When a circle group $S^1$ is acting continuously on a paracompact topological space $X$, an important invariant of the group action is the equivariant cohomology ring $H^*_S(X; k)$ where $k$ is a field of arbitrary characteristic. This cohomology ring is the cohomology of the space $X_{S^1}$ which is the total space of the Borel fibering ([1,3])

$$X \to X_{S^1} \to B_{S^1}.$$ 

The spectral sequence $E_r$, $1 \leq r \leq \infty$, of this fibering is such that $E_\infty$ is the sum of subquotients

$$F^q/F^{q-1} \simeq E^{pq}_\infty, q \geq 0,$$

where $F^{q-1} \subset F^q \subset H^*_S(X; k)$ is a filtration of the module $H^*_S(X; k)$ over $k[t] = H^*(B_{S^1}; k)$ where $t$ is a generator of $H^2(B_{S^1}; k)$.

We now state the result of this paper. We assume that

$$\dim_k H^q(X; k) < \infty \text{ for } q \geq 0.$$

**Theorem.**

As graded modules over the polynomial ring $k[t]$ the cohomology module $H^*_S(X; k)$ is isomorphic to the module $E_\infty$ of the spectral sequence.

When $Y \subseteq X$ is a closed invariant subspace, the corresponding statement on $H^*_S(X, Y; k)$ is equally valid.

The case of $H^1_S(X, Y; k)$ is similar to the case of $H^*_S(X; k)$ and we focus on the latter.

The localization theorem for equivariant cohomology will not be used in this paper. Hence the field $k$ may be of any characteristic.

We will define a mapping of sets

$$E : H^*_S(X; k) \to E_\infty$$

which is not a module homomorphism. We define $E(0) = 0$ and if

$$x \in F^q, x \not\in F^{q-1}, q \geq 0,$$

then $E(x)$ is the image of $x$ by the module homomorphism

$$F^q \to F^q/F^{q-1} \cong E^{pq}_\infty$$

associated to the spectral sequence. Each $E^{pq}_\infty$ lies in the image of $E$ and $E(x) \neq 0$ for $x \neq 0$, but $E$ is not injective. The mapping $E$ has the following four properties where $x_j$ are homogeneous elements of $H^*_S(X; k)$.

1. If $E(x_1)E(x_2) \neq 0$, then $E(x_1x_2) = E(x_1)E(x_2)$
If \( \tau a E(x_1) \neq 0 \), then \( E(t^a x_1) = t^a E(x_1), a \geq 1 \).

(3) If \( E(x_1) \in E_{\infty}^{s_q} \) with \( q \geq 0 \), then \( E(t^a x_1) \in E_{\infty}^{s_q} \) with \( s \leq q \) for \( a \geq 1 \).

(4) If \( x_1 \neq 0 \) and \( t^a E(x_1) = 0 \) and \( E(x_1) \in E_{\infty}^{s_q}, q \geq 0 \), then \( E(t^a x_1) \in E_{\infty}^{s_q} \) with \( s < q \).

We shall use the following lemma of T. Chang and the author.

**Lemma.** ([2])

The \( k[t]\)-module \( E_r^{p,q}, 2 \leq r \leq \infty \), is generated as a module by the linear subspace \( E_r^{p+q} \).

We first prove a key lemma.

**Lemma.**

Let \( x \in E_{\infty}^{p+q} \) be such that \( t^a x = 0 \) for some \( a \geq 1 \). Then there is an \( u \in H_{S_1}^{p+q}(X; k) \) with \( E(u) = x \) and \( t^a u = 0 \).

**Proof.**

If \( q = 0 \) so that \( x \in E_{\infty}^{p+q} \subset F^p \subset \infty \), this is evident. Thus we may assume that \( q > 0 \). Choose \( v \in H_{S_1}^{p+q}(X; k) \) such that \( E(v) = x \). As \( t^a E(v) = t^a x = 0 \), whereas \( t^a v \neq 0 \) in general, we have \( t^a v \in E_{\infty}^{p+q} \) for some \( q_1 < q \), by property (4).

As \( E_{\infty}^{p+q} \) is generated over \( k[t] \) by \( E_{\infty}^{p+q} \), there is some \( v_1 \in H_{S_1}^{p+q}(X; k) \) with \( E(v_1) \in E_{\infty}^{p+q} \) and \( t^{a+k_1} E(v_1) = E(t^a v_1) \neq 0 \), (in general), where \( k_1 > 0 \).

It is convenient to draw a picture of \( E_{\infty} \).

As \( E(t^a v) - E(t^{a+k_1} v_1) = 0 \), it follows that \( E(t^a v - t^{a+k_1} v_1) \in E_{\infty}^{p+q} \) with \( q_2 < q_1 \). Thus there is some \( v_2 \in H_{S_1}^{p+q}(X; k) \) with \( E(v_2) \in E_{\infty}^{p+q} \) and, with \( k_2 > k_1, t^{a+k_2} E(v_2) = E(t^a v - t^{a+k_1} v_1) \). We then have

\[
E(t^a v - t^{a+k_1} v_1 - t^{a+k_2} v_2) \in E_{\infty}^{p+q}
\]

with \( q_3 < q_2 < q_1 < q \).

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We go on in this manner until we get $g_j \leq 0$. We then get

$$E(t^av - (t^{a+k_1}v_1 + t^{a+k_2}v_2 + \cdots + t^{a+k_j}v_j)) = 0,$$

where $0 < k_1 < k_2 \cdots < k_j$, and hence,

$$t^av = t^{a+k_1}v_1 + t^{a+k_2}v_2 + \cdots + t^{a+k_j}v_j.$$

We now define $u \in H^{p+q}_{S_1}(X; k)$ by the equation

$$v = t^kv_1 + t^{k_2}v_2 + \cdots + t^{k_j}v_j + u.$$

We then have $t^au = 0$ and as $v_1, v_2, \cdots, v_j \in F^q \subseteq F^{q-1}$ and $v \notin F^{q-1}$, we obtain $x = E(v) = E(u)$ where $t^au = 0$.

We now prove the theorem together with the following lemma.

**Lemma.**

For each $q \geq 0$ the exact sequence

$$0 \rightarrow F^{q-1} \rightarrow F^q \rightarrow E_{\infty}^{pq} \rightarrow 0$$

is a split exact sequence of graded $k[t]$ modules.

**Proof.**

Choose elements

$$\alpha_1, \ldots, \alpha_\alpha, \beta_1, \ldots, \beta_\beta \in E_{\infty}^{pq}$$

such that the cyclic $k[t]$-modules generated by $\alpha_j$ are torsion modules of dimension $d_j \geq 1$ over $k$, and the submodules generated by the $\beta_j$ are free modules, and such that $E_{\infty}^{pq}$ is the direct sum of those $a+b$ submodules.

Let $\alpha_j' \in H^{q}_{S_1}(X; k)$ be such that $t^a\alpha_j = 0$ and $E(\alpha_j') = \alpha_j$, and let $\beta_j' \in H^{q}_{S_1}(X; k)$ be such that $E(\beta_j') = \beta_j$. Then the $a+b$ cyclic submodules of $H^{q}_{S_1}(X; k)$ generated by the $\alpha_j'$ and the $\beta_j'$ form a direct sum in $F^q \subseteq H^q_{S_1}(X; k)$, and this sum maps isomorphically onto $E_{\infty}^{pq}$ under the homomorphism $F^q \rightarrow E_{\infty}^{pq}$.

The proof of the theorem follows by using the split sequences of this lemma for all $q \geq 0$. 

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REFERENCES

