $B_{\Lambda_t}$ is a time-changed Brownian motion.

**Theorem 2.2.3 (Monroe).** Every càdlàg $\mathbb{F}$-semimartingale $S = \{S_t, t \geq 0\}$ can be written as a time-changed Brownian motion, if there is a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, an $\mathcal{F}_t$- Brownian motion $W = \{W_t, t \geq 0\}$ and a càdlàg family of stopping times $\Lambda = \{\Lambda_t, t \geq 0\}$, such that $S_t = W_{\Lambda_t}$.

For a general time-changed process $Y = \{Y_t, t \geq 0\}$ and $W = \{W_t, t \geq 0\}$ a Brownian motion on $\mathbb{R}$, we have that $Y_t = W_{\Lambda_t}$ is a conditionally Gaussian process. That is,

$$P(Y_t \leq y|\Lambda_t) = \Phi\left(\frac{y}{\sqrt{\Lambda_t}}\right)$$

Where $\Phi$ is the cumulative probability distribution function of a standard normal variable. So $Y$ is Gaussian, conditional on the time change process.

There are different types of time change processes, we will look at subordinators and absolutely continuous processes.

### 2.2.1 Subordinator

A subordinator is a Lévy process on $\mathbb{R}$ that is non-decreasing and non-negative. So for a subordinator $\Lambda = \{\Lambda_t, t \geq 0\}$ we have $\Lambda_t \geq 0$ a.s. for each $t \geq 0$ and $\Lambda_{t_1} \leq \Lambda_{t_2}$ a.s whenever $t_1 \leq t_2$.

Let the Lévy triplet of $\Lambda$ be $(a, 0, \rho)$, then, for all $u \geq 0$,

$$E[\exp(u\Lambda_t)] = \exp(tl(u))$$

where $l(u) = au + \int_{(a,\infty)}(e^{uy} - 1)\rho(dy)$, $a \geq 0$ and the Lévy measure $\rho$ satisfies

$$\rho(-\infty, 0) = 0 \quad \text{and} \quad \int_0^\infty (1 \wedge y)\rho(dy) < \infty$$

The function $l(\cdot)$ is called the **Laplace exponent** of $\Lambda$. 

Example 2.2.4. Let $N = \{N_t, t \geq 0\}$ be a Poisson process of intensity $\lambda$. This can be represented as follows. For a sequence of independent exponential random variables $\tau_i, i = 1, 2, \ldots$ with parameter $\lambda$ and $T_n = \sum_{i=1}^{n} \tau_i$, we have

$$N_t = \sum_{n \geq 1} 1_{t \geq T_n}$$

The Poisson process is a Lévy process. Also it is nonnegative, since $N_t$ is defined on $\mathbb{N} \cup \{0\}$ and non-decreasing, since $N_t$ is a constant on each interval $[T_n, T_{n+1})$ and increases by one at each $T_n$. Hence the Poisson process is a subordinator.

Example 2.2.5. Let $\Lambda$ be a subordinator, with Lévy triplet $(a,0,\rho)$. Then $\Lambda$ is also a absolutely continuous time change if we can write it as,

$$\Lambda_t = at$$

where $a > 0$. In this case the triplet is given by $(a,0,0)$.

In general, we can write a subordinator $\Lambda$ as a non-decreasing semimartingale (see Definition A.3.7). That is, let $\Lambda$ be a subordinator with Lévy triplet $(a,0,\rho)$, then by the definition of a subordinator, condition (i)-(ii) in Definition A.3.7 are satisfied, also we have $B \equiv 0$, for all $t \geq 0$, hence condition (iii) is satisfied. Finally, let $\alpha_t = at - \int_0^t \int_{0 < x \leq 1} x \rho(dx)ds$, for all $t \geq 0$. When $\alpha$ is non-decreasing, we have

$$\Lambda_t = \alpha_t + \int_{(0,\infty)} xJ(t, dx)$$

where $J$ is the jump measure of the possible positive jumps of $\Lambda$.

We will use subordinators to time-change. The following result is from Theorem 4.2 in [CT04].

Theorem 2.2.6. Let $X = \{X_t, t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy triplet $(\beta, A, \nu)$ and characteristic exponent $\psi_X(\cdot)$ given by (2.1) and let $\Lambda = \{\Lambda_t, t \geq 0\}$ be a subordinator, with Lévy triplet $(a,0,\rho)$ and Laplace exponent $l(\cdot)$. Let $\Lambda$ be independent of $X$. Define a new stochastic process $Y = \{Y_t, t \geq 0\}$ by

$$Y_t := X_{\Lambda_t}$$

for each $t \geq 0$. Then $Y$ is a Lévy process with triplet $(\beta_Y, A_Y, \nu_Y)$, given by

$$\beta_Y = a\beta + \int_{(0,\infty)} \rho(ds) \int_{|x| \leq 1} xP_X(dx)$$
The proof of finding the Lévy triplet

Define the measure $\nu_y(B) = \alpha \nu(B) + \int_{(0,\infty)} P_X(B) \rho(ds), \ B \in \mathcal{B}(\mathbb{R}^d)$

where $P_X$ is the probability distribution of $X$. Moreover, the Lévy characteristic function is given by

$$\Psi_Y := E[\exp(iuY_t)] = \exp(t\psi_X(u)), \ u \in \mathbb{R}^d. \quad (2.3)$$

**Proof.** (i) $Y$ has independent increments. Denote $\mathcal{F}^\Lambda = \mathcal{F}^\Lambda_T$ to be the filtration generated by $\Lambda$ at the finite time horizon $T$ and let $t_0 < t_1 < \cdots < t_n$ be a partition of $[0, T]$. Then by the independent increments of $X$ and $\Lambda$,

$$E[\prod_{i=1}^n \exp(iu_i(X_{t_i} - X_{t_{i-1}}))] = E[E[\prod_{i=1}^n \exp(iu_i(X_{t_i} - X_{t_{i-1}}))|\mathcal{F}^\Lambda]]$$

$$= E[\prod_{i=1}^n E[\exp(iu_i(X_{t_i} - X_{t_{i-1}}))|\mathcal{F}^\Lambda]] \overset{(*)}{=} E[\prod_{i=1}^n \exp((\Lambda_{t_i} - \Lambda_{t_{i-1}})\psi_X(u_i))]$$

$$= \prod_{i=1}^n E[\exp((\Lambda_{t_i} - \Lambda_{t_{i-1}})\psi_X(u_i))] = \prod_{i=1}^n E[\exp(iu_i(X_{t_i} - X_{t_{i-1}}))]$$

Where in (*) we applied the independence between $X$ and $\Lambda$. That is, for a bounded function $f$, we have $E[E[f(X_{\Lambda_t})|\mathcal{F}^\Lambda]] = E[g(\Lambda_t)]$, where $g(t) = E[f(X_t)]$.

(ii) Stationarity of increments. Similar to (i).

(iii) For every $\epsilon > 0, \delta > 0$, we have

$$P(|X_{\Lambda_t} - \Lambda_s| > \epsilon) \leq P(|X_{\Lambda_t} - X_{\Lambda_s}| > \epsilon|\Lambda_t - \Lambda_s| < \delta) + P(|\Lambda_t - \Lambda_s| \geq \delta)$$

Since $X$ is a Lévy process, it is uniformly continuous in probability, so the first term can be made arbitrarily small. The second term tends to zero when $s \to t$, because $\Lambda$ is continuous in probability. Hence

$$P(|Y_t - Y_s| > \epsilon) \to 0 \quad as \ s \to t$$

Hence by Definition 2.1.1 $Y$ is a Lévy process. Moreover, the function (2.3) is obtained by conditioning on $\mathcal{F}^\Lambda$,

$$E[\exp(iuY_t)] = \exp(t\psi_X(u))$$

The proof of finding the Lévy triplet $(\beta_Y, A_Y, \mu_Y)$, is taken from the proof of Theorem 30.1 in Sato (1999).

Define the measure $\gamma$ by $\gamma(\{0\}) = 0$ and

$$\gamma(B) = \int_{(0,\infty)} P_X(B) \rho(ds), \ B \in \mathcal{B}(\mathbb{R}^d)$$
2.2. TIME CHANGED PROCESS

We need to show that
\[ \int_{|x| \leq 1} |x|^2 \gamma(dx) < \infty \quad \text{and} \quad \int_{|x| \geq 1} |x|^2 \gamma(dx) < \infty \]

By Lemma 30.3 in [Sat99], there is a \( C_1 = C_1(\epsilon) \) such that for any \( t \geq 0 \),
\[ P(|X_t| > \epsilon) \leq C_1 t \tag{2.4} \]
and there is a \( C_2 \) such that for any \( t \geq 0 \),
\[ E[|X_t|^2 | |X_t| \leq 1] \leq C_2 t \tag{2.5} \]

Let \( D = \{ x : |x| \leq 1 \} \). Then (2.4) and (2.5) yields, respectively
\[ \int_{|x| > 1} \gamma(dx) = \int_{(0,\infty)} P(|X_s| > 1) \rho(ds) < \infty \]
\[ \int_D |x|^2 \gamma(dx) = \int_{(0,\infty)} \rho(ds) \int_D |x|^2 P_X(dx) < \infty \]

Furthermore, we have
\[ l(\psi_X(u)) = a\psi_X(u) + \int_{(0,\infty)} (e^{\psi_X(u)s} - 1) \rho(ds) \]

Let \( g(u, x) = e^{iu} - 1 - iux1_D(x) \), then
\[ \int_{(0,\infty)} (e^{\psi_X(u)s} - 1) \rho(ds) = \int_{(0,\infty)} \rho(ds) \int_{\mathbb{R}^d} (e^{iu} - 1) P_X(dx) \]
\[ = \int_{(0,\infty)} \rho(ds) \int_{\mathbb{R}^d} g(u, x) P_X(dx) + i \int_{(0,\infty)} \rho(ds) \int_{\mathbb{R}^d} u x 1_D(x) P_X(dx) \]
\[ = \int_{\mathbb{R}^d} g(u, x) \gamma(dx) + iu \int_{(0,\infty)} \rho(ds) \int_D x P_X(dx) \]

Hence
\[ a\psi_X(u) + \int_{(0,\infty)} (e^{\psi_X(u)s} - 1) \rho(ds) = i\beta a u - \frac{1}{2} u^2 a A + a \int_{\mathbb{R}^d} g(u, x) \nu(dx) \]
\[ + \int_{\mathbb{R}^d} g(u, x) \gamma(dx) + iu \int_{(0,\infty)} \rho(ds) \int_D x P_X(dx) \]
\[ = iu(\beta a + \int_{(0,\infty)} \rho(ds) \int_D x P_X(dx)) \]
\[ - \frac{1}{2} u^2 a A + \int_{\mathbb{R}^d} g(u, x)(av(dx) + \gamma(dx)) \]

Example 2.2.7. Let $X$ in Theorem 2.2.6 be a $d$-dimensional Brownian motion (w.r.t its natural filtration) with Lévy triplet $(0, A, 0)$, such that

$$\psi_X(u) = -\frac{1}{2} u' Au, \quad u \in \mathbb{R}^d$$

Let $\Lambda$ be a subordinator, independent of $X$ with Laplace exponent $l(\cdot)$ and triplet $(a, 0, \rho)$. Then $Y_t = X_{\Lambda_t}$ has characteristic function

$$\Psi_Y(u) = \exp(tl(\psi_X(u))) = \exp(tl(-\frac{1}{2} u' Au)), \quad u \in \mathbb{R}^d$$

and the Lévy triplet $(\beta_Y, A_Y, \nu_Y)$ given by

$$\beta_Y = \int_{(0, \infty)} \rho(ds) \int_{|x| \leq 1} x P_X(dx)$$

$$A_Y = aA$$

$$\nu_Y(B) = \int_{(0, \infty)} P_X(B) \rho(ds), \quad B \in \mathcal{B} (\mathbb{R}^d)$$

where $P_X$ is the probability distribution of the Brownian motion.

Example 2.2.8. Let $\Lambda$ be a subordinator, $\Lambda_0 = 0$ and let $W$ be a standard Brownian motion with respect to its natural filtration, independent of the subordinator. Let $\mathcal{G}_t = \sigma \{ \Lambda_s, W_{\Lambda_s}, s \leq t \}$. We want to show that the Lévy process $X_t = \mu \Lambda_t + \sigma W_{\Lambda_t}$ $($$\mu, \sigma \in \mathbb{R}$$)$ is a time-changed Brownian motion with respect to $\mathcal{G}_t$, for all $t \geq 0$. We find the characteristic function of $X$,

$$E[e^{iuX_t}] = E[E[e^{iuX_t} | \mathcal{F}^\Lambda_T]] = E[E[e^{iu(\mu \Lambda_t + \sigma W_{\Lambda_t})} | \mathcal{F}^\Lambda_T]]$$

$$= E[e^{iu\mu \Lambda_t - \frac{1}{2} u^2 \sigma^2 \Lambda_t}] = e^{t(l(iu\mu - \frac{1}{2} u^2 \sigma^2))}$$

where $\mathcal{F}^\Lambda$ is the $\sigma$-algebra generated by $\Lambda$ and $T$ the finite time horizon. The Lévy triplet $(\beta_X, A_X, \nu_X)$ of $X$, is given by

$$\beta_X = a \mu + \int_{(0, \infty)} \rho(ds) \int_{|x| \leq 1} x P_W(dx)$$

$$A_X = a \sigma^2$$

$$\nu_X(B) = \int_{(0, \infty)} P_W(B) \rho(ds), \quad B \in \mathcal{B} (\mathbb{R}^d)$$

where $P_W$ is the probability distribution of the Brownian motion, with drift $\mu$ and volatility $\sigma^2$. Hence $X_t$ is a time-changed Brownian motion with drift $\mu$ and volatility $\sigma^2$ with respect to $\mathcal{G}_t$, for all $t \geq 0$. 
2.2. TIME CHANGED PROCESS

2.2.2 Absolutely continuous time change

The second type of time change we want to look at is the absolutely continuous time change. Let \( \Lambda_t = \int_0^t \lambda_s ds \) be a stochastic process on \( \mathbb{R} \), where \( \lambda \) is a positive and integrable process. Note that \( \lambda \) may exhibit jumps.

The stochastic process \( Y_t = X_{\Lambda_t} \), where \( X \) is a Lévy process on \( \mathbb{R}^d \) with characteristic function \( \Psi_X \) and \( \Lambda \) absolutely continuous time change, has for all \( t \geq 0, u \in \mathbb{R}^d \) the characteristic function

\[
\Psi_{Y_t}(u) = E[\exp(iu'X_{\Lambda_t})] = E[E[\exp(iu'X_s)|\Lambda_t = s]]
\]

If we assume independence of \( X \) and \( \Lambda \), we obtain the characteristic function

\[
\Psi_{Y_t}(u) = E[\exp(-\Lambda_t \psi_X(u))]
\]

Then we can write it in terms of the Laplace transform of \( \Lambda \)

\[
\Psi_{Y_t}(u) = \mathcal{L}_{\Lambda_t}(\psi_X(u))
\]

where \( \mathcal{L}_{\Lambda_t}(u) := E[\exp(-u \int_0^t \lambda_s ds)] \).

Remark. The two classes of time change, subordinators and absolutely continuous time-change, are two different classes, but they intersect. Let \( \Lambda \) be the subordinator from Example 2.2.5, i.e. \( \Lambda_t = at \). Then the subordinator is also a absolutely continuous process, as we can write

\[
\Lambda_t = at = \int_0^t \lambda_s ds
\]

On the other hand, let \( \Lambda \) be a absolutely continuous time-change, and suppose \( \lambda \) is constant, then we can write \( \Lambda \) as in Example 2.2.5.

The advantage of time change processes that are absolutely continuous is that we can determine the characteristic function and the Laplace function. In view of the paper by Carr and Wu (2004), we know that absolutely continuous time-change lead to Affine models, and these models are tractable. There are several ways to specify the affine function of \( \Lambda \) and the corresponding Laplace function, see Table 2 in [CW04].

The Affine model is a class of processes \( X \) where, \( X \) is affine with respect to a Markovian process \( Y \), in the sense that we can write

\[
X_t = A - B'Y_t
\]
where \( A, B \) are deterministic functions. The Markovian process \( Y \) is said to be of affine term structure.

Recall that a Markov process \( Y = \{ Y_t, t \geq 0 \} \) on \( \mathbb{R}^d \), is a stochastic process with respect to \( \mathcal{F} \) the \( \sigma \)-algebra generated by \( Y \), such that, for \( 0 \leq s \leq t, \)

\[
E[f(Y_t)|\mathcal{F}_s] = E[f(Y_t)|Y_s]
\]

where \( f \) is a bounded Borel function from \( \mathbb{R}^d \) to \( \mathbb{R} \).

In finance affine models are commonly used to relate zero-coupon bond prices to a spot rate model. That is, the zero-coupon bond prices process \( P \) is affine with respect to a spot rate model. For example the Vasicek model

\[
dr = (b - ar)dt + \sigma dW
\]

is a spot rate model with affine term structure.

We will give a summary of the Affine activity rate model.

Let \( \Lambda_t = \int_0^t \lambda_s ds \) and let \( \lambda \) be a function of a Markov process \( Z = \{ Z_t, t \geq 0 \} \) on \( \mathbb{R}^d \). The dynamics of \( Z \) is given by

\[
dZ_t = \mu(Z_t)dt + \sigma(Z_t)dW_t, \quad Z_0 = z_0
\]

where \( W \) is a \( d \)-dimensional Brownian motion and \( \mu \) the drift vector and \( \sigma \) is the diffusion matrix, both \( d \)-dimensional. All technical conditions are assumed to hold.

The Laplace function of \( \Lambda \) is an exponential-affine function of \( Z \) if

\[
L_{\Lambda_t}(u) := E[exp(-u\Lambda_t)] = exp(b'_t z_0 + c_t), \quad b \in \mathbb{R}^d, c \in \mathbb{R} \quad (2.9)
\]

Moreover if \( \lambda, \mu(Z) \) and \( \sigma(Z)\sigma(Z)' \) are all affine with respect to \( Z \), then we obtain that \( L_{\Lambda} \) is exponential-affine in \( z_0 \). That is, if we can write the processes in the following form,

\[
\begin{align*}
\lambda_t &= b'_\lambda Z_t + c_{\lambda} \\
\mu(Z_t) &= a - \kappa Z_t \\
[\sigma(Z_t)\sigma(Z_t)']_{ij} &= \begin{cases} 
\alpha_i + \beta_i Z_t, & \text{if } i = j \\
0, & \text{if } i \neq j
\end{cases}
\end{align*}
\quad (2.10)
\]

where \( c_{\lambda}, \alpha \in \mathbb{R}, b_{\lambda}, a, \beta \in \mathbb{R}^d \) and \( \kappa \in \mathbb{R}^d \times \mathbb{R}^d \), then (2.9) is satisfied.

Moreover, \( b \) and \( c \) from (2.9) is given by the following ordinary differential equations

\[
\begin{align*}
b'_t &= ub_{\lambda} - \kappa' b_t - \frac{\beta b_t^2}{2}, \quad b_0 = 0 \\
c'_t &= uc_{\lambda} - b'_t a - \frac{b_t \alpha b_t}{2}, \quad c_0 = 0
\end{align*}
\]
The affine process is tractable, in that the coefficients $b$, $c$ are known explicitly by the ordinary differential equation. Hence, the Laplace function and thus the characteristic function of the time-changed Lévy process remains completely determine.
2.3 Time changed Brownian motion

Now we narrow the results to the time changed Brownian motions. That is, we consider the Lévy process $X$ of the previous sections to be a Brownian motion. Hence $X_{\Lambda_t} = W_{\Lambda_t} =: B_t$ for all $t \geq 0$.

In this thesis we will consider the case where $d = 1$. We also assume for convenience that $\Lambda_0 = 0$. Let $T$ be the finite time horizon.

We need to introduce the filtration we will work under.

Let $F^\Lambda_t = \sigma\{\Lambda_t, t \in [0, T]\}$ (2.11) be the $\sigma$-algebra generated by $\Lambda_t, t \in [0, T]$ and

$$F_t = \sigma\{B_s, s \leq t\} \vee \mathcal{N}$$

the filtration generated by the time-changed Brownian motion $B$ at time $t$.

Let $\mathbb{F} = \{F_t, t \in [0, T]\}$. Then we define $\mathbb{H} = \{\mathcal{H}_t, t \in [0, T]\}$ by

$$\mathcal{H}_t := F_t \vee F^\Lambda_t \vee \mathcal{N}$$

and $\mathbb{G} = \{G_t, t \in [0, T]\}$ by

$$G_t := F_t \vee F^\Lambda_T \vee \mathcal{N}$$

Here $\mathbb{H}$ is the filtration generated by the noise with initial knowledge of the time-change process and $\mathbb{G}$ is generated by the noise and the whole time-change process. Moreover, we see that

$$G_T = \mathcal{H}_T = F_T, \quad G_0 = F^\Lambda_T$$

while $F_0$ and $\mathcal{H}_0$ are trivial.

By Theorem 2.1.9 in [App09] we conclude the following result

**Lemma 2.3.1.** The filtration $\mathbb{G}$ is right-continuous.

So we can say that the filtration $\mathbb{G}$ satisfies the usual conditions, it is complete (i.e. contains all the P-zero sets of $\mathbb{F}$) and right-continuous.

We can in a way regard the information flow $\mathbb{F}$ as partial with respect to the information flow $\mathbb{G}$, even though the filtration is of perfect information of the time-changed Brownian motion. We have that the filtration $\mathbb{G}$ includes information about the future since we have knowledge of the whole time change process, so we will see that the market under $\mathbb{G}$ is in some sense complete in Chapter 3. From a financial point of view, filtration $\mathbb{F}$ will be more interesting for the our study. However it can be difficult to find the local risk minimizing strategy under $\mathbb{F}$ as we will see in Chapter 3, therefore we will also consider the strategy under filtration $\mathbb{H}$. 
Remark. Note that if a process is $\mathcal{F}$-adapted it is also $\mathcal{H}$- and $\mathcal{G}$-adapted. Moreover, if a process is $\mathcal{H}$-adapted it is $\mathcal{G}$-adapted.

For simplicity we will denote $\mathbb{K}$ to be the filtration for results where we can apply both $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{G}$.

Assume now that $\Lambda = \{\Lambda_t, t \in [0,T]\}$ is either a subordinator or an absolutely continuous time-change. Denote $\Lambda_\Theta$, $\Theta$ a Borel set on $[0,T]$, to be a positive random measure generated by $\Lambda_{(s,t]}(\omega) := \Lambda_t(\omega) - \Lambda_s(\omega)$, $\omega \in \Omega$ (see e.g. [KF61] for more information on measures). Moreover, we assume $E[\Lambda_T] < \infty$.

Now we introduce the noise driving the stochastic dynamics in the market model we shall consider.

**Definition 2.3.2.** Let $B$ be a signed random measure on the Borel sets of $[0,T]$, such that

(i) $P(B(\Theta) \leq x|\mathcal{F}_T^\Lambda) = P(B(\Theta) \leq x|\Lambda_\Theta) = \Phi\left(\frac{x}{\sqrt{\Lambda_\Theta}}\right)$, $x \in \mathbb{R}$, $\Theta \subseteq [0,T]$

(ii) $B(\Theta_1)$ and $B(\Theta_2)$ are conditionally independent given $\mathcal{F}_T^\Lambda$ for any disjoint sets $\Theta_1, \Theta_2 \subseteq [0,T]$.

Here $\Phi$ stands for the cumulative probability distribution function of a standard normal variable and "conditional independent given $\mathcal{F}_T^\Lambda$" means that we have a.s.

$$P(B(\Theta_1) \in \Gamma_1, B(\Theta_2) \in \Gamma_2|\mathcal{F}_T^\Lambda) = P(B(\Theta_1) \in \Gamma_1|\mathcal{F}_T^\Lambda)P(B(\Theta_2) \in \Gamma_2|\mathcal{F}_T^\Lambda)$$

for $\Gamma_1, \Gamma_2$, Borel sets on $[0,T]$ and $\Theta_1, \Theta_2 \subseteq [0,T]$ such that $\Theta_1 \cap \Theta_2 = \emptyset$.

We have that the random measure $B$ is a Gaussian random measure conditional on $\Lambda$. Moreover, $B$ is related to the time-change Brownian motion. That is, given $\Lambda$, the characteristic function of $B(\Theta)$ for any $\Theta \subseteq [0,T]$ is given by

$$E[\exp(iuB(\Theta))|\mathcal{F}_T^\Lambda] = \exp\left(\frac{1}{2}u^2\Lambda_\Theta\right), \quad u \in \mathbb{R} \quad (2.12)$$

The following result is from Theorem 3.1 [Ser72].

**Theorem 2.3.3.** Let $W = \{W_t, t \in [0,T]\}$ be a Brownian motion on $(\Omega, \mathcal{F}, P)$ independent of $\Lambda$. Then $B$ satisfies (i)-(ii) in Definition 2.3.2 and (2.12) if and only if, for any $t \in [0,T]$, $B_t := B((0,t])$, is such that

$$B_t \overset{d}{=} W_{\Lambda_t} \quad (2.13)$$
Proposition 2.3.4. \( B \) is a martingale with respect to
(i) \( G \), (ii) \( F \), and (iii) \( H \)

Proof. (i) For all \( s \leq t \),
\[
E[B_t - B_s | G_s] = E[B((s,t)) | G_s] = E[B((s,t)) | F_s \vee \mathcal{F}^A_T] \overset{(\star)}{=} E[B((s,t)) | \mathcal{F}^A_T] \overset{(**)}{=} 0
\]

Where we used:
(**): from condition (i) in Definition 2.3.2, we have
\[
E[B((s,t)) | \mathcal{F}^A_T] = 0 \quad \text{for all} \quad 0 \leq s \leq t
\]

(*) : from conditional independent (ii) in Definition 2.3.2 we have, for
\( F \in F_s \), \( G \in F_{\Lambda T} \),
\[
E[1_{A} | F_s \vee \mathcal{F}^A_T]] = E[1_{A} | \mathcal{F}^A_T] = E[1_{A} | G] = E[B((s,t)) | \mathcal{F}^A_T]
\]
By a monotone class argument, this extends to
\[
E[1_{A} E[B((s,t)) | \mathcal{F}^A_T]] = E[1_{A} E[B((s,t)) | \mathcal{F}^A_T]]
\]
for \( A \in G_s = F_s \vee F^A_T \).

(ii) By the tower property and the fact that \( B_s \) is \( F_s \)-measurable, we have for all \( s \leq t \),
\[
E[B_t | F_s] = E[E[B_t | G_s] | F_s] = E[B_s | F_s] = B_s
\]

(iii) The proof is similarly to (ii). \( B \) is \( H \)-adapted, so by the tower property and (i) the result follows.

Proposition 2.3.5. Let \( X \) be a continuous stochastic process defined on the probability space \( (\Omega, \mathcal{F}, P) \), with \( X_0 = 0 \). Suppose that
(i) \( X \) is a \( (G, P) \)-martingale,
(ii) \( \langle X \rangle_t = \Lambda_t \), for all \( t \in [0, T] \).
Then there is a Brownian motion \( W \) with respect to \( \mathcal{F}^W_t \), independent of \( \Lambda \), such that for each \( t \in [0, T] \), \( X_t = W_{\Lambda_t} \). Hence \( X \) is a time-changed Brownian motion with respect to \( G \).
2.3. TIME CHANGED BROWNIAN MOTION

Proof. By the martingale representation theorem (see Theorem 3.5 in [DS14]) there exists a \( \phi \) such that

\[
X_t = \int_0^t \phi_s dB'_s
\]

where \( B' \) is a time-changed \((G, P)\)-Brownian motion. By assumption

\[
\langle X \rangle_t = \Lambda_t
\]

Apply Itô’s formula for continuous martingales to \( f(x) = e^{iux}, \ u \in \mathbb{R} \),

\[
d(e^{iux_t}) = iue^{iux_t} \phi_t dB'_t - \frac{1}{2} u^2 e^{iux_t} d\Lambda_t
\]

Fix \( 0 \leq \theta \leq t \),

\[
e^{iux_t} - e^{iux_\theta} = \int_\theta^t iue^{iux_s} \phi_s dB'_s - \frac{1}{2} \int_\theta^t u^2 e^{iux_s} d\Lambda_s
\]

\[
e^{iux_t - x_\theta} - 1 = \int_\theta^t iue^{iux_s - x_\theta} \phi_s dB'_s - \frac{1}{2} \int_\theta^t u^2 e^{iux_s - x_\theta} d\Lambda_s
\]

Then taking conditional expectation given \( \mathcal{F}^\Lambda_T \), we obtain

\[
E[e^{iux_t - x_\theta} | \mathcal{F}^\Lambda_T] - 1 = E[\int_\theta^t iue^{iux_s - x_\theta} \phi_s dB'_s | \mathcal{F}^\Lambda_T] - \frac{1}{2} u^2 E[\int_\theta^t e^{iux_s - x_\theta} d\Lambda_s | \mathcal{F}^\Lambda_T]
\]

By the tower property,

\[
E[\int_\theta^t iue^{iux_s - x_\theta} \phi_s dB'_s | \mathcal{F}^\Lambda_T] = E[E[\int_\theta^t iue^{iux_s - x_\theta} \phi_s dB'_s | \mathcal{G}_\theta] | \mathcal{F}^\Lambda_T] = 0
\]

Clearly \( \Lambda \) is \( \mathcal{F}^\Lambda_T \)-measurable, so

\[
E[e^{iux_t - x_\theta} | \mathcal{F}^\Lambda_T] = 1 - \frac{1}{2} u^2 \int_\theta^t E[e^{iux_s - x_\theta} | \mathcal{F}^\Lambda_T] d\Lambda_s
\]

Let

\[
Z_t = E[e^{iux_t - x_\theta} | \mathcal{F}^\Lambda_T], \quad t \in [0, T]
\]

so we have

\[
dZ_t = -\frac{1}{2} u^2 Z_t d\Lambda_t, \quad Z_\theta = 1
\]

Hence by Itô’s formula for continuous processes, we have, for all \( 0 \leq \theta \leq t \),

\[
Z_t = \exp(-\frac{1}{2} u^2 (\Lambda_t - \Lambda_\theta))
\]
This is the characteristic function of the conditional normal distribution given $\mathcal{F}_T^\Lambda$, thus $X$ is normal distributed with mean 0 and variance $\Lambda$ given $\mathcal{F}_T^\Lambda$. This means that there exists a standard Brownian motion $W$, s.t. $W_{\Lambda t} = X_t$. Furthermore, let $Q$ be the law of $X$ and let $B$ with law $P$ be given by Definition 2.3.2. Since $X$ and $B$ has the same characteristic function, we get that $P = Q$ (see e.g. Theorem 9.5.1 in [Dud02]). Hence $X$ is conditional independent given $\mathcal{F}_T^\Lambda$.

So $X$ satisfies both conditions (i)-(ii) in Definition 2.3.2 and equation (2.12), so by Theorem 2.3.3 we get the result.
2.4 The market model

Let $(\Omega, F, P)$ be a complete probability space as before. The market model is given by the riskless asset

$$dS_t^{(0)} = r_t S_t^{(0)} dt, \quad S_0^{(0)} = 1$$

and one risky asset, with dynamics given by

$$dS_t^{(1)} = \alpha_t S_t^{(1)} dt + \sigma_t S_t^{(1)} dB_t, \quad S_0^{(1)} = x > 1 \quad (2.14)$$

where $B$ is a time-changed Brownian motion from (2.13), i.e. $B_t = W_{\Lambda_t}$.

Market models driven by time-changed Brownian motions are considered a useful tool to describe the stochastic volatility in the market. A class of popular stochastic volatility models driven by time-changed Brownian motions can be found in [BNS02], [Hes93], [HW87] and [SS91]. In common these volatility models have the following dynamics of the price process

$$dS_t = \mu_t S_t dt + \sigma_t \lambda_t S_t dW_t^{(1)}$$
$$d\lambda_t = \alpha_t \lambda_t dt + \beta_t dW_t^{(2)}$$

where $W^{(1)}$, $W^{(2)}$ are Brownian motions. In the case where $W^{(1)}$ and $W^{(2)}$ are independent, the process $B_t = \int_0^t \lambda_s dW_s^{(1)}$ is a conditional Brownian motion as in Definition 2.3.2, and our framework can be applied.

To ensure the existence of a strong solution and to allow further analysis, we assume for the market model that the processes $\alpha, \sigma$ and $r$ are càglàd $F$-adapted and stochastic, $\sigma_t \neq 0$ $P \times dt - a.e.$ and

$$E \left[ \int_0^T (|\alpha_t - r_t| dt + \sigma_t^2 d\langle B \rangle_t^K) \right] < \infty$$

Here $K$ represents any filtration $F$, $H$, $G$. From (2.14) we see that $S^{(1)}$ is a $K$-semimartingale. Note that the processes $\alpha, \sigma$ and $r$ are also $H$- and $G$-adapted.

Here we set $\langle B \rangle^K$ to be the predictable quadratic variation of the $K$-martingale $B$ with respect to the filtration $K$ (see Appendix A.3). Note that

$$\langle B \rangle^G = \langle B \rangle^H = \Lambda_t$$

Here we have used Doob-Meyer Theorem, that is $B_t^2 - \langle B \rangle^K$ is a unique $K$-martingale. While $\langle B \rangle^F$ is more difficult to calculate, since $\Lambda$ is not a $F$-adapted, and hence $B_t^2 - \Lambda_t$ is in general not a $F$-martingale. We will however see in Section 3.1.2, that we can write $\langle B \rangle^F$ as a projection of $\Lambda$. 
From now on we will use discounted prices. That is, we use the riskless asset as numéraire, so the discounted riskless asset has price 1 at all times and the discounted risky asset is given by $S = \frac{s_t}{s_0}$. So the dynamics of the $\mathbb{K}$-semimartingale $S$ is given by

$$dS_t = (\alpha_t - r_t)S_t dt + \sigma_t S_t dB_t, \quad S_0 = x \tag{2.15}$$

**Proposition 2.4.1.** The solution to (2.15) with respect to $\mathbb{K}$, is given by

(i) $S_t = x \exp\left( \int_0^t (\alpha_s - r_s) ds - \int_0^t \frac{1}{2} \sigma_s^2 d\langle B_s \rangle \mathbb{K} + \int_0^t \sigma_s dB_s \right)$

when $\Lambda$ is a absolutely continuous time change.

(ii) $S_t = xe^{Y_t}$

where

$$dY_t = \left\{ (\alpha_t - r_t) + \sigma_t \tilde{\beta}_Y - \frac{1}{2} \sigma_t^2 A_Y \right\} dt + \sigma_t A_Y dW_t + \int_{|x| \geq 1} \log(1 + \sigma_t x) \tilde{N}(dt, dx), \quad Y_0 = 1$$

when $\Lambda$ is a subordinator.

**Proof.** (i) Let $\Lambda$ be a absolutely continuous time change, then $S$ is a continuous $\mathbb{K}$-semimartingale. So we can apply Itô’s formula\(^1\) for continuous semimartingales, and we get the result.

(ii) Let $\Lambda$ be a subordinator. From Example 2.2.7 we know that $B$ has the Lévy triplet $(\beta_Y, A_Y, \mu_Y)$ given by (2.6) and by the Lévy-Itô decomposition we have

$$dB_t = \beta_Y dt + A_Y dW_t + \int_{|x| \geq 1} xN(dt, dx) + \int_{|x| < 1} x\tilde{N}(dt, dx)$$

where $W = \{W_t, t \in [0, T]\}$ is a standard Brownian motion.

The dynamics of $B$ can be rewritten in a simpler form, in the case when $B$ is square integrable (see p.133 Appelbaum (2004)). We have

$$E[B_t^2] = E[E[B_t^2|\mathcal{F}_T]] = E[\Lambda_t] < \infty$$

So

$$dB_t = \tilde{\beta}_Y dt + A_Y dW_t + \int_{|x| \geq 1} xN(dt, dx)$$

\(^1\)See Appendix A.5
where
\[
\tilde{\beta}_Y = \beta_Y - \int_{|x|<1} x \mu_Y(dx)
\]
Let
\[
dS_t = (\alpha_t - r_t)S_t dt + \sigma_t S_t dB_t = S_t dX_t
\]
where
\[
dX_t = (\alpha_t - r_t)dt + \sigma_t dB_t
\]
Assume \(\inf\{\Delta X_t, t \in [0, T]\} > -1\) a.s. Then by Proposition A.5.4 where
\[
G_t = (\alpha_t - r_t) + \tilde{\beta}_Y
\]
\[
F_t = \sigma_t A_Y
\]
\[
H(t, x) = \sigma_t x
\]
we get that the solution of the \(\mathbb{K}\)-semimartingale \(S\) is the Doléans-Dade exponential of \(X, \mathcal{E}_X\) (see Appendix A.5.3). \qed
Chapter 3

The Local Risk Minimizing strategy

In an incomplete market perfect hedging is in general not possible. Local risk-minimizing (LRM) is a way to treat the problem of hedging in such markets.

The main features of the LRM strategy is finding a portfolio \( \varphi \) that is not self-financing such that the discounted value process at time of maturity \( T \) is equal to the discounted contingent claim\(^1\), i.e. \( V_T(\varphi) = F \). This we can do by allowing a small cost process. Then the local risk minimization aims to minimize such cost.

This measure of riskiness by a quadratic criterion of a strategy was first described by Föllmer and Sonderman (1986) in the case where the discounted price process is a martingale, and later extended by Schweizer (1988) to the case of general semimartingales.

Denote \( \varphi = (\eta, \xi) \) to be the portfolio, where \( \eta \) is the number of units in the riskless asset and \( \xi \) is the number of units in the risky asset.

Define the stochastic process \( V_t \) to be the discounted value of the portfolio at time \( t \), given by

\[
V_t(\varphi) = \eta_t + \xi_t S_t, \quad t \in [0, T]
\]

(3.1)

Risk Minimizing

If the discounted asset \( S \) is a local \( \mathbb{K} \)-martingale, we want to find an admissible hedging strategy for the contingent claim \( F \).

\(^1\)In this market we consider a contingent claim whose payoff is given by an \( \mathcal{F}_T \)-measurable random variable \( F \) such that \( E[|F|^2] < \infty \) (or equivalent \( F \in L^2(\Omega, \mathcal{F}_T, P) \)).
The portfolio $\varphi$ is a $\mathcal{K}$-trading strategy if $\xi$ is a $\mathcal{K}$-predictable processes such that $\xi \in L^2_{\mathcal{K}}(S)^2$ and $\eta$ is a $\mathcal{K}$-adapted process such that the value process $V(\varphi)$ is right-continuous and $E[V(\varphi)^2] < \infty$.

**Definition 3.0.1.** Define the cost process by

$$C_t(\varphi) := V_t(\varphi) - \int_0^t \xi_u dS_u \quad (3.2)$$

such that it is right-continuous and square-integrable.

**Remark.** The term $\int_0^t \xi_u dS_u$ is called gain process.

Thus the cost process represents the difference between the value process of an $\mathcal{K}$-trading strategy and the self-financing evaluation of the gain process.

A $\mathcal{K}$-trading strategy $\varphi = (\eta, \xi)$ is called $\mathcal{K}$-self-financing if the cost process is constant over time. It is called $\mathcal{K}$-mean self-financing if the cost process is a $\mathcal{K}$-martingale under $P$.

Since we want to minimize the cost process, we define the $\mathcal{K}$-risk process by

$$R_t(\varphi) := E[(C_T(\varphi) - C_t(\varphi))^2|\mathcal{K}_t]$$

The idea behind risk minimizing is to look among all trading strategies with $V_T(\varphi) = F$ for the one which minimizes the risk process.

**Definition 3.0.2.** An $\mathcal{K}$-trading strategy $\varphi$ is called $\mathcal{K}$-risk-minimizing if for any $\mathcal{K}$-trading strategy $\tilde{\varphi}$ such that $V_T(\tilde{\varphi}) = V_T(\varphi)$ $P$-as, we have

$$R_t(\varphi) \leq R_t(\tilde{\varphi}) \quad P\text{-as for every } t \in [0, T]$$

---

\(^2\)The space $L^2_{\mathcal{K}}(S)$ denotes all $\mathcal{K}$-predictable processes $\xi$ such that $(E[\int_0^T \xi_u^2 d\langle S \rangle_u])^{1/2} < \infty$. 
3.1 The Local Risk Minimizing problem

In the case where the discounted price process $S$ is not a local $\mathbb{K}$-martingale, but a $\mathbb{K}$-semimartingale, Schweizer (1988) has proved that the contingent claim $F$ does not admits a risk-minimizing hedging strategy. Therefore Schweizer extended the theory to LRM strategies.

Before introducing the formal definition of local risk minimizing under both the filtration $\mathbb{G}$ and $\mathbb{F}$, we introduce some useful concepts where we can apply all three filtrations $\mathbb{G}$, $\mathbb{H}$ and $\mathbb{F}$.

**Assumption.** We will now assume that the time-change process $\Lambda$ is absolutely continuous, i.e. $\Lambda_t = \int_0^t \lambda_s ds$. Hence the discounted price process $S$ is continuous (recall Proposition 2.4.1 (i)).

From the semimartingale decomposition of Definition A.3.6, we consider the two processes $A^K = \{A_t, t \in [0, T]\}$ and $M^K = \{M_t, t \in [0, T]\}$, where $A^K$ is $\mathbb{K}$-predictable of finite variation with $A^K_0 = 0$, and $M^K$ is a locally square integrable local $\mathbb{K}$-martingale, with $M^K_0 = 0$, such that $S$ can be written as

$$S = S_0 + M^K + A^K \quad (3.3)$$

We say that the semimartingale $S$ satisfies the **structure conditions** under the filtration $\mathbb{K}$ (for short $(\mathbf{SC})^K$) if there exists a $\mathbb{K}$-predictable process $\theta = \{\theta_t, t \in [0, T]\}$ such that

$$dA^K_t = \theta_t d\langle M^K \rangle_t \quad \text{and} \quad \int_0^T \theta^2_t d\langle M^K \rangle_t < \infty \quad \text{P-a.s.}$$

We define the mean-variance tradeoff process by

$$K_t = \int_0^t \theta^2_s d\langle M^K \rangle_s$$

We need to define the space of $\mathbb{K}$-predictable processes $\phi$ such that

$$\Theta(\mathbb{K}) = \{\phi \text{ $\mathbb{K}$-predictable} : E[\int_0^T \phi^2_t d\langle M^K \rangle_t + (\int_0^T |\phi_u| dA_t)^2] < \infty\}$$

where the processes $M$ and $A$ are the decomposition of the $\mathbb{K}$-semimartingale $S$. When $\xi \in \Theta(\mathbb{K})$, the stochastic integrals in (3.2) is well defined.
3.1. THE LOCAL RISK MINIMIZING PROBLEM

3.1.1 The LRM under $\mathcal{G}$, via MMM

Consider the filtration $\mathcal{G}$, i.e. full information of the time change process. In this case the discounted price process $S$ is a continuous $\mathcal{G}$-semimartingale. The main material and the general definitions in this section are taken from [FS91] and [Sch99] if not otherwise specified.

Recall the discounted price process, given by (2.15), that is

$$S_t = S_0 + \int_0^t S_s (\alpha_s - r_s) ds + \int_0^t S_s \sigma_s dB_s$$

Moreover, $S$ admits the $\mathcal{G}$-semimartingale decomposition (3.3), i.e.,

$$S = S_0 + M^{\mathcal{G},\mathbb{P}} + A^{\mathcal{G},\mathbb{P}}$$

By comparing the two equations above, we obtain

$$A^{\mathcal{G},\mathbb{P}}_t := \int_0^t S_s (\alpha_s - r_s) ds$$
$$M^{\mathcal{G},\mathbb{P}}_t := \int_0^t S_s \sigma_s dB_s$$

(3.4)

Then

$$d\langle M^{\mathcal{G},\mathbb{P}} \rangle_t = (\sigma_t S_t)^2 d\langle B \rangle_t = \sigma_t^2 S_t^2 d\Lambda_t$$

For simplicity we denote quadratic variation by $\langle \rangle = \langle \rangle^\mathcal{G}$ and also $A^{\mathcal{G},\mathbb{P}} = A^\mathbb{P}, M^{\mathcal{G},\mathbb{P}} = M^\mathbb{P}$ for the rest of this section.

If $S$ satisfies the (SC)$^\mathcal{G}$, there exists a $\mathcal{G}$-predictable process $\theta$ such that

$$(\alpha_t - r_t)S_t dt = \sigma_t^2 S_t^2 \theta_t d\Lambda_t$$

Recall the previous assumption on the market model, $\sigma_t \neq 0$ $P \times dt$ - a.e., the solution of $S$ given by Proposition 2.4.1 (i), is nonnegative, so $S_t > 0$ for all $t \in [0, T]$ and $\Lambda_t = \int_0^t \lambda_s ds > 0$ for $t \in (0, T]$, $\Lambda_0 = 0$. Then we obtain, for $t \in [0, T]$

$$\theta_t = \frac{\alpha_t - r_t}{\sigma_t^2 S_t \lambda_t}$$

(3.5)

with $\theta_0 = 0$ and

$$\int_0^T (\theta_t \sigma_t S_t)^2 d\Lambda_t = \int_0^T \frac{(\alpha_t - r_t)^2}{\sigma_t^2 \lambda_t} dt < \infty \text{ P-a.s.}$$

**Notation.** Let $X$ be a $\mathbb{K}$-semimartingale and $H$ a bounded $\mathbb{K}$-predictable process, then we denote the Itô-type the stochastic process $H \cdot X$ by

$$H \cdot X = (\int_0^t H_s dX_s)_{t \in [0, T]}$$
Definition 3.1.1. The portfolio \( \varphi = (\eta, \xi) \) is a \( \mathbb{G} \)-trading strategy if \( \xi \) is a \( \mathbb{G} \)-predictable process, \( \xi \in \Theta(\mathbb{G}) \) and \( \eta \) is \( \mathbb{G} \)-adapted such that \( V = \xi S + \eta \) has right-continuous paths and \( E[V_t(\varphi)^2] < \infty \) for every \( t \in [0, T] \).

A \( \mathbb{G} \)-trading strategy \( \varphi \) is called \( \mathbb{G} \)-local risk-minimizing if the remaining risk \( R_t(\varphi) \) is minimal under all infinitesimal perturbations of the strategy at time \( t \).

Definition 3.1.2. A trading strategy \( \Delta = (\delta, \epsilon) \) is called a small perturbation if it satisfies the following conditions

(i) \( \delta \) is bounded
(ii) \( \int_0^T |\delta_s dA_s^P| \) is bounded
(iii) \( \delta_T = \epsilon_T = 0 \)

and for any \( (s, t] \subset [0, T] \),

\[ \Delta_{(s,t]} := (\delta_{(s,t]}, \epsilon_{(s,t)}) \]

Definition 3.1.3. Let \( \varphi \) be a trading strategy, \( \Delta \) a small perturbation and \( \tau \) a partition of \( [0, T] \), and define

\[ r^\tau(\varphi, \Delta) := \sum_{t_i, t_i+1 \in \tau} \frac{R_{t_i}(\varphi + \Delta_{(t_i,t_{i+1})}) - R_{t_i}(\varphi)}{E[(M^P)_{t_{i+1}} - (M^P)_{t_i} | \mathcal{G}_t]} 1_{(t_i,t_{i+1})}(t) \]

Then \( \varphi \) is called locally risk-minimizing if

\[ \liminf_{n \to \infty} r^{\tau_n}(\varphi, \Delta) \geq 0 \]

\( P \)-a.e. on \( \Omega \times [0, T] \), for every \( \Delta \) as defined above and \( \lim_{n \to \infty} \tau_n = [0, T] \).

The above definition can be shown to be equivalent to the following properties of the associated cost process (see [FS91]).

Recall, if the cost process is a \((\mathbb{G}, P)\)-martingale, then the \( \mathbb{G} \)-trading strategy \( \varphi \) is mean-self-financing.

Theorem 3.1.4. Assume \( S \) satisfies the \((SC)^{\mathbb{G}} \) and \( E[K_T] < \infty \). Let \( F \) be a contingent claim in \( L^2(\Omega, \mathcal{F}_T, P) \) and let \( \varphi \) be a \( \mathbb{G} \)-trading strategy. Then \( \varphi \) is a \( \mathbb{G} \)-locally risk-minimizing strategy if and only if \( \varphi \) is mean-self-financing and the martingale \( C(\varphi) \) is orthogonal to \( M^P \).

Definition 3.1.5. The \( \mathbb{G} \)-trading strategy \( \varphi \) is called a \( \mathbb{G} \)-optimal strategy if \( \varphi \) is mean-self-financing and the martingale \( C(\varphi) \) is orthogonal to \( M^P \).
To find a solution to the LRM problem, Föllmer and Schweizer introduced in [FS91] the Föllmer-Schweizer (FS) decomposition.

**Proposition 3.1.6.** Let $F \in L^2(\Omega, \mathcal{F}_T, P)$, then $F$ admits a $\mathcal{G}$-optimal strategy $\varphi$, with $V_T(\varphi) = F$ if and only if $F$ admits the following decomposition,

$$ F = F_0 + \int_0^T \xi_t^{FS} dS_t + L_t^{FS}, \text{ P-a.s.} \quad (3.6) $$

where $F_0 \in L^2(\Omega, \mathcal{F}_T^\Lambda, P)$, $\xi_t^{FS} \in \Theta(\mathcal{G})$ and $L_t^{FS}$ is a $\mathcal{G}$-martingale orthogonal to $M_P$, with $L_0 = 0$.

The strategy $\varphi = (\eta, \xi)$ is given by

$$ \xi = \xi_t^{FS}, \quad C_t(\varphi) = F_0 + L_t^{FS} $$

with

$$ V_t(\varphi) = F_0 + \int_0^t \xi_i^{FS} dS_i + L_t^{FS} $$

The decomposition (3.6) is called the **FS decomposition of F**

Since the $\mathcal{G}$-semimartingale $S$ is continuous, the Föllmer-Schweizer decomposition of $F$ under $P$ can be obtained as the Galtchouk-Kunita-Watanabe (GKW) decomposition (see Appendix A.6) under the minimal martingale measure (MMM). This comes from the fact that the minimal martingale measure "preserves orthogonality" when $S$ is continuous. By preserving orthogonality we mean that given any square integrable martingale under $P$ orthogonal to the martingale part of $S$, $M_P$, it is still a martingale under MMM, orthogonal to $S$.

**Remark.** If the semimartingale $S$ is discontinuous the only way to obtain the LRM strategy is by calculating directly the FS decomposition as in Theorem 3.5.4 in [Van10].

**Minimal Martingale Measure**

We extend the theory on MMM from Chapter 3 in [FS91], to our framework. That is, to construct a MMM $Q$, for a given continuous market model with a discounted price, we introduce the Radon-Nikodym density process $Z$ that describes a change of measure from $P$ to an equivalent martingale measure $Q$. Such change of measure applied to Lévy noises is structure preserving (e.g. see Di Nunno and Karlsen [DK16]). Then we will show that for the
equivalent martingale measure $Q$ with Radon-Nikodym density process $Z$, there exists a $\mathbb{G}$-predictable process $\theta$ such that

$$dA^P = \theta d\langle M^P \rangle$$

Finally we find an explicit solution to the density process $Z$, such that the associated equivalent martingale measure $Q$ becomes minimal.

Recall that for an equivalent martingale measure $Q$ with respect to $P$, we can define the process $Z = \{Z_t, t \in [0, T]\}$ by

$$Z_T = \frac{dQ}{dP}$$

and, for all $t \in [0, T],$

$$Z_t = E[\frac{dQ}{dP} | \mathcal{G}_t]$$

which is positive $(\mathbb{G}, P)$-martingale and it is the so-called Radon-Nikodym density process.

**Theorem 3.1.7 (Girsanov Theorem).** Let $S$ be a $\mathbb{G}$-semimartingale under $P$ with decomposition $S = S_0 + M^P + A^P$, and let $B$ be the time-changed $(\mathbb{G}, P)$-Brownian motion. Let $Z = \{Z_t, t \in [0, T]\}$ be a positive $(\mathbb{G}, P)$-martingale with $E[Z_T^2] < \infty$, and define the probability measure $Q$ by

$$\frac{dQ}{dP} = Z_T$$

Define the processes

$$M^Q_t := M^P_t - \int_0^t \frac{1}{Z_s} d\langle Z, M^P \rangle_s$$

$$B^Q_t := B_t - \int_0^t \frac{1}{Z_s} d\langle Z, B \rangle_s$$

Then $M^Q_t$ is a continuous $(\mathbb{G}, Q)$-martingale, and $B^Q_t$ is a continuous $(\mathbb{G}, Q)$-martingale and a time-changed $(\mathbb{G}, Q)$-Brownian motion.

**Proof.** We know that $M^P$ and $Z$ are $(\mathbb{G}, P)$-martingales. By integration by part we have, for all $t \in [0, T],$

$$d(M^Q_t Z_t) = M^Q_t dZ_t + Z_t dM^Q_t + dM^Q_t dZ_t$$

$$= M^Q_t dZ_t + Z_t dM^P_t$$
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So $M^Q Z$ is a $(\mathcal{G}, P)$-martingale, hence $M^Q$ is a $(\mathcal{G}, Q)$-martingale. In the same way we can show that $B^Q$ is a $(\mathcal{G}, Q)$-martingale. We have, for all $t \in [0, T]$,
\[
d(Z_t B^Q_t) = B^Q_t dZ_t + Z_t dB_t
\]
Hence $B^Q$ is a continuous $(\mathcal{G}, Q)$-martingale. Also
\[
\langle B^Q \rangle_t = \langle B \rangle_t = \Lambda_t, \quad \text{for all } t \in [0, T]
\]
Hence since $B^Q$ is a continuous $(\mathcal{G}, Q)$-martingale and since $\langle B^Q \rangle = \Lambda$, by Proposition 2.3.5, $B^Q$ is a time-changed $(\mathcal{G}, Q)$-Brownian motion.

In what follows we want to determined the Radon-Nikodym density process $Z$ such that $Q$ is a minimal martingale measure.

By Girsanov’s theorem, we have, for all $t \in [0, T]$,
\[
S_t = S_0 + M^P_t + A^P_t = S_0 + M^Q_t + \int_0^t \frac{1}{Z_s} d\langle Z, M^P \rangle_s + A^P_t
\]
Since $Q$ ia an equivalent martingale measure, $S$ is a $(\mathcal{G}, Q)$-martingale, hence
\[
A^P_t = -\int_0^t \frac{1}{Z_s} d\langle Z, M^P \rangle_s
\]
From Kunita-Watanabe inequality (see Proposition 1.1.10 [Pha09]) we obtain that $\langle Z, M^P \rangle \ll \langle M^P \rangle$. That is, for a bounded process $\alpha = \{\alpha_t, t \in [0, T]\}$,
\[
\int_0^t \alpha_s d\langle Z, M^P \rangle_s \leq \sqrt{\langle Z \rangle_t} \int_0^t \alpha_s d\langle M^P \rangle_s
\]
so if $\int_0^t \alpha_s d\langle M^P \rangle_s = 0$, then $\int_0^t \alpha_s d\langle Z, M^P \rangle_s = 0$, for all $t \in [0, T]$.
Hence by Radon-Nikodym theorem, there exists a $\mathcal{G}$-predictable process $\beta = \{\beta_t, t \in [0, T]\}$ such that
\[
d\langle Z, M^P \rangle = \beta d\langle M^P \rangle
\]
Moreover, let the $\mathcal{G}$-predictable process $\theta = \{\theta_t, t \in [0, T]\}$, be given by
\[
\theta = -\frac{\beta}{Z}
\]
such that
\[
dA^P = \theta d\langle M^P \rangle
\] (3.7)
Remark. Let the $\mathcal{G}$-predictable process $\phi \in \Theta(\mathcal{G})$, then note that

$$E[\int_0^T \phi_t^2 d\langle M^P \rangle] \leq E[\int_0^T \phi_t^2 d\langle M^P \rangle + (\int_0^T |\phi_t| dA_t)^2] < \infty$$

Moreover, $E[\int_0^T \phi_t^2 d\langle M^P \rangle] < \infty$ implies

$$\int_0^T \phi_t^2 d\langle M^P \rangle < \infty, \text{ P-a.s.}$$

Now suppose the predictable process in (3.7) is in the space $\Theta(\mathcal{G})$, which implies

$$\int_0^T \theta_t^2 d\langle M^P \rangle < \infty \text{ P-a.s.}$$

Then $S$ satisfies the $(\text{SC})^G$.

**Definition 3.1.8.** Suppose $S$ satisfies $(\text{SC})^G$, and let $Q$ be the equivalent martingale measure, that is $\frac{dQ}{dP} \in L^2(\Omega, \mathcal{F}_T, P)$. Then $Q$ is called minimal if

(i) $Q = P$ on $\mathcal{G}_0$, where $\mathcal{G}_0 = \mathcal{F}_T^\Lambda$

(ii) for any square integrable $(\mathcal{G}, P)$-martingale $L$, with $L_0 = 0$, orthogonal to $M^P$, $L$ is still a martingale under $Q$.

Moreover also orthogonality is preserved in this case, namely any square integrable $(\mathcal{G}, P)$-martingale, orthogonal to $M^P$ under $P$ is also orthogonal to $S$ under the minimal martingale measure (see e.g. p.22 [Van10]).

**Theorem 3.1.9.** Suppose $Q$ is a minimal martingale measure. Then $Q$ exists if and only if

$$Z_t = \exp(\int_0^t \theta_s dM^P_s - \int_0^t \theta_t^2 d\langle M^P \rangle_s)$$

(3.8)

is a square-integrable martingale under $P$ (in that case, $Q$ is given by $\frac{dQ}{dP} = Z_T$), where the $\mathcal{G}$-predictable process $\theta$ is given by (3.5).

Moreover, $Q$ preserves orthogonality, i.e. any martingale under $P$ orthogonal to $M^P$ is also a martingale under $Q$ orthogonal to $S$.

**Proof.** Suppose $Q$ exists, and let $S = S_0 + M^P + A^P$ be the Doob Meyer decomposition under $(\mathcal{G}, P)$. Let $\tilde{Z}$ be a square-integrable $(\mathcal{G}, P)$-martingale associated to the equivalent martingale measure, then by the GKW decomposition under $P$,

$$\tilde{Z}_t = \tilde{Z}_0 + \int_0^t \beta_s dM^P_s + L_t$$
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where $L$ is a square-integrable martingale under $P$ orthogonal to $M^P$, $L_0 = 0$ and $\beta \in \Theta(G)$. This gives

$$\int_0^t \frac{1}{Z_s} d\langle \tilde{Z}, M^P \rangle_s = \int_0^t \frac{1}{Z_s} \beta_s d\langle M^P \rangle_s$$

By Girsanov’s Theorem, the process

$$M^P_t - \int_0^t \frac{1}{Z_s} d\langle \tilde{Z}, M^P \rangle_s = M^P_t - \int_0^t \frac{1}{Z_s} \beta_s d\langle M^P \rangle_s$$

is a $(\mathcal{G}, Q)$-martingale. On the other hand, we know that $S$ is a martingale under $Q$. Define

$$\theta = \frac{1}{Z} \beta$$

Since $\tilde{Z} > 0$ and $\beta \in \Theta(G)$, we have

$$E[\int_0^T (\frac{1}{Z_t} \beta_t)^2 d\langle M^P \rangle_t] \leq E[\int_0^T \beta_t^2 d\langle M^P \rangle_t] < \infty$$

which implies

$$\int_0^T \theta_s^2 d\langle M^P \rangle_s < \infty, \quad P\text{-a.s.}$$

From the definition of the MMM, condition (i) gives that $\tilde{Z}_0 = 1$ and condition (ii) says that $L$ is a martingale under $Q$, so $\langle L, \tilde{Z} \rangle = 0$ and hence $L \equiv 0$. Thus

$$\tilde{Z}_t = 1 + \int_0^t \tilde{Z}_s \theta_s dM^P_s$$

Hence $\tilde{Z} = Z$.

Now assume $Z$ is given by (3.8) and $Z_T = \frac{dQ}{dP}$. We want to show that $Q$ is minimal. Let $L$ be a square-integrable $(\mathcal{G}, P)$-martingale with $\langle L, M^P \rangle = 0$ under $P$. We then get $\langle L, Z \rangle = 0$, since for all $t \in [0, T]$

$$\langle L, Z \rangle_t = \int_0^t Z_s \theta_s d\langle L, M^P \rangle$$

Thus $L$ is a local martingale under $Q$. Moreover, since

$$\sup_{0 \leq t \leq T} |L_t| \in L^2(\Omega, \mathcal{F}, P)$$
and $Z_t \in L^2(\Omega, \mathcal{F}, P)$, we have

$$\sup_{0 \leq t \leq T} |L_t| \in L^1(\Omega, \mathcal{F}, Q)$$

So $L$ is a local $Q$-martingale and $E_Q[\sup_{0 \leq t \leq T} |L_t|] < \infty$ for every $t \in [0, T]$, hence $L$ is a martingale under $Q$.

Now we want to show that $L$ also satisfies $\langle L, S \rangle = 0$ under $Q$. Since $S$ and $A^P$ are continuous, we have

$$\langle L, S \rangle = [L, S] = [L, M^P] + [L, A] = [L, M^P]$$

under $Q$. But $M^P$ is continuous, so we have

$$[L, M^P] = \langle L, M^P \rangle = 0$$

under $P$. This implies that $[L, M^P] = 0$ also under $Q$ (see the proof of Theorem 1 in [FS91]).

Let us return to the problem of computing the $\mathbb{G}$-optimal strategy (see Definition 3.1.5).

Define the process $\hat{V} = \{\hat{V}_t, t \in [0, T]\}$ by

$$\hat{V}_t := E_Q[F|\mathcal{G}_t], \ t \in [0, T]$$

(3.9)

where the notation $E_Q[\cdot|\mathcal{G}_t]$ denotes the conditional expectation with respect to $\mathcal{G}_t$ under $Q$. Since $F \in L^2(\Omega, \mathcal{F}_T, P)$ and $\frac{dQ}{dP} \in L^2(\Omega, \mathcal{F}_T, P)$, we obtain that $F \in L^1(\Omega, \mathcal{F}_T, Q)$, thus the process $V^Q$ is well-defined.

The process $\hat{V}$ admits the following GKW decomposition under $S$ and $Q$,

$$\hat{V}_t = \hat{V}_0 + \int_0^t \hat{\xi}_u dS_u + \hat{L}_t, \ \text{Q-a.s.}$$

(3.10)

where $\hat{V}_0 \in L^2(\Omega, \mathcal{F}_0^\wedge, Q)$, $\hat{\xi} \in L^2_{\mathbb{G}}(S)$ and $\hat{L}$ is a $(\mathbb{G}, Q)$-martingale orthogonal to $S$ under $Q$ with $\hat{L}_0 = 0$.

**Theorem 3.1.10.** Suppose $S$ satisfies (SC)$^\mathbb{G}$. Let $\hat{V}$ be given by (3.9) and let $Q$ be the MMM. If

(i) $F$ admits the FS decomposition, given by (3.6)

or
(ii) \( \hat{\xi} \in \Theta(G) \) and \( \hat{L} \) is a \((G,P)\)-martingale orthogonal to \( M^P \) under \( P \), then (3.10), for \( t=T \), gives the FS decomposition of \( F \) and \( \hat{\xi} \) determines the \( G \)-optimal strategy for \( F \).

**Proof.** Assume (i). Since the \((G,P)\)-martingale \( L^{FS} \) of the FS decomposition is orthogonal to \( M^P \) under \( P \), and \( Q \) is the MMM, \( L^{FS} \) is a \((G,Q)\)-martingale orthogonal to \( S \) under \( Q \). Then by projection of \( G_t \) under \( Q \)

\[
E_Q[F|G_t] = F_0 + \int_0^t \xi_u^{FS} dS_u + L_t^{FS}, \quad Q\text{-a.s.} \tag{3.11}
\]

Hence

\[
\hat{V}_t = F_0 + \int_0^t \xi_u^{FS} dS_u + L_t^{FS}, \quad P\text{-a.s.} \tag{3.12}
\]

By the uniqueness of the GKW decomposition (3.12) coincides with (3.10). Now, assuming (ii). Then clearly (3.10) is given for \( P\text{-a.s.} \), and for \( t = T \), it admits the FS decomposition of \( F \).

Hence, for both (i) and (ii), by Proposition 3.1.6 we get the \( G \)-optimal strategy \( \varphi = (\eta, \xi) \), given by

\[
\xi = \tilde{\xi}, \quad \eta_t = V_t - \tilde{\xi}_t S_t
\]

\[\square\]

In a summary, we have shown that finding the LRM strategy is deduce to finding the GKW decomposition of the claim \( F \) under the MMM \( Q \). This is very useful since the density process \( Z \) of \( Q \) with respect to \( P \), given by (3.8), can be written explicitly in terms of the dynamics of the price process \( S \), given that \((SC)^G\) is satisfied.

Moreover, by the unique solution of the portfolio we can obtain the cost process (recall Definition 3.0.1). That is, the associated cost process, for all \( t \in [0,T] \), is given by

\[
C_t = V_t(\varphi) - \int_0^t \xi_s dS_s = V_0 + L_t^{FS}
\]

where \( V_0 \) is a \( \mathcal{F}_T^A \)-measurable random variable and \( L \) is a \( G \)-martingale, both orthogonal to \( M^P \).
Now we can see that the local risk minimizing strategy with respect to the filtration $\mathcal{G}$, is "in a sense complete".

To see this, let use first consider the case when we are in a complete market and $S$ is a $\mathcal{G}$-martingale, then the claim $F$ admits an Itô representation

$$F = F_0 + \int_0^T \xi_s^F S dS_s$$

where $F_0$ is $\mathcal{G}_0$-measurable orthogonal to the integrand and $\xi^F \in L^2_{\mathcal{G}}(S)$. Then the strategy is given by

$$\xi = \xi^F, \quad \eta = V - \xi \cdot S, \quad V_t = F_0 + \int_0^t \xi_s^F dS_s$$

Moreover, the strategy is self-financing, i.e. the cost process is known from the start since it is $\mathcal{G}_0$-measurable,

$$C_t = C_T = F_0$$

Thus, this representation leads to a strategy which produces the claim $H$ from the initial amount $C_0 = F_0$.

Now back to the LRM with respect to $\mathcal{G}$. Recall that $\mathcal{G}_0 = \mathcal{F}^\Lambda_T$ is the initial information. We have

$$F = F_0 + \int_0^T \xi_s^F S dS_s + L^F_T$$

where $F_0$ is a $\mathcal{G}_0(= \mathcal{F}^\Lambda_T)$-measurable random variable and $L^F$ a $\mathcal{G}$-martingale orthogonal to $S$. The strategy is

$$\xi = \xi^F, \quad \eta = V - \xi \cdot S$$

with

$$V_t = F_0 + \int_0^t \xi_s^F S dS_s + L^F_t$$

$$C_t = F_0 + L^F_t$$

On the other hand, this means that

$$\eta_t = F_0 + L_t$$

Hence in reality since $\eta$ is the asset of the riskless asset, this means that the martingale $L^F$ vanishes. So this gives

$$C_t = C_T = F_0$$
where $F_0$ is a $\mathcal{G}_0 (= \mathcal{F}^1_T)$-measurable random variable. But $F_0$ is a known process at time $t = 0$, since we have knowledge of the whole time change process. Hence, the strategy is in a sense self-financing, that is, the cost processes is a know process, hence the market is in a sense complete under the filtration $\mathcal{G}$.

**Explicit solution**

In a special case where we can write the value process of a portfolio $\varphi$ in the form

$$V_t(\varphi) = f(t, S_t) =: V_t^*$$

for a function $f(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$, we can find explicit solution for $V_t^*$ and the solution of the associated portfolio $\varphi = (\eta, \xi)$.

**Proposition 3.1.11.** Let $F$ be the contingent claim, $F \in L^2(\Omega, \mathcal{F}_T, P)$. Then the local risk minimizing strategy $\varphi = (\eta, \xi)$ is given by

$$\begin{align*}
\eta_t &= f(t, S_t) - \xi_t \cdot S_t \\
\xi_t &= \frac{\partial f}{\partial x}(t, S_t) + \zeta_t
\end{align*}$$

where

$$L = -\zeta \cdot S$$

**Proof.** Since $f(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$ we can use Itô’s formula

$$V_t^* = V^*_0 + \int_0^t \frac{\partial f}{\partial t}(u, S_u)du + \int_0^t \frac{\partial f}{\partial x}(u, S_u)dS_u + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(u, S_u)S^2_u\sigma^2_u\lambda_u du$$

Moreover, $V^*$ is a $(\mathcal{G}, Q)$-martingale, so we can deduce that $f$ satisfies a partial differential equation (PDE) and

$$V_t^* = V_0^* + \int_0^t \frac{\partial f}{\partial x}(u, S_u)dS_u$$

Hence from the GKW decomposition of $V^*$ under $Q$, we get

$$\begin{align*}
\xi_t &= \frac{\partial f}{\partial x}(t, S_t) + \zeta_t \\
L &= -\zeta \cdot S
\end{align*}$$

Using Itô’s formula, we deduce that the function $f(t, x)$ is the solution to the following PDE

$$\begin{align*}
\frac{\partial f}{\partial t}(t, S_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t)S^2_t\sigma^2_t\lambda_t &= 0 \\
f(T, S_T) &= F
\end{align*} \quad (3.13)$$
Example 3.1.12. If we have a European put option, with stick price $K$ and payoff function $F = (K - S_T)^+$, then

$$f(t, x) = E_Q[(K - x \frac{S_T}{S_t})^+]$$
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3.1.2 The LRM under $\mathbb{F}$, via MMM

We consider the local risk minimizing strategy in the case of $\mathbb{F}$, i.e. the filtration generated by the time-changed Brownian motion. We will use a partial information approach. This approach is discussed by the authors in \cite{CCR14a} and \cite{CCR14b}.

The main idea in the papers by Ceci et al. (2014), is that for the two filtrations $\mathbb{H}$ and $\mathbb{F}$, $\mathcal{F}_t \subseteq \mathcal{H}_t$, for all $t \in [0,T]$, we can for the $\mathbb{H}$-martingales $M$ and $L$ and for $F \in L^2(\Omega, \mathcal{H}_T, \mathbb{P})$, find a GKW decomposition of $F$, i.e.

$$F = F_0 + \int_0^T \xi_s dM_s + L_T, \quad \mathbb{P} \text{-a.s.}$$

where $\xi$ is a $\mathbb{F}$-predictable process.

This is achieved by projecting the results obtained for the $\mathbb{H}$-predictable case onto the $\mathbb{F}$-predictable one, using the so-called predictable dual projection. Moreover, we will also see that the quadratic variation $\langle B \rangle^\mathbb{F}$ can be obtained by projection $\Lambda$.

Under partial information, given the portfolio $\varphi = (\eta, \xi)$, we have that $\xi$ is $\mathbb{F}$-adapted and $\eta$ is $\mathbb{H}$-adapted. This means that the investors can only access the information flow $\mathbb{F}$ about trading in the risky asset. From this, the technical definitions on the portfolio $\varphi = (\eta, \xi)$ such as trading and optimal strategies, will depend on both the two filtrations $\mathbb{H}$ and $\mathbb{F}$.

For example we can apply partial information when working with incomplete information credit risk models where investors may have a delayed observation of the process driving the default risk. Then $\mathcal{F}_t = \mathcal{H}_{(t-\tau)^+}$ where $\tau \in (0,T)$ is a fixed delay.

In the papers \cite{CCR14a} and \cite{CCR14b} the authors aim at providing the existence and uniqueness of the BSDEs under partial information, while in this section, we will apply this theory to find the LRM strategy with a measure change. That is, under the MMM with respect to $\mathbb{H}$, we will find a GKW decomposition under partial information of the claim $F$.

We recall that the contingent claim $F \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Under the filtration $\mathbb{F}$ the claim $F$ only admits the FS decomposition and the GKW decomposition under partial information, therefore we will see that the FS decomposition under $\mathbb{P}$ is not equivalent to the GKW decomposition under a MMM. We will still provide the GKW decomposition under partial information under the MMM $\hat{\mathbb{Q}}$, and will therefore need to assume that the claim $F \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \cap L^2(\Omega, \mathcal{F}_T, \hat{\mathbb{Q}})$.

We need to clarify that, even though, we apply the partial information approach, our model is completely different than in the papers \cite{CCR14a}.
In these papers the authors define a subfiltration $\mathcal{F}$ of the filtration $\mathcal{H}$ of full information. That is, there is an filtration of partial information $\mathcal{F}$ with respect to the filtration $\mathcal{H}$. While in our setting, the filtration $\mathcal{F}$ may be regarded as a subfiltration of $\mathcal{H}$, but it is not partial, since it is of perfect information of the time-changed Brownian motion.

The reason we want to apply the partial information approach to our case, is not that we have partial information, but rather that we have different measurability properties on the market model under the two different filtrations $\mathcal{F}$ and $\mathcal{H}$.

We have to remark that in the papers [CCR14a] and [CCR14b], the claim is $\mathcal{H}_T$-measurable. Recall in our case we have that the claim $\mathcal{F}$ is $\mathcal{F}_T = \mathcal{H}_T = \mathcal{G}_T$-measurable.

Before proceeding with the partial information approach and the corresponding local risk minimizing strategy, we recall some useful properties and define the decompositions of the semimartingale $S$ with respect to $\mathcal{F}$ and $\mathcal{H}$.

Recall that in our work, we have the information flow $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{G}$, given by, respectively,

$$
\mathcal{F}_t = \sigma\{B_s, s \leq t\} \lor \mathcal{N}, \quad t \in [0, T]
$$

$$
\mathcal{H}_t = \mathcal{F}_t \lor \mathcal{F}_t^\Lambda \lor \mathcal{N}, \quad t \in [0, T]
$$

$$
\mathcal{G}_t = \mathcal{F}_t \lor \mathcal{F}_T^\Lambda \lor \mathcal{N}, \quad t \in [0, T]
$$

Clearly, we have

$$
\mathcal{F}_t \subseteq \mathcal{H}_t \subseteq \mathcal{G}_t, \quad t \in [0, T]
$$

Note that $\mathcal{H}_0$ and $\mathcal{F}_0$ are both trivial, while $\mathcal{G}_0 = \mathcal{F}_T^\Lambda$.

The reason we define the LRM under $\mathcal{H}$, is that we can adopt useful properties from $\mathcal{G}$, by double expectation. Moreover, the time change process $\Lambda$ is $\mathcal{H}$-adapted.

Recall, since $B$ is a conditional Gaussian random process, by (2.12), we have

$$
E[exp(\int_t^T iudB_s)|\mathcal{F}_T^\Lambda] = exp(\int_t^T \frac{1}{2} u^2 d\Lambda_s)
$$
Now since $B$ is $\mathbb{H}$-adapted, then $B$ is also $\mathbb{G}$-adapted and

$$E[\exp(\int_t^T i u dB_s)|\mathcal{H}_t] = E[E[\exp(\int_t^T i u dB_s)|\mathcal{G}_t]|\mathcal{H}_t]$$
$$= E[E[\exp(\int_t^T i u dB_s)|\mathcal{F}_T^\Lambda]|\mathcal{H}_t]$$
$$= E[\exp(\int_t^T \frac{1}{2} u^2 d\Lambda_s)|\mathcal{H}_t]$$

(3.14)

Recall the discounted stock price $S$ given by (2.15), where the coefficient $\alpha$, $r$ and $\sigma$ are $\mathbb{F}$-adapted processes. Moreover, since the coefficient are $\mathbb{F}$-adapted, they are also $\mathbb{H}$-adapted.

Furthermore, we recall that the stock price process $S$ is both a $\mathbb{H}$- and $\mathbb{F}$-semimartingale.

By (3.3), the $\mathbb{H}$-semimartingale admits the following decomposition,

$$S = S_0 + N + R$$

(3.15)

where $N = \{N_t, t \in [0, T]\}$ is a $\mathbb{H}$-martingale, $N_0 = 0$ and $R = \{R_t, t \in [0, T]\}$ is a $\mathbb{H}$-adapted process of finite variation, with $R_0 = 0$. Comparing the $\mathbb{H}$-semimartingale to the dynamics given by (2.15), we obtain $N_t = \int_0^t S_s \sigma_s dB_s$ and $R_t = \int_0^t S_s (\alpha_s - r_s) ds$.

We recall the space $\Theta(\mathbb{H})$, i.e.

$$\Theta(\mathbb{H}) = \{ \phi \text{ $\mathbb{H}$-predictable} : E\left[ \int_0^T \phi_u^2 d\langle N \rangle_u^H + \left( \int_0^T |\phi_u| d\langle R \rangle_u^H \right)^2 \right] < \infty \}$$

Suppose the $\mathbb{H}$-semimartingale $S$ satisfies the structure condition (SC)$^H$, then there exists a $\mathbb{H}$-predictable process $\theta^H$ such that $\int_0^T \theta^2_t d(\langle N \rangle)_t < \infty$ and

$$S_t = S_0 + N_t + \int_0^t \theta^H_s d\langle N \rangle^H_s$$

(3.16)

The $\mathbb{F}$-semimartingale $S$ admits a decomposition

$$S = S_0 + M + A$$

where $M = \{M_t, t \in [0, T]\}$ is a $\mathbb{F}$-martingale and $A = \{A_t, t \in [0, T]\}$ is a $\mathbb{F}$-adapted process of finite variation, with $M_0 = 0, A_0 = 0$. Again,
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comparing to (2.15), we obtain $M_t = \int_0^t S_s \sigma_s dB_s$ and $A_t = \int_0^t S_s (\alpha_s - r_s) ds$. Correspondingly

$$\Theta(F) = \{ \phi \ F\text{-predictable : } E[ \int_0^T \phi_u^2 d\langle M \rangle_u^F + (\int_0^T |\phi_u| dA_u)^2 ] < \infty \}$$

Note that, even though any $F$-adapted process is also $\mathbb{H}$-adapted, we have

$$\langle M \rangle^F \neq \langle N \rangle^H$$

since $\langle B \rangle^F \neq \langle B \rangle^\mathbb{H} = \Lambda$.

This means that we do not have in general $\Theta(F) \subset \Theta(\mathbb{H})$. Instead, we define the space

$$\Theta(F, \mathbb{H}) = \{ \phi \in \Theta(\mathbb{H}) : \phi \text{ is } F\text{-predictable } \}$$

Clearly, $\Theta(F, \mathbb{H}) \subset \Theta(\mathbb{H})$.

Moreover, we define the space $L^2_{F, \mathbb{H}}(N) := \{ \phi \in L^2_{\mathbb{H}}(N) : \phi \text{ } F\text{-predictable} \}$.

The following definitions are from the paper [CCR14b].

We need a new definition on orthogonality, so that we can define the FS decomposition under partial information with respect to $F$.

**Definition 3.1.13.** Two square integrable $\mathbb{H}$-martingales $X = \{X_t, t \in [0, T]\}$ and $Y = \{Y_t, t \in [0, T]\}$ are weakly $F$-orthogonal if, for all $F$-predictable processes $\phi = \{\phi_t, t \in [0, T]\}$ such that $\phi \in L^2_{F, \mathbb{H}}(Y)$, we have

$$E[X_T \int_0^T \phi_t dY_t] = 0$$

(3.17)

**Remark.** The $\mathbb{H}$-orthogonal condition between two $\mathbb{H}$-martingales implies weak $F$-orthogonality. Let $X$ and $Y$ be square integrable $\mathbb{H}$-orthogonal $\mathbb{H}$-martingales, i.e. $\langle X, Y \rangle^\mathbb{H} = 0$. Set $Z = \int_0^t \phi_t dY_t$, where $\phi$ is a $F$-predictable process, then for $t \in [0, T],$

$$\langle X, Z \rangle^\mathbb{H}_t = \int_0^t \phi_s d\langle X, Y \rangle^\mathbb{H}_s = 0$$

This implies that $XZ$ is an $\mathbb{H}$-martingale null at zero, hence for all $t \in [0, T],$

$$E[X_t Z_t] = 0$$

Note that the converse is not true, i.e. weak $F$-orthogonal does not imply $\mathbb{H}$-orthogonal.
3.1. THE LOCAL RISK MINIMIZING PROBLEM

Without further mention, we let S be the $\mathbb{H}$-semimartingale for this section, given by (3.15).

Now we define the FS decomposition under partial information with respect to the filtration $\mathbb{F}$. We will call it the **FS decomposition under $\Theta(\mathbb{F}, \mathbb{H})$**.

**Definition 3.1.14.** Let $F \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. $F$ admits the FS decomposition under $\theta(\mathbb{F}, \mathbb{H})$, if there exists a random variable $F_0 \in L^2(\Omega, \mathcal{H}_0, \mathbb{P})$, $\xi^F \in \Theta(\mathbb{F}, \mathbb{H})$ and $L$ is a $\mathbb{H}$-martingale weakly $\mathbb{F}$-orthogonal to $N$ with $L_0 = 0$, such that

$$F = F_0 + \int_0^T \xi^F_t dS_t + L_T, \quad P\text{-a.s.} \quad (3.18)$$

We adapt Definition 3.1.1 and Definition 3.1.5 on trading and optimal strategy with respect to $\mathbb{G}$, respectively, to trading and optimal strategy under partial information with respect to $\mathbb{F}$.

**Definition 3.1.15.** The portfolio $\varphi = (\eta, \xi)$ is a $\mathbb{F}$-trading strategy if $\xi$ is a $\mathbb{F}$-predictable process, $\xi \in \Theta(\mathbb{F}, \mathbb{H})$ and $\eta$ is a $\mathbb{H}$-adapted process such that $V(\varphi) := \xi S + \eta$ has right-continuous paths and $V_t(\varphi) \in L^2(\Omega, \mathcal{H}_t, \mathbb{P})$ for each $t \in [0, T]$.

Recall the $\mathbb{F}$-trading strategy $\varphi$ is **mean-self-financing** if the associated cost process $C_t(\varphi)$ is a $(\mathbb{H}, \mathbb{P})$-martingale.

**Proposition 3.1.16.** Let $F$ be a contingent claim in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and let $\varphi$ be a $\mathbb{F}$-trading strategy. Then $\varphi$ is a local risk-minimizing strategy if and only if $\varphi$ is mean-self-financing and the $\mathbb{H}$-martingale $C(\varphi)$ is weakly $\mathbb{F}$-orthogonal to $N$.

From the latter we get the following definition.

**Definition 3.1.17.** The $\mathbb{F}$-trading strategy $\varphi$ is called a $\mathbb{F}$-optimal strategy if $\varphi$ is mean-self-financing and the cost process $C(\varphi)$ is a $\mathbb{H}$-martingale weakly $\mathbb{F}$-orthogonal to $N$.

**Proposition 3.1.18.** A contingent claim $F$ admits an $\mathbb{F}$-optimal strategy $\varphi = (\eta, \xi)$ with $V_T(\varphi) = F$ $P$-as if and only if $F$ can be written as the FS decomposition under $\Theta(\mathbb{F}, \mathbb{H})$, given by (3.18). Then $\varphi$ is given by

$$\xi_t = \xi^F_t$$

$$C_t(\varphi) = F_0 + L_t$$
Hence if the FS decomposition under $\Theta(F,H)$, given by (3.18) holds,

$$V_t(\varphi) = C_t(\varphi) + \int_0^t \xi_s dS_s$$

and

$$\eta_t = V_t(\varphi) - \xi_t^F S_t$$

Proof. Suppose the portfolio $\varphi = (\eta, \xi)$ is a $F$-optimal strategy, with $V_T(\varphi) = F$, then

$$F = V_T(\varphi) = C_T(\varphi) + \int_0^T \xi_s dS_s = C_0(\varphi) + \int_0^T \xi_s dS_s + (C_T(\varphi) - C_0(\varphi))$$

(3.19)

Since $\varphi$ is optimal, the cost process is a $H$-martingale weakly $F$-orthogonal to $N$, hence $C_T(\varphi) - C_0(\varphi)$ is a $H$-martingale weakly $F$-orthogonal to $N$, that in addition is null at zero. Hence (3.19) is the FS decomposition under $\Theta(F,H)$ of $F$, with $\xi_t = \xi_t^F$ and $L = C(\varphi) - C_0(\varphi)$.

Now we assume that the FS decomposition under $\Theta(F,H)$, (3.18) holds. We choose

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + L_t - \xi_t^F S_t - \int_0^t \xi_s^F dS_s$$

Then, the portfolio $\varphi = (\eta, \xi)$ is such that the cost process is given by

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \xi_s^F dS_s = F_0 + L_t$$

Notice that $C_T(\varphi) = F_0 + L_T$, hence $C_t(\varphi)$ a $H$-martingale weakly $F$-orthogonal to $N$. This implies that $\varphi$ is $F$-optimal.

Predictable dual projection

We define the useful concept of $F$-predictable dual projection. With the predictable dual projection, we will be able to project the given $H$-predictable process $\xi^H$ onto a $F$-predictable process. So if we know the GKW decomposition of the claim $F$ under the filtration $H$, we can use the predictable dual projection to define the GKW decomposition under partial information with respect to $F$. See chapter 4 in [CCR14a] for more information on predictable dual projection.
3.1. THE LOCAL RISK MINIMIZING PROBLEM

Recall, a càdlàg $H$-adapted process $X = \{X_t, t \in [0,T]\}$ is of integrable variation if for

$$V_t := \sup_{\pi} \sum_{i=0}^{n(\pi)-1} |X_{t_{i+1}} - X_{t_i}|$$

where $\pi = \{t_0, \ldots, t_{n(\pi)}\}$ is a partition of $[0,t]$, we have

$$E[V_T] < \infty$$

**Definition 3.1.19.** Let $X = \{X_t, t \in [0,T]\}$ be a càdlàg $H$-adapted process of integrable variation. Then there exists a unique $\mathbb{F}$-predictable process $X_{p,F} = \{X_{t,F}, t \in [0,T]\}$ of integrable variation, such that

$$E[\int_0^T \varphi_t dX_{t,F}] = E[\int_0^T \varphi_t dX_t]$$

for every $\mathbb{F}$-predictable process $\varphi = \{\varphi_t, t \in [0,T]\}$. The process $X_{p,F}$ is called the $\mathbb{F}$-predictable dual projection of $X$.

The following result is from Proposition 4.3 in [CCR14a].

**Proposition 3.1.20.** The $\mathbb{F}$-predictable dual projection defined above exist and is unique.

By the $\mathbb{F}$-predictable dual projection, we find a representation of the predictable quadratic variation of $B$ with respect to $\mathbb{F}$.

First, we notice by Dubins-Schwarz theorem (Theorem 2.2.2), that every martingale can be written as a time-changed Brownian motion. We know that the $B$ is a $\mathbb{F}$-martingale, so there exists a Brownian motion $\bar{W}$ such that $B_t = \bar{W}_{\langle B \rangle_F}$.

On the other hand, by Serfozo Theorem 2.3.3 $B_t \overset{d}{=} W_{\Lambda_t}$.

By the $\mathbb{F}$-predictable dual projection, we can connect the two above statements.

**Proposition 3.1.21.** Let $\Lambda$ be the time change process and let $\langle B \rangle_F$ be the predictable quadratic variation of $B$ with respect to $\mathbb{F}$. Then $\langle B \rangle_F$ is the $\mathbb{F}$-predictable dual projection of $\Lambda$, i.e.,

$$E[\int_0^T \varphi_t d\langle B \rangle_F] = E[\int_0^T \varphi_t d\Lambda_t]$$

for every $\mathbb{F}$-predictable process $\varphi$. 
Proof. Without loss of generality, we restrict the proof to the case where \( \varphi = 1_{(s,t]}(u)1_C, C \in \mathcal{F}_s \), that is prove the following equality
\[
E[1_C(\Lambda_t - \Lambda_s)] = E[1_C(\langle B \rangle_t^\mathbb{F} - \langle B \rangle_s^\mathbb{F})]
\]
Since \( \Lambda \) is a càdlàg (any continuous process is càdlàg), increasing, integrable \( \mathbb{H} \)-adapted process, by Proposition 3.1.20, there exists a unique increasing, integrable \( \mathbb{F} \)-predictable dual projection process. It remains to show that the unique process is \( \langle B \rangle^\mathbb{F} \).
Let \( \tilde{\Lambda}_t = E[\Lambda_t | \mathcal{F}_t] \), for \( t \in [0, T] \), then
\[
E[1_C(\Lambda_t - \Lambda_s)] = E[E[1_C(\Lambda_t - \Lambda_s)|\mathcal{F}_s]] = E[1_C(E[\Lambda_t|\mathcal{F}_s] - \tilde{\Lambda}_s)]
\]
Moreover, \( \tilde{\Lambda} \) is a \( \mathbb{F} \)-submartingale,
\[
E[\tilde{\Lambda}_t|\mathcal{F}_s] = E[E[\Lambda_t|\mathcal{F}_t]|\mathcal{F}_s] = E[\Lambda_t|\mathcal{F}_s] \geq E[\Lambda_s|\mathcal{F}_s] = \tilde{\Lambda}_s
\]
since \( \Lambda \) is increasing.
Then by the Doob-Meyer Theorem, there exists a unique increasing, integrable \( \mathbb{F} \)-predictable process \( X \) such that \( \tilde{\Lambda} - X \) is a \( \mathbb{F} \)-martingale. Then for \( C \in \mathcal{F}_s \),
\[
E[1_C(\tilde{\Lambda}_t - \tilde{\Lambda}_s)] = E[1_C(X_t - X_s)]
\]
On the other hand,
\[
B_t^2 = E[B_t^2|\mathcal{F}_t] = E[E[B_t^2|\mathcal{F}_s]|\mathcal{F}_t] = E[\Lambda_t|\mathcal{F}_t] = \tilde{\Lambda}_t
\]
So again by Doob-Meyer Theorem, we have that
\[
B_t^2 - \langle B \rangle_t^\mathbb{F}
\]
is a unique \( \mathbb{F} \)-martingale.
Hence, \( X = \langle B \rangle^\mathbb{F} \), and the result follows. \( \square \)

In the sequel we will need the important result, from section 4.2 in [CCR14a].
That is,

**Proposition 3.1.22.** Denote the \( \mathbb{H} \)-adapted process \( O = \{O_t, t \in [0, T]\} \), such that \( O^p, F = \int_0^\cdot \xi_s^\mathbb{H} d\langle N \rangle_s^\mathbb{H}, \xi^\mathbb{H} \in \Theta(\mathbb{H}) \). Then
\[
O^p, F \ll \langle N \rangle^{p, F}
\]
where the notation \( p, F \) is the predictable dual projection.
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Proof. Define the measures \( \mu^F((s,t] \times C) := E[1_C(O_t^{p,F} - O_s^{p,F})] \) for \( C \in \mathcal{F}_s \), where \( O = \int_0^t \xi^H_s d\langle N \rangle_s \), and \( \mu^H((s,t] \times D) := E[1_D((N)_t^{p,F} - (N)_s^{p,F})] \) for \( D \in \mathcal{H}_s \). Here \( \xi^H \) is the integrand in the GKW decomposition of \( F \) with respect to \( \mathbb{H} \).

Then \( \mu^F \ll \mu^H \) on the \( \mathbb{F} \)-predictable \( \sigma \)-field. To see this, let \( \psi \) be a nonnegative \( \mathbb{F} \)-predictable process such that \( E[\int_0^T \psi_t d\langle N \rangle_t] = 0 \). Then \( E[\int_0^T \psi_t d\langle N \rangle_t] = 0 \), and this gives \( \psi = 0 \) \( \mathbb{P} \)-a.s. \( t \in [0,T] \). Hence

\[
E[\int_0^T \psi_t dO_t] = E[\int_0^T \psi_t dO_t] = E[\int_0^T \varphi_t d\xi^H_t d\langle N \rangle_t] = 0
\]

Then it remains to show that \( E[\int_0^T \varphi_t dO_t] = E[\int_0^T \varphi_t d\xi^H_t d\langle N \rangle_t] \) for every \( \mathbb{F} \)-predictable process \( \varphi \). Consider \( \varphi_u = 1_{(s,t]}(u)1_C, C \in \mathcal{F}_s \). Then

\[
E[\int_0^T \varphi_u d\xi^H_t d\langle N \rangle_t] = E[\int_0^t \xi^F_u d\langle N \rangle_u^{p,F}] = \int_s^t \int_C \xi^F_u d\mu^H
\]

\[
= \mu^F((s,t] \times C) = E[1_C(O_t^{p,F} - O_s^{p,F})]
\]

\[
= E[1_C \int_s^t \xi^F_u d\langle N \rangle_u^{p,F}] = E[\int_0^T \varphi_u dO_u]
\]

Galtchouk-Kunita-Watanabe decomposition

We need some preliminary results, to show that the GKW decomposition under partial information exists. So for this subsection we will assume that the \( \mathbb{H} \)-semimartingale \( S \) is a \( \mathbb{H}, \mathbb{P} \)-martingale, i.e. \( S = N \).

Assume the claim \( F \) admits the unique GKW decomposition with respect to \( \mathbb{H} \), i.e.

\[
F = \tilde{F}_0 + \int_0^T \xi^H_t dN_t + \tilde{L}_T, \quad \mathbb{P} \text{-a.s.} \quad (3.21)
\]

where \( \tilde{F}_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}), \xi^H \in L^2(\mathbb{H}, N) \), and \( \tilde{L} \) is a \( \mathbb{H}, \mathbb{P} \)-martingale with \( \tilde{L}_0 = 0 \), \( \mathbb{H} \)-orthogonal to \( N \).

We call the GKW decomposition under partial information with respect to \( \mathbb{F} \), for the GKW decomposition under \( \Theta(\mathbb{F}, \mathbb{H}) \).

**Proposition 3.1.23.** Let \( F \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) and let the decomposition \((3.21)\) of \( F \) be given. Then \( F \) admits a unique GKW decomposition under \( \Theta(\mathbb{F}, \mathbb{H}) \),

\[
F = F'_0 + \int_0^T \xi^F_t dN_t + L'_T, \quad \mathbb{P} \text{-a.s.} \quad (3.22)
\]
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with

\[ F'_0 = E[\tilde{F}_0|\mathcal{F}_0] \]

\[ \xi^F = \frac{dO^{p,F}}{d\langle N \rangle^{p,\mathbb{F}}} \]

where \( O^{p,F} = (\int_0^T \xi^H_t d\langle N \rangle^H_t)^{p,F} \) and the notation \( ^{p,F} \) refers to the \( \mathbb{F} \)-predictable dual projection and \( L' \) is a \((\mathbb{H}, P)\)-martingale \( \mathbb{F} \)-weakly orthogonal to \( N \), \( L'_0 = 0 \). Moreover, \( \xi^F \in L^2_{\mathbb{F}^p}(\mathbb{N}) \).

Proof. Let (3.21) be given. We take the conditional expectation of \( F \) with respect to \( \mathcal{F}_0 \),

\[ E[F|\mathcal{F}_0] = E[\tilde{F}_0 + \int_0^T \xi^H_t dN_t + \tilde{L}_T|\mathcal{F}_0] \]

\[ = E[E[\tilde{F}_0 + \int_0^T \xi^H_t dN_t|\mathcal{H}_0]|\mathcal{F}_0] + E[E[\tilde{L}_T|\mathcal{H}_0]|\mathcal{F}_0] \]

\[ = E[\tilde{F}_0|\mathcal{F}_0] + E[\tilde{L}_0|\mathcal{F}_0] = E[\tilde{F}_0|\mathcal{F}_0] \]

where we used the tower property and that \( (\tilde{F}_0 + \int_0^T \xi^H_t dN_t) \) and \( \tilde{L} \) are \((\mathbb{H}, P)\)-martingales.

Let \( F'_0 = E[\tilde{F}_0|\mathcal{F}_0] \). Clearly, \( \tilde{F}_0 - F'_0 \in L^2(\Omega, \mathcal{H}_0, P) \).

We need to show

\[ E[(\int_0^T (\xi^H_t - \xi^F_t) dN_t)(\int_0^T \varphi_t dN_t)] = 0 \]

respectively,

\[ E[(\int_0^T \xi^H_t dN_t)(\int_0^T \varphi_t dN_t)] = E[(\int_0^T \xi^F_t dN_t)(\int_0^T \varphi_t dN_t)] \]

which is equivalent to the following,

\[ E[\int_0^T \xi^H_t \varphi_t d\langle N \rangle^H_t] = E[\int_0^T \xi^F_t \varphi_t d\langle N \rangle^H_t] \tag{3.23} \]

We have the equality in (3.23) if,

\[ \xi^F = \frac{dO^{p,F}}{d\langle N \rangle^{p,\mathbb{F}}} \]

That is, \( \xi^F \) is the \( \mathbb{F} \)-predictable Radon-Nikodym derivative of the process \( O^{p,F} \) with respect to \( \langle N \rangle^{p,\mathbb{F}} \), recall Proposition 3.1.22.
Moreover, since $\tilde{L}$ is $\mathbb{H}$-orthogonal to $N$, it is also weakly $\mathbb{F}$-orthogonal to $N$. Then we can define the $(\mathbb{H}, P)$-martingale $\tilde{L}'$ weakly $\mathbb{F}$-orthogonal to $N$ by,

$$L_t' = \tilde{F}_0 - F_0' + \int_0^T (\xi_t^H - \xi_t^F) dN_t + \tilde{L}_t$$

Furthermore, we have $\xi^F \in L^2_{\mathbb{H}}(N)$. To see this let $\varphi = \xi^F$ in (3.23), so we get

$$E\left[\int_0^T (\xi_t^F)^2 d\langle N \rangle_t^H\right] = E\left[\int_0^T \xi_t^F \xi_t^H d\langle N \rangle_t^H\right]$$

Then by Cauchy-Schwarz inequality we obtain

$$E\left[\int_0^T (\xi_t^F)^2 d\langle N \rangle_t^H\right] \leq (E\left[\int_0^T (\xi_t^F)^2 d\langle N \rangle_t^H\right])^{\frac{1}{2}} (E\left[\int_0^T (\xi_t^H)^2 d\langle N \rangle_t^H\right])^{\frac{1}{2}}$$

Hence, since $\xi^H \in L^2_{\mathbb{H}}(N)$, we get

$$E\left[\int_0^T (\xi_t^F)^2 d\langle N \rangle_t^H\right] \leq E\left[\int_0^T (\xi_t^H)^2 d\langle N \rangle_t^H\right] < \infty$$

\[\square\]

**Remark.**

$$F_0' = E[F_0|\mathcal{F}_0] = E[E[F|\mathcal{H}_0]|\mathcal{F}_0] = E[F|\mathcal{F}_0]$$

**Minimal martingale measure**

Recall the minimal martingale measure $Q$ from Section 3.1. The importance of the measure $Q$, is that it preserves orthogonality, that is, a square-integrable martingale under $P$ orthogonal to the martingale part of $S$, is still a square-integrable martingale under $Q$. This gives that the GKW decomposition under $Q$ of the claim $F$ with respect to $\mathcal{G}$, is equivalent to the FS decomposition of $F$ under $P$.

We will observe that under partial information, we can not apply the same result, since the $\mathbb{H}$-martingale $L'$ in the FS decomposition of $F$ under partial information, is only weakly orthogonal to the martingale part of $S$, and not in general a $\mathbb{H}$-martingale under the MMM.

We recall the definition of the minimal martingale measure with respect to $\mathbb{H}$.

**Definition 3.1.24.** An equivalent martingale measure $\tilde{Q}$ where

$$\frac{d\tilde{Q}}{dP} \in L^2(\Omega, \mathcal{F}_T, P)$$

is a minimal martingale measure if $Q = P$ on $\mathcal{H}_0$ and if every $(\mathbb{H}, P)$-martingale, $\mathbb{H}$-orthogonal to the martingale part of $S, N$, is also an $(\mathbb{H}, \tilde{Q})$-martingale.
CHAPTER 3. THE LOCAL RISK MINIMIZING STRATEGY

Define the equivalent measure $\tilde{Q}$ by

$$
\frac{d\tilde{Q}}{dP} = \tilde{Z}_T := \exp\left(\int_0^T \theta^H_s dN_s - \frac{1}{2} \int_0^T (\theta^H_s)^2 d\langle N \rangle^H_s \right)
$$

(3.25)

where $\theta^H$ is a $\mathbb{H}$-predictable process given by (3.16). Then the process $\tilde{Z} = \{\tilde{Z}_t, t \in [0, T]\}$, given by $\tilde{Z}_t = E[\tilde{Z}_T | \mathcal{H}_t], t \in [0, T]$ is a $(\mathbb{H}, P)$-martingale, since for all $t \in [0, T]$ we have

$$
E[\tilde{Z}_T | \mathcal{H}_t] = E[\exp\left(\int_0^T \theta^H_s dN_s - \frac{1}{2} \int_0^T (\theta^H_s)^2 d\langle N \rangle^H_s \right) | \mathcal{H}_t] \tilde{Z}_t
$$

where we used that by (3.14),

$$
E[\exp\left(\int_0^T \theta^H_s dN_s \right) | \mathcal{H}_t] = E[E[\exp\left(\int_0^T \theta^H_s dN_s \right) | \mathcal{G}_t] | \mathcal{H}_t] = E[\exp\left(\frac{1}{2} \int_0^T (\theta^H_s)^2 d\langle N \rangle^H_s \right) | \mathcal{H}_t]
$$

Moreover $\tilde{Z}$ is square-integrable, since $\int_0^T (\theta^H_t)^2 d\langle N \rangle^H_t < \infty$, we get

$$
E[|\tilde{Z}_t|^2] \leq E[\exp\left(\int_0^T (\theta^H_t)^2 d\langle N \rangle^H_t \right)] < \infty
$$

Then by Theorem 3.1.9 the minimal martingale measure $\tilde{Q}$ exists and is given by (3.25).

Now, we want to show that the GKW decomposition of $F$ under $\tilde{Q}$, gives a GKW decomposition under $\Theta(\mathbb{F}, \mathbb{H})$ of $F$. Since $S$ is a $(\mathbb{H}, \tilde{Q})$-martingale, we can define the GKW decomposition of $F$ with respect to $\mathbb{H}$ under $\tilde{Q}$. Then we can apply result form the above subsection, i.e. the case where $S$ was assumed to be a $(\mathbb{H}, P)$-martingale.

Assume that the contingent claim $F \in L^2(\Omega, \mathcal{F}_T, P) \cap L^2(\Omega, \mathcal{F}_T, \tilde{Q})$.

We have that $S$ is a $(\mathbb{H}, \tilde{Q})$ martingale and the claim $F$ admits the unique GKW decomposition with respect to $\mathbb{H}$ under $\tilde{Q}$, i.e.

$$
F = \tilde{F}_0 + \int_0^T \xi^H_u dS_u + \tilde{L}_T, \quad \tilde{Q}\text{-a.s.}
$$

(3.26)

where $\tilde{F}_0 \in L^2(\Omega, \mathcal{H}_0, \tilde{Q}), \xi^H \in \Theta(\mathbb{H}, \tilde{Q}),$ and $\tilde{L}$ is a $(\mathbb{H}, \tilde{Q})$-martingale with $\tilde{L}_0 = 0$, $\mathbb{H}$-orthogonal to $S$ under $\tilde{Q}$.
3.1. THE LOCAL RISK MINIMIZING PROBLEM

Then by Proposition 3.1.23, the claim $F$ admits a GKW decomposition under $\Theta(F, H)$,

$$F = F_0' + \int_0^T \xi^F_u dS_u + L'_T, \quad \tilde{Q}\text{-a.s.} \tag{3.27}$$

if we have

$$V'_0 = E_{\tilde{Q}}[\tilde{V}_0|\mathcal{F}_0]$$

$$\xi^F = \frac{dO^{p,F}}{d(S)^{p,F}}$$

and that $L'$ is a $(H, \tilde{Q})$-martingale weakly $\mathbb{F}$-orthogonal to $S$, $L'_0 = 0$, where $O^{p,F} = (\int_0^T \xi^H_u d\langle S\rangle^H_u)^{p,F}$ and the notation $^{p,F}$ refers to the $(H, \tilde{Q})$-predictable dual projection.

So we need to show that

$$E_{\tilde{Q}}[\int_0^T \xi^H_t \varphi_t d\langle N\rangle^H_t] = E_{\tilde{Q}}[\int_0^T \xi^F_t \varphi_t d\langle N\rangle^H_t] \tag{3.28}$$

respectively

$$E[\tilde{Z}_T, \int_0^T \xi^H_t \varphi_t d\langle N\rangle^H_t] = E[\tilde{Z}_T, \int_0^T \xi^F_t \varphi_t d\langle N\rangle^H_t]$$

Recall the $\mathbb{F}$-predictable projection, that is, $X^p$ is the $\mathbb{F}$-predictable projection of a $\mathbb{H}$-adapted process $X$, if $X^p = E[X_\tau | \mathcal{F}_\tau]$ for any $\mathbb{F}$-stopping time $\tau$. Moreover, $E[\int_0^T X_t dY_t] = E[\int_0^T X^p_t dY_t]$.

Then we notice the $\mathbb{F}$-predictable projection $\tilde{Z}^p$ of $\tilde{Z}$,

$$\tilde{Z}^p_t = E[\tilde{Z}_i | \mathcal{F}_i] = E[E[\tilde{Z}_T | \mathcal{H}_i] | \mathcal{F}_i] = E[\tilde{Z}_T | \mathcal{F}_i]$$

So

$$E[\tilde{Z}_T, \int_0^T \xi^H_t \varphi_t d\langle N\rangle^H_t] = E[\int_0^T \tilde{Z}^p_t \xi^H_t \varphi_t d\langle N\rangle^H_t]$$

Now, since $\{\tilde{Z}^p \varphi\}$ are $\mathbb{F}$-predictable, we have

$$E[\int_0^T \tilde{Z}^p_t \xi^H_t \varphi_t d\langle N\rangle^H_t] = E[\int_0^T \tilde{Z}^p_t \xi^F_t \varphi_t d\langle N\rangle^H_t]$$

where $\xi^F = \frac{dO^{p,F}}{d(N)^{p,F}}$. Hence we have equality in equation (3.28).
CHAPTER 3. THE LOCAL RISK MINIMIZING STRATEGY

The $\mathbb{F}$-optimal strategy

Even if we cannot determine the value process as the process $\hat{V}_t = E_{\tilde{Q}}[F|\mathcal{H}_t]$, in the case $F$ admits a FS decomposition under $\Theta(\mathbb{F}, \mathbb{H})$, we can still determine the $\mathbb{F}$-optimal strategy in terms of the solution of the GKW decomposition with respect to $\mathbb{H}$. This is based on the similarly result, Proposition 3.12 in [CCR14b], where in the case of [CCR14b] it is given in terms of a BSDE.

The idea is, if the claim $F$ admits a FS decomposition under $\Theta(\mathbb{F}, \mathbb{H})$, we can determine the $\mathbb{F}$-optimal strategy. So if $F$ admits

$$F = F_0 + \int_0^T \xi^F_t dS_t + L_T, \quad \text{P-a.s.}$$

we can find a $\mathbb{F}$-optimal $\varphi = (\eta, \xi)$.

Note that by the decomposition $S = S_0 + N + R$, we have

$$F - \int_0^T \xi^F_t dR_t = F_0 + \int_0^T \xi^F_t dN_t + L_T$$

where we observe that the last part is looks like a GKW decomposition under $\Theta(\mathbb{F}, \mathbb{H})$ of a random variable.

Set $G = F - \int_0^T \xi^F_t dR_t$, then $G$ admits a GKW decomposition with respect to $\mathbb{H}$, i.e.,

$$G = G_0 + \int_0^T \xi^H_t dN_t + \tilde{L}_T, \quad \text{P-a.s.} \quad (3.29)$$

where $G_0$ is $\mathcal{H}_0$-measurable, $\xi^H \in \Theta(\mathbb{H})$ and $\tilde{L}$ is a $(\mathbb{H}, P)$-martingale $\mathbb{H}$-orthogonal to $N$.

Clearly, we need to assume that $\xi^F = \frac{d\tilde{Q}}{d\mathbb{P}} \xi^H + (N)^\mathbb{H}$, where $O = \int_0^T \xi^H d(N)^\mathbb{H}$, such that the GKW decomposition of $G$ with respect to $\mathbb{H}$ can be written as the GKW decomposition under $\Theta(\mathbb{F}, \mathbb{H})$.

We formulate the idea to the following result.

**Proposition 3.1.25.** Assume that the claim $F$ under filtration $\mathbb{H}$ can be written as

$$F = G_0 + \int_0^T \xi^F_t dR_t + \int_0^T \xi^H_t dN_t + \tilde{L}_T, \quad \text{P-a.s.} \quad (3.30)$$
where $G_0$ is $\mathcal{H}_0$-measurable, $\tilde{L}$ is $\mathbb{H}$-orthogonal to $N$, $\tilde{L}_0 = 0$ and $\xi^F = \frac{d\mathbb{P}}{d\mathbb{F}}\xi^\mathbb{H}$. Then the $\mathbb{F}$-optimal strategy $\varphi = (\eta, \xi)$, the value process and the cost process are given by

$$
\xi = \xi^F
$$

$$
V_t(\varphi) = G_0 + \int_0^t \xi^F_s dR_s + \int_0^t \xi^\mathbb{H}_s dN_s + \tilde{L}_t \quad (3.31)
$$

$$
C_t(\varphi) = G_0 + \int_0^t (\xi^\mathbb{H}_s - \xi^F_s) dN_s + \tilde{L}_t \quad (3.32)
$$

\textbf{Proof.} By Proposition 3.1.23 $G$ given by (3.29) admits a GKW decomposition under $\Theta(\mathbb{F}, \mathbb{H})$, that is

$$
G = G_0' + \int_0^T \xi^F_t dN_t + L_T, \quad \text{P.-a.s.} \quad (3.33)
$$

where $G_0' = E[G_0|\mathcal{F}_0]$, $\xi^F = \frac{d\mathbb{P}}{d\mathbb{F}}\xi^\mathbb{H}$ and $L$ is a $\mathbb{H}$-martingale weakly $\mathbb{F}$-orthogonal to $N$, $L_0 = 0$. Since $G = F - \int_0^T \xi^F$, we get

$$
F = G_0' + \int_0^T \xi^F_t dS_t + L_T \quad (3.34)
$$

Hence $F$ admits a FS decomposition under $\Theta(\mathbb{F}, \mathbb{H})$. By Proposition 3.1.18 this is equivalent to the portfolio $\varphi = (\eta, \xi)$ being $\mathbb{F}$-optimal, given by

$$
\xi = \xi^F
$$

$$
V_t(\varphi) = G_0' + \int_0^t \xi^F_t dS_t + L_t
$$

$$
C_t(\varphi) = G_0' + L_t
$$

The cost process under partial information with information $\mathbb{F}$, given by (3.32), contains an orthogonal random process and it is difficult to find an explicit expression for this process. Moreover, the cost process is not necessarily measurable with respect to $\mathbb{F}$, to see this, consider the case where the martingale $L$ is given by the time-change process.
3.2 Backward Stochastic Differential Equation

The quadratic hedging strategy such as the local risk minimizing hedging strategy can be tackled by backward stochastic differential equations (BS-DEs).

We refer to a paper by Jeanblanc et al. (2012) in which the relation between the BSDEs and quadratic hedging strategies in the context of general semimartingales are discussed.

The advantage of using BSDEs to find the LRM strategy, is that we do not need a change of measure, so we do not have to find and apply the minimal martingale measure.

Instead we will use the existence and uniqueness of the BSDEs to ensure that the final value $F$ admits a unique FS decomposition. By this we will determine the value of the portfolio.

In context of our two information flows $\mathcal{G}$ and $\mathcal{F}$, we will apply different BSDEs depending on the measurability properties.

Before introducing the BSDEs, we will define the non-anticipating stochastic integration and some useful representation.

Stochastic integrals and representation theorems

Given the martingale structure of the time-changed Brownian motion, we can construct a (Itô type) non-anticipating stochastic integration according to the classical scheme.

Non-anticipating stochastic integration with respect to general martingales is treated in [Di 02], and this theory can be adapted in the setup of this thesis.

We construct the integration scheme around $\mathcal{G}$.

**Definition 3.2.1.** The measurable function $\varphi^n$, $n \in \mathbb{N}$ on the form

$$\varphi^n_t = \sum_{i=1}^{n} \varphi^h_i 1_{(t_i,t_{i+1})}(t), \quad 0 \leq t \leq T$$

where $\varphi^h_i$ a $\mathcal{G}_{t_i}$-measurable random variable with

$$E[(\varphi^h_i)^2(\Lambda_{t_{i+1}} - \Lambda_{t_i})] < \infty$$

and $(t_i,t_{i+1}]$ a partitions of $[0,T]$, is called a **simple integrand**.

Furthermore, **general integrands** $\varphi$ are identified as following limit

$$\varphi = \lim_{n \to \infty} \varphi^n$$
of a sequence \((\varphi^n)_{n \geq 1}\) of simple integrands such that \(E[\int_0^T (\varphi^n_t)^2 d\Lambda_t] < \infty\), for each \(n\).

We define the non-anticipating integral \(I\) as the limit
\[
I(\varphi) = \int_0^T \varphi dB_t := \lim_{n \to \infty} \int_0^T \varphi^n dB_t
\]
with convergence in \(L^2(\Omega, \mathcal{F}, P)\).

Then, we have
\[
E[(\int_0^T \varphi dB_t)^2] = E[\int_0^T \varphi^2 dB_t] = \int_0^T \varphi^2 dB_t
\]

**Definition 3.2.2.** Define the spaces of integrands
\[
\mathcal{I}_G := \{\varphi \in L^2(\Omega, \mathcal{F}, P), \text{G-predictable} : (E[\int_0^T \varphi^2 d\Lambda_s])^{\frac{1}{2}} < \infty\}
\]
\[
\mathcal{I}_F := \{\varphi \in L^2(\Omega, \mathcal{F}, P), \text{F-predictable} : (E[\int_0^T \varphi^2 d\langle B \rangle_s])^{\frac{1}{2}} < \infty\}
\]

Recall from Proposition 3.1.21 that for a \(\mathcal{F}\)-predictable \(\varphi\),
\[
E[\int_0^T \varphi^2 d\langle B \rangle_s]^2 = E[\int_0^T \varphi^2 d\Lambda_s]
\]

Hence, we notice that
\[
\mathcal{I}_F = \{\varphi \in \mathcal{I}_G : \varphi \text{ is F-predictable}\}
\]

**Remark.** We see that the non-anticipative stochastic integral \(I\) is the mapping \(I : \mathcal{I}_G \to L^2(\Omega, \mathcal{F}, P)\).

We will give the integral and martingale representation theorems with respect to \(G\). In the case of filtration \(F\), the representation takes a different form. First consider \(B\) to be a martingale with respect to \(G\).

**Theorem 3.2.3 (Integral Representation).** Let \(\xi \in L^2(\Omega, \mathcal{F}_T, P)\). Then there exists a unique \(\phi \in \mathcal{I}_G\) such that
\[
\xi = E[\xi | \mathcal{F}^\Lambda_T] + \int_0^T \phi_s dB_s
\]
where \(E[\xi | \mathcal{F}^\Lambda_T] \in L^2(\Omega, \mathcal{F}^\Lambda_T, P)\).

**Proof:** See proof of Theorem 3.3 in [DS14]
Theorem 3.2.4 (Martingale Representation). Assume $M = \{M_t, t \in [0, T]\}$ is a $(\mathcal{G}, P)$-martingale. Then there exists a unique $\phi \in \mathcal{I}_G$ such that

$$M_t = E[M_T | \mathcal{F}_T^A] + \int_0^t \phi_s dB_s$$

where $E[M_T | \mathcal{F}_T^A] \in L^2(\Omega, \mathcal{F}_T^A, P)$.

Proof. This proof is an extension of the proof of the classical Martingale Representation Theorem, see e.g. Section 4.3 in Øksendal (2003). By the Integral Representation Theorem, for a fixed $t \in [0, T]$ there exists a unique $\phi(t) \in \mathcal{I}_G$ such that

$$M_t = E[M_t | \mathcal{F}_T^A] + \int_0^t \phi(t)_s dB_s$$

Since

$$E[M_T | \mathcal{F}_T^A] = E[E[M_T | \mathcal{G}_t] | \mathcal{F}_T^A] = E[M_t | \mathcal{F}_T^A]$$

we get

$$M_t = E[M_T | \mathcal{F}_T^A] + \int_0^t \phi(t)_s dB_s$$

Assume now that $0 \leq t_1 < t_2$. Then

$$M_{t_1} = E[M_{t_1} | \mathcal{F}_t^A] + \int_0^{t_1} \phi(t_1)_s dB_s$$

On the other hand, since $M$ is a martingale, we have

$$M_{t_1} = E[M_{t_2} | \mathcal{G}_{t_1}]$$

$$= E[M_T | \mathcal{F}_t^A] + E[\int_0^{t_2} \phi(t_2)_s dB_s | \mathcal{G}_{t_1}]$$

$$= E[M_T | \mathcal{F}_t^A] + \int_0^{t_1} \phi(t_2)_s dB_s$$

Hence, by comparing the two later equations, we get

$$0 = E[(\int_0^{t_1} \phi(t_1)_s - \phi(t_2)_s dB_s)^2] = E[\int_0^{t_1} (\phi(t_1)_s - \phi(t_2)_s)^2 d\Lambda_s]$$

So $\phi(t_1) = \phi(t_2) \text{ P-a.s. for } t \in [0, t_1]$.

Define $\phi_t := \phi(t) \in \mathcal{I}_G$ and

$$M_t = E[M_T | \mathcal{F}_T^A] + \int_0^t \phi_s dB_s$$

$\square$
3.2. BACKWARD STOCHASTIC DIFFERENTIAL EQUATION

Now assume B is a martingale with respect to $\mathbb{F}$. Then we observe that the Integral Representation theorem takes a different form.

**Theorem 3.2.5 (Integral Representation).** Assume $\xi \in L^2(\Omega, \mathcal{F}_T, P)$. Then there exists a unique $\varphi \in \mathcal{I}_F$ such that

$$
\xi = \xi^0 + \int_0^T \varphi_s dB_s
$$

where $\xi^0$ is a random variable in $L^2(\Omega, \mathcal{F}_T, P)$ orthogonal to the integral part.

**BSDE preliminaries**

In this section, we introduce the notation and make an introduction to the BSDEs when considering the information flow $\mathbb{G}$. The following theory on BSDE driven by martingales is taken from [DS14].

Define the space

$$
S^G_2 := \{X: \Omega \times [0,T] \to \mathbb{R} : \mathbb{G}\text{-adapted, càdlàg and } E\left[ \sup_{0 \leq t \leq T} |X_t|^2 \right] < \infty \}
$$

Consider the BSDE

$$
\begin{cases}
-dX_t = f_t(\lambda_t, X_t, \varphi_t)dt - \varphi_t dB_t \\
X_T = H
\end{cases}
$$

(3.37)

where $H$ is the terminal condition and $f$ the generator.

**Definition 3.2.6.** The couple $(H, f)$ are standard parameters, with respect to $\mathbb{G}$ if

(A) $H \in L^2(\Omega, \mathcal{F}_T, P)$

(B) $f: [0, T] \times [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that

- $f(\lambda, x, \varphi)$ is $\mathbb{G}$-adapted for all $x \in S^G_2, \varphi \in \mathcal{I}_G$
- $E\left[ \int_0^T |f_t(\lambda_t, 0, 0)|^2 dt \right] < \infty$
- $f$ satisfies a uniformly Lipschitz condition in $(x, \varphi)$, i.e. there exists a positive constant $K$ such that

$$
|f_t(x^{(1)}, \varphi^{(1)}) - f_t(x^{(2)}, \varphi^{(2)})| \leq K(|x^{(1)} - x^{(2)}| + |\varphi^{(1)} - \varphi^{(2)}|\sqrt{\lambda}), \quad dt \times dP \text{ a.e.}
$$

for all $\lambda \in [0, \infty)$ and $x^{(1)}, x^{(2)}, \varphi^{(1)}, \varphi^{(2)} \in \mathbb{R}$
Definition 3.2.7. A solution to the BSDE (3.37) is the pair \((X, \varphi) \in S^G_2 \times \mathcal{L}_G\) satisfying

\[
X_t = H + \int_t^T f_s(\lambda_s, X_s, \varphi_s) \, ds - \int_t^T \varphi_s \, dB_s
\]

Theorem 3.2.8. Given the couple \((H, f)\) satisfying (A) and (B) in Definition 3.2.6, there exists a unique solution \((X, \varphi)\) to the BSDE (3.37).
3.2. BACKWARD STOCHASTIC DIFFERENTIAL EQUATION

3.2.1 The LRM under $G$, via BSDEs

First we consider the case under filtration $G$, i.e. when we have complete information on the time change process. The BSDE in this case is driven by the time-changed Brownian motion.

The relation between the BSDE and the FS decomposition, in the case the BSDEs are driven by martingales, is addressed by [DKV15]. We will use the setup of section 4 in [DKV15], and apply it to our framework.

Recall the discounted stock price $S$ given by (3.3) has decomposition,

$$S = S_0 + M + A,$$

under $(G,P)$. 

Consider a process $\tilde{\xi} \in \Theta(G)$ and let $F$ be the contingent claim. We want to find the BSDE for the process $V$, defined as

$$V_t = E[F - \int_t^T \tilde{\xi}_s dA_s | G_t], \quad t \in [0,T]$$

(3.38)

We apply the GKW decomposition under $P$, to the random variable $F - \int_0^T \tilde{\xi}_s dA_s \in L^2(\Omega,\mathcal{F}_T, P)$, that is there exists a $\xi^* \in \Theta(G)$ such that

$$F - \int_0^T \tilde{\xi}_s dA_s = E[F - \int_0^T \tilde{\xi}_s dA_s | \mathcal{F}_T^M] + \int_0^T \xi^*_s dM_s + L_T$$

(3.39)

where $L$ is a $G$-martingale orthogonal to $M$, $L_0 = 0$. Taking conditional expectation in (3.39), we obtain,

$$E[F - \int_0^T \tilde{\xi}_s dA_s | G_t] = E[V_0 + \int_0^T \xi^*_s dM_s + L_T | G_t]$$

$$= V_0 + \int_0^t \xi^*_s dM_s + L_t$$

where $V_0 = E[F - \int_0^T \tilde{\xi}_s dA_s | \mathcal{F}_T^M]$. This gives

$$V_t = V_0 + \int_0^t \tilde{\xi}_s dA_s + \int_0^t \xi^*_s dM_s + L_t$$

(3.40)

By the Martingale Representation theorem, there exist a unique $\phi \in \mathcal{L}_G$ such that

$$L_t = E[L_T | \mathcal{F}_T^M] + \int_0^t \phi_s dB_s$$

In addition we know that $L_0 = 0$, so

$$L_t = \int_0^t \phi_s dB_s$$
CHAPTER 3. THE LOCAL RISK MINIMIZING STRATEGY

Recall, the processes $A$ and $M$ are given by (3.4), that is,

$A_t = \int_0^t S_s(\alpha_s - r_s)ds$

and $M_t = \int_0^t S_s\sigma_s dB_s$, for all $t \in [0, T]$.

So by (3.40), we obtain the following BSDE

$$\begin{cases}
dV_t = \tilde{\xi}_t S_t(\alpha_t - r_t)dt + (\xi^*_t S_t\sigma_t + \phi_t)dB_t \\
V_T = F
\end{cases} \tag{3.41}$$

**Proposition 3.2.9.** There exists a unique solution $(V, \varphi)$ to the BSDE (3.41), where $V$ is given by (3.38) and $\varphi_t = \xi^*_t S_t\sigma_t + \phi_t$ for every $t \in [0, T]$.

Before we provide the proof, we recall a useful fundamental inequality

$$(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + x_2^2 + \cdots + x_n^2)$$

for any $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

**Proof.** Since the BSDE (3.41) has the same frame as the BSDE (3.37), we know there exist a unique solution $(V, \varphi)$. So it remains to show that $V$ given by (3.38) and $\varphi_t = \xi^*_t S_t\sigma_t + \phi_t$ satisfies Definition 3.2.7 and Theorem 3.2.8.

By Definition 3.2.7 there exists a solution if $(V, \varphi) \in S_G^2 \times I_G$. Let $\varphi_t = \xi^*_t S_t\sigma_t + \phi_t$, $t \in [0, T]$.

Then

$$|\varphi|^2 \leq 2|\xi^*_t S_t\sigma_t|^2 + 2|\phi_t|^2$$

We know that $\xi^*_t \in \Theta(G)$ and $\phi \in I_G$, so clearly $\varphi \in I_G$, since

$$E[\int_0^T |\varphi_t|^2 d\Lambda_t] \leq 2E[\int_0^T |\xi^*_t S_t\sigma_t|^2 d\Lambda_t] + 2E[\int_0^T |\phi_t|^2 d\Lambda_t] < \infty$$

Furthermore

$$|V_t| = |E[F - \int_t^T \tilde{\xi}_s dA_s | G_t]|$$

$$\leq E[|F|] + \int_t^T |\tilde{\xi}_s dA_s | G_t]$$

$$\leq E[|F|] + \int_0^T |\tilde{\xi}_s dA_s | G_t] =: m_t \tag{3.42}$$

where $m$ is a square-integrable $G$-martingale. By Doob’s inequality, we get

$$E[\sup_{0 \leq t \leq T} |V_t|^2] \leq E[\sup_{0 \leq t \leq T} |m_t|^2] \leq 4E[|m_T|^2]$$

$$= 4E[|F|^2] + \int_0^T |\tilde{\xi}_s dA_s|^2] \tag{3.43}$$

$$\leq 8E[|F|^2] + 8E[\int_0^T |\tilde{\xi}_s dA_s|^2]$$
Since $F \in L^2(\Omega, \mathcal{F}_T, P)$ and $\tilde{\xi} \in \Theta(\mathcal{G})$, we have
\[
E[\sup_{0 \leq t \leq T} |V_t|^2] \leq 8E[|F|^2] + 8E[\int_0^T |\tilde{\xi}_s|d|A|_s]^2] < \infty
\]
By Theorem 3.2.8 the pair $(V, \varphi)$ is unique if the couple $(F, f)$ where
\[
f_t(V, \varphi) = \tilde{\xi}_t S_t (\alpha_t - r_t)
\]
are standard parameters. Clearly, by the definition of $F$ it satisfies (A) and
since $\tilde{\xi} \in \Theta(\mathcal{G})$, also (B) is satisfied in Definition 3.2.6.
Hence the pair $(V, \varphi)$ is the unique solution to the BSDE (3.41).

We want to find a FS decomposition of $F$ and we need therefore to fix the
component $\tilde{\xi}$ in the BSDE (3.41).

The following result is from Proposition 14 in Schweizer (1994).

**Proposition 3.2.10.** Assume $S$ satisfies $(SC)^G$. If for some constant $K$ we
have the following inequality
\[
\frac{|\alpha_t - r_t|}{\sqrt{\sigma_t^2 \lambda_t}} \leq K \quad P\text{-a.s.} \tag{3.44}
\]
for all $t \in [0, T]$, then $\xi^* = \tilde{\xi}$ in $L^2_\mathcal{G}(M)$.

**Proof.** See Appendix B. \qed

**Remark.** Assuming the inequality (3.44) holds and that $S$ satisfies $(SC)^G$, we
see that the mean-variance trade process is bounded, that is
\[
K_T = \int_0^T \theta_t^2 d\langle M \rangle_t = \int_0^T \frac{(\alpha_t - r_t)^2}{\sigma_t^2 \lambda_t} dt \leq K^2
\]
Under the assumptions $S$ satisfying $(SC)^G$ and inequality (3.44) holds, the
set of equations (3.41) are equivalent to
\[
\begin{cases}
    dV_t = \tilde{\xi}_t dS_t + \phi_t dB_t \\
    V_T = F
\end{cases} \tag{3.45}
\]
The process $V$ and the associated BSDE (3.45) directly provides the local
risk minimizing strategy. That is, the component $\tilde{\xi}$ is the number of units in
the risky asset and $V$ is the discounted value of the portfolio.
Proposition 3.2.11. Assume inequality (3.44) holds. Suppose there is a solution to the BSDE (3.45). That is, for $t \in [0, T]$, 

$$V_t = F - \int_t^T \tilde{\xi}_s dS_s - (L_T - L_t)$$  \hspace{1cm} (3.46)

where $L$ is a $\mathcal{G}$-martingale orthogonal to $M$, $L_0 = 0$. Then $F$ admits a unique FS decomposition and hence $\varphi = (\eta, \xi)$ is a $\mathcal{G}$-optimal strategy.

Moreover, the value process and the cost process is given by

$$V_t(\varphi) = V_t$$
$$C_t(\varphi) = V_0 + L_t$$

Proof. By (3.46) we obtain

$$F = V_T = V_0 + \int_0^T \tilde{\xi}_s dS_s + L_T, \hspace{0.5cm} \text{P-a.s.}$$

and by the uniqueness of the BSDE, $F$ admits a unique FS decomposition. Then by Proposition 3.1.6 the existence of the FS decomposition is equivalent to $\varphi$ being $\mathcal{G}$-optimal. Let $\varphi = (\eta, \xi)$. We choose

$$\xi = \tilde{\xi}$$

and

$$\eta_t = V_0 + L_t - \tilde{\xi}_t S_t + \int_0^t \tilde{\xi}_s dS_s$$

Then, we obtain the value process,

$$V_t(\varphi) = \eta_t + \xi S_t = V_0 + \int_0^t \tilde{\xi}_s dS_s + L_t$$

Hence $V_t(\varphi) = V_t$. Thus, the associated cost is given by

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \xi_s dS_s = V_0 + L_t$$

□
Moreover, by the orthogonality property of \( L \) and \( M \), that is \( \langle L, M \rangle = 0 \), we obtain
\[
\langle \int_0^t \phi_s dB_s, \int_0^t S_s \sigma_s dB_s \rangle = \int_0^t \phi_s S_s \sigma_s d\langle B \rangle_s = \int_0^t \phi_s S_s \sigma_s \lambda_s ds = 0 \quad (3.47)
\]
Hence
\[
\phi_t S_t \sigma_t \lambda_t = 0, \quad \text{P-a.s for } t \in [0, T]
\]
However, recall the market assumptions, \( \sigma_t \neq 0, S_t \geq 1 \), for \( t \in [0, T] \) and \( \Lambda_t > 0 \), for \( t \in (0, T] \), with \( \Lambda_0 = 0 \). Hence the \( \mathcal{G} \)-martingale becomes zero for all \( t \in [0, T] \). So again we obtain \( C_t = V_0 \), for all \( t \in [0, T] \), hence also by the BSDE method the filtration \( \mathcal{G} \) is in a "sense complete", as discussed on page 35.
3.2.2 The LRM under $\mathbb{F}$, via BSDEs

Recall the information flow $\mathbb{F}$ that is generated by the time-changed Brownian motion.

In the case of filtration $\mathbb{F}$, the BSDE will take another form than with respect to $\mathbb{G}$. First we will see that with respect to filtration $\mathbb{H}$, the BSDE are driven by a càdlàg martingale. Then, we will apply the partial information approach, disused in [CCR14a] and [CCR14b], to find a representation of the BSDE driven by a càdlàg $\mathbb{H}$-martingale under partial information with respect to $\mathbb{F}$.

We need to give some technical definition and results, on the BSDE driven by càdlàg martingales under the filtration $\mathbb{H}$ and BSDE driven by càdlàg martingales under the partial information with respect to $\mathbb{F}$ and $\mathbb{H}$. We will use the theory on BSDE under partial information from [CCR14b].

Recall the filtrations $\mathbb{F}$ and $\mathbb{H}$, given by $\mathcal{F}_t = \sigma\{B_s, s \leq t\} \vee \mathcal{N}$, $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{F}_t^\Lambda \vee \mathcal{N}$ for $t \in [0, T]$, respectively. So $\mathcal{F}_t \subseteq \mathcal{H}_t$, $t \in [0, T]$.

Recall the $\mathbb{H}$-semimartingale $S$ is given by (3.3), i.e., $S = S_0 + N + R$. If $S$ satisfies the $(SC)^\mathbb{H}$, then there exists a $\mathbb{H}$-predictable process $\theta^\mathbb{H}$ such that

$$\int_0^T (\theta^\mathbb{H}_t)^2 d\langle N \rangle^\mathbb{H}_t < \infty$$

and

$$S_t = S_0 + N_t + \int_0^t \theta^\mathbb{H}_s d\langle N \rangle^\mathbb{H}_s$$

For the rest of this section we assume that the $\mathbb{H}$-semimartingale satisfies the $(SC)^\mathbb{H}$.

Under filtration $\mathbb{H}$ the BSDE takes form

$$X_t = F + \int_t^T g_s(X_s, \phi_s)d\langle N \rangle^\mathbb{H}_s - \int_t^T \phi_s dN_s - L_T + L_t$$

where

(i) $N$ is a $(\mathbb{H}, P)$-martingale,
(ii) $\langle N \rangle^\mathbb{H}$ its quadratic variation
(iii) and $L$ is a square integrable $\mathbb{H}$-martingale $\mathbb{H}$-orthogonal to $N$, with $L_0 = 0$.

That is a BSDEs driven by càdlàg martingales. See Carbone et al. [CFS08] or Appendix B for the description of this BSDEs and the solutions.
3.2. **BACKWARD STOCHASTIC DIFFERENTIAL EQUATION**

Recall the spaces $\mathcal{I}_G$ and $\mathcal{I}_F$, given by (3.35). We define the space

$$\mathcal{I}_H := \{ \varphi \in \mathcal{I}_G : \varphi \text{ is } \mathbb{H}\text{-predictable} \}$$

Clearly, we have $\mathcal{I}_F \subseteq \mathcal{I}_H \subseteq \mathcal{I}_\mathbb{H}$. We need to define the standard parameter of the BSDE under partial information. Let the pair $(F,f)$ be standard parameter with respect to $\mathbb{H}$ (see Definition B.0.1), then they are standard parameters under partial information, if $f$ is $\mathbb{F}$-predictable and $f_t(0,0) \in \mathcal{I}_F$. We denote this pair by $(F,f,\mathbb{F})$.

Recall the definition of **weak orthogonality** given in Definition 3.17, that is, two square integrable $\mathbb{H}$-martingales $X,Y$ are weakly $\mathbb{F}$-orthogonal if

$$E[X_T \int_0^T \varphi_s dY_s] = 0$$

for every $\mathbb{F}$-predictable process $\varphi$ such that $\varphi \in \mathcal{I}_F$.

**Definition 3.2.12.** A solution of the BSDE

$$X_t = F + \int_t^T f_s(X_s, \phi_s)d\langle N \rangle^H_s - \int_t^T \phi_s dN_s - L_T + L_t$$

with data $(F,f,\mathbb{F})$ under partial information, is the triplet $(X,\phi,L)$ such that

$$E[\sup_{0 \leq t \leq T} |X_t|^2] < \infty \quad \text{and} \quad E[\int_0^T |\phi_t|^2d\langle N \rangle^H_t] < \infty$$

and $L$ is a $\mathbb{H}$-martingale weakly $\mathbb{F}$-orthogonal to $N$ with $L_0 = 0$.

The existence of the representation of the BSDE under partial information is given in Lemma 2.10 in [CCR14b]. In this result they need the key assumption that the predictable quadratic variation $\langle N \rangle^H$ is bounded. However, since we are not looking for a general function $f$, as the standard parameter under partial information, we only need to assume boundedness on the mean-variance process. We assume the following inequality,

$$\frac{|\alpha_t - r_t|}{\sqrt{\sigma_t^2 \lambda_t}} \leq K$$

for some constant $K \geq 0$. Then it follows that the mean-variance process is bounded

$$K_T = \int_0^T (\theta_t^H)^2d\langle N \rangle_t \leq K^2$$
Proposition 3.2.13. Assume the inequality \((3.48)\) holds. Let \(\psi \in \mathcal{I}_F\). Then the BSDE
\[
X_t = F + \int_t^T \psi_s \theta^H_s d\langle N \rangle^\text{H}_s - \int_t^T \phi_s dN_s - L_T + L_t
\]
has a solution under partial information, with \((F, f, \mathcal{F})\), in the sense of Definition 3.2.12, where \(f_t(Y, \psi) = \psi_t \theta^H_t, t \in [0, T]\).

See the Proof of Lemma 2.10 in [CCR14b] for the general proof. Here the inequality \((3.48)\) is used to provide that \(X\) satisfies the condition \(E[\sup_{0 \leq t \leq T} |X_t|^2] < \infty\).

We will use the setup of Section 4 in [DKV15], similar to the theory in Section 3.2.1, extended to the case of partial information.

Consider a process \(\xi^F \in \mathcal{I}_F\) and let \(F\) be the contingent claim. Define the process \(\tilde{V}\) as
\[
\tilde{V}_t = E[F - \int_t^T \xi^F_s \theta^H_s d\langle N \rangle^\text{H}_s | \mathcal{H}_t]
\]
Apply the GKW decomposition with respect to \(\mathbb{H}\) and \(P\), to the random variable \(F - \int_0^T \xi^F_s \theta^H_s d\langle N \rangle^\text{H}_s \in L^2(\Omega, \mathcal{F}_T, P)\).
That is, there exists a unique process \(\xi^H \in \Theta(\mathbb{H})\) such that
\[
F - \int_0^T \xi^F_s \theta^H_s d\langle N \rangle^\text{H}_s = \tilde{U}_0 + \int_0^T \xi^H_s dN_s + \tilde{L}_T
\]
where \(\tilde{U}_0\) is \(\mathcal{H}_0\)-measurable and \(\tilde{L}\) is a \(\mathbb{H}\)-martingale \(\mathbb{H}\)-orthogonal to \(N\), with \(\tilde{L}_0 = 0\).
So
\[
\tilde{V}_t = \int_0^t \xi^F_s \theta^H_s d\langle N \rangle^\text{H}_s + E[\tilde{U}_0 + \int_0^T \xi^H_s dN_s + \tilde{L}_T | \mathcal{H}_t]
\]
= \(\tilde{V}_0 + \int_0^t \xi^F_s \theta^H_s d\langle N \rangle^\text{H}_s + \int_0^t \xi^H_s dN_s + \tilde{L}_t\)
where \(\tilde{V}_0 = E[F - \int_0^T \xi^F_s \theta^H_s d\langle N \rangle^\text{H}_s]\).
This gives the BSDE
\[
\begin{cases}
d\tilde{V}_t = \xi^F_t \theta^H_t d\langle N \rangle^\text{H}_t + \xi^H_t dN_t + d\tilde{L}_t \\
\tilde{V}_T = F
\end{cases}
\]
where \(d\langle N \rangle^\text{H} = S^2 \sigma^2 d\Lambda\).
Note that the BSDE \((3.51)\) has the same frame as the BSDE driven by càdlàg martingale, see Appendix B. Hence we know there exist a solution.
3.2. BACKWARD STOCHASTIC DIFFERENTIAL EQUATION

To find a solution to the set of equations (3.51), we need the $\mathbb{F}$-predictable dual projection. Recall Definition 3.1.19 on $\mathbb{F}$-predictable dual projection. That is, the $\mathbb{F}$-predictable process $X^{p,F}$ is the $\mathbb{F}$-predictable dual projection of the $\mathbb{H}$-adapted process $X$ of integrable variation, if

$$E[\int_0^T \varphi_s dX^{p,F}_s] = E[\int_0^T \varphi_s dX_s]$$

for every $\mathbb{F}$-predictable process $\varphi$.

**Proposition 3.2.14.** Let inequality (3.48) hold and let $(\tilde{V}, \xi^H, \tilde{L})$ be the solution to the problem with respect to $\mathbb{H}$,

$$\tilde{V}_t = F - \int_t^T \xi^F_s \theta^H_s \langle N \rangle^H_s - \int_t^T \xi^H_s dN_s - \tilde{L}_T + \tilde{L}_t$$

where $\tilde{L}$ is a $\mathbb{H}$-martingale $\mathbb{H}$-orthogonal to $N$ and

$$\xi^F = \frac{d(\xi^H d(N)^H_s)}{d((N)^H_s)^{p,F}}$$

Then there exist a solution $(V, \xi, L) = (\tilde{V}, \xi^H, \tilde{L} + M)$ to the BSDE under partial information, where $M = \int_0^T (\xi^H_s - \xi^F_s) dN_s$ is a $\mathbb{H}$-martingale weakly $\mathbb{F}$-orthogonal to $N$.

**Proof.** Set $M_t = \int_0^t (\xi^H_s - \xi^F_s) dN_s$, for every $t \in [0, T]$, then

$$\tilde{V}_t = F - \int_t^T \xi^F_s \theta^H_s \langle N \rangle^H_s - \int_t^T \xi^H_s dN_s - \tilde{L}_T + \tilde{L}_t$$

$$= F - \int_t^T \xi^F_s \theta^H_s d\langle N \rangle^H_s - \int_t^T \xi^H_s dN_s - \int_t^T (\xi^H_s - \xi^F_s) dN_s - \tilde{L}_T + \tilde{L}_t$$

(3.52)

It is sufficient to prove that $M$ is a $\mathbb{H}$-martingale weakly $\mathbb{F}$-orthogonal to $N$. We have

$$E[\int_0^T \varphi_s \xi^H_s d\langle N \rangle^H_s] = E[\int_0^T \varphi_s \xi^F_s d\langle N \rangle^H_s]$$

Hence

$$E[(\int_0^T (\xi^H_s - \xi^F_s) dN_s)(\int_0^T \varphi_s dN_s)] = 0$$

Now we can express the $\mathbb{F}$-optimal strategy, given by Definition 3.1.17, in terms of the solution of the problem under filtration $\mathbb{H}$.
CHAPTER 3. THE LOCAL RISK MINIMIZING STRATEGY

Proposition 3.2.15. Assume inequality (3.48) holds. Let \((\tilde{V}, \xi^H, \tilde{L})\) be the solution to the problem with respect to \(\mathbb{H}\), i.e.

\[
\tilde{V}_t = F - \int_t^T \theta^H_s \xi^F_s d\langle N \rangle^H_s - \int_t^T \xi^H_s dN_s - \tilde{L}_T + \tilde{L}_t
\]

where \(\tilde{L}\) is a \(\mathbb{H}\)-martingale \(\mathbb{H}\)-orthogonal to \(N\) and

\[
\xi^F = \frac{d\langle \xi^H \rangle^p}{d\langle N \rangle^p}
\]

Then the \(\mathbb{F}\)-optimal strategy \(\varphi = (\eta, \xi)\), the value process and the cost process, for \(t \in [0, T]\), is given by

\[
\xi_t = \xi^F_t, \quad V_t(\varphi) = \tilde{V}_t, \quad C_t(\varphi) = \tilde{V}_0 + \tilde{L}_t + \int_0^t (\xi^F_s - \xi^H_s) dN_s
\]

respectively.

Proof. By Proposition 3.2.14, we get that the triplet

\((V, \xi, L) = (\tilde{V}, \xi^F, \tilde{L} + M)\)

where \(M_t = \int_0^t (\xi^H_s - \xi^F_s) dN_s\), is a solution to the BSDE under partial information, i.e.,

\[
V_t = F - \int_t^T \xi^F_s \theta^H_s d\langle N \rangle^H_s - \int_t^T \xi^F_s dN_s - L_T + L_t
\]

Moreover, we have

\[
F = V_0 + \int_0^T \xi^F_s \theta^H_s d\langle N \rangle^H_s + \int_0^T \xi^F_s dN_s + L_T
\]

\[
= V_0 + \int_0^T \xi^F_s dS_s + L_T, \quad \text{P-a.s.}
\]

Hence \(F\) admits a FS decomposition under partial information, and by Proposition 3.1.18, this is equivalent to the strategy \(\varphi = (\eta, \xi)\) being \(\mathbb{F}\)-optimal and given by

\[
\xi_t = \xi^F_t
\]

\[
C_t(\varphi) = V_0 + L_t
\]

\[
V_t(\varphi) = C_t(\varphi) + \int_0^t \xi_s dS_s = V_0 + \int_0^t \xi^F_s dS_s + L_t
\]

We get that the cost process contains an orthogonal random process, and it is difficult to find an explicit expression for this process.
3.3  Conclusion and further research

We have considered an incomplete continuous market driven by a time-changed Brownian motion, where the random change of time is a absolutely continuous time change. For this market we characterize the LRM hedging strategy for the contingent claim $F$, with respect to two different filtrations, $\mathcal{G}$ and $\mathcal{F}$.

For the LRM hedging strategy with respect to the filtration $\mathcal{G}$, we found a FS decomposition of the claim $F$ and the portfolio value, considering two approaches. In the first approach via MMM, we showed the connection between the GKW decomposition and the FS decomposition of the claim $F$, recall Theorem \ref{thm:mmm}. Via the BSDE approach we showed the connection between the FS decomposition of the claim $F$ and the BSDE driven by the noise with the final value equal the claim $F$, recall Proposition \ref{prop:bsde}. We notice that the solutions of the LRM problem via MMM and via BSDEs coincide. Which is what we would expect, since we are looking at the same problem, with different approaches.

In the case of information flow $\mathcal{F}$, we did not have the same measurability properties as in the case of $\mathcal{G}$. So we also considered the case of filtration $\mathcal{H}$, since among other useful properties of $\mathcal{H}$, the time change process is measurable to $\mathcal{H}$. Considering the filtration $\mathcal{F}$ with a partial information approach with respect to $\mathcal{H}$, a partial solution of the portfolio value and the FS decomposition of the claim $F$ was presented, recall Proposition \ref{prop:partial}.

Comparing the LRM problem via MMM in context of the two different filtration $\mathcal{F}$ and $\mathcal{G}$, we obtain different solutions for the associated portfolio value depending on the measurability properties. While for $\mathcal{G}$, we could apply the MMM to find the connection between the GKW decomposition and the FS decomposition, this result could not be applied with the partial information approach, i.e. in the case of $\mathcal{F}$.

Comparing the solutions of the BSDEs, with respect to the information flows $\mathcal{G}$ and $\mathcal{F}$, we clearly have different solutions also in this case. We propose two BSDEs related to the two filtrations, where the terminal condition is the same. Since the two filtrations have different measurability properties, the two BSDEs lead to two different solutions.

For both methods in the case of $\mathcal{G}$, we obtain an known orthogonal random variable as cost and hence can reduce the risk, since we have knowledge of the future. While in the case of $\mathcal{F}$ the cost is adapted to the filtration $\mathcal{H}$ and consists of a unknown orthogonal process that is difficult to calculate. In addition, the random process has a weaker orthogonality condition than the random variable.
The theory on the LRM hedging strategy considered in this thesis in narrowed down to a specific setting, i.e. the market driven by a time-changed Brownian motion, with absolutely continuous time change. While considering the subordinator to time change would also be interesting. We would not in general be considering a continuous market model, and we would not be able to apply the theory the LRM strategy which we have considered. Then one can use BSDEs driven by the time-changed noise, to characterize the LRM hedging strategy.
Appendix A

In the chapter we will give some definitions and results about general stochastic processes, Lévy processes, martingales and quadratic variation. The main theory in this section is taken from the book by Phillip Protter [Pro05].

A.1 General processes

Definition A.1.1 (Adapted). A process $X = \{X_t, t \geq 0\}$ is called adapted if $X_t$ is $\mathcal{F}_t$-measurable for all $t$.

Definition A.1.2 (Predictable). The process $X = \{X_t, t \geq 0\}$ is predictable if $X$, considered as a mapping from $\mathbb{R}^+ \times \Omega \to \mathbb{R}$, is measurable with respect to the smallest $\sigma$-algebra generated by all adapted left-continuous mappings form $\mathbb{R}^+ \times \Omega \to \mathbb{R}$.

A stochastic process $X = \{X_t, t \geq 0\}$ is càdlàg if all paths are right-continuous with left-limits. That is for almost every $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right-continuous, i.e. $\lim_{s \to t, s > t} X_s = X_t$ and $\lim_{s \to t, s < t} X_s$ exists and is finite.

Remark. Any adapted process that is left-continuous is a predictable process. Hence if a process $X$ is càdlàg adapted it is predictable.

Definition A.1.3 (Modification). Let $X = \{X_t, t \geq 0\}$ and $Y = \{Y_t, t \geq 0\}$ be two stochastic processes, both defined on $(\Omega, \mathcal{F}, P)$. Then $Y$ is a modification of $X$ if

$$P(X_t \neq Y_t) = 0$$

for each $t \geq 0$. It follows that $X$ and $Y$ have the same finite-dimensional distribution.
A.2 The Lévy-Itô decomposition

Let $X = \{X_t, t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ and $\nu$ its Lévy measure. Then

- $\nu$ is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ such that
  \[ \int_{|x| \leq 1} |x|^2 \nu(dx) < \infty, \quad \int_{|x| \geq 1} \nu(dx) < \infty \]

- the jump measure of $X$, denoted by $N$, is a Poisson random measure on $[0, \infty) \times \mathbb{R}^d$ with intensity measure $\nu(dx)dt$

- there exists a vector $\beta$ and a $d$-dimensional Brownian motion $B = \{B_t, t \geq 0\}$ with covariance matrix $A$,

such that $X$ admits a Lévy-Itô decomposition

\[ X_t = \beta t + B_t + \int_{|x| \geq 1} xN(t, dx) + \int_{|x| < 1} x\tilde{N}(t, dx) \]

where

\[ \int_{|x| < 1} x\tilde{N}(t, dx) := \lim_{\epsilon \to 0} \int_{\epsilon \leq |x| < 1} x\tilde{N}(t, dx) = \lim_{\epsilon \to 0} \int_{\epsilon \leq |x| < 1} x(N(t, dx) - t\nu(dx)) \]

(where the limit is taken in $L^2(\Omega, \mathcal{F}, P)$).

Remark. We can also show that the $\int_{|x| < 1} x\tilde{N}(t, dx)$ converge P-a.s. That is, consider a decreasing sequence $(\epsilon_n)_{n \geq 1} \to 0$ and let

\[ Y_n = \int_{\epsilon_{n+1} \leq |x| < 1} x\tilde{N}(t, dx) - \int_{\epsilon_n \leq |x| < 1} x\tilde{N}(t, dx) \]

Then all $Y_i$ have mean zero and $\sum Var Y_i < \infty$. Hence, by Kolmogorov’s three series theorem (see [Kal06], Theorem 3.18), $\sum Y_i$ converges P-a.s., which means that $\int_{|x| < 1} x\tilde{N}(t, dx)$ converges P-a.s. for $t \in [0, T]$ as $\epsilon \to 0$.

A.3 Martingales, local martingales and semi-martingales

Definition A.3.1 (Martingale). Let $M = \{M_t, t \geq 0\}$ be an adapted process such that $E[|M_t|] < \infty$ for all $t \geq 0$. Then $M$ is a martingale if, for all $0 \leq s \leq t < \infty$

\[ E[M_t|\mathcal{F}_s] = M_s \]
Remark. Let $M = \{M_t, t \geq 0\}$ be an adapted process, with $E[|M_t|] < \infty$ for all $t \geq 0$. If for every $0 \leq s \leq t < \infty$, $E[M_t|F_s] \geq M_s$, then $M$ is called a submartingale. The process $M$ is a supermartingale in the case $E[M_t|F_s] \leq M_s$, for every $0 \leq s \leq t < \infty$.

Remark. A martingale $M$ is said to be square-integrable if $E[|M_t|^2] < \infty$ for each $t \geq 0$.

Definition A.3.2 (Local martingale). An adapted, càdlàg process $X = \{X_t, t \geq 0\}$ is a local martingale if there exists a sequence of increasing stopping times, $T_n, n \in \mathbb{N}$, with $\lim_{n \to \infty} T_n = \infty$ such that $X_{t \wedge T_n}1_{T_n > 0}$ is a uniformly integrable martingale for each $n$.

Recall $X = \{X_t, t \geq 0\}$ is a uniformly integrable martingale if and only if $Y = \lim_{t \to \infty} X_t$ a.s. exists, $E[|Y|] < \infty$ and $X$ is a martingale, where $X_\infty = Y$.

Remark. If $M = \{M_t, t \geq 0\}$ is a local martingale $M$ such that for every $t \geq 0$, $E[\sup_{s \leq t} |M_s|] < \infty$, then $M$ is a martingale.

Example A.3.3. Any càdlàg martingale is a local martingale. Let $X$ be a càdlàg martingale and $T_n \equiv n, n \in \mathbb{N}$ a increasing stopping time. Then the process $X_{t \wedge \tau_n}$, for all $t \geq 0$ is a uniformly integrable martingale, hence $X$ is a local martingale.

Theorem A.3.4 (The Doob-Meyer decomposition). Let $Y = \{Y_t, t \geq 0\}$ be a submartingale, then there exists a unique predictable, integrable, increasing process $A = \{A_t, t \geq 0\}$ with $A_0 = 0$ such that the process

$$M_t = Y_t - Y_0 - A_t$$

is a martingale for each $t \geq 0$, $M_0 = 0$.

Remark. In the case where the submartingale is given by $Y_t = M_t^2$, then $\langle M \rangle_t = A_t$ for each $t \geq 0$, where $\langle M \rangle$ is called the Meyer’s angle-bracket process. Then by Doob-Meyer decomposition we obtain for $t \geq 0$,

$$M_t^2 - \langle M \rangle_t$$

is a martingale. This we will see in the next section is the definition of the predictable quadratic variation.

Definition A.3.5 (Doob’s inequality). Let $M = \{M_t, t \geq 0\}$ be a square integrable martingale, then

$$E[\sup_{0 \leq t \leq T} |M_t|^2] \leq 4 \sup_{0 \leq t \leq T} E[|M_t|^2]$$
Definition A.3.6 (Semimartingales). We say that $X = \{X_t, t \geq 0\}$ is a semimartingale if it is an adapted process such that, for every $t \geq 0$, we can write $X$ as the following

$$X_t = X_0 + M_t + A_t$$

where $M = \{M_t, t \geq 0\}$ is a local martingale, with $M_0 = 0$ and $A = \{A_t, t \geq 0\}$ is an adapted process of finite variation, with $A_0 = 0$.

Remark. By Proposition 2.7.1 in [App09] we have that every Lévy process is a semimartingale.

Definition A.3.7. Let $X$ be a Lévy process, then we can write $X$ as a non-decreasing semimartingale, if $X \geq 0$ with Lévy triplet $(\beta, A, \nu)$ satisfying

(i) $\nu(-\infty, 0) = 0$
(ii) $\int_{(0<x \leq 1)} x \nu(dx) < \infty$
(iii) $A_\infty = 0$
(iv) There is a process $\alpha = \{\alpha_t, t \geq 0\}$ with $\alpha_t = \beta t - \int_0^t \int_{0<x \leq 1} x \nu(dx)ds$

that is nondecreasing.

Moreover

$$X_t = \alpha_t + \int_{(0,\infty)} xN(t, dx)$$

A.4 Quadratic variation

Definition A.4.1. Let $X$ be a semimartingale, then the quadratic variation of $X$ is defined by

$$[X] := [X, X] = X^2 - 2 \int X_- dX$$

Definition A.4.2. The quadratic covariation of two semimartingales $X$ and $Y$, also called bracket process of $X$ and $Y$, is defined by

$$[X,Y] = XY - \int X_- dY - \int Y_- dX$$

Definition A.4.3. Let $X$ be a semimartingale such that its quadratic variation process $[X]$ is locally integrable. Then the predictable variation of $X$, denoted $\langle X \rangle := \langle X, X \rangle$ exists and is defined such that

$$[X] - \langle X \rangle$$

is a local martingale.
Remark. If $X$ is continuous, then

$$[X] = \langle X \rangle$$

Example A.4.4. Let $B = \{B_t, t \geq 0\}$ be a time-changed Brownian motion with respect to $H_t = \sigma\{B_s, \Lambda_s, s \leq t\}, t \in [0, T]$, with the time-change process $\Lambda_t = \int_0^t \lambda_s ds, t \in [0, T]$, then $B$ has continuous paths, hence

$$[B]_t = \langle B \rangle_t = \Lambda_t, \quad t \geq 0$$

Example A.4.5. Let $\pi = \{\pi_t, t \geq 0\}$ be a Poisson process with intensity $\lambda$. Then

$$[\pi]_t = \sum_{s \leq t} (\Delta \pi_s)^2 = \pi_t$$

Moreover

$$[\pi]_t - \langle \pi \rangle_t = \pi_t - \lambda t$$

is a martingale, hence a local martingale (with respect to its natural filtration).

In Protter (2003), there are two different types of orthogonality, weakly orthogonal and strongly orthogonal. For simplicity we call strong orthogonal for orthogonal.

Definition A.4.6 (Orthogonality). Two square-integrable martingales $M$ and $N$ are said to be orthogonal if their quadratic covariation process $[M, N]$ is a local martingale. Moreover

$$\langle M, N \rangle = 0$$

Definition A.4.7 (Weak orthogonality). Two square-integrable martingales $M$ and $N$ are weakly orthogonal if $E[M_\infty N_\infty] = 0$.

A.5 Itô’s formulas

Theorem A.5.1 (Itô’s formula for semimartingales with jumps). Let $C^k(\mathbb{R}^d)$ denote the class of $k$ times continuously differentiable functions on $\mathbb{R}^d$. Let $X$ be a semimartingale with jumps in $\mathbb{R}^d$, and fix any function $f \in C^2(\mathbb{R}^d)$. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s^-) dX_s + \frac{1}{2} \int_0^t f''(X_s) d[X]^c_s$$

$$+ \sum_{0 < s \leq t} (f(X_s) - f(X_{s}^-) - f'(X_{s}^-) \Delta X_s)$$
Where \([X]_t^c\) is the continuous part of \([X]_t\), that is,

\[
[X]_t^c = [X]_t - X_0^2 - \sum_{0 < s \leq t} (\Delta X_s)^2
\]

In the same way we define the Itô formula in the case where the semimartingale is continuous.

\[
f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s
\]

**Example A.5.2.** Let \(X = \{X_t, t \geq 0\}\) be a Lévy-Itô process on \(\mathbb{R}\), that is a process on the form

\[
dX_t = G_t dt + F_t dW_t + \int_{|x| \geq 1} K(t, x) N(dt, dx) + \int_{|x| < 1} H(t, x) \tilde{N}(dt, dx)
\]

where \(W = \{W_t, t \geq 0\}\) is a Brownian motion and \(N = \{N_t, t \geq 0\}\) a Poisson random measure, and \(B\) and \(N\) are independent. Let \(f \in C^2(\mathbb{R}^2)\). Then Itô’s formula is

\[
 df(X_t) = f'(X_t) G_t dt + f'(X_t) F_t dW_t + \frac{1}{2} f''(X_t) F_t^2 dt \\
+ \int_{|x| < 1} \left\{ f(X_t^- + H(t, x)) - f(X_t^-) \right\} \tilde{N}(dt, dx) \\
+ \int_{|x| \geq 1} \left\{ f(X_t^- + K(t, x)) - f(X_t^-) \right\} N(dt, dx) \\
+ \int_{|x| \geq 1} \left\{ f(X_t^- + H(t, x)) - f(X_t^-) - H(t, x) f'(X_t^-) \right\} \nu(dx) dt
\]

**Definition A.5.3 (Doléans-Dade exponential).** The stochastic exponential or Doléans-Dade exponential of a Lévy process \(X = \{X_t, t \geq 0\}\), is denoted by \(\mathcal{E}_X = \{\mathcal{E}_X(t), t \geq 0\}\) and it is defined as

\[
\mathcal{E}_X(t) = \exp\{X_t - \frac{1}{2} [X]_t^c \} \prod_{0 \leq s \leq t} [1 + \Delta X_s] e^{-\Delta X_s}
\]

for each \(t \geq 0\).

The following result is taken from Section 5.1 in [App09].

**Proposition A.5.4.** Let \(Z = \{Z_t, t \geq 0\}\) be an adapted process and let \(X = \{X_t, t \geq 0\}\) be a Lévy-Itô process on \(\mathbb{R}\). Suppose we have the stochastic differential equation

\[
dZ_t = Z_t dX_t \quad Z_0 = 1
\]
Assume that \( \inf \{ \Delta X_s, t > 0 \} > -1 \) a.s.. Then the solution of \( Z \) is given by,

\[
Z_t = \mathcal{E}_X(t) = e^{Y_t}
\]

where

\[
dY_t = \left\{ G_t - \frac{1}{2} F_t^2 \right\} dt + F_t dB_t
\]

\[
+ \int_{|x| \geq 1} \log(1 + K(t, x)) N(dt, dx)
\]

\[
+ \int_{|x| < 1} \log(1 + H(t, x)) \tilde{N}(dt, dx)
\]

\[
+ \int_{|x| < 1} \left\{ \log(1 + H(t, x)) - H(t, x) \right\} \nu(dx) dt
\]

**Proof.** Let \( X \) be a Lévy-Itô process as in Example A.5.2 and \( \mathcal{E}_X \) the Doléans-Dade exponential of \( X \). We verify that

\[
Z_t = e^{Y_t}
\]

Applying Itô’s formula (see Example A.5.2), we obtain for each \( t \geq 0 \),

\[
d(e^{Y_t}) = e^{Y_t-} \left\{ G_t dt + F_t dB_t + \int_{|x| \geq 1} \left\{ \log(1 + H(t, x)) - H(t, x) \right\} \nu(dx) dt \right\}
\]

\[
+ \int_{|x| \geq 1} \left\{ \exp(Y_t- + \log(1 + K(t, x))) - \exp(Y_t-) \right\} N(dt, dx)
\]

\[
+ \int_{|x| \geq 1} \left\{ \exp(Y_t- + \log(1 + H(t, x))) - \exp(Y_t-) \right\} \tilde{N}(dt, dx)
\]

\[
+ \int_{|x| \geq 1} \left\{ \exp(Y_t- + \log(1 + H(t, x))) - \exp(Y_t-) \right\} \nu(dx) dt
\]

\[
- \log(1 + H(t, x)\exp(Y_t-)) \nu(dx) dt
\]

\[
e^{Y_t} \left\{ G_t dt + F_t dB_t + \int_{|x| < 1} H(t, x) \tilde{N}(dt, dx) + \int_{|x| \geq 1} K(t, x) N(dt, dx) \right\}
\]

Hence \( e^{Y_t} \) is the solution of \( dZ_t = Z_t - dX_t \).

Now, we need to show that \( \mathcal{E}_X(t) \) from Definition A.5.3 corresponds to \( e^{Y_t} \).

Let

\[
\prod_{0 \leq s \leq t} [1 + \Delta X_s] e^{-\Delta X_s} = C_t + D_t
\]

where

\[
C_t = \prod_{0 \leq s \leq t} [1 + \Delta X_s] e^{-\Delta X_s} 1_{(\Delta X_s \geq 1)}
\]

and
\[ D_t = \prod_{0 \leq s \leq t} [1 + \Delta X_s] e^{-\Delta X_s} 1_{\{\Delta X_s < 1\}} \]

From the assumption \( \inf \{\Delta X_s, t > 0\} > -1 \text{ a.s.} \), the process \( D \) is a.s. finite and we can write

\[
D_t = \exp\left( \sum_{0 \leq s \leq t} (\log[1 + \Delta X_s] - \Delta X_s) 1_{\{\Delta X_s < 1\}} \right)
\]

\[ = \exp\left( \int_0^t \int_{|x| < 1} (\log[1 + H(s, x)] - H(s, x)) N(ds, dx) \right) \]

\[ = \exp\left( \int_0^t \int_{|x| < 1} (\log[1 + H(s, x)] - H(s, x)) (\tilde{N}(ds, dx) + \nu(dx)ds) \right) \]

(A.1)

Moreover, since \( X \) is càdlàg, we have \( \sharp\{0 \geq s \geq t : |\Delta X_s| \geq 1\} < \infty \), so the process \( C \) is a.s. a finite process and

\[
C_t = \exp\left( \sum_{0 \leq s \leq t} (\log[1 + \Delta X_s] - \Delta X_s) 1_{\{\Delta X_s \geq 1\}} \right)
\]

\[ = \exp\left( \int_0^t \int_{|x| \geq 1} (\log[1 + K(s, x)] - K(s, x)) N(ds, dx) \right) \]

(A.2)

Hence

\[ E_X(t) = \exp\left\{ X_t - \frac{1}{2} [X, X]^c_t \right\} \prod_{0 \leq s \leq t} [1 + \Delta X_s] e^{-\Delta X_s} \]

\[ = \exp\left\{ \int_0^t (G_s - \frac{1}{2} F^2_s) ds + \int_0^t F_s dW_s + \int_0^t \int_{|x| \geq 1} K(s, x) N(ds, dx) \right\} (C_t + D_t) \]

(A.3)

\[ = e^{Y_t} \]

\[ \square \]

A.6 GKW decomposition

The following result is taken from Theorem 1.2.10 in [Pha09]. We define the GKW decomposition for a random variable.

Definition A.6.1 (Galtchouk-Kunita-Watanabe decomposition). Let \( M = \{M_t, t \geq 0\} \) be a continuous local \((\mathbb{K}, P)\)-martingale and let \( \zeta \in \mathbb{R}^d \)
L^2(\Omega, K_T, P). Then there exists a \( \mathbb{K} \)-predictable process \( \xi \) such that \( E[\int_0^T |\xi_s|^2 d\langle M \rangle_s] < \infty \) and \( L \) a càdlàg local \((\mathbb{K}, P)\)-martingale orthogonal to \( M \), \( L_0 = 0 \) such that

\[
\zeta = E[\xi | K_0] + \int_0^T \xi_s dM_s + L_T, \quad P - a.s
\]
Appendix B

In this appendix we will provide some theory about BSDEs and a proof, to support the theory in Section 3.2.

The theory for backward stochastic differential equations driven by càdlàg martingales is taken from Carbone et al. (2008). In the case of information flow $\mathbb{H}$ the BSDE takes form

$$Y_t = H + \int_t^T g_s(Y_s, \phi_s) d\langle B \rangle_s - \int_t^T \phi_s dB_s - N_T + N_t \quad \text{(B.1)}$$

where

(i) $B$ is a $(\mathbb{H}, P)$-martingale,
(ii) $\langle B \rangle^H$ its quadratic variation
(iii) and $N$ is a square integrable $\mathbb{H}$-martingale orthogonal to $B$, with $N_0 = 0$.
and where $H$ is the terminal condition and $g$ the generator.

That is a BSDEs driven by càdlàg martingales. The definition for standard parameter $(H, g)$ is the same as in Definition 3.2.6.

**Definition B.0.1.** The pair $(H, g)$ are standard parameters with respect to $\mathbb{H}$ if (A) $H \in L^2(\Omega, \mathcal{F}_T, P)$ and (B) $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a $\mathbb{H}$-adapted mapping such that for $t \in [0, T]$, $g_t(0, 0) \in \mathcal{L}_{\mathbb{H}}$ and $g$ is uniformly Lipschitz.

Then the BSDEs of type (B.1) admits a solution, which is characterized by the triplet $(Y, \phi, N)$.

**Definition B.0.2.** The triplet $(Y, \phi, N)$ satisfying (B.1) P-a.s. such that

$$E[\sup_{0 \leq t \leq T} |Y_t|^2] < \infty \quad \text{and} \quad E[\int_0^T |\phi_t|^2 d\langle B \rangle_t] < \infty$$

and $N$ is a martingale orthogonal to $B$ with $N_0 = 0$. 

Then the existence and uniqueness of this solution when \((H, g)\) are standard parameters, is provided by Theorem 2.1 in \([\text{CFS08}]\) given \(\beta = 0\).

**Theorem B.0.3.** Let \((H, g)\) be standard parameters. Then \((B.1)\) has a unique solution \((Y, \phi, N)\).

In the following, we provide the proof of Proposition 3.2.10.

We are in the case of information flow \(\mathbb{G}\) and adopt the setup of Section 3.1. Consider the equation set

\[
\begin{align*}
-dV_t &= f_t(V_t, \varphi_t)dt - \varphi_t dB_t \\
V_T &= F
\end{align*}
\]

where \(F\) is the terminal condition and \(f\) the generator, with respect to \(\mathbb{G}\). Recall the definitions of spaces \(I_{\mathbb{G}}\) and \(S_{\mathbb{G}}^2\), given by (3.35) and (3.36), respectively.

**Proof of Proposition 3.2.10**

*Proof.* We follow the proof of Proposition 14 in \([\text{Sch94}]\).

In this proof we will apply Banach fixed point theorem, that states that every contraction mapping on a non-empty complete metric space has a unique fixed point.

Define a map \(\Psi(V, \xi, L) = (\tilde{V}, \tilde{\xi}, \tilde{L})\) with norm

\[
\| (V, \xi, L) \|_\alpha = \alpha \left( E \left[ \sup_{0 \leq t \leq T} |V_t|^2 \right] \right)^{\frac{1}{2}} + \left( E \left[ \int_0^T |\xi_t|^2 \, d\langle M \rangle_t + \langle L \rangle_T \right] \right)^{\frac{1}{2}}
\]

where

\[
\tilde{V}_t = E[F - \int_t^T \xi_t dA_t | \mathbb{G}_t]
\]

and \(\tilde{\xi}\) and \(\tilde{L}\) are given by

\[
F - \int_0^T \xi_t dA_t = \tilde{V}_0 + \int_0^T \tilde{\xi}_t dM_t + \tilde{L}_T
\]

Then clearly

\[
\tilde{V}_t = F - \int_t^T \xi_s dA_s - \int_t^T \tilde{\xi}_s dM_s - (\tilde{L}_T - \tilde{L}_t)
\]
Recall the inequality \[3.44\], i.e.,

\[
\frac{|\alpha_t - r_t|}{\sqrt{\sigma_t^2 \lambda_t}} \leq K \quad \text{P-a.s for all } t
\]

for some constant \(K\).

We want to show that by the inequality \(3.44\), the map \(\Psi\) has a unique fixed point in \(B^2\) (Banach space) for every \(F \in L^2(\Omega, \mathcal{F}_T, P)\).

We need to show that \(\Phi\) is a contraction on \((B^2, \|\cdot\|_\alpha)\) for a suitable \(\alpha\).

Suppose we have another triplet \((\tilde{V}', \tilde{\xi}', \tilde{L}')\), such that

\[
\tilde{V}'_t = E\left[F - \int_t^T \tilde{\xi}_s dA_s | \mathcal{G}_t\right]
\]

and

\[
F - \int_0^T \xi_t dA_t = \tilde{V}'_0 + \int_0^T \tilde{\xi}_t dM_t + \tilde{L}'_T
\]

Then we have

\[
\tilde{V}_t - \tilde{V}'_t = -\int_t^T (\xi_t - \xi'_t) dA_s - \int_t^T (\tilde{\xi} - \tilde{\xi}'_t) dM_t - (\tilde{L}_T - \tilde{L}'_T)
\]

\[
|\tilde{V}_t - \tilde{V}'_t| \leq E[\int_t^T |\xi'_s - \xi_s| d|A_s| | \mathcal{G}_t] \leq E[\int_0^T |\xi'_s - \xi_s| d|A_s| | \mathcal{G}_t] =: \tilde{m}_t
\]

where \(\tilde{m}\) is a square integrable \(\mathbb{G}\)-martingale, so by Doob’s inequality, we obtain

\[
E\left[\sup_{0 \leq t \leq T} |\tilde{V}_t - \tilde{V}'_t|^2\right] \leq E\left[\sup_{0 \leq t \leq T} |\tilde{m}_t|^2\right] \leq 4E[|\tilde{m}_T|^2]
\]

\[
= 4E\left[(\int_0^T |\xi'_t - \xi_t| d|A_t|)^2\right] \quad (B.4)
\]

Note that

\[
\int_0^T |\xi_t| d|A_t| = \int_0^T |\xi_t \theta_t| d\langle M \rangle_t
\]

\[
\leq \int_0^T (\xi_t^2)^{\frac{1}{2}} (\theta_t^2)^{\frac{1}{2}} d\langle M \rangle_t \leq (K_T)^{\frac{1}{2}} (\int_0^T \xi_t^2 d\langle M \rangle_t)^{\frac{1}{2}}
\]

Recall the mean-variance tradeoff process \(K_T = \int_0^T \theta_t^2 d\langle M \rangle_t\) and by the inequality \(3.44\), we obtain

\[
K_T = \int_0^T \theta_t^2 d\langle M \rangle_t = \int_0^T \frac{(\alpha_t - r_t)^2}{\sigma_t^2 \lambda_t} dt \leq K^2
\]
So we have
\[
(E\sup_{0\leq t\leq T} |\tilde{V}_t - \tilde{V}'_t|^2)^{\frac{1}{2}} \leq (4E(\int_0^T |\xi_t - \xi'_t|d|M_t|)^2))^{\frac{1}{2}} \leq 2(E[K_T\int_0^T |\xi_t - \xi'_t|^2d(M_t)])^{\frac{1}{2}} \leq 2K\|\xi - \xi'\|_{L_2(M)} \tag{B.5}
\]
Moreover,
\[
E[\int_0^T |\tilde{\xi}_t - \tilde{\xi}'_t|^2d(M_t) + \langle \tilde{L} - \tilde{L}' \rangle_T] = E[\int_0^T (\tilde{\xi}_t - \tilde{\xi}'_t)dM_t + \tilde{L}_T - \tilde{L}'_T|^2]
\]
since, by the orthogonality property we obtain
\[
E[(\tilde{L}_T - \tilde{L}'_T)\int_0^T (\tilde{\xi}_t - \tilde{\xi}'_t)dM_t] = 0
\]
So by \([B.3]\)
\[
\int_0^T (\tilde{\xi}_t - \tilde{\xi}'_t)dM_t + \tilde{L}_T - \tilde{L}'_T = \int_0^T (\xi_t - \xi'_t)dA_t + (\tilde{V}_0' - \tilde{V}_0)
\]
\[
= \int_0^T (\xi_t - \xi_t)dA_t - E[\int_0^T (\xi_t - \xi_t)dA_t|\mathcal{F}_T^A]
\]
and so we obtain
\[
(E[\int_0^T |\tilde{\xi}_t - \tilde{\xi}'_t|^2d(M)_t + \langle \tilde{L} - \tilde{L}' \rangle_T])^{\frac{1}{2}} = (E[\int_0^T (\tilde{\xi}_t - \tilde{\xi}'_t)dM_t + \tilde{L}_T - \tilde{L}'_T|^2])^{\frac{1}{2}} \leq (E[\int_0^T (\xi'_t - \xi_t)dA_t]^2)^{\frac{1}{2}} \leq K\|\xi' - \xi\|_{L_2(M)} \tag{B.7}
\]
Combining \((B.5)\) and \((B.7)\), we have
\[
\|\Psi(V, \xi, L) - \Psi(V', \xi', L')\|_\alpha \leq (2\alpha + 1)K\|\xi' - \xi\|_{L_2(M)} \tag{B.8}
\]
\[
\leq (2\alpha + 1)K\|(V, \xi, L) - (V', \xi', L')\|_\alpha
\]
Hence the inequality \((3.44)\) implies that \(\Psi\) is a contraction on \((\mathcal{B}^2, \|\cdot\|_\alpha)\) for \(0 < \alpha < \frac{1 - K}{2K}\).
Bibliography


