Random walks on discrete quantum groups and associated categories

Bas Petrus Antonius Jordans

Dissertation presented for the degree of Philosophiæ Doctor

Department of Mathematics
University of Oslo
2016
## Contents

**Introduction** .................................................. iii
  Acknowledgements ........................................... viii

1 **Preliminaries** ............................................... 1
  1.1 Conventions and notation ................................ 1
  1.2 q-calculus .................................................. 2
  1.3 Operator algebras .......................................... 4
  1.4 Quantum groups ........................................... 11
    1.4.1 Compact quantum groups and their representations . 11
    1.4.2 Discrete quantum groups ............................. 14
    1.4.3 Actions ............................................... 18
  1.5 $C^*$-tensor categories .................................. 19
  1.6 Lie group theory .......................................... 25
    1.6.1 Lie groups and Lie algebras ......................... 26
    1.6.2 Drinfeld–Jimbo $q$-deformations .................... 27

2 **Categories of SU($N$)-type** ............................... 31
  2.1 Preliminaries on categories of SU($N$)-type .......... 31
  2.2 Hecke algebras ............................................ 35
  2.3 Computations in Rep(SU$_\mu(N)$) ....................... 40
  2.4 Representations of Hecke algebras ....................... 47
  2.5 Categories generated by Hecke algebras ................. 56
  2.6 Two characterisations of SU($N$)-type categories ....... 63

3 **The basics of random walks** ............................... 71
  3.1 Classical theory ......................................... 71
  3.2 Random walks on quantum groups ......................... 75

4 **The Martin boundary for discrete quantum groups** ....... 83
  4.1 Random walks on the center .............................. 83
    4.1.1 Random walks on the torus and center .............. 84
    4.1.2 The Martin compactification ......................... 89
  4.2 Partial results on the Martin boundary of SU$_q(N)$ . . 96
    4.2.1 Preliminaries on SU$_q(N)$ ......................... 96
    4.2.2 Clebsch–Gordan coefficients ......................... 104
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2.3</td>
<td>The Martin kernel for SU_q(N)</td>
<td>107</td>
</tr>
<tr>
<td>4.2.4</td>
<td>Commutation relations in M(SU_q(3),μ)</td>
<td>118</td>
</tr>
<tr>
<td>4.3</td>
<td>Convergence to the boundary for quantum groups</td>
<td>128</td>
</tr>
<tr>
<td>4.4</td>
<td>Convergence to the boundary for SU_q(Z)</td>
<td>136</td>
</tr>
<tr>
<td>4.4.1</td>
<td>SU_q(2) and its Martin boundary</td>
<td>136</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Regularity</td>
<td>141</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>5</th>
<th>A categorical approach to the Martin boundary</th>
<th>151</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Boundary convergence and monoidal equivalence</td>
<td>151</td>
</tr>
<tr>
<td>5.2</td>
<td>Random walks on C*-tensor categories</td>
<td>160</td>
</tr>
<tr>
<td>5.3</td>
<td>The categorical Martin boundary</td>
<td>166</td>
</tr>
<tr>
<td>5.4</td>
<td>Categorical convergence to the boundary</td>
<td>174</td>
</tr>
<tr>
<td>5.5</td>
<td>Correspondence with quantum groups</td>
<td>180</td>
</tr>
<tr>
<td>5.5.1</td>
<td>Duality between G-C*-algebras and categories</td>
<td>180</td>
</tr>
<tr>
<td>5.5.2</td>
<td>The correspondence with discrete quantum groups</td>
<td>184</td>
</tr>
</tbody>
</table>

An introduction for the layman | 201 |

Index of symbols | 207 |

Index of subjects | 211 |

Bibliography | 213 |
Introduction

The term “quantum group” is not so old. It was first introduced by Drinfeld [Dri87] in the middle eighties. At that time the main motivation for studying these algebras came from physics. Quantum groups were used to generate solutions to the quantum Yang–Baxter equation in order to apply the inverse scattering method and solve integrable models. An important class of quantum groups was formed by the one-parameter deformations of the universal enveloping algebras of semisimple complex Lie algebras. These were suggested by Drinfeld [Dri87] and Jimbo [Jim85] and worked out by Levendorskii, Soibelman and Vaksman [VS88, Vak89, VS90, LS91]. At the same time Woronowicz [Wor87] independently came up with the definition of the compact quantum group $SU_q(2)$ and he started his study of compact matrix quantum groups. A slight generalisation of this lead to what are now called compact quantum groups.

Our interest, however, is not in physics. But also from the viewpoint of mathematics there are good reasons to study quantum groups. Going further back in time, in the mid thirties there was the quest for an extended version of Pontryagin duality [Pon34]. Pontryagin duality gives a way to recover a locally compact abelian group from its category of representations. In fact, it proves more: the theorem states that the dual $\hat{G}$ of a locally compact abelian group $G$ is again a locally compact abelian group and the double dual $\hat{\hat{G}}$ is isomorphic to the original group $G$. In order to generalise this result Tannaka [Tan38] and Krein [Kre49] proved reconstruction theorems. Tannaka showed how to reconstruct a locally compact group from its category of representations and Krein classified the class of categories that occur as the representation category of locally compact groups. These are very nice results and have been generalised in a lot of directions, nowadays there are many Tannaka–Krein type reconstruction theorems. For example the Doplicher–Roberts theorem [DR89], Deligne’s theorem [Del90] and Woronowicz’s duality [Wor88] to name but a few. However, one thing remained unsatisfactory about the Tannaka–Krein reconstruction theorem, namely the dual object $\text{Rep}(G)$ is in a different category, it does not have the structure of a group. Several years after Tannaka and Krein, the mathematicians Kac–Vainerman and Enock–Schwartz (see [ES92]) independently solved this problem by defining a class of objects nowadays known as Kac algebras. This class is self-dual and contains all locally compact groups. This theory of Kac algebras can be seen as the first topological theory of quantum groups. There was however one drawback to this approach, namely that the theory was rather technical and missed interesting examples (so not of the form groups or objects dual to groups). For example the Drinfeld–Jimbo $q$-deformed enveloping algebras were not of this form. Eventually, building upon the work of many people, Kustermans–Vaes [KV00] defined the locally compact quantum groups, a
self-dual category which contained locally compact groups and compact quantum groups. In this thesis the main interest will be on two special cases of these locally compact quantum groups, namely the compact and the discrete quantum groups. For these the theory is easier and has been well-developed.

Let us describe the philosophy behind quantum groups. First of all, it should be mentioned that there is not just one definition of a quantum group. There are several approaches to quantum groups. The biggest distinction is between the purely algebraic and the topological point of view. We will stick to the topological viewpoint. The main motivation (for the operator algebraic point of view) comes from the Gelfand–Naimark theorem [GN43]. They proved that every commutative C*-algebra is isomorphic to the algebra of continuous functions on a topological space. A noncommutative C*-algebra can therefore be thought of as the functions on a “noncommutative space” or a “quantum space”. In noncommutative geometry a noncommutative space is formed by a (noncommutative) C*-algebra equipped with extra structure to account for the geometry. The same holds for quantum groups, they should be thought of as the “algebra of functions on a quantum space”. This algebra of functions then comes equipped with extra structure to account for the group multiplication and inverse. So, contrary to what the name suggests, a quantum group is not a special sort of group. But given a compact group one can construct a compact quantum group out of it and conversely every commutative compact quantum group arises from a compact group.

One other major field of interest in this thesis is the theory of random walks. The term “random walks” is a very general one: it refers to mathematical models that describe a point moving randomly through a space. The space can have all kinds of structure, for example it can be a topological space, a lattice, a graph or a group to name but a few. Also the walk itself can be different with the biggest distinction between discrete jumps or paths that move in continuous time. Random walks can be very useful to distinguish or describe the structure of a space. For example random walks can be used to detect amenability of groups [Woe00], determine the structure of graphs by identifying certain subtrees in a graph, determine structure of groups (via for instance the Renewal theorem) and many more results are known. We are concerned with the study of probabilistic boundaries of random walks on discrete (quantum) groups. Such boundaries are used to describe the behaviour of random walks as time tends to infinity. Let us be a bit more precise. Let \((X, P)\) be a discrete Markov chain, so \(X\) is a discrete space and \(P\) is the matrix of transition probabilities. Some natural questions arise: which functions \(h: X \to \mathbb{C}\) are \(P\)-harmonic, meaning \(P h = h\)? And what is the asymptotic behaviour of the random walk as time goes to infinity? So, let \(\Omega := X^\mathbb{N}\) be the space of all infinite paths in \(X\) and let \(X_n: \Omega \to X\) be the \(n\)-th coordinate projection. Can we describe the behaviour of \(X_n\) as \(n \to \infty\)?

It turns out that the answers to these two questions are related. Martin [Mar41] defined a compactification \(\hat{M}(X, P)\) of \(X\) with respect to \(P\) (nowadays called the Martin compactification) and a boundary \(M(X, P) := \hat{M}(X, P) \setminus X\). He proved that every positive harmonic function can be represented by an integral over this boundary, hereby partially answering the first question. This result is the probabilistic analogue of the theorem that
any analytic function on the disc can be represented by an integral over the circle. Some years later Doob [Doo59] and Hunt [Hun60] independently showed existence of a measurable function $X_\infty : \Omega \to M(X, P)$ such that the coordinate maps $X_n$ converge to $X_\infty$. Note that this function $X_\infty$ takes values in the Martin boundary, therefore nowadays the terminology “convergence to the boundary” is used. Moreover they strengthened Martin’s result and proved by means of this convergence result that positive harmonic functions can uniquely be represented on a smaller subset of the Martin boundary, the so-called minimal boundary. Let $\nu^1$ be the measure on the Martin boundary which represents the constant function 1. It can be shown that if $h$ is a bounded harmonic function, then the corresponding representing measure $\nu^h$ is absolutely continuous with respect to $\nu^1$. The Martin boundary together with this measure $\nu^1$ is called the Poisson boundary: a measure theoretic boundary of $X$. The Poisson boundary describes all bounded harmonic functions on $X$.

In the early nineties Biane [Bia91, Bia92a, Bia92b, Bia94] started the study of noncommutative random walks on duals of compact groups. His idea was to work on the group von Neumann algebra $L(G)$ and act with operators of the form $P_\varphi := (\varphi \otimes 1)\Delta$, where $\Delta$ is the comultiplication dual to the multiplication on the group and $\varphi$ is some state on $L(G)$. These operators $P_\varphi$ form the analogues of Markov operators used in Markov chains on discrete spaces. He used these to define a noncommutative analogue of a Martin kernel, which lead to a Martin boundary. Biane computed the Martin boundary of $\hat{SU}(2)$, which turned out to be the two-dimensional sphere.

Izumi [Izu02] took this one step further and brought the fields of quantum groups and random walks together. The operators $P_\varphi$ considered by Biane easily translated to discrete quantum groups, which gave rise to Markov operators $P_\varphi$ and hence random walks on discrete quantum groups. The space of $P_\varphi$-harmonic elements is called the Poisson boundary and forms the noncommutative analogue of the Poisson boundary for classical random walks. In spirit of the Gelfand–Naimark theorem, this Poisson boundary should be thought of as the space of functions on a noncommutative Poisson boundary. Izumi immediately computed the first example: the Poisson boundary of $\hat{SU}_q(2)$ is $L^\infty(T\setminus SU_q(2))$, the weak closure of a Podleš sphere. Later the most general result was obtained by Tomatsu [Tom07] identifying the Poisson boundary of a random walk on any coamenable compact quantum group with commutative fusion rules. Other quantum groups for which the Poisson boundary has been identified include: $SU_q(N)$ [INT06]; free orthogonal and free unitary quantum group [VVV08, VVV10] and quantum automorphism groups [DRVV10]. As in the classical case these Poisson boundaries capture some of the structure of the underlying quantum group. For example Izumi’s main motivation for developing the Poisson boundary was to study the (non)minimality of actions of compact quantum groups on von Neumann algebras. He considered actions on infinite tensor products and proved that the relative commutant of the action is isomorphic to the space of harmonic elements of a Markov operator $P$.

Neshveyev and Tuset [NT04] built further on this story and defined the Martin boundary for noncommutative random walks on discrete quantum groups. Again they computed the fundamental example of the Martin boundary of random walks on $SU_q(2)$, this bound-
ary is isomorphic to $C(S^2_{0,q})$, the Podleś sphere. Classically the Martin boundaries are much harder to compute than Poisson boundaries. In the quantum case this is not different. Some Martin boundaries of random walks on other quantum groups have been identified, but in some way they were always related to $SU_q(2)$. See for example the free orthogonal quantum groups [VV07], [VVV08] and [DRVV10].

Since there are very few examples known, one of the goals of this PhD project is to obtain more concrete examples of Martin boundaries of random walks on discrete quantum groups. Given a random walk defined by a state $\varphi_\mu$ on a discrete quantum group $l^\infty(\hat{G})$, there exists a classical random walk on the set of irreducible representations $\text{Irr}(G)$. The Martin boundary $M(\text{Irr}(G), \mu)$ of this classical random walk can be embedded into the center $Z(M(\hat{G}, \mu))$. So a first step in understanding the Martin boundary $M(\hat{G}, \mu)$ is to understand the Martin boundary of this classical random walk. The hope is then that $M(\hat{G}, \mu)$ forms a $C^*$-algebra over the topological space $M(\text{Irr}(G), \mu)$. The next step would be to determine the fibers over each point in this classical Martin boundary. Biane [Bia91] worked on a related problem for $SU(N)$, in which he partially computed the Martin boundary for random walks on $\text{Irr}(SU(N))$. Motivated by his methods we compute the largest part of the Martin boundary $M(\text{Irr}(G_q), \mu)$ for classical random walks on the lattice of the irreducible representations for $q$-deformed Lie groups $G_q$. However, in the second step trying to identify the fibers over each point in $M(\text{Irr}(G_q), \mu)$, the story ends. When $G_q = SU_q(N)$, it becomes very hard to determine the asymptotic behaviour of the Martin kernel acting on matrix units, so to identify $K_\mu(m^q_{ij})$ in the quotient space $l^\infty(\hat{G})/C_0(\hat{G})$ for matrix units $m^q_{ij}$. There are some natural candidates for possible Martin boundaries, most notably some quantum flag manifolds [DCN15], but these turned out not to work, at least not in the most natural way.

In [NT04] it is proved that, analogous to the classical case in which any harmonic element is represented by a measure on the Martin boundary, in the quantum case any positive harmonic element can be represented by a linear functional on the boundary. However, there is only existence and not uniqueness or a natural choice for such a representing functional. Classically, convergence to the boundary resolves this issue. In the overview paper [NT03] there is a conjecture what convergence to the boundary should correspond to in the quantum world. However, the problem with proving boundary convergence in the noncommutative setting is that the “commutative proof” is very hard to translate. Classically stopping times and martingale convergence theorems are used to obtain a very delicate proof of almost everywhere convergence. At this point it is not clear how to modify the stopping times, up- and downcrossings in a noncommutative way. We are also not able to prove the conjecture, but we will verify that the conjecture of convergence to the boundary, as proposed in [NT03], holds for $SU_q(2)$. Our approach is very computational, but it shows that there is exponentially fast convergence, which is a lot faster than what occurs classically.

We now discuss the last part of the title of the thesis: the categories. Given a compact quantum group $G$, all unitary finite dimensional representations of $G$ can be assembled into a $C^*$-tensor category. Woronowicz’s duality theorem [Wor88] gives a procedure how
to reconstruct a compact quantum group from its category of representations and describes which categories occur as a representation category of compact quantum groups. In the paper discussing this Tannaka–Krein duality, Woronowicz also posed the problem of classifying all quantum groups $G$ which have the same representation theory as $SU(N)$, so having the same fusion rules and dimensions. This and the same problem for other Lie groups is very hard and many people have worked on it, see for example [McM84, Was88a, Was88b, Ban96, Hai00, Bic03, Pin07, NY16] and references therein. If one restricts the class of quantum groups to the non-Kac, compact quantum groups, then the problem becomes manageable. Indeed, Neshveyev and Yamashita [NY14c, NY14b] defined a Markov operator for C$^*$-tensor categories and associated to it a Poisson boundary. They used this in a crucial way to obtain the desired classification. Their result is that all quantum groups with the same representation theory as $SU(N)$ are given by certain twists of $SU_q(N)$. As an intermediate step in this classification procedure it is necessary to classify all C$^*$-tensor categories that are monoidally equivalent to $\text{Rep}(SU(N))$. Kazhdan–Wenzl [KW93] obtained such a classification in the purely algebraic setting, in [Jor14] we modified their argument to work in the C$^*$-tensor case. Moreover we give a sufficient condition when a “twist” of $\text{Rep}(SU_q(N))$ can be embedded in a given C$^*$-tensor category, hereby making the connection with Pinzari’s work ([Pin07]) in which she gives an intrinsic characterization of $SU_q(N)$. Our work on these problems can be found in Chapter 2.

This definition of a categorical Poisson boundary for random walks on C$^*$-tensor categories did not come out of the blue. De Rijdt and Vander Vennet [DRVV10] established a concrete method to compute the Poisson and Martin boundary for a quantum group out of the Poisson and Martin boundary of a monoidally equivalent quantum group. Later Neshveyev and Yamashita [NY14a] gave a correspondence between certain algebras with $G$-$\hat{G}$-actions and C$^*$-tensor categories related to $\text{Rep}(G)$. The Poisson boundary $H^\infty(\hat{G},\mu)$ of a random walk on a discrete quantum group is exactly such an algebra, so there exists a C$^*$-tensor category corresponding to $H^\infty(\hat{G},\mu)$. It shows that the Poisson boundary is intrinsic to the underlying representation category $\text{Rep}(G)$ and does not depend on the actual choice of fiber functor realising the compact quantum group and hence motivates the definition of a categorical Poisson boundary. In a similar fashion also the Martin boundary $M(\hat{G},\mu)$ admits $G$-$\hat{G}$-actions and Neshveyev–Yamashita’s correspondence can be applied to the Martin boundary. It is therefore natural to expect that there exists a categorical Martin compactification and boundary. This turns out to be the case. We give a definition and show that under the correspondence of [NY14a] the Martin boundary $M(\hat{G},\mu)$ of a random walk on a discrete quantum group can be reconstructed from the categorical Martin boundary $M(\text{Rep}(G),\mu)$ of the random walk on the representation category $\text{Rep}(G)$.

A natural question to ask is how convergence to the boundary behaves under monoidal equivalence, or to be more precise if it can be formulated in the categorical setting. Fortunately this question has a positive answer. It yields a broader range of examples of compact quantum groups for which convergence to the boundary holds.

The thesis is organised as follows. We start with some background material regarding
operator algebras, category theory, quantum groups and Lie groups. The thesis is fairly self-contained, but we do not go all the way back to the basics. In each of these preliminary sections there will be a list of standard references to the literature. The first chapter mainly serves to fix notation and to give an overview of the theorems and tools that will be used later on. In this chapter there are no new results.

The second chapter is essentially (a slightly extended version of) the paper [Jor14]. Here we follow Kazhdan–Wenzl’s ideas and give a classification of all $C^*$-tensor categories with the same fusion rules as $\text{SU}(N)$.

Chapter 3 is again a chapter containing preparatory material. The first section gives an overview of the results on discrete Markov chains in the classical case. The focus is on theorems regarding the Poisson and Martin boundary. In the second section we build up the theory of noncommutative random walks on discrete quantum groups. Again there are no new results here.

The largest chapter of this thesis, Chapter 4 contains several new results on the Martin boundary for random walks on discrete quantum groups. In the first section we compute parts of the Martin boundary of classical random walks on the center of $L^\infty(\hat{G}_q)$ for $q$-deformations of semisimple simply-connected compact Lie groups. In the second section one can find the partial results for the computation of the full Martin boundary for random walks on $\text{SU}_q(N)$. The third section is devoted to the conjecture of convergence to the boundary. It is a mixture of new results and results already contained in the literature. After this we verify that random walks on $\text{SU}_q(2)$ satisfy convergence to the boundary.

In the final chapter we put some of the results in a categorical framework. We start with the more concrete picture of link algebras to transport the property of convergence to the boundary from one quantum group to a monoidally equivalent one. Then we move on and define the Martin compactification and Martin boundary for random walks on $C^*$-tensor categories. In Section 5.4 convergence to the boundary is defined for such random walks. We conclude by showing that whenever the category under consideration is of the form $\mathcal{C} = \text{Rep}(G)$ for a compact quantum group $G$, then the quantum and categorical pictures are equivalent, meaning that one can pass from one to the other and back. The material presented in Chapter 5 and Sections 4.3, 4.4 can also be found in the preprint [Jor16].

Finally, there is an introduction for the layman. In this short chapter we try to explain in a nontechnical way what this thesis is about and what kind of problems we tried to solve.

**Acknowledgements**

The last three years I had the pleasure of doing a PhD here at the university of Oslo. This time has now almost come to an end and I would like to thank some people.

First and foremost I would like to thank Sergey Neshveyev for giving me the opportunity to do a PhD here in Oslo. The project you let me work on was very interesting and gave me the opportunity to work with some beautiful mathematics. In addition you
were always a great source of information and inspiration and you put me back on track whenever I got stuck at a problem.

I want to thank Lars Tuset for being my second supervisor. In addition I want to express my gratitude to all the other people in the operator algebras group. In particular I would like to mention Marco and Sara for welcome breaks and good talks over a coffee or a dinner, but most of all for being good friends.

When writing the chapter “Introduction for the Layman” I received some useful feedback from Torgeir and Sara. Thank you for that.

Moreover I am very grateful to everyone at the two sport clubs I joined: OSI Aikido and OSI Ultimate. Moving to a new country was not always easy and I am very glad that you embraced me as a member. I got to know a lot of friends and I spend countless nights with you training in the sports centre. Doing sports after a day of mathematics was the perfect way for me to relax my mind while having a good physical workout.

Last, but definitely not least, I would like to express my gratitude to my parents Tonnie and Maria and my sister Carlijn and brother Luuk for unconditionally supporting me.

Thanks!

The research presented in this thesis has received funding from the European Research Council under the European Unions Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement no. 307663 (PI: S. Neshveyev).
INTRODUCTION
Chapter 1

Preliminaries

In this chapter we build up the essential tools. At the beginning of each subsection we will give a number of standard references to the literature. For the elementary results the references will be omitted.

1.1 Conventions and notation

To avoid confusion we fix the most elementary notation in this short section.

- $\mathbb{N}$ denotes the natural numbers and equals the set $\{0, 1, 2, \ldots\}$, so including 0;
- $\mathbb{Z}$ denotes the integers and $\mathbb{Z}_+ := \mathbb{N}$;
- $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the rational, real and respectively complex numbers;
- $\iota$ will be used to denote the identity map;
- In a Hilbert space the inner product is denoted by $\langle \cdot, \cdot \rangle$, which is assumed to be antilinear in first entry and linear in the second;
- $B(\mathcal{H}, \mathcal{K})$, $B_0(\mathcal{H}, \mathcal{K})$ and $B_{00}(\mathcal{H}, \mathcal{K})$ denote respectively the bounded, compact and finite rank operators $\mathcal{H} \to \mathcal{K}$;
- $B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H})$, $B_0(\mathcal{H}) := B_0(\mathcal{H}, \mathcal{H})$ and $B_{00}(\mathcal{H}) := B_{00}(\mathcal{H}, \mathcal{H})$;
- $\otimes$ will be used to denote various kinds of tensor products. For finite dimensional vector spaces it is the usual (algebraic) tensor product. For Hilbert spaces, the completed Hilbert space tensor product, for $C^*$-algebras the minimal tensor product and in tensor categories it will be the tensor product of the category;
- $\otimes_{\text{alg}}$ will be used to indicate the algebraic tensor product whenever there is a danger of confusion;
- If $X$ is a normed space, $X^*$ is the set of all bounded linear functionals on $X$;
- If $A$ is a $C^*$-algebra, $A_+$ is the set of all positive elements of $A$. 
Sweedler’s sumless notation will be used whenever convenient. If an element $x \in A \otimes B$ is of the form $x = \sum_i x_i^{(1)} \otimes x_i^{(2)}$, we write for example $x_{12} = x_{1,2} = x \otimes 1$ in $A \otimes B \otimes C$ or $x_{13} = x_{1,3} = \sum_i x_i^{(1)} \otimes 1 \otimes x_i^{(2)}$ in $A \otimes D \otimes B$. Strictly speaking this notation does not make sense, but it is a very useful and short notation for doing computations.

Given two functions $f, g : X \to \mathbb{C}$ and a point $x_0 \in X$, we write

(i) $f$ is $O(g)$ as $x \to x_0$ if there exists $C > 0$ and an open neighbourhood $U$ of $x_0$ such that $|f(x)| \leq C|g(x)|$ for all $x \in U$.

Assume in addition that $g$ has no zeroes in a neighbourhood of $x_0$, then we write

(ii) $f$ is $o(g)$ as $x \to x_0$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$;

(iii) $f(x) \sim g(x)$ as $x \to x_0$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$, in this case we say $f$ is of order $g$ as $x \to x_0$.

These first two are the so-called big $O$-notation and little $o$-notation. In particular $f$ is $o(1)$ as $x \to x_0$ if and only if $\lim_{x \to x_0} f(x) = 0$. We abuse the notation and write $f = g + o(h)$ if $f - g$ is of order $o(h)$. Similar notation is used for $O(h)$.

### 1.2 q-calculus

There is a whole theory of $q$-calculus involving $q$-differentiation, $q$-integration and $q$-special functions. Books have been written on this topic (see for instance [KC02]). In this thesis we only need the basic arithmetic properties, for this we follow [KS97, Ch. 2]. There are two different sorts of $q$-numbers and we will need them both.

**Notation 1.2.1.** For $q \notin \{-1, 0, 1\}$ denote

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad (n \in \mathbb{Z}).$$

For $q \neq 1$ put

$$[[n]]_q := \frac{1 - q^n}{1 - q}, \quad (n \in \mathbb{Z})$$

and $[[n]]_1 := n$. We refer to both quantities as $q$-numbers.\(^1\) The corresponding $q$-factorials are defined as

$$[1]_q! := 1, \quad [n]_q! := [n]_q[[n-1]]_q! \quad (n \geq 2),$$
$$[[1]]_q! := 1, \quad [[n]]_q! := [[n]]_q[[n-1]]_q! \quad (n \geq 2).$$

This leads to the $q$-binomials

$$\left[n\atop m\right]_q := \frac{[n]_q!}{[m]_q![n-m]_q!}.$$\(^1\)There is no distinction in name of $[n]_q$ and $[[n]]_q$.
The following lemma is easy to prove and we omit the proof. The lemma will be freely used throughout the thesis.

**Lemma 1.2.2.** The $q$-numbers satisfy the following identities:

(i) $[n]_q = q^{n-1} + q^{n-3} + \ldots + q^{-n+1}$;

(ii) $(m + n)q = q^m[m]_q + q^{-m}[n]_q = q^{-n}[m]_q + q^m[n]_q$;

(iii) $[[n]]_q = (1 + q + \ldots + q^{a-1})$;

(iv) $[[n]]_q = q^{(n-1)/2}[n]_q^{1/2}$.

**Lemma 1.2.3.** Assume that $0 < q < 1$. The $q$-numbers $[n]_q$ satisfy the following asymptotics:

$$
\frac{[n]_q}{[m]_q} \sim \frac{q^{-n}}{q^{-m}} = q^{m-n}, \quad \text{as both } m, n \to \infty;
$$

$$
\frac{[n]_q}{[m]_q} \sim \frac{q^n}{q^m} = q^{m-n}, \quad \text{as both } m, n \to -\infty.
$$

More precisely,

$$
[n]_q = \frac{-q^{-n}}{q - q^{-1}} (1 + o(1)), \quad \text{as } n \to \infty;
$$

$$
\frac{1}{[n]_q} = -q^n (q - q^{-1})(1 + o(1)), \quad \text{as } n \to \infty;
$$

$$
\frac{[n]_q}{[m]_q} = q^{m-n}(1 + O(q^{2m}) + O(q^{2n})) = q^{m-n}(1 + o(1)), \quad \text{as both } m, n \to \infty;
$$

$$
\frac{[n]_q}{[m]_q} = q^{n-m}(1 + O(q^{2m}) + O(q^{2n})) = q^{n-m}(1 + o(1)), \quad \text{as both } m, n \to -\infty.
$$

Moreover the following estimate holds

$$
\left| \frac{[n]_q}{[m]_q} - \frac{q^{-n}}{q^{-m}} \right| \leq q^n \frac{q^m}{1 - q^{2m}}, \quad \text{if } m \geq n.
$$

Also there exists $0 < c_1 < c_2 < \infty$ such that for all $n \in \mathbb{N}\{0\}$ it holds that $c_1 < [n]_q q^n < c_2$ and $c_1 < \frac{1}{[n]_q} q^{-n} < c_2$.

**Proof.** We have

$$
\frac{[n]_q}{[m]_q} \left( \frac{q^{-n}}{q^{-m}} \right)^{-1} = \frac{(q^n - q^{-n})q^n}{(q^m - q^{-m})q^m} = \frac{q^{2n} - 1}{q^{2m} - 1}.
$$

Recall that $q \in (0, 1)$, so taking the limit $m, n \to \infty$ gives $\lim_{m,n \to \infty} \frac{q^{2n} - 1}{q^{2m} - 1} = 1$ and shows the first assertion. The second one is proved similarly. To prove the $o$-statements, define $g$ by $[n]_q = \frac{-q^{-n}}{q - q^{-1}} (1 + g(n))$, so

$$
g(n) = \frac{q^n - q^{-n}}{q - q^{-1}} - q^{-n} - 1 = -q^{2n} + 1 - 1 = -q^{2n}.
$$
which is $o(1)$. The second one can be proved analogous. For the third define $h$ by
\[
\frac{|n|}{|m|} = q^{m-n}(1 + h(m, n)),
\]
then
\[
h(m, n) = \frac{|n|}{|m|} q^n - 1 = \frac{q^{2n} - 1}{q^{2m} - 1} - 1 = \frac{q^{2m} - q^{2m}}{q^{2m} - 1} = o(1), \quad \text{as } n, m \to \infty.
\]
and the last one is similar again. For the estimate we have
\[
\frac{|n|}{|m|} q^n - q^{-m} = \frac{(q^n - q^{-m})q^{-m} - (q^n - q^{-m})q^{-m}}{(q^n - q^{-m})q^{-m}} = \frac{q^{n-m} - q^{-n+m}}{1 - q^{-2m}} = q^m \frac{q^n - q^{-n+2m}}{q^{2m} - 1}.
\]
Thus for $m \geq n$
\[
\left| \frac{|n|}{|m|} q^n - q^{-m} \right| = q^m \frac{q^n - q^{-n+2m}}{1 - q^{2m}} \leq q^n \frac{q^m}{1 - q^{2m}}.
\]
Let $f(n) := |n|q^n = \frac{1 - q^{2n}}{1 - q}$ and thus
\[
c'_1 := \frac{1 - q^2}{q - 1} \leq \frac{1 - q^{2n}}{q - 1 - q} \leq \frac{1}{q - 1 - q} =: c'_2.
\]
Doing the same thing for $\frac{1}{|m|} q^{-n}$ gives $c''_1$ and $c''_2$. Setting $c_1 := \frac{1}{2} \min\{c'_1, c''_1\}$ and $c_2 := \max\{c'_2, c''_2\} + 1$ gives the result.

\[\Box\]

### 1.3 Operator algebras

Here we review some of the tools of operator algebras that are used throughout the thesis. We will not go all the way back to the basics, for this we refer to the literature. There are many good textbooks on operator algebras for instance [Bla06], [Mur90] and [Tak02, Tak03a, Tak03b].

The algebra $B(\mathcal{H})$ has several topologies. We list the ones that are important for us. Let $(x_i)_i \subset B(\mathcal{H})$ be a net and $x \in \mathcal{H}$. Then $x_i$ converges to $x$ in

(i) norm or uniform topology if $\|x_i - x\| \to 0$;

(ii) strong operator topology if $\|(x_i - x)\xi\| \to 0$ for all $\xi \in \mathcal{H}$;

(iii) strong* operator topology if $\|(x_i - x)\xi\| + \|(x_i^* - x^*)\xi\| \to 0$ for all $\xi \in \mathcal{H}$;

(iv) weak operator topology if $\langle \eta, (x_i - x)\xi \rangle \to 0$ for all $\eta, \xi \in \mathcal{H}$;

(v) $\sigma$-weak operator topology if $|\sum_n (\xi_n, (x_i - x)\eta_n)| \to 0$ for all sequences $(\xi_n)_n$ and $(\eta_n)_n$ in $\mathcal{H}$ with $\sum_n \|\xi_n\|^2 < \infty$ and $\sum_n \|\eta_n\|^2 < \infty$.

We write respectively $\lim_i x_i = x$, $s\lim_i x_i = x$, $s^*\lim_i x_i = x$, $w\lim_i x_i = x$ and
\(\sigma\text{-}\lim_i x_i = x\). The following implications hold:

\[
\lim_i x_i = x \quad \Rightarrow \quad s^*\text{-}\lim_i x_i = x \quad \Rightarrow \quad s\text{-}\lim_i x_i = x.
\]

There are more topologies around, but we will not use them, so they are not listed.

Recall that if \(M\) is a von Neumann algebra, then the weak and strong operator topologies on \(M\) in general depend on the embedding of \(M\) into \(B(H)\) for some Hilbert space \(H\). The restriction of the topologies to bounded sets are independent of such an embedding.

Given \(S \subset B(H)\) denote its commutant by \(S' := \{x \in B(H) : xs = sx\text{ for all }s \in S\}\).

Let \(A\) be a C\(^*\)-algebra. A linear functional \(\varphi : A \to \mathbb{C}\) is called

- (i) faithful whenever \(\varphi(x^*x) = 0\) if and only if \(x = 0\);
- (ii) positive whenever \(\varphi(x^*x) \geq 0\) for all \(x \in A\);
- (iii) tracial whenever \(\varphi(xy) = \varphi(yx)\) for all \(x, y \in A\);
- (iv) a state if \(\varphi\) is positive and of norm 1.

Let \(M\) be a von Neumann algebra and \(\varphi : M \to \mathbb{C}\) a linear functional, then \(\varphi\) is called

- (v) normal if \(\varphi(\lim_i x_i) = \varphi(\sup(x_i))\) for every bounded increasing net \((x_i)_{i \in M_+}\).

Here \(\sup(x_i)\) is the least upper bound of the net \((x_i)_{i \in M_+}\).

For a positive linear functional \(\varphi\) the following Cauchy–Schwarz inequality holds:

\[|\varphi(x^*y)|^2 \leq \varphi(x^*x)\varphi(y^*y), \quad \text{for all } x, y \in A.\]

Any positive linear functional on a C\(^*\)-algebra is automatically norm-continuous. The GNS representation of a positive linear functional \(\varphi\) is denoted by \((\pi_\varphi, H_\varphi, \xi_\varphi)\).

The following result is folklore.

**Lemma 1.3.1.** Suppose \(M\) is a von Neumann algebra, \((x_i)_{i \in M} \subset M\) is a bounded net and \(\varphi \in M_+^*\) is a faithful positive linear functional. Then \(s\text{-}\lim_i x_i = x\) if and only if \(\lim_i \varphi((x_i - x)^*(x_i - x)) = 0\).

**Proof.** Let \(\xi_\varphi\) be the unit vector of the GNS representation of \(\varphi\). Then strong convergence implies that \(\lim_i \varphi((x_i - x)^*(x_i - x)) = \lim_i ||(x_i - x)\xi_\varphi||^2 = 0\).

Conversely, consider the GNS representation \((H_\varphi, \pi_\varphi, \xi_\varphi)\) of \(M\) with respect to \(\varphi\). Embed \(M \subset B(H_\varphi)\) using \(\pi_\varphi\). For any \(x' \in M' \subset B(H_\varphi)\) we get

\[
\lim_i ||(x_i - x)x'\xi_\varphi||^2 = \lim_i ||x'(x_i - x)\xi_\varphi||^2 \leq \limsup_i ||x'||^2||(x_i - x)\xi_\varphi||^2
\]

\[= \limsup_i ||x'||^2\varphi((x_i - x)^*(x_i - x)) = 0.\]

The vector \(\xi_\varphi\) is separating for \(\pi_\varphi(M)\), because \(\varphi\) is faithful. Hence \(\xi_\varphi\) is cyclic for \(M'\), which in turn means that \(||(x_i - x)\xi|| \to 0\) for the dense collection of vectors \(\xi \in \{x'\xi_\varphi : x' \in M'\} \subset H_\varphi\). Thus \(s\text{-}\lim_i x_i = x\). \(\blacksquare\)
Example 1.3.2. Suppose we are in the commutative situation. So let \( \mu \) be a probability measure on a space \( X \). Then \( L^\infty(X, \mu) \) acts on \( L^2(X, \mu) \) by point-wise multiplication of functions. The functional \( \int_X \cdot \, d\mu \) is a faithful normal state on \( L^\infty(X, \mu) \). Assume that \( (f_n)_n \subset L^\infty(X, \mu) \) is a uniformly bounded sequence of functions that converges in strong topology to \( f \). By Lemma 1.3.1 this is equivalent to

\[
\lim_n \int_X |f_n - f|^2 \, d\mu = \lim_n \int_X (f_n - f)^*(f_n - f) \, d\mu = 0.
\]

Thus \( f_n \) converges to \( f \) in square mean if and only if \( \text{s}^* \lim_n f_n = f \).

A representation of a \( C^* \)-algebra \( A \) on a Hilbert space \( \mathcal{H} \) is a \( * \)-homomorphism \( \pi : A \to B(\mathcal{H}) \). A representation is called faithful if \( \pi(x) = 0 \) implies \( x = 0 \). The Gelfand–Naimark theorem asserts that any \( C^* \)-algebra can be concretely represented on a Hilbert space, meaning that for every \( C^* \)-algebra \( A \) there exists a Hilbert space \( \mathcal{H} \) together with a faithful representation \( \pi : A \to B(\mathcal{H}) \). Such a representation is constructed by taking the direct sum of all GNS representations \( (\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi) \), where \( \varphi \) ranges over all states of \( A \). The obtained representation \( (\pi_u, \mathcal{H}_u) := \bigoplus_\varphi (\pi_\varphi, \mathcal{H}_\varphi) \) is sometimes called the universal representation of \( A \). The enveloping von Neumann algebra equals \( \pi_u(A)' \subset B(\mathcal{H}_u) \). If \( \pi : A \to B(\mathcal{H}) \) is a faithful representation, we sometimes suppress \( \pi \) and write \( A \subset B(\mathcal{H}) \). Suppose \( (\pi, \mathcal{H}) \) and \( (\tilde{\pi}, \tilde{\mathcal{H}}) \) are two faithful representations of \( A \), then \( \tilde{\pi} \circ \pi^{-1} : \pi(A) \to \tilde{\pi}(A) \) is a \( * \)-isomorphism.

Suppose \( (\pi, \mathcal{H}) \) is a faithful representation of \( A \). An element \( x \in \pi(A)' \) is called a multiplier if \( x\pi(A) \subset \pi(A) \) and \( \pi(A)x \subset \pi(A) \). Denote \( M(A) := \{ x \in B(\mathcal{H}) : x \text{ is a multiplier} \} \). \( M(A) \) forms a \( C^* \)-subalgebra of \( \pi(A)' \), containing \( A \) as a closed ideal and is independent of the choice of the representation \( (\pi, \mathcal{H}) \). The algebra \( M(A) \) is called the multiplier algebra of \( A \). Obviously if \( A \) is unital, then \( M(A) = A \). However if \( A \) is not unital, the multiplier algebra is much bigger than \( A \). The multiplier algebra is the analogue of the Stone–Čech compactification of a locally compact space.

Notation 1.3.3. Given a collection of Banach spaces \( \{X_i\}_{i \in I} \) for some index set \( I \). We use the following conventions:

\[
\prod_{i \in I} X_i := \{(x_i)_i : x_i \in X_i\};
\]

\[
\bigoplus_{i \in I} X_i := \{(x_i)_i \in \prod_{i \in I} X_i : x_i \neq 0 \text{ for at most finitely many } i\};
\]

\[
c_0 \bigoplus_{i \in I} X_i := \{(x_i)_i \in \prod_{i \in I} X_i : (\|x_i\|_i) \in c_0(I)\};
\]

\[
l^\infty \bigoplus_{i \in I} X_i := \{(x_i)_i \in \prod_{i \in I} X_i : \sup_{i \in I} \|x_i\| < \infty\}.
\]

Note that the last two algebras are Banach spaces, with respect to the norm \( \|(x_i)_i\| := \sup_{i \in I} \|x_i\| \). The norm-closure of \( \bigoplus_{i \in I} X_i \) in \( l^\infty \bigoplus_{i \in I} X_i \) equals \( c_0 \bigoplus_{i \in I} X_i \). Furthermore if all \( X_i = A_i \) are \( C^* \)-algebras, then \( c_0 \bigoplus_{i \in I} A_i \) and \( l^\infty \bigoplus_{i \in I} A_i \) are \( C^* \)-algebras. Moreover if all \( A_i \) are unital for the multiplier algebra the equality \( M(c_0 \bigoplus_{i \in I} A_i) = l^\infty \bigoplus_{i \in I} A_i \) holds.
1.3. OPERATOR ALGEBRAS

For the construction of infinite tensor products of von Neumann algebras we follow [Tak03b, §XIV.1]. This construction makes use of infinite tensor products of C*-algebras, which in turn needs inductive limits.

**Definition 1.3.4.** Given a sequence of C*-algebras \((A_n)_{n=0}^{\infty}\) together with \(\ast\)-homomorphisms \(\pi_n: A_n \to A_{n+1}\), there exists an inductive limit

\[ A := \lim_{\rightarrow} (A_n, \pi_n), \]

with inclusion maps \(\iota_n: A_n \to A\). This C*-algebra \(A\) can be described by the following universal property. If \(B\) is a C*-algebra with maps \(\alpha_n: A_n \to B\) such that for each \(n \in \mathbb{N}\) the diagram

\[ \begin{array}{ccc} A_n & \xrightarrow{\alpha_n} & B \\ \pi_n \downarrow & & \downarrow \alpha_{n+1} \\ A_{n+1} & \xrightarrow{\iota_n} & A \end{array} \]

commutes, then there exists a unique map \(\alpha: A \to B\) such that the following diagram commutes for every \(n \in \mathbb{N}\)

\[ \begin{array}{ccc} A_n & \xrightarrow{\alpha_n} & B \\ \iota_n \downarrow & & \downarrow \alpha \\ A & \xrightarrow{\alpha} & B \end{array} \]

**Definition 1.3.5.** Given two C*-algebras \(A_1\) and \(A_2\) define a norm \(\|\cdot\|_{\text{min}}\) on the algebraic tensor product \(A_1 \otimes_{\text{alg}} A_2\) by

\[ \|x\|_{\text{min}} := \sup\{\|\pi_1 \otimes \pi_2(x)\|: \pi_i \text{ is a representation of } A_i\}. \]

The completion of \(A_1 \otimes_{\text{alg}} A_2\) with respect to \(\|\cdot\|_{\text{min}}\) is called the minimal tensor product and will be denoted by \(A_1 \otimes_{\text{min}} A_2\). The norm satisfies \(\|x \otimes y\|_{\text{min}} = \|x\| \|y\|\). Moreover \(\|\cdot\|_{\text{min}}\) is the smallest C*-norm on \(A_1 \otimes_{\text{alg}} A_2\).

If \(M_1 \subset B(H_1)\) and \(M_2 \subset B(H_2)\) are two von Neumann algebras, the von Neumann tensor product is defined as

\[ M_1 \overline{\otimes} M_2 := (M_1 \otimes_{\text{alg}} M_2)^{\prime\prime}. \]

In general \(M_1 \overline{\otimes} M_2\) is larger than \(M_1 \otimes_{\text{min}} M_2\).

Given C*-algebras \(A_i, B_i\) and \(\ast\)-homomorphisms \(\alpha_i: A_i \to B_i\) for \(i = 1, 2\), then \(\alpha_1 \otimes \alpha_2: A_1 \otimes_{\text{alg}} A_2 \to B_1 \otimes_{\text{alg}} B_2\) can be extended to a \(\ast\)-homomorphism \(\beta_1 \otimes \alpha_2: A_1 \otimes A_2 \to B_1 \otimes B_2\). If \(\alpha_1\) and \(\alpha_2\) are injective then so is \(\alpha_1 \otimes \alpha_2\). Moreover, if \(A_i\) are concretely represented on a Hilbert space, then \((A_1 \otimes A_2)^{\prime\prime} = A_1^{\prime\prime} \overline{\otimes} A_2^{\prime\prime}.

**Definition 1.3.6.** The C*-algebraic tensor product of a sequence \((A_n)_{n=0}^{\infty}\) of unital C*-algebras is defined as the inductive limit

\[ \bigotimes_{n=0}^{\infty} A_n := \lim_{\rightarrow} (A_n, \pi_n), \]
CHAPTER 1. PRELIMINARIES

where $A'_n := A_1 \otimes \cdots \otimes A_n$ and $\pi_n(x) := x \otimes 1$. Consider the inclusion maps $\iota_n: A'_n \to \bigotimes_n A_n$. If $x \in A_1 \otimes \cdots \otimes A_n$, denote $\cdots \otimes 1 \otimes 1 \otimes x := \lim_n \iota_n(x)$.

Given a sequence $(M_n)_{n=0}^{\infty}$ of von Neumann algebras with normal states $\omega_n: M_n \to \mathbb{C}$, define the state $\omega = \bigotimes_n \omega_n$ on the $C^*$-algebra $A := \bigotimes_n M_n$ by

$$\omega(x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \otimes \cdots) := \omega_1(x_1) \cdots \omega_n(x_n)$$

and extension to $A$. The GNS construction applied to $\omega$ gives a cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ of $A$. Define the von Neumann infinite tensor product of $(M_n, \omega_n)_{n=0}^{\infty}$ as

$$M := \bigotimes_{n=0}^{\infty} (M_n, \omega_n) := (\pi_\omega(A))''.$$

The commutant is taken in $B(\mathcal{H}_\omega)$. The functional $\omega$ extends to a state on $M$. The state $\omega$ is faithful on $M$ if for each $i$ the state $\omega_i$ is faithful on $\pi_\omega_i(M_i)$. In a similar way one can define tensor products of the form $\bigotimes_{-\infty}^{-1}$, these will be relevant for us.

Note that this construction of the infinite tensor product depends heavily on the choice of the sequence of states $(\omega_n)_n$, different choices can give non-isomorphic algebras see for example [Tak03b, Thm. XVIII.1.1].

**Lemma 1.3.7** (Noncommutative martingale convergence theorem). Let $M$ be a von Neumann algebra with a normal state $\varphi$. By construction of the infinite tensor product for any $n \in \mathbb{N}$ there is the embedding

$$i_n: \bigotimes_{-n}^{-1} (M, \varphi) \hookrightarrow \bigotimes_{-\infty}^{-1} (M, \varphi), \quad x \mapsto \cdots \otimes 1 \otimes 1 \otimes x.$$

Define slice maps

$$E'_n: \bigotimes_{-\infty}^{-1} (M, \varphi) \to \bigotimes_{-n}^{-1} (M, \varphi),$$

$$E'_n(\cdots \otimes 1 \otimes 1 \otimes x_m \otimes \cdots \otimes x_1) := \varphi(x_m) \cdots \varphi(x_{n+1}) x_n \otimes \cdots \otimes x_1, \quad (m > n).$$

Then $E_n := i_n \circ E'_n$ is the unique $\varphi$-preserving conditional expectation onto $i_n(\bigotimes_{-n}^{-1} (M, \varphi))$. The maps $E_n$ satisfy:

(i) for every $x \in \bigotimes_{-\infty}^{-1} (M, \varphi)$ it holds that $x = s^* \lim_n E_n(x)$;

(ii) if the sequence $(x_n)_n \subset \bigotimes_{-\infty}^{-1} (M, \varphi)$ satisfies $E_n(x_{n+1}) = x_n$ for all $n$, then there exists a unique $x \in \bigotimes_{-\infty}^{-1} (M, \varphi)$ such that $x_n = E_n(x)$ for all $n$.

**Proof.** The formulation of this lemma is from [Izu02, Lem. 3.4.], the proof can be found in [Con75, Lem. 2].

For KMS states we follow [BR87, BR97].
**1.3. OPERATOR ALGEBRAS**

**Definition 1.3.8.** A $C^*$-dynamical system consists of a pair $(A, \tau)$, where $A$ is a $C^*$-algebra and $\tau: \mathbb{R} \to \text{End}(A)$, $t \mapsto \tau_t$ is a group homomorphism such that $t \mapsto \tau_t(a)$ is norm-continuous for each $a \in A$.

Similarly a $W^*$-dynamical system consists of a pair $(M, \tau)$, where $M$ is a von Neumann algebra and $t \mapsto \tau_t \in \text{End}(M)$ is a group homomorphism such that $t \mapsto \tau_t$ is weakly continuous.

An element $a \in A$ is analytic for $\tau$ if there exists a function $f: \mathbb{C} \to A$ such that

(i) $f(t) = \tau_t(a)$ for all $t \in \mathbb{R}$;

(ii) the function $\mathbb{C} \to \mathbb{C}$, $z \mapsto \eta(f(z))$ is analytic for all $\eta \in A^*$.

Denote the set of $\tau$-analytic elements by $A^\tau$. For $a \in A^\tau$ and $z \in \mathbb{C}$ write $\tau_z(a) := f(z)$.

The space $A^\tau$ is a norm-dense $*$-subalgebra of $A$. Moreover $A^\tau$ is $\tau$-invariant, i.e. $\tau_t(a) \in A^\tau$ for all $a \in A^\tau$ and $t \in \mathbb{R}$. The same definition for analyticity applies in the von Neumann case, we write $M^\tau$ for the $\tau$-analytic elements of $M$. In fact, it suffices to only take normal linear functionals instead of the full dual $M^*$.

**Definition 1.3.9.** A state $\varphi$ on a $C^*$-algebra $A$ is a $\tau$-KMS state at value $\beta \in \mathbb{R}$ if $\varphi(\tau_t(a)b) = \varphi(b\tau_t(a))$ for all $a, b$ in a norm dense, $\tau$-invariant $*$-subalgebra of $A^\tau$.

A state $\varphi$ on a von Neumann algebra $M$ is a $\tau$-KMS state at value $\beta \in \mathbb{R}$ if $\varphi$ is normal and $\varphi(\tau_t(a)b) = \varphi(b\tau_t(a))$ for all $a, b$ in a $\sigma$-weakly dense, $\tau$-invariant $*$-subalgebra of $M^\tau$.

We say $\varphi$ is a KMS state if it is a KMS state at value $-1$.

We finish this section with some modular theory. A von Neumann algebra $M \subset B(\mathcal{H})$ is in standard form if there exists a unit vector $\xi \in \mathcal{H}$ which is cyclic and separating for $M$. Any von Neumann algebra with separable predual $M^*$ can be brought in standard form.

Let $M \subset B(\mathcal{H})$ with $\xi \in \mathcal{H}$ be in standard form. Define the (unbounded) antilinear operator

$$S_0: M\xi \to \mathcal{H}, \quad S_0(a\xi) := a^*\xi.$$ 

The domain $\text{Dom}(S_0) = M\xi$ is dense in $\mathcal{H}$. It can be shown that $S_0$ is closable. Denote its closure by $S$. Use the polar decomposition to write $S = J\Delta^{\frac{1}{2}}$. Then $J$ is an antilinear invertible isometry, called the modular conjugation, satisfying $J^* = J$ and $J^2 = \iota$. Moreover $J\Delta J = \Delta^{-1}$. The operator $\Delta$ is positive, it is called the modular operator of $(M, \xi)$. The modular group is defined as

$$\sigma_t(x) := \Delta_t^{it}\Delta^{-it}, \quad (x \in M).$$

**Theorem 1.3.10 (Tomita–Takesaki).** Using the notation introduced above the following holds:

(i) $M' = JMJ$;

(ii) $\sigma_t(M) \subset M$ and $\Delta^{it}M'\Delta^{-it} \subset M'$ for every $t \in \mathbb{R}$.
Lemma 1.3.11. Let \( \eta: M \to \mathbb{C} \) be a state on a von Neumann algebra \( M \). Consider the GNS representation \( (\pi_\eta, \mathcal{H}_\eta, \xi_\eta) \). Then \( (M \subset \mathcal{B}(\mathcal{H}_\eta), \xi_\eta) \) is in standard form. Denote the induced modular group by \( \{\sigma_t^\eta\}_t \). Takesaki showed that \( \varphi \) is a KMS state for \( \sigma^\varphi \) and moreover \( \sigma^t \) is the only continuous \( \mathbb{R} \)-action for which this is true [Tak03a, §VIII.1].

Let \( \varphi \) be a faithful normal state on a von Neumann algebra \( M \). Consider the GNS representation \( (\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi) \). Then \( (M \subset \mathcal{B}(\mathcal{H}_\varphi), \xi_\varphi) \) is in standard form. Denote the induced modular group by \( \{\sigma_t^\varphi\}_t \). Takesaki showed that \( \varphi \) is a KMS state for \( \sigma^\varphi \) and moreover \( \sigma^t \) is the only continuous \( \mathbb{R} \)-action for which this is true [Tak03a, §VIII.1].

**Lemma 1.3.11.** Let \( \eta: A \to \mathbb{C} \) be a state on a \( C^* \)-algebra \( A \). Then \( \eta \) extends to a normal faithful state on the von Neumann algebra \( M := \pi_\eta(A)' \). Let \( \sigma^n = \sigma \) be the associated modular group. Then \( \eta: M \to \mathbb{C} \) is a \( \sigma \)-KMS state and

(i) the sesquilinear form \( \langle \cdot, \cdot \rangle_\eta: A \times A \to \mathbb{C}, (a,b)_\eta := \eta(b\sigma_{-\frac{t}{2}}(a^*)) = \langle \xi_\eta, bJ_\eta a\xi_\eta \rangle \) is a semi-inner product;

(ii) the sesquilinear form \( \langle \cdot, \cdot \rangle_\eta: M \times M \to \mathbb{C}, (a,b)_\eta := \eta(b\sigma_{-\frac{t}{2}}(a^*)) = \langle \xi_\eta, bJ_\eta a\xi_\eta \rangle \) is an inner product;

(iii) the linear functionals \( \langle \cdot, c \rangle_\eta: M \to \mathbb{C} \) and \( (c, \cdot)_\eta: M \to \mathbb{C} \) are positive if \( c \in M \) is positive;

This form above is a modification of the well-known semi-inner product \( \langle \cdot, \cdot \rangle_\eta: A \times A \to \mathbb{C}, (a,b)_\eta := \eta(a^*b) \). The modular group ensures that positivity result in (iii) holds. Note that \( \sigma_{-\frac{t}{2}}(b^*) \) need not be defined for all \( b \in A \), but the inner product \( \langle \xi_\eta, bJ_\eta a\xi_\eta \rangle \) is well-defined for all \( a, b \in A \).

**Proof of Lemma 1.3.11.** By the remarks above we only need to prove (i)–(iii). Clearly \( \langle \cdot, \cdot \rangle_\eta \) is sesquilinear. The KMS condition implies that \( \eta(ab) = \eta(ba_{-\frac{t}{2}}(a)) \) and also \( \eta(a) = \eta(\sigma_t(a)) \) for any \( t \). Recall that \( J_\xi_\eta = \xi_\eta \). To show that \( \langle \cdot, \cdot \rangle_\eta \) is positive and antisymmetric we calculate

\[
(a,a)_\eta = \langle \xi_\eta, aJa_\xi_\eta \rangle = \langle a^*\xi_\eta, JJa^*\xi_\eta \rangle = \langle a^*\xi_\eta, J^2\Delta^{\frac{1}{2}}a^*\xi_\eta \rangle = \langle a^*\xi_\eta, \Delta^{\frac{1}{2}}a^*\xi_\eta \rangle \geq 0;
\]

\[
(a,b)_\eta = \langle \xi_\eta, bJa_\xi_\eta \rangle = \langle \xi_\eta, b(J_\eta a)J_\xi_\eta \rangle = \langle \xi_\eta, Ja_\eta bJ_\xi_\eta \rangle = \langle J^*\xi_\eta, aJa_\xi_\eta \rangle = \langle \xi_\eta, aJa_\xi_\eta \rangle.
\]

So \( \langle \cdot, \cdot \rangle_\eta \) is a semi-inner product on \( A \). Since \( \eta \) is faithful on \( M \), it defines an inner product on \( M \).

For (iii) consider the positive elements \( a^*a \) and \( b^*b \), then

\[
(a^*a, b^*b)_\eta = \langle \xi_\eta, b^*bJa^*a_\xi_\eta \rangle = \langle \xi_\eta, b^*J(J_\eta b)J_\eta a^*a_\xi_\eta \rangle = \langle \xi_\eta, b^*Ja^*aJa_\eta J_\eta \rangle
\]

\[
= \langle aJa_\eta, aJa_\xi_\eta \rangle \geq 0,
\]

as desired. \( \square \)
Lemma 1.3.12 ([AC82, Prop. 3.1]). Let $B_1$ and $B_2$ be two $C^*$-algebras with KMS states $\eta_1$ and respectively $\eta_2$ and a completely positive contraction $T: B_1 \to B_2$ such that $\eta_2 T = \eta_1$. Write $N_i := \pi_{\eta_i}(B_i)^\prime\prime$. Then there exists a unique completely positive normal unital map $T^*: N_2 \to N_1$ such that $\eta_1 T^* = \eta_2$ and $(x, T^* y)_{\eta_1} = (Tx, y)_{\eta_2}$ for all $x \in B_1$ and $y \in B_2$.

1.4 Quantum groups

There are many good presentations of the theory of compact and discrete quantum groups, see for instance [NT13], [KT99], [Tim08]. Here we give a brief overview of the concepts and the notation we will use. We mostly follow [NT13], but some conventions are different.

1.4.1 Compact quantum groups and their representations

Definition 1.4.1. A compact quantum group is a pair $G = (\mathcal{C}(G), \Delta)$ consisting of a unital $C^*$-algebra $\mathcal{C}(G)$ and a $*$-homomorphism $\Delta: \mathcal{C}(G) \to \mathcal{C}(G) \otimes \mathcal{C}(G)$ called the comultiplication satisfying the properties

(i) $(\iota \otimes \Delta) \Delta = (\Delta \otimes \iota) \Delta$ as $*$-homomorphisms $\mathcal{C}(G) \to \mathcal{C}(G) \otimes \mathcal{C}(G) \otimes \mathcal{C}(G)$;

(ii) the linear spaces span\{(a \otimes 1) \Delta(b) : a, b \in \mathcal{C}(G)\} and span\{(1 \otimes a) \Delta(b) : a, b \in \mathcal{C}(G)\}$ are norm-dense in $\mathcal{C}(G) \otimes \mathcal{C}(G)$.

The first property is called coassociativity and the second the cancellation property.

Here $G$ is a symbolic notation, it suggests that the $C^*$-algebra $\mathcal{C}(G)$ can be thought of as functions on a group $G$. In general this is not the case.

Any compact semigroup with cancellation is a compact group. So the cancellation property on a compact semigroup implies the existence of an inverse. This is exactly why for compact quantum groups this cancellation property is enforced. Requiring the existence of a coinverse would be a severe restriction to the theory.

Example 1.4.2. The easiest examples of compact quantum groups are the following two

(i) If $G$ is a compact group then $\mathcal{C}(G)$, the $C^*$-algebra of continuous functions on $G$, forms a compact quantum group with comultiplication $\Delta(f)(g, h) := f(gh) \in \mathcal{C}(G \times G) \cong \mathcal{C}(G) \otimes \mathcal{C}(G)$.

(ii) Given a discrete group $\Gamma$. Let $\mathcal{C}(G) := \mathcal{C}_r^*(\Gamma)$ be the reduced group $C^*$-algebra of $\Gamma$ and denote its canonical generators by $\lambda_\gamma$. Define a comultiplication by $\Delta(\lambda_\gamma) := \lambda_\gamma \otimes \lambda_\gamma$. Note that $\Delta$ is cocommutative.

Let $G$ be a compact quantum group. There exists a unique state $h: \mathcal{C}(G) \to \mathbb{C}$, called the Haar state which satisfies

$$(h \otimes \iota) \Delta(a) = h(a) 1 = (\iota \otimes h) \Delta(a).$$

This is called right invariance, respectively left invariance of the Haar state. We assume that the Haar state $h$ is faithful, so we work with reduced compact quantum groups.
Definition 1.4.3. A unitary representation of $G$ on a Hilbert space $\mathcal{H}$ is a unitary element $U \in M(C(G) \otimes B_0(\mathcal{H}))$, such that

$$(\Delta \otimes \iota)U = U_{13}U_{23}. \quad (1.4.1)$$

We will unless mentioned otherwise only deal with unitary representations and therefore we will usually omit the adjective “unitary”.

Definition 1.4.4. If $U$ and $V$ are representations, a linear map $T \in B(\mathcal{H}_U, \mathcal{H}_V)$ is called an intertwiner if $(1 \otimes T)U = V(1 \otimes T)$. Denote the space of intertwiners by

$$\text{Hom}_G(U, V) := \{T \in B(\mathcal{H}_U, \mathcal{H}_V) : (1 \otimes T)U = V(1 \otimes T)\}.$$ 

Usually the subscript $G$ will be omitted. If $\text{Hom}(U, U) = \mathbb{C}\mathcal{I}_{\mathcal{H}_U}$, the representation $U$ is called irreducible. Two representations $U$ and $V$ are (unitarily) equivalent if $\text{Hom}(U, V)$ contains a unitary intertwiner, in which case we write $U \cong V$. The set of equivalence classes of irreducible representations is denoted by $\text{Irr}(G)$. We fix representatives $U_s$ for every $s \in \text{Irr}(G)$. The equivalence class of the trivial representation, the representation on $\mathbb{C}$, is denoted by $0$. The dimension of $U$ is by definition the dimension of $\mathcal{H}_U$ and is denoted by $\text{dim}(U)$. Sometimes, due to the leg numbering we put the $s$ as a superscript, so we write for example $U_{12}^s := (U_s)_{12}$.

Every irreducible representation is finite dimensional and every representation decomposes into a direct sum of irreducible representations. So in most cases it suffices to deal with finite dimensional representations. For two finite dimensional representations $U$ and $V$ define the tensor product representation on $\mathcal{H}_U \otimes \mathcal{H}_V$ by $U \times V := U_{12}V_{13}$. Note that in general $U \times V \not\cong V \times U$. For $s, t \in \text{Irr}(G)$ we write $s \otimes t$ when we want to indicate the tensor product $U_s \times U_t$. For instance, $\text{Hom}(s \otimes t, r) = \text{Hom}(U_s, U_t, U_r)$.

Notation 1.4.5. If $\mathcal{H}$ is a Hilbert space, identify the dual space $\mathcal{H}^*$ with the complex conjugate Hilbert space $\bar{\mathcal{H}}$. So $\bar{\xi} \in \bar{\mathcal{H}}$ acts as $\bar{\xi}(\zeta) = \langle \xi, \zeta \rangle$. Consider the map $j : B(\mathcal{H}) \to B(\bar{\mathcal{H}})$, $j(T)\bar{\xi} := \bar{T}^*\xi$. So under this identification $j$ maps $T$ to its dual operator.

Definition 1.4.6. Let $U$ be a finite dimensional representation. The contragredient representation $U^c$ on $\mathcal{H}_U$ is defined by

$$U^c := (\iota \otimes j)(U^{-1}) \in C(G) \otimes B(\bar{\mathcal{H}}).$$

Then $U^c \in C(G) \otimes B(\bar{\mathcal{H}})$ is invertible and satisfies $(\Delta \otimes \iota)(U^c) = U_{13}^cU_{23}^c$, although it need not be unitary. There exists a unique positive invertible operator $\rho_U \in \text{Hom}(U, U^c)$ such that

$$\text{Tr}(x\rho_U) = \text{Tr}(x\rho_U^{-1}), \quad \text{for all } x \in \text{Hom}(U, U). \quad (1.4.2)$$

The conjugate representation $\bar{U}$ of $U$ is defined as

$$\bar{U} := (1 \otimes j(\rho_U^{-1}))U^c(1 \otimes j(\rho_U)^{-\frac{1}{2}}) \in C(G) \otimes B(\bar{\mathcal{H}}_U)$$

this is again a unitary representation. The quantum dimension of $U$ equals $\text{Tr}(\rho_U)$ and is denoted by $d_U$.
Since $\rho_U$ is positive and invertible, it is diagonalizable. Write $n = \dim(U)$. Let $\{x_i\}_{i=1}^n$ be the eigenvalues of $\rho_U$ listed with multiplicity. The defining relation (1.4.2) of $\rho_U$ implies in particular that $\sum_{i=1}^n x_i = \Tr(\rho_U) = \Tr(\rho_U^{-1}) = \sum_{i=1}^n x_i^{-1}$. Therefore by Cauchy–Schwarz it follows that

$$\dim(U) = \Tr(\iota_U) = \sum_{i=1}^n x_i^{\frac{1}{2}} x_i^{-\frac{1}{2}} \leq \left(\sum_{i=1}^n x_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^n x_i^{-1}\right)^{\frac{1}{2}} = \Tr(\rho_U) = d_U. \tag{1.4.3}$$

Furthermore Cauchy–Schwarz tells us that equality holds if and only if there exists a $c \in \mathbb{C}$ such that $cx_i = x_i^{-1}$ for all $i$. Hence $d_U = \dim(U)$ if and only if $\rho_U$ is the identity.

The conjugate representation satisfies

$$\overline{U} \oplus \overline{V} = \overline{U} \oplus \overline{V}, \quad \overline{U} = U, \quad \overline{U} \times \overline{V} \cong \overline{V} \times \overline{U}.$$ 

Frobenius reciprocity holds and can be formulated as

$$\Hom(U_1, U_3 \times \overline{U}_2) \cong \Hom(U_1 \times U_2, U_3) \cong \Hom(U_2, \overline{U}_1 \times U_3),$$

where $U_1$, $U_2$ and $U_3$ are representations. In particular if $U$ is irreducible, then so is $\overline{U}$. We write $s \in \Irr(G)$ for the unique representative satisfying $\overline{U}_s = U_s$.

**Definition 1.4.7.** For a finite dimensional representation $U$ and $t \in \Irr(G)$ denote $m_U^t := \dim(\Hom(U, U))$ for the multiplicity of the representation $U$ in $U$. We abbreviate

$$m_{s_1, \ldots, s_n} := m_{U_{s_1} \times \cdots \times U_{s_n}}, \quad \text{for } s_1, \ldots, s_n, t \in \Irr(G).$$

**Lemma 1.4.8.** For $s_1, \ldots, s_n, t \in \Irr(G)$ the multiplicities satisfy the inequalities

$$m_{s_1, \ldots, s_n}^t \leq \frac{\dim(U_{s_1}) \cdots \dim(U_{s_n})}{\dim(U_t)},$$

$$\sum_{t \in \Irr(G)} m_{s_1, \ldots, s_n}^t \leq \dim(U_{s_1}) \cdots \dim(U_{s_n}).$$

**Proof.** For $s_1, \ldots, s_n \in \Irr(G)$ it holds that $\bigoplus_{r} m_{s_1, \ldots, s_n}^r U_r \cong U_{s_1} \times \cdots \times U_{s_n}$. Thus

$$\sum_{r \in \Irr(G)} m_{s_1, \ldots, s_n}^r \dim(U_r) = \dim(U_{s_1}) \cdots \dim(U_{s_n}),$$

from which the first inequality follows immediately. As each representation has dimension $\geq 1$ also the second estimate follows from this identity.

**Notation 1.4.9.** The matrix coefficients of $G$ are defined as

$$\mathbb{C}[G] := \{(\iota \otimes \omega)(U) : U \text{ f.d. representation, } \omega \in B(\mathcal{H}_U)^*\}. $$

If $\xi, \zeta \in \mathcal{H}$, denote the matrix unit $m_{\xi \cdot} := \langle \zeta, \cdot \rangle \xi \in B(\mathcal{H})$. For each $s \in \Irr(G)$ we fix once and for all an orthonormal basis $\{\xi_{s,i}\}_{i=1}^{\dim(s)}$ in $\mathcal{H}_s$ such that $\rho_s$ acts diagonal with
respective to this basis. Abbreviate \( m_{ij}^s := m_{\xi^i,\xi^j} \in B(\mathcal{H}) \), so \( m_{ij}^s(\xi_k) = \delta_{jk}\xi_i \). Write \( U_s = \sum_{i,j} u_{ij}^s \otimes m_{ij}^s = (u_{ij}^s)_{i,j} \) and \( \rho_s = \sum_i (\rho_s)_{ii} m_{ii}^s \).

Suppose \( \xi, \xi', \zeta, \zeta', \eta, \eta' \in \mathcal{H} \). Recall the map \( j \) defined in Notation 1.4.5. The matrix units satisfy

\[
\langle m_{\xi,\zeta}(\eta), \eta' \rangle = \langle \xi, \eta \rangle \langle \zeta, \eta' \rangle = \langle \eta, \xi \rangle \langle \zeta, \eta' \rangle = \langle \eta, m_{\xi,\zeta}(\eta') \rangle;
\]

\[
m_{\xi,\zeta} \circ m_{\xi',\zeta'}(\eta) = \langle \xi, \zeta' \rangle \eta \rangle = \langle \zeta, \xi' \rangle \eta \rangle = m_{\xi,\zeta}(\eta);
\]

\[
j(m_{\xi,\zeta})(\bar{\eta}) = m_{\xi,\zeta}(\eta) = \xi, \eta \rangle \zeta = \langle \eta, \xi \rangle \zeta = \langle \bar{\xi} \rangle \bar{\eta} \zeta = m_{\bar{\xi},\bar{\zeta}}(\eta)
\]

and thus

\[
(m_{\xi,\zeta})^* = m_{\bar{\xi},\bar{\zeta}}, \quad m_{\xi,\zeta} \circ m_{\xi',\zeta'} = \langle \zeta, \xi' \rangle m_{\xi,\zeta'}, \quad j(m_{\xi,\zeta}) = m_{\bar{\xi},\bar{\zeta}}. \tag{1.4.4}
\]

Note that (1.4.1) reads as \( \Delta(u_{ij}^s) = \sum_{k=1}^{\dim(U_s)} u_{ik}^s \otimes u_{kj}^s \). The matrix coefficients satisfy the following orthogonality relations:

\[
h((u_{kl}^s)(u_{ij}^s)^*) = \delta_{kl} \bar{\delta}_{ji} \frac{(\rho_s)^{-1}_{ik}}{d_s}, \quad h(u_{kl}^s(u_{ij}^s)^*) = \delta_{ij} \bar{\delta}_{kl} \frac{(\rho_s)_{ji}}{d_s}. \tag{1.4.5}
\]

**Definition 1.4.10.** The Woronowicz characters is the family of linear functionals \( \{f_z\}_{z \in \mathbb{C}} \) on \( \mathbb{C}[G] \) defined on a matrix coefficient by \( f_z(u_{ij}^s) := (\rho_z^s)_{ij} \) or equivalently by the identity

\[
(f_z \otimes \iota)(U) = \rho_U^z
\]

for all finite dimensional representations \( U \).

For every \( z \in \mathbb{C} \), the functional \( f_z : \mathbb{C}[G] \to \mathbb{C} \) is a homomorphism, or in other words a character. Moreover \( \tilde{f}_z = f_{-z} \) and \( (f_z \otimes f_{z'}) \Delta = f_z + f_{z'} \).

**1.4.2 Discrete quantum groups**

**Definition 1.4.11.** A Hopf \( \ast \)-algebra is a pair \( (H, \Delta) \) consisting of a unital \( \ast \)-algebra \( H \) and a unital \( \ast \)-homomorphism \( \Delta : H \to H \otimes H \), the comultiplication, satisfying \( (\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta \), together with linear maps \( S : H \to H \) and \( \varepsilon : H \to \mathbb{C} \) such that the identities

\[
(\varepsilon \otimes \iota)\Delta(a) = (\iota \otimes \varepsilon)\Delta(a) = a \quad \text{and} \quad m(S \otimes \iota)\Delta(a) = m(\iota \otimes S)\Delta(a) = \varepsilon(a)1, \tag{1.4.6}
\]

hold for all \( a \in H \). Here \( m : H \otimes H \to H \) is the multiplication map. \( S \) is called the antipode and \( \varepsilon \) the counit.

The maps \( S \) and \( \varepsilon \) are uniquely determined by (1.4.6). Moreover \( \varepsilon \) is a \( \ast \)-homomorphism, \( S \) is an antihomomorphism and they satisfy the following relations

\[
\varepsilon(S(a)) = \varepsilon(a); \quad \Delta S = (S \otimes S)\sigma \Delta; \quad S(S(a^*)) = a, \quad (a \in H).
\]
Here $\sigma : H \otimes H \to H \otimes H$ is the flip map, defined as $a \otimes b \mapsto b \otimes a$. Hopf algebras form the algebraic framework for quantum groups, take a look at for example [Kas95] or [Maj95]

The pair $(\mathbb{C}[G], \Delta)$ is a Hopf algebra with maps $S$, $\varepsilon$ defined by the identities

$$(S \otimes i)(U) = U^* \quad \text{and} \quad (\varepsilon \otimes i)(U) = 1.$$ 

In more detail, if $U_s$ is an irreducible unitary representation then $S(u_{ij}^s) = (u_{ji}^s)^*$ and $\varepsilon(u_{ij}^s) = \delta_{ij}$. Moreover $\mathbb{C}[G]$ is dense in $C(G)$. Define the unitary antipode $R$ as

$$(R \otimes j)(U) = \hat{U},$$

so explicitly $R(u_{ij}^s) = \rho_{ii}^{-\frac{1}{2}} \rho_{jj}^{rac{1}{2}} (u_{ij}^s)^*$. Recall that $\rho$ is diagonal in the basis $\{\xi_i\}$, used to define the matrix coefficients.

Discrete quantum groups can be defined abstractly, this was done by van Daele [VD96]. On the other hand any discrete quantum group is dual to a compact quantum group [VD98]. We use this duality to define discrete quantum groups. There are several equivalent ways towards this dual discrete quantum group of $G$, in particular the comultiplication can be defined in many ways. Consider the space of linear functionals $\mathbb{C}[G]'$. For every finite dimensional representation $U$ of $G$ we get a representation $\pi_U$ of $\mathbb{C}[G]'$ on $H_U$ by

$$\pi_U : \mathbb{C}[G]' \to B(H_U), \quad \omega \mapsto (\omega \otimes i)U.$$  

The collection $\{\pi_U\}_{s \in \text{Irr}(G)}$ defines an isomorphism $\mathbb{C}[G]' \cong \prod_s B(H_s) =: \mathcal{U}(\hat{G})$. Similarly $(\mathbb{C}[G]^\otimes n)' \cong \prod_{s_1, \ldots, s_n} B(H_{s_1} \otimes \cdots \otimes H_{s_n}) =: \mathcal{U}(\hat{G}^n)$. Define a unital $*$-morphism $\hat{\Delta} : \mathbb{C}[G]' \to (\mathbb{C}[G]^\otimes 2)'$ dual to the multiplication on $G$ by

$$\hat{\Delta}(\omega)(a \otimes b) := \omega(ab), \quad \text{for } \omega \in \mathbb{C}[G]' \text{ and } a, b \in \mathbb{C}[G].$$

Similarly define a multiplication and involution by

$$(\omega \nu)(a) := (\omega \otimes \nu)\Delta(a), \quad \omega^*(a) := \bar{\omega}(S(a)) = \bar{\omega}(S(a)^*).$$

The unit in $\mathbb{C}[G]'$ is given by the counit $\varepsilon$. Then $(\mathbb{C}[G]', \hat{\Delta})$ satisfies the “same axioms” as a Hopf $*$-algebra. It is not a Hopf algebra, because the elements can be unbounded. The role of the counit is played by $\varepsilon(\omega) = \omega(1)$ and antipode $\hat{S}(\omega) = \omega(S(\cdot))$ whenever $\omega \in \mathbb{C}[G]'$. Via the isomorphisms above this leads to a map $\hat{\Delta} : \mathcal{U}(\hat{G}) \to \mathcal{U}(\hat{G}^2)$. We will generally use $\mathcal{U}(\hat{G})$ instead of $\mathbb{C}[G]'$. Let $\pi_s : \mathcal{U}(\hat{G}) \to B(H_s)$ denote the projection on the matrix block corresponding to $s$. Equivalently one can define the comultiplication $\hat{\Delta}$ by

$$(\pi_s \otimes \pi_t)(\hat{\Delta}(a)) T = T \pi_r(a), \quad \text{for all } T \in \text{Hom}(r, s \otimes t) \text{ and } a \in \prod_{s \in \text{Irr}(G)} B(H_s).$$  

Another way goes via the multiplicative unitary. For this consider the GNS representation $(H_h, \xi_h, \pi_h)$ associated to the Haar state $h$. We will write $L^2(G)$ for $H_h$. If $x \in C(G)$ write $\Lambda(x) = x\xi_h \in L^2(G)$ for the corresponding vector in the GNS construction. Since $h$
is faithful, we can identify $C(G)$ with $\pi_h(C(G))$ inside $B(L^2(G))$. Denote by $L^\infty(G)$ the von Neumann algebra generated by $\pi_h(C(G))$ in $B(L^2(G))$. By density of $C(G)$ inside $L^2(G)$ the map

$$\langle \xi \otimes a \xi_h \rangle \mapsto \Delta(a)(\xi \otimes \xi_h)$$  \hspace{1cm} (1.4.9)$$

extends to a unitary operator in $B(L^2(G) \otimes L^2(G))$. The adjoint $W$ of (1.4.9) is called the multiplicative unitary and defines a unitary (not necessarily finite dimensional) representation of $G$ on $L^2(G)$, the so-called left regular representation. Given $\xi, \zeta \in \mathcal{H}$ denote the functional $\omega_{\xi,\zeta}: B(\mathcal{H}) \to \mathbb{C}$, $\omega_{\xi,\zeta}(T) := \langle \xi, T\zeta \rangle$. If $U_s$ is an irreducible representation, then for every $\zeta \in \mathcal{H}$ the map

$$\theta_\zeta: \mathcal{H} \to L^2(G), \quad \xi \mapsto (d_\zeta)^{\frac{1}{2}}(\Lambda \otimes \omega_{\rho_s^{-\frac{1}{2}}, 1}(U_s^*))$$

intertwines $U_s$ and $W$. If $\zeta = \xi^\delta$, then

$$\theta_{\xi^\delta}(\xi^i_j) = d_\xi^\delta(\rho_s^{-\frac{1}{2}})_{jj} \sum_{i=1}^{\dim U_s} u^s_{ij} m_{\xi_h,\xi^i_j}(\xi^i_j) = \delta_{s,i} d_\xi^\delta(\rho_s^{-\frac{1}{2}})_{jj} \Lambda(u^s_{ij}).$$

From the orthogonality relations it follows that $\theta_\zeta$ is isometric if $\zeta$ is a unit vector. Moreover if $\zeta, \zeta' \in \mathcal{H}$ are two orthogonal vectors the corresponding images of $\theta_\zeta$ and $\theta_{\zeta'}$ are orthogonal. Thus by picking orthonormal bases we obtain a canonical inclusion $\mathcal{H} \otimes \mathcal{H} \hookrightarrow L^2(G)$, it corresponds to the space of matrix coefficients of $U_s$. By identifying $B(\mathcal{H}_s) \cong \mathcal{H}_s \otimes \mathcal{H}_s$ we obtain an inclusion $\bigoplus_s B(\mathcal{H}_s) \hookrightarrow L^2(G)$. Taking all irreducible representations exhausts $L^2(G)$. This gives the Peter–Weyl decomposition for compact quantum groups. Using these identifications the comultiplication on $\hat{G}$ can be described in terms of $W$, see (1.4.12) below. Moreover observe that $W$ can be expressed as

$$W = \bigoplus_{s \in \text{Irr}(G)} U_s = \sum_s \sum_{i,j} u^s_{ij} \otimes m^s_{ij}. \hspace{1cm} (1.4.10)$$

The multiplicative unitary satisfies the pentagon equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$.

**Notation 1.4.12.** Let $G = (C(G), \Delta)$ be a compact quantum group. The discrete quantum group dual to $G$ is the virtual object indicated by $(\hat{G}, \hat{\Delta})$. We write

$$c_0(\hat{G}) := \bigoplus_{s \in \text{Irr}(G)} B(\mathcal{H}_s), \quad c_0(\hat{G}) := c_0 \bigoplus_{s \in \text{Irr}(G)} B(\mathcal{H}_s),$$

$$l^\infty(\hat{G}) := \bigoplus_{s \in \text{Irr}(G)} B(\mathcal{H}_s), \quad \mathcal{U}(\hat{G}) := \prod_{s \in \text{Irr}(G)} B(\mathcal{H}_s).$$

Also $l^\infty(\hat{G})$ is represented on $L^2(G)$. Indeed, given an element $x \in B(\mathcal{H}_s) \subset l^\infty(\hat{G})$, consider the corresponding functional $\omega \in \mathbb{C}[G]'$, so $x = (\omega \otimes \iota)(U_s)$. Then $x$ acts as $x\Lambda(a) = \Lambda((\omega S^{-1} \otimes \iota)\Delta(a))$. The multiplicative unitary satisfies $W \in M(C(G) \otimes l^\infty(\hat{G}))$. $W$ encodes all information of the quantum group $G$. Namely its matrix coefficients span
both \( \mathbb{C}[G] \) and \( c_{00}(\hat{G}) \) and the comultiplications are given by
\[
\Delta(a) = W^*(1 \otimes a)W \in M(C(G) \otimes C(G)), \quad (a \in C(G)); \quad (1.4.11)
\]
\[
\hat{\Delta}(x) = W(x \otimes 1)W^* \in \mathcal{L}_\infty(\hat{G}) \otimes \mathcal{L}_\infty(\hat{G}), \quad (x \in \mathcal{L}_\infty(\hat{G})). \quad (1.4.12)
\]
We will not need it, but also the antipode and counit can be expressed in terms of \( W \) (cf. [KS97, Prop. 11.37]).

This picture of duality can been seen from a broader point of view. Baaj–Skandalis [BS93] developed a very general theory of duality of operator algebras using such multiplicative unitaries.

**Remark 1.4.13.** The left regular representation will be one of the few infinite dimensional representations we will consider. Therefore, unless stated otherwise, all other representations of compact quantum groups are assumed to be finite dimensional.

**Example 1.4.14.** We continue part (ii) of Example 1.4.2. The elements \( \{\lambda_\gamma\}_{\gamma \in \Gamma} \) are one-dimensional representations of the compact quantum group \( G \). They span a dense subset of \( C(G) \). Hence by the orthogonality relations these are all finite dimensional irreducible representations. Thus \( \text{Irr}(G) \cong \Gamma \) and \( \mathbb{C}[G] \subset C(G) = C_r^*(\Gamma) \) equals the ordinary group algebra of \( \Gamma \). It follows that \( U(G) = \mathbb{C}[G]^r \) equals the functions on \( \Gamma \). The discrete quantum group can be identified as \( \mathcal{L}_\infty(\hat{G}) \cong \mathcal{L}_\infty(\Gamma) \) with pointwise multiplication and involution. The comultiplication is given by \( \hat{\Delta}(f)(\gamma, \gamma') := f(\gamma \gamma') \in \mathcal{L}_\infty(\Gamma \times \Gamma) \cong \mathcal{L}_\infty(\Gamma) \otimes \mathcal{L}_\infty(\Gamma) \).

Recall the elements \( \rho_s \) defined in Definition 1.4.6. We write \( \rho \in U(\hat{G}) \) for the element that satisfies \( \pi_s(\rho) = \rho_s \) for every \( s \).

The comultiplication \( \hat{\Delta} \) has unique right- and left-invariant weights, denoted \( \hat{\psi} \) and respectively \( \hat{\phi} \) with domains \( c_{00}(\hat{G}) \). Invariance means that they satisfy
\[
(\hat{\psi} \otimes \iota)\hat{\Delta}(x) = \hat{\psi}(x)1; \quad (\iota \otimes \hat{\phi})\hat{\Delta}(x) = \hat{\phi}(x)1, \quad \text{for all } x \in c_{00}(\hat{G}). \quad (1.4.13)
\]
Note that if \( x \in c_{00}(\hat{G}) \), then \( \hat{\Delta}(x) \) is in general not an element of \( c_{00}(\hat{G}) \) \( \otimes \) \( c_{00}(\hat{G}) \).

However, \( (y \otimes 1)\hat{\Delta}(x), (1 \otimes y)\hat{\Delta}(x) \in c_{00}(\hat{G}) \otimes_{\text{alg}} c_{00}(\hat{G}) \) whenever \( y \in c_{00}(\hat{G}) \), so (1.4.13) makes sense. These weights can be written down explicitly as
\[
\hat{\psi}(x) = \sum_{s \in \text{Irr}(G)} d_s \text{Tr}(\pi_s(xp^{-1})); \quad \hat{\phi}(x) = \sum_{s \in \text{Irr}(G)} d_s \text{Tr}(\pi_s(xp)), \quad (x \in c_{00}(\hat{G})). \quad (1.4.14)
\]
Note that \( \hat{\psi} \) and \( \hat{\phi} \) are unbounded on \( c_{00}(\hat{G}) \). The modular groups are given by
\[
\sigma_\iota^\hat{\psi}(x) = \rho^{-it}x\rho^{it}; \quad \sigma_\iota^\hat{\phi}(x) = \rho^{it}x\rho^{-it} \quad (x \in c_{00}(\hat{G})). \quad (1.4.15)
\]

**Definition 1.4.15.** The Fourier transform is the mapping
\[
\mathcal{F} : \mathbb{C}[G] \rightarrow U(\hat{G}), \quad a \mapsto h(a).$

Here \( h \) is the Haar state, so \( h(\cdot a) \) defines a linear functional by \( (h(\cdot a))(b) := h(ba) \) and we view \( \mathcal{U}(\hat{G}) \cong \mathbb{C}[G] \).

The Fourier transform satisfies (see [NT04, Lem. 1.1])

\[
\hat{\mathcal{F}}(u_{ij}^s) = d^{-1}_s f_{-\frac{1}{2}}(u_{ni}^s)f_{-\frac{1}{2}}(u_{jj}^s)j_s(m_{ji}^s),
\]

where \( f_{-\frac{1}{2}} \) is the Woronowicz character and \( j_s \) the map defined in Notation 1.4.5. In particular it follows that \( \hat{\mathcal{F}}(\mathbb{C}[G]) \subset c_{00}(\hat{G}) \). Moreover the Plancherel formula holds

\[
\hat{\psi}(\hat{\mathcal{F}}(a)^*\hat{\mathcal{F}}(b)) = h(a^*b), \quad (a, b \in \mathbb{C}[G]).
\]

### 1.4.3 Actions

Actions of compact and discrete quantum groups give any \( C\)-algebra a lot of extra structure. This extra structure is important when placing the theory in a categorical framework. More information about actions can be found in the papers [Boc95], [Li09] and [Wan99].

**Definition 1.4.16.** Let \( G, \hat{G} \) be a compact respectively discrete quantum group and \( B \) a unital \( C\)-algebra. A **left action** of \( G \) (resp. \( \hat{G} \)) on \( B \) is a unital \( \ast \)-homomorphism \( \alpha : B \rightarrow C(G) \otimes B \) (resp. \( \alpha : B \rightarrow M(c_0(\hat{G}) \otimes B) \)) satisfying

\[
(\iota \otimes \alpha)\alpha = (\Delta \otimes \iota)\alpha \quad \text{(respectively } (\iota \otimes \alpha)\alpha = (\hat{\Delta} \otimes \iota)\alpha \text{)}
\]

and the space \( \alpha(B)(C(G) \otimes 1) \) is norm dense in \( C(G) \otimes B \) (resp. \( \alpha(B)(c_0(\hat{G}) \otimes 1) \) is dense in \( c_0(\hat{G}) \otimes B \)). **Right actions** are defined similarly.

Occasionally we use the terminology that \( B \) is a (left) \( G\)-\( C\)-algebra instead of \( \alpha : B \rightarrow C(G) \otimes B \) is a left action. This density condition is required for the same reason as the density condition for the comultiplication on \( C(G) \).

**Definition 1.4.17.** Let \( G, \hat{G} \) be a compact respectively discrete quantum group and \( N \) a von Neumann algebra. A **right action** of \( G \) (resp. \( \hat{G} \)) on \( N \) is an injective normal unital \( \ast \)-homomorphism \( \beta : N \rightarrow N \bar{\otimes} L^\infty(G) \) (resp. \( \beta : N \rightarrow N \bar{\otimes} L^\infty(\hat{G}) \)) satisfying

\[
(\beta \otimes \iota)\beta = (\iota \otimes \Delta)\beta \quad \text{(respectively } (\beta \otimes \iota)\beta = (\iota \otimes \hat{\Delta})\beta \text{)}.
\]

**Left actions** are defined similarly.

We do not need to put the density condition \( \beta(N)(1 \otimes L^\infty(G)) \) is dense in \( N \bar{\otimes} L^\infty(G) \) in the definition, because that is automatically fulfilled for von Neumann algebraic actions ([Vae01, Thm. 2.6]).

**Definition 1.4.18.** Let \( \alpha \) be a right action of \( G \) on \( A \). The **fixed point algebra** is defined as \( A^\alpha := \{ x \in C(G) : \alpha(x) = x \otimes 1 \} \). The action \( \alpha \) is called **ergodic** if \( A^\alpha = \mathbb{C}_1A \).

For us the most important actions are the adjoint actions.
Notation 1.4.19. The multiplicative unitary $W$ defines the \textit{left adjoint action} of $G$ on $\hat{G}$ and the \textit{right adjoint action} of $\hat{G}$ on $G$ by
\begin{align*}
\alpha_l: l^\infty(\hat{G}) &\to C(G) \otimes l^\infty(\hat{G}), & x \mapsto W^*(1 \otimes x)W; \\
\alpha_r: C(G) &\to M(C(G) \otimes c_0(\hat{G})), & a \mapsto W(a \otimes 1)W^*.
\end{align*}
This left action $\alpha_l$ extends to an action on the level of von Neumann algebras
\begin{align*}
\alpha_l: l^\infty(\hat{G}) &\to L^\infty(G) \otimes l^\infty(\hat{G}).
\end{align*}
Moreover $\alpha_l$ can be further extended to a left action of $G$ on $\bigotimes_{-n}^{-1} l^\infty(\hat{G})$ by
\begin{align*}
\alpha_l: \bigotimes_{-n}^{-1} l^\infty(\hat{G}) &\to L^\infty(G) \otimes \left( \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \right), & x \mapsto W_{1,n+1}^* \cdots W_{1,2}^*(1 \otimes x)W_{1,2} \cdots W_{1,n+1}.
\end{align*}
For $x \in \bigotimes_{-n}^{-1} l^\infty(\hat{G})$, the following limit can be shown to exist in norm (see \cite[Lem. 3.1]{Izu02})
\begin{align*}
\lim_{n \to \infty} W_{1,-n}^* \cdots W_{1,-1}^*(1 \otimes x)W_{1,-n} \cdots W_{1,-1}.
\end{align*}
This defines an action on the infinite tensor product. The leg numbering here is different than elsewhere, the $1$ refers to $L^\infty(G)$ and the $-j$ to the $-j$-th component of $\bigotimes_{-\infty}^{-1}$.

Again this action is denoted by $\alpha_l$.

1.5 \textbf{C$^*$-tensor categories}

We assume basic knowledge of category theory. Most of the definitions below are quite long, but the essential element one has to keep in mind is that in a tensor category one is able to take “tensor products” of objects and morphisms and a C$^*$-category has all the properties of a C$^*$-algebra. A good overview of tensor categories is given in \cite{Mug10}, for general category theory we refer to \cite{ML98}. This section is based on \cite[§2.1]{NT13}.

\textbf{Definition 1.5.1.} A category $\mathcal{C}$ is called a \textbf{C$^*$-category} if the following conditions are satisfied:

\begin{enumerate}
\item \textbf{(i)} $\text{Hom}_\mathcal{C}(U, V)$ is a Banach space for all objects $U, V \in \text{Ob}(\mathcal{C})$, the map
\begin{align*}
\text{Hom}_\mathcal{C}(V, W) \times \text{Hom}_\mathcal{C}(U, V) &\to \text{Hom}_\mathcal{C}(U, W), & (S, T) &\mapsto ST
\end{align*}
is bilinear and satisfies $\|ST\| \leq \|S\| \|T\|;$
\item \textbf{(ii)} we are given an antilinear contravariant functor $*: \mathcal{C} \to \mathcal{C}$, called the \textbf{involution}, which is the identity on objects (so if $T \in \text{Hom}_\mathcal{C}(U, V)$, then $T^* \in \text{Hom}_\mathcal{C}(V, U)$) and satisfies the properties
\begin{enumerate}
\item $T^{**} = T$ for every morphism $T$;
\item $\|T^*T\| = \|T\|^2$ for every morphism $T$;
\end{enumerate}
\end{enumerate}
3. $T^*T$ is positive in the $\mathbb{C}^*$-algebra $\text{End}_\mathbb{C}(U)$ for every $T \in \text{Hom}_\mathbb{C}(U, V)$.

Note that because of the functor $*$ we can speak of projections, unitaries, self-adjoint morphisms etcetera.

**Definition 1.5.2.** A category $\mathcal{C}$ is called a tensor category or monoidal category if the following conditions are satisfied:

(i) we are given a bilinear bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(U, V) \mapsto U \otimes V$ called the tensor product and natural isomorphisms

$$\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W),$$

called the associativity morphisms, such that the pentagon diagram

$$
\begin{array}{ccc}
((U \otimes V) \otimes W) \otimes X & \xrightarrow{\alpha_{12,3,4}} & U \otimes (V \otimes (W \otimes X)) \\
(g \otimes h) \otimes i & \xrightarrow{\alpha_{1,23,4}} & (U \otimes (V \otimes W)) \otimes X \\
U \otimes ((V \otimes W) \otimes X) & \xrightarrow{\lambda \otimes \alpha} & U \otimes (V \otimes (W \otimes X))
\end{array}
$$

commutes. Here we abbreviated $\alpha_{12,3,4} := \alpha_{U \otimes V, W, X}$, etcetera;

(ii) there exists an object $1 = 1_\mathcal{C}$ (the unit) and natural isomorphisms

$$\lambda_U : 1 \otimes U \to U, \quad \rho_U : U \otimes 1 \to U$$

such that $\lambda_1 = \rho_1$ and the triangle diagram

$$
\begin{array}{ccc}
(U \otimes 1) \otimes V & \xrightarrow{\alpha} & U \otimes (1 \otimes V) \\
\rho \otimes \lambda & & \lambda \otimes \rho
\end{array}
$$

commutes.

**Definition 1.5.3.** A category $\mathcal{C}$ is a $\mathbb{C}^*$-tensor category if $\mathcal{C}$ is both a $\mathbb{C}^*$-category and tensor category. Moreover $\mathcal{C}$ should satisfy the additional compatibility conditions:

$$(S \otimes T)^* = S^* \otimes T^*$$

and the associativity morphisms $\alpha$ and natural isomorphisms $\lambda$ and $\mu$ are unitary.

**Assumption 1.5.4.** We always assume that our $\mathbb{C}^*$-tensor categories have the following extra structure:

(i) $\mathcal{C}$ has finite direct sums, thus for all objects $U$ and $V$ there exists an object $W$ and isometries $u \in \text{Hom}_\mathbb{C}(U, W)$ and $v \in \text{Hom}_\mathbb{C}(V, W)$ such that $uu^* + vv^* = \iota_W$;
(ii) \( \mathcal{C} \) has subobjects, thus for every projection \( p \in \text{End}_\mathcal{C}(U) \) there exists an object \( V \) and isometry \( v \in \text{Hom}_\mathcal{C}(V,U) \) such that \( vv^* = p \) and \( v^*v = \iota_V \);

(iii) the class of objects is a set.

These first two conditions are very mild. If a category does not have subobjects or finite direct sums, one can always complete the category by adding extra objects and morphisms (cf. Remark 1.5.11 below). The third assumption is to avoid any difficulties in set theoretical notions.

In addition we often (but not always) assume that

(iv) \( \text{End}_\mathcal{C}(1) = \mathcal{C} \iota \cong \mathbb{C} \), so the unit is simple.

This last assumption, unfortunately, is not stable under certain constructions which we shall perform.

The direct sum of \( U \) and \( V \) as in (i) will be denoted by \( W := U \oplus V \). If \( V \) is a subobject of \( U \) defined by a projection \( p \) as in (ii), then we write \( V \subset U \). Moreover via the isometry \( v \) we can restrict morphisms \( T \in \text{End}(U) \) that commute with \( p \) to the object \( V \). We write \( T|_V := v^*Tv \in \text{End}(V) \).

A \((\mathcal{C}^*\text{-})\) tensor category will be called \textit{strict} if

\[
(U \otimes V) \otimes W = U \otimes (V \otimes W), \quad 1 \otimes U = U = U \otimes 1
\]

and the associativity morphisms \( \alpha \) and the morphisms \( \lambda \) and \( \rho \) are the identity morphisms.

**Example 1.5.5.** One of the main examples of \( \mathcal{C}^*\)-tensor categories is given by the category of finite dimensional Hilbert spaces. The class of examples which is most important to us is \( \text{Rep}(G) \), the category of finite dimensional unitary representations of \( G \), where \( G \) is a compact (quantum) group. Morphisms in this category are given by intertwiners.

**Definition 1.5.6.** An object \( U \) is called \textit{simple} if \( \text{End}(U) = \mathcal{C} \iota \cong \mathbb{C} \). \( \mathcal{C} \) is called \textit{semisimple} if every object is a direct sum of finitely many simple objects. Let \( \text{Irr}(\mathcal{C}) \) denote the equivalence classes of the simple objects in \( \mathcal{C} \). For each \( s \in \text{Irr}(\mathcal{C}) \) fix a representative \( U_s \).

We can define functors of \( \mathcal{C}^*\)-tensor categories. Again this is a long definition, but the main part is that the functor preserves all structure of the category.

**Definition 1.5.7.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories and \( F : \mathcal{C} \to \mathcal{D} \) a functor.

(i) If \( \mathcal{C} \) and \( \mathcal{D} \) are tensor categories, the functor \( F \) together with an isomorphism \( F_0 : 1_\mathcal{D} \to F(1_\mathcal{C}) \) and natural isomorphisms

\[
F_2 : F(U) \otimes F(V) \to F(U \otimes V)
\]

is called a \textit{tensor functor} if the diagrams

\[
\begin{array}{ccc}
(F(U) \otimes F(V)) \otimes F(W) & \xrightarrow{F_2 \otimes 1} & F(U \otimes V) \otimes F(W) \\
\downarrow \alpha_D & & \downarrow F(\alpha_C) \\
F(U) \otimes (F(V) \otimes F(W)) & \xrightarrow{\iota \otimes F_2} & F(U) \otimes F(V \otimes W) \end{array}
\]

\[
\begin{array}{ccc}
\end{array}
\]

\[
\begin{array}{ccc}
\end{array}
\]
(ii) if $C$ and $D$ are $C^*$-categories $F$ is called unitary if $F(T)^* = F(T^*)$ for all morphisms $T$;

(iii) if $C$ and $D$ are $C^*$-tensor categories $F$ is a unitary tensor functor if $F$ is both a tensor functor of monoidal categories, a unitary tensor functor of $C^*$-categories and $F_2: F(U) \otimes F(V) \to F(U \otimes V)$ and $F_0$ are unitary morphisms.

For two functors $F,G: C \to D$ denote by $\text{Nat}(F,G)$ the set of natural transformations $F \to G$. Recall that a natural transformation $\eta: F \to G$ consists of a collection of morphisms $(\eta_X)_{X \in \text{Ob}(C)}$ natural in $X$, meaning that for every morphism $f \in \text{Hom}_C(X,Y)$ the following diagram commutes

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
\eta_X & \downarrow & \eta_Y \\
G(X) & \xrightarrow{G(f)} & G(Y)
\end{array}
$$

So the requirement that $F_2$ are natural isomorphisms means precisely that the tensor product structure is preserved under $F_2$, thus that the diagram

$$
\begin{array}{ccc}
F(U_1) \otimes F(V_1) & \xrightarrow{F_2} & F(U_1 \otimes V_1) \\
\downarrow_{F(S) \otimes F(T)} & & \downarrow_{F(S \otimes T)} \\
F(U_2) \otimes F(V_2) & \xrightarrow{F_2} & F(U_2 \otimes V_2)
\end{array}
$$

commutes for all morphisms $S: U_1 \to U_2$, $T: V_1 \to V_2$.

**Definition 1.5.8.** Given two ($C^*$-) tensor categories $C$ and $D$ be with two tensor functors $F,G: C \to D$. A natural isomorphism $\eta: F \to G$ is called monoidal if the two diagrams

$$
\begin{array}{ccc}
F(U) \otimes F(V) & \xrightarrow{F_2} & F(U \otimes V) \\
\downarrow_{\eta \otimes \eta} & & \downarrow_\eta \\
G(U) \otimes G(V) & \xrightarrow{G_2} & G(U \otimes V)
\end{array}
$$

$$
\begin{array}{ccc}
F(1_C) & \xrightarrow{F_0} & F(1_C) \\
\downarrow_\eta & & \downarrow_\eta \\
G(1_C) & \xrightarrow{G_0} & G(1_C)
\end{array}
$$

commute.

**Definition 1.5.9.** Let $C$ and $D$ be two tensor categories. A tensor functor $F: C \to D$ is called a monoidal equivalence if there exists a tensor functor $G: D \to C$ such that
If deal with strict categories. with a new category which is a $C^*$-category except from the existence of direct sums and subobjects, then $D$ that if $C$ is a monoidal equivalence $F: C \to D$. If $C$ and $D$ are $C^*$-tensor categories and $F$, $G$ and the natural isomorphisms can be chosen unitary then $D$ and $C$ are unitarily monoidally equivalent.

A tensor functor $F: C \to D$ is called fully faithful if $F$ defines an isomorphism between $\text{Hom}_C(U, V)$ and $\text{Hom}_D(F(U), F(V))$ for every pair of objects $U, V \in \text{Ob}(C)$. $F$ is called essentially surjective if for every object $U \in \text{Ob}(D)$ there exists an object $V \in \text{Ob}(C)$ such that $U$ is isomorphic to $F(V)$. Then $F$ is a monoidal equivalence if and only if $F$ is fully faithful and essentially surjective.

Remark 1.5.10. Any $C^*$-tensor category can be strictified [ML98, §XI.3]. This means that if $C$ is a (non-strict) $C^*$-tensor category, then there exists a strict $C^*$-tensor category $D$ such that $C$ and $D$ are unitarily monoidally equivalent. So unless stated otherwise we deal with strict categories.

Remark 1.5.11. If $C$ is a category which satisfies all the requirements of a $C^*$-tensor category except from the existence of direct sums and subobjects, then $C$ can be completed to a new category which is a $C^*$-tensor category (see, [NT13, §2.5]). For this define $C'$ with

$$\text{Ob}(C') := \{(U_1, \ldots, U_n) : n \geq 1, U_i \in \text{Ob}(C)\};$$
$$\text{Hom}_{C'}((U_1, \ldots, U_m), (V_1, \ldots, V_n)) := \bigoplus_{i,j} \text{Hom}_{C}(U_i, V_j).$$

Now $(U_i)_i \oplus (V_j)_j := (U_1, \ldots, U_m, V_1, \ldots, V_n)$ and $(U_i)_i \otimes (V_j)_j$ is given by the tuple consisting of the lexicographical ordering of $U_i \otimes V_j$. Let $C''$ be the category with

$$\text{Ob}(C'') := \{(U, p) : U \in \text{Ob}(C'), p \in \text{End}_{C'}(U) \text{ projection}\}$$
$$\text{Hom}_{C''}((U, p), (V, q)) := q \text{Hom}_{C'}(U, V)p.$$

The tensor product of objects is given by $(U, p) \otimes (V, q) := (U \otimes V, p \otimes q)$. The involution, direct sums and tensor products of morphisms on $C'$ and $C''$ are defined in the obvious way. Then $C''$ is a $C^*$-tensor category. It is clear that there exists a unitary tensor functor $i: C \to C''$. We call $C''$ the direct sum and subobject completion of $C$.

This completion $C''$ is universal in the following sense: if $D$ is a $C^*$-tensor category and $F: C \to D$ is a unitary tensor functor, then $F$ extends uniquely (up to unitary monoidal isomorphism) to a unitary tensor functor $F'': C'' \to D$. To construct this functor define

$$F'((U_1, \ldots, U_n)) := F(U_1) \oplus \ldots \oplus F(U_n), \quad F'((T_{ij})) := (F(T_{ij}))_{ij}.$$ 

If $(U, p) \in \text{Ob}(C'')$, then $F'(p)$ is a projection in $\text{End}_D(F(U))$, so there exists $V \in \text{Ob}(D)$ and an isometry $v \in \text{Hom}_D(V, F(U))$ such that $vv^* = F(p)$ and $v^*v = \iota_V$. Define $F''((U, p)) := V$ and let

$$F''(p'Tp) := v^*F'(p'Tp)v = v^*F'(T)v, \quad \text{for } p'Tp \in \text{Hom}_{C''}((U, p), (U', p')).$$
The tensor and involutive structure are again defined in the obvious way.

Note that in both steps of this extension of \( F \) one has to make a choice of objects, a different choice leads to an isomorphic functor. Furthermore if both \( \mathcal{C} \) and \( \mathcal{D} \) are not necessarily closed under direct sums and subobjects and \( F : \mathcal{C} \to \mathcal{D} \) is a unitary tensor functor, then \( F \) extends to a functor \( \mathcal{C}'' \to \mathcal{D}'' \). This extension is constructed by applying the universal property to \( i \circ F \), where \( i : \mathcal{D} \to \mathcal{D}'' \) is the inclusion. The properties “fully faithful” and “essentially surjective” are preserved under this extension of tensor functors.

**Definition 1.5.12.** Let \( \mathcal{C} \) be a strict \( C^* \)-tensor category and \( U \in \text{Ob}(\mathcal{C}) \). An object \( \bar{U} \in \text{Ob}(\mathcal{C}) \) is called **conjugate** to \( U \) if there exist morphisms \( R \in \text{Hom}(1, \bar{U} \otimes U) \) and \( \bar{R} \in \text{Hom}(1, U \otimes \bar{U}) \) such that the compositions

\[
U \xrightarrow{i \otimes R} U \otimes U \xrightarrow{R^* \otimes w} U \quad \text{and} \quad \bar{U} \xrightarrow{i \otimes \bar{R}} \bar{U} \otimes U \xrightarrow{R \otimes \bar{w}} \bar{U}
\]

are the identity morphisms. We say that the pair \((R, \bar{R})\) **solves the conjugate equations for** \( U \). If every object in \( \mathcal{C} \) has a conjugate object then \( \mathcal{C} \) is **rigid**. Suppose \( U \in \text{Ob}(\mathcal{C}) \), decompose \( U \cong \bigoplus_k U_k \) into simple objects. If \((R, \bar{R})\) is of the form

\[
R = \sum_k (\bar{w}_k \otimes w_k) R_k, \quad \bar{R} = \sum_k (w_k \otimes \bar{w}_k) \bar{R}_k,
\]

where \( \|R_k\| = \|\bar{R}_k\| \) and \( w_k \in \text{Hom}(U_k, U) \) and \( \bar{w}_k \in \text{Hom}(\bar{U}_k, U) \) are isometries such that \( \sum_k w_k w_k^* = t_U \) and \( \sum_k \bar{w}_k \bar{w}_k^* = t_{\bar{U}} \), then \((R, \bar{R})\) is called a **standard solution** of the conjugate equations.

**Assumption 1.5.13.** All \( C^* \)-tensor categories are assumed to be rigid.

Suppose that \( U \in \text{Ob}(\mathcal{C}) \), by assumption \( U \) has a conjugate. It can be proved that in this case \( U \) also admits a standard solution of the conjugate equations. Moreover, if both \((R, \bar{R})\) and \((R', \bar{R}')\) are standard solutions for \((U, \bar{U})\) respectively \((U', \bar{U}')\), then there exists a unitary \( T \in \text{Hom}(\bar{U}, \bar{U}') \) such that \( R' = (T \otimes i) R \) and \( \bar{R}' = (i \otimes T) \bar{R} \). In addition \( \text{End}_C(U) \) is finite dimensional. For each object \( U \in \text{Ob}(\mathcal{C}) \) fix a pair of standard solutions \((R_U, \bar{R}_U)\).

**Lemma 1.5.14 (Frobenius reciprocity).** Let \( U \in \text{Ob}(\mathcal{C}) \) with solutions of the conjugate equations \((R, \bar{R})\), then

\[
\text{Hom}(U \otimes V, W) \to \text{Hom}(V, \bar{U} \otimes W), \quad T \mapsto (i_U \otimes T)(R \otimes \iota_V)
\]

is a linear isomorphism with inverse

\[
S \mapsto (\bar{R}^* \otimes \iota_W)(\iota_U \otimes S).
\]

Similarly, \( \text{Hom}(V \otimes U, W) \cong \text{Hom}(V, W \otimes \bar{U}) \).

For each \( U_s \) fix a conjugate object \( \bar{U}_s \). By Frobenius reciprocity \( \bar{U}_s \) is again simple, thus isomorphic to \( U_t \) for some \( t \in \text{Irr} \mathcal{C} \). We define a map \( \text{Irr} \mathcal{C} \to \text{Irr} \mathcal{C} \), \( s \mapsto \bar{s} \), where \( \bar{s} \) is defined by the identity \( \bar{U}_s = U_{\bar{s}} \). The class \( 0 \in \text{Irr} \mathcal{C} \) indicates the unit object, thus \( U_0 = \)
1.6. LIE GROUP THEORY

As before, the multiplicity of \( U_i \) in \( U \otimes V \) is denoted by \( m_{U,V}^i = \dim(\text{Hom}_C(U_i, U \otimes V)) \). Thus \( U \otimes V \cong \bigoplus_i m_{U,V}^i U_i \), where \( m_{U,V}^i \) means the direct sum of \( m_{U,V}^i \) copies of \( U_i \).

**Lemma 1.5.15.** Suppose that \( C \) has a simple unit. Let \( U \in \text{Ob}(C) \) and \((R, \bar{R})\) be any solution of the conjugate equations for \( U \). Define maps \( \varphi_U, \psi_U : \text{End}_C(U) \to \text{End}(1) \cong \mathbb{C} \) in the following way. For \( T \in \text{End}_C(U) \) let \( \varphi_U(T) \) be the composition

\[
1 \xrightarrow{R} \bar{U} \otimes U \xrightarrow{\psi T} \bar{U} \otimes U \xrightarrow{R^*} 1
\]

and respectively let \( \psi_U(T) \) be

\[
1 \xrightarrow{R} \bar{U} \otimes U \xrightarrow{T \otimes \epsilon} \bar{U} \otimes U \xrightarrow{\bar{R}} 1.
\]

Then \((R, \bar{R})\) is standard if and only if \( \varphi_U = \psi_U \). Furthermore, if \((R, \bar{R})\) is standard, then the maps \( \varphi_U \) and \( \psi_U \) are tracial, positive, faithful and do not depend on the choice of the standard solution.

This lemma justifies the following definition.

**Definition 1.5.16.** Suppose that \( C \) has a simple unit, \( U \in \text{Ob}(C) \) and \((R, \bar{R})\) is a standard solution of the conjugate equations for \( U \). Identify \( \text{End}_C(1) \cong \mathbb{C} \) and define \( \text{Tr}_U(T) := \varphi_U(T) \in \mathbb{C} \), where \( T \in \text{End}_C(U) \) and \( \varphi_U \) is as in (1.5.1). This functional \( \text{Tr}_U \) is called the categorical trace of \( U \). By the previous lemma it is independent of the choice of standard solutions. The intrinsic dimension is defined as \( d_i(U) := \|R\|^2 = \text{Tr}_U(\iota_U) \). The functional \( \text{tr}_U := d_i(U)^{-1} \text{Tr}_U \) is called the normalised categorical trace.

If \( C = \text{Rep}(G) \) for some compact quantum group \( G \), then it can be shown that \( R_s \) and \( \bar{R}_s \) given by

\[
R_s(1) := \sum_i \bar{\xi}_i^s \otimes \rho_s^{-\frac{1}{2}} \xi_i^s \quad \quad \bar{R}_s(1) := \sum_i \rho_s \frac{1}{2} \xi_i^s \otimes \bar{\xi}_i^s
\]

form standard solutions for \( U_s \). In particular, if \( x \in \text{End}_{\text{Rep}(G)}(U_s) = C_{t_s} \subset B(\mathcal{H}_s) \subset l^\infty(\hat{G}) \) it holds that \( \text{Tr}(xp_s^{-1}) = \text{Tr}_{U_s}(x) \), where \( \text{Tr} \) is the trace on \( B(\mathcal{H}_s) \) and \( \text{Tr}_{U_s} \) the categorical trace of \( U_s \). More interesting is the case when \( U \) is nonsimple, by decomposing \( U \cong \bigoplus_k U_k \) into simple objects we get again that \( \text{Tr}(xp^{-1}) = \text{Tr}_U(x) \). Indeed, using the notation of Definition 1.5.12 we see

\[
\text{Tr}_U(x) = \sum_k R_k^*(\bar{w}_k^* \otimes w_k^*) (t \otimes x) (\bar{w}_k \otimes w_k) R_k = \sum_k \text{Tr}(w_k^* x w_k p_k^{-1}) = \sum_k \text{Tr}(\pi_k(xp^{-1})).
\]

In addition (1.5.2) shows that \( d_i(U) = d_U \), so the intrinsic dimension of a representation equals the quantum dimension.

## 1.6 Lie group theory

Lie theory is a vast field, so in this section we will be very brief and only state the results we will need. This section mainly serves to fix the notation regarding Lie groups, their Lie
algebras, root systems and $q$-deformations. For more information regarding Lie groups and Lie algebras consult for example [FH91] and [BtD85]. More background information on $q$-deformed Lie groups can be found in many books on quantum groups, see for instance [CP95], [KS97] and [NT13]. We stay closest to [NT13].

1.6.1 Lie groups and Lie algebras

Let $G$ be a semisimple simply connected compact Lie group with Lie algebra $\mathfrak{g}_\mathbb{R}$ of rank $r$. Denote the weight lattice $\mathfrak{g}_\mathbb{Z}$ by $\Delta$. Denote the weight lattice $\mathfrak{g}_{\mathbb{C}}$ by $\mathfrak{g}$ and denote by $\Delta \subset \mathfrak{h}^*$ the roots of $G$. Note that $\mathfrak{h}$ is the complexification of $\mathfrak{t}$, where $\mathfrak{t}$ is the Lie algebra of a maximal torus $T \subset G$. Recall that $\alpha$ is a root if and only if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, where

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$ 

The adjoint representation $\text{ad}: \mathfrak{g} \to \text{End}(\mathfrak{g})$ is defined as $\text{ad}(X)(Y) := [X, Y]$. The Killing form on $\mathfrak{g}$ is indicated by $(\cdot, \cdot)$. This is a bilinear, symmetric, nondegenerate, negative definite ad-invariant map $(\cdot, \cdot): \mathfrak{g}_{\mathbb{R}} \times \mathfrak{g}_{\mathbb{R}} \to \mathbb{R}$. Here ad-invariance means that $(\text{ad}(X)Y, Z) = -(Y, \text{ad}(X)Z)$.

Given $\alpha \in \mathfrak{h}^*$ there is a unique $h_\alpha \in \mathfrak{h}$ such that $(h_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. This element is well-defined, because $(\cdot, \cdot)$ is nondegenerate. Define a dual form on $\mathfrak{h}^*$, again written as $(\cdot, \cdot)$, by $(\alpha, \beta) := (h_\alpha, h_\beta)$. We assume that the Killing form is properly normalised, in the sense that $(\alpha, \alpha) = 2$ for any short root. Abbreviate $d_\alpha := \frac{1}{2}(\alpha, \alpha)$ and renormalise $H_\alpha := \frac{1}{d_\alpha} h_\alpha$.

Consider the real dual $\mathfrak{t}^*$ of $\mathfrak{t}$. The roots $\alpha \in \Delta$ determine hyperplanes in $i\mathfrak{t}^*$ by $\Omega_\alpha := \{\beta \in i\mathfrak{t}^*: (\alpha, \beta) = 0\}$. $\Omega_\alpha$ is called the wall corresponding to $\alpha$. The connected components of $i\mathfrak{t}^* \setminus \bigcup_{\alpha \in \Delta} \Omega_\alpha$ are the (open) Weyl chambers. Fix (once and for all) one of these connected components, we denote it by $C_+$ and call it the positive Weyl chamber. $\overline{C_+}$, the closure of $C_+$, is referred to as the closed Weyl chamber. Indicate the positive roots by $\Delta_+ := \{\alpha \in \Delta : (\alpha, \beta) > 0 \text{ for all } \beta \in C_+\}$ and select the corresponding positive simple roots $\alpha_1, \ldots, \alpha_r$. These roots define the fundamental weights $\{\omega_j\}_{j=1}^r$ in $\mathfrak{h}^*$ by the equations $2\frac{(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{i,j}$. Observe that $\text{span}_\mathbb{C}\{\omega_j : j = 1, \ldots, r\} = \mathfrak{h}^*$ and $\text{span}_\mathbb{R}\{\omega_j : j = 1, \ldots, r\} = i\mathfrak{t}^*$. If $x \in \mathfrak{h}^*$ we generally write $x = x_1 \omega_1 + \ldots + x_r \omega_r$. From the defining relation of the fundamental weights it follows that $x \in C_+$ if and only if $x_i > 0$ for all $i$.

Denote the weight lattice $P := \text{span}_\mathbb{Z}\{\omega_j : j = 1, \ldots, r\}$ and the cone of positive weights $P_+ := \overline{C_+} \cap P$. Also the roots span a lattice, the root lattice $R := \text{span}_\mathbb{Z}(\Delta)$. It is a sublattice of the weight lattice $P$. The element $\rho := \omega_1 + \ldots + \omega_r$ plays a special role. It can be shown that $2\rho = \sum_{\alpha \in \Delta_+} \alpha$ and thus $2\rho \in R$.

The Cartan matrix $(a_{ij})_{i,j=1}^r$ has entries $a_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. It satisfies $a_{ij} \in \mathbb{Z}$. Moreover $\alpha_j = \sum_i a_{ij} \omega_i$. The roots $\alpha \in \Delta$ determine reflections $s_\alpha$ in the hyperplanes $\Omega_\alpha$ by $s_\alpha(\beta) := \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \alpha$. Write $s_i := s_{\alpha_i}$, it follows that $s_i(\alpha_j) = \alpha_j - a_{ij} \alpha_i$. The group generated by these reflections $\{s_\alpha\}_{\alpha \in \Delta}$ is the Weyl group $W$ of $G$.

The Cartan matrix essentially determines $G$. Meaning that if $G$ and $G'$ are two semisimple
1.6. LIE GROUP THEORY

simply connected compact Lie groups with the same Cartan matrix, then \( G \) and \( G' \) are isomorphic. If \( \alpha \) is a root, the space \( \mathfrak{g}_\alpha \) is one-dimensional. Let \( E_\alpha \in \mathfrak{g}_\alpha \) such that \( (E_\alpha, E^*_\alpha) = \frac{2}{(\alpha, \alpha)} \). Write \( F_\alpha := E^*_\alpha \). We denote \( H_i := H_{\alpha_i}, E_i := E_{\alpha_i} \) and \( F_i := F_{\alpha_i} \) for \( i = 1, \ldots, r \). The universal enveloping algebra is defined as \( U\mathfrak{g} := T\mathfrak{g}/J\mathfrak{g} \), where \( T\mathfrak{g} \) denotes the tensor algebra of \( \mathfrak{g} \) and \( J\mathfrak{g} \) is the ideal

\[
J\mathfrak{g} := \text{span}\{T\mathfrak{g} \otimes (X \otimes Y - Y \otimes X - [X,Y]) \otimes T\mathfrak{g} : X, Y \in \mathfrak{g}\}.
\]

The universal enveloping algebra can also be described as the universal associative algebra generated by the elements \( E_i, F_i, H_i, i = 1, \ldots, r \) satisfying the relations

\[
\begin{align*}
[H_i, H_j] &= 0, & [E_i, F_i] &= H_i, & [E_i, F_j] &= 0, & \text{if } i \neq j \\
[H_i, E_j] &= a_{ij} E_j, & [H_i, F_j] &= -a_{ij} F_j,
\end{align*}
\]

\[
(\text{ad}(E_i))^{1-a_{ij}}(E_j) = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_i^{1-a_{ij}-k} E_j E_i^k = 0, & \text{if } i \neq j,
\]

\[
(\text{ad}(F_i))^{1-a_{ij}}(F_j) = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_i^{1-a_{ij}-k} F_j F_i^k = 0, & \text{if } i \neq j.
\]

Here \([\cdot, \cdot] \) denotes the commutator bracket. These last two relations are called the Serre relations. By the highest weight classification of representations it follows that there is a bijection between irreducible representations of \( \mathfrak{g} \) and the positive weights. We identify \( \text{Irr}(\mathfrak{g}) \) with \( \mathbb{P}_+ \). For a weight \( \lambda \in \mathbb{P}_+ \) the corresponding irreducible representation is denoted by \( V_\lambda \). If \( V \) is a representation of \( \mathfrak{g} \), the weight space corresponding to \( \mu \in \mathbb{P} \) is denoted \( V(\mu) := \{ \xi \in V : H_i \xi = (\mu, \alpha_i) \xi \} \).

1.6.2 Drinfeld–Jimbo \( q \)-deformations

As above we let \( G \) be a semisimple simply connected compact Lie group. Assume \( q \neq 0 \). The goal is to define a \( q \)-deformation of \( G \). The key in this procedure is Woronowicz’s Tannaka–Krein duality [Wor88]. It shows how to reconstruct a compact quantum group from a \( C^* \)-tensor category with some additional structure. The definition of \( G_q \) goes in three steps:

1. define a \( q \)-deformation \( U_q(\mathfrak{g}) \) of the universal enveloping algebra \( U\mathfrak{g} \);

2. construct from this a \( C^* \)-tensor category;

3. apply Woronowicz’s theorem to obtain a compact quantum group \( G_q \).

Let us briefly explain these three ingredients. To define a \( q \)-deformation of \( U\mathfrak{g} \) there are several choices one has to make and therefore there are many slightly different algebras around in the literature. Also in this thesis we do not stick to one single choice. We restrict ourselves to \( q \in (0, 1) \). Denote \( d_i := \frac{1}{2}(\alpha_i, \alpha_i) \) and \( q_i := q^{d_i} \).
The following is the Hopf algebra defined by [KS97, Eq. (29)–(33)], it is called the quantized universal enveloping algebra. Define $\tilde{U}(g)$ as the universal algebra with generators $E_i, F_i, K_i, K_i^{-1}, i = 1, \ldots, r$ satisfying the following defining relations:

$$
\begin{align*}
K_iK_j &= K_jK_i, & K_iK_i^{-1} &= K_i^{-1}K_i = 1, \\
K_iE_jK_i^{-1} &= q_i^{a_{ij}/2}E_j, & K_iF_jK_i^{-1} &= q_i^{-a_{ij}/2}F_j, \\
E_iF_j - F_jE_i &= \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}}, \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ 1 - \frac{a_{ij}}{k} \right] q_i^{1-a_{ij}-k} E_i^{1-a_{ij}-k} F_j^k + q_i^{-1-a_{ij}-k} F_j^k E_i^{1-a_{ij}-k} &= 0, & i \neq j,
\end{align*}
$$

Note that the first two lines of these relations are obtained from the defining relations of $Ug$ by formally taking $K_i := q_i^{1/2}$. The last two are obtained by substituting a $q$-binomial instead of the normal binomial coefficient. The algebra $\tilde{U}(g)$ becomes a Hopf $\ast$-algebra when equipped with the extra structure

$$
\begin{align*}
\hat{\Delta}(K_i) &= K_i \otimes K_i, & \hat{\Delta}(E_i) &= E_i \otimes K_i + K_i^{-1} \otimes E_i, & \hat{\Delta}(F_i) &= F_i \otimes K_i + K_i^{-1} \otimes F_i; \\
\hat{\epsilon}(K_i) &= 1, & \hat{\epsilon}(E_i) = \hat{\epsilon}(F_i) &= 0; \\
\hat{S}(K_i) &= K_i^{-1}, & \hat{S}(E_i) &= -q_i E_i, & \hat{S}(F_i) &= -q_i^{-1} F_i; \\
K_i^\ast &= K_i, & E_i^\ast &= F_i.
\end{align*}
$$

The enveloping algebra of the torus is denoted $\tilde{U}(g)$ and equals the subalgebra of $\tilde{U}(g)$ generated by $K_i$ and $K_i^{-1}, i = 1, \ldots, r$. Obviously $\tilde{U}(h)$ is commutative.

The representations of $\tilde{U}(g)$ can be turned into a $C^\ast$-tensor category. Let $V$ be a $\tilde{U}(g)$-module. For $\lambda \in P$ denote $V(\lambda) := \{ \xi \in V : K_i \xi = q_i^{(\lambda, \alpha_i)/2} \xi \text{ for all } i \}$. If $\xi \in V(\lambda)$, then $\xi$ is said to be a vector of weight $\lambda$. We write $\text{wt}(\xi) = \lambda$. The module $V$ is called admissible\footnote{Sometimes in the literature the terminology “type 1” is used.} if

$$
V = \bigoplus_{\lambda \in P} V(\lambda).
$$

Denote by $C_q(g)$ the $C^\ast$-category of finite dimensional admissible unitary representations. It is easy to check that it is a tensor category with conjugates. A version of Woronowicz’s theorem states the following.

**Theorem 1.6.1** ([NT13, Thm. 2.3.13]). Assume $(U, \hat{\Delta})$ is a Hopf $\ast$-algebra and $C \subset \text{Rep}(U)$ is a full $C^\ast$-tensor category with conjugates and $1 = \hat{\epsilon}$. Let $A$ be the subspace of $U^\ast$ spanned by the matrix coefficients of all representations $\pi \in C$. Then $A$ is a Hopf
\section*{1.6. LIE GROUP THEORY}

\(\ast\)-algebra with

\[(ab)(\omega) = (a \otimes b)\hat{\Delta}(\omega), \quad a^*(\omega) = a(\hat{S}(\omega)^*), \quad \Delta(a)(\omega \otimes \nu) = a(\omega \nu), \quad (a, b \in \mathcal{A}, \omega, \nu \in \mathcal{U}).\]

Furthermore \((\mathcal{A}, \Delta) = (\mathbb{C}[G], \Delta)\) for some compact quantum group \(G\). If for every \(\pi \in \text{Ob}(\mathcal{C})\) we define \(U_\pi \in \mathcal{A} \otimes B(\mathcal{H}_\pi)\) by \((\omega \otimes \iota)(U_\pi) = \pi(\omega)\), then the functor \(F: \mathcal{C} \to \text{Rep}(G)\) defined by \(F(\pi) := U_\pi\) on objects and \(F(T) := T\) on morphisms and with \(F_2 := \iota\), is a unitary monoidal equivalence of categories.

\textbf{Definition 1.6.2.} The Hopf algebra \(\hat{U}_q(\mathfrak{g})\) satisfies the requirements of the theorem and therefore there exists a compact quantum group \(G_q = (C(C_q), \Delta)\) such that \((C[G_q], \Delta) = (\mathcal{A}, \Delta)\) where \(\mathcal{A}\) is obtained from \(\hat{U}_q(\mathfrak{g})\) as described above. Let \(C[G_q]\) be the \(\ast\)-envelope of \(C[G_q]\). It can be shown that the compact quantum group \((C[G_q], \Delta)\) is coamenable and thus \(C[G_q]\) is the unique completion of \(\mathbb{C}[G_q]\) to a \(\ast\)-algebra turning it into a compact quantum group (cf. \([NT13, \S 2.7]\)). \(G_q\) is called the Drinfeld–Jimbo \(q\)-deformation of \(G\).

The \(q\)-deformed quantum group \(G_q\) resembles many of the original properties of \(G\). Especially the representation theory is well-understood.

Let \(V \neq 0\) be a \(\hat{U}_q(\mathfrak{g})\)-module. A nonzero vector \(\xi \in V\) is called a highest weight vector of weight \(\lambda \in P_+\) if \(\xi \in V(\lambda), E_i \xi = 0\) for all \(i\) and \(V = \hat{U}_q(\mathfrak{g})\xi\). If such a vector \(\xi \in V\) exists, then \(V\) is called a highest weight module of weight \(\lambda\). Any such module is admissible and for any \(\lambda \in P_+\) there exists a unique irreducible highest weight module \(V_\lambda\) of weight \(\lambda\).

\textbf{Theorem 1.6.3.} The following holds:

(i) for every \(\lambda \in P_+\) the module \(V_\lambda\) is finite dimensional and unitarisable;

(ii) if \(V\) is a finite dimensional admissible \(\hat{U}_q(\mathfrak{g})\)-module, then it decomposes into a direct sum of modules \(V_\lambda\), in particular it is completely reducible and unitarisable;

(iii) the dimensions of \(V_\lambda\) and the weight spaces \(V_\lambda(\mu)\) are equal to the dimensions in the classical case;

(iv) the multiplicities \(m_{\lambda, \mu}^\nu\) in the decomposition

\[V_\lambda \otimes V_\mu \cong \bigoplus_{\nu \in P_+} V_\nu \oplus \ldots \oplus V_\nu\]

are the same as in the classical case.

Part (i) of the above theorem can be made more explicit. For every highest weight vector \(\xi \in V_\lambda\) there exists a unique inner product on \(V_\lambda\) such that \(\langle \xi, \xi \rangle = 1\) and \((V_\lambda, \langle \cdot, \cdot \rangle)\) is a unitary representation of \(\hat{U}_q(\mathfrak{g})\). We write \(\mathcal{H}_\lambda\) for this inner product space (note that it depends on the choice of a highest weight vector \(\xi\), which is unique up to a scalar). Using this the discrete dual \(l^\infty(G_q)\) can be identified with \(l^\infty(\bigoplus_{\lambda \in P_+} B(\mathcal{H}_\lambda))\). Via this
CHAPTER 1. PRELIMINARIES

identification we also have $\tilde{U}_q(\mathfrak{g}) \subset \mathcal{U}(\hat{G}_q)$.

Define the map $R \to \tilde{U}_q(\mathfrak{g})$, $\alpha \mapsto K_\alpha$, where

$$K_\alpha := K_{\alpha_1} \cdots K_{\alpha_r}^n, \quad \text{if } \alpha = n_1 \alpha_1 + \cdots + n_r \alpha_r.$$  

(1.6.10)

It can be shown that the Woronowicz character $f_1 = \rho \in \prod_{\lambda \in P_+} B(\mathcal{H}_\lambda)$ is given by

$$\rho_{\mathcal{H}_\lambda} = \pi_{\lambda}(K_{-4\rho}).$$  

This implies that

$$d_\lambda = \text{Tr}(\pi_{\lambda}(K_{-4\rho})).$$  

(1.6.11)

The assignment (1.6.10) can be extended to $P \to \mathcal{U}(\hat{G}_q)$ by defining $K_\omega$ via the identity

$$\pi_{\lambda}(K_\omega) \xi := q^{(\mu,\omega)/2} \xi$$  

for all $\xi \in V_\lambda(\mu)$.

**Definition 1.6.4.** Suppose $\pi: \tilde{U}_q(\mathfrak{g}) \to B(\mathcal{H})$ is a representation on $\mathcal{H}$. The **conjugate representation** $\bar{\pi}$ on the conjugate Hilbert space $\mathcal{H}$ is defined as follows. Let $\xi \in \mathcal{H}$, an element $X \in \tilde{U}_q(\mathfrak{g})$ acts as $X\bar{\xi} = \hat{R}(X^*)\xi$. Here $\hat{R}$ is the unitary antipode, given by

$$\hat{R}(K_\alpha) := K^{-1}_\alpha, \quad \hat{R}(E_i) := -E_i, \quad \hat{R}(F_i) := -F_i.$$  

In particular, as $\hat{R}(K_\omega) = K_{-\omega}$, it follows that for weight vectors $\text{wt}(\xi) = -\text{wt}(\bar{\xi})$.

Let $w_0$ be the longest element in the Weyl group of $\mathfrak{g}$ and $\lambda \in P_+$, then $w_0(\lambda)$ is the lowest weight of $\pi_\lambda$. Hence $-w_0(\lambda)$ is the highest weight of $\pi_\lambda$. So define $\bar{\lambda} := -w_0(\lambda)$, it follows by the highest weight classification of irreducible representations that $(\bar{\pi}_\lambda, \mathcal{H}_{\bar{\lambda}}) \cong (\pi_\lambda, \mathcal{H}_\lambda)$. Therefore there exists up to a scalar a unique linear map map $\mathcal{H}_\lambda \to \mathcal{H}_{\bar{\lambda}}$ intertwining the representations $\bar{\pi}_\lambda$ and $\pi_\lambda$. Let $J: \mathcal{H}_\lambda \to \mathcal{H}_{\bar{\lambda}}$ be the corresponding antilinear isometry, it is unique up to a scalar of modulus one. If $\xi_\lambda$ is a highest weight vector of $\mathcal{H}_\lambda$ define $J(\bar{\xi}_\lambda) := \xi_{-\lambda}$, where $\xi_{-\lambda}$ is a lowest weight vector of $\mathcal{H}_{\bar{\lambda}}$ and extend $J$ by equivariance. By decomposing a representation into irreducible components we extend $J$ to arbitrary finite dimensional representations.
Chapter 2

Categories of SU($N$)-type

This chapter is based on the paper [Jor14]. At some points the arguments of the paper have been expanded and more details are given. The results on the other hand are the same. In addition some of the preliminaries stated in the paper are reformulated and can be found in the first chapter of this thesis.

The aim of this chapter is to give an intrinsic characterisation of C$^*$-tensor categories with the same fusion rules as SU($N$) and to classify all such categories. Such a classification is an intermediate step in the classification of all non-Kac compact quantum groups with the same representation theory as SU($N$) [NY14b].

2.1 Preliminaries on categories of SU($N$)-type

In this section we introduce the specific class of C$^*$-tensor categories we are interested in, namely the SU($N$)-type categories. We also discuss “twists” of such categories.

Before we define a SU($N$)-type category let us first say something about the representations of the special unitary group SU($N$). Details can be found in lots of books, e.g. [FH91]. To avoid trivialities we always assume that $N \geq 2$. The fundamental (or defining) representation of SU($N$) is the representation on $V := \mathbb{C}^N$ by letting the group elements act on vectors of $V$ in the straightforward way. By the highest weight classification of irreducible representations of SU($N$), the irreducible representations can be identified with tuples in the set

$$\text{Irr}(\text{SU}(N)) \cong \Lambda_N := \{\lambda = (\lambda_1, \ldots, \lambda_{N-1}) \in \mathbb{N}^{N-1} : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N-1}\}. \quad (2.1.1)$$

For Lie groups it is more common to label the irreducible representations by Greek letters instead of Roman letters. We adopt this convention and label the irreducible representations of SU($N$) usually by $\lambda$, $\mu$ or $\nu$. We denote by $V_\lambda$ the irreducible representation corresponding to $\lambda$. For $\lambda \in \mathbb{N}^{N-1}$ let $|\lambda| := \lambda_1 + \ldots + \lambda_{N-1}$. It can be shown that any irreducible representation $V_\lambda$ is contained in the tensor product $V^{\otimes |\lambda|}$. Another special fact for SU($N$) is that the $N$-th anti-symmetric tensor power $\Lambda^N V$ is isomorphic to the trivial representation. Thus there exists a nonzero map $\mathbb{C} \to V^{\otimes N}$ intertwining the trivial representation and the $N$-th tensor power of the defining representation. Given two irreducible representations $V_\lambda$ and $V_\mu$, we decompose their tensor product representation into...
irreducible representations as

\[ V_\lambda \otimes V_\mu = \bigoplus_{\nu \in \Lambda_N} m_{\lambda,\mu}^\nu V_\nu, \]

where \( m_{\lambda,\mu}^\nu := \dim \Hom(V_\nu, V_\lambda \otimes V_\mu) \) denote the multiplicities and \( m_{\lambda,\mu}^\nu V_\nu := V_\nu \oplus \ldots \oplus V_\nu \), with \( m_{\lambda,\mu}^\nu \) copies.

**Assumption 2.1.1.** All \( C^*\)-tensor categories in this chapter have a simple unit \( 1 \).

**Definition 2.1.2.** Two objects \( U, V \in \text{Ob}(C) \) in a \( C^*\)-tensor category \( C \) are isomorphic if there exists an isomorphism in \( \Hom(U, V) \). We write \([U]\) for the equivalence class of objects isomorphic to \( U \). Denote by \( K^+[C] \) the fusion semiring of \( C \), it is the universal semiring generated by the equivalence classes \([U]\) of objects \( U \in \text{Ob}(C) \) with sum and product given by

\[ [U] + [V] := [V \oplus V], \quad [U][V] := [U \otimes V]. \]

Note that there is no need to define a subtraction as we define a semiring.

**Definition 2.1.3.** A \( C^*\)-tensor category of \( SU(N)\)-type, or simply a \( SU(N)\)-type category, is a rigid \( C^*\)-tensor category \( C \) such that the semirings \( K^+[C] \) and \( K^+[\text{Rep}(SU(N))] \) are isomorphic. In particular since simple objects can not be further decomposed, this isomorphism maps simple objects onto simple objects. Therefore we can index the equivalence classes of simple objects of a \( SU(N)\)-type category by the set \( \Lambda_N \). An object \( X \in C \) which corresponds to the fundamental representation \( [\mathbb{C}^N] \) of \( SU(N) \) is called the fundamental object of \( C \).

From now on we fix a \( SU(N)\)-type category \( C \) with fundamental object \( X \) and for every \( \lambda \in \Lambda_N \) we fix a simple object \( X_\lambda \) corresponding to \( \lambda \).

**Example 2.1.4.** The object \( X_{(k)} := X_{(k,0,\ldots,0)} \) corresponds to \( S^k(V) \), the \( k \)-th symmetric tensor power of the fundamental representation of \( SU(N) \) on \( V \). For \( 1 \leq k \leq N - 1 \), the object \( X_{(1^k)} := X_{(1,\ldots,1,0,\ldots,0)} \) corresponds to \( \wedge^k(V) \), the \( k \)-th antisymmetric tensor power of the fundamental representation.

The conjugate object \( \bar{X} \cong X_{(1^{N-1})} \). Indeed, by the fusion rules of \( SU(N) \) it follows that \( X \otimes X_{(1^{N-1})} \cong X_{(1^N)} \oplus X_{(21^{N-2})} \). Since \( X_{(1^N)} \cong 1 \), we obtain that \( \Hom(1, X \otimes X_{(1^{N-1})}) \neq \{0\} \), from which the claim follows. We often write \( X_{(1^N)} \) to identify the \( 1 \) in \( X^{\otimes N} \).

**Notation 2.1.5.** In a not necessarily strict \( SU(N)\)-type category denote the objects \( X^{\otimes 1} := X \) and \( X^{\otimes n} := X \otimes X^{\otimes n-1} \) for \( n \geq 2 \). Unwrapping this recursive definition gives \( X^{\otimes n} = X \otimes (X \otimes (\cdots (X \otimes X) \cdots)) \).

**Lemma 2.1.6.** Suppose \( X \) is a fundamental object of a \( SU(N)\)-type category \( C \). Then

\[ X^{\otimes n} = \bigoplus_{\lambda \in \Lambda_N} m^\lambda_{X^{\otimes n}} X_\lambda \]

and the multiplicities satisfy \( m^\lambda_{X^{\otimes n}} = 0 \) if \( |\lambda| \neq n \) (mod \( N \)). In particular if \( m \neq n \) (mod \( N \), then \( \Hom(X^{\otimes m}, X^{\otimes n}) = \{0\} \).
2.1. PRELIMINARIES ON CATEGORIES OF SU(N)-TYPE

Proof. For $\text{Rep}(\text{SU}(N))$ Identity (2.1.2) follows for $X = V$ and $X_\lambda = V_\lambda$ from [FH91, Prop. 15.25]. Since $\mathcal{C}$ is a SU($N$)-type category it satisfies the same fusion rules as SU($N$).

It is possible to obtain a new $C^*$-tensor category from an existing one by changing the associativity morphisms. This can be done using twists. We only define twists in the case of a special type of categories, but a twist can be defined more general for example for compact quantum groups, see for instance [NT13, NY15].

**Definition 2.1.7.** Let $\mathcal{C}$ be a strict $C^*$-tensor category and $X \in \text{Ob}(\mathcal{C})$. Assume that $\text{Hom}_\mathcal{C}(X^\otimes m, X^\otimes n) = \{0\}$ if $m \neq n \pmod{N}$. Let $\hat{\mathcal{C}}$ be the category with objects $\{1, X, X^\otimes 2, \ldots\}$ and morphisms $\text{Hom}_\mathcal{C}(X^\otimes m, X^\otimes n) := \text{Hom}_\mathcal{C}(X^\otimes m, X^\otimes n)$. Let $\rho$ be an $N$-th root of unity. Put $\omega(a, b) := \left\lfloor \frac{a+b}{N} \right\rfloor - \left\lfloor \frac{a}{N} \right\rfloor$ for $a, b \in \mathbb{N}$. Define the morphisms

$$\alpha_{X^\otimes a, X^\otimes b, X^\otimes c}^\rho := \rho^{\omega(a,b)c} \cdot \alpha_{X^\otimes a, X^\otimes b, X^\otimes c} : (X^\otimes a \otimes X^\otimes b) \otimes X^\otimes c \to X^\otimes a \otimes (X^\otimes b \otimes X^\otimes c).$$ (2.1.3)

It can be checked (see Lemma 2.1.9 below) that the morphisms $\alpha^\rho$ satisfy the pentagon axiom and the collection $\{\alpha_{X,Y,Z}^\rho\}_{X,Y,Z}$ is natural in $X, Y, Z$. Therefore $\alpha^\rho$ define new associativity morphisms on $\hat{\mathcal{C}}$. Completing $\hat{\mathcal{C}}$ with respect to subobjects and direct sums and extending $\alpha^\rho$ to this completion gives new associativity morphisms for the $C^*$-tensor category generated by $\hat{\mathcal{C}}$. We denote this category by $\mathcal{C}^\rho$. If $\mathcal{C}$ is generated by $X$ (that is $\mathcal{C}$ is the direct sum and subobject completion of the full subcategory with objects $\{1, X, X^\otimes 2, \ldots\}$), we denote the category we obtain in this way by $\mathcal{C}^\rho$.

At first sight these associativity morphisms might seem artificial, but one can prove that the functionals $\rho^{\omega(a,b)c}$ defined above represent all classes in $H^3(\mathbb{Z}/ZN, \mathbb{T})$, see e.g. [NY15, Prop. A.3]. This cocycle property is exactly needed to make the pentagon diagram commutative.

**Remark 2.1.8.** Note that $\rho^{\omega(a,b)c} \rho^{\omega(a,b)c} = (\rho^c)^{\omega(a,b)c}$ for all $a, b$ and $c$. So if $\mathcal{C}$ is generated by $X$, we immediately obtain that $(\mathcal{C}^\rho)^d \cong \mathcal{C}(\rho^d)$.

**Lemma 2.1.9.** The morphisms $\alpha^\rho$ defined in (2.1.3) satisfy the pentagon axiom and are natural.

Proof. Since $\alpha$ are associativity morphisms, commutativity of the diagram

$$
\begin{array}{ccc}
((X^\otimes a \otimes X^\otimes b) \otimes X^\otimes c) \otimes X^\otimes d & \xrightarrow{\alpha_{1,2,3,4}^\rho} & ((X^\otimes a \otimes X^\otimes b) \otimes (X^\otimes c \otimes X^\otimes d)) \\
\alpha_{1,2,3}^\rho & & \alpha_{1,2,3,4}^\rho \\
(X^\otimes a \otimes (X^\otimes b \otimes X^\otimes c)) \otimes X^\otimes d & \xrightarrow{1 \otimes \alpha_{2,3,4}^\rho} & (X^\otimes a \otimes X^\otimes b) \otimes (X^\otimes c \otimes X^\otimes d) \\
\alpha_{1,2,3}^\rho & & \\
X^\otimes a \otimes ((X^\otimes b \otimes X^\otimes c) \otimes X^\otimes d) & \xrightarrow{1 \otimes \alpha_{2,3,4}^\rho} & X^\otimes a \otimes (X^\otimes b \otimes (X^\otimes c \otimes X^\otimes d))
\end{array}
$$

is equivalent to

$$\rho^{\omega(b,c)d} \rho^{\omega(a,b+c)d} = \rho^{\omega(a,b)(c+d)} \rho^{\omega(a+b,c)d}.$$
For which in turn it is sufficient to prove that
\[ \omega(b, c)d + \omega(a, b + c)d + \omega(a, b)c - \omega(a, b)(c + d) - \omega(a + b, c)d \equiv 0 \pmod{N}. \]

This congruency (which is in fact an equality) can readily be verified.

For naturality recall that by assumption \( \text{Hom}_C(X^\otimes m, X^\otimes n) = \{0\} \) if \( m \not\equiv n \pmod{N} \).

Suppose that for \( i = 1, 2, 3 \) the morphisms \( T_i \in \text{Hom}(X^\otimes a_i, X^\otimes b_i) \) are nonzero, then \( a_i \equiv b_i \pmod{N} \). Say \( a_i + k_i N = b_i \). Then
\[
\omega(b_1, b_2) = \frac{a_1 + a_2 + (k_1 + k_2)N}{N} - \frac{a_1 + k_1 N}{N} - \frac{a_2 + k_2 N}{N}
= \frac{a_1 + a_2}{N} + (k_1 + k_2) - \frac{a_1}{N} - k_1 - \frac{a_2}{N} - k_2
= \omega(a_1, a_2)
\]
and hence \( \rho^{\omega(a_1, a_2)a_3} = \rho^{\omega(b_1, b_2)b_3} \). Now naturality of \( \alpha^\rho \) follows from the fact that \( \alpha \) is an associativity morphism. ☐

**Lemma 2.1.10.** Suppose that \( \mathcal{C} \) is a strict \( \mathcal{C}^\ast \)-tensor category generated by an object \( X \) and \( \rho \) is an \( N \)-th root of unity for some \( N \geq 2 \). Let \( \alpha \) and \( \alpha^\rho \) be the associativity morphisms in \( \mathcal{C} \) respectively in \( \mathcal{C}^\rho \). Consider for \( m, n \geq 1 \) the associativity morphism \( \alpha_{m,n} : X^\otimes m \otimes X^\otimes n \to X^\otimes m+n \) in \( \mathcal{C} \), defined by the following inductive relations
\[
\alpha_{m,n} := \begin{cases} 
\iota_{X^\otimes X}, & \text{if } m = n = 1; \\
(\iota \otimes \alpha_{m-1,1})\alpha_{X^\otimes X^\otimes m-1, X}, & \text{if } m \geq 2, n = 1; \\
\alpha_{m+1,n-1} \circ (\alpha_{m,1} \otimes \iota^{\otimes n-1}) \circ \alpha_{X^\otimes m,X^\otimes n-1}^{-1}, & \text{if } m \geq 1, n \geq 2.
\end{cases}
\]

Define similarly the morphisms \( \alpha^\rho_{m,n} \) in \( \mathcal{C}^\rho \). Then it holds that
\[
\alpha^\rho_{m,n} = \rho^{n \left\lfloor \frac{m}{N} \right\rfloor} \alpha_{m,n}.
\]

**Proof.** We prove this lemma by induction on \( m \) and \( n \). If \( m = n = 1 \), the statement is trivial. Suppose that \( n = 1 \). Note that because \( N \geq 2 \) it holds that \( \left\lfloor \frac{1}{N} \right\rfloor = 0 \). So by definition of the twist we obtain
\[
\alpha^\rho_{X, X^\otimes m-1, X} = \rho^{\left\lfloor \frac{m}{N} \right\rfloor - \left\lfloor \frac{m-1}{N} \right\rfloor} \alpha_{X, X^\otimes m-1, X}
\]
as a map \( (X^\otimes m) \otimes X \to X \otimes ((X^\otimes m-1) \otimes X) \). Proceeding by induction on \( m \) it follows that
\[
\alpha^\rho_{m,1} = (\iota \otimes \alpha^\rho_{m-1,1})\alpha^\rho_{X, X^\otimes m-1, X}
= \rho^{\left\lfloor \frac{m-1}{N} \right\rfloor + \left\lfloor \frac{m-1}{N} \right\rfloor} (\iota \otimes \alpha_{m-1,1}) \alpha_{X, X^\otimes m-1, X}
= \rho^{\left\lfloor \frac{m}{N} \right\rfloor} \alpha_{m,1}
\]
and the lemma is proved for \( n = 1 \). Now suppose that \( n > 1 \). By the definition and
induction hypothesis, it holds that
\[\alpha_{m,n}^n = (\rho^{(n-1)}(\frac{m+1}{q})\alpha_{m+1,n-1})(\rho^{\lfloor \frac{m}{N} \rfloor} \alpha_{m,1} \otimes l^{\otimes n-1})(\rho^{-(\lfloor \frac{m+1}{N} \rfloor - \lfloor \frac{n}{N} \rfloor)(n-1)} \alpha_{X^{\otimes m},X^{\otimes n-1}})\]

as desired. \qed

\section{Hecke algebras}

In this section we briefly recall some results about Hecke algebras which are used later when considering SU\((N)\)-type categories. More about Hecke algebras can be found in for example [Wen88].

\textbf{Definition 2.2.1.} Given \(n \in \mathbb{N}\) and \(q \in \mathbb{C}\), define the Hecke algebra \(H_n(q)\) to be the unital algebra generated by the \(n-1\) elements \(g_1, \ldots, g_{n-1}\) which satisfy the following three relations:

\[g_i g_j = g_j g_i \quad \text{if} \quad |i - j| \geq 2; \quad (2.2.1)\]

\[g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for} \quad i = 1, \ldots, n - 2; \quad (2.2.2)\]

\[g_i^2 = (q - 1) g_i + q \quad \text{for} \quad i = 1, \ldots, n - 1. \quad (2.2.3)\]

Note that if \(q \neq 0\) we have

\[g_i \left( \frac{1 - q}{q} + \frac{1}{q} g_i \right) = \frac{1 - q}{q} g_i + \frac{1}{q} ((q - 1)g_i + q) = 1.\]

So for \(q \neq 0\) the elements \(g_i\) have inverses. We denote these by \(g_i^{-1} := \frac{1 - q}{q} + \frac{1}{q} g_i\). Observe that if \(q = 1\) relation (2.2.3) reads as \(g_i^2 = 1\) hence \(H_n(1) = \mathbb{C}[S_n]\), the group algebra of the symmetric group on \(n\) elements. So for \(q = 1\) we obtain a map \(S_n \to H_n(q)\), but also for general \(q \in \mathbb{C}\) we can define such a map.

\textbf{Definition 2.2.2.} An elementary transposition of \(S_n\) is an element of the form \(\sigma_i := (i, i+1)\). Any element \(\pi \in S_n\) can be written as a product of elementary transpositions \(\pi = \sigma_{i_1} \cdots \sigma_{i_k}\). For a permutation \(\pi\) choose such a product of shortest length. The corresponding \(k\) is referred to as the \textit{length} of \(\pi\), we put \(l(\pi) := k\). A product of shortest length is called a \textit{reduced expression} for \(\pi\). For the identity element \(e \in S_n\), we put \(g_e := 1\). If \(\pi \in S_n\) and \(\pi \neq e\), we define \(g_\pi := g_{i_1} \cdots g_{i_k} \in H_n(q)\). From the the lemma below it follows that the element \(g_\pi\) is well-defined.

\textbf{Lemma 2.2.3.} Suppose that \(\pi \in S_n\). Define \(d_\pi(i) := |\{1 \leq j < i : \pi(j) > \pi(i)\}|\). Then \(l(\pi) = \sum_{i=1}^{n} d_\pi(i)\). Put

\[C_{i,j} := \begin{cases} 1, & \text{if } i \geq j; \\ \sigma_i \cdots \sigma_{j-1}, & \text{if } i < j. \end{cases}\]
Then $C_{n-d_a(n).n} \cdot C_{3-d_a(3).3} C_{2-d_a(2),2}$ is a reduced expression for $\pi$. Any two reduced expressions for $\pi$ can be transformed in one another by only using the transformations

\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 1; \]

\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \ldots, n - 2. \]

**Proof.** The proofs of these statements can be found in [Gar02, §1.1].

We can embed $H_n(q)$ into $H_{n+1}(q)$ via the homomorphism $H_n(q) \ni g_i \mapsto g_i \in H_{n+1}(q)$. Iterating this procedure we obtain embeddings $i_{m,n} : H_m(q) \to H_n(q)$ for $m \leq n$. The inductive limit of $(H_n(q), i_{m,n})$ is denoted by $H_\infty(q)$. Similarly the shift map $\Sigma : H_n(q) \to H_{n+1}(q)$, $g_i \mapsto g_{i+1}$ yields another embedding. Unless stated otherwise we use the first embedding.

Note that (2.2.3) can be rewritten as $(g_i + 1)(g_i - q) = 0$. So $g_i$ has exactly two spectral values: $-1$ and $q$. In accordance with Kazhdan and Wenzl we define the idempotents $e_i := \frac{g_i - g_{i+1}}{q}$. Then

**Lemma 2.2.4.** Let $\sigma_{m,n} \in S_{m+n}$ be the permutation defined by

\[ \sigma_{m,n}(i) := \begin{cases} i + n, & \text{if } 1 \leq i \leq m; \\ i - m, & \text{if } m + 1 \leq i \leq m + n. \end{cases} \]

Then

\[ g_{\sigma_{m,n}} = (g_n g_{n-1} \cdots g_1)(g_{n+1} g_n \cdots g_2) \cdots (g_{n+m-n+1} g_{n+m-2} \cdots g_m) \]

\[ = (g_n g_{n+1} \cdots g_{n+m-1})(g_{n+1} g_n \cdots g_{n+m-2}) \cdots (g_1 g_2 \cdots g_m) \]

and

\[ g_{\sigma_{m,n}} g_i = \begin{cases} g_{i+n} g_{\sigma_{m,n}}, & \text{if } 1 \leq i \leq m - 1; \\ g_{i-m} g_{\sigma_{m,n}}, & \text{if } m + 1 \leq i \leq m + n. \end{cases} \quad (2.2.4) \]

**Proof.** The explicit formulas for $g_{\sigma_{m,n}}$ follow from Lemma 2.2.3. Suppose that $m = 1$ and $i > 1$, then

\[ (g_n \cdots g_1)g_i = g_n \cdots (g_i g_{i-1} g_i)g_{i-2} \cdots g_1 = g_n \cdots (g_{i-1} g_i g_{i-1})g_{i-2} \cdots g_1 = g_{i-1}(g_n \cdots g_1). \]

Now for $m > 1$ and $i > m$ the statement follows from the case $m = 1$, induction on $m$ and the explicit formula of $g_{\sigma_{m,n}}$. For $i < m$ we use the other expression of $g_{\sigma_{m,n}}$. Assume $n = 1$, we obtain

\[ (g_1 \cdots g_m)g_i = g_1 \cdots (g_i g_{i-1} g_i)g_{i-2} \cdots g_m = g_1 \cdots (g_{i-1} g_i g_{i-1})g_{i-2} \cdots g_m = g_{i-1}(g_1 \cdots g_m). \]

Again the general case follows from this case, induction on $n$ and the explicit formula of $g_{\sigma_{m,n}}$. ☐

\[ ^1 \text{Note that the other choice of idempotents } e'_i := \frac{1+g_i}{q+q} \text{ is also used in the literature.} \]
**Definition 2.2.5.** Suppose $q > 0$ or $|q| = 1$. Define an involution on $H_n(q)$ by $e_i^* := e_i$ and by antilinear extension. This involution is called the **standard involution** of $H_n(q)$. From now on we assume that for these values of $q$ the Hecke algebra $H_n(q)$ is equipped with this standard involution. Note that the idempotents $e_i$ become the spectral projections corresponding to the spectral value $-1$ of $g_i$. Furthermore if $q > 0$ the elements $g_i$ become self-adjoint.

Recall the $q$-numbers defined in Notation 1.2.1. The results of the following two lemmas are similar to [Pin07, §2]. We use some different conventions, so for convenience we give a proof.

**Lemma 2.2.6.** Denote $A_n := \sum_{\sigma \in S_n} g_{\sigma}$. If $q^n \neq 1$ for $m = 1, \ldots, n$ denote $E_n := ([|n|/q])^{-1} A_n$. The following holds:

1. $A_n = (1 + g_{n-1} + g_{n-2}g_{n-1} + \ldots + g_1 \cdots g_{n-1})A_{n-1}$
   \[= (1 + g_1 + g_2g_1 + \ldots + g_{n-1} \cdots g_1)\Sigma(A_{n-1})
   \[= \Sigma(A_{n-1})(1 + g_1 + g_1g_2 + \ldots + g_1 \cdots g_{n-1});
   \]

2. $A_n g_i = g_i A_n = q A_n$, for $i = 1, \ldots, n - 1$;

3. $E_n$ is a minimal idempotent in $H_n(q)$. If $q \in \mathbb{R}$ or $|q| = 1$ and $q^n \neq 1$ for $m = 1, \ldots, n$, it is a projection.

**Proof.** (i) From Lemma 2.2.3 it follows that every element $\pi \in S_n$ can uniquely be written as $\pi = \sigma_j \sigma_{j+1} \cdots \sigma_{n-1} \pi'$ for some $j \in \{1, \ldots, n\}$ and $\pi' \in S_{n-1}$. For a permutation $\sigma$ denote $\sigma S_n := \{\sigma \sigma' : \sigma' \in S_n\}$. By Lemma 2.2.3,

\[
S_{n+1} = S_n \cup \sigma_n S_n \cup \ldots \cup (\sigma_1 \cdots \sigma_n) S_n,
\]

hence by induction the first equality follows. Also $\pi^{-1} = \pi'^{-1} \sigma_{n-1} \cdots \sigma_j$, therefore

\[
\sum_{\pi \in S_n} g_{\pi} = \sum_{\pi \in S_n} g_{\pi^{-1}} = \sum_{\pi' \in S_{n-1}} g_{\pi'} \left( \sum_{j=1}^{n} g_{n-1} \cdots g_{j+1} g_j \right).
\]

Now by induction the second equality in (i) follows. The third and fourth equalities can be proved similarly (use the map $\sigma_i \mapsto \sigma_{i-1}$).

(ii) If $n = 1$ there is nothing to prove, for $n = 2$ we can directly verify using relation (2.2.3)

\[
A_2 g_1 = g_1 A_2 = (1 + g_1) g_1 = g_1 + (q - 1) g_1 + q = q (g_1 + 1) = q A_2.
\]

Suppose that $n > 1$ and $i < n - 1$, then by induction

\[
A_n g_i = (1 + g_{n-1} + g_{n-2}g_{n-1} + \ldots + g_1 \cdots g_{n-1}) A_{n-1} g_i
   \[= (1 + g_{n-1} + g_{n-2}g_{n-1} + \ldots + g_1 \cdots g_{n-1}) q A_{n-1} = q A_n.
   \]

if $i = n - 1$ consider $(1 + g_1 + g_2g_1 + \ldots + g_{n-1} \cdots g_1) \sigma(A_{n-1})$. For $g_i A_n$ use the other formulas of (i).
(iii) By (ii) we have $E_n^2 = \left( [[[n]]_q]^{-1} \sum_{\pi \in S_n} q^{l(\pi)} E_n \right)^2$. So we need to show that $\sum_{\pi \in S_n} q^{l(\pi)} = [[[n]]_q]^{-1}$. Again we prove this by induction. The case $n = 1$ is trivial. We immediately obtain $l(\sigma_i \cdots \sigma_n \sigma) = l(\sigma) + n - i + 1$ for $\sigma \in S_n$. So by (2.2.5)

$$\sum_{\pi \in S_{n+1}} q^{l(\pi)} = (1 + q + \ldots + q^n) \sum_{\pi \in S_n} q^n = [[[n+1]]_q][[n]_q]^{-1} = [[[n+1]]_q]^{-1},$$

and thus $E_n$ is an idempotent. Minimality is immediate from (ii). Clearly, if $q > 0$, then $g_i^* = g_i$, and $E_n$ is self-adjoint. Suppose that $|q| = 1$, then $g_i^* = \overline{q} - (q+1) e_i = \frac{q-q}{q+1} + \frac{q+1}{q+1} g_i$. Note if $|q| = 1$, then $\frac{q-q}{q+1} + \frac{q+1}{q+1} = \overline{q}$. Therefore $g_i^* E_n = \overline{q} E_n$ and

$$E_n^* E_n = \left( [[[n]]_q]^{-1} \sum_{\sigma \in S_n} g_\sigma^* E_n \right) \left( [[[n]]_q]^{-1} \sum_{\sigma \in S_n} q^{l(\sigma)} E_n \right) = [[[n]]_q]^{-1} [[[n]]_q] E_n = E_n.$$

Hence $E_n^* = E_n^+ E_n = E_n$ and $E_n$ is self-adjoint.

**Notation 2.2.7.** Define maps $\alpha$ and $\beta$ on the generators by

$$\alpha: H_n(q) \to H_n(q), \quad g_i \mapsto q - 1 - g_i;$$

$$\beta: H_n(q) \to H_n(q^{-1}), \quad g_i \mapsto -q^{-1} g_i.$$

A simple computation shows that $\alpha$ and $\beta$ respect the defining relations of the Hecke algebras (cf. Identities (2.2.1)–(2.2.3)) and thus that $\alpha$ and $\beta$ are Hecke algebra morphisms. Furthermore $\alpha \circ \alpha = id$ and $\beta \circ \beta = id$. Note also that $\alpha(e_i) = e'_i = \frac{1+q}{q+1}$, where $e'_i$ is the other choice of idempotents, as discussed in Section 2.2.

**Lemma 2.2.8.** Suppose $q \neq 0$. Denote $B_n := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} g_\sigma$. If $q^n \neq 1$ for $m = 1, \ldots, n$ let $F_n := \left( [[[n]]_{\frac{1}{q}}]^{-1} \right) B_n$. The following holds:

(i) $B_n = (1 - q^{-1} g_{n-1} + q^{-2} g_{n-2} + \ldots + (-q)^{-(n-1)} g_1) B_{n-1}$

$$= B_{n-1} (1 - q^{-1} g_{n-1} + q^{-2} g_{n-2} + \ldots + (-q)^{-(n-1)} g_1)$$

$$= \Sigma(B_{n-1})(1 - q^{-1} g_1 + q^{-2} g_2 + \ldots + (-q)^{-(n-1)} g_1) \Sigma(B_{n-1})$$

$$= q^n (1 - q g_{n-1} + \ldots + (-q)^{-n} g_{n-1}) B_{n-1};$$

(ii) $B_n g_i = g_i B_n = -B_n$, for $i = 1, \ldots, n - 1$;

(iii) $F_n$ is a minimal idempotent in $H_n(q)$. If $q \in \mathbb{R} \setminus \{0\}$, or $|q| = 1$ and $q^n \neq 1$ for $m = 1, \ldots, n$ it is a projection;

(iv) $\alpha(B_n) = q^{-n(n-1)/2} A_n$.

**Proof.** It is immediate that $\beta(A_n) = B_n$. Thus all assertions except from the last equality in item (i) and item (iv) follow from the previous lemma. To prove (i), first note that if
\[ g_i^{-1} \cdots g_n^{-1} B_{n-1} = \frac{1}{q} g_i^{-1} \cdots g_{n-2} g_{n-1} B_{n-1} + \frac{1-q}{q} g_i^{-1} \cdots g_{n-2} B_{n-1} \]

Thus we obtain that (2.2.6) equals

\[ (g_1^{-1} \cdots g_n^{-1}) (1-q) g_i^{-1} \cdots g_{n-2} g_{n-1} B_{n-1} + \frac{1-q}{q} g_i^{-1} \cdots g_{n-2} B_{n-1} \]

\[ = \frac{1-q}{q} g_i^{-1} \cdots g_{n-2} g_{n-1} B_{n-1} + \frac{1-q}{q} (1-q) g_i^{-1} \cdots g_{n-2} B_{n-1} \]

\[ = \left( q^{-(n-i)} g_i \cdots g_{n-1} + \frac{1-q}{q} q^{-(n-i-1)} g_{i+1} \cdots g_{n-1} \right. \]

\[ + \frac{1-q}{q} q^{-(n-i-2)} (1) g_{i+2} \cdots g_{n-1} + \frac{1-q}{q} q^{-(n-i-3)} (1) g_{i+3} \cdots g_{n-1} \]

\[ + \cdots + \frac{1-q}{q} (1) B_{n-1} \]

Therefore

\[ (g_1^{-1} \cdots g_n^{-1}) + (-q)^{-1} g_2^{-1} \cdots g_{n-1} + \cdots + (-q)^{-2} g_{n-1} + (-q)^{-(n-1)} B_{n-1} \]

\[ = \left( q^{-(n-i)} g_i \cdots g_{n-1} + \frac{1-q}{q} q^{-(n-i-1)} g_{i+1} \cdots g_{n-1} \right. \]

\[ + (-q)^{-1} q^{-(n-i)} g_{i+2} \cdots g_{n-1} \]

\[ + \frac{1-q}{q} q^{-(n-i-3)} (1) g_{i+3} \cdots g_{n-1} + \cdots + (-q)^{-(n-1)} B_{n-1}. \] (2.2.6)

Gathering all terms \( g_i \cdots g_{n-1} \), the constant in front of \( g_i \cdots g_{n-1} \) becomes

\[ \frac{1-q}{q} \left( (-1)^i q^{-(n-i)} + (-q)^{-1} (-1)^{-1} q^{-(n-i)} + \cdots + (-q)^{-2} q^{-(n-i)} \right) + (-q)^{-(n-i)} \]

\[ = (1-q)(-1)^i q^{-(n+1-i)} + q^{-(n+1-i)} + \cdots + q^{-(n-1)} - (-1)^i q^{-(n-1)} \]

\[ = (-1)^i (-q)^{-1} q^{-(n+1-i)} + (-1)^i q^{-(n-1)} - (-1)^i q^{-(n-1)} \]

\[ = (-1)^{i+1} q^{-(n-i)}. \]

In the second last equality above we use the fact that we have an alternating sum. We thus obtain that (2.2.6) equals

\[ \left( (-1)^{i+1} q^{-(n-i)} g_i \cdots g_{n-1} + (-1)^{2+1} q^{-(n-2)} g_2 \cdots g_{n-1} + \cdots + (-1)^{n+1} q^{-(n-n)} \right) B_{n-1} \]

\[ = (-1)^{n-1} \left( (-q)^{-(n-1)} g_1 \cdots g_{n-1} + (-q)^{-(n-2)} g_2 \cdots g_{n-1} + \cdots + 1 \right) B_{n-1}, \]

which by the first equality of item (i) in this lemma gives the desired result.

We proved (iv) by induction. The case \( n = 2 \) is easy, as

\[ \alpha(B_2) = \alpha(1-q^{-1} g_1) = 1 - q^{-1} (q - 1 - g_1) = q^{-1} (1 + g_1) = q^{-2(2-1)/2} A_2. \]

To prove the induction step, first note that \( \alpha(-qg_i^{-1}) = \alpha(-1+q-g_i) = -1+q-q+1+g_i = g_i \). Therefore using part (i) of this lemma, the induction hypothesis and Lemma 2.2.6, we
CHAPTER 2. CATEGORIES OF SU(N)-TYPE

get
\[ \alpha(B_{n+1}) = \alpha(q^{-n}(1 - qg_n^{-1} + \ldots + (-q)^{n}g_1^{-1}\ldots g_n^{-1})B_n) \]
\[ = q^{-n}(1 + \alpha(-qg_n^{-1}) + \ldots + \alpha((-q)^{n}g_1^{-1}\ldots g_n^{-1}))q^{-n(n-1)/2}A_n \]
\[ = q^{-(n+1)n/2}(1 + g_n + \ldots + g_1\ldots g_n)A_n \]
\[ = q^{-(n+1)n/2}A_{n+1}, \]

as desired. \qed

Definition 2.2.9. A trace on the Hecke algebra \( H_\infty(q) \) is a linear functional \( \text{tr}: H_\infty(q) \to \mathbb{C} \) such that \( \text{tr}(ab) = \text{tr}(ba) \) for all \( a, b \in H_\infty(q) \) and \( \text{tr}(1) = 1 \). The trace is called a Markov trace if there exists an \( \eta \in \mathbb{C} \) such that for all \( n \in \mathbb{N} \) and \( x, y \in H_n(q) \subset H_\infty(q) \) the equality \( \text{tr}(xen^y) = \eta \text{tr}(xy) \) holds. We refer to this identity as the Markov property.

It is known that for each value \( \eta \in \mathbb{C} \) there exists a (unique) Markov trace with \( \text{tr}(e_1) = \eta \), for a proof of this result see [Jon87, Thm. 5.1]. In fact, the trace property comes for free.

Lemma 2.2.10. Suppose \( \text{tr} \) is a Markov trace on \( H_\infty(q) \) and \( \varphi: H_\infty(q) \to \mathbb{C} \) is a functional with the Markov property such that \( \text{tr}(e_1) = \varphi(e_1) \), then \( \text{tr} = \varphi \). In particular \( \varphi \) is tracial.

Proof. Any element \( x \in H_n(q) \subset H_\infty(q) \) can be written as \( x = x_1 + x_2e_{n-1}x_3 \) for some \( x_1, x_2, x_3 \in H_{n-1}(q) \). Now for \( \psi = \text{tr} \) and \( \psi = \varphi \) it holds that
\[ \psi(x) = \psi(x_1) + \psi(x_2e_{n-1}x_3) = \psi(x_1) + \psi(e_1)\psi(x_2x_3) \]
and the lemma follows by induction on \( n \) and the fact that \( H_\infty(q) = \bigcup_n H_n(q) \). \qed

2.3 Computations in \( \text{Rep}(\text{SU}_\mu(N)) \)

In this section we make some computations in the category \( \text{Rep}(\text{SU}_\mu(N)) \) (for \( \mu \in (0, 1] \)) which are needed later on. The results are analogous to [Pin07, §5], but in that paper a different representation of the Hecke algebra in \( \text{End}_{\text{Rep}(\text{SU}_\mu(N))}(\mathcal{H}^\otimes n) \) is used.\(^2\)

Since the representation category of a deformed Lie group is very similar to the representation category of the Lie group itself (cf. Section 1.6), it is immediate that \( \text{Rep}(\text{SU}_\mu(N)) \) is a SU(N)-type category.

Notation 2.3.1. Consider the \( \text{C}^* \)-tensor category \( \text{Hilb}_f \), with objects all finite dimensional Hilbert spaces and the collection of morphisms between two objects is given by all linear maps between the corresponding Hilbert spaces. Let \( \mathcal{H} := \mathbb{C}^N \in \text{Ob}(\text{Hilb}_f) \) and let \( \{\psi_i\}_{i=1}^N \) be an orthonormal basis in \( \mathcal{H} \). Jimbo [Jim86a, §4] and Woronowicz [Wor88, §4]

\(^2\)Contrary to the rest of the thesis, in this Chapter we use \( \mu \) to denote the deformation parameter of \( \text{SU}(N) \). This is due to the fact that \( q \) is already used for the Hecke algebra. Later these \( q \) and \( \mu \) will be related to each other, see Section 2.5.
2.3. **COMPUTATIONS IN** $\text{Rep}(\text{SU}_\mu(N))$

defined the following representation of the Hecke algebra. Let $q := \mu^2$. Define the map $T \in \text{End}(\mathcal{H} \otimes \mathcal{H})$ by

$$T(\psi_i \otimes \psi_j) := \begin{cases} (q - 1)\psi_i \otimes \psi_j + \mu \psi_j \otimes \psi_i, & \text{if } i < j; \\ q\psi_i \otimes \psi_j, & \text{if } i = j; \\ \mu \psi_j \otimes \psi_i, & \text{if } i > j. \end{cases}$$

A straightforward computation shows that

$$\eta: H_n(q) \to \text{End}(\mathcal{H}^{\otimes n}), \quad g_i \mapsto \iota^{i-1} \otimes T \otimes \iota^{n-i-1}$$

defines a representation of $H_n(q)$. If it is necessary to keep track of $n$ we write $\eta_n$ for this representation. The action of the idempotents $e_i$ corresponds to the linear map

$$c := \frac{q-T}{q+1} (\psi_i \otimes \psi_j) = \begin{cases} \frac{1}{q+1} (\psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i), & \text{if } i < j; \\ 0, & \text{if } i = j; \\ \frac{1}{q+1} (q\psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i), & \text{if } i > j. \end{cases}$$

To define the category $\text{Rep}(\text{SU}_\mu(N))$ we also need an embedding $C \hookrightarrow \mathcal{H}^{\otimes N}$ corresponding to the morphism intertwining the trivial representation of $\text{SU}(N)$ on $C$ with the $N$-th tensor power of the standard representation on $\mathcal{H}^{\otimes N}$. Up to a normalisation the following element in $\mathcal{H}^{\otimes N}$ plays the role of this embedding $C \hookrightarrow \mathcal{H}^{\otimes N}$

$$S := \sum_{\sigma \in S_N} (-\mu)^{-l(\sigma)} \psi_{\sigma(N)} \otimes \cdots \otimes \psi_{\sigma(1)}. \quad (2.3.2)$$

Here $l(\sigma)$ denotes the length of $\sigma$, see Definition 2.2.2. We write $S$ both for the vector defined in (2.3.2) and for the map $C \hookrightarrow \mathcal{H}^{\otimes N}, c \mapsto cS$. This element $S$ can be considered as the $q$-deformed determinant. Then $\text{Rep}(\text{SU}_\mu(N))$ is generated as a $C^*$-tensor category by $\mathcal{H}$, $T$ and $S$ ([Wor88, §4]).

Let us compute $\|S\|$. As $\{\psi_i\}_{i=1}^N$ is an orthonormal basis in $\mathcal{H}$, for $\sigma, \sigma' \in S_N$ it follows that

$$\langle \psi_{\sigma(N)} \otimes \cdots \otimes \psi_{\sigma(1)}, \psi_{\sigma'(N)} \otimes \cdots \otimes \psi_{\sigma'(1)} \rangle = \delta_{\sigma, \sigma'}.$$

So

$$\|S\|^2 = \langle S, S \rangle = \sum_{\sigma \in S_N} (-\mu)^{-2l(\sigma)},$$

which equals $[[N]]_\frac{1}{\mu}$ by the computation made in the proof of Lemma 2.2.6.

Recall the labelling of the simple objects as introduced in Definition 2.1.3. The representation $\eta$ acts as follows.

**Lemma 2.3.2.** For the representation $\eta: H_n(q) \to \text{End}(\mathcal{H}^{\otimes n})$ the morphism $\eta(e_1) \in \text{End}(\mathcal{H}^{\otimes 2})$ is the projection onto $\mathcal{H}_{\{1,2\}}$.

**Proof.** Using (2.3.1) we obtain for $i < j$ and a constant $a \in \mathbb{C}$
\[ \eta(e_1)(\psi \otimes \psi_j + a\psi_j \otimes \psi_i) = \frac{1-\mu a}{q+1} (\psi \otimes \psi_j - \mu \psi_j \otimes \psi_i); \]
\[ \eta(e_1)(\psi_i \otimes \psi_i) = 0. \]

In particular putting \( a = -\mu \) respectively \( a = \frac{1}{\mu} \), shows that
\[ \eta(e_1)(\psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i) = \psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i \]
\[ \eta(e_1)(\psi_i \otimes \psi_j + \frac{1}{\mu} \psi_j \otimes \psi_i) = 0. \]

This means that \( \eta(e_1) \) is the orthogonal projection onto
\[ U := \text{span}\left( \{ \psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i : 1 \leq i < j \leq N \} \right). \]

Since \( g_1 e_1 = e_1 g_1 \), we have \( \eta(g_1) U \subset U \). Thus \( U \) is a subobject of \( \mathcal{H}^\otimes 2 \) in \( \text{Rep}(SU_\mu(N)) \).

Now note that \( \mathcal{H}^\otimes 2 = \mathcal{H}_{\{1\}} \oplus \mathcal{H}_{\{2\}} \). Recall that \( V_\lambda \) was defined to be the irreducible representation of \( SU(N) \) corresponding to \( \lambda \). By Theorem \( 1.6.3 \) the dimensions of \( \mathcal{H}_\lambda \) are the same as the dimensions of \( V_\lambda \). Therefore \( \dim(\mathcal{H}_{\{1\}}) = \frac{1}{2} N(N-1) \) and \( \dim(\mathcal{H}_{\{2\}}) = \frac{1}{2} N(N+1) \). Note that \( \dim(U) = \frac{1}{2} N(N-1) \), hence \( U = \mathcal{H}_{\{1\}} \).

**Remark 2.3.3.** Recall the Hecke algebra morphism \( \alpha \) of Notation 2.2.7. It is immediate that \( \eta \circ \alpha \) is also a representation of \( H_n(q) \) on \( \mathcal{H}^\otimes n \). This is exactly the representation which Pinzari considers in [Pin07, §4]. Explicitly \( \eta \circ \alpha(g_i) = \iota^{\otimes i-1} \otimes T' \otimes \iota^{n-i-1} \), where

\[ T'(\psi_i \otimes \psi_j) := ((q-1)i - T)(\psi_i \otimes \psi_j) = \begin{cases} -\mu \psi_j \otimes \psi_i, & \text{if } i < j; \\ -\psi_i \otimes \psi_j, & \text{if } i = j; \\ (q-1)\psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i, & \text{if } i > j. \end{cases} \]

For later use we prove the following identities in \( \text{Rep}(SU_\mu(N)) \).

**Proposition 2.3.4.** In \( \text{Rep}(SU_\mu(N)) \) the following relations hold:

\[ S = \eta(B_N)(\psi_N \otimes \cdots \otimes \psi_1); \] (2.3.3)
\[ S^* S = [[N]]_q^! \iota; \] (2.3.4)
\[ SS^* = \eta(B_N); \] (2.3.5)
\[ (S^* \otimes \iota)(\iota \otimes S) = (-\mu)^{-[N-1]}[[N-1]]_q^! \iota; \] (2.3.6)
\[ (S^* \otimes \iota^{\otimes N-1})(\iota^{\otimes N-1} \otimes S) = (-\mu)^{-(N-1)}\eta(B_{N-1}); \] (2.3.7)
\[ \eta(g_1 \cdots g_N)(S \otimes \iota) = \mu^{N+1}(\iota \otimes S). \] (2.3.8)

Here \( B_n \in H_n(q) \) is as in Lemma 2.2.8.

**Proof.** As stated before, these identities are closely related to the identities proved by Pinzari in [Pin07, §5], in fact one can deduce the relations above from the identities of [Pin07]. We do this first and then we also show how one can compute everything directly.
2.3. COMPUTATIONS IN $\text{Rep}(\text{SU}_\mu(N))$

We denote Pinzari’s deformed determinant by $\tilde{S} := \sum_{\sigma \in S_N} (-\mu)^{l(\sigma)} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(N)}$. Let $r: S_N \to S_N$ be defined by $r(\sigma)(i) := \sigma(N + 1 - i)$. Then by Lemma 2.2.3

$$l(r(\sigma)) = |\{(i, j) : i < j, r(\sigma)(i) > r(\sigma)(j)\}|$$

$$= |\{(i, j) : i < j, \sigma(N + 1 - i) > \sigma(N + 1 - j)\}|$$

and thus $l(\sigma) + l(r(\sigma)) = N(N - 1)/2$. Therefore we obtain

$$\tilde{S} = \sum_{\sigma \in S_N} (-\mu)^{l(\sigma)} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(N)}$$

$$= \sum_{\sigma \in S_N} (-\mu)^{N(N-1)/2-l(r(\sigma))} \psi_{r(\sigma)(N)} \otimes \cdots \otimes \psi_{r(\sigma)(1)}$$

$$= (-\mu)^{N(N-1)/2} \sum_{\sigma \in S_N} (-\mu)^{-l(\sigma)} \psi_{\sigma(N)} \otimes \cdots \otimes \psi_{\sigma(1)}$$

$$= (-\mu)^{N(N-1)/2} S.$$

With this identity and the properties of $\alpha$ (see Notation 2.2.7), we can derive equations (2.3.3)–(2.3.8) from the results in [Pin07, §5]. For example using [Pin07, Lem. 5.1 b)] gives

$$\eta(B_N) \psi_N \otimes \cdots \otimes \psi_1 = \mu^{-N(N-1)}(\eta \circ \alpha(A_N)) \psi_N \otimes \cdots \otimes \psi_1$$

$$= \mu^{-N(N-1)}(-\mu)^{N(N-1)/2} \tilde{S}$$

$$= \mu^{-N(N-1)}(-\mu)^{N(N-1)/2}(-\mu)^{N(N-1)/2}S = S.$$

Or by [Pin07, Lem. 5.4]

$$(S^* \otimes \iota)(\iota \otimes S) = (-\mu)^{-N(N-1)}(\tilde{S}^* \otimes \iota)(\iota \otimes \tilde{S})$$

$$= (-\mu)^{-N(N-1)}\mu^{N-1}[N-1]_q!\iota$$

$$= (-\mu)^{N-1}[N-1]_q! \iota.$$

The other identities can be verified in a similar way, the details are left to the reader.

To compute everything directly, so without using Pinzari’s result, we start with a general identity. Suppose that $1 \leq i_1 < i_2 < \ldots < i_n \leq N$ and $1 \leq j \leq n - 1$, then

$$\eta(g_{n-j} \cdots g_j)(\psi_{i_n} \otimes \cdots \otimes \psi_{i_1}) = \mu^{n+1-j}(\psi_{i_n} \otimes \cdots \otimes \psi_{i_{n+2-j}} \otimes \psi_{i_{n-j}} \otimes \cdots \otimes \psi_{i_1} \otimes \psi_{i_{n+1-j}}).$$

Now suppose that $\theta \in S_n$. From Lemma 2.2.3 we have the reduced expression $\theta = (\theta^{-1})^{-1} = (C_{c_{n,n}} \cdots C_{c_{3,3}} C_{c_{2,2}})^{-1}$, where $c_i = i - d_{\theta^{-1}(i)}$. This gives in combination with (2.3.9) and the fact $l(\theta) = l(\theta^{-1})$, that the following identity holds

$$\eta(g_{\theta})(\psi_{i_n} \otimes \cdots \otimes \psi_{i_1}) = \mu^{l(\theta)}(\psi_{i_{\theta^{-1}(n)}} \otimes \cdots \otimes \psi_{i_{\theta^{-1}(1)}}).$$

Suppose again $1 \leq i_1 < i_2 < \ldots < i_n \leq N$, define $S_{i_{n-1}, \ldots, i_1} := \sum_{\sigma \in S_n} (-\mu)^{-l(\sigma)} \psi_{i_{\sigma(n)}} \otimes \cdots \otimes \psi_{i_{\sigma(1)}}$.
\psi_{i_{\sigma(1)}}. By (2.3.10) and Lemma 2.2.8 we get
\[
\eta(B_n)(\psi_{p-1(n)} \otimes \cdots \otimes \psi_{p-1(i)}) = \mu^{-l(\theta)} \eta(B_n) \eta(g_0)(\psi_{i_n} \otimes \cdots \otimes \psi_{i_1})
\]
\[
= (-\mu)^{-l(\theta)} \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} \eta(g_{\sigma})(\psi_{i_n} \otimes \cdots \otimes \psi_{i_1})
\]
\[
= (-\mu)^{-l(\theta)} \sum_{\sigma \in S_n} (-\mu)^{-l(\sigma)} (\psi_{i_{\sigma(p-1)}(n)} \otimes \cdots \otimes \psi_{i_{\sigma(p-1)(i)})}
\]
\[
= (-\mu)^{-l(\theta)} \sum_{\sigma \in S_n} (-\mu)^{-l(\sigma)} (\psi_{i_{\sigma(n)}} \otimes \cdots \otimes \psi_{i_{\sigma(i)}})
\]
\[
= (-\mu)^{-l(\theta)} S_{i_{n},...,i_{1}}.
\]  
(2.3.11)

Setting \( n = N, (i_1, \ldots, i_N) = (1, \ldots, N) \) and \( \theta = id \) gives \( S_{i_{n},...,i_{1}} = S \) and proves (2.3.3).
Equation (2.3.4) is immediate from the norm of \( S \).
Instead of proving (2.3.5), we prove a stronger statement which we will use later in the proof of this proposition. Using the notation introduced above, we show that
\[
\sum_{N \geq i_n > \cdots > i_1 \geq 1} S_{i_{n},...,i_{1}} S_{i_{n},...,i_{1}}^* = \eta(B_n).
\]  
(2.3.12)

Indeed, suppose that \( j_1, \ldots, j_n \in \{ i_1, \ldots, i_n \} \). Order the tuple \( (j_n, \ldots, j_1) \) in decreasing order so we obtain \( k_n \geq \ldots \geq k_2 \geq k_1 \). Then let \( p \) be minimal such that \( k_p = j_1 \). Then
\[
\eta(g_{n-p+1} \cdots g_{n-1})(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = \mu^{p-1} \psi_{k_n} \otimes \cdots \otimes \psi_{k_{p-1}} \otimes \psi_{k_p-1} \otimes \cdots \otimes \psi_{k_1} \otimes \psi_{j_1}.
\]

Iterating this procedure, it follows that there exists a \( \sigma \in S_n \) and \( c \in \mathbb{R} \setminus \{ 0 \} \) such that
\[
\psi_{j_n} \otimes \cdots \otimes \psi_{j_1} = c \eta(g_{\sigma})(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}).
\]
Suppose that \( j_{l'} = j_{l''} \) for some \( l' \neq l'' \), then \( k_l = k_{l+1} \) for some \( l \). We thus have
\[
\eta(B_n)(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = -\eta(B_n) \eta(g_{n-l})(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = -q \eta(B_n)(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}),
\]
where the first equality follows by Lemma 2.2.8 and the second from the action of \( \eta(g_{n-1}) \) on \( (\psi_{i_n} \otimes \cdots \otimes \psi_{i_1}) \). Recall \( q > 0 \), so in particular \( q \neq -1 \). Therefore \( \eta(B_n)(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = 0 \). Now
\[
\eta(B_n)(\psi_{j_n} \otimes \cdots \otimes \psi_{j_1}) = c \eta(B_n) \eta(g_{\sigma})(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1})
\]
\[
= c(-1)^{l(\sigma)} \eta(B_n)(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = 0.
\]
So
\[
\ker(\eta(B_n)) = \sum_{N \geq i_n > \cdots > i_1 \geq 1} \{ \psi_{\sigma(i_n)} \otimes \cdots \otimes \psi_{\sigma(i_1)} : \sigma \in S_n, N \geq i_n > \cdots > i_1 \geq 1 \}.
\]
Note that \( S^*_{i_{n},...,i_{1}}(\psi_{j_n} \otimes \cdots \otimes \psi_{j_1}) = 0 \) if there does not exist a \( \sigma \in S_n \) such that \( i_k = j_{\sigma(k)} \)
for all \(k = 1, \ldots, n\). Thus also

\[
\ker \left( \sum_{i_n > \ldots > i_1} S_{i_n, \ldots, i_1} \right)^\perp \subseteq \text{span}(\{\psi_{\sigma(i_n)} \otimes \cdots \otimes \psi_{\sigma(i_1)} : \sigma \in S_n, \ N \geq i_n > \ldots > i_1 \geq 1\}).
\]

Now by the fact that \(l(\sigma) = l(\sigma^{-1})\) and (2.3.11) we conclude

\[
\sum_{i_n > \ldots > i_1} S_{i_n, \ldots, i_1} S_{i_n, \ldots, i_1}^* (\psi_{\sigma(n)} \otimes \cdots \otimes \psi_{\sigma(1)}) = (-\mu)^{-l(\sigma)} S_{j_n, \ldots, j_1}
\]

\[
= \eta(B_n)(\psi_{\sigma(n)} \otimes \cdots \otimes \psi_{\sigma(1)}).
\]

Since the vectors \(S_{i_n, \ldots, i_1}\) and \(S_{j_n, \ldots, j_1}\) are orthogonal if \((i_n, \ldots, i_1) \neq (j_n, \ldots, j_1)\), it follows that \(\sum_{i_n > \ldots > i_1} S_{i_n, \ldots, i_1} S_{i_n, \ldots, i_1}^* \) and \(\eta(B_n)\) act the same on the space

\[
\text{span}(\{\psi_{\sigma(i_n)} \otimes \cdots \otimes \psi_{\sigma(i_1)} : \sigma \in S_n, \ N \geq i_n > \ldots > i_1 \geq 1\}).
\]

Hence (2.3.12) holds. The choice \(n = N\) and \((i_N, \ldots, i_1) = (N, \ldots, 1)\) gives (2.3.5). For the proof of (2.3.6) and (2.3.7) we introduce the following tensors

\[
S_j^{(1)} := \sum_{\sigma \in S_N, \sigma(1) = j} (-\mu)^{-l(\sigma)} \psi_{\sigma(N-1)} \otimes \cdots \otimes \psi_{\sigma(1)};
\]

\[
S_j^{(2)} := \sum_{\sigma \in S_N, \sigma(1) = j} (-\mu)^{-l(\sigma)} \psi_{\sigma(N)} \otimes \cdots \otimes \psi_{\sigma(2)}.
\]

Note that it is immediate that

\[
S = \sum_{j=1}^N \psi_j \otimes S_j^{(1)} = \sum_{j=1}^N S_j^{(2)} \otimes \psi_j.
\]

For \(\sigma \in S_{N-1}\) and \(j \leq N\) define \(p(\sigma) \in S_N\) by

\[
p(\sigma)(i) := \begin{cases} 
  j & \text{if } i = 1; \\
  \sigma(i-1) & \text{if } \sigma(i-1) < j; \\
  \sigma(i-1) + 1 & \text{if } \sigma(i-1) > j.
\end{cases}
\]

Then \(l(p(\sigma)) = l(\sigma) + j - 1\) and \(p: S_{N-1} \to \{\theta \in S_N : \theta(1) = j\}\) is a bijection. For the tuple \((i_{N-1}, \ldots, i_1) := (N, \ldots, j + 1, j - 1, \ldots, 1)\) we then obtain that

\[
S_{i_{N-1}, \ldots, i_1} = \sum_{\sigma \in S_{N-1}} (-\mu)^{-l(\sigma)} \psi_{\sigma(N-1)} \otimes \cdots \otimes \psi_{\sigma(1)}
\]

\[
= \sum_{\sigma \in S_N, \sigma(1) = j} (-\mu)^{-l(\sigma) + j - 1} \psi_{\sigma(N)} \otimes \cdots \otimes \psi_{\sigma(2)}
\]

\[
= (-\mu)^{j-1} S_j^{(2)}. \quad (2.3.13)
\]
Furthermore we have that the map

\[ s: \{ \sigma \in S_N : \sigma(N) = j \} \to \{ \sigma \in S_N : \sigma(1) = j \}, \quad s(\sigma)(i) := \begin{cases} j & \text{if } i = 1; \\ \sigma(i - 1) & \text{if } i > 1, \end{cases} \]

is a bijection and one easily checks that \( l(s(\sigma)) = l(\sigma) - (N + 1) + 2j \). It follows that

\[ S_j^{(1)} = (-\mu)^{-N(1)+2j} S_j^{(2)}. \tag{2.3.14} \]

Since \( \mathcal{H} \) is an irreducible object in \( \text{Rep}(\text{SU}_N) \), the morphism \( (S^* \otimes \iota)(\iota \otimes S) \) acts as a scalar. Suppose that \( \{ \psi_i \}_{i=1}^N \) is an orthonormal basis with respect to the inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H} \). We obtain

\[ \langle \psi_i, (S^* \otimes \iota)(\iota \otimes S)\psi_j \rangle_{\mathcal{H}} = \left( \sum_{k=1}^N \psi_k \otimes S_k^{(1)} \otimes \psi_i, \sum_{k=1}^N \psi_j \otimes S_k^{(2)} \otimes \psi_k \right)_{\mathcal{H}^\otimes N+1} \]

\[ = \langle S_j^{(1)}, S_j^{(2)} \rangle_{\mathcal{H}^\otimes N-1}. \]

To compute this scalar \( (S^* \otimes \iota)(\iota \otimes S) \) it thus suffices to compute \( \langle S_N^{(1)}, S_N^{(2)} \rangle \). For this we have

\[ \langle S_N^{(1)}, S_N^{(2)} \rangle = (-\mu)^{N+1-2N} \langle S_N^{(1)}, S_N^{(1)} \rangle \]

\[ = (-\mu)^{-N+1-2N} \sum_{\sigma, \theta \in S_N} (-\mu)^{-l(\sigma)-l(\theta)} \langle \psi_{\sigma(N-1)}, \psi_{\theta(N-1)} \rangle \cdots \langle \psi_{\sigma(1)}, \psi_{\theta(1)} \rangle \]

\[ = (-\mu)^{-N+1-2N} \sum_{\sigma, \theta \in S_N} (-\mu)^{-2l(\sigma)} \sum_{\sigma \in S_N, \sigma(N) = N} \]

\[ = (-\mu)^{-N+1-2N} \sum_{\sigma \in S_N, \sigma(N) = N} q^{-2l(\sigma)} = (-\mu)^{-N+1}[N-1]_{\frac{1}{q}}!, \]

which establishes \( 2.3.6 \).

Suppose that \( \xi_1 \in \mathcal{H} \), then

\[ (S^* \otimes \iota^{\otimes N-1})(\iota^{\otimes N-1} \otimes S)(\xi_1 \otimes \cdots \otimes \xi_{N-1}) \]

\[ = \sum_{i,j=1}^N (S_j^{(2)*} \otimes \psi_j^* \otimes \iota^{\otimes N-1})(\xi_1 \otimes \cdots \otimes \xi_{N-1} \otimes \psi_i \otimes S_i^{(1)} \rangle \]

\[ = \sum_{j=1}^N S_j^{(2)*}(\xi_1 \otimes \cdots \otimes \xi_{N-1}) \cdot S_j^{(1)}. \]

Applying to this expression the identities \( 2.3.14 \), \( 2.3.13 \) and \( 2.3.12 \) it follows that

\[ (S^* \otimes \iota^{\otimes N-1})(\iota^{\otimes N-1} \otimes S) = \sum_{j=1}^N S_j^{(1)} S_j^{(2)*} = \sum_{j=1}^N (-\mu)^{-(N-1)} S_{N-j+1,j-1,...,j-1} \]

\[ = (-\mu)^{-(N-1)} \eta(B_{N-1}). \]
Thus (2.3.7) holds.

To prove (2.3.8) we use (2.3.3) and Lemma 2.2.4. We obtain the following
\[ \eta(g_1 \cdots g_N)(S \otimes \psi_i) = \eta(g_1 \cdots g_N)\eta(B_N)(\psi_N \otimes \cdots \otimes \psi_1 \otimes \psi_i) \]
\[ = \eta(\Sigma(B_N))(g_1 \cdots g_N)(\psi_N \otimes \cdots \otimes \psi_1 \otimes \psi_i) \]
\[ = \mu^{N-1} \eta(\Sigma(B_N))(\psi_i \otimes \psi_N \otimes \cdots \otimes \psi_1) \]
\[ = \mu^{N+1}(\psi_i \otimes S), \]
which concludes the proof of this proposition. \( \Box \)

**Remark 2.3.5.** From relations (2.3.6) and (2.3.7) it follows directly that the pair \((R, \bar{R})\) given by
\[ R := \mu^{(N-1)/2}([[N - 1]]_{\frac{1}{q}})^{-1/2}S, \quad \bar{R} := (-1)^{N-1} \mu^{(N-1)/2}([[N - 1]]_{\frac{1}{q}})^{-1/2}S \]
solves the conjugate equations for \( \mathcal{H} \) in \( \text{Rep}(\text{SU}_\mu(N)) \). Since \( \mathcal{H} \) is simple and \( \|R\| = \|\bar{R}\| \)
it is a standard solution.

## 2.4 Representations of Hecke algebras

Suppose that \( \mathcal{C} \) is a strict \( \text{SU}(N) \)-type category, with fundamental object \( X \). The aim of this section is to show that one can extract a constant \( q \) from \( \mathcal{C} \) such that there exists a representation of the Hecke algebra \( H_n(q) \) into \( \text{End}(X^{\otimes n}) \). This section is closely related to [KW93, §4]. Once we established this representation, we show that this representation essentially only depends on the constant \( q \) and not on the other information of the category \( \mathcal{C} \). To obtain this result the Markov traces are used.

In this section \( \mathcal{C} \) is a strict \( \text{SU}(N) \)-type category.

**Notation 2.4.1.** Recall that \( V = \mathbb{C}^N \) is the fundamental representation of \( \text{SU}(N) \), in \( \text{Rep}(\text{SU}(N)) \) the object \( V_{\{12\}} \) is a subrepresentation of \( V^{\otimes 2} \). Therefore there exists a projection \( a \in \text{End}(X^{\otimes 2}) \) and a morphism \( v \in \text{Hom}(X_{\{12\}}, X^{\otimes 2}) \) such that \( v^*v = id_{X_{\{12\}}} \) and \( vv^* = a \). We say that \( a \) is the projection of \( X^{\otimes 2} \) onto \( X_{\{12\}} \). Define the elements \( a_k := i^{\otimes (k-1)} \otimes a \in \text{End}(X^{\otimes (k+1)}) \). We denote for \( m < n \) the map \( \iota_{m,n} : \text{End}(X^{\otimes m}) \rightarrow \text{End}(X^{\otimes n}), \quad T \mapsto T \otimes i^{\otimes (n-m)} \).

Clearly if \( k < m < n \) then \( \iota_{m,n}^* i_{k,m} = i_{k,n} \), thus \( (\text{End}(X^{\otimes n}), \iota_{m,n}) \) has an algebraic inductive limit. Denote this limit by \( M_\mathcal{C} \). If \( k < n \) we also write \( a_k \) for the element \( i_{k+1,n}(a) \). Denote by \( \Sigma \) the map \( \Sigma(a_k) := a_{k+1} \).

**Lemma 2.4.2.** Suppose \( 1 \leq n \leq N - 1 \). Consider the projection \( p_n \in \text{End}(X^{\otimes n}) \) onto the subobject \( X_{\{1n\}} \subset X^{\otimes n} \). Then \( (p_n \otimes i) \wedge (i \otimes p_n) = p_{n+1} \). Moreover \( (p_N \otimes i) \wedge (i \otimes p_N) = 0 \).

**Proof.** By the fusion rules of \( \text{SU}(N) \) we have \( X_{\{1n\}} \otimes X \cong X_{\{1n+1\}} \oplus X_{\{2n-1\}} \) and \( X \otimes X_{\{1n\}} \cong X_{\{1n+1\}} \oplus X_{\{21n-1\}} \). Since \( X_{\{1n+1\}} \subset X \otimes X_{\{1n\}} \) and \( X_{\{1n+1\}} \subset X_{\{1n\}} \otimes X \), we get
We argue by contradiction to show that the second case holds. Let for \( r_0 := \nu \otimes p \). For \( (2.4.3) \) we first compute

\[
G_n = (G_{n-1} \sigma(G_{n-1}))^2 = (\sigma(G_{n-1})G_{n-1})^2, \quad \text{for all } n.
\]

The same argument as before can be used to show that \( (p_N \otimes \nu) \neq (\nu \otimes p_N) \). Since \( X \otimes X_{\{1\}^N} \cong X \cong X_{\{1\}^N} \otimes X \) and \( X \) is simple, it follows that \( (p_N \otimes \nu) \wedge (\nu \otimes p_N) = 0 \).

**Lemma 2.4.3.** Let \( G_n := (a_1 \cdots a_{n-1}) \cdots (a_1a_2)a_1 \). Suppose that the projection \( \alpha \in \text{End}(X^{\otimes 2}) \) defined previously satisfies the braid relations, meaning that

\[
a_1a_2a_1 = a_2a_1a_2.
\]

Then \( G_n \) satisfies the identities

\[
a_iG_n = G_na_i = G_n, \quad \text{for } i = 1, \ldots, n-1;
\]

\[
G_n = (G_{n-1} \sigma(G_{n-1}))^2 = (\sigma(G_{n-1})G_{n-1})^2, \quad \text{for all } n.
\]

Furthermore for \( n \leq N \), the morphism \( G_n : X^{\otimes n+1} \to X^{\otimes n+1} \) is the projection onto the object \( X_{\{1\}^n} \) and \( G_{N+1} = 0 \).

**Proof.** This is similar to the proof of Lemma 2.2.6, but we give the details. From (2.4.1) we immediately see that \( a_iG_{i+1}a_i = a_{i+1}a_iG_{i+1} \) for all \( i \geq 1 \). By definition it follows that \( G_n = (a_1 \cdots a_{n-1})G_{n-1} \). From (2.4.1) it is immediate that

\[
(a_{j-1} \cdots a_1)(a_2 \cdots a_j) = (a_1 \cdots a_j)(a_{j-2} \cdots a_1)
\]

and therefore by induction it follows that \( (a_{n-1} \cdots a_1)\sigma(G_{n-1}) = G_n \).

First let us show that \( G_n^* = G_n \). Indeed, this holds for \( n = 2 \). For \( n \geq 2 \) and \( i < n \) we have \( (a_i \cdots a_1)(a_{n-1} \cdots a_1) = (a_{n-1} \cdots a_1)(a_{i+1} \cdots a_2) \). Thus by induction

\[
G_n^* = ((a_1 \cdots a_{n-1})G_{n-1})^* = G_{n-1}(a_{n-1} \cdots a_1) = (a_{n-1} \cdots a_1)\sigma(G_{n-1}) = G_n.
\]

Obviously (2.4.2) is true for \( n = 2 \). Let us prove the other cases by induction

\[
G_n a_i = (a_1 \cdots a_{n-1})G_{n-1}a_i = (a_1 \cdots a_{n-1})G_{n-1} = G_n, \quad \text{if } i \leq n-2;
\]

\[
G_n a_{n-1} = (a_{n-1} \cdots a_1)\sigma(G_{n-1})a_{n-1} = (a_{n-1} \cdots a_1)\sigma(G_{n-1}) = G_n.
\]

The case \( a_i G_n \) can proved similarly. Hence (2.4.2) holds. From this identity it immediately follows that \( G_n^2 = G_n \), thus \( G_n \) is a projection.

For (2.4.3) we first compute

\[
G_n\sigma(G_n)a_1 = G_n\sigma(G_{n-1}a_{n-1} \cdots a_1)a_1 = G_n\sigma(G_{n-1})(a_n \cdots a_2a_1) = G_n(a_n \cdots a_1) = G_{n+1}.
\]
From (2.4.2) we now obtain that

$$(G_n\sigma(G_n))^2 = (G_n\sigma(G_n)a_1)(a_2 \cdots a_{n-1}) \cdots (a_1a_2)a_1\sigma(G_n) = G_{n+1},$$  \hspace{1cm} (2.4.4)

thus (2.4.3) holds.

By the fusion rules of $\text{SU}(N)$ it follows that $V_{\{1^N\}} \subset V_{\{1^1\}} \otimes V_{\{1^2\}} \otimes V_{\{3^{n-1}\}},$ for $i = 1, \ldots, n-1.$ For $n = 2, 3, \ldots, N$ let $p_n \in \text{End}(X_{\{1^n\}})$ be the projection onto $X_{\{1^n\}}.$ Then it follows that $p_n^2 = a_1 a_n = p_n$ and thus $p_n G_n = G_n p_n = p_n.$ So $p_n$ is a subprojection of $G_n.$ Now we proceed by induction to $n$ to show that $G_n = p_n.$  It is clearly true for $n = 2,$ so let $2 < n < N$ and assume $G_n = p_n.$ From Lemma 2.4.2 we obtain $(p_n \otimes \iota) \wedge (\iota \otimes p_n) = p_{n+1}.$ Therefore by the induction hypothesis and (2.4.4)

$$G_{n+1} p_{n+1} = G_{n+1}((p_n \otimes \iota) \wedge (\iota \otimes p_n))$$

$$= G_{n+1}(s^* \lim_m ((p_n \otimes \iota)(\iota \otimes p_n))^m)$$

$$= G_{n+1}(s^* \lim_m (G_n \sigma(G_n))^m)$$

$$= G_{n+1}(s^* \lim_m (G_{n+1})^m)$$

$$= G_{n+1}.$$  

Combining this with the fact that $p_{n+1}$ is a subprojection of $G_{n+1}$ we thus obtain $G_{n+1} = G_{n+1} p_{n+1} = p_{n+1}.$

By Lemma 2.4.2 it holds $(p_N \otimes \iota) \wedge (\iota \otimes p_N) = 0$ and thus

$$G_{N+1} = \lim_m G_{N+1}^m = \lim_m ((G_N \sigma(G_N))^{2m} = \lim_m ((p_N \otimes \iota)(\iota \otimes p_N))^{2m}$$

$$= (p_N \otimes \iota) \wedge (\iota \otimes p_N) = 0,$$

as desired. □

**Lemma 2.4.4.** Let $a \in \text{End}(X_{\{2\}})$ be the projection onto $X_{\{1^2\}} \subset X_{\{1^2\}}.$ Put $a_1 := a \otimes \iota$ and $a_2 := \iota \otimes a.$ Then there exists a $\gamma \in (0, 1]$ such that

$$a_1 a_2 a_1 - \gamma a_1 = a_2 a_1 a_2 - \gamma a_2.$$  \hspace{1cm} (2.4.5)

This is a slightly stronger statement than what it is proved in [KW93, Prop. 4.2] this is due to the fact that $a$ is a projection and not only an idempotent, we follow the proof by Kazhdan and Wenzl.

**Proof of Lemma 2.4.4.** By the fusion rules of $\text{SU}(N)$ we have

$$X_{\{2\}} \cong \begin{cases} X_{\{2,1\}} \oplus X_{\{2,1\}} \oplus X_{\{3\}} & \text{if } N = 2; \\ X_{\{1^3\}} \oplus X_{\{2,1\}} \oplus X_{\{2,1\}} \oplus X_{\{3\}} & \text{if } N \geq 3. \end{cases}$$

Therefore

$$\text{End}(X_{\{2\}}) \cong \begin{cases} M_2(\mathbb{C}) \oplus \mathbb{C} & \text{if } N = 2; \\ \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} & \text{if } N \geq 3. \end{cases}  \hspace{1cm} (2.4.6)$$
We now only consider the case $N \geq 3$, the case $N = 2$ is similar. From the fusion rules of $\text{SU}(N)$ it follows that $X_{(1,1)}$ is a subobject of $X_{(1,2)} \otimes X$, meaning that there exists a projection $p \in \text{End}(X_{(1,2)} \otimes X)$ and a morphism $v \in \text{Hom}(X_{(1,1)}, X_{(1,2)} \otimes X)$ such that $v^* v = \text{id}_{X_{(1,1)}}$ and $vv^* = p$. Similarly there exists $w \in \text{Hom}(X_{(1,2)}, X^{\otimes 2})$ such that $w^* w = \text{id}_{X_{(1,2)}}$ and $ww^* = a$. Then

$$a_1|_{X_{(1,1)}} = v^*(w^* \otimes \iota) a_1 (w \otimes \iota)v = v^*(w^* \otimes \iota)(ww^* \otimes \iota)(w \otimes \iota)v = v^*\text{id}_{X_{(1,2)} \otimes X}v = \text{id}_{X_{(1,1)}}.$$ 

So $a_1$ acts on $X_{(1,1)}$ as the identity. Similarly $a_2|_{X_{(1,1)}} = \text{id}_{X_{(1,1)}}$. The object $X_{(3)}$ is not a subobject of $X_{(1,2)} \otimes X$ and $X \otimes X_{(1,2)}$, which implies $a_1|_{X_{(3)}} = a_2|_{X_{(3)}} = 0$. We have $\dim(\text{Hom}(X_{(2,1)}, X_{(1,2)} \otimes X)) = \dim(\text{Hom}(X_{(2,1)}, X \otimes X_{(1,2)})) = 1$, thus in $\text{End}(X_{(2,1)} \otimes X_{(2,1)})$ the morphisms $a_i$ act as rank 1 projections. So using the isomorphism (2.4.6) there exist rank 1 projections $f_i \in M_2(\mathbb{C})$ such that the projection $a_i \in \text{End}(X^{\otimes 3})$ corresponds to $(1, f_i, 0) \in \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$. Since $\text{ran}(f_i f_j f_i) \subset \text{ran}(f_i)$ there exists a $\gamma_1 \in \mathbb{C}$ such that $f_1 f_2 f_1 = \gamma_1 f_1$. For the same reasons there exists $\gamma_2 \in \mathbb{C}$ such that $f_2 f_1 f_2 = \gamma_2 f_2$. Therefore

$$\gamma_1 f_1 f_2 = f_1 (f_2 f_1 f_2) = (f_1 f_2 f_1) f_2 = \gamma_2 f_1 f_2. \tag{2.4.7}$$

Hence either $\gamma_1 = \gamma_2$ or $f_1 f_2 = 0$ in the latter case we can set $\gamma_1 = \gamma_2 = 0$. Put $\gamma_c := \gamma_1$. Because $f_i$ are projections and thus positive, it must hold that $\gamma_c \in [0, 1]$. Since $a_i$ corresponds to $(1, f_i, 0)$, (2.4.7) gives (2.4.5).

It remains to show that $\gamma_c \neq 0$. We show this by contradiction. Suppose $\gamma_c = 0$, then the projections $a_i$ satisfy

$$a_i a_j a_i = a_j a_i a_j, \quad \text{if } |i - j| = 1; \quad a_i a_j = a_j a_i \quad \text{if } |i - j| \geq 2.$$ 

In particular the projections $a_i$ satisfy the braid relations. We apply Lemma 2.4.3. Put $G_n := (a_1 \cdots a_{n-1}) \cdots (a_1 a_2) a_1$. Then $G_n \in \text{End}(X^{\otimes n})$ is the projection onto $X_{(1^n)}$. Let $(R, \tilde{R})$ be the standard solutions of the conjugate equations for $X$. From Example 2.1.4 we know that $\tilde{X} \cong X_{(1^{N-1})}$ and thus by the fusion rules $R: 1 \to X_{(1^{N-1})} \otimes X = X_{(1^N)} \oplus X_{(2^{N-2})}$. Note $X_{(2^{N-2})} \not\cong 1 = X_{(1^N)}$. Moreover the multiplicity $m^{X_{(1^N)}}_{X_{(1^N)}} = 1$, so there is a unique copy of $X_{(1^N)}$ inside $X^{\otimes N}$. It follows that $R$ extends to a map $\tilde{R}: 1 \to X_{(1^N)} \subset X^{\otimes N}$ and moreover $\ker(\tilde{R}^*) = \ker(G_N)$. Now consider the space

$$Z := \{ \alpha \in \text{End}(X^{\otimes N+1}) : \ker(G_N) \otimes X \subset \ker(\alpha), \text{Im}(\alpha) \subset X \otimes X_{(1^N)} \}.$$ 

As $\ker(G_N)$ is complemented by the space $X_{(1^N)}$ and $\dim(\text{Hom}(X_{(1^N)} \otimes X, X \otimes X_{(1^N)})) = 1$, it follows that $\dim(Z) = 1$. Clearly the morphism $S := (\iota \otimes R)(\tilde{R}^* \otimes \iota) \in Z$. Since $(R, \tilde{R})$ solve the conjugate equations for $X$ we see that $S^2 = S$. Also note that $\sigma(G_N)G_N \in Z$. Hence there exists a constant $c$ such that $\sigma(G_N)G_N = cS$. Applying Lemma 2.4.3 gives

$$c^2 S = (cS)^2 = (\sigma(G_N)G_N)^2 = G_{N+1} = 0$$

and thus $c = 0$. We conclude that $G_N \sigma(G_N) = (\sigma(G_N)G_N)^* = (cS)^* = 0$. On the other
hand it holds that $G_N R = R$ and $R^* G_N = R^*$, therefore

$$(\tilde{R}^* \otimes \iota)(\iota \otimes R) = (\tilde{R}^* \otimes \iota) G_N \sigma(G_N)(\iota \otimes R) = 0.$$ 

But this is a contradiction and thus $\gamma_C \neq 0$.  

**Notation 2.4.5.** Put $\gamma_C$ to be the constant obtained from $C$ as in the previous lemma. Pick $q_C$ such that $\gamma_C = \frac{q_C}{(1 + q_C)^2}$, i.e. such that $q_C + q_C^{-1} = \gamma_C^{-1} - 2$. From this it is clear that $q_C$ is uniquely determined up to $q_C \leftrightarrow q_C^{-1}$. Therefore to fix a unique $q_C$ we select $q_C \in \{ z \in \mathbb{C} : |z| \leq 1, \text{Im}(z) \geq 0 \} \cup \{ z \in \mathbb{C} : |z| < 1, \text{Im}(z) < 0 \}$. If it is clear which category $C$ is considered we omit the subscript $c$ in $q_C$ and $\gamma_C$.

**Remark 2.4.6.** At this point it is not clear why $q_C$ is indeed an invariant of the category. A priori it might be dependent on the choice of $X$. However this constant is indeed independent, we say more about this issue later (cf. Remark 2.6.8).

**Lemma 2.4.7.** For a strict $\text{SU}(N)$-type category $C$ it holds that $q_C \in (0, 1] \cup \{ e^{i\alpha} : 0 < \alpha < \frac{2\pi}{3} \}$.

**Proof.** The function $(0, 1] \rightarrow [2, \infty), q \mapsto q + q^{-1}$ is a bijection, so for $\gamma \in (0, 1/4]$ it holds $q \in (0, 1]$. If $\gamma \in (\frac{1}{4}, 1]$, then write $\gamma = \frac{1}{4} \cos^{-2}(\alpha/2)$ for a unique $\alpha \in (0, \frac{2\pi}{3}]$. We have

$$\gamma = (e^{i\alpha/2} + e^{-i\alpha/2})^{-2} = \frac{e^{i\alpha}}{(1 + e^{i\alpha})^2},$$

which implies that $q = e^{i\alpha}$.  

**Corollary 2.4.8.** The map

$$\theta_n : H_n(q_C) \rightarrow \text{End}(X^\otimes n), \quad e_i \mapsto a_i$$

extends to a $*$-representation of the Hecke algebra $H_n(q_C)$.

**Proof.** Since $g_i = q - (q + 1)e_i$, in the Hecke algebra $H_n(q)$ the relations (2.2.1)–(2.2.3) can equivalently be described in terms of the idempotents $e_i$ by

$$e_i e_j = e_j e_i, \quad \text{if } |i - j| \geq 2;$$

$$e_i e_{i+1} e_i - \frac{q}{(1 + q)^2} e_i = e_{i+1} e_i e_{i+1} - \frac{q}{(1 + q)^2} e_{i+1}, \quad \text{for } i = 1, \ldots, n - 2;$$

$$e_i^2 = e_i, \quad \text{for } i = 1, \ldots, n - 1.$$

From this characterisation, the fact that $a$ is a projection satisfying (2.4.5) and the choice of $q$ it is immediate that the map (2.4.8) extends to a representation of $H_n(q)$. Since $e_i$ is self-adjoint in $H_n(q)$ and $a_i$ is self-adjoint in $\text{End}(X^\otimes n)$ the map is $*$-preserving.  

**Lemma 2.4.9.** If $q_C = e^{i\alpha}$ for some $0 < \alpha < \pi$, then $q_C$ is a root of unity.
Proof. We can write \( q = e^{2\pi i \beta} \) for some \( 0 < \beta < \frac{1}{2} \). A representation of \( H_n(q) \) into a C*-algebra is a C*-representation if the idempotents \( e_i \) are mapped to projections. Such a representation is called trivial if it is a direct sum of representations \( \pi_1 \) and \( \pi_0 \) where \( \pi_1: e_i \mapsto \text{id} \) for all \( i \) and \( \pi_0: e_i \mapsto 0 \) for all \( i \). If \( q \) is not a root of unity, then there exists an \( m \in \mathbb{N} \setminus \{0\} \) such that \( m - 1 < \frac{1}{\beta} < m \). Now [Wen88, Prop. 2.9] implies that there exist no nontrivial C*-representations of \( H_n(q) \) for \( n > ((m + 1)/2)^2 \). However by Corollary 2.4.8 for each \( n \) we do have a nontrivial C*-representation. Hence \( q \) must be a root of unity.

Notation 2.4.10. Recall the embeddings \( i_{m,n}: \text{End}_C(X^\otimes m) \to \text{End}_C(X^\otimes n) \) and the inductive limit \( M_C \) defined in Notation 2.4.1. The representations \( \theta_n: H_n(q_C) \to \text{End}(X^\otimes n) \) obtained from Corollary 2.4.8 satisfy \( i_{m,n} \circ \theta_m(x) = \theta_n \circ i_{m,n}(x) \) for all \( m, n \) and \( x \in H_m(q_C) \). Thus the collection \( \{ \theta_n \}_{n} \) extends to a representation of the inductive limits \( \theta_C: H_\infty(q_C) \to M_C \). We denote \( \theta_C(x) = \theta_n(x) = i_{m,n}(\theta_m(x)) \in \text{End}(X^\otimes n) \) for \( x \in H_m(q_C) \subset H_\infty(q_C) \). Again we write just \( \theta \) if no confusion is possible.

Proposition 2.4.11. Let \( R: 1 \to \tilde{X} \otimes X, \bar{R}: 1 \to X \otimes \tilde{X} \) be a standard solution of the conjugate equations. The categorical trace \( \text{Tr}_C \) on \( C \) induces a Markov trace \( \text{tr}_C \) on \( H_\infty(q) \) via

\[
\text{tr}_C(x) := ||R||^{-2n} \text{Tr}_\otimes^n(\theta_C(x)), \quad (x \in H_n(q)).
\]

(2.4.12)

Proof. Recall that if \( (R, \bar{R}) \) and \( (S, \bar{S}) \) are standard solutions for \( U \) respectively \( V \) then \( (\iota \otimes R \otimes \iota)S \) and \( (\iota \otimes \bar{S} \otimes \iota)\bar{R} \) are a standard solution for \( U \otimes V \) [NT13, Thm. 2.2.16]. It follows immediately that \( \text{Tr}_{U \otimes V} = \text{Tr}_U(\iota \otimes \text{Tr}_V) = \text{Tr}_V(\text{Tr}_U \otimes \iota) \). Suppose \( x \in H_n(q) \) and \( n < m \), then in \( \text{End}(X^\otimes m) \)

\[
(\iota^\otimes m \otimes \text{Tr}_X)(\theta(x)) = (\iota^\otimes m \otimes \bar{R}^*)(\theta(x) \otimes \iota)(\iota^\otimes m \otimes \bar{R}) = ||R||^2 \theta(x).
\]

We conclude that \( \text{tr}_C \) is independent of \( n \).

Since \( \text{Tr}_{X^\otimes n} \) is tracial, it only remains to check that \( \text{tr}_C \) has the Markov property. Since \( X \) is simple, there exists a scalar \( \lambda \in \mathbb{C} \) such that \( (\iota \otimes \text{Tr})(a_1) = \lambda a_1 \). Then \( (\iota^\otimes n \otimes \text{Tr})(a_n) = \lambda^\otimes n a_n \) and also

\[
\text{tr}_C(e_1) = ||R||^{-4} \text{Tr}_{X^\otimes 2}(\theta(e_1)) = ||R||^{-4} \text{Tr}_X((\iota \otimes \text{Tr}_X)(a_1)) = ||R||^{-4} \text{Tr}_X(\lambda a_1) = ||R||^{-2} \lambda.
\]

Now suppose \( x, y \in H_n(q) \), then

\[
\text{tr}_C(xe_n y) = ||R||^{-2n-2} \text{Tr}_{X^\otimes n+1}(\theta(xe_n y)) = ||R||^{-2n-2} \text{Tr}_{X^\otimes n} \circ (\iota^\otimes n \otimes \text{Tr}_X)(\theta(x) \theta(e_n) \theta(y)) = ||R||^{-2n-2} \text{Tr}_{X^\otimes n}(\theta(x)(\iota^\otimes n \otimes ((\iota \otimes \text{Tr}_X)(a_1))) \theta(y)) = ||R||^{-2n-2} \text{Tr}_{X^\otimes n}(\lambda \theta(xy)) = \text{tr}_C(e_1) \text{tr}_C(xy).
\]

Hence \( \text{tr}_C \) is a Markov trace. \( \blacksquare \)
Lemma 2.4.12. Suppose that $q_C$ is a primitive root of unity of order $k \in \{3, 4, \ldots\} \cup \{\infty\}$. Then for $n \leq \min(N, k-1)$ the morphism $\theta(F_n) \in \text{End}(X^{\otimes n})$ is the projection onto $X_{1^n}$. For $m \leq k-1$ the morphism $\theta(E_m) \in \text{End}(X^{\otimes m})$ is the projection onto $X_{1^m}$.

Note that the assumption $n + 1 \leq \min(N, k-1)$ is used to make sure that $X_{1^n}$ is a nonzero object and that $F_{n+1} = [[n+1]]_{\mathbb{Z}}^{-1} B_{n+1}$ is well-defined, since $[[n+1]]_{\mathbb{Z}}^{-1} = 0$ if and only if $n + 1 \geq k$.

Proof of Lemma 2.4.12. This proof is similar to the proof of Lemma 2.4.3, but there are some minor differences so we work out the case $F_n$. For $n \leq N$ let $p_n \in \text{End}(X^{\otimes n})$ be the projection onto $X_{1^n}$, then by Lemma 2.4.2 $p_n = (p_{n-1} \otimes \iota) \wedge (\iota \otimes p_{n-1})$. We adopt the induction hypothesis that $\theta(F_n) = p_n$ is the projection onto $X_{1^n}$. This is obviously true for $n = 1, 2$. To prove the induction step, assume $2 \leq n < \min(N, k-1)$. By Lemma 2.2.8 we obtain that $\sigma(F_n) = F_{n+1}$. Thus by the induction hypothesis

$$\theta(p_{n+1}) = \theta(F_{n+1}) = \theta(F_n) \text{ and the induction is complete.}$$

So $\theta(F_{n+1})$ is a subprojection of $p_{n+1}$. It remains to show that $\theta(F_{n+1}) p_{n+1} = p_{n+1}$. From the identity $p_{n+1} = (p_n \otimes \iota) \wedge (\iota \otimes p_n)$ and the induction hypothesis it follows that $\theta(F_n) p_{n+1} = \theta(\sigma(F_n)) p_{n+1} = p_{n+1}$. Combining this with Lemma 2.2.8 gives that

$$\theta(F_{n+1}) p_{n+1} = [[n+1]]_{\mathbb{Z}}^{-1} \theta(F_n(1-q^{-1} g_n + \ldots + (-q)^{-n} g_n \cdots g_1)) \theta(F_n) p_{n+1}$$

$$= [[n+1]]_{\mathbb{Z}}^{-1} \theta(F_n(1-q^{-1} g_n - \ldots - q^{-n} g_n)) p_{n+1} = p_{n+1}$$

So $p_{n+1} = \theta(F_{n+1})$ and the induction is complete. The second case with $E_n$ is proved similarly.

\[ \square \]

Lemma 2.4.13. The constant $q_C$ is positive, thus not a nontrivial root of unity.

Proof. We argue by contradiction, so suppose $q$ is a primitive root of unity of order $2 \leq k < \infty$. We first show that $\text{tr}_C(g_1) = q^{m-n}$ for some $m < k$. To start note that by Lemma 2.2.8

$$\| R \|^{-2} (\iota^{\otimes n} \otimes \text{Tr}_X)(\theta(B_{n+1})) = (1 - \text{tr}_C(g_1) q^{-1} + \ldots + q^{-n}) \theta(B_n)$$

$$= (1 - \text{tr}_C(g_1) q^{-1}[[n]]_{\mathbb{Z}}) \theta(B_n).$$
Observe that $\frac{q^m}{[m]_q} = \frac{q^n}{[n]_q}$ and $\theta(B_1) = t \neq 0$. So if $\text{tr}_C(g_1) \neq \frac{q^n}{[n]_q}$ for all $n < k$ the above calculation shows that $\theta(B_n) \neq 0$ for $n = 1, \ldots, k$. Note that Lemma 2.2.8 implies that

$$B_k^2 = [k]_q! B_k = \frac{1 - q^{-k}}{1 - q^{-1}} [k - 1]_q! B_k = 0,$$

hence $B_k = 0$, which is a contradiction. Therefore there exists a $m < k$ such that $\text{tr}_C(g_1) = \frac{q^m}{[m]_q}$. We show that $m \neq 1$. Indeed, by definition $B_2 = 2q^1 e_1$. Therefore $\theta(B_2) = 2q^1 a_1$ which is nonzero because $q \neq -1$ (see Lemma 2.4.7), so $m \in \{2, 3, \ldots, k - 1\}$. Consider the projection $\theta(E_{k-m+1})$. Note that $k - m + 1 \in \{2, \ldots, k - 1\}$. By assumption $q$ is a primitive root of order $k$, therefore by Lemma 2.4.12 we obtain that $\theta(E_{k-m+1})$ is the projection on the nonzero object $X_{\{k-m+1\}} \subset X^{\otimes k-m+1}$. On the other hand we have

$$||R||^{-2} (\iota^{\otimes k-m} \otimes \text{Tr}_X)(\theta(E_{k-m+1}))$$

$$= [k - m + 1]_q (1 + \text{tr}_C(g_1)(1 + \ldots + q^{k-m-1})) \theta(E_{k-m})$$

$$= [k - m + 1]_q (1 + \frac{q^m}{[m]_q} [k - m]_q) \theta(E_{k-m}).$$

(2.4.13) As

$$1 + \frac{q^m}{[m]_q} [k - m]_q = 1 + q^m (1 - q)(1 - q^{k-m}) (1 - q^m)(1 - q) = 1 + \frac{q^m - 1}{1 - q^m} = 0,$$

(2.4.13) implies by faithfulness of ($\iota^{\otimes k-m} \otimes \text{Tr}_X$) that $\theta(E_{k-m+1}) = 0$, which is a contradiction.

We need one last result on the decomposition of Hecke algebras in irreducible components. Recall the definition of $\Lambda_N$, the set of the labels of irreducible representations of $\text{SU}(N)$, see (2.1.1).

**Proposition 2.4.14** ([KW93, Prop. 3.1]). Assume $q > 0$, $\mu \in \mathbb{C}$ and $\text{tr}$ is the Markov trace with $\text{tr}(g_1) = \mu$. Put $I_n^\mu := \{x \in H_n(q) : \text{tr}(xy) = 0, \text{for all } y \in H_n(q)\}$. Each $\lambda \in \Lambda_n$ defines a Young diagram and hence a representation $W_\lambda$ of $S_n$. Denote $d(\lambda) := \dim(W_\lambda)$. Let $V_{m,\lambda}$ be the irreducible representation of $\text{SU}(m)$ labelled by $\lambda \in \Lambda_m$ and let $\mathcal{H}_m := \mathbb{C}^m$ be the fundamental representation of $\text{SU}(m)$. Denote $d_m^{\mu}(\lambda) := \dim(\text{Hom}(V_{m,\lambda}, \mathcal{H}_m^{\otimes \mu}))$. Then $I_n^\mu$ is a two-sided ideal in $H_n(q)$ and the following statements hold:

(i) the Hecke algebra $H_n(q)$ is semisimple and isomorphic to $\mathbb{C}[S_n]$, in particular

$$H_n(q) \cong \bigoplus_{\lambda \in \Lambda_{n+1}, |\lambda| = n} M_{d(\lambda)}(\mathbb{C});$$

(ii) if $\mu \neq \frac{q^m}{[m]_q}$ for all $m \in \{1, \ldots, n-1\}$, then $I_n^\mu = \{0\}$;

(iii) if $\mu = \frac{q^m}{[m]_q}$ for some $m \in \{1, \ldots, n-1\}$, then

$$H_n(q)/I_n^\mu \cong \bigoplus_{\lambda \in \Lambda_m} M_{d_m^{\mu}(\lambda)}(\mathbb{C}).$$
Observe that [KW93] proves more. They also deal with the case \( q \) is a root of unity which is not necessary for us, because of Lemma 2.4.13. Now we have all the prerequisites to prove that the representation of the Hecke algebra is independent of the category \( C \) in the following sense.

**Theorem 2.4.15** (Kazhdan–Wenzl). If \( C \) is a strict \( SU(N) \)-type category, then \( q_C \in (0, 1] \) and the Markov trace satisfies \( \text{tr}_C(g_1) = \frac{q_{C}^{2}}{[N]_{q_{C}}} \). Therefore the kernel of the representation \( \theta_C : H_n(q_C) \rightarrow \text{End}(X^{\otimes n}) \) depends only on \( q_C \). Furthermore \( \theta_C(H_n(q_C)) = \text{End}(X^{\otimes n}) \).

**Proof.** Since \( \|\theta(x^\ast x)\| = \|\theta(x)\|^2 \), it holds that \( \theta(x) = 0 \) if and only if \( \theta(x^\ast x) = 0 \). Because the categorical trace \( \text{Tr}_{X^{\otimes n}} \) is faithful, we obtain that

\[
\ker(\theta : H_n(q) \rightarrow \text{End}(X^{\otimes n})) = \{ x \in H_n(q) : \text{Tr}_{X^{\otimes n}}(\theta(x^\ast x)) = 0 \}
\]

\[
= \{ x \in H_n(q) : \text{tr}_C(x^\ast x) = 0 \}. \tag{2.4.14}
\]

To characterise the kernel of \( \theta \) by Proposition 2.4.11 and Lemma 2.2.10 it is thus sufficient to show that \( \text{tr}_C(g_1) \) can be computed in terms of \( q \).

Let \( I_{N+1}^\theta \) be as in Proposition 2.4.14. We claim that \( I_{N+1}^{\text{tr}(g_1)} = \ker(\theta) \). Indeed, if \( x \in I_{N+1}^{\text{tr}(g_1)} \), then in particular \( \text{tr}(xx^\ast) = 0 \), so by (2.4.14) it follows \( \theta(x^\ast x) = 0 \) and thus \( \theta(x) = 0 \).

For the converse, let \( x, y \in H_{N+1}(q_C) \) and assume \( \theta(x) = 0 \). Since \( yy^\ast \leq \|y\|^21 \) it follows \( x^\ast(y^\ast y)x \leq \|x\|^2x^\ast x \). The categorical trace \( \text{Tr}_{X^{\otimes N+1}} \) is a positive linear functional, therefore

\[
\text{Tr}_{X^{\otimes N+1}}(|\theta(x)\theta(y)|^2) \leq \|y\|^2 \text{Tr}_{X^{\otimes N+1}}(\theta(x^\ast x)) = \|y^\ast y\| \text{tr}(x^\ast x) = 0.
\]

By faithfulness of \( \text{Tr}_{X^{\otimes N+1}} \) this implies \( \theta(x^\ast y^\ast yx) = 0 \), thus since \( \text{End}(X^{\otimes N+1}) \) is a \( \ast \)-

algebra, \( \|\theta(xy)\|^2 = \|\theta(x^\ast y^\ast yx)\| = 0 \) and hence \( \theta(xy) = 0 \). Thus \( \text{tr}(xy) = 0 \).

By Lemma 2.4.13 we may assume that \( q > 0 \). Suppose \( \text{tr}_C(g_1) \neq \frac{q_{m}}{[m]_{q}} \) for \( 1 \leq m \leq N \). By Proposition 2.4.14 we have

\[
\dim(H_{N+1}(q)) = \sum_{\lambda \in \Lambda_{N+2}, |\lambda| = N+1} \dim(W_{\lambda})^2,
\]

where \( W_{\lambda} \) is as in Proposition 2.4.14. Similarly we have

\[
\dim(\text{End}(X^{\otimes N+1})) = \sum_{\lambda \in \Lambda_{N}} \dim(\text{Hom}(V_{\lambda}, H^{\otimes N+1}))^2,
\]

where \( H \) is the fundamental representation of \( SU(N) \) and \( V_{\lambda} \) is the irreducible representation of the group \( SU(N) \) labelled by \( \lambda \). Note that if \( \dim(\text{Hom}(V_{\lambda}, H^{\otimes N+1})) \neq 0 \), then \( \lambda \in \Lambda_N \) and \( |\lambda| \in \{1, N+1\} \). The dimensions \( \dim(W_{\lambda}) \) and \( \dim(\text{Hom}(V_{\lambda}, H^{\otimes N+1})) \) satisfy the same recursion relations (compare [KW93, Lem. 1.2 a]) and [Wen88, Eq. (2.6)]). In particular if we define

\[
A := \{ \lambda \in \Lambda_{N+2} : |\lambda| = N+1 \} \setminus \{(1, \ldots, 1), (2, 1, \ldots, 1, 0)\}.
\]
it follows that
\[
\sum_{\lambda \in \Lambda} \dim(W_\lambda)^2 = \sum_{\lambda \in \Lambda, |\lambda|=N+1} \dim(\text{Hom}(V_\lambda, \mathcal{H}^{\otimes N+1}))^2.
\]
Since \(\dim(W_{(1,\ldots,1)}) > 0\), \(\dim(W_{(2,1,\ldots,0)}) > 0\) and \(\dim(\text{Hom}(V_{(1,0,\ldots,0)}, \mathcal{H}^{\otimes N+1})) = 1\) we conclude
\[
\dim(H_{N+1}(q)) > \dim(\text{End}(X^{\otimes N+1})).
\tag{2.4.15}
\]
As computed in the beginning of this proof, \(\ker(\theta) = I_{N+1}^{1\text{tr}(g_1)}\). Therefore again Proposition 2.4.14 implies that \(\ker(\theta) = \{0\}\), but then \(\theta: H_{N+1}(q) \to \text{End}(X^{\otimes N+1})\) is an embedding, which is a contradiction with (2.4.15). We conclude that \(\text{tr}_C(g_1) = \frac{q^m}{[[m]]_q}\) for some \(1 \leq m \leq N\).

Suppose \(\text{tr}_C(g_1) = \frac{q^m}{[[m]]_q}\) for some \(1 \leq m < N\). Lemma 2.4.12 implies that \(\theta(F_{m+1})\) is the projection onto \(X_{\{1^{m+1}\}}\) in particular \(\theta(F_{m+1}) \neq 0\). However, by Lemma 2.2.8 we obtain
\[
[[m+1]]_q! \text{tr}_C(F_{m+1}) = \text{tr}_C(((1 + (-q)^{-1} g_m + \ldots + (-q)^{-m} g_1 \cdots g_m) B_m)
= \text{tr}_C((1 - \text{tr}_C(g_1)(q^{-1} + \ldots + q^{-m}) B_m
= \text{tr}_C((1 - \text{tr}_C(g_1) q^{-1}[[m]]_q) B_m).
\]

From the assumption \(\text{tr}_C(g_1) = \frac{q^m}{[[m]]_q}\) we get
\[
1 - \text{tr}_C(g_1) q^{-1}[[m]]_q = 1 - q^m \frac{1 - q}{1 - q^m} \frac{1 - q^{-m}}{1 - \frac{1}{q}} = 1 - \frac{1 - q}{1 - q^m} \frac{q^m - 1}{q - 1} = 0,
\]
but then \(\text{tr}_C(F_{m+1}) = 0\), which is again a contradiction. Hence \(\text{tr}_C(g_1) = \frac{q^N}{[[N]]_q}\).

Applying Proposition 2.4.14 (iii) shows, by equality of dimensions, that \(H_n(q)/I_n^{1\text{tr}(g_1)} \cong \text{End}(X^{\otimes n})\) for every \(n \in \{1,2,\ldots\}\). Since \(\ker(\theta) = I_n^{1\text{tr}(g_1)}\) the representation \(\theta: H_n(q) \to \text{End}(X^{\otimes n})\) is surjective.

Remark 2.4.16. Combining the preceding theorem and [Pin07, Prop. 4.1] it follows that for \(n > N\) the kernel \(\ker(\theta_C: H_n(q) \to \text{End}(X^{\otimes n}))\) equals the ideal generated by the element \(B_{N+1} \in H_n(q)\).

### 2.5 Categories generated by Hecke algebras

In this section we give a number of technical requirements on \(\text{C}^*\)-tensor categories which allow us to prove that a category that satisfies these assumptions is in fact unitarily monoidally equivalent to a twist of \(\text{Rep}(\text{SU}_\mu(N))\). In the next section we will use this result to show that all \(\text{SU}(N)\)-type categories are equivalent to a twist of \(\text{Rep}(\text{SU}_\mu(N))\). Furthermore we show that in a special case these categories admit a braiding.

**Assumption 2.5.1.** Assume \(C\) is a strict \(\text{C}^*\)-tensor category generated by an object \(X\) that satisfies the following requirements:
We let \( \mu \) if it satisfies Assumption 2.5.1. Remark 2.5.2. will be dropped. The fact that the representation \( \rho \) is a root of unity of order \( \omega \). Suppose that \( \omega \) is the associativity morphisms of a category \( (\mathcal{C}, \otimes, \mathbb{I}) \) satisfied the requirements of Assumption 2.5.1 and \( \omega \). This requirement defines a representation \( \theta_n : H_n(qc) \to \text{End}(X_\otimes^n) \) by \( e_i \mapsto \iota^{\otimes i-1} \otimes a \otimes i^{\otimes n-i-1} \).

(ii) \( \theta_n : H_n(qc) \to \text{End}(X_\otimes^n) \) is surjective;

(iii) \( \ker(\theta_n : H_n(qc) \to \text{End}(X_\otimes^n)) = \ker(\eta_n : H_n(qc) \to \text{End}(\mathcal{H}_\otimes^n)) \), here \( \eta_n \) is as in Notation 2.3.1.

(iv) there exists an integer \( N_C \geq 2 \) and a morphism \( \nu \in \text{Hom}(\mathbb{I}, X_\otimes^{N_C}) \) such that \( \nu^* \nu = \iota \) and \( \nu^* \nu = \theta(\mathcal{F}_{N_C}) \);

(v) there exists a \( N_C \)-th root of unity \( \omega \) such that \( \theta(g_{N_C} \cdots g_1)(\iota \otimes \nu) = \omega \nu \theta(N_C+1)/(\nu \otimes \iota) \);

(vi) \( \text{Hom}(X_\otimes^m, X_\otimes^n) = \{0\} \), if \( m \neq n \) (mod \( N_C \)).

We let \( \mu_C \in (0, 1], \mu_C := q_c^{1/2} \). If it is clear which category is considered, the subscript \( c \) will be dropped.

Remark 2.5.2. The results in Proposition 2.3.4 show that \( \text{Rep}(\text{SU}_\mu(N)) \) satisfies the conditions (i) and (iii)–(vi) of the assumption above. The fact that the representation \( \eta : H_n(q) \to \text{End}(\mathcal{H}_\otimes^n) \) is surjective follows from Theorem 2.4.15. So \( \text{Rep}(\text{SU}_\mu(N)) \) satisfies Assumption 2.5.1.

Notation 2.5.3. If \( C \) is strict, then the twisted category \( C^\rho \) (see Definition 2.1.7) is in general not strict. We define \( \theta_n(g_i) \in \text{End}_{C^\rho}(X_\otimes^n) \) to be the composition

\[
X_\otimes^n \xrightarrow{\alpha} X_\otimes^{i-1} \otimes (X_\otimes^2 \otimes X_\otimes^{n-i-1}) \xrightarrow{\beta} X_\otimes^{i-1} \otimes (X_\otimes^2 \otimes X_\otimes^{n-i-1}) \xrightarrow{\alpha^{-1}} X_\otimes^n.
\]

Where \( \alpha \) is the appropriate associativity morphism in \( C^\rho \) and \( \beta := \iota^{\otimes i-1} \circ (\theta_2(g_1) \otimes i^{\otimes n-i-1}) \).

As shown in the next proposition the constant \( \omega \) behaves nicely with respect to twisting the associativity morphisms of a category \( C \). This proposition will be of importance, because in some cases it implies that we can restrict ourselves to the case \( \omega = 1 \).

Proposition 2.5.4. Suppose that \( C \) satisfies the requirements of Assumption 2.5.1 and \( \rho \) is a root of unity of order \( N_C \), then in \( C^\rho \) the equality \( (\nu^* \otimes \iota)\theta(g_{N_C}) \cdots \theta(g_1)(\iota \otimes \nu) = \rho^{-1} \omega \mu_C^{N_C+1} \iota \) holds. In particular if \( \check{C} \) is the strictification of \( C^\omega \), then in \( \check{C} \) it holds that \( (\nu^* \otimes \iota)\theta(g_{N_C}) \cdots \theta(g_1)(\iota \otimes \nu) = \mu_C^{N_C+1} \iota \).

Proof. Since in general \( C^\rho \) is not strict, consider \( (\nu^* \otimes \iota)\theta(g_N) \cdots \theta(g_1)(\iota \otimes \nu) \) which equals the composition

\[
X = X \otimes \mathbb{I} \xrightarrow{\iota \otimes \nu} X \otimes X_\otimes^N \xrightarrow{\alpha_{N}^\rho} X_\otimes^{i-1} \otimes (X_\otimes^2 \otimes X_\otimes^{n-i-1}) \xrightarrow{\beta} X_\otimes^{i-1} \otimes (X_\otimes^2 \otimes X_\otimes^{n-i-1}) \xrightarrow{\alpha^{-1}} X_\otimes^n
\]

\[
X \otimes (X_\otimes^2 \otimes X_\otimes^{n-2}) \xrightarrow{(\theta(g_1) \otimes i^{\otimes n-2})} X_\otimes^{n-1} \otimes X_\otimes^2 \xrightarrow{\alpha_{n-1}^\rho} X_\otimes^{n-1} \otimes (X_\otimes^2 \otimes X_\otimes^{n-1}) \xrightarrow{\alpha^{-1}} X_\otimes^n
\]

\[
X_\otimes^n \otimes X \xrightarrow{\nu \otimes \iota} \mathbb{I} \otimes X = X.
\]
Here $\alpha_i^\rho$ are associativity morphisms in $C^\rho$. The composition of these morphisms $\alpha_{N+1}^\rho \circ \cdots \circ \alpha_2^\rho \circ \alpha_1^\rho$ equals the associativity morphism $\alpha^\rho$: $X \otimes X^{\otimes N} \to X^{\otimes N} \otimes X$, which by Lemma 2.1.10 acts as multiplication by $\rho^{-1}(\xi) = \rho^{-1}$. In $C$ the associativity morphisms are trivial. Thus if we replace $\alpha^\rho$ by the associativity morphisms $\alpha$ of $C$, in $C$ the composition (2.5.6) equals $\mu_{C}^{N+1} \omega_C \iota$ by requirement (v) of Assumption 2.5.1. Hence in $C^\rho$ the morphism (2.5.1) equals $\rho^{-1} \mu_{C}^{N+1} \omega_C \iota$, as desired.

**Notation 2.5.5.** Introduce the constant $\delta_C := (\omega_C \mu_C^{N+1})^{-\frac{1}{N}}$. Denote

$$T_{m,n} := \delta_C^{mn} \theta_C(g_{\sigma_{m,n}}) \in \text{End}(X^{\otimes m+n}).$$

Observe the crucial property $T_{1,N} = (\omega_C \mu_C^{N+1})^{-1} \theta(g_{N} \cdots g_1)$, this implies that $T_{1,N} \iota \nu = \nu \otimes \iota$. The following proposition is similar to [KW93, Prop. 2.2 a)].

**Proposition 2.5.6.** Suppose that $C$ satisfies Assumption 2.5.1 and $\omega_C = \pm 1$, then the collection of morphisms $\{T_{m,n}\}_{m,n \in \mathbb{N}}$ defines a braiding on the category $C$. Explicitly,

$$T_{k,m+n} = (\iota^m \otimes T_{k,n})(T_{k,m} \otimes \iota^m); \quad (2.5.2)$$

$$T_{k+m,n} = (T_{k,n} \otimes \iota^m)(\iota^k \otimes T_{m,n}); \quad (2.5.3)$$

$$(\beta \otimes \alpha)T_{k,m} = T_{l,n}(\alpha \otimes \beta), \quad \text{for all } \alpha \in \text{Hom}(X^\otimes k, X^\otimes l), \beta \in \text{Hom}(X^\otimes m, X^\otimes n). \quad (2.5.4)$$

Note that the case $\omega = -1$ can only occur when $N$ is even, because $\omega$ is an $N$-th root of unity.

**Proof of Proposition 2.5.6.** From the explicit formulas in Lemma 2.2.4 we obtain the identities

$$\Sigma^m(g_{\sigma_{k,m}})g_{\sigma_{k,m+n}} = g_{\sigma_{k,m+n}}, \quad g_{\sigma_{k,n}} \Sigma^k(g_{\sigma_{m,n}}) = g_{\sigma_{k+m,n}},$$

from which (2.5.2) and (2.5.3) immediately follow. Denote the morphism $\nu_{m,n} := \iota^m \otimes \nu \otimes \iota^m \in \text{Hom}(X^{\otimes m+n}, X^{\otimes m+N+n})$. The collection $\{T_{m,n}\}_{m,n}$ satisfies the following relations

$$(T_{m,N} \otimes \iota^m)\nu_{m,n} = \nu_{0,m+n}; \quad (2.5.5)$$

$$(T_{N,m} \otimes \iota^m)\nu_{0,m+n} = \nu_{m,n}; \quad (2.5.6)$$

$$\nu^*_{m,n}(T_{N,m} \otimes \iota^m) = \nu^*_{0,m+n}; \quad (2.5.7)$$

$$\nu^*_{0,m+n}(T_{m,N} \otimes \iota^m) = \nu^*_{m,n}. \quad (2.5.8)$$

Indeed, the case $m = 1$ of (2.5.5) follows immediately from

$$(\nu^* \otimes \iota)\eta(g_N \cdots g_1)(\iota \otimes \nu) = \omega \mu^{N+1} \iota \quad (2.5.9)$$

and the definition of $T_{1,N}$. The case $m > 1$ can be proved using induction and (2.5.3). For (2.5.6) observe that taking the adjoint of (2.5.9) gives

$$(\iota \otimes \nu^*)\eta(g_1 \cdots g_N)(\nu \otimes \iota) = \omega \mu^{N+1} \iota.$$
2.5. CATEGORIES GENERATED BY HECKE ALGEBRAS

Here it is crucial that \( \omega = \pm 1 \), otherwise we would have the factor \( \mathbb{J} \). From this Equation (2.5.6) follows for \( m = 1 \) and the general case can again be proved by induction. The identities (2.5.7) and (2.5.8) follow from respectively (2.5.5) and (2.5.6) by taking conjugates. Again the requirement \( \omega = \pm 1 \) is implicitly used.

By assumption on \( \mathcal{C} \) the map \( \theta: H_n(q) \rightarrow \text{End}(X^\otimes n) \) is surjective. Combination with Lemma 2.2.4 gives immediately that for all \( \alpha \in \text{End}(X^\otimes m) \) and \( \beta \in \text{End}(X^\otimes n) \)

\[
T_{m,n}(\alpha \otimes \beta) = (\beta \otimes \alpha)T_{m,n}. \tag{2.5.10}
\]

It remains to show that (2.5.4) also holds for homomorphisms \( \alpha \in \text{Hom}(X^\otimes k, X^\otimes l) \) and \( \beta \in \text{Hom}(X^\otimes m, X^\otimes n) \). We may assume that \( k = l + pN \) and \( m = n + qN \) for some \( p, q \in \mathbb{Z} \). We proceed by induction on \( p \) and \( q \). The basis case \( p = q = 0 \) is exactly (2.5.10). So first suppose \( p \geq 1, q = 0 \), \( \alpha \in \text{Hom}(X^\otimes k, X^\otimes l) \) and \( \beta \in \text{Hom}(X^\otimes m, X^\otimes n) \). Then \( (\nu \otimes \alpha) \in \text{Hom}(X^\otimes k, X^\otimes l) \) and \( \beta \in \text{Hom}(X^\otimes m, X^\otimes n) \). Using the induction hypothesis, (2.5.3) and (2.5.6) we have

\[
\nu_{m,l}(\beta \otimes \alpha)T_{k,m} = (\beta \otimes \nu \otimes \alpha)T_{k,m} = T_{l+m,n}(\nu \otimes \alpha \otimes \beta) = (T_{N,m} \otimes \iota^\otimes)(\nu \otimes \alpha \otimes \beta) = (\nu \otimes \iota^\otimes)(T_{l+m,n})T_{l,m}(\alpha \otimes \beta) = \nu_{m,l}T_{l,m}(\alpha \otimes \beta).
\]

Since the map \( \text{Hom}(X^\otimes r, X^\otimes s) \rightarrow \text{Hom}(X^\otimes u+v, X^\otimes s) \), \( \gamma \mapsto \nu_{u,v} \circ \gamma \) is injective, it follows by induction that \( (\beta \otimes \alpha)T_{k,m} = T_{l,m}(\alpha \otimes \beta) \). Now suppose \( p < 0 \), then \( (\nu^* \otimes a) \in \text{Hom}(X^\otimes k+N, X^\otimes l) \) with a similar argument as above involving the relations (2.5.2) and (2.5.7) one can show that

\[
(\beta \otimes \alpha)T_{k,m}\nu_{0,k+m}^* = T_{l,n}(\alpha \otimes \beta)\nu_{0,k+m}^*.
\]

Injectivity of the map \( \text{Hom}(X^\otimes u+v, X^\otimes s) \rightarrow \text{Hom}(X^\otimes u+N+v, X^\otimes s) \), \( \gamma \mapsto \gamma \circ \nu_{u,v}^* \) closes the induction on \( p \). Induction on \( q \) is similar and thus (2.5.4) holds.

The proof of the following theorem uses the ideas of monoidal algebras as described by Kazhdan–Wenzl in [KW93, §2].

**Theorem 2.5.7.** Suppose that \( \mathcal{C} \) satisfies the requirements of Assumptions 2.5.1. Then \( \mathcal{C} \) is unitarily monoidally equivalent to \( \text{Rep}(\text{SU}_\kappa(N))^{\text{inv}} \).

**Proof.** From Proposition 2.5.4 and Remark 2.1.8 it follows that it suffices to consider the case \( \omega_C = 1 \). The idea of the proof of this theorem is to extend the isomorphisms \( \text{End}(X^\otimes n) \rightarrow \text{End}(H^\otimes n) \) to \( \text{Hom}(X^\otimes k, X^\otimes l) \rightarrow \text{Hom}(H^\otimes k, H^\otimes l) \) by embedding \( \text{Hom}(X^\otimes k, X^\otimes l) \) into \( \text{End}(X^\otimes p) \) for some large \( p \in \mathbb{N} \) using the maps \( \alpha \mapsto \alpha \otimes \nu \) and \( \alpha \mapsto \alpha \otimes \nu^* \). For this, suppose \( k, l, m, n, p \in \mathbb{N} \) such that \( p = m + kN = n + lN \). We define some subspaces and maps for \( \mathcal{C} \). Note that these constructions can of course also
be performed in $\text{Rep}(\text{SU}_\mu(N))$. Define the map

\[
H_p^{m,n} : \text{Hom}(X^{\otimes m}, X^{\otimes n}) \to \text{End}(X^{\otimes p}),
\]

\[
\alpha \mapsto (\nu^{\otimes l} \otimes \iota^{\otimes n})\alpha((\nu^*)^{\otimes k} \otimes \iota^{\otimes m}) = \nu^{\otimes l} \otimes (\nu^*)^{\otimes k} \otimes \alpha.
\]

Then clearly $H_p^{m,n}$ is linear. Define the subspace $\Sigma_p^{m,n} \subset \text{End}(X^{\otimes p})$ to be

\[
\Sigma_p^{m,n} := \{ \beta \in \text{End}(X^{\otimes p}) : ((\nu \nu^*)^{\otimes l} \otimes \iota^{\otimes n})\beta = \beta((\nu \nu^*)^{\otimes k} \otimes \iota^{\otimes m}) = \beta \}.
\]

The following lemmas are similar to [KW93, §2.2], the extra thing we have to check is compatibility of the $*$-structure.

**Lemma 2.5.8.** $H_p^{m,n}$ is an isomorphism of $\text{Hom}(X^{\otimes m}, X^{\otimes n})$ onto $\Sigma_p^{m,n}$. Furthermore for $\alpha \in \text{Hom}(X^{\otimes m}, X^{\otimes n})$ and $\beta \in \text{Hom}(X^{\otimes n}, X^{\otimes r})$ the following identities hold

\[
H_p^{m,n}(\alpha)^* = H_p^{m,n}(\alpha^*), \quad H_p^{m,r}(\beta) \circ H_p^{m,n}(\alpha) = H_p^{m,r}(\beta \circ \alpha).
\]

**Proof.** Indeed, it is immediate that $H_p^{m,n}$ is injective. Since $\nu \nu^* = \iota$ we have

\[
((\nu \nu^*)^{\otimes l} \otimes \iota^{\otimes n})H_p^{m,n}(\alpha) = ((\nu \nu^*)^{\otimes l} \otimes \iota^{\otimes n})\alpha((\nu^*)^{\otimes k} \otimes \iota^{\otimes m}) = H_p^{m,n}(\alpha).
\]

Similarly $H_p^{m,n}(\alpha)((\nu \nu^*)^{\otimes k} \otimes \iota^{\otimes m}) = H_p^{m,n}(\alpha)$. For surjectivity suppose that $\beta \in \Sigma_p^{m,n}$. Define $\alpha := ((\nu^*)^{\otimes l} \otimes \iota^{\otimes n})\beta(\nu \nu^* \otimes \iota^{\otimes m})$. Clearly $\alpha \in \text{Hom}(X^{\otimes m}, X^{\otimes n})$. Since $\beta \in \Sigma_p^{m,n}$ we have

\[
H_p^{m,n}(\alpha) = ((\nu \nu^*)^{\otimes l} \otimes \iota^{\otimes n})\beta((\nu \nu^*)^{\otimes k} \otimes \iota^{\otimes m}) = \beta.
\]

Moreover this calculation shows that $(H_p^{m,n})^{-1}(\beta) = ((\nu^*)^{\otimes l} \otimes \iota^{\otimes n})\beta(\nu \nu^* \otimes \iota^{\otimes m})$. Furthermore

\[
H_p^{m,n}(\alpha)^* = ((\nu \nu^*)^{\otimes l} \otimes \iota^{\otimes n})\alpha((\nu^*)^{\otimes k} \otimes \iota^{\otimes m})^* = (\nu \nu^* \otimes \iota^{\otimes m})\alpha^*(\nu^*)^{\otimes l} \otimes \iota^{\otimes n}) = H_p^{m,n}(\alpha^*).
\]

Write $m_1 = m$, $m_2 = n$ and $m_3 = r$ and suppose $p = m_i + k_i N$ for some $k_i \in \mathbb{N}$. Then we have

\[
H_p^{m_2,m_3}(\beta) \circ H_p^{m_1,m_2}(\alpha) = (\nu^{\otimes k_3} \otimes \iota^{\otimes m_3})\beta((\nu^*)^{\otimes k_2} \otimes \iota^{\otimes m_2})(\nu \nu^* \otimes \iota^{\otimes m_2})\alpha((\nu^*)^{\otimes k_1} \otimes \iota^{\otimes m_1})
\]

\[
= (\nu^{\otimes k_3} \otimes \iota^{\otimes m_3})\beta\alpha((\nu^*)^{\otimes k_1} \otimes \iota^{\otimes m_1})
\]

\[
= H_p^{m_1,m_3}(\beta \circ \alpha)
\]

and the lemma is proved. \(\Box\)

For each $p$, let $\psi_p : \text{End}(X^{\otimes p}) \to \text{End}(\mathcal{H}^{\otimes p})$ be a $*$-isomorphism making the diagram

\[
\begin{array}{ccc}
H_p(q) & \xrightarrow{\theta_p} & \text{End}(X^{\otimes p}) \\
\downarrow{\eta_p} & & \downarrow{\psi_p} \\
\text{End}(\mathcal{H}^{\otimes p}) & & \\
\end{array}
\]

commute. Such an isomorphism exists, because by assumption and Theorem 2.4.15
2.5. CATEGORIES GENERATED BY HECKE ALGEBRAS

\( \theta_p : H_p(q) \to \text{End}(X^{\otimes p}) \) and \( \eta_p : H_p(q) \to \text{End}(\mathcal{H}^{\otimes p}) \) are surjective and \( \ker(\theta_p) = \ker(\eta_p) \).

Let us write \( \kappa := \|S\|^{-1}S \), where \( S : 1 \to \mathcal{H}^{\otimes N} \) is the intertwiner defined in (2.3.2). Because \( \nu \nu^* = \theta(F_N) \) and \( \kappa \kappa^* = \|S\|^{-2}SS^* = \eta(F_N) \), we have \( \psi_N(\nu \nu^*) = \|S\|^{-2}SS^* = \kappa \kappa^* \).

Define for \( m \equiv n \) (mod N) the map \( \psi_{m,n} \) which is the composition

\[
\text{Hom}_C(X^{\otimes m}, X^{\otimes n}) \xrightarrow{H_{m,n}^{p,C}} \sum_{p \in \mathcal{D}} \psi_p \sum_{p \in \text{Rep}(SU_\mu(N))} (H_{m,n}^{p,\text{Rep}(SU_\mu(N))})^{-1} \xrightarrow{\text{Hom}_{\text{Rep}(SU_\mu(N))}(\mathcal{H}^{\otimes m}, \mathcal{H}^{\otimes n})}.
\]

We omit the indices \( C \) and \( \text{Rep}(SU_\mu(N)) \) from \( H_{m,n}^{p} \).

**Lemma 2.5.9.** The morphisms \( \psi_{m,n} \) are well-defined (independent of \( p \)) isomorphisms of linear spaces and satisfy

\[
\psi_{m,n}(\alpha^*) = \psi_{n,m}(\alpha^*), \quad \psi_{n,r}(\beta) \circ \psi_{m,n}(\alpha) = \psi_{m,r}(\beta \circ \alpha).
\]  

(2.5.13)

**Proof.** Let us check that \( \psi_{m,n} \) is independent of \( p \) and therefore well-defined. For this suppose \( m, n, p, k, l, p', k', l' \in \mathbb{N} \) such that \( p = m + kN = n + lN \) and \( p' = m + k'N = n + l'N \). Say \( p < p' \), let \( j \) be such that \( k' = k + j \) and \( l' = l + j \). It is immediate from the definition that \( H_{p,n}^{m,m} = H_{p',n}^{m,m} \). So it is sufficient to show that \( H_{p',n}^{m,m} \circ \psi_p = \psi_{p'} \circ H_{p'}^{m,n} \).

This is the case, namely let \( \alpha \in \text{End}(X^{\otimes p}) \), then

\[
H_{p'}^{m,n} \circ \psi_p(\alpha) = (\kappa^{\otimes j} \otimes \iota^{\otimes p}) \psi_p(\alpha)(\kappa^* \otimes \iota^{\otimes p}) = \psi_{p'}((\nu \nu^*)^{\otimes j} \otimes \iota^{\otimes p}) (\nu^{\otimes j} \otimes \iota^{\otimes p}) \psi_{p'} = \psi_{p'} \circ H_{p'}^{m,n}(\alpha).
\]

Clearly \( \psi_p \) restricts to an isomorphism \( \Sigma_{m,n}^{\otimes} \rightarrow \Sigma_{m,n}^{\otimes} \). This implies that the maps \( \psi_{m,n} \) are isomorphisms. Identity (2.5.13) follows directly from (2.5.11). \( \Box \)

With these isomorphisms \( (\psi_{m,n})_{m,n} \) we are able to define a unitary tensor functor from \( C \) to \( \text{Rep}(SU_\mu(N)) \). For this consider the full subcategory \( \tilde{C} \) of \( C \) with objects \( \text{Ob}(\tilde{C}) := \{X^{\otimes n} : n \in \mathbb{N} \} \) and \( \mathcal{D} \) the full subcategory of \( \text{Rep}(SU_\mu(N)) \) objects \( \text{Ob}(\mathcal{D}) := \{\mathcal{H}^{\otimes n} : n \in \mathbb{N} \} \). The completion of \( \tilde{C} \) and \( \mathcal{D} \) with respect to direct sums and subobjects equal respectively \( C \) and \( \text{Rep}(SU_\mu(N)) \). Define \( \tilde{F} : \tilde{C} \to \mathcal{D} \) by \( X^{\otimes n} \mapsto \mathcal{H}^{\otimes n} \) on objects, \( \tilde{F}(\alpha) := \psi_{m,n}(\alpha) \) for morphisms \( \alpha \in \text{Hom}(X^{\otimes m}, X^{\otimes n}) \) and \( \tilde{F}_0 = id, \tilde{F}_2 = id \). \( \tilde{F}(\alpha) \) is well-defined, because by assumption \( m \equiv n \) (mod \( N \)) if \( \alpha \neq 0 \).

**Lemma 2.5.10.** \( \tilde{F} \) is a unitary tensor functor.

**Proof.** Let \( \alpha \in \text{Hom}(X^{\otimes m_1}, X^{\otimes m_2}) \) and \( \beta \in \text{Hom}(X^{\otimes m_2}, X^{\otimes m_3}) \) be nonzero morphisms. For \( i = 1, 2, 3 \) select an integer \( k_i \in \mathbb{N} \) such that \( p = m_i + k_i N \). Then

\[
\tilde{F}(\beta) \circ \tilde{F}(\alpha) = ((\nu^*)^{\otimes k_3} \otimes \iota^{\otimes m_3}) \psi((\nu^{\otimes k_3} \otimes \iota^{\otimes m_3}) (\beta ((\nu^*)^{\otimes k_2} \otimes \iota^{\otimes m_2}))(\kappa^{\otimes k_2} \otimes \iota^{\otimes m_2})) \circ ((\nu^*)^{\otimes k_2} \otimes \iota^{\otimes m_2}) \psi((\nu^{\otimes k_2} \otimes \iota^{\otimes m_2}) (\alpha ((\nu^*)^{\otimes k_1} \otimes \iota^{\otimes m_1})))(\kappa^{\otimes k_1} \otimes \iota^{\otimes m_1}).
\]
Since $\psi_N(\nu\nu^*) = \kappa\kappa^*$ and $\nu^*\nu = \iota$ this composition equals
\[
((\kappa^*)^\otimes k_3 \otimes i^\otimes m_3)\psi((\nu^\otimes k_3 \otimes i^\otimes m_3)2(\nu^* \nu^*)\nu^\otimes k_2 \otimes i^\otimes m_2)\alpha((\nu^*)^\otimes k_1 \otimes i^\otimes m_1))(\kappa^\otimes k_1 \otimes i^\otimes m_1)
= ((\kappa^*)^\otimes k_3 \otimes i^\otimes m_3)\psi((\nu^\otimes k_3 \otimes i^\otimes m_3)2\alpha((\nu^*)^\otimes k_1 \otimes i^\otimes m_1))(\kappa^\otimes k_1 \otimes i^\otimes m_1)
= \tilde{F}(\beta \circ \alpha),
\]
so $\tilde{F}$ is a functor. From (2.5.13) it is clear that $\tilde{F}$ is unitary. Because the categories $\mathcal{C}$ and $\text{Rep}(\text{SU}_\mu(N))$ are strict, it only remains to check the property $\tilde{F}(\alpha \otimes \beta) = \tilde{F}(\alpha) \otimes \tilde{F}(\beta)$ to conclude that $\tilde{F}$ is a tensor functor. For this suppose $\alpha \in \text{Hom}(X^\otimes m_1, X^\otimes m_2)$, $\beta \in \text{Hom}(X^\otimes m_3, X^\otimes m_4)$ and $m_1 + k_1 N = m_2 + k_2 N = p_1$, $m_3 + k_3 N = m_4 + k_4 N = p_2$. Here it is needed that $\omega_2 = 1$ so that we are able to apply the identities of the proof of Proposition 2.5.6. Using (2.5.5)–(2.5.8) it follows that there exists $x_1 \in H_{k_1N+m_1}(q)$ and $x_2 \in H_{k_2N+m_2}(q)$ such that
\[
((\nu^*)^\otimes k_3 \otimes i^\otimes m_1) = (i^\otimes m_1 \otimes (\nu^*)^\otimes k_3)\theta(x_1); \quad \theta(x_1)(\nu^\otimes k_3 \otimes i^\otimes m_1) = (i^\otimes m_1 \otimes \nu^\otimes k_3);
(\nu^\otimes k_4 \otimes i^\otimes m_2) = \theta(x_2)(i^\otimes m_2 \otimes \nu^\otimes k_4); \quad ((\nu^*)^\otimes k_4 \otimes i^\otimes m_2)\theta(x_2) = (i^\otimes m_2 \otimes (\nu^*)^\otimes k_4).
\]
Therefore using the fact that $\psi(fg) = \psi(f)\psi(g)$ and $\psi$ makes the diagram (2.5.12) commute, we obtain
\[
\tilde{F}(\alpha \otimes \beta) = ((\kappa^*)^\otimes k_2+k_4 \otimes i^\otimes m_2+m_4)\psi((\nu^\otimes k_1+k_3 \otimes i^\otimes m_1+m_3)(\alpha \otimes \beta)
((\nu^*)^\otimes k_1+k_3 \otimes i^\otimes m_1+m_3))(\kappa^\otimes k_2+k_4 \otimes i^\otimes m_2+m_4)
= ((\kappa^*)^\otimes k_2+k_4 \otimes i^\otimes m_2+m_4)\psi((\nu^\otimes k_1+k_3 \otimes i^\otimes m_1+m_3)(\alpha \otimes \beta)
((\nu^*)^\otimes k_1+k_3 \otimes i^\otimes m_1+m_3))(\kappa^\otimes k_2+k_4 \otimes i^\otimes m_2+m_4)
= ((\kappa^*)^\otimes k_2+k_4 \otimes i^\otimes m_2+m_4)\psi((\nu^\otimes k_1+k_3 \otimes i^\otimes m_1+m_3)(\alpha \otimes \beta)
((\nu^*)^\otimes k_1+k_3 \otimes i^\otimes m_1+m_3))(\kappa^\otimes k_2+k_4 \otimes i^\otimes m_2+m_4)
= \tilde{F}(\alpha) \otimes \tilde{F}(\beta),
\]
which concludes the proof of the lemma. 

Clearly $\tilde{F}$ is essentially surjective. Note that Lemmas 2.5.9 and 2.5.10 imply that $\tilde{F}$ is a fully faithful unitary tensor functor. Taking the completions of $\tilde{C}$ and $D$ with respect to direct sums and subobjects gives us the categories $\mathcal{C}$ and $\text{Rep}(\text{SU}_\mu(N))$. Under this completion $\tilde{F}$ extends uniquely (up to natural unitary isomorphism) to a unitary tensor functor $F: \mathcal{C} \to \text{Rep}(\text{SU}_\mu(N))$. Then $F$ is again fully faithful and essentially surjective. This implies that $F$ is a unitary monoidal equivalence, in other words $\mathcal{C}$ is unitarily monoidally equivalent to $\text{Rep}(\text{SU}_\mu(N))$. 

\[\Box\]
2.6 Two characterisations of SU(N)-type categories

Now most of the preparatory work is done to prove the main results. We characterise all SU(N)-type categories and give a condition when it is possible to embed $\text{Rep}(\text{SU}_\mu(N))$ in a given $C^*$-tensor category. It is shown that all SU(N)-type categories can be classified by a pair $(q, \omega)$ where $q \in (0, 1]$ and $\omega$ is a $N$-th root of unity. The requirement for existence of an embedding is given by six identities which basically state that if a category satisfies those requirements, there exist a representation of the Hecke algebra, and the twist and solutions of the conjugate equations can be explicitly computed. The proofs of both theorems consist of showing that Assumption 2.5.1 is satisfied allowing to apply Theorem 2.5.7.

**Definition 2.6.1.** Let $C$ be a strict SU(N)-type category. Since in $\text{Rep}(\text{SU}(N))$ the trivial representation $\mathbb{C}$ is a subrepresentation of $V^\otimes N$, there exist a morphism $\nu: 1 \mapsto X^\otimes N$, such that $\nu^* \nu = id_1$ and $\nu \nu^* \in \text{End}(X^\otimes N)$ is a projection. The *twist* $\tau_C$ of $C$ is defined as the scalar by which one multiplies in the following composition\(^3\)

$$x = x \otimes 1 \xrightarrow{\nu \otimes X} x \otimes X^\otimes N \xrightarrow{\theta(g_N \cdots g_1)} X^\otimes N \otimes x \xrightarrow{\nu^* \otimes X} 1 \otimes x = x.$$ 

Since $x$ is simple this is indeed a scalar. Also $\tau_C$ is clearly independent of the choice of $\nu$. Again, a priori it is not clear why $\tau_C$ is independent of the choice of $X$. Fortunately this is the case as we will show later (cf. Remark 2.6.8).

**Lemma 2.6.2.** The following holds: $\theta(g_N \cdots g_1)(\iota \otimes \nu) = \tau_C(\nu \otimes \iota)$.

**Proof.** Note that $\nu \nu^* \in \text{End}(X^\otimes N)$. From Theorem 2.4.15 we obtain that there exists a $x \in H_N(q)$ such that $\nu \nu^* \theta(x)$. By Lemma 2.2.4 it therefore follows that $\theta(g_N \cdots g_1)(\iota \otimes \nu \nu^*)(\iota \otimes \nu) = (\nu \nu^* \otimes \iota)\theta(g_N \cdots g_1)$. Since $\nu \nu^* = \iota$ we have

$$\theta(g_N \cdots g_1)(\iota \otimes \nu) = \theta(g_N \cdots g_1)(\iota \otimes \nu \nu^*)(\iota \otimes \nu) = (\nu \nu^* \otimes \iota)\theta(g_N \cdots g_1)(\iota \otimes \nu)$$

and the result follows. \(\square\)

Observe that identity (2.3.8) of Proposition 2.3.4 implies that the twist of $\text{Rep}(\text{SU}_\mu(N))$ equals $\mu^{N+1}$.

**Notation 2.6.3.** Since for a strict SU(N)-type category $C$ the constant $q_C \in (0, 1]$, define $\mu_C \in (0, 1]$ to be the positive square root of $q_C$.

**Corollary 2.6.4.** Suppose that $C$ is a strict SU(N)-type category, then there exists an $N$-th root of unity $\omega_C$ such that $\tau_C = \omega_C \mu_C^{N+1}$.

**Proof.** First note that Lemma 2.2.4 implies that $g_{\sigma_{k,N}} = g_{\sigma_{k-1,N}} \Sigma^{k-1}(g_{\sigma_{1,N}})$, where $\Sigma$ denotes the shift map. Combination with the identity $\theta(g_N \cdots g_1)(\iota \otimes \nu) = \tau_C(\nu \otimes \iota)$ gives

$$\theta(g_{\sigma_{k,N}})(\iota \otimes \nu) = \tau_C \theta(g_{\sigma_{k-1,N}}) (\iota \otimes \nu \otimes \iota).$$

\(^3\)Note that this twist differs a factor $(-1)^N$ from the twist defined in [KW93].
Thus in particular
\[ \omega_{\mu} \text{ a braiding if } X \text{ the object } \theta \text{ (2.3.8)). } \]
The mistake in the proof, is that it is claimed that \( \omega \) of unity. By induction we obtain for all \( k \in \mathbb{N} \)
\[ \theta(g_{\sigma_{k,N}})(\nu \otimes \nu) = \tau_{C}^{k}(\nu \otimes \nu^{\otimes k}). \]
Thus in particular
\[ \theta(g_{\sigma_{N,N}})(\nu \otimes \nu) = \tau_{C}^{N}(\nu \otimes \nu^{\otimes N}). \]
Multiplying both sides by \( (\nu^{*} \otimes \nu^{\otimes N}) \) gives \( (\nu^{*} \otimes \nu^{\otimes N}) \theta(g_{\sigma_{N,N}})(\nu \otimes \nu) = \tau_{C}^{N} \nu^{\otimes N}. \) Combination with Lemma 2.4.12 gives that as a morphism in \( \text{End}(X^{\otimes 2N}) \) we have
\[ (\theta(F_{N}) \otimes \theta(F_{N})) \theta(g_{\sigma_{N,N}})(\theta(F_{N}) \otimes \theta(F_{N})) = \tau_{C}^{N}(\theta(F_{N}) \otimes \theta(F_{N})). \]
Theorem 2.4.15 shows that the representations \( \theta \) and \( \eta \) are equivalent. In particular this implies that \( \tau_{C}^{N} = \tau_{\text{Rep}(SU_{\mu}(N))}^{N} = (\mu^{N+1})^{N}, \) which proves the corollary.

Remark 2.6.5. In [KW93, Prop. 5.2] it is asserted that \( \tau_{C} = (-1)^{N} \omega \) for some \( N \)-th root of unity \( \omega \). This is not true as for example the explicit calculation for \( SU_{\mu}(N) \) shows (cf. (2.3.8)). The mistake in the proof, is that it is claimed that \( \theta(g_{\sigma_{N,N}}) \) acts as \( (-1)^{N^{2}} \) on the object \( X_{1} \otimes X_{1} \).

Now all the technical work has been done to give a classification of \( SU(N) \)-type categories.

Theorem 2.6.6. If \( C \) is a \( SU(N) \)-type category with fundamental object \( X \). Then the category \( (\text{Rep}(SU_{\mu}(N)))^{\otimes N} \) is unitarily monoidally equivalent to \( C \). Furthermore \( C \) admits a braiding if \( \omega_{C} = \pm 1 \).

Proof. By Corollary 2.4.8 we have a representation of the Hecke algebra \( H_{n}(q_{C}) \rightarrow \text{End}_{C}(X^{\otimes n}) \). By Theorem 2.4.15 this representation is surjective and depends only on \( q_{C} \). As the representation \( \eta: H_{2}(q_{C}) \rightarrow \text{End}_{\text{Rep}(SU_{\mu_{C}(N))})(H^{\otimes 2}) \) satisfies that \( \eta(e_{1}) \) is the projection onto \( H_{12} \) (cf. Lemma 2.3.2), we obtain that \( q_{\text{Rep}(SU_{\mu_{C}(N))}} = q_{C} \). Then again by Theorem 2.4.15 \( \ker(\eta) = \ker(\theta) \). Lemmas 2.1.6, 2.6.2, 2.4.12 and Corollary 2.6.4 show that the other requirements of Assumption 2.5.1 are satisfied. Now Proposition 2.5.6 and Theorem 2.5.7 give the result.

Remark 2.6.7. It can be shown [NY15, Rem. 4.4] that in general a \( SU(N) \)-type category is not braided; such a category \( C \) admits a braiding if and only if \( \omega_{C} = \pm 1 \).

Remark 2.6.8. Now we can also prove why the constants \( q_{C} \) and \( \tau_{C} \) are independent of the chosen generator \( X \) of the category \( C \). By [McM84, Cor. 1] all automorphisms of the fusion semiring \( K^{+}[\text{Rep}(SU(N))] \) are in one-to-one correspondence with symmetries of the Dynkin diagram of \( SU(N) \). This diagram, consisting of \( N - 1 \) nodes \( \{1, 2, \ldots, N - 1\} \) where the nodes \( i \) and \( i + 1 \) are connected by a single edge, has exactly two symmetries, namely the identity and the map given on the nodes by \( i \mapsto N - i \). So we only have to show that \( q_{C} \) and \( \tau_{C} \) are invariant under this second, nontrivial, map. This map induces an automorphism of \( U_{\mu}(su_{N}) \), the quantum enveloping Hopf algebra of \( su_{N} \), given on the generators by \( E_{i} \mapsto E_{N-i}, F_{i} \mapsto F_{N-i}, K_{i}^{\pm} \mapsto K_{N-i}^{\pm} \). In \( \text{Rep}(SU_{\mu}(N)) \) it thus maps
every object to a conjugate object. Therefore it is sufficient to show that if we would have chosen $\tilde{X}$ instead of $X$ as generating object, the resulting constants $q_c$ and $\tau_c$ are the same. This is implicitly proved in [NY15, §4.2]. The idea is the following, suppose that in $\mathcal{C}$ the associativity morphisms are given by a cocycle $\varphi \in H^3(\mathbb{Z}/N\mathbb{Z}, T)$, thus $\alpha: (X^{\otimes a} \otimes X^{\otimes b}) \otimes X^{\otimes c} \to X^{\otimes a} \otimes (X^{\otimes b} \otimes X^{\otimes c})$ acts as multiplication by $\varphi(a, b, c)$ (in our case $\varphi$ is of the form $\varphi(a, b, c) = \omega_C^{(\frac{ab}{N}-1)\frac{(a+b)}{2}-\frac{c}{2}}$). We write $\text{Rep}(\text{SU}_\mu(N))^\varphi$ for the category $\text{Rep}(\text{SU}_\mu(N))$ with these new associativity morphisms. Then $\mathcal{C} \cong \text{Rep}(\text{SU}_\mu(N))^\varphi$. The map $X \mapsto \tilde{X}$ corresponds to changing the cocycle $\varphi$ to the new one given by $\psi(a, b, c) := \varphi(-a, -b, -c)$. So one obtains an isomorphism $\theta: \text{Rep}(\text{SU}_\mu(N))^\varphi \to \mathcal{C} \to \text{Rep}(\text{SU}_\mu(N))^\psi$. The question is now whether this isomorphism acts trivially on $H^3(\mathbb{Z}/N\mathbb{Z}, T)$. This is indeed the case, since $\varphi = \partial f$, $\psi = \partial g$, where $f(a, b) = \omega^{-\frac{ab}{N}}$ and $g(a, b) = \omega^{\frac{ab}{N}}$ are maps $f, g: \mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to T$. Now a direct computation shows that $fg^{-1}$ factors through $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and thus $\varphi$ and $\psi$ are equivalent cocycles. Thus $\theta$ acts trivially and $\omega_C$ and $q_c$ are invariant under $X \mapsto \tilde{X}$.

Another (more elementary) method of proving that those constants are invariant is by explicitly computing everything. This can be done in the following way. We adopt the notation as in [NT13, §2.2] and denote $F: \mathcal{C} \to \mathcal{C}$ for the contravariant tensor functor

$$\text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{C}), \quad U \mapsto \bar{U}; \quad \text{Hom}(U, V) \to \text{Hom}(\bar{V}, \bar{U}), \quad T \mapsto T^\vee,$$

where $T^\vee := (\iota \otimes R_U)(\iota \otimes T \otimes \iota)(R_U \otimes \iota)$. Define $F_2(U, V): \bar{V} \otimes \bar{U} \to \bar{U} \otimes \bar{V}$ by the identity

$$(F_2(U, V) \otimes \iota \otimes \iota)(\iota \otimes R_U \otimes \iota)R_V = R_U \otimes V.$$

Put $a_t := F_2^*(X, X)a_p F_2(X, X)$. Then it can be checked that the $a_t$ satisfy the relations of $e_i$ in the Hecke algebra $\mathcal{H}_n(q_c)$ and thus we get a representation $\theta^e: \mathcal{H}_n(q_c) \to \text{End}_C(X^{\otimes n})$. Hence $q_c$ is invariant. Now for $\tau_c$ we define

$$\nu^e := (F_2^2(X, X) \otimes \iota^{\otimes N-2}) \cdots (F_2^2(X^{\otimes N-2}, X) \otimes \iota)F_2^2(X^{\otimes N-1}, X)(\nu^*)^\vee: 1 \to \bar{X}^{\otimes N}.$$

Then one can verify that $\nu^e$ plays the role of $\nu$ and

$$(\nu^{e*} \otimes \iota)\theta^e(g_N \cdots g_1)(\iota \otimes \nu^e) = \tau_c \iota,$$

whence $\tau_c$ is invariant under the transformation $X \mapsto \tilde{X}$.

**Remark 2.6.9.** The above theorem says that all SU($N$)-type categories can be described by a pair $(q, \omega)$, where $q \in (0, 1]$ and $\omega$ is a $N$-th root of unity. Namely we have shown that a SU($N$)-type category $\mathcal{C}$ is isomorphic to $((\text{Rep}(\text{SU}_3(N))))^\omega$. Now one might wonder if each pair $(q, \omega)$ of this form can be realised by a compact quantum group. This is indeed the case, see [NY15].

Inspired by [Pin07, Thm. 6.2] we have the following condition for the existence of an embedding of a twist of $(\text{Rep}(\text{SU}_\mu(N)))^\omega$ in a C*-tensor category $\mathcal{D}$. We use the notation as introduced in Notation 2.3.1.
Theorem 2.6.10. Suppose that $\mathcal{D}$ is a strict $C^*$-tensor category such that there exists an object $X \in \text{Ob}(\mathcal{D})$, morphisms $\nu \in \text{Hom}(1, X^{\otimes N})$, $a \in \text{End}(X^{\otimes 2})$, a constant $\mu \in (0, 1]$ and an $N$-th root of unity $\omega$ satisfying the following properties:

\begin{align*}
a &= a^* = a^2; \\
(a \otimes \nu)(a \otimes \nu) &= \frac{q}{(1 + q)^2}(a \otimes \nu)(a \otimes \nu) - \frac{q}{(1 + q)^2}(a \otimes \nu); \\
\nu^*\nu &= \iota; \\
\nu
\nu^* &= \theta(F_N); \\
(\nu^* \otimes \iota)(\nu \otimes \iota) &= \omega(-\mu)^{-(N-1)}[[N]]^{-1}\iota; \\
\theta(g_N \cdots g_l)(\nu \otimes \iota) &= \omega\mu^{N+1}(\nu \otimes \iota). \tag{2.6.6}
\end{align*}

Here $q := \mu^2$ and $\theta : H_n(q) \to \text{End}_{\mathcal{D}}(X^{\otimes n})$ is the representation of the Hecke algebra as in Corollary 2.4.8. Let $\mathcal{C}$ be the sub $C^*$-tensor category of $\mathcal{D}$ generated by the object $X$ and morphisms $\nu$ and $a$. Then $\mathcal{C}$ is a $\text{SU}(N)$-type category and there exists a unique (up to natural unitary isomorphism) unitary tensor functor $F : (\text{Rep}(\text{SU}_\mu(N)))^\omega \to \mathcal{D}$ such that $F(H) = X$ and $F(S) = ([[N]]_q)_{1/2}^\omega$, $F(T) = q - (q + 1)a$.

Proof. First note that Equations (2.6.1) and (2.6.2) together with Corollary 2.4.8 imply that we have a $*$-representation $\theta : H_n(q) \to \text{End}_{\mathcal{D}}(X^{\otimes n})$. Therefore the identities (2.6.4) and (2.6.6) make sense. We would like to use Theorem 2.5.7, for this we only need to check three conditions: equality of the kernels of $\theta$ and $\eta$, surjectivity of the representation $\theta : H_n(q) \to \text{End}_{\mathcal{C}}(X^{\otimes n})$ and $\text{Hom}(X^{\otimes m}, X^{\otimes n}) = \{0\}$ if $m \neq n \pmod{N}$. Let us start with the easiest one: the last one.

For this note that $\text{Hom}_{\mathcal{C}}(X^{\otimes m}, X^{\otimes n})$ is generated by $a$ and $\nu$. So if $\alpha \in \text{Hom}_{\mathcal{C}}(X^{\otimes m}, X^{\otimes n})$, then $\alpha = 0$, or $\alpha$ is a linear combination of words consisting of the letters $\nu \otimes k \otimes \nu \otimes l$, $\nu \otimes k \otimes \nu^* \otimes l$ and $\theta(x)$ for $k, l \in \mathbb{N}$ and $x \in H_\infty(q)$. It is sufficient to consider individual words. If $x \in H_p(q) \subset H_\infty(q)$, then $\theta(x) \in \text{End}(X^{\otimes p})$ and for $k, l \in \mathbb{N}$ we have $\nu \otimes k \otimes \nu \otimes l \in \text{Hom}(X^{k+1}, X^{k+N+1})$, $\nu \otimes k \otimes \nu^* \otimes l \in \text{Hom}(X^{k+N+1}, X^{k+l})$. Induction on the length of a word gives the result.

To be able to prove the other two remaining requirements we first compute $(\nu^* \otimes \nu)(\nu \otimes \nu)$. In the upcoming computations we need the identity

\[ \theta(g_i)\nu = \theta(g_i)\nu^*\nu = \theta(g_i)F_N\nu = -\theta(F_N)\nu = -(\nu^*\nu) = -\nu, \quad \text{for } i = 1, \ldots, N - 1, \tag{2.6.7} \]

which follows from (2.6.3), (2.6.4) and Lemma 2.2.8. This also implies that $\nu^*\theta(g_i) = -\nu^*$. \hfill \Box

Lemma 2.6.11. For $k = 1, 2, \ldots, N - 1$ the following equality holds

\[ (\nu^* \otimes \nu)(\nu \otimes \nu) = \omega^k \frac{[[N - k]]_{1/2}[[k]]_{1/2}^2}{[[N]]_{1/2}^2} (-\mu)^{-k(N-k)}\theta(F_k). \tag{2.6.8} \]

Proof. We prove this by induction. The case $k = 1$ is exactly assumption (2.6.5) of the
2.6. TWO CHARACTERISATIONS OF SU(N)-TYPE CATEGORIES

theorem. To prove the induction step consider the morphism

\[ T := (\nu^* \otimes \iota^k)(\iota^k \otimes k \otimes \nu \otimes \iota)(\iota^k \otimes \nu^* \otimes \iota)(\iota^k \otimes \nu). \]

By the induction hypothesis and the assumption of this theorem, this morphism equals

\[ T = \omega^{k-1} \frac{[N-k+1] \frac{1}{v} ![k-1] \frac{1}{v} !}{[[N]] \frac{1}{v} ![k-1] \frac{1}{v} !} (-\mu)^{(k-1)(N-k+1)} \omega(-\mu)^{-(N-1)[N]\frac{1}{v} !} (\theta(F_{k-1}) \otimes \iota) \]

\[ = \omega^{k-1} \frac{[N-k+1] \frac{1}{v} ![k-1] \frac{1}{v} !}{[[N]] \frac{1}{v} ![N] \frac{1}{v} !} (-\mu)^{k-2-Nk-2k+2} (\theta(F_{k-1}) \otimes \iota). \]

(2.6.9)

On the other hand as \( \nu \mu^* = \theta(F_N) \), \( \theta(g_i) \nu = -\nu \) and \( \nu^* \theta(g_i) = -\nu^* \) we have

\[ T = (\nu^* \otimes \iota^k)(\iota^k \otimes \theta(F_N) \otimes \iota)(\iota^k \otimes \nu) \]

\[ = \frac{[N] \frac{1}{v} ![N-k+1] \frac{1}{v} ![k-1] \frac{1}{v} !}{[[N]] \frac{1}{v} ![N] \frac{1}{v} !} \theta(1 + (q-1)g_k + \ldots + (q)^{-(N-1)}g_{k+N-2} \ldots g_k) \]

\[ (\iota^k \otimes \theta(F_{k-1}) \otimes \iota)(\iota^k \otimes \nu) \]

\[ = \frac{[N] \frac{1}{v} ![N-k+1] \frac{1}{v} ![k-1] \frac{1}{v} !}{[[N]] \frac{1}{v} ![N] \frac{1}{v} !} \theta(1 + q^{-1} + \ldots + q^{-(N-k)} + (q)^{-(N-1-k)}g_{N-k} \ldots g_k) \]

\[ + \ldots + (q)^{-(N-1)}g_{k+N-2} \ldots g_k)(\iota^k \otimes \nu) \]

\[ = \frac{[N] \frac{1}{v} ![N-k+1] \frac{1}{v} ![N] \frac{1}{v} !}{[[N]] \frac{1}{v} ![N] \frac{1}{v} !} \theta(1 + (q-1)g_1 + \ldots + (q)^{-(k-2)}g_{k-2} \ldots g_1)(\nu^* \otimes \iota^k)\theta(g_N \ldots g_k)(\iota^k \otimes \nu) \]

(2.6.10)

Now note that by the assumptions and induction hypothesis

\( (\nu^* \otimes \iota^k)\theta(g_N \ldots g_k)(\iota^k \otimes \nu) = (\nu^* \otimes \iota^k)\theta(g_{N+1}^{-1} \cdot \ldots \cdot g_{k+N-1}^{-1})\theta(g_k \ldots g_1)(\iota^k \otimes \nu) \)

\[ = \theta(g_1^{-1} \cdot \ldots \cdot g_{k-1}^{-1})(\nu^* \otimes \iota^k)\omega \mu^{N+1} (\iota^k \otimes \nu \otimes \iota) \]

\[ = \omega \mu^{N+1} \omega^{k-1} \frac{[N-k+1] \frac{1}{v} ![N] \frac{1}{v} ![k-1] \frac{1}{v} !}{[[N]] \frac{1}{v} ![N] \frac{1}{v} !} (-\mu)^{-(k-1)(N-k+1)} \theta(g_1^{-1} \cdot \ldots \cdot g_{k-1}^{-1})(\theta(F_{k-1}) \otimes \iota). \]

Since

\( (q)^{-(N+1-k)} \mu^{N+1} (-\mu)^{-(N-1-k)} = (-1)^k (N+1-k) \mu^{-k(N-k)} \),

identity (2.6.10) equals

\[ (2.6.10) = \frac{([N] \frac{1}{v} ![N-k+1] \frac{1}{v} ![N] \frac{1}{v} ![k-1] \frac{1}{v} !}{[[N]] \frac{1}{v} ![N] \frac{1}{v} !} \theta(g_1^{-1} \cdot \ldots \cdot g_{k-1}^{-1} + (q)^{-(k-2)}g_{k-1}^{-1} + \ldots + (q)^{-(N-1)}g_{k-1}^{-1})(\theta(F_{k-1}) \otimes \iota). \]

(2.6.11)
If we now combine both expressions of \( T \), (2.6.9) and (2.6.11), we get
\[
[[N]]^{-1}[[N + 1 - k]]_q^1 (\nu^* \otimes \nu^k)(\nu^k \otimes \nu)
= (-1)^{k(N+1-k)}\mu^{-k(N-k)} \omega^k \frac{[[N - k + 1]]_q^{1/2}[[k - 1]]_q^{1/2}}{[[N]]_q^{1/2}[[N]]_q^{1/2}} \times \theta(g_1^{-1} \cdots g_{k-1}^{-1} + (-q)^{-1}g_2^{-1} \cdots g_{k-1}^{-1} + \ldots + (-q)^{-k+1}((\theta(F_{k-1}) \otimes \nu)\)
which by Lemma 2.2.8 equals
\[
(-1)^{k-1}(-1)^{k(N+1-k)+1} \mu^{-k(N-k)} \omega^k \frac{[[N - k + 1]]_q^{1/2}[[k - 1]]_q^{1/2}}{[[N]]_q^{1/2}[[N]]_q^{1/2}} \theta(F_k).
\]
From this (2.6.8) follows immediately and the lemma is proved. \( \Box \)

**Lemma 2.6.12.** The representation \( \theta \) satisfies
\[
\ker (\theta: H_n(q) \to \text{End}(X^\otimes n)) = \ker (\eta: H_n(q) \to \text{End}(\mathcal{H}^\otimes n)).
\]

**Proof.** From the above lemma it follows in particular that
\[
(\nu^* \otimes \nu^0) (\nu \otimes \nu^N \otimes \nu) = \overline{\omega}(-\mu)^{-(N-1)}[[N]]^{1/2} \theta(F_{N-1})
\]
and thus that the morphisms
\[
R := \overline{\omega}(-1)^{N-1}[[N]]^{1/2} \mu(N-1)/2 \nu, \quad \tilde{R} := [[N]]^{1/2} \mu(N-1)/2 \nu
\]
satisfy the conjugate equations for \( X \). Define a map
\[
\varphi^{(n)}: \text{End}(X^{\otimes n}) \to \text{End}(X^{\otimes n-1}), \quad \alpha \mapsto (\iota^{\otimes n-1} \otimes \nu^*)(\alpha \otimes \iota^{\otimes N-1})(\iota^{\otimes n-1} \otimes \nu)
\]
and the functional \( \varphi_n := \varphi^{(1)} \circ \cdots \circ \varphi^{(n-1)} \circ \varphi^{(n)} \). Now let \( (R', \tilde{R}') \) be a standard solution of the conjugate equations of \( X \). The map
\[
\text{End}(X^{\otimes n}) \to \text{End}(X^{\otimes n-1}), \quad \alpha \mapsto (\iota^{\otimes n-1} \otimes \tilde{R}'^*)(\alpha \otimes \iota^{\otimes N-1})(\iota^{\otimes n-1} \otimes \tilde{R}')
\]
is a partial trace induced by a standard solution, so it is tracial and faithful. There exists an invertible morphism \( T \in \text{Hom}(X', X) \) such that \( R = (T^{-1} \otimes \iota)R' \) and \( \tilde{R} = (\iota \otimes T^*)\tilde{R}' \) ([NT13, Prop. 2.2.4]). From this it is immediate that \( \varphi^{(n)} \) and thus \( \varphi_n \) are also faithful. Using the involution, equation (2.6.5) can be rewritten as
\[
(\iota \otimes \nu^*)(\nu \otimes \iota) = \overline{\omega}(-\mu)^{-(N-1)}[[N]]^{-1} \nu.
\]
Combination with (2.6.6) and (2.6.7) gives that

\[ \varphi^{(2)} \circ \theta(g_1) = (\iota \otimes \nu^*)\theta(g_1)(\iota \otimes \nu) \]
\[ = (-1)^{N-1}(\iota \otimes \nu^*)\theta(g_i \cdots g_1)(\iota \otimes \nu) \]
\[ = (-1)^{N-1}\omega^k_{N+1}(\iota \otimes \nu^*)(\nu \otimes \iota) \]
\[ = (-1)^{N-1}\omega^k_{N+1}\varphi(-\mu)^{(N-1)}[[N]]^{-1}\iota \]
\[ = q^{[N]}_{[1]}\iota = q^{[N]}_{[1]} \iota. \]

In particular \( \varphi^{(2)} \circ \theta(g_1) \) is a scalar in \( \text{End}(X) \) and thus \( \varphi^{(2)} \circ \theta(e_1) \) is a scalar as well. Therefore if \( x, y \in H_{n-1}(q) \)

\[ \varphi^{(n)}(\theta(xe_{n-1}y)) = (\iota^{\otimes n-1} \otimes \nu^*)\theta(x)(\iota^{\otimes n-2} \otimes \theta(e_1))\theta(y)(\iota^{\otimes n-1} \otimes \nu) \]
\[ = \theta(x)(\iota^{\otimes n-1} \otimes \nu^*)(\iota^{\otimes n-2} \otimes \theta(e_1))(\iota^{\otimes n-1} \otimes \nu)\theta(y) \]
\[ = \varphi_{2}(\theta(e_1)) \cdot \theta(xy). \]

So \( \varphi \circ \theta \) defines a faithful functional with the Markov property on \( H_n(q) \). According to Lemma 2.2.10 this functional must be tracial and hence we obtain a Markov trace \( \text{tr}_C := \varphi \circ \theta: H_n(q) \rightarrow \mathbb{C} \). Markov traces are characterized by their value on the generator \( g_i \). Recall from Theorem 2.4.15 that \( \text{tr}_{\text{Rep}(SU_\theta(N))}(g_1) = \frac{q^{N}}{[[N]]_{\eta}} \). It follows that \( \text{tr}_C = \text{tr}_{\text{Rep}(SU_\theta(N))} \) and thus \( \text{ker}(\theta) = \ker(\eta) \).

To prove surjectivity of \( \theta: H_n(q) \rightarrow \text{End}(X^{\otimes n}) \) we need the following lemma. Recall the notation \( \nu_{k,l} := \iota^{\otimes k} \otimes \nu \otimes \iota^{\otimes l} \) and \( \nu_{k,l}^{*} := \iota^{\otimes k} \otimes \nu^* \otimes \iota^{\otimes l} \).

**Lemma 2.6.13.** Let \( x \in H_p(q) \) and \( k, l, m, n \in \mathbb{N} \) satisfying \( k + l + N = m + n + N = p \), then there exist elements \( x_1, x_2 \in H_{\infty}(q) \) such that

\[ \nu_{k,l}^{*} \theta(x)\nu_{m,n} = \theta(x_1) \]
\[ \nu_{k,l} \theta(x)\nu_{m,n}^{*} = \theta(x_2). \]

**Proof.** First we prove the second assertion. Note that by (2.6.6) there exist \( y_1 \in H_{k+N}(q) \) and \( y_2 \in H_{m+N}(q) \) such that

\[ \nu_{k,l} = \theta(y_1)\nu_{0,k+l}, \quad \nu_{m,n}^{*} = \nu_{0,m+n}^{*}\theta(y_2). \]  

(2.6.12)

Then by (2.6.4)

\[ \nu_{k,l}^{*} \theta(x)\nu_{m,n} = \theta(y_1)\nu_{0,k+l}^{*}\theta(x)\nu_{0,m+n}^{*}\theta(y_2) = \theta(y_1)(\iota^{\otimes N} \otimes \theta(x))(\nu \nu^* \otimes \iota_{k+l})\theta(y_2) \]
\[ = \theta(y_1)\Sigma^N(x)F_Ny_2, \]

where \( \Sigma \) still denotes the shift map. Now the first case. Similar to (2.6.12) there exist \( y_1 \) and \( y_2 \) such that

\[ \nu_{k,l}^{*} \theta(x)\nu_{m,n} = \nu_{0,k+l}^{*}\theta(y_1)\theta(x)\theta(y_2)\nu_{0,m+n}. \]

So we can assume to deal with the case \( \nu_{0,k}^{*}\theta(x)\nu_{0,k} \) and \( x \in H_{N+k}(q) \). Now observe that \( \nu_{0,k}^{*}\theta(x)\nu_{0,k} = \nu_{0,k}^{*}\theta(F_NxF_N)\nu_{0,k} \). By surjectivity of the representation \( \eta: H_k(q) \rightarrow \)
End(\(H^{\otimes k}\)) there exists an element \(y \in H_k(q)\) such that \((S^* \otimes \iota^\otimes k)\eta(F_NxF_N)(S \otimes \iota^\otimes k) = \eta(y)\), here \(S\) and \(\eta\) are as in Notation 2.3.1. This implies that
\[
\eta(F_N\Sigma^N(y)) = SS^* \otimes \eta(y) = (S \otimes \iota^\otimes k)\eta(y)(S^* \otimes \iota^\otimes k) = (SS^* \otimes \iota^\otimes k)\eta(F_NxF_N)(SS^* \otimes \iota^\otimes k) = \eta(F_NxF_N).
\]
Because by Lemma 2.6.12 the representations \(\eta\) and \(\theta\) have the same kernel, it follows that \(\theta(F_NxF_N) = \theta(F_N\Sigma^N(y))\). Combining all these gives
\[
\nu^*_{0,k}\theta(x)\nu_{0,k} = \nu^*_{0,k}\theta(F_NxF_N)\nu_{0,k} = \nu_{0,k}\theta(F_N\Sigma^N(y))\nu_{0,k} = \theta(y)\nu^*_{0,k}\nu_{0,k}\nu_{0,k} = \theta(y)
\]
and concludes the lemma.

To prove that \(\theta: H_n(q) \to \text{End}_C(X^{\otimes n})\) is surjective let \(\alpha \in \text{End}_C(X^{\otimes n})\). Then \(\alpha\) is a linear combination of words consisting of the letters \(\theta(x)\) for \(x \in H_\infty(q)\) and \(\nu_{k,l}, \nu_{m,n}\). Let \(\beta = \beta_1 \cdots \beta_r\) be such a word and \(\beta_i\) the letters. Then \(\beta \in \text{End}_C(X^{\otimes n})\) and thus
\[
|i : \beta_i = \nu_{k,l} \text{ some } k,l| = |i : \beta_i = \nu^*_{k,l} \text{ some } k,l|.
\]
We now apply induction on the length \(r\). If \(r = 1\), then the above sets must be empty and thus \(\beta = \theta(x)\) for some \(x \in H_n(q)\). Suppose \(r > 1\) and not all \(\beta_i\) are of the form \(\theta(x)\) for \(x \in H_\infty(q)\), then there must exist \(1 \leq i < j \leq r\) such that either \(\beta_i = \nu_{k,l}, \beta_j = \nu_{m,n}^*\) for some \(k,l,m,n\) and \(\beta_s = \theta(x_s)\) for all \(i < s < j\), \(x_s \in H_\infty(q)\) or \(\beta_i = \nu_{k,l}^*, \beta_j = \nu_{m,n}\) for some \(k,l,m,n\) and \(\beta_s = \theta(x_s)\) for all \(i < s < j\), \(x_s \in H_\infty(q)\). In both cases we can apply Lemma 2.6.13 to reduce \(\beta_i \beta_{i+1} \cdots \beta_j\) to \(\theta(x)\) for some \(x \in H_\infty(q)\). In this way we obtain a word of length < \(r\) and by induction \(\beta \in \theta(H_n(q))\). Hence \(\theta: H_n(q) \to \text{End}_C(X^{\otimes n})\) is surjective and the conclusion follows from Theorem 2.5.7.
Chapter 3
The basics of random walks

Random walks form one of the pillars of the rest of the thesis. We start by giving an overview of the classical theory and then move on to random walks on discrete quantum groups.

3.1 Classical theory

For classical random walks we mainly follow [Woe00], but there are several other good sources e.g., [KSK76] and [Rev84]. Again we omit references, but proofs of all results stated below can be found in these books.

Definition 3.1.1. A discrete Markov chain consists of a pair \((X, P)\) where \(X\) is a discrete countable space and \(P = (p(x, y))_{x,y \in X}\) is a matrix which satisfies the properties \(p(x, y) \in [0, 1]\) for all \(x, y \in X\) and \(\sum_{y \in X} p(x, y) = 1\) for all \(x \in X\). We say that \(P\) defines a (classical) random walk on \(X\). The scalars \(p(x, y)\) are the transition probabilities that the random walk jumps from \(x\) to \(y\). The matrix \(P\) is called the Markov kernel.

The path space consists of all paths of infinite length in \(X\)

\[
\Omega := X^\mathbb{N} := \{\omega = (\omega_n)_{n=0}^\infty : \omega_n \in X\}.
\]

Denote the \(n\)-th coordinate projection

\[
X_n : \Omega \to X, \quad \omega = (\omega_m)_m \mapsto \omega_n.
\]

For \(x \in X\) we obtain a probability measure \(\mathbb{P}_x\) on \(\Omega\) defined on the cylinders by

\[
\mathbb{P}_x(\{\omega \in \Omega : X_0(\omega) = x, X_1(\omega) = x_1, \ldots, X_n(\omega) = x_n\}) := p(x, x_1)p(x_1, x_2)\cdots p(x_{n-1}, x_n).
\]

Definition 3.1.2. The Markov kernel acts on functions as follows. Let \(f : X \to \mathbb{C}\), define

\[
(Pf)(x) := \sum_{y \in X} f(y)p(x, y).
\]

A function \(h : X \to \mathbb{C}\) is called
(i) \(P\)-harmonic if \(Ph = h\);

(ii) \(P\)-superharmonic if \(h \geq 0\) and \(Ph \leq h\);

(iii) \(P\)-minimal harmonic if \(h\) is \(P\)-harmonic, \(h \geq 0\) and whenever \(h_1\) is another \(P\)-harmonic function satisfying \(0 \leq h_1 \leq h\) there exists a constant \(c \geq 0\) such that \(h = ch_1\).

Example 3.1.3. Let \(\Gamma\) be a discrete group and \(\mu\) a probability measure on \(\Gamma\). The matrix \(P\) with entries \(p(x, y) := \mu(yx^{-1})\) defines a Markov kernel on \(\Gamma\). The kernel \(P\) acts on functions as convolution by \(\mu\). Indeed,

\[(Pf)(x) = \sum_y p(x, y)f(y) = \sum_y \mu(yx^{-1})f(y) = \sum_y \mu(y)f(yx).\]

Definition 3.1.4. Suppose that \(\{p(x, y)\}_{x, y \in X}\) defines a random walk on \(X\). Let

\[p^n(x, y) := \begin{cases} 
\delta_{x, y} & \text{if } n = 0; \\
p(x, y), & \text{if } n = 1; \\
\sum_{z \in X} p^{n-1}(x, z)p(z, y), & \text{if } n > 1.
\end{cases}\]

This is the probability that the random walk is at \(y\) after \(n\) steps when started at \(x\). The random walk is called transient if \(\sum_{n=1}^\infty p^n(x, y) < \infty\) for all \(x, y \in X\). It is called irreducible if for all \(x, y \in X\) there exists an \(n \in \mathbb{N}\) such that \(p^n(x, y) > 0\).

Transience means that for any pair \((x, y)\) the expected number of times the random walk hits \(y\) when starting at \(x\) is finite. Irreducibility exactly means that every point can eventually be reached from any other point with positive probability.

Assumption 3.1.5. For the remainder of this section we assume that \(P\) is transient and irreducible.

Recall that if \(X\) is a locally compact Hausdorff space, a compact space \(Y\) is a compactification of \(X\) if \(X\) is homeomorphic to a dense subspace of \(Y\). A compactification \(Y_1\) of \(X\) is smaller than \(Y_2\) if the identity map \(\iota_X\) extends to a continuous surjection \(Y_2 \to Y_1\).

Definition 3.1.6. Suppose \(P\) is transient, then the Green kernel

\[g(x, y) := \sum_{n=0}^\infty p^n(x, y), \quad \text{for } x, y \in X\]

is well-defined. Fix a reference point \(x_0\). If in addition \(P\) is irreducible, we define the Martin kernel \(k\) as

\[k(x, y) := \frac{g(x, y)}{g(x_0, y)}, \quad \text{for } x, y \in X.\]

The Martin compactification \(\tilde{M}(X, P)\) of \(X\) with respect to \(P\) and \(x_0\) is the smallest compactification of \(X\) to which all Martin kernels \(\{k(x, \cdot) : x \in X\}\) extend continuously and such that all indicator functions \(\{\delta_x\}_{x \in X}\) are continuous. The Martin boundary equals
3.1. CLASSICAL THEORY

\[ M(X, P) := \tilde{M}(X, P) \setminus X. \] The Martin kernels when extended to the boundary are again denoted by \( k(x, \cdot) \).

**Remark 3.1.7.** Usually we omit \( P \) and write \( \tilde{M}(X) \) and \( M(X) \) for the compactification and boundary. The choice of another starting point gives different Martin kernels, but homeomorphic boundaries. Indeed, given a different point \( x'_0 \) we get

\[ k_{x'_0}(x, y) = \frac{k_{x_0}(x, y)}{k_{x_0}(x'_0, y)}. \]

where \( k_{x_0} \) and \( k_{x'_0} \) denote the Martin kernels with reference points \( x_0 \) and respectively \( x'_0 \).

The Martin compactification can alternatively be defined as the spectrum of the C*-algebra generated by \( C_0(X) \) and \( \{k(x, \cdot) : x \in X\} \). This point of view will be useful in the quantum setting which we discuss in the next section.

Another way to think about the Martin boundary is as the space of all equivalence classes of paths \((\omega_n)_n\) in \( X \) tending to infinity (so leaving every finite set) such that \( \lim_n k(x, \omega_n) \) exists. Two sequences \((\omega_n)_n\) and \((\omega'_n)_n\) are equivalent if and only if \( \lim_n k(x, \omega_n) = \lim_n k(x, \omega'_n) \) for every \( x \in X \).

The following example essentially summarizes the Ney–Spitzer theorem. We will come back to that theorem later (cf. Subsection §4.1.2).

**Example 3.1.8.** Consider Example 3.1.3 with \( \Gamma = \mathbb{Z}^d \) for some \( d > 0 \). Assume that \( \mu \) is a finitely supported measure and that \( P \) is transient and generating. The mean of \( \mu \) is defined as

\[ m(\mu) := \sum_{x \in \mathbb{Z}^d} \mu(x) x \in \mathbb{R}^d. \]

If \( m(\mu) \neq 0 \), where 0 indicates the zero vector in \( \mathbb{R}^d \), then \( M(\mathbb{Z}^d, P) \cong S^{d-1} \) via the embedding

\[ \mathbb{Z}^d \hookrightarrow \{x \in \mathbb{R}^d : \|x\| \leq 1\}, \quad x \mapsto \frac{1}{1 + \|x\|} x. \]

While if \( m(\mu) = 0 \), then \( M(\mathbb{Z}^d, P) \cong \{\text{pt}\} \).

**Theorem 3.1.9** (Poisson–Martin representation). *Given a finite measure \( \nu \) on \( \tilde{M}(X, P) \), the function

\[ f(x) = \int_{\tilde{M}(X, P)} k(x, \cdot) d\nu \]

is \( P \)-superharmonic. Conversely, if \( f \) is \( P \)-superharmonic there exists a bounded measure \( \nu^f \) on \( \tilde{M}(X, P) \) such that

\[ f(x) = \int_{\tilde{M}(X, P)} k(x, \cdot) d\nu^f. \]

If \( f \) is \( P \)-harmonic, then \( \nu^f \) is supported on the Martin boundary \( M(X, P) \).

Notice that this representation (3.1.2) need not be unique.

**Proposition 3.1.10.** The following holds:
Consider the set \( B := \{ f : X \to [0, \infty) : f \text{ superharmonic}, \ f(x_0) = 1 \} \). The extremal elements of \( B \) are precisely all kernels \( \{ k(\cdot, x) : x \in X \} \) and all minimal harmonic functions;

(ii) If \( h \) is minimal harmonic, then the representing measure \( \nu^h \) of Theorem 3.1.9 is unique and supported on a point. So there exists \( \xi \in M(X, P) \) and \( c > 0 \) such that \( h = c k(\cdot, \xi) \).

This result allows for the uniqueness of the representation of harmonic measures. Denote \( M_{\text{min}}(X, P) := \{ \xi \in M(X, P) : k(\cdot, \xi) \text{ is minimal harmonic} \} \).

Theorem 3.1.11. \( M_{\text{min}}(X) \) is a Borel measurable subset of \( M(X) \). Moreover, given a positive harmonic function \( h \), there exists a unique measure \( \nu^h \) such that \( \nu^h(M(X) \setminus M_{\text{min}}(X)) = 0 \) and \( \nu^h \) represents \( h \) as in (3.1.2).

Recall that \( x_0 \in X \) is the reference point used to define the Martin kernel.

Theorem 3.1.12 (Convergence to the boundary). There exists a Borel measurable function \( X_\infty : \Omega \to M_{\text{min}} \) such that for every \( x \in X \) the sequence \( (X_n)_n \) converges \( P_x \)-almost everywhere to \( X_\infty \). For \( x \in X \) define a probability measure on \( M_{\text{min}}(X) \subset M(X) \) by

\[
\nu_x(B) := \mathbb{P}_x(\{ \omega \in \Omega : X_\infty(\omega) \in B \}).
\]

These measures are related to each other as follows

\[
\nu_x(B) = \int_B k(x, \cdot) \, d\nu_{x_0}.
\]

This result implies that

\[
1(x) = 1 = \nu_x(M_{\text{min}}) = \int_{M_{\text{min}}} k(x, \cdot) \, d\nu_{x_0},
\]

so \( \nu_{x_0} \) equals the unique measure \( \nu^1 \) representing the constant function 1. The measure space \( (M(X), \nu_{x_0}) \) is called the Poisson boundary of \( (X, P) \).

Corollary 3.1.13. If \( h : X \to \mathbb{R} \) is a bounded harmonic function, then \( \nu^h \ll \nu^1 \) and there exists a bounded measurable function \( f \in L^\infty(M, \nu_0) \) such that

\[
h(x) = \int_{M(X)} k(x, \cdot) \, d\nu_{x_0} = \int_{M(X)} f \, d\nu_x.
\]

This function \( f \) is \( \nu_{x_0} \)-almost everywhere unique.

Conversely if \( f \in L^\infty(M, \nu_0) \), then the function \( h \) defined by (3.1.4) is bounded harmonic and satisfies

\[
\lim_n h(X_n) = f(X_\infty) \quad \mathbb{P}_{x_0} \text{-almost everywhere}.
\]

Lemma 3.1.14. The sequence of random variables \( (X_n)_n \) converges to \( X_\infty \) \( \mathbb{P}_x \)-almost everywhere if and only if \( g \circ X_n \) converges to \( g \circ X_\infty \) \( \mathbb{P}_x \)-almost everywhere for every continuous function \( g : \tilde{M}(X, P) \to \mathbb{R} \).
3.2. RANDOM WALKS ON QUANTUM GROUPS

Proof. The “only if” implication is trivial by continuity of \( g \). Conversely, by construction of the Martin boundary, the Martin kernels \( \{k(x,\cdot)\}_{x \in X} \) are continuous functions on the Martin compactification \( \tilde{M}(X,P) \). Fix \( x \in X \) arbitrary. Define for \( y \in X \)

\[
A'_y := \{ \omega \in \Omega : \lim_n k(y, X_n(\omega)) \text{ does not exist, or } \lim_n k(y, X_n(\omega)) \neq k(y, X_\infty(\omega)) \}.
\]

By assumption there exists \( A_y \supset A'_y \) measurable with \( P_x(A_y) = 0 \). Put \( A := \bigcup_{y \in X} A_y \).

Since \( X \) is countable, \( P_x(A) \leq \sum_{y} P_x(A_y) = 0 \). By choice of \( A \) it now follows that \( \lim_{n \to \infty} k(x,\cdot) \circ X_n(\omega) = k(x,\cdot) \circ X_\infty(\omega) \), for all \( \omega \in \Omega \setminus A \).

By assumption \( X_\infty \) takes values in the boundary and the Martin kernels separate the points of the boundary, hence it follows that \( \lim_n X_n(\omega) = X_\infty(\omega) \) for all \( \omega \in \Omega \setminus A \). \( \Box \)

3.2 Random walks on quantum groups

Here we present the set-up of random walks on discrete quantum groups and we state the general results about the Martin boundary of such random walks. It is mainly based on the papers [Izu02] and [NT04].

In this section \( G = (C(G), \Delta) \) denotes a compact quantum group with discrete dual \( (\hat{G}, \hat{\Delta}) \).

Definition 3.2.1. Given a normal linear functional \( \varphi : l^\infty(\hat{G}) \to C \) define a convolution operator \( P_\varphi \) by

\[
P_\varphi := (\varphi \otimes \iota)\hat{\Delta} : l^\infty(\hat{G}) \to l^\infty(\hat{G}).
\]

\( P_\varphi \) is called the Markov operator with respect to \( \varphi \). We say that \( P_\varphi \) defines a random walk on \( \hat{G} \).

Clearly \( P_\varphi(c_0(\hat{G})) \subset c_0(\hat{G}) \). Moreover, if \( \varphi \) is a normal state, \( P_\varphi \) is a unital completely positive map. This class of operators turns out to be too large and we need to consider a smaller subclass which behaves in a nicer way.

Recall the operator \( \rho \in U(\hat{G}) \) defined in Subsection 1.4.2.

Definition 3.2.2. Let \( U \) be a finite dimensional representation of \( G \). Define the state

\[
\varphi_U : B(H_U) \to C, \quad T \mapsto \frac{\text{Tr}(T \pi_U(\rho^{-1}))}{d_U}.
\]

Write \( \varphi_s := \varphi_U \circ \pi_s \). Obviously \( \varphi_s \circ \pi_s \) defines a state on \( l^\infty(\hat{G}) \), which we again denote by \( \varphi_s \). Denote \( \mathcal{C} := \text{span}\{ \varphi_s : s \in \text{Irr}(G) \} \), where we take the norm-closure. Clearly, for any \( \varphi \in \mathcal{C} \) there exists a finite (complex) measure \( \mu \) on \( \text{Irr}(G) \) such that \( \varphi = \sum_{s \in \text{Irr}(G)} \mu(s) \varphi_s \).

In that case we write \( \varphi = \varphi_\mu \).

The orthogonality relations imply that

\[
\varphi_s(x)1_s = (h \otimes \iota)(U_s^*(1 \otimes x)U_s).
\]
So $\varphi_s$ is a $\alpha|_{B(H_s)}$-invariant state. In fact, it is the unique $\alpha|_{B(H_s)}$-invariant state ([NT04, §1.4]). We get

$$\varphi_s(x) = \varphi_s(x)\varphi_s(1_s) = (h \otimes \varphi_s)(W^*(1 \otimes x)W).$$  \hfill (3.2.1)

Note that (1.4.14) translates to

$$\hat{\psi}(x) = \sum_{s \in \text{Irr}(\hat{G})} d_s^2 \varphi_s(x).$$

Lemma 3.2.3 ([NT04, Prop. 2.1]). Let $\varphi$ be a normal linear functional on $l^\infty(\hat{G})$, then $P_\varphi(Z(l^\infty(\hat{G}))) \subset Z(l^\infty(\hat{G}))$ if and only if $\varphi \in \mathcal{C}$.

Because of this Lemma we will mainly focus on Markov operators defined by states of the form $\varphi_\mu$ for some probability measure $\mu$. We write $P_\mu := P_{\varphi_\mu}$.

Remark 3.2.4. In the literature, there is another quite common convention. Namely noncommutative random walks defined by the states $\psi_s := d_s^{-1} \text{Tr}(\pi_s(\cdot \rho))$, and $\hat{P}_\mu := \sum_s \mu(s)(\iota \otimes \psi_s)\hat{\Delta}$. Thus slicing in the right leg of the comultiplication with a different state. Of course all results that hold for $P_\mu$ also hold for $\hat{P}_\mu$ and conversely, but one has to be aware on how to translate them. In this thesis we will only work with $P_\mu$.

Example 3.2.5. Let $\Gamma$ be a discrete group, this defines a compact quantum group $G$ with $C(G) = C^*_r(\Gamma)$. Then $l^\infty(\hat{G}) = l^\infty(\Gamma)$ is the discrete quantum group (see Example 1.4.14).

Let $\mu$ be a probability measure on $\Gamma = \text{Irr}(G)$. Since all irreducible representations of $G$ are one-dimensional, $\rho$ equals the identity and thus $\varphi_\mu : l^\infty(\Gamma) \to \mathbb{C}$ equals $\varphi_\mu(f) := \sum_{\gamma \in \Gamma} f(\gamma)\mu(\gamma)$. It follows that

$$P_\varphi(f)(x) = ((\varphi \otimes \iota)\hat{\Delta}(f))(x) = \sum_{y \in \Gamma} \mu(y)f(y^{-1}x) = \sum_{y \in \Gamma} \mu(y^{-1})f(yx),$$

as we had before in Example 3.1.3 but with measure $\mu'$ instead of $\mu$, where $\mu'(y) := \mu(y^{-1})$.

Definition 3.2.6. Let $I = \bigoplus_s I_s$ be the identity in $l^\infty(\hat{G})$. Since $\hat{\Delta}(I) = I \otimes I$ and $\varphi_\mu$ is a state, it follows that $P_\mu(I) = I$. Define scalars $p_\mu(s, t) \in [0, 1]$ by $P_\mu(I_t)I_s = p_\mu(s, t)I_s$, then it follows that $\sum_{t \in \text{Irr}(G)} p_\mu(s, t) = 1$ and $p_\mu(s, t) \geq 0$. Therefore $\{p_\mu(s, t)\}_{s, t \in \text{Irr}(G)}$ defines a discrete Markov chain on $\text{Irr}(G)$. It is called the random walk on the center.

The involution on $l^\infty(\hat{G})^*$ does not behave well on $\mathcal{C}$. Namely $\varphi_\mu^*(x) = d_s^{-1} \text{Tr}(\pi_s(x \rho))$, thus $\mathcal{C}$ is in general not a *-algebra with respect to the involution *. Fortunately there does exist another involution. Define the measure $\mu^*$ by $\mu^*(s) := \mu(\overline{s})$. Denote $\hat{\varphi}_s := \varphi_{\mu^*}$ and extend this linearly to the states $\varphi_\mu$.

Recall if $\varphi$ and $\psi$ are two functionals on $l^\infty(\hat{G})$, their product is defined by $\varphi \psi := (\varphi \otimes \psi)\hat{\Delta}$.

Lemma 3.2.7 ([NT04, Lem. 2.4, Cor. 2.5]). The following properties hold:

(i) $\varphi_U \varphi_V = \varphi_{U \otimes V} = \sum_r \frac{d_U}{d_U d_V} m^r_{U, V} \varphi_r$, for all finite dimensional unitary representations $U, V$ of $G$;
3.2. RANDOM WALKS ON QUANTUM GROUPS

(ii) \( P_\varphi P_\psi = P_{\varphi \psi} \);

(iii) \( \bar{\varphi}_\mu = \varphi_{\bar{\mu}} \);

(iv) the map \( \varphi_\mu \mapsto \bar{\varphi}_\mu \) is anti-multiplicative;

(v) \( p_{\varphi_\mu}(s,t) = \frac{d_s}{d_t} m_{t_1,s}^t \);

(vi) \( p_{\mu}(s,t) = \left( \frac{d_s}{d_t} \right)^2 p_{\mu}(t,s) \);

Definition 3.2.8. The measure \( \mu \) or the operator \( P_\mu \) is called \textit{generating} if for all \( s \in \text{Irr}(G) \) there exists an \( n \in \mathbb{N} \) such that \( \varphi^n_\mu(I_s) > 0 \). \( P_\mu \) or \( \mu \) is called \textit{transient} if the discrete Markov chain on \( \text{Irr}(G) \) with kernel \( \{p_\mu(s,t)\}_{s,t \in \text{Irr}(G)} \) is transient.

It is easy to show that \( p_\mu(s,t) \) is irreducible if and only if \( P_\mu \) is generating. Moreover statement (vi) of the preceding lemma implies that \( \mu \) is transient if and only if \( \bar{\mu} \) is transient and that \( \mu \) is generating if and only if \( \bar{\mu} \) is generating.

Transience almost automatically holds for “true” quantum groups as is shown by the following result.

Lemma 3.2.9 ([NT04, Thm. 2.6]). Suppose \( \mu \) is a probability measure on \( \text{Irr}(G) \). Define constants \( c_{n,r}(\mu) \) by the identity \( \varphi^n_\mu = \sum_{r,t} c_{n,r}(\mu) \varphi_r \varphi_t \), then

\[
c_{n,r}(\mu) = \sum_{t_1, \ldots, t_n \in \text{Irr}(G)} \mu(t_1) \cdots \mu(t_n) m_{t_1, \ldots, t_n}^{r} \frac{d_r}{d_{t_1} \cdots d_{t_n}}.
\]

(3.2.2)

Put \( \lambda := \sum_r \mu(r) \frac{\dim(U_r)}{d_r} \). The following inequalities hold for any \( n \in \mathbb{N} \)

\[
\sum_{r \in \text{Irr}(G)} c_{n,r}(\mu) d_r^{-1} \leq \lambda^n;
\]

\[
p_{\varphi^n_\mu}(s,t) \leq \frac{d_s}{d_t} \frac{\dim(U_s)}{\dim(U_t)} \lambda^n, \quad \text{for any } s, t \in \text{Irr}(G).
\]

In particular if there exists \( s \in \text{supp}(\mu) \) such that \( \dim(U_s) < d_s \), then

\[
\lim_{n \to \infty} \sum_{r \in \text{Irr}(G)} c_{n,r}(\mu) d_r^{-1} = 0; \quad \sum_{n=1}^{\infty} p^n_\mu(s,t) < \infty,
\]

in which case \( \mu \) is transient.

The statements regarding \( p^n_\varphi \) are from [NT04, Thm. 2.6]. The identities of \( c_{n,r} \) are an easy consequence of the computations of the proof of that theorem.

Proof of Lemma 3.2.9. We prove (3.2.2) by induction on \( n \). If \( n = 1 \), this is trivially true as \( m_{t_1}^r = \delta_{t_1,r} \). For \( n > 1 \) we have

\[
\varphi^{n+1}_\mu = \varphi^n_\mu \varphi_\mu = \sum_{r,t} c_{n,r}(\mu) \mu(t) \varphi_r \varphi_t
\]

\[
= \sum_{r,t} \left( \sum_{t_1, \ldots, t_n} \mu(t_1) \cdots \mu(t_n) m_{t_1, \ldots, t_n}^{r} \frac{d_r}{d_{t_1} \cdots d_{t_n}} \right) \mu(t) \left( \sum_s m_{t,s}^{r} \frac{d_s}{d_t} \varphi_s \right)
\]
\[= \sum_s \left( \sum_{r,t_1,\ldots,t_n} \mu(t_1) \cdots \mu(t_n) \mu(t_{n+1}) \frac{d_s}{d_{t_1} \cdots d_{t_n} d_{t_{n+1}}} m_{r,t_1,\ldots,t_n} \right) \varphi_s \]

which completes the induction. Use the estimates of Lemma 1.4.8 to obtain

\[\sum_r c_{n,r}(\mu) d_r^{-1} = \sum_{t_1,\ldots,t_n} \mu(t_1) \cdots \mu(t_n) \left( \sum_r m_{t_1,\ldots,t_n} \right) \frac{1}{d_{t_1} \cdots d_{t_n}} \leq \sum_{t_1,\ldots,t_n} \mu(t_1) \cdots \mu(t_n) \frac{\dim(U_{t_1}) \cdots \dim(U_{t_n})}{d_{t_1} \cdots d_{t_n}} \]

\[= \left( \sum_t \mu(t) \frac{\dim(U_t)}{d_t} \right)^n.\]

For the estimate of \(p_{\varphi,\mu}^n\), observe that Lemma 3.2.7 together with (3.2.2) implies that

\[p_{\varphi,\mu}^n(s, t) = p_{\varphi,\mu}^n(s, t) = \sum_r c_{n,r}(\mu) \frac{d_t}{d_r d_s} m_{r,s}^t \]

\[= \sum_{r_1, \ldots, r_n} \mu(r_1) \cdots \mu(r_n) \frac{1}{d_{r_1} \cdots d_{r_n}} \frac{d_t}{d_s} m_{r_1,\ldots,r_n,s} \]

\[\leq \sum_{r_1, \ldots, r_n} \mu(r_1) \cdots \mu(r_n) \frac{1}{d_{r_1} \cdots d_{r_n}} \frac{d_t}{d_s} \frac{\dim(U_{r_1}) \cdots \dim(U_{r_n}) \dim(U_s)}{\dim(U_t)} = \frac{d_t}{d_s} \frac{\dim(U_s)}{\dim(U_t)} \chi^n.\]

The last part follows from the observation that if \(d_s > \dim(U_s)\) for some \(s \in \text{supp}(\mu)\), then \(0 < \sum_t \mu(t) \frac{\dim(U_t)}{d_t} < 1\). \(\blacksquare\)

**Definition 3.2.10 ([NT04]).** Suppose \(\mu\) is transient then the following operator, the Green kernel, makes sense

\[G_\mu : c_0(\hat{G}) \to l^\infty(\hat{G}), \quad x \mapsto \sum_{n=0}^\infty P_\mu^n(x).\]

If in addition \(\mu\) is generating, the Martin kernel of \(\mu\) given by

\[K_\mu : c_0(\hat{G}) \to l^\infty(\hat{G}), \quad x \mapsto G_\mu(x)(G_\mu(I_0))^{-1}\]

is well-defined. Recall that \(I_0\) is the identity in the trivial representation. By definition, the Martin compactification \(M(\hat{G}, \mu)\) of \(\hat{G}\) with respect to \(\mu\) is the \(C^*\)-subalgebra of \(l^\infty(\hat{G})\) generated by \(c_0(\hat{G})\) and \(K_\mu(c_0(\hat{G}))\). The Martin boundary \(M(\hat{G}, \mu)\) of \(\hat{G}\) is the quotient \(C^\ast\)-algebra \(M(\hat{G}, \mu)/c_0(\hat{G})\).

At this point it might seem unnatural to define the Martin boundary by means of \(K_\mu\) instead of \(K_\mu\). The reason to do this is to ensure compatibility with the classical random walk \(\{p_\mu(s, t)\}_{s, t \in \text{Irr}(\hat{G})}\) on the center, see Remark 3.2.13 below.
Define the Poisson boundary called $\varphi$ discussed in Definition 1.3.6. Here the tensor product is the infinite tensor product of von Neumann algebras as discussed in Definition 1.3.12. Let $\varphi$ be a state on $l^\infty(\hat{G})$. An element $h \in l^\infty(\hat{G})$ is called $\varphi$-harmonic if $P_\varphi(h) = h$. If $x \geq 0$ and $P_\varphi(x) \leq x$, then $x$ is called $\varphi$-superharmonic. Define the Poisson boundary $H^\infty(\hat{G}, P_\varphi) := \{ h \in l^\infty(\hat{G}) : P_\varphi(h) = h \}$. This is a von Neumann algebra with product $x \cdot y := s^* \lim_n P_\varphi^n(xy)$. If $\varphi = \varphi_\mu$ for some $\mu$ probability measure on $\text{Irr}(G)$ we write $H^\infty(\hat{G}, \mu)$ for the Poisson boundary.

Note that in the literature different manifestations of this product can be found. However, all products are the same, because $H^\infty(\hat{G}, \varphi)$ is an operator system in $l^\infty(\hat{G})$. By a result of Choi and Effros [CE77], it admits at most one product turning it into a von Neumann algebra. The nontrivial part is therefore to show existence of such a product. Originally Izumi [Izu02, §2.5] used an ultrafilter and Cesàro-summation to define the product (see also [Izu12, Appendix] for a discussion on the product on $H^\infty(\hat{G}, \varphi)$).

Definition 3.2.12. Let $\varphi$ be a state on $(l^\infty(\hat{G}), \hat{\Delta})$. Denote for $n \in \mathbb{N}$ the unital $*$-homomorphisms

$$j_n : l^\infty(\hat{G}) \to \bigotimes_{-\infty}^{-1}(l^\infty(\hat{G}), \varphi), \quad x \mapsto \cdot \otimes 1 \otimes \hat{\Delta}^{n-1}(x).$$

Here the tensor product is the infinite tensor product of von Neumann algebras as discussed in Definition 1.3.6.

Remark 3.2.13. This algebra $\bigotimes_{-\infty}^{-1}(l^\infty(\hat{G}), \varphi)$ turns out to be the proper generalisation of the algebra of functions on the path space of the random walk. The maps $j_n$ correspond to the duals of the coordinate maps. It is so-called “quantum Markov chain” [AFL82, §2].

The correspondence with random walks on the center is as follows. Given a measure $\mu$ on $\text{Irr}(G)$, consider the classical random walk $\{p_\mu(s, t)\}_{s, t \in \text{Irr}(G)}$ on the discrete space $\text{Irr}(G)$. Let $\Omega$ be the associated path space and let $\mathbb{P}_0$ be the probability measure given by (3.2.3) with starting point 0 corresponding to the trivial representation. As stated in [NT04, Prop. 2.2], there exists an embedding $j : L^\infty(\Omega, \mathbb{P}_0) \to \bigotimes_{-\infty}^{-1}(l^\infty(\hat{G}), \varphi)$ defined by

$$j^\infty(a_{-n} \otimes \cdots \otimes a_{-1}) := j_n(a_{-n}) \cdots j_1(a_{-1}), \quad \text{for } a_{-n}, \ldots, a_{-1} \in Z(l^\infty(\hat{G})).$$

(3.2.3)

The expression $(a_{-n} \otimes \cdots \otimes a_{-1})$ defines a function in $L^\infty(\Omega, \mathbb{P}_0)$ in the following way. If $a \in Z(l^\infty(\hat{G}))$, then $a$ can be written as $a = \sum_s a(s) I_s$ for some scalars $a(s) \in \mathbb{C}$. So we identify $Z(l^\infty(\hat{G})) \cong l^\infty(\text{Irr}(G))$. Given elements $a_i \in Z(l^\infty(\hat{G}))$ define a function by

$$(a_{-n} \otimes \cdots \otimes a_{-1})(\omega) := a_{-n}(\omega_n) \cdots a_{-1}(\omega_1) = a_{-n}(X_n(\omega)) \cdots a_{-1}(X_1(\omega)), \quad (\omega \in \Omega).$$

Since $\hat{M}(\text{Irr}(G), \mu)$ is compact the continuous functions on $\hat{M}(\text{Irr}(G), \mu)$ are bounded. From Lemma 3.2.7, we get

$$K_\mu(I_t)I_s = G_\mu(I_t)G_\mu(I_0)^{-1}I_s = g_\mu(s, t)g_\mu(s, 0)^{-1}I_s = \left(\frac{d_t}{d_s}\right)^2 g_\mu(t, s)g_\mu(0, s)^{-1}I_s = d_t^2k_\mu(t, s)I_s.$$
So define an embedding
\[
\kappa: C(\tilde{M}(\text{Irr}(G), \mu)) \hookrightarrow Z(\ell^\infty(\hat{G}));
\]
\[
1_s \mapsto I_s, \\
k_\mu(t, \cdot) \mapsto \frac{1}{d_t^\mu} K_\mu(I_t).
\]

This prescription fully describes the embedding, because by construction of the Martin compactification the functions \(\{1_s\}_{s \in \text{Irr}(G)}\) and \(\{k_\mu(t, \cdot)\}_{t \in \text{Irr}(G)}\) generate the algebra of continuous functions on the Martin compactification (cf. Remark 3.1.7). Combining these identifications gives the correspondence
\[
j^\infty(f \circ X_n) = j_n(\kappa(f)), \quad \text{for } f \in C(\tilde{M}(\text{Irr}(G), \mu)).
\]

This embedding \(\kappa\) is compatible with integration with respect to \(\mathbb{P}_0\) in the following sense
\[
\int_\Omega \cdot \mathbb{dP}_0 = \varphi^\infty \circ j^\infty.
\]

Indeed, it suffices to verify the identity on functions of the form \((a_{-n} \otimes \cdots \otimes a_{-1}) \in L^\infty(\Omega, \mathbb{P}_0)\). We have
\[
\varphi^\infty \circ j^\infty(a_{-n} \otimes \cdots \otimes a_{-1}) = \varphi^\infty(j_n(a_{-n}) \cdots j_1(a_{-1}))
\]
\[
= (\varphi \otimes \cdots \otimes \varphi)(\hat{\Delta}^{n-1}(a_{-n}) \cdots (1 \otimes \hat{\Delta}(a_{-2}))(1 \otimes \hat{\Delta}(a_{-1}))
\]
\[
= (\varphi \otimes \cdots \otimes \varphi \otimes \hat{\epsilon})(\hat{\Delta}^{n-1}(a_{-n}) \cdots (1 \otimes \hat{\Delta}(a_{-2}))(1 \otimes \hat{\Delta}(a_{-1}))
\]
\[
= \cdots = \hat{\epsilon} \circ P_\varphi(P_\varphi(\cdots (P_\varphi(a_{-n}) a_{-(n-1)}) \cdots a_{-2}) a_{-1}).
\]

As \(\hat{\epsilon}(I_s) = \delta_{s,0}\) and
\[
P_{\varphi_\mu}(a) = \sum_t a(t) P_{\varphi_\mu}(I_t) = \sum_{s,t} a(t) P_{\varphi_\mu}(I_t) I_s = \sum_{s,t} a(t) p_\mu(s,t) I_s, \quad (a \in Z(\ell^\infty(\hat{G}))).
\]

we see that
\[
(3.2.7) = \sum_{s_1, \ldots, s_n} a_{-n}(s_n) \cdots a_{-1}(s_1) p_\mu(0, s_1) \cdots p_\mu(s_{n-1}, s_n)
\]
\[
= \int_\Omega a_{-n}(X_n(\omega)) \cdots a_{-1}(X_1(\omega)) d\mathbb{P}_0 = \int_\Omega a_{-n} \otimes \cdots \otimes a_{-1} d\mathbb{P}_0,
\]
as claimed.

Lemma 3.2.14 ([Izu02, Lem. 2.2]). The Markov operator \(P_\mu\) satisfies for any \(x \in \ell^\infty(\hat{G})\)
\[
\hat{\Delta}P_\mu(x) = (P_\mu \otimes \iota)\hat{\Delta}(x); \quad \alpha_1(P_\mu(x)) = (\iota \otimes P_\mu)\alpha_1(x).
\]
Hence the restrictions of the comultiplication and left adjoint action to the Poisson boundary satisfy
\[ \hat{\Delta}(H^\infty(\hat{G}, \mu)) \subset H^\infty(\hat{G}, \mu) \hat{\otimes} l^\infty(\hat{G}); \quad \alpha_l(H^\infty(\hat{G}, \mu)) \subset L^\infty(G) \hat{\otimes} H^\infty(\hat{G}, \mu) \]
and thus define actions on the Poisson boundary.

Similarly the following result holds for the Martin compactification and boundary.

**Proposition 3.2.15** ([NT04, Thm. 3.5]). The actions \( \hat{\Delta} \) and \( \alpha_l \) satisfy the identities
\[ \hat{\Delta}(K_\psi(x)) = (K_\psi \otimes \iota)(\hat{\Delta}(x))( (K_\psi \otimes \iota)\hat{\Delta}(I_0))^{-1}; \quad \alpha_l(K_\psi(x)) = (\iota \otimes K_\psi)\alpha_l(x) \]
and the following inclusions hold
\[ \hat{\Delta}(\hat{M}(\hat{G}, \mu)) \subset M(\hat{M}(\hat{G}, \mu) \otimes c_0(\hat{G})); \quad \alpha_l(\hat{M}(\hat{G}, \mu)) \subset C(G) \otimes \hat{M}(\hat{G}, \mu). \]

Moreover \( \hat{\Delta} \) and \( \alpha_l \) define actions on the Martin compactification which factor through the Martin boundary.

These right and left adjoint actions obtained from the above lemma and proposition are again denoted by \( \hat{\Delta} \) and \( \alpha_l \).

**Proposition 3.2.16** ([NT04, Thm. 3.3]). For any \( \mu \)-superharmonic element \( x \in l^\infty(\hat{G}) \) there exists a bounded positive linear functional \( \omega_x : \hat{M}(\hat{G}, \mu) \to \mathbb{C} \) such that \( (x, y)_\psi = \omega_x K_\psi(y) \) for all \( y \in c_0(\hat{G}) \).

Conversely if \( \omega : \hat{M}(\hat{G}, \mu) \to \mathbb{C} \) is a bounded positive linear functional, then there exists a unique superharmonic element \( x_\omega \in l^\infty(\hat{G}) \) such that \( (x_\omega, y)_\psi = \omega K_\psi(y) \) for all \( y \in c_0(\hat{G}) \). If \( x_\omega \) is \( \mu \)-harmonic, then \( \omega|_{c_0(\hat{G})} = 0 \).

Moreover, if \( \text{supp}(\mu) \) is finite, then \( x_\omega \) is harmonic if and only if \( \omega|_{c_0(\hat{G})} = 0 \).

We say that the above functional \( \omega_x \) represents the element \( x \). One of the goals in this thesis is to give a good expression of such functionals \( \omega_h \) whenever \( h \) is harmonic. If \( x \) is the unit element, then it is not hard to find a state representing 1 (cf. Proposition 3.2.18 below). However, if \( h \) is an arbitrary harmonic element, we need a convergence to the boundary type of result. This will be discussed in the next chapter.

**Lemma 3.2.17** ([NT04, §3.3]). Consider the modular group \( \{\sigma^\psi_t\}_t \) of the right Haar weight \( \psi \). For any \( z \in \mathbb{C} \) the following identities hold:

1. \( \sigma^\psi_z P_\psi = P_\psi \sigma^\psi_z \);
2. \( \sigma^\psi_z K_\psi = K_\psi \sigma^\psi_z \).

Moreover, if \( t \in \mathbb{R} \), then \( \sigma^\psi_t(c_0(\hat{G})) \subset c_0(\hat{G}) \).

**Proof.** Recall \( \sigma^\psi_z(x) = \rho^{-iz}x\rho^{iz} \) (see (1.4.15)). Whence
\[ \hat{\Delta}(\sigma^\psi_z(x)) = \hat{\Delta}(\rho^{-iz}x\rho^{iz}) = (\rho^{-iz} \otimes \rho^{-iz})\hat{\Delta}(x)(\rho^{iz} \otimes \rho^{iz}) = (\sigma^\psi_{-z} \otimes \sigma^\psi_z)\hat{\Delta}(x). \]
Also

\[ \varphi_s(\sigma_x^{\hat{\psi}}(x)) = \frac{1}{d_s} \text{Tr}(\pi_s(\rho^{-iz} x \rho^{iz} \rho^{-1})) = \frac{1}{d_s} \text{Tr}(\pi_s(x \rho^{iz} \rho^{-1})) = \frac{1}{d_s} \text{Tr}(\pi_s(x \rho^{-1})) = \varphi_s(x), \]

which proves (i). Since \( \sigma_x^{\hat{\psi}}(I_0) = I_0 \), the second is an immediate consequence of (i). To prove the final statement, note that \( \rho \) is self-adjoint. So if \( x \in l^\infty(\hat{G}) \) and \( t \in \mathbb{R} \), then

\[
\|\pi_s(\sigma_t^{\hat{\psi}}(x))\| = \|\rho_s^{-it}\pi_s(x)\rho_s^{it}\| \leq \|\rho_s^{-it}\||\pi_s(x)||\rho_s^{it}\| = \|\pi_s(x)\|,
\]

which completes the proof. \( \blacklozenge \)

It follows from this lemma that \( \{\sigma_t^{\hat{\psi}}\}_t \) defines a strongly continuous one-parameter automorphism group on \( \tilde{M}(\hat{G}, \varphi) \). This restriction to the Martin compactification will be denoted by \( \gamma = \{\gamma_t\}_t \). As \( \gamma_t(c_0(\hat{G})) \subset c_0(\hat{G}) \) the automorphism group \( \gamma \) factors through the Martin boundary. We denote this again by \( \gamma \).

**Proposition 3.2.18** ([NT04, Thm. 3.10]). Let \( \nu \) be a weak* limit point of the sequence \( \{\varphi^n_{\mu}|_{\tilde{M}(\hat{G}, \varphi^n_{\mu})}\}_{n=1}^\infty \). Then \( \nu \) is a \( \gamma \)-KMS state representing the unit \( 1 \in H^\infty(\hat{G}, \mu) \).

Note that this in particular implies that \( \sigma_t^{\nu} = \sigma_t^{\hat{\psi}} \), so the modular groups coincide.
Chapter 4

The Martin boundary for discrete quantum groups

This chapter deals with several questions regarding the Martin boundary of random walks on discrete quantum groups. It can be divided into two parts. The first goal is to get more examples of Martin boundaries for random walks on discrete quantum groups. An interesting class of examples are the $q$-deformations $G_q$ for Lie groups $G$. The Martin boundary for random walks on $\hat{SU}_q(2)$ has been computed, so the next class of examples to study is $\hat{SU}_q(N)$ of which already $\hat{SU}_q(3)$ is interesting. For this is one first needs to understand the random walks on the center of $l^\infty(G_q)$. This is what we do in Section 4.1. The second part of this chapter is devoted to convergence to the boundary. This conjecture gives the quantum analogue of the classical convergence behaviour of paths with respect to random walks. We prove a number of general results and show that random walks on $\hat{SU}_q(2)$ satisfy convergence to the boundary.

4.1 Random walks on the center

A first step in understanding the Martin boundary for random walks on discrete quantum groups is to understand the center of the Martin compactification and boundary. The random walk on the center is a classical random walk and its Martin boundary is a commutative subalgebra in the center of the noncommutative random walk on the whole discrete quantum group. We analyse this classical Martin boundary for $q$-deformed Lie groups.

In this section $G$ is a semisimple simply connected compact Lie group of rank $r$ and $q \in (0, 1)$. We consider the $q$-deformed Lie group $G_q$ as defined in Section 1.6.

The main idea of computing the Martin boundary on the center is by relating the random walk on the center to a random walk on the torus, these methods are inspired by the works [Bia91] and [Col04]. The dual of the torus of $G_q$ can be identified with a lattice $\mathbb{Z}^r$. For random walks on $\mathbb{Z}^r$ the Martin boundary has been described by the Ney–Spitzer theorem. This gives us the tools to give a partial description of the Martin boundary of the random walk on the center.
4.1.1 Random walks on the torus and center

**Notation 4.1.1.** Recall the map (1.6.10). If $\xi$ is a vector of weight $\lambda$ it holds that $K_\alpha \xi = q^{(\alpha,\lambda)/2} \xi$. This motivates the definition of the following map

$$
\theta: \hat{U}_q(\mathfrak{h}) \rightarrow F(P), \quad \theta(K_\alpha)(\lambda) := q^{(\alpha,\lambda)/2},
$$

(4.1.1)

where $F(P)$ denotes the functions on the weight lattice $P$.

Note that $\theta(K_\alpha)$ is unbounded if $\alpha \neq 0$. Indeed, if $\alpha = t_1\alpha_1 + \ldots + t_r\alpha_r$ then for some $i$ the scalar $t_i \neq 0$. Suppose $t_i > 0$, put $x^{(\alpha)} := -n\omega_i$. Since $0 < q < 1$, we have

$$
\lim_{n \rightarrow \infty} \theta(K_\alpha)(x^{(\alpha)}) \lim_n = q^{(\alpha,-n\omega_i)} = \lim_n q^{-nt_i} = \infty.
$$

A similar argument applies if $t_i < 0$. The functions $\theta(K_\alpha)$ are not in $L^\infty(P)$, but they are affiliated to the von Neumann algebra $L^\infty(P)$.

**Lemma 4.1.2.** Suppose that $t \in P_+$ and $\alpha$ is a root, then the following equality holds

$$
\text{Tr}(\pi_t(K_\alpha)) = \sum_{s \in P} \dim V_t(s) q^{(s,\alpha)/2}.
$$

**Proof.** Let $s_j$ be the weights of $V_t$ listed with multiplicities and select an orthonormal basis $(\xi_j)_j$ in $V_t$ satisfying $\xi_j \in V_t(s_j)$. This gives

$$
\text{Tr}(\pi_t(K_\alpha)) = \sum_j (q^{(s_j,\alpha)/2} \xi_j, \xi_j) = \sum_{s \in P} \dim V_t(s) q^{(s,\alpha)/2},
$$

as desired. \(\Box\)

**Notation 4.1.3.** Given a measure $\mu$ on $P_+$ define a measure $\mathbb{P}_\mu$ on $P$ by

$$
\mathbb{P}_\mu(s) := \sum_{t \in P_+} \mu(t) d_t^{-1} \dim V_t(s) q^{(s,-2\rho)}.
$$

**Lemma 4.1.4.** Let $\mu$ be a probability measure on $P_+$, then $\mathbb{P}_\mu$ is also a probability measure. If $\mu$ is finitely supported, then the state $\varphi_{\mu}$ induces the Markov operator $P^{(t)}_{\mu}(\cdot) := (\varphi_{\mu} \otimes t) \hat{\Delta}$ on the torus $\hat{U}_q(\mathfrak{h})$ which corresponds via the duality (4.1.1) to the convolution on $P$ with $\mathbb{P}_\mu$, i.e.

$$
\theta(P^{(t)}_{\mu}(K_\alpha))(x) = \sum_{s \in P} \theta(K_\alpha)(x - s) \mathbb{P}_\mu(s).
$$

(4.1.2)

The condition that $\mu$ has finite support is needed, because the functions $\theta(K_\alpha)$ are unbounded.

**Proof.** $\mathbb{P}_\mu$ is a probability measure, because by Lemma 4.1.2

$$
\sum_{s \in P} \mathbb{P}_\mu(s) = \sum_{s \in P} \sum_{t \in P_+} \mu(t) \left( \sum_{r \in P} \dim V_t(r) q^{(r,-2\rho)} \right)^{-1} \dim V_t(s) q^{(s,-2\rho)} = \sum_{t \in P_+} \mu(t) = 1.
$$

Since $\hat{\Delta}: \hat{U}_q(\mathfrak{h}) \rightarrow \hat{U}_q(\mathfrak{h}) \otimes \hat{U}_q(\mathfrak{h})$, it follows that the restriction $P^{(t)}_{\mu} := P_{\mu}|_{\hat{U}_q(\mathfrak{g})}$ satisfies the property $P^{(t)}_{\mu} (\hat{U}_q(\mathfrak{h})) \subset \hat{U}_q(\mathfrak{h})$. 


To prove Identity (4.1.2), by linearity in \( \mu \) it suffices to deal with the case that \( \mu \) is a point measure. Thus assume that \( \varphi_\mu = \varphi_t \) for some \( t \in P_+ \). Recall that \( f_1 \) is implemented by \( K_{-4\rho} \). From the definition of the contragredient representation, it follows immediately that if \( V \) is a representation of \( G \), then \( \dim V(s) = \dim \hat{V}(-s) \). Using Lemma 4.1.2 and the map \( \theta \), one therefore obtains

\[
\theta(P_\mu^{(t)}(K_\alpha))(x) = \theta((\varphi_t \otimes \iota)\hat{\Delta}(K_\alpha))(x)
\]
\[
= d_t^{-1} \text{Tr} (\pi_t(K_\alpha K_{-4\rho})) \cdot \theta(K_\alpha)(x)
\]
\[
= d_t^{-1} \sum_{s \in P} \dim V_t(s) q^{(s,\alpha+4\rho)/2} q^{(\alpha,x)/2}
\]
\[
= d_t^{-1} \sum_{s \in P} \dim V_t(-s) q^{(-s,\alpha+4\rho)/2} q^{(\alpha,x)/2}
\]
\[
= d_t^{-1} \sum_{s \in P} \dim V_t(s) q^{(s,-2\rho)} q^{(\alpha,x-s)/2}
\]
\[
= \sum_{s \in P} \theta(K_\alpha)(x - s) P_t(s),
\]

as desired. \( \Box \)

As \( P \cong \mathbb{Z} \) is a discrete group, Example 3.1.3 can be applied to the weight lattice \( P \). Recall if \( \mathbb{P} \) and \( \mathbb{P}' \) are measures on \( \mathbb{Z} \), their convolution is defined by \( \mathbb{P} * \mathbb{P}'(x) := \sum_y \mathbb{P}(x - y) \mathbb{P}'(y) \). A direct computation shows that \( p_\mu^\mu(0, x) = p_{\mathbb{P} * \mathbb{P}'}(0, x) \).

**Definition 4.1.5.** If \( \mu \) is a probability measure on \( P_+ \), the probability measure \( \mathbb{P}\mu \) induces a Markov kernel \( \mu^{(t)}_\mu \) on the weight lattice \( P \) by

\[
\mu^{(t)}_\mu(s, t) := \mathbb{P}_\mu(t - s), \quad (s, t \in P).
\]

Because of the duality of Lemma 4.1.4 we call this the random walk on the torus. There is a more conceptual way to introduce this random walk on the torus, the idea is the same as for the random walk on the center.

**Lemma 4.1.6.** For \( \lambda \in P_+ \) and \( \nu \in P \) let \( T^\lambda_\nu : \mathcal{H}_\lambda \rightarrow \mathcal{H}_\nu \) be the orthogonal projection onto the weight space \( \mathcal{H}_\nu(\lambda) := V_\lambda(\nu) = \{ \xi \in \mathcal{H}_\lambda : \pi_\lambda(K_\alpha)\xi = q^{(\alpha,\nu)/2}\xi \text{ for all } \alpha \} \). Write \( T_\nu := \sum T^\lambda_\nu \in \ell^\infty(\hat{G}_q) \). Then \( P_\mu(T_\nu)T_{\nu'} = P_\mu^{(t)}(\nu', \nu)T_{\nu'} \). \( (4.1.3) \)

**Proof.** We start by computing \( \hat{\Delta}(T^\lambda_\nu) \). Suppose \( r_1, r_2 \in P_+ \) and \( m^{\lambda}_{r_1, r_2} \neq 0 \). Denote by \( i^{(r_1, r_2)}_\lambda : \mathcal{H}_\lambda \rightarrow \mathcal{H}_{r_1} \otimes \mathcal{H}_{r_2} \) an embedding. We get

\[
\hat{\Delta}(T^\lambda_\nu) i^{(r_1, r_2)}_\lambda = i^{(r_1, r_2)}_\lambda T^\lambda_\nu = T^\nu_{r_2} i^{(r_1, r_2)}_\lambda = \sum_{\nu' \in P} (T^\nu_{r_2} \otimes T^\nu_{\nu'}) i^{(r_1, r_2)}_\lambda.
\]

It follows that

\[
\hat{\Delta}(T_\nu) = \sum_{\lambda \in P_+} \hat{\Delta}(T^\lambda_\nu) = \sum_{r_1, r_2 \in P_+} \sum_{\nu' \in P} T^\nu_{r_2} \otimes T^\nu_{\nu'}.
\]
CHAPTER 4. THE MARTIN BOUNDARY FOR DQG’S

This gives

$$P_r(T_\nu) = (\varphi_r \otimes \iota) \hat{\Delta}(T_\nu)$$

$$= \sum_{r_1, r_2 \in P, \nu' \in P} \varphi_{r_2}(T_{r_1 \cdot \nu'}) T_{r_2'}$$

$$= \sum_{r_1, r_2 \in P, \nu' \in P} \delta_{r, r_1} d_r^{-1} \dim(V_{r_1}(\nu - \nu')) q^{(\nu - \nu', -2\rho)} T_{r_2'}$$

$$= \sum_{r \in P} \sum_{\nu' \in P} d_r^{-1} \dim(V_r(\nu - \nu')) q^{(\nu - \nu', -2\rho)} T_{r_2'}.$$

So

$$P_\mu(T_\nu) T_{\nu'} = \sum_{r \in P} \mu(r) \left( \sum_{t \in P} \sum_{\nu'' \in P} d_r^{-1} \dim(V_r(\nu - \nu'')) q^{(\nu - \nu'', -2\rho)} T_{r_2'} \right) T_{\nu'}$$

$$= \sum_{r \in P} \mu(r) d_r^{-1} \dim(V_r(\nu - \nu')) q^{(\nu - \nu', -2\rho)} T_{\nu'}$$

$$= \mathbb{P}_\mu(\nu - \nu') T_{\nu'}$$

$$= p^{(t)}_\mu(\nu', \nu) T_{\nu'},$$

as desired. \(\Box\)

From this lemma we see that under the embedding \(l^\infty(P) \hookrightarrow l^\infty(\hat{G}_q), \delta_\nu \mapsto T_\nu\) the random walk on the weight lattice \(P\) defined by \(p^{(t)}_\mu\) corresponds to the random walk on the discrete quantum group \(l^\infty(\hat{G}_q)\) defined by \(P_\mu\).

Lemma 4.1.7. The following identity holds:

$$p^{(t)}_\mu(s, t) = q^{(t-s, -4\rho)} p^{(t)}_\mu(s, t), \quad (s, t \in P).$$

Proof. Recall that \(\bar{s} = -w_0 s\) (where \(w_0\) is the longest element of the Weyl group) and that the dimensions of the weight spaces are invariant under the action of the Weyl group. We now compute

$$\mathbb{P}_\mu(s) = \sum_{t \in P} \mu(t) d_t^{-1} \dim(V_t(s)) q^{(s, -2\rho)}$$

$$= \sum_{t \in P} \mu(t) d_t^{-1} \dim(V_t(\bar{s})) q^{(s, -2\rho)}$$

$$= \sum_{t \in P} \mu(t) d_t^{-1} \dim(V_t(-s)) q^{(-s, -2\rho)} q^{(s, -4\rho)}$$

$$= q^{(s, -4\rho)} \mathbb{P}_\mu(-s),$$

which gives the result. \(\Box\)

As described in Definition 3.2.6 there is also a random walk on the center. We decorate that kernel with an additional \((c)\) to distinguish it from the random walk on the torus.
So the defining relation reads as

\[ P_\mu(I_t)I_s = p_\mu^{(c)}(s, t)I_s. \]

**Lemma 4.1.8.** Let \( \rho \) be half the sum of the positive roots, then the multiplicities satisfy

\[ m_{r,s}^t = \sum_{w \in W} \det(w) \dim V_r((t + \rho) - w(s + \rho)), \quad (r, s, t \in P_+). \]

**Proof.** This is [FH91, Exercise 25.31], we therefore outline a proof. Define

\[ f : \mathfrak{h}^* \to \mathbb{C}, \quad f(x) := \prod_{\alpha \in \Delta} (e^{\alpha(x)} - 1). \]

Steinberg’s and Kostant’s multiplicity formulas [FH91, Prop. 25.29] and respectively [FH91, Prop. 25.21] give

\[ m_{r,s}^t = \sum_{w, w' \in W} \det(ww') f(w(s + \rho) + w'(r + \rho) - t - 2\rho) \]
\[ = \sum_{w \in W} \det(w) \sum_{w' \in W} \det(w') f(w'(r + \rho) - ((t + \rho) - w(s + \rho) + \rho)) \]
\[ = \sum_{w \in W} \det(w) \dim V_s((t + \rho) - w(s + \rho)), \]

as required. \( \Box \)

**Proposition 4.1.9.** Suppose that \( \mu \) is a probability measure on \( P_+ \). The following relation holds for all \( s, t \in P_+ \)

\[ p_\mu^{(c)}(s, t) = \frac{d_t}{d_s} \sum_{w \in W} \det(w) q^{((t+\rho)-w(s+\rho),2\rho)} P_\mu^{(t)}(w(s + \rho) - \rho, t). \]

**Proof.** We combine Lemmas 3.2.7 and 4.1.8 to obtain

\[ p_\mu^{(c)}(s, t) = \sum_{r \in P_+} \mu(r) \frac{d_t}{d_r d_s} m_{r,s}^t \]
\[ = \frac{d_t}{d_s} \sum_{r \in P_+} \mu(r) \frac{1}{d_r} \sum_{w \in W} \det(w) \dim V_r((t + \rho) - w(s + \rho)) \]
\[ = \frac{d_t}{d_s} \sum_{w \in W} \det(w) \sum_{r \in P_+} \mu(r) \frac{1}{d_r} \dim V_r((t + \rho) - w(s + \rho)) q^{((t+\rho)-w(s+\rho),-2\rho)} q^{((t+\rho)-w(s+\rho),2\rho)} \]
\[ = \frac{d_t}{d_s} \sum_{w \in W} \det(w) q^{((t+\rho)-w(s+\rho),2\rho)}, \]

as desired. \( \Box \)
Lemma 4.1.10. Suppose that $\mu$ is a probability measure on $P_+$. The following identities hold:

\[ p_{\varphi_{k+l}}^{(c)}(s, t) = \sum_{r \in P} p_{\varphi_{k+l}}^{(c)}(s, r)p_{\varphi_{k+l}}^{(c)}(r, t) = (p_{\mu}^{(c)})^{k+l}(s, t); \]
\[ p_{\varphi_{k+l}}^{(t)}(s, t) = \sum_{r \in P} p_{\varphi_{k+l}}^{(t)}(s, r)p_{\varphi_{k+l}}^{(t)}(r, t) = (p_{\mu}^{(t)})^{k+l}(s, t). \]

Proof. We use Lemma 3.2.7. For the first one we have

\[ p_{\varphi_{k+l}}^{(c)}(s, t)I_s = P_{\varphi_{k+l}}(I_s) = P_{\varphi_{k+l}}(P_{\varphi_{l}}(I_t))I_s = \sum_r P_{\varphi_{k+l}}(P_{\varphi_{l}}(I_t))I_s = \sum_r p_{\varphi_{k+l}}^{(c)}(s, r)p_{\varphi_{k+l}}^{(c)}(r, t)I_s. \]

By induction it immediately follows that $\sum_r p_{\varphi_{k+l}}^{(c)}(s, r)p_{\varphi_{k+l}}^{(c)}(r, t) = (p_{\mu}^{(c)})^{k+l}(s, t)$. The second identity can be proved by the same argument upon replacing $p_{\mu}^{(c)}$ by $p_{\mu}^{(t)}$ and $I_s$, $I_t$ and $I_r$ by the projections of the form $T_r$ of Lemma 4.1.6.

Proposition 4.1.11. Suppose that $\mu$ is a measure on $P_+$ with $\text{supp}(\mu) \neq \{0\}$. Put $\lambda := \sum t \mu(t) \frac{\dim(V_t)}{d_t}$, then $\lambda < 1$ and

\[ (p_{\mu}^{(c)})^n(s, t) \leq \frac{d_t \dim V_s}{d_s \dim V_t} \lambda^n, \quad \text{for all } s, t \in P_+ \text{ and } n \in \mathbb{N}; \]
\[ (p_{\mu}^{(t)})^n(s, t) \leq q^{(t-s, -2\rho)} \lambda^n, \quad \text{for all } s, t \in P \text{ and } n \in \mathbb{N}. \]

In particular $p_{\mu}^{(c)}$ and $p_{\mu}^{(t)}$ are both transient. Furthermore, if $p_{\mu}^{(c)}$ is irreducible, then $p_{\mu}^{(t)}$ is irreducible.

Proof. We already know that $d_t > \dim(V_t)$ if $t \neq 0$ (see (1.4.3) and (1.6.11)), hence $\lambda < 1$. The first inequality is already proved in Lemma 3.2.9. As $p_{\mu}^{(t)}$ is translation invariant we may take $s = 0$. From Lemmas 4.1.10 and 3.2.9 it follows that

\[ (p_{\mu}^{(t)})^n(0, t) = \sum_{r \in P_+} c_{n, r}(\mu) d_r^{-1} \dim V_r(t) q^{(t, -2\rho)} \]
\[ = \sum_{r \in P_+} \sum_{t_1, \ldots, t_n \in P_+} \frac{\mu(t_1) \cdots \mu(t_n)}{d_{t_1} \cdots d_{t_n}} m_{t_1, \ldots, t_n} \dim V_r(t) q^{(t, -2\rho)} \]
\[ \leq \sum_{t_1, \ldots, t_n \in P_+} \frac{\mu(t_1) \cdots \mu(t_n)}{d_{t_1} \cdots d_{t_n}} \dim V_{t_1 \cdots \otimes t_n}(t) q^{(t, -2\rho)} \]
\[ \leq \sum_{t_1, \ldots, t_n \in P_+} \frac{\mu(t_1) \cdots \mu(t_n)}{d_{t_1} \cdots d_{t_n}} \dim V_{t_1} \cdots \dim V_{t_n} q^{(t, -2\rho)} \]
\[ = q^{(t, -2\rho)} \lambda^n. \]
Here by \( \dim V_{t_1 \otimes \cdots \otimes t_n}(t) \) we mean the dimension of the weight space of weight \( t \) in the representation \( V_{t_1} \otimes \cdots \otimes V_{t_n} \).

Regarding the irreducibility, if \( p_\mu^{(c)} \) is irreducible, then for any \( s \in P_+ \) there exists an \( n \in \mathbb{N} \) such that \( (p_\mu^{(c)})^n(0, s) > 0 \). Lemmas 3.2.7 and 3.2.9 imply

\[
(p_\mu^{(c)})^n(0, s) = \sum_{t \in P_+} c_{n,t}(\mu) \frac{d_s}{d_{t0}} m_{t,0}^s = c_{n,s}(\mu).
\]

Thus for every \( s \in P_+ \) there exists an \( n \) such that \( c_{n,s}(\mu) > 0 \). Now let \( s \in P \) be arbitrary, select \( w \in W \) such that \( w(s) \in P_+ \) and \( n \in \mathbb{N} \) such that \( c_{n,w(s)}(\mu) > 0 \). Then

\[
\mathbb{P}_\mu^n(s) = \sum_{r \in P_+} c_{n,r}(\mu)d_r^{-1} \dim V_r(s) q^{(s,-2\rho)} \geq c_{n,w(s)}(\mu) d_{w(s)}^{-1} \dim V_{w(s)}(s) q^{(s,-2\rho)} > 0,
\]

since \( p_\mu^{(t)}(s, t) = \mathbb{P}_\mu(t - s) \) the kernel \( p_\mu^{(t)} \) is irreducible.

**Notation 4.1.12.** If \( \text{supp}(\mu) \neq \{0\} \), the Green kernels are well-defined by the preceding lemma. Denote these by \( \{g_\mu^{(c)}(s, t)\}_{s,t \in P_+} \) and \( \{g_\mu^{(t)}(s, t)\}_{s,t \in P} \). If in addition \( p_\mu^{(c)} \) is generating the Martin kernels exist. We take as reference point the trivial representation \( 0 \in \text{Irr}(G_q) \). Denote these Martin kernels by \( \{k_\mu^{(c)}(s, t)\}_{s,t \in P_+} \) and \( \{k_\mu^{(t)}(s, t)\}_{s,t \in P} \). Note that if we let \( \mathbb{G}_\mu(s) := \sum_{n=0}^{\infty} \mathbb{P}_\mu^n(s) \), then we have \( g_\mu^{(t)}(0, s) = \mathbb{G}_\mu(s) \).

The following corollary is an immediate consequence of Proposition 4.1.9 and Lemma 4.1.10.

**Corollary 4.1.13.** The following relation between the Green kernels holds

\[
g_\mu^{(c)}(s, t) = \frac{d_t}{d_s} \sum_{w \in W} \det(w) q^{((t+\rho)-(w(s+\rho))2\rho)} g_\mu^{(t)}(w(s + \rho) - \rho, t).
\]

### 4.1.2 The Martin compactification

The correspondence obtained in the previous subsection now allows to apply the Ney–Spitzer theorem to the random walk on the torus and transport it to the random walk on the center. This gives a partial description of the Martin compactification of the random walk on \( P_+ \).

**Notation 4.1.14.** We identify the weight lattice as \( P \cong \mathbb{Z}^r \subset \mathbb{R}^r \) by mapping \( \omega_i \mapsto e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \) where the 1 is on the \( i \)-th spot. Via this identification \( P_+ \cong \mathbb{N}^r \). The standard inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^r \) given by \( \langle e_i, e_j \rangle := \delta_{ij} \) restricts to \( P \). It satisfies \( \langle \omega_i, \omega_j \rangle = \delta_{ij} \). Write

\[
S_+^{r-1} := \{ x \in \mathbb{R}^r : \|x\| = 1, x_i \geq 0 \text{ for all } i \};
\]

\[
\text{int}(S_+^{r-1}) := \{ x \in \mathbb{R}^r : \|x\| = 1, x_i > 0 \text{ for all } i \}
\]

for the positive sector of the \((r - 1)\)-sphere and its interior. The boundary equals \( \partial S_+^{r-1} := S_+^{r-1} \setminus \text{int}(S_+^{r-1}) \). The norm \( \|\cdot\| \) indicates the \( l^2 \) norm on \( \mathbb{R}^r \). Thus if \( t = (t_1, \ldots, t_r) \), then
Suppose that
\[\text{Definition 4.1.16.}\]
To be able to use the Ney–Spitzer theorem, we need to introduce some functions.

(iii) Let 
\[\text{Definition 4.1.16.}\]

\[\text{Lemma 4.1.15.}\] The space \(\overline{P}_+\) has the following properties:

(i) \(\overline{P}_+\) is compact;

(ii) \(P_+\) is open in \(\overline{P}_+\);

(iii) \(P_+\) is dense in \(\overline{P}_+\).

**Proof.** (i) We prove that \(\overline{P}_+\) is sequentially compact. For this let \((x_n)_n \subset \overline{P}_+\) be a sequence. If \(|\{n : x_n \in S_{t+}^{-1}\}| = \infty\), then because \(S_{t+}^{-1}\) is compact, the sequence \((x_n)_n\) contains a convergent subsequence. If this is not the case, then by taking a subsequence, we may assume that \(x_n \in P_+\) for all \(n\). If there exists an \(M > 0\) such that \(|\{n : \|x_n\| \leq M\}| = \infty\), then because the set \(\{s \in P_+ : \|s\| \leq M\}\) is finite, the sequence \((x_n)_n\) contains a constant subsequence. However, if for all \(M\) we have \(|\{n : \|x_n\| \leq M\}| < \infty\), then there exists a subsequence \((x_{n_m})_m\) such that \(\|x_{n_m}\| \geq m\) for all \(m\). Now consider \(\frac{x_{n_m}}{\|x_{n_m}\|} \in S_{t+}^{-1}\). Since \(S_{t+}^{-1}\) is compact the sequence \((x_{n_m})_m\) contains a further subsequence \((x_{n_{m_k}})_k\) for which 
\[\frac{x_{n_{m_k}}}{\|x_{n_{m_k}}\|}\] converges to some \(y \in S_{t+}^{-1}\). Hence \(\text{b-lim}_k x_{n_{m_k}} = y\).

(ii) Every singleton \(\{t\}\) for \(t \in P_+\) is open in \(\overline{P}_+\), hence \(P_+\) is open in \(\overline{P}_+\).

(iii) Let \(y = y_1 e_1 + \ldots + y_r e_r \in S_{t+}^{-1}\). Put \(x_n := \lfloor ny_1 \rfloor \omega_1 + \ldots + \lfloor ny_r \rfloor \omega_r\). Then \(\|x_n\| \uparrow \infty\) and \(\text{lim}_n \frac{x_n}{\|x_n\|} = y\), that is \(\text{b-lim}_n x_n = y\).

So in the topological space \(\overline{P}_+\), the subspace \(S_{t+}^{-1}\) is really the boundary of \(P_+\). Write
\[\partial P_+ := S_{t+}^{-1} \cup P_+ \cup S_{t+}^{-1} = \overline{P}_+\.

To be able to use the Ney–Spitzer theorem, we need to introduce some functions.

**Definition 4.1.16.** Suppose that \(d \geq 2\) and \(\mathbb{P}\) is a probability measure on \(\mathbb{Z}^d\). Define 
\[\tau_{\mathbb{P}} : \mathbb{R}^d \to [0, \infty] \text{ by } \tau_{\mathbb{P}}(c) := \sum_{\xi \in \mathbb{Z}^d} e^{\langle c, \xi \rangle} \mathbb{P}(\xi).\]
Here \(\langle \cdot, \cdot \rangle\) denotes the standard inner product on \(\mathbb{R}^d\). Let 
\[T_{\mathbb{P}} := \{u \in \mathbb{R}^d : \tau_{\mathbb{P}}(u) \leq 1\}\] and \(\partial T_{\mathbb{P}} := \{u \in T_{\mathbb{P}} : \tau_{\mathbb{P}}(u) = 1\}\). The **gradient** of \(\tau_{\mathbb{P}}\) equals
\[
\text{grad } \tau_{\mathbb{P}} : \mathbb{R}^d \to \mathbb{R}^d, \quad \text{grad } \tau_{\mathbb{P}}(c) := \sum_{\xi \in \mathbb{Z}^d} \xi e^{\langle c, \xi \rangle} \mathbb{P}(\xi).
\]

The **mean** of \(\mathbb{P}\) is defined as 
\[m_{\mathbb{P}} := \sum_{\xi \in \mathbb{Z}^d} \xi \mathbb{P}(\xi).\]
If \(\mu\) is a probability measure on \(P_+\) and we consider the associated \(\mathbb{P}_\mu\) on \(P \cong \mathbb{Z}^r\), then we drop \(\mathbb{P}\) from the notation, so for example we write 
\[\tau_{\mu} := \tau_{\mathbb{P}_\mu}, T_{\mu} := T_{\mathbb{P}_\mu}.\] If in addition it is clear which \(\mu\) is considered, we also remove the subscript \(\mu\).
Lemma 4.1.17. Suppose that \( d \geq 2 \) and \( \mathbb{P} \) is a probability measure on \( \mathbb{Z}^d \) which has the following properties:

(i) for all \( \xi \in \mathbb{Z}^d \) there exists an \( n \geq 0 \) such that \( \mathbb{P}^n(\xi) > 0 \);

(ii) \( m_\mathbb{P} \neq 0 \);

(iii) \( \mathbb{P} \) is finitely supported.

Then

(i) the set \( \mathcal{T}_\mathbb{P} \) is compact and convex;

(ii) the gradient \( \text{grad} \, \mathcal{T}_\mathbb{P} \) exists everywhere on \( \mathcal{T}_\mathbb{P} \) and does not vanish on the boundary \( \partial \mathcal{T}_\mathbb{P} \);

(iii) the map \( \partial \mathcal{T}_\mathbb{P} \to S^{d-1} \), \( u \mapsto \frac{\text{grad} \, \mathcal{T}_\mathbb{P}(u)}{\|\text{grad} \, \mathcal{T}_\mathbb{P}(u)\|} \) is a homeomorphism.

The inverse of the above homeomorphism is denoted by \( c_\mathbb{P} : S^{d-1} \to \mathcal{T}_\mathbb{P} \) or \( c \) if it is clear from the context which measure \( \mathbb{P} \) is considered. This homeomorphism can continuously be extended to \( c : \mathbb{R}^d \setminus \{0\} \to \mathcal{T}_\mathbb{P} \) by \( x \mapsto c(\frac{x}{\|x\|}) \). This extension is again denoted by \( c \).

Proof. See [Hen63, §2.5]. See also [NS66, §1] for a discussion on requirement (iii), it can be slightly relaxed.

\[ \Box \]

Lemma 4.1.18. Let \( C \) be a Weyl chamber in \( \text{it}^* \) and \( x \in \text{it}^* \). Denote by \( \overline{C} \) the closure of \( C \). The following holds:

(i) if \( x = 0 \), then \( \sum_{w \in W} w(x)e^{(w(x),y)} = 0 \);

(ii) if \( y \in \overline{C} \), then \( \sum_{w \in W} w(x)e^{(w(x),y)} \in \overline{C} \);

(iii) if \( x \neq 0 \) and \( y \in C \), then \( \sum_{w \in W} w(x)e^{(w(x),y)} \in C \);

(iv) if \( y \in \Omega_\alpha \), then \( \sum_{w \in W} w(x)e^{(w(x),y)} \in \Omega_\alpha \).

Proof. Assertion (i) is trivial. For the other assertions, since we average over the Weyl group, we can assume without loss of generality that \( x \in \overline{C}_+ \setminus \{0\} \). We can also assume that \( y \in \overline{C}_+ \). Indeed, if \( y \in C \) for an arbitrary Weyl chamber \( C \), then there exists an element \( w' \in W \) such that \( w'C = C_+ \). We get

\[
\sum_{w \in W} w(x)e^{(w(x),y)} = \sum_{w \in W} w(x)e^{(w'(w(x)),(w'(y)))} = w'^{-1}\left( \sum_{w \in W} w(x)e^{(w(x),(w'(y)))} \right),
\]

which reduces everything to the case \( y \in \overline{C}_+ \).

Recall, if \( z \in \text{it}^* \), then \( z \in \overline{C}_+ \) if and only if \( (z,\alpha_i) \geq 0 \) for all \( i = 1, \ldots, r \). Similarly \( z \in C_+ \) if and only if \( (z,\alpha_i) > 0 \) for all \( i = 1, \ldots, r \). We compute

\[
\left( \sum_{w \in W} w(x)e^{(w(x),y)}, \alpha_i \right) = \sum_{w \in W} (x, w^{-1}(\alpha_i))e^{(x,w^{-1}(y))} = \sum_{w \in W} (x, w(\alpha_i))e^{(x,w(y))}. \quad (4.1.4)
\]

\[ \Box \]
The element \( s_i \in W \) is the reflection in the root \( \alpha_i \), so \( w(\alpha_i) = \alpha \) if and only if \( w(s_i(\alpha_i)) = -\alpha \). Moreover \( \Delta = \Delta_+ \sqcup -\Delta_+ \), here \( \sqcup \) indicates the disjoint union. Thus we obtain

\[
(4.1.4) = \sum_{\alpha \in \Delta_+} \left( \sum_{w \in W} (x, \alpha)e^{(x,w(y))} + \sum_{w \in W} (x,-\alpha)e^{(x,w(y))} \right) = \sum_{\alpha \in \Delta_+} (x, \alpha) \left( \sum_{w \in W} e^{(x,w(y))} - e^{(x,w(s_i(\alpha)))} \right).
\]

We know \( s_i(\omega_j) = \omega_j - \frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} \alpha_i = \omega_j - \delta_{ij} \alpha_i \). Therefore if \( y = y_1 \omega_1 + \ldots + y_r \omega_r \), then \( s_i(y) = y - y_i \alpha_i \). Hence

\[
\sum_{w \in W} e^{(x,w(y))} - e^{(x,w(s_i(y)))} = \sum_{w \in W} e^{(x,w(y))} - e^{(x,w(y-y_i \alpha_i))} = (1 - e^{-(x,y_i \alpha_i)}) \sum_{w \in W} e^{(x,w(y))}.
\]

A study of \( f := (1 - e^{-(x,y_i \alpha)}) \) gives the result. Namely, if \( y \in \overline{C}_+ \), then \( y_i \geq 0 \) for all \( i \) and hence for all \( i \) and all \( \alpha \in \Delta_+ \) the scalar product \( (x, y_i \alpha_i) \geq 0 \). Therefore \( f \geq 0 \), thus (4.1.4) is positive and assertion (ii) holds. If \( y \in \overline{C}_+ \), then \( y_i > 0 \) for all \( i \). By assumption \( x \neq 0 \), therefore for some \( \alpha \in \Delta_+ \) the inequality \( (x, y_i \alpha) > 0 \) holds and thus for this \( \alpha \) the quantity \( f > 0 \), hence (4.1.4) is strictly positive for all \( i \) and thus (iii) holds. Recall that \( y \in \Omega_{\alpha_i} \) if and only if \( (y, \alpha_i) = 0 \), that is if and only if \( y_i = 0 \). Thus \( (x, y_i \alpha_i) = 0 \), whence \( f = 0 \) and thus (4.1.4) =0, hence (iv) holds.

It is automatic that the mean of \( \mathbb{P}_\mu \) is nonzero.

**Corollary 4.1.19.** If \( \text{supp}(\mu) \neq \{0\} \), then \( \mathfrak{m}_\mu \in C_+ \). In particular \( \mathfrak{m}_\mu \neq 0 \).

**Proof.** The dimensions of the weight spaces of a representation are invariant under the action of the Weyl group. Therefore

\[
\sum_{s \in P} s\mathbb{P}_\mu(s) = \sum_{t \in P_+} \mu(t) \frac{1}{d_t} \sum_{s \in P} s \dim V_t(s) q^{(s,-2\rho)}
= \sum_{t \in P_+} \mu(t) \frac{1}{d_t} \frac{1}{|W|} \sum_{w \in W} \sum_{s \in P} w(s) \dim V_t(w(s)) q^{(w(s),-2\rho)}
= \sum_{t \in P_+} \mu(t) \frac{1}{d_t} \frac{1}{|W|} \sum_{s \in P} \dim V_t(s) \sum_{w \in W} w(s) e^{(w(s),-2\log(q)\rho)}.
\]

Since \( 0 < q < 1 \), \( \log(q) < 0 \) and thus \( -2\log(q)\rho \in C_+ \). By Lemma 4.1.18 it follows that each term \( \sum_{w \in W} w(s) q^{(w(s),-2\rho)} \in \overline{C}_+ \). Since \( \text{supp}(\mu) \neq \{0\} \), there exists \( t, s \in P_+ \setminus \{0\} \) such that \( \mu(t) > 0 \) and \( \dim V_t(s) > 0 \). Now the third assertion of Lemma 4.1.18 implies that for this \( s \) one has

\[
\sum_{w \in W} w(s) e^{(w(s),-2\log(q)\rho)} \in C_+.
\]
4.1. RANDOM WALKS ON THE CENTER

Hence $m_{\mu} = \sum_{s \in P} sP_{\mu}(s) \in C_+$.

The formulation of the following result is taken from [Woe00, Thm. 25.15].

Proposition 4.1.20 (Ney–Spitzer). Suppose that $\mathbb{P}$ is a probability measure on $\mathbb{Z}^d$ for $d \geq 2$. This defines a random walk as described in Example 3.1.3. Denote the associated Green kernel by $g(x,y)$. Suppose that $\mathbb{P}$ satisfies the conditions of Lemma 4.1.17, then the Green kernel $g$ has the following asymptotic behaviour

$$g(\zeta, \xi) \sim \| \text{grad} \tau_{\mathbb{P}}(c_{\mathbb{P}}(\xi)) \|^{-1} \sqrt{\det Q_{\zeta}} (2\pi \| \xi \|)^{-(d-1)/2} e^{c_{\mathbb{P}}(\xi, \zeta - \xi)},$$

as $\| \xi \| \to \infty$ and $\frac{\xi}{\| \xi \|} \to x$ for some $x \in S^{d-1}$.

Here $Q_y$ is some $(d-1) \times (d-1)$-matrix. The precise definition is unimportant for us, the only property which is relevant is that $y \mapsto \det Q_y$ is continuous and bounded away from zero, see [Woe00, Lem. 25.14].

Proof of Proposition 4.1.20. For the proof see [Woe00, Thm. 25.15] or the original paper [NS66, Thm. 1.2].

Notation 4.1.21. Define a map $T: P \to P$ by $T\omega_i := \alpha_i$ and extend it to $\mathfrak{h}^* \to \mathfrak{h}^*$. For $x = x_1\alpha_1 + \ldots + x_r\alpha_r \in \mathfrak{h}^*$ write $H_x := x_1H_1 + \ldots + x_rH_r \in \mathfrak{h}$. For $s \in P_+$ denote by $\chi_s$ the character of the irreducible representation of $G$ labelled by $s$.

If $\mu$ is a probability measure on $P_+$ extend the Martin kernel $k_{\mu}^{(c)}(s,\cdot)$ to the boundary of $\overline{P}_+$ in the following way

$$k_{\mu}^{(c)}(s,t) := \begin{cases} \frac{g_{\mu}^{(c)}(s,t)}{g_{\mu}^{(c)}(0,t)} & \text{if } t \in P_+; \\ \frac{1}{d_\lambda} \chi_s(\exp(H_{-2\log(q)\rho + Tc_{\mu}(t)})) & \text{if } t \in S^{d-1}_+, \end{cases}$$

where $c_{\mu}: S^{d-1}_+ \to \partial \mathcal{T}_{\mu}$ is the function induced by $\mathbb{P}_{\mu}$ as defined in Lemma 4.1.17. In the proofs below we usually just write $c$ instead of $c_{\mu}$.

Lemma 4.1.22. Let $C$ be a Weyl chamber in $i^*$ and suppose $x \neq 0$. Then

(i) $x \in \overline{C}$ if and only if $Tc_{\mu}(x) - 2\log(q)\rho \in \overline{C};$

(ii) $x \in C$ if and only if $Tc_{\mu}(x) - 2\log(q)\rho \in C;$

(iii) $x \in \Omega_\alpha$ if and only if $Tc_{\mu}(x) - 2\log(q)\rho \in \Omega_\alpha$.

Proof. Recall $c(x)$ is the unique element in $\mathcal{T}_{\mu}$ such that $\frac{x}{\|x\|} = \frac{\text{grad} \tau_{\mu}(c(x))}{\|\text{grad} \tau_{\mu}(c(x))\|}$. From the definition of $\text{grad} \tau_{\mu}(c(x))$ it follows that

$$\text{grad} \tau_{\mu}(c(x)) = \frac{1}{|W|} \sum_{t \in P} \sum_{\lambda \in P_+} \mu(\lambda) \frac{1}{d_\lambda} \dim V_{\lambda}(t) \sum_{w \in W} w(t)e^{(w(t), Tc_{\mu}(x) - 2\log(q)\rho)}.$$}

Since $\frac{1}{|W|}, \mu(\lambda), d_\lambda^{-1}$ and $\dim V_{\lambda}(t)$ are all nonnegative, from Lemma 4.1.18 we get that

$$Tc(x) - 2\log(q)\rho \in \overline{C} \Rightarrow \text{grad} \tau_{\mu}(c(x)) \in \overline{C};$$

$$Tc(x) - 2\log(q)\rho \in C \Rightarrow \text{grad} \tau_{\mu}(c(x)) \in C.$$
Since $x$ is a positive multiple of $\text{grad} \, \tau_{\mu}(c(x))$ it thus follows that

$$Tc(x) - 2 \log(q)\rho \in \bar{C} \Rightarrow x \in \bar{C};$$
$$Tc(x) - 2 \log(q)\rho \in C \Rightarrow x \in C,$$

which proves the “if” part of implications (i) and (ii).

For the converse note that $C = it^* \setminus (\bigcup_{C^* \neq C} C^*) = \bigcap_{C^* \neq C}(it^* \setminus C^*)$. Implication (4.1.5) can also be stated as

$$x \notin \bar{C}' \Rightarrow Tc(x) - 2 \log(q)\rho \notin \bar{C}'$$

and thus

$$x \in C \Rightarrow x \notin \bar{C}' \quad \text{for all } C' \neq C$$
$$\Rightarrow Tc(x) - 2 \log(q)\rho \notin \bar{C}' \quad \text{for all } C' \neq C$$
$$\Rightarrow Tc(x) - 2 \log(q)\rho \in C,$$

which proves “only if” of (ii). Since the function $c: it^* \to T_{\mu}$ is continuous, the “only if” of (i) follows immediately.

For part (iii) note that $\Omega_{\alpha} = \{x \in it^* : (\alpha, x) = 0\}$. Write $I_{\alpha}$ for the collection of Weyl chambers $C$ such that $(x, \alpha) > 0$ for all $x \in C$. Then $\Omega_{\alpha} = (\bigcap_{C \in I_{\alpha}} C) \cup - (\bigcap_{C \in I_{\alpha}} C)$. Now the equivalence follows from (i).

**Lemma 4.1.23.** For $x \in it^*$ the following are equivalent:

(i) $x \in \Omega_{\alpha}$ for some $\alpha \in \Delta$;

(ii) $\sum_{w \in W} \det(w)e^{(w(\rho),x)} = 0$.

**Proof.** Recall the identity

$$\sum_{w \in W} \det(w)e^{(w(\rho),x)} = e^{-(\rho,x)} \prod_{\alpha \in \Delta_+} (e^{(\alpha,x)} - 1),$$

see for instance [FH91, Lem. 24.3]. Since $e^{-(\rho,x)} > 0$ one has $\sum_{w \in W} \det(w)e^{(w(\rho),x)} = 0$ if and only if $(\alpha, x) = 0$ for some $\alpha \in \Delta_+$. Thus if and only if $x \in \Omega_{\alpha}$ for some $\alpha \in \Delta$.

**Lemma 4.1.24.** Suppose that $\mu$ is a finitely supported probability measure on $P_+$ with $\text{supp}(\mu) \neq \{0\}$ such that $p^{(c)}_{\mu}$ is irreducible, then the Martin kernel $k^{(c)}_{\mu}(s, \cdot)$ is continuous on $P_+ \cup \text{int}(S_+^{r-1})$.

**Proof.** Fix a weight $s \in P_+$. Since $P_+$ is discrete, it is immediate that $k^{(c)}_{\mu}(s, \cdot)$ is continuous on $P_+$. Also since $x \mapsto H_x$ is continuous as a map $\mathfrak{h}^* \to \mathfrak{h}$, and the functions $c, \exp, \chi_s$ are continuous, the Martin kernel is continuous on $S_+^{r-1}$. It remains to show that if $x \in \text{int}(S_+^{r-1})$ and $(t_n)_n \subset P_+$ with $b\text{-lim}_n t_n = x$, then $\lim_{n} k^{(c)}_{\mu}(s, t_n) = k^{(c)}_{\mu}(s, x)$. Since the function $c$ is a homeomorphism and $x \mapsto \sqrt{\det Q_x}$ is continuous, it follows from
Proposition 4.1.20 that for any \( w \in W \) and \( s \in P_+ \)

\[
\lim_{n \to \infty} \frac{g_{t}^{(t)}(w(s + \rho) - \rho, t_n)}{||t_n||^{-(r-1)/2}e^{c(t_n), t_n}} = ||\text{grad } \tau(c(x))||^{-1}\sqrt{\det Q_x(2\pi)^{-(r-1)/2}e^{c(x), w(s+\rho) - \rho}}.
\]

(4.1.6)

This is finite and nonzero by assumptions on \( \mu \) and properties of \( \tau \) and \( \det Q_x \). Note that \( \langle \omega_i, \omega_j \rangle = \delta_{ij} = (\alpha_i, \omega_j) = (T\omega_i, \omega_j) \) and thus \( T \) satisfies \( \langle x, y \rangle = (Tx, y) \) for all \( x, y \in \mathfrak{h}^* \). Combination of (4.1.6) and the relation between the random walks on the torus and center (cf. Lemma 4.1.13) gives

\[
\lim_{n \to \infty} k_{t}^{(c)}(s, t_n) = \lim_{n \to \infty} \frac{g_{t}^{(c)}(s, t_n)}{g_{t}^{(c)}(0, t_n)}
\]

\[
= \lim_{n \to \infty} \frac{d_s}{d_n} \sum_{w \in W} \det(w)q^{(t_n+\rho)-w(s+\rho), 2\rho} g_{t}^{(t)}(w(s + \rho) - \rho, t_n)
\]

\[
= \lim_{n \to \infty} \frac{\sum_{w \in W} \det(w)q^{(-w(s+\rho), 2\rho)} ||\text{grad } \tau(c(x))||^{-1}\sqrt{\det Q_x(2\pi)^{-(r-1)/2}e^{c(x), w(s+\rho) - \rho}}}{d_s \sum_{w \in W} \det(w)q^{(-w(s+\rho), 2\rho)} e^{c(x), w(s+\rho)}}
\]

\[
= \frac{1}{d_s} \sum_{w \in W} \det(w)e^{(w(s+\rho), -2\log(q)\rho + Tc(x))}.
\]

(4.1.7)

Since \( x \in \text{int}(S_+^{r-1}) \) we conclude from Lemma 4.1.22 that \( -2\log(q)\rho + Tc(x) \in C_+ \). Now Lemma 4.1.23 implies that \( \sum_{w \in W} \det(w)e^{(w(\rho), -2\log(q)\rho + Tc(x))} \neq 0 \). By the Weyl character formula (see e.g. [FH91, Ch. 24] or [BtD85, §VI.1]) Identity (4.1.7) equals exactly \( \frac{1}{dx} \chi_s(\exp(H-2\log(q)\rho + Tc(x))) \).

\[ \Box \]

**Remark 4.1.25.** The above result does not prove that the Martin kernel extends continuously to \( P_+ \cup S_+^{r-1} \). The problem is that if \( u \in \partial S_+^{r-1} \), that is if \( u \in \Omega_\alpha \) for some \( \alpha \), then the denominator \( \sum_{w \in W} \det(w)e^{(w(\rho), -2\log(q)\rho + Tc(x))} \) vanishes (cf. Lemma 4.1.23) and we no longer can invoke the Weyl character formula.

Biane [Bia91] had a similar problem when computing the Martin boundary for the standard representation in the classical case of SU(N). He could also only show continuity on the interior of \( S_+^{N-2} \).

In case of the standard representation of \( SU_q(N) \), so for \( \mu = \delta_{\omega_1} \), the Martin boundary has been computed by Collins [Col04] and this is indeed \( S_+^{N-2} \). Unfortunately, his proof cannot easily be adapted to arbitrary representations of \( SU_q(N) \) or other Lie groups due to the fact that many combinatorial properties of the standard representation are used.

**Lemma 4.1.26.** The extended kernels \( \{ k_{t}^{(c)}(s, \cdot) \}_{s \in P_+} \) separate the points of \( \partial P_+ = S_+^{r-1} \).

**Proof.** The Peter–Weyl theorem shows that the characters of the irreducible represent-
ations form an orthonormal basis in the $L^2$-space of class functions. Therefore by the Stone–Weierstrass theorem the characters $\{\chi_s\}_{s \in P_+}$ separate the conjugacy classes of $G$. Assume that $x, y \in \partial P_+$ with $x \neq y$. To complete the proof it thus suffices to show that the group elements $h_x := \exp(H_{-2\log(q)\rho + Tc(x)})$ and $h_y := \exp(H_{-2\log(q)\rho + Tc(y)})$ lie in different conjugacy classes of $G$.

From Lemma 4.1.22 we see that both $Tc(x) - 2\log(q)\rho$, $Tc(y) - 2\log(q)\rho \in \hat{C}_+$. It is known (see [BtD85, Lem. V.4.3]) that if $z_1, z_2 \in \hat{C}_+$ and there exists $w \in W$ such that $wz_1 = z_2$, then $z_1 = z_2$. Therefore every $w \in W$ satisfies $w(Tc(x) - 2\log(q)\rho) \neq Tc(y) - 2\log(q)\rho$. The Weyl group also acts on the full group $G$ and it follows that $h_x$ and $h_y$ are in different orbits of the Weyl group. Now recall the known fact ([BtD85, Lem. IV.2.5]) that two elements of the torus $g, h \in T$ are conjugated in $G$ if and only if they are in the same orbit of $W$ acting on $G$.

\[ \square \]

**Theorem 4.1.27.** Let $\mu$ be a finitely supported measure on $P_+$ with $\text{supp}(\mu) \neq \{0\}$ such that the random walk $p^{(c)}_\mu$ is irreducible, then $\text{int}(S_{\mu}^{-1})$ is a subspace of the Martin boundary of the classical random walk on $P_+$ with Markov kernel $\{p^{(c)}_\mu(s, t)\}_{s, t \in P_+}$ induced by the quantum random walk on the center of the discrete dual of $G_q$.

**Proof.** From Lemma 4.1.15 it follows that $\overline{P_+}$ is compact. By Lemma 4.1.24, the Martin kernels extend continuously from $P_+$ to $P_+ \cup \text{int}(S_{\mu}^{-1})$ and Lemma 4.1.26 shows that the kernels separates the points of $S_{\mu}^{-1}$.

Note that this subspace is independent of the choice of the measure $\mu$.

### 4.2 Partial results on the Martin boundary of $\text{SU}_q(N)$

In the previous section we determined most of the structure of the Martin boundary $M(\text{Irr}(G_q), \mu)$. Since there is an embedding $M(\text{Irr}(G_q), \mu) \hookrightarrow Z(M(\hat{G}_q, \mu))$ one expects that $M(\hat{G}_q, \mu)$ forms a $C_0$-algebra over $M(\text{Irr}(G_q), \mu)$. Here we restrict ourselves to $G = \text{SU}(N)$ and we try to determine the fibers over the points $x \in M(\text{Irr}(\text{SU}_q(N)), \mu)$. This turns out to be a very hard task. We are not able to complete this and we can only unveil a small part of the structure of the full Martin boundary $M(\text{SU}_q(N), \mu)$.

#### 4.2.1 Preliminaries on $\text{SU}_q(N)$

We start with some additional definitions of $q$-deformed enveloping algebras. Fix for the rest of this section $q \in (0, 1)$ and $N \in \mathbb{N} \setminus \{0, 1\}$. Consider the Lie algebra $\mathfrak{su}_N$, its complexification equals $\mathfrak{sl}_N$. The latter has Cartan matrix $(a_{ij})_{i, j=1}^{N-1}$, where

\[
a_{ij} = \begin{cases} 
2 & \text{if } i = j; \\
-1 & \text{if } i = j \pm 1; \\
0 & \text{otherwise},
\end{cases}
\]
so the matrix equals

\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \vdots \\
0 & -1 & 2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 2 & -1 \\
0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}.
\]

As described in the preliminaries (cf. Subsection 1.6.2) we obtain a Hopf $\ast$-algebra $\tilde{U}_q(\mathfrak{su}_N)$. For convenience we sometimes use the notation $X^+_i := E_i$ and $X^-_i := F_i$. With this notation the comultiplication, counit, antipode and unitary antipode are given by

\[
\begin{align*}
\hat{\Delta}(K_\omega) &= K_\omega \otimes K_\omega, & \hat{\Delta}(X^+_i) &= X^+_i \otimes K_{\alpha_i} + K_{\alpha_i}^{-1} \otimes X^+_i, \\
\varepsilon(K_\omega) &= 1, & \varepsilon(X^+_i) &= 0, \\
S(K_\omega) &= K_\omega^{-1}, & S(X^+_i) &= -q^\pm 1 X^+_i, \\
R(K_\omega) &= K_\omega^{-1}, & R(X^+_i) &= -X^+_i.
\end{align*}
\]

Consider the Lie group $GL(N)$. It has compact form $U(N) \subset GL(N)$. Note that $U(N)$ is not simple. We apply the Drinfeld-Jimbo $q$-deformation, for this we follow [KS97, §7.3.1]. Define $\tilde{U}_q(\mathfrak{u}_N)$ to be the algebra with generators $K_i, K_i^{-1}, i = 1, \ldots, N$ and $E_i, F_i, i = 1, \ldots, N - 1$ subject to the following relations

\[
\begin{align*}
K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\
K_i E_j K_i^{-1} &= q^{(\delta_{ij} - \delta_{i,j+1})/2} E_j, & K_i F_j K_i^{-1} &= q^{(-\delta_{ij} + \delta_{i,j+1})/2} F_j, \\
E_i F_j - F_j E_i &= \delta_{ij} K_i^{2} K_{i+1}^{2} - K_i^{-2} K_{i+1}^{2} / q - q^{-1}, \\
E_i E_j - E_j E_i &= F_i F_j - F_j F_i, & \text{if } |i - j| \geq 2, \\
E_i^2 E_{i+1} - (q + q^{-1}) E_i E_{i+1} E_i + E_{i+1} E_i^2 &= 0, \\
F_i^2 F_{i+1} - (q + q^{-1}) F_i F_{i+1} F_i + F_{i+1} F_i^2 &= 0.
\end{align*}
\]

These last three identities are the Serre relations explicitly written out. Write again $X^+_i := E_i$ and $X^-_i := F_i$. The algebra $\tilde{U}_q(\mathfrak{u}_N)$ also becomes a Hopf $\ast$-algebra when equipped with the additional structure

\[
\begin{align*}
K_i^\ast &= K_i, & (X_i^\pm)^\ast &= X_i^\mp, \\
\hat{\Delta}(K_i) &= K_i \otimes K_i, & \hat{\Delta}(X_i^\pm) &= X_i^\pm \otimes K_i K_{i+1}^{-1} + K_i^{-1} K_{i+1} \otimes X_i^\pm, \\
\varepsilon(K_\omega) &= 1, & \varepsilon(X_i^\pm) &= 0, \\
S(K_\omega) &= K_\omega^{-1}, & S(X_i^\pm) &= -q^\pm 1 X_i^\pm, \\
R(K_\omega) &= K_\omega^{-1}, & R(X_i^\pm) &= -X_i^\pm.
\end{align*}
\]

\footnote{[KS97] write $\tilde{U}_q(\mathfrak{gl}_N)$ instead of $\tilde{U}_q(\mathfrak{u}_N)$ since they do not define a $\ast$-structure.}
Define a map $\psi: \bar{U}_q(\mathfrak{su}_N) \to \bar{U}_q(\mathfrak{u}_N)$ by

$$
\psi(K_i) := K_iK_{i+1}^{-1}, \quad \psi(E_i) := E_i, \quad \psi(F_i) := F_i \quad \text{for } i = 1, \ldots, N - 1. \quad (4.2.1)
$$

One easily verifies that $\psi$ preserves the defining relations of $\bar{U}_q(\mathfrak{su}_N)$ and thus that $\psi$ is an injective Hopf $*$-algebra morphism. Therefore if $\pi$ is a representation of $\bar{U}_q(\mathfrak{u}_N)$, the composition $\psi \circ \pi$ defines a representation of $\bar{U}_q(\mathfrak{su}_N)$.

**Definition 4.2.1.** A Gelfand–Tsetlin tableau or GT tableau for short is an array of the form

$$
\mathbf{r} = (r_{i,j})_{i,j} = \begin{pmatrix}
    r_{11} & r_{12} & \cdots & r_{1,N-1} & r_{1,N} \\
    r_{21} & r_{22} & \cdots & r_{2,N-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{N-1,1} & r_{N-1,2} & \cdots & r_{N,1} \\
\end{pmatrix}
$$

satisfying the condition

$$
r_{ij} \geq r_{i+1,j} \geq r_{i,j+1} \quad \text{for all } i, j. \quad (4.2.2)
$$

Denote $\mu_i(\mathbf{r}) := \sum_{j=1}^i r_{N+1-i,j}$ for the sum of all entries in the $i$-th row counted from below. Set $\mu_0(\mathbf{r}) := 0$. If it is clear which tableau is considered we remove $\mathbf{r}$ and simply write $\mu_i = \mu_i(\mathbf{r})$. We denote the $i$-th row of $\mathbf{r}$ (counted from above) by $\mathbf{r}_i := (r_{i,1}, \ldots, r_{i,N+1-i})$.

These conventions are the same as in [CP08]. Also [KS97, §7.3] introduces GT tableaux, but the labelling of the indices is turned around. The entry $r_{i,j}$ of [CP08] corresponds to $m_{j,N+1-i}$ of [KS97], or equivalently $m_{k,l}$ corresponds to $r_{N+1-l,k}$.

Let $\pi^{\mathbf{u}_N}_\lambda$ be a representation of $\bar{U}_q(\mathfrak{u}_N)$ on $V$ and $\lambda \in \mathbb{Z}^N$. A nonzero vector $\xi \in V$ has weight $\lambda = \lambda_1 e_1 + \ldots + \lambda_N e_N$ if $\pi^{\mathbf{u}_N}_\lambda(K_i) \xi = q^{\lambda_i/2} \xi$ for all $i$. The admissible unitary irreducible finite dimensional representations of $\bar{U}_q(\mathfrak{u}_N)$ are classified by their highest weight. The set of highest weights is given by $\{\lambda_1 e_1 + \ldots + \lambda_N e_N \in \mathbb{Z}^N : \lambda_i \geq \lambda_{i+1} \}$ (see [KS97, §7.3]). The representation corresponding to $\lambda$ is denoted by $\pi^{\mathbf{u}_N}_\lambda$, it acts on the space $V^{\mathbf{u}_N}_\lambda$.

**Lemma 4.2.2** ([Jim86b], [UTS90, Thm. 2.11]). Let $\lambda = a_1 e_1 + \ldots + a_N e_N$ be a highest weight of $\bar{U}_q(\mathfrak{u}_N)$. The set of GT tableaux $\{\mathbf{r} : r_{1i} = a_i, \ i = 1, \ldots, N\}$ forms an orthonormal basis in $V^{\mathbf{u}_N}_\lambda$. The generators of $\bar{U}_q(\mathfrak{u}_N)$ act on this basis as follows

$$
\pi^{\mathbf{u}_N}_\lambda(K_i) \mathbf{r} = q^{(\mu_i(\mathbf{r}) - \mu_{i-1}(\mathbf{r}))/2} \mathbf{r}, \quad i = 1, \ldots, N;
$$

$$
\pi^{\mathbf{u}_N}_\lambda(E_i) \mathbf{r} = \sum_{j=1}^i A_{i,j}(\mathbf{r}) \mathbf{r}^{i,j}, \quad i = 1, \ldots, N - 1;
$$

$$
\pi^{\mathbf{u}_N}_\lambda(F_i) \mathbf{r} = \sum_{j=1}^i A_{i,j}(r_{1,j}) \mathbf{r}_{i,j}, \quad i = 1, \ldots, N - 1.
$$

Here $\mathbf{r}^{i,j}$ and $\mathbf{r}_{i,j}$ are the arrays obtained by replacing the entry $r_{N+1-i,j}$ of $\mathbf{r}$ by respectively...
$r_{N+1-i,j} + 1$ and $r_{N+1-i,j} - 1$. The quantity $A_{k,j}(r)$ equals

$$A_{k,j}(r) := \left( -\frac{\prod_{i=1}^{k+1}[r_{N-k,i} - r_{N+1-k,j} - i + j]q \prod_{i=1}^{k-1}[r_{N+2-k,i} - r_{N+1-k,j} - i + j - 1]q}{\prod_{i=1}^{k}[r_{N+1-k,i} - r_{N+1-k,j} - i + j]q[r_{N+1-k,i} - r_{N+1-k,j} - i + j - 1]q} \right)^{1/2}.$$ 

If $r^{i,j}$ or $r_{i,j}$ are not GT tableaux (meaning they do not fulfil Condition (4.2.2)), these arrays are omitted from the respective sums.

**Notation 4.2.3.** For $\lambda = a_1e_1 + \ldots + a_ne_N$, the condition that $r_{1i} = a_i$ for all $i$, is stated as $r_{1\cdot} = \lambda$.

**Corollary 4.2.4.** For $\lambda = a_1e_1 + \ldots + a_ne_N$ consider the GT tableaux

$$r_\lambda := \begin{pmatrix} a_1 & a_2 & \ldots & a_{N-1} & a_N \\ a_1 & a_2 & \ldots & a_{N-1} \\ \vdots & \vdots & \ddots \\ a_1 & a_2 \end{pmatrix}, \quad s := \begin{pmatrix} a_1 & a_2 & \ldots & a_{N-1} & a_N \\ a_2 & a_3 & \ldots & a_N \\ \vdots & \vdots & \ddots \\ a_{N-1} & a_N \end{pmatrix}.$$ 

Then $r_\lambda$ is a highest weight vector of $V_\lambda$ and $s$ is a lowest weight vector of $V_\lambda$.

**Proof.** It immediately follows that $\mu_i(r_\lambda) - \mu_{i-1}(r_\lambda) = a_i$. Hence by Lemma 4.2.2 $\pi^{u_N}_\lambda(K_i)r_\lambda = q^{a_i/2}r_\lambda$. Similarly $\mu_i(s) - \mu_{i-1}(s) = a_{N-i}$. \(\Box\)

Consider the Hopf algebra morphism $\psi$ of (4.2.1). The composition $\pi_\lambda := \pi_{\lambda N}^u \circ \psi$ gives a representation of $\hat{U}_q(\mathfrak{su}_N)$ on $V_{\lambda N}^{u_N}$. The highest weight vector $r_\lambda$ is a generating vector for $\pi_{\lambda N}^{u_N}$. Since $\psi(E_i) = E_i$ and $\psi(F_i) = F_i$, the vector $r_\lambda$ is also generating for $\pi_\lambda$ and thus $\pi_\lambda$ is irreducible. Different (inequivalent) representations $\pi_\lambda$ of $\hat{U}_q(\mathfrak{su}_N)$ can yield equivalent representations of $\hat{U}_q(\mathfrak{su}_N)$ (basically this is due to the determinant which can be nontrivial in $GL(N)$ and always equals 1 in $SL(N)$). This occurs precisely for pairs $\lambda = a_1e_1 + \ldots + a_ne_N$ and $\lambda' = a'_1e_1 + \ldots + a'_ne_N$ for which there exists an $a \in \mathbb{Z}$ such that $a_i = a'_i + a$ for all $i$. Furthermore if $\pi$ is an irreducible highest weight representation of $\hat{U}_q(\mathfrak{su}_N)$, then it is of the form $\pi = \pi_{\lambda N}^{u_N}$ for some highest weight $\lambda$ of $\mathfrak{u}_N$ ([Wor88, Thm. 1.5]). It follows that the weight lattice $P$ of $\mathfrak{su}_N$ can be identified as $\mathbb{Z}^N/\langle e_1 + \ldots + e_N = 0 \rangle$. The positive cone $P_+$ corresponds to

$$\{a_1e_1 + \ldots + a_ne_N \in \mathbb{Z}^N : a_1 \geq \ldots \geq a_N \}/\langle e_1 + \ldots + e_N = 0 \rangle.$$ 

Recall the fundamental weights $\omega_i$ (cf. Subsection 1.6.1). Define elements $e_i \in P$, $i = 1, \ldots, N$ as follows

$$e_1 := \omega_1, \quad e_i := \omega_i - \omega_{i-1}, \quad \text{if } i \neq 1, N, \quad e_N := -\omega_N.$$ 

This gives the explicit identification of $P$. We have proved the following lemma.
Lemma 4.2.5. If $\lambda = a_1 e_1 + \ldots + a_N e_N \in P_+$ is a positive weight of $\hat{U}_q(\mathfrak{su}_N)$. The GT tableaux $\{ r : r_{ii} = a_i, \ i = 1, \ldots, N \}$ form an orthonormal basis in $V^\omega_{\mathfrak{su}_N}$. The generators of $\hat{U}_q(\mathfrak{su}_N)$ act on this basis as follows

\[
\pi_\lambda(K_i) r = q^{(\mu_i(r) - \mu_{i+1}(r) - \mu_{i-1}(r))/2} r, \quad i = 1, \ldots, N;
\]

\[
\pi_\lambda(E_i) r = \sum_{j=1}^i A_{i,j}(r) r^{i,j}, \quad i = 1, \ldots, N - 1;
\]

\[
\pi_\lambda(F_i) r = \sum_{j=1}^i A_{i,j}(r_{i,j}) r_{i,j}, \quad i = 1, \ldots, N - 1.
\]

Here $r^{i,j}$, $r_{i,j}$ and $A_{i,j}(r)$ are as in Lemma 4.2.2.

For $\hat{U}_q(\mathfrak{su}_N)$ we write $V_\lambda := V^\omega_{\mathfrak{su}_N}$. If we want to emphasize the Hilbert space structure which makes $\{ r : r_{1,1} = \lambda \}$ an orthonormal basis in $V_\lambda$, we use $\mathcal{H}_\lambda$ to indicate this space. The operators $K_\omega$ act as $\pi(K_\omega)\xi = q^{\text{wt}(\xi)/2}\xi$ whenever $\omega \in P$ and $\xi$ is a weight vector.

Corollary 4.2.6. The following identity holds:

\[\text{wt}(r) = \sum_{i=1}^{N-1} (2\mu_i(r) - \mu_{i+1}(r) - \mu_{i-1}(r)) \omega_i.\]

Proof. For $\mathfrak{su}_N$ it holds $(\alpha_i, \alpha_i) = 2$. Therefore the defining relation of the fundamental weights implies $(\omega_i, \alpha_j) = \frac{\langle \omega_i, \alpha_j \rangle}{(\alpha_i, \alpha_i)} = \delta_{ij}$. Thus

\[
\left( \sum_{i=1}^{N-1} 2(\mu_i(r) - \mu_{i+1}(r) - \mu_{i-1}(r)) \omega_i, \alpha_j \right) = 2\mu_j(r) - \mu_{j+1}(r) - \mu_{j-1}(r).
\]

In which case for $i = 1, \ldots, N - 1$ it holds

\[q^{(\text{wt}(r), \alpha_i)/2} r = \pi_\lambda(K_i) r = q^{(2\mu_i(r) - \mu_{i+1}(r) - \mu_{i-1}(r))/2} r.
\]

Since the functionals $\{ \omega \mapsto (\omega, \alpha_j) \}_{j=1}^{N-1}$ separate the points of $P$ and $q$ is not a root of unity, the claim follows.

Recall that $(\cdot, \cdot)$ denotes the Killing form and $\rho := \omega_1 + \ldots + \omega_{N-1}$. By elementary computations one readily proves the following lemma.

Lemma 4.2.7. The following identities hold for $SU(N)$:

(i) $(e_i, e_j) = \delta_{i,j} - \frac{1}{N};$

(ii) $(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i = j - 1, j + 1; \\ 0 & \text{otherwise}; \end{cases}$

(iii) $(\omega_i, \alpha_j) = \delta_{i,j};$
4.2. PARTIAL RESULTS ON THE MARTIN BOUNDARY OF $\text{SU}_q(N)$

(iv) $(\omega_i, e_j) = \begin{cases} 1 - \frac{i}{N} & \text{if } i \geq j; \\ -\frac{i}{N} & \text{if } i < j; \end{cases}$

(v) $(\omega_i, \omega_j) = \min(i, j) - \frac{ij}{N}$;

(vi) $(e_i, 2\rho) = N + 1 - 2i$;

(vii) $(\omega_i, 2\rho) = i(N-i)$;

(viii) $(\alpha_i, 2\rho) = 2$.

**Lemma 4.2.8.** Suppose that $r$ is a GT tableau with $r_1. = \lambda = \lambda_1 e_1 + \ldots + \lambda_N e_N$. The following identities hold:

(i) $(\text{wt}(r), \alpha_j) = 2\mu_j(r) - \mu_{j+1}(r) - \mu_{j-1}(r)$;

(ii) $(\text{wt}(r), \omega_j) = \mu_j(r) - \frac{1}{N} \mu_N(r)$;

(iii) $(\text{wt}(r), e_j) = \mu_j(r) - \mu_{j-1}(r) - \frac{1}{N} \mu_N(r)$;

(iv) $(\lambda, \omega_j) = (\lambda_1 + \ldots + \lambda_j) - \frac{j}{N} (\lambda_{j+1} + \ldots + \lambda_N) = (r_{1,1} + \ldots + r_{1,j}) - \frac{j}{N} \mu_N(r)$;

(v) $(\lambda, e_j) = \lambda_j - \frac{1}{N} (\lambda_1 + \ldots + \lambda_{N-1}) = r_{1,j} - \frac{1}{N} \mu_N(r)$.

**Proof.** Part (i) is immediate from Corollary 4.2.6 and $(\alpha_i, \omega_j) = \delta_{i,j}$. A straightforward computation gives

$$(\text{wt}(r), \omega_j) = \sum_{i=1}^{N-1} (2\mu_i - \mu_{i-1} - \mu_{i+1})(\omega_i, \omega_j)$$

$$= \sum_{i=1}^{j} (i - \frac{ij}{N})(2\mu_i - \mu_{i-1} - \mu_{i+1}) + \sum_{i=j+1}^{N-1} (j - \frac{ij}{N})(2\mu_i - \mu_{i-1} - \mu_{i+1})$$

$$= \left(2(1 - \frac{j}{N}) + (2 - \frac{2j}{N})\right)\mu_1 + \left((4 - \frac{4j}{N}) - (1 - \frac{j}{N}) - (3 - \frac{3j}{N})\right)\mu_2 + \ldots$$

$$+ \left((2(j - 1) - \frac{2(j-1)j}{N}) - (j - 2 - \frac{2j}{N}) - (j - \frac{2j}{N})\right)\mu_{j-1}$$

$$+ \left((2j - \frac{2jj}{N}) - (j - \frac{j-1}{N}) - (j - \frac{j+1}{N})\right)\mu_j$$

$$+ \left((2j - \frac{2(j+1)j}{N}) - (j - \frac{j}{N}) - (j - \frac{j+2}{N})\right)\mu_{j+1} + \ldots$$

$$+ \left((2j - \frac{2(N-2)j}{N}) - (j - \frac{(N-3)j}{N}) - (j - \frac{(N-1)j}{N})\right)\mu_{N-2}$$

$$+ \left((2j - \frac{2(N-1)j}{N}) - (j - \frac{(N-2)j}{N})\right)\mu_{N-1} - \left(j - \frac{(N-1)j}{N}\right)\mu_N$$

$$= \mu_j - \frac{j}{N} \mu_N,$$

which proves (ii). As $e_j = \omega_j - \omega_{j-1}$ one has

$$(\text{wt}(r), e_j) = \mu_j(r) - \frac{j}{N} \mu_N(r) - \mu_{j-1}(r) + \frac{j-1}{N} \mu_N(r) = \mu_j(r) - \mu_{j-1}(r) - \frac{1}{N} \mu_N(r).$$
To prove the fourth assertion, let \( r_\lambda \) be the highest weight vector of \( V_\lambda \). Then \( \text{wt}(r_\lambda) = \lambda \) and the first rows of the tableaux \( r_\lambda \) and \( r \) coincide, that is \( (r_\lambda)_{i,j} = r_{1,i} \). Thus \( \mu_N(r) = \mu_N(r_\lambda) \). From Corollary 4.2.4 we see that \( \mu_i(r_\lambda) = \lambda_1 + \ldots + \lambda_i \), therefore

\[
(\lambda, \omega_j) = (\text{wt}(r_\lambda), \omega_j) = \mu_j(r_\lambda) - \frac{j}{N} \mu_N(r_\lambda) = \lambda_1 + \ldots + \lambda_j - \frac{j}{N} \mu_N(r) = (r_{1,1} + \ldots + r_{1,j}) - \frac{j}{N} \mu_N(r).
\]

The last equality follows from (iv) and the fact that \( e_j = \omega_j - \omega_{j-1} \).

We rephrase some of the results of Chapter 1 in terms of GT tableaux. If \( r \) and \( s \) are GT tableaux, we write \( m_{r,s}^\lambda \) for the matrix units defined by the vectors \( r \) and \( s \) in the representation \( V_\lambda \) whenever \( \lambda = r_{1,.} = s_{1,.} \). Similarly we use \( u_{r,s}^\lambda \) to indicate the corresponding matrix coefficient.

Every GT tableau \( r \) with \( r_{1,.} = \omega_1 \) is of the form \( r = (r_{ij})_{ij} \), where

\[
r_{ij} = \begin{cases} 
1 & \text{if } 1 \leq i \leq k \text{ and } j = 1; \\
0 & \text{otherwise}
\end{cases}
\]

for some \( 1 \leq k \leq N \). Note that the weight of such a vector \( r \) equals \( e_{N+1-k} \).

In order to uniquely define the map \( J \) of Definition 1.6.4 for \( SU_q(N) \), let

\[
J: \mathcal{H}_\lambda \rightarrow \mathcal{H}_{\bar{\lambda}}, \quad r_\lambda \mapsto r_{-\lambda}
\]

Here \( r_{-\lambda} \) is the weight vector of the lowest weight of the representation \( V_{-\lambda} \) indicated by the GT tableau. Then \( J \) can be extended by equivariance. It is easy to verify that \( J \) is isometric and satisfies

\[
J(\xi \otimes \zeta) = J(\zeta) \otimes J(\xi).
\]

This identity follows basically from the fact that \( \pi_1 \otimes \pi_2 \cong \pi_2 \otimes \pi_1 \) or equivalently from \( (\hat{R} \otimes \hat{R}) \hat{\Delta} = \hat{\Delta}^{op} \hat{R} \). Indeed, if \( r_\lambda \) and \( r_\mu \) are the highest weight vectors of \( V_\lambda \) and respectively \( V_\mu \), then \( J(r_\lambda \otimes r_\mu) = J(r_{\lambda+\mu}) = r_{-\lambda-\mu} = r_{-\lambda} \otimes r_{-\mu} \).

Now

\[
X(J(\xi) \otimes J(\eta)) = X(1)J(\xi) \otimes X(2)J(\eta) = J(\hat{R}(X^*_1)\xi) \otimes J(\hat{R}(X^*_2)\eta)
\]

\[
= J(\hat{R}(X^*_2)\xi) \otimes J(\hat{R}(X^*_1)\eta) = J(\hat{R}(X^*_1)\eta) \otimes \hat{R}(X^*_2)\xi)
\]

\[
= J(\hat{R}(X^*)(\eta \otimes \xi)) = XJ(\eta \otimes \xi)
\]

and the claim follows by equivariance.

**Lemma 4.2.9.** The following formulas hold for \( SU_q(N) \):

1. \( f_z(u_{r,s}^\lambda) = \delta_{r,s} q^{(\text{wt}(r)-2\rho)} \);
2. \( \mathcal{F}(u_{r,s}^\lambda) = d_\lambda^{-1} q^{(\text{wt}(r)+\text{wt}(s),\rho)} m_{r,s}^\lambda \) and \( \mathcal{F}(u_{r,s}^\lambda) = d_\lambda^{-1} q^{-\text{wt}(r)+\text{wt}(s),\rho} m_{r,s}^\lambda \), here \( \mathcal{F} \) indicates the Fourier transform and \( J \) is the map defined in Definition 1.6.4.
(iii) if \( \varphi = \varphi_\mu \) is a state on \( l^\infty(SU_q(N)) \), then

\[
G_\varphi(I_0) = \sum_{\lambda,r} g_\varphi(\lambda,0)d_\lambda q^{(\text{wt}(r),-2\rho)} \mathcal{F}(u^\lambda_{r,r}) = \sum_{\lambda,r} g_\varphi(\lambda,0)d_\lambda q^{(\text{wt}(r),2\rho)} \mathcal{F}(u^\lambda_{J(r),J(r)}),
\]

here \( I_0 \in B(H_0) \) denotes the identity in the trivial representation;

(iv) \( d_\lambda = \sum_{r,r_1 = \lambda} q^{(\text{wt}(r),-2\rho)} \);

(v) For \( k = 1, \ldots, N \) let \( r_{e_{N+1-k}} \) be the GT tableau (4.2.3) and \( r_{-e_{N+1-k}} \) be the GT tableau with entries

\[
(r_{-e_k})_{i,j} = \begin{cases} 0 & \text{if } 1 \leq i \leq k \text{ and } j = N + 1 - i; \\ 1 & \text{otherwise} \end{cases}
\]

then \( J(r_{e_k}) = (-1)^{i-1}r_{-e_i} \), where \( J \) is as in Definition 1.6.4.

**Proof.** (i) follows immediately from the definition of \( f_\varepsilon \) (cf. Definition 1.4.10) and the fact that \( \rho_\lambda \) is given by \( \pi_\lambda(K_{-\rho}) \).

The first formula in (ii) follows from (1.4.16) combined with (i). The second is immediate from the first as \( J^2 = I \).

Observe that (ii) can be rewritten as

\[
m^\lambda_{J(r),J(s)} = d_\lambda q^{-(\text{wt}(r)+\text{wt}(s),\rho)} \mathcal{F}(u^\lambda_{r,s}).
\]

Using this we obtain

\[
G_\varphi(I_0) = \sum_{\lambda} G_\varphi(I_0)I_\lambda = \sum_{\lambda} g_\varphi(\lambda,0)I_\lambda = \sum_{\lambda} \sum_{r,r_1 = \lambda} g_\varphi(\lambda,0)m^\lambda_{r,r} = \sum_{\lambda} \sum_{r,r_1 = \lambda} g_\varphi(\lambda,0)m^\lambda_{J(r),J(r)} + \sum_{\lambda} \sum_{r,r_1 = \lambda} g_\varphi(\lambda,0)d_\lambda q^{(\text{wt}(r),2\rho)} \mathcal{F}(u^\lambda_{r,r}),
\]

which is exactly the first equality in (iii). The other equality follows by making the substitutions \( \lambda \to \bar{\lambda} \) and \( r \to J(r) \).

(iv) As \( \rho_\lambda \) equals \( \pi_\lambda(K_{-\rho}) \) we get

\[
d_\lambda = \text{tr}(\rho_\lambda) = \sum_{r,r_1 = \lambda} \langle r, \pi_\lambda(K_{-\rho})r \rangle = \sum_{r,r_1 = \lambda} q^{(\text{wt}(r),-2\rho)}.
\]

(v) Let \( \lambda \in \{e_1, \ldots, e_N\} \). By construction we get \( \text{wt}(r_\lambda) = \lambda \). Thus by definition of \( J \) it follows that \( J(r_{e_1}) = r_{-e_1} \). The remaining \( k \geq 2 \) can be proved by induction on \( k \). Indeed,

\[
J(r_{e_k}) = J((A_{k-1,1}(r_{ek}))^{-1} \pi_{\omega_1}(F_{k-1})r_{e_{k-1}}) = - (A_{k-1,1}(r_{ek}))^{-1} \pi_{\omega_{N-1}}(E_{k-1})J(r_{e_{k-1}}) = - (A_{k-1,1}(r_{ek}))^{-1} \pi_{\omega_{N-1}}(E_{k-1})(-1)^{k-1}r_{-e_{k-1}} = (-1)^{k-1}(A_{k-1,1}(r_{ek}))^{-1}A_{k-1,k-1}(r_{-e_{k-1}})r_{-e_k}.
\]
A direct computation shows that \( A_{k-1,1}(r_{e_k}) = A_{k-1,k-1}(r_{-e_{k-1}}) = 1 \).

4.2.2 Clebsch–Gordan coefficients

We take a step back and we introduce the Clebsch–Gordan coefficients for representations of \( q \)-deformations of generic semisimple simply connected compact Lie groups. We use the notation of Section 1.6. Before specializing again to \( SU_q(N) \) we state some of the elementary properties of the Clebsch–Gordan coefficients.

**Notation 4.2.10.** Consider the \( q \)-deformation \( G_q \) of a compact semisimple simply connected Lie group and \( q \in (0, 1) \). For each positive weight \( \lambda \in P_+ \) fix an orthonormal basis of \( V_\lambda \) consisting of weight vectors by \( \{ \xi_i^\lambda \}_{i=1}^{\dim V_\lambda} \). The representation \( \pi_\lambda \otimes \pi_\mu \) decomposes into irreducible components as \( \bigoplus_{\nu \in P_+} m_{\lambda,\mu}^{\nu} \pi_\nu \), where \( m_{\lambda,\mu}^{\nu} \) indicates the multiplicity. Therefore there are two orthonormal bases of \( V_\lambda \otimes V_\mu \), namely \( \{ \xi_i^\lambda \otimes \xi_j^\mu \ : \ i = 1, \ldots, \dim V_\lambda, \ j = 1, \ldots, \dim V_\mu \} \) and \( \{ i\xi_k^\nu : \nu \in P_+, l = 1, \ldots, m_{\lambda,\mu}^{\nu}, k = 1, \ldots, \dim V_\nu \} \). These bases are related by the Clebsch–Gordan coefficients

\[
C_q(\lambda, \mu, \nu; \xi_i^\lambda; \xi_j^\mu; \xi_k^\nu) := \langle \xi_i^\lambda \otimes \xi_j^\mu; i\xi_k^\nu \rangle.
\]

(4.2.5)

The highest weights \( \lambda, \mu \) and \( \nu \) of the representations \( V_\lambda, V_\mu \) and \( V_\nu \) are contained in the labels of the vectors, so we abbreviate (4.2.5) by \( C_q(\xi_i^\lambda, \xi_j^\mu, i\xi_k^\nu) \). The following orthogonality relations hold for the Clebsch–Gordan coefficients ([KS97, Eq. (7.18), (7.19)]):

\[
\sum_{i,j} C_q(\xi_i^\lambda, \xi_j^\mu, i\xi_k^\nu) C_q(\xi_i^\lambda, \xi_j^\mu, i\xi_k^\nu') = \delta_{\nu,\nu'} \delta_{i,i'} \delta_{k,k'};
\]

(4.2.6)

\[
\sum_{\nu,k,l} C_q(\xi_i^\lambda, \xi_j^\mu, i\xi_k^\nu) C_q(\xi_i^\lambda, \xi_j^\mu, i\xi_k^\nu) = \delta_{i,i'} \delta_{j,j'}.
\]

(4.2.7)

Moreover ([KS97, Eq. (7.20), (7.21)]

\[
u_{i,i'}u_{j,j'}^\mu = \sum_{\nu,k,k'} C_q(\xi_i^\lambda, \xi_j^\nu, i\xi_k^\nu) C_q(\xi_i^\lambda, \xi_j^\nu, i\xi_k^\nu) u_{k,k'}^\nu;
\]

(4.2.8)

\[
u_{j,j'} = \sum_{i,i',k,k'} C_q(\xi_i^\lambda, \xi_j^\nu, i\xi_k^\nu) C_q(\xi_i^\lambda, \xi_j^\nu, i\xi_k^\nu) u_{i,i'}^\mu u_{j,j'}^\nu, \quad \text{if} \ m_{\lambda,\mu}^{\nu} \neq 0.
\]

(4.2.9)

**Lemma 4.2.11.** If \( C_q(\xi_i^\lambda, \xi_j^\mu, i\xi_k^\nu) \neq 0 \), then \( \text{wt}(\xi_i^\lambda) + \text{wt}(\xi_j^\mu) = \text{wt}(\xi_k^\nu) \).

**Proof.** Let \( \alpha \in P \), then

\[
K_\alpha(\mathbf{r} \otimes \mathbf{s}) = K_\alpha(\mathbf{r}) \otimes K_\alpha(\mathbf{s}) = q^{(\text{wt}(\mathbf{r}) + \text{wt}(\mathbf{s}), \alpha)/2} \mathbf{r} \otimes \mathbf{s}.
\]

Hence by self-adjointness of \( K_\alpha \)

\[
q^{(\text{wt}(\mathbf{r}) + \text{wt}(\mathbf{s}), \alpha)/2} C^\prime_q(\mathbf{r}, \mathbf{s}, \mathbf{t}) = \langle K_\alpha(\mathbf{r} \otimes \mathbf{s}), \mathbf{t} \rangle = \langle \mathbf{r} \otimes \mathbf{s}, K_\alpha(\mathbf{t}) \rangle = q^{(\text{wt}(\mathbf{t}), \alpha)/2} C_q(\mathbf{r}, \mathbf{s}, \mathbf{t}).
\]

As \( C_q(\mathbf{r}, \mathbf{s}, \mathbf{t}) \neq 0 \) and \( q^{(\text{wt}(\mathbf{r}) + \text{wt}(\mathbf{s}), \alpha)} \) is real-valued it follows that \( q^{(\text{wt}(\mathbf{r}) + \text{wt}(\mathbf{s}), \alpha)/2} = q^{(\text{wt}(\mathbf{t}), \alpha)/2} \) for all \( \alpha \in P \). Since \( q \) is not a root of unity we get \( \text{wt}(\mathbf{r}) + \text{wt}(\mathbf{s}) = \text{wt}(\mathbf{t}) \).
For arbitrary $q$-deformed Lie groups $G_q$ the Clebsch–Gordan coefficients are very hard to explicitly write down. For $SU_q(N)$ the formulas are known if $\lambda = \omega_i$ for $i = 1, \ldots, N-1$. Even in these cases the expressions are already quite involved. Since for every $\lambda \in P_+$ there is an $n$ such that $U_\lambda$ is contained in $U_{q\omega_n}$ we restrict ourselves to $\lambda = \omega_1$. In that case many things simplify. From now on we work with $SU_q(N)$. We change again to the notation of GT tableaux.

**Notation 4.2.12.** The standard representation $V_{\omega_i}$ has the property that $m_{\omega_i, \mu} \in \{0, 1\}$ for all $\mu, \nu \in P_+$. Therefore we can omit the subscript $l$ in (4.2.5). Let $s_{1, \cdot} = \lambda$ and $r_{1, \cdot} = \mu$. If $r$ is of the form (4.2.3) we write $C_q(k, s, t) := C_q(\omega_1, \lambda, \mu; r, s, t)$. For

$$M = (m_1, \ldots, m_i) \in \mathbb{N}^i,$$

with $1 \leq m_j \leq N + 1 - j$ (4.2.10)

denote by $r^M$ and $r_M$ the tableaux with entries respectively

$$(r^M)_{jk} := \begin{cases} r_{jk} + 1 & \text{if } k = m_j, 1 \leq j \leq i; \\ r_{jk} & \text{otherwise,} \end{cases} \quad (r_M)_{jk} := \begin{cases} r_{jk} + 1 & \text{if } k = m_j, 1 \leq j \leq i; \\ r_{jk} & \text{otherwise.} \end{cases}$$

If $i = 1$ and $M = (m_1) = (j)$, we abbreviate $r^M =: r^{(j)}$ and $r_M =: r_{(j)}$. Note that if $r$ is a GT tableau $r^M$ or $r_M$ need not be GT tableaux.

**Lemma 4.2.13.** If $M \in \mathbb{N}^1$ satisfies (4.2.10) and $r^M$ is a GT tableau, then $\text{wt}(r^M) = \text{wt}(r) + e_{N+1-i}$. Similarly if $r_M$ is a GT tableau, then $\text{wt}(r_M) = \text{wt}(r) - e_{N+1-i}$.

**Proof.** Suppose $M \in \mathbb{N}^1$ fulfills (4.2.10) and $r^M$ is a GT tableau, then

$$\mu_j(r^M) = \begin{cases} \mu_j(r) + 1 & \text{if } 1 \leq N + 1 - j \leq i; \\ \mu_j(r) & \text{if } i < N + 1 - j \leq N \end{cases} = \begin{cases} \mu_j(r) + 1 & \text{if } N + 1 - i \leq j \leq N; \\ \mu_j(r) & \text{if } 1 \leq j < N + 1 - i. \end{cases}$$

Therefore, if we write $\omega_N = 0$,

$$\text{wt}(r^M) = \sum_{j = 1}^{N-1} (2\mu_j(r^M) - \mu_{j-1}(r^M) - \mu_{j+1}(r^M))\omega_j$$

$$= \sum_{j = 1}^{N-i-1} (2\mu_j(r) - \mu_{j-1}(r) - \mu_{j+1}(r))\omega_j + (2\mu_{N-i}(r) - \mu_{N-i-1}(r) - \mu_{N-i+1}(r) - 1)\omega_{N-i}$$

$$+ (2\mu_{N-i+1}(r) + 2 - \mu_{N-i}(r) - \mu_{N-i+2}(r) - 1)\omega_{N-i}$$

$$+ \sum_{j = N-i+2}^{N-1} (2\mu_j(r) + 2)\omega_j$$

$$= \sum_{j = 1}^{N-1} (2\mu_j(r^M) - \mu_{j-1}(r^M) - \mu_{j+1}(r^M))\omega_j - \omega_{N-i} + \omega_{N-i+1}$$

$$= \text{wt}(r) + e_{N+1-i}$$

As $r^M_M = r$, the previous calculation yields $\text{wt}(r) = \text{wt}(r^M) = \text{wt}(r_M) + e_{N+1-i}$. 

\[\square\]
Lemma 4.2.14 ([CP08, Eq. (4.4)–(4.6)],[KS97, §7.3.7]). The following explicit formulas hold:

\[
C_q(i, r, r^M) = \prod_{a=1}^{i-1} \begin{vmatrix} (1,0) & r_{a,0} & r_{a,0} + e_m & \cdots & r_{a,0} + e_m \\ (1,0) & r_{a+1,0} & r_{a+1,0} + e_m & \cdots & r_{a+1,0} + e_m \\ \end{vmatrix}
\times \prod_{a=1}^{i-1} \begin{vmatrix} (1,0) & r_{i,0} & r_{i,0} + e_m & \cdots & r_{i,0} + e_m \\ (0,0) & r_{i+1,0} & r_{i+1,0} + e_m & \cdots & r_{i+1,0} + e_m \\ \end{vmatrix},
\]

where

\[
\begin{align*}
&\left(1, 0\right) r_{a,0} \quad r_{a,0} + e_j \\
&\left(1, 0\right) r_{a+1,0} \quad r_{a+1,0} + e_k \\
&\left(0, 0\right) r_{a+1,0} \quad r_{a+1,0} + e_j
\end{align*}
\]

\[
= q^{\left(-r_{a,j} + r_{a+1,k} - k + j\right)} \prod_{i=1}^{N-a} \left[r_{a,i} - r_{a+1,k - i + j}\right]^{j} \prod_{i=1}^{N-a} \left[r_{a+1,i} - r_{a,j - i + j - 1}\right]^{j}.
\]

(4.2.12)

\[
\begin{align*}
&\left(1, 0\right) r_{a,0} \quad r_{a,0} + e_j \\
&\left(0, 0\right) r_{a+1,0} \quad r_{a+1,0} + e_j
\end{align*}
\]

\[
= q^{\left(-1 + j - \sum_{i=1}^{N-a} r_{a,i} - \sum_{i=1, i \neq j}^{N-a} r_{a,i}\right)} \prod_{i=1}^{N-a} \left[r_{a+1,i} - r_{a,j - i + j - 1}\right]^{j}.
\]

(4.2.13)

Here \(0\) is a vector with only zeroes of the appropriate length, \(e_j\) is a vector of the appropriate length with a 1 on position \(j\) and zeroes everywhere else.

If a GT tableau \(s\) is not of the form \(r^M\) for some \(M \in \mathbb{N}^i\) satisfying (4.2.10) and a GT tableau \(r\), then \(C_q(i, r, s) = 0\).

Moreover \(C_q(r, i, r^M) = C_{\frac{1}{r}}(r, i, r^M)\).

[CP08] obtained their formulas from [KS97]. In [KS97, Eq. (7.45), see their errata] the exponent of (4.2.13), is of the form \(q^{(-1 + j + \ldots)}\), contrary to the formula [CP08, Eq. (4.6)], where it is of the form \(q^{(1 - j + \ldots)}\).

More general Clebsch–Gordan coefficients can be found (see e.g. [AS94]). However these become increasingly more complicated and are not suitable for the computations we want to perform.

Adopt the notation of Lemma 4.2.9 part (v). From Lemma 4.2.14 and Equation (4.2.4) we get

\[
C_q(J(r_{eN+1-k}), J(r), J(r^M)) = \langle J(r_{eN+1-k}) \otimes J(r), J(r^M) \rangle = \langle J(r \otimes r_{eN+1-k}), J(r^M) \rangle
\]

\[
= \langle r \otimes r_{eN+1-k}, r^M \rangle = C_q(r, r_{eN+1-k}, r^M) = \frac{C_q}{k}(k, r, r^M)
\]

\[
= C_{\frac{1}{r}}(k, r, r^M).
\]

(4.2.14)

The last equality follows, because for these entries \(k, r\) and \(r^M\), \(C_{\frac{1}{r}}\) is real-valued. If instead of \(r_{eN+1-k}\) one takes a general GT tableau \(s\), this identity need not be valid, some extra \(-1\) signs might be needed, see [AS94].
4.2.3 The Martin kernel for $SU_q(N)$

To identify the Martin boundary defined by a state $\varphi$ we need to compute for any $\lambda \in P_+$ the image $K_{\varphi}(m^\lambda_{r,s})$, where $r, s$ are weight vectors of $V_\lambda$. For this we follow the ideas of [Bia94] and [NT04, Pf. of Prop. 4.9]. For $a \in C(SU_q(N))$ denote by $Q_a$ the convolution with $a$, that is

\[ Q_a : \bigoplus_{\lambda \in P_+} B(\mathcal{H}_\lambda) \to \bigoplus_{\lambda \in P_+} B(\mathcal{H}_\lambda), \quad Q_a := (\iota \otimes a)\hat{\Lambda}. \]

Then $P_\varphi Q_a = Q_a P_\varphi$ and $\mathcal{F}(ab) = Q_a \mathcal{F}(b)$. Therefore $P_\varphi(\mathcal{F}(ab)) = Q_a P_\varphi(\mathcal{F}(b))$. Hence also $G_\varphi(\mathcal{F}(ab)) = Q_a G_\varphi(\mathcal{F}(b))$. Combining this identity with (4.2.9) allows the use of induction.

A first step is to compute $G_\varphi(\mathcal{F}(u^\omega_{i,i}))$ from $G_\varphi(I_0)$. Here $u^\omega_{i,i}$ denotes the matrix coefficient corresponding to the weight vectors of weight $e_i$ of the standard representation $V_{\omega_i}$. Using the observations above and the identities of Lemmas 4.2.13 and 4.2.9 we get

\[ G_\varphi(\mathcal{F}(u^\omega_{i,i,N+1-N-i+1})) = Q_{u^\omega_{i,i,N+1-N-i+1}}(G_\varphi(I_0)) = \sum_{\lambda,r} g_\varphi(\lambda,0) d_{\lambda} q^{(wt(r),-2\rho)} \mathcal{F}(u^\lambda_{r,r}) \]

\[ = \sum_{\lambda,r} g_\varphi(\lambda,0) d_{\lambda} q^{(wt(r),-2\rho)} \mathcal{F}(u^\omega_{i,i,N+1-N-i+1}u^\lambda_{r,r}) \]

\[ = \sum_{\lambda,r} g_\varphi(\lambda,0) d_{\lambda} q^{(wt(r),-2\rho)} \mathcal{F}\left( \sum_{s,t} C_q(i,r,s) C_q(i,r,t) u^s_{s,t} \right) \]

\[ = \sum_{\lambda,r} g_\varphi(\lambda,0) d_{\lambda} q^{(wt(r),-2\rho)} \mathcal{F}\left( \sum_{\mu,M,M'} C_q(i,r,M) C_q(i,r,M') u^{\mu}_{r,M,r,M'} \right) \]

\[ = \sum_{\lambda,r} g_\varphi(\lambda,0) d_{\lambda} q^{(wt(r),-2\rho)} \sum_{\lambda,M,M'} C_q(i,r,M) C_q(i,r,M') d^{-1}_{\mu} q^{(wt(M)+wt(M'),\rho)} m^{\mu}_{J(M),J(M')} \]

\[ = \sum_{\lambda,r} g_\varphi(\lambda,0) \sum_{\mu,M,M'} C_q(i,r,M) C_q(i,r,M') d_{\mu} q^{(-wt(r)+wt(r)+\epsilon_{N+1-N-i+2\rho})} m^{\mu}_{J(M),J(M')} \]

\[ = \sum_{\lambda,r} g_\varphi(\lambda,0) \sum_{\mu,M,M'} C_q(i,r,M) C_q(i,r,M') d_{\mu} q^{(\epsilon_{N+1-N-i+2\rho})} m^{\mu}_{J(M),J(M')} \].

The sum in (4.2.15) is over all weights $\mu$ of the form $\mu = \lambda + e_j$ for some $j = 1, \ldots, N$ and all $M, M' \in \mathbb{N}^*$ satisfying (4.2.10) with $(r^M)_{1.} = (r^{M'})_{1.} = \mu$. Thus if $M = (m_1, \ldots, m_i)$, $M' = (m'_1, \ldots, m'_i)$ and $\mu = \lambda + e_j$, then $m_1 = m'_1 = j$.

The question is therefore how these Clebsch–Gordan coefficients behave asymptotically.

We specialise to the case $i = 1$.

If $i = 1$ and $M \in \mathbb{N}^*$, then $M$ is of the the form $M = (j)$ for some $1 \leq j \leq N$. Furthermore if $r^{(j)}_{1.} = r^{(j)}_{1.} = \mu$, then $j = j'$ and $\mu = \lambda + e_j$. Identity (4.2.16) reduces to

\[ G_\varphi(\mathcal{F}(u^\omega_{1,N})) = \sum_{\lambda,r} g_\varphi(\lambda,0) \sum_{j=1}^{N} C_q(1,r,(j)) d_{\lambda} q^{(\epsilon_{N+1-N-i+2\rho})} m^{\mu}_{J(r^{(j)}),J(r^{(j)})}. \]

(4.2.17)
Lemma 4.2.15. Let \((\lambda_n)_n\) be a sequence of positive weights and \(\nu \in P\). Write \(\lambda_n = n\lambda_1e_1 + \ldots + n\lambda_{N-1}e_{N-1}\). Assume that \(n\lambda_i - n\lambda_{i+1} \to \infty\) as \(n \to \infty\). The following asymptotics hold
\[
\frac{d\lambda_n}{d_{\lambda+n}} = q^{(\nu,2\rho)}(1 + o(1)), \quad \text{as } n \to \infty.
\]

Proof. First note that if \(n\) is large enough, then \(\lambda_n + \nu \in P_+\). We use the quantum analogue of Weyl’s dimension formula, see e.g., [Fuc95, Eq. (4.4.14)]. For \(SU_q(N)\) this formula reads as
\[
d_\lambda = \prod_{\alpha > 0} \frac{[\lambda + \rho,\alpha]_q}{[\rho,\alpha]_q},
\]
where the product is taken over all positive roots. This leads to
\[
\frac{d_{\lambda+n}}{d\lambda} = \prod_{\alpha > 0} \frac{[\lambda + \nu,\rho,\alpha]_q}{[\lambda + \rho,\alpha]_q} = \prod_{\alpha > 0} q^{-\nu,\alpha}[\lambda + \rho,\alpha]_q + q^{\lambda+\rho,\alpha}[\nu,\alpha]_q\]
\[
= \prod_{\alpha > 0} \left( q^{-\nu,\alpha} + q^{\lambda+\rho,\alpha} [\nu,\alpha]_q \right).
\]
Since \(n\lambda_i - n\lambda_{i+1} \to \infty\), it follows that \((\lambda_n + \rho,\alpha) \to \infty\) for all \(\alpha\). Hence
\[
\lim_n \frac{d_{\lambda+n}}{d\lambda_n} = \prod_{\alpha > 0} q^{-\nu,\alpha} = q^{\nu,\sum_{\alpha > 0} \alpha} = q^{\nu,\rho},
\]
which concludes the proof. \(\Box\)

Remark 4.2.16. Given a sequence \((n\mathbf{r})_n\) of GT tableau, let \(\lambda_n := n\mathbf{r}_1\). Write \(n\mathbf{r} = (n_{i,j})_{i,j}\) and \(\lambda_n = n\lambda_1e_1 + \ldots + n\lambda_{N-1}e_{N-1}\) (so \(n\lambda_N = 0\)). We want to estimate
\[
C_q(1, n\mathbf{r}, n\mathbf{r}^{(j)})^2 \frac{d\lambda_n}{d\lambda_{n+e_j}} q^{(e_N, 2\rho)}, \quad \text{as } n \to \infty.
\]

Assume that the sequence \((n\mathbf{r})_n\) has the property that \(\lim_n |n_{r_{2,i}} - n_{r_{1,j}}| = \infty\) and \(\lim_n |n\lambda_i - n\lambda_{i+1}| = \infty\) for all \(i\) and \(j\).

The quantum dimension \(d_\lambda\) explicitly can be computed in terms of Schur functions cf. [KS97, p. 442] and [Mac79, §1.3]. These become very complicated, so we only work with the asymptotics. We remove the subscript \(n\) from the GT tableaux. By Lemma 1.2.3
\[
C_q(1, \mathbf{r}, \mathbf{r}^{(j)})^2 \frac{d\lambda}{d_{\lambda+e_j}} q^{(e_N, 2\rho)}
\]
\[
= q^{-1+j+N-1 \sum_{i=1}^{N} r_{2,i}- \sum_{i=1, i \neq j}^{N} r_{1,i}} \prod_{i=1}^{N-1} [r_{2,i} - r_{1,j} - i + j - 1]_q \prod_{i=1}^{N} [r_{1,i} - r_{1,j} - i + j]_q q^{N+1-2j-N+1}(1 + o(1))
\]
4.2. PARTIAL RESULTS ON THE MARTIN BOUNDARY OF SU$_q$(N)

4.2.8

leading term of (4.2.20) equals

\[ i,j \]

If (4.2.18) and (4.2.19) Equation (4.2.17) can be written as

\[ \lambda \]

Now we make some naive (invalid!) computations. Assume that (4.2.18) holds for all

\[ n \]

From the requirements on GT tableaux it follows that

\[ (r_{2,j} + \ldots + r_{2,N-1}) - (r_{1,j+1} + \ldots + r_{1,N}) \]

is positive. We must give the asymptotic behaviour of this quantity. By Lemma

\[ 4.2.8 \]

\[ (r_{2,j} + \ldots + r_{2,N-1}) - (r_{1,j+1} + \ldots + r_{1,N}) \]

\[ = (r_{1,1} + \ldots + (r_{1,j} + 1)) - (r_{2,1} + \ldots + r_{2,j-1}) + \mu_{N-1}(r^{(j)}) - \mu_{N}(r^{(j)}) \]

\[ = (r_{1,1} - r_{2,1}) + \ldots + (r_{1,j-1} - r_{2,j-1}) + (\text{wt}(r^{(j)}), \omega_{N-1}) + (r_{1,j} + 1) - \frac{1}{N} \mu_{N}(r^{(j)}) \]

\[ = (r_{1,1} - r_{2,1}) + \ldots + (r_{1,j-1} - r_{2,j-1}) - (\text{wt}(J(r^{(j)})), \omega_{1}) + (\lambda + e_{j}, e_{j}). \quad (4.2.19) \]

Now we make some naive (invalid!) computations. Assume that (4.2.18) holds for all

\[ n \]

sequences of GT tableaux \( (n r)_{n} \), so not only the ones with \( |n r_{2,i} - n r_{1,j}| \to \infty \). By means of (4.2.18) and (4.2.19) Equation (4.2.17) can be written as

\[ G_{\varphi}(F(u^\varphi)_{N,N}) = \sum_{\lambda, r} g_{\varphi}(\lambda, 0) \sum_{j=1}^{N} q^{2((r_{1,1} - r_{2,1}) + \ldots + (r_{1,j-1} - r_{2,j-1}))}(1 + o(1)) \]

\[ \times (1 + o(1))m_{J(r^{(j)}), J(r^{(j)})}^{1+e_{j}} \]

\[ = \sum_{\lambda, r} g_{\varphi}(\lambda, 0) \sum_{j=1}^{N} q^{2((r_{1,1} - r_{2,1}) + \ldots + (r_{1,j-1} - r_{2,j-1}))}(1 + o(1)) \]

\[ \times q^{2(\lambda + e_{j})} \pi_{\lambda}(K^{-4\omega_{1}})m_{J(r^{(j)}), J(r^{(j)})}^{1+e_{j}} \]

\[ = \sum_{r, \lambda} \sum_{j=1}^{N} g_{\varphi}(\lambda - e_{j}, 0)q^{2((r_{1,1} - r_{2,1}) + \ldots + (r_{1,j-1} - r_{2,j-1}))}(1 + o(1)) \]

\[ \times q^{2(\lambda)\pi_{\lambda}(K^{-4\omega_{1}})m_{J(r), J(r)}}. \quad (4.2.20) \]

In the last step we changed the summation from \( \lambda \) to \( \lambda - e_{j} \) and \( r \) to \( r^{(j)} \).

If \( (n r)_{n} \) is a sequence of GT tableaux with the property that \( |n r_{2,i} - n r_{1,j}| \to \infty \) for all \( i, j \). Then for large \( n \), \( n r^{(j)} \) is always a GT tableau. Moreover for \( j > 1 \) the factor

\[ q^{2((n r_{1,1} - n r_{2,1}) + \ldots + (n r_{1,j-1} - n r_{2,j-1}))} \to 0. \]

Therefore we expect that modulo \( c_{0}(\text{SU}_{q}(N)) \) the leading term of (4.2.20) equals

\[ \sum_{r, \lambda} g_{\varphi}(\lambda - e_{1}, 0)q^{2(\lambda)\pi_{\lambda}(K^{-4\omega_{1}})m_{J(r), J(r)}} \]

\[ = \sum_{\lambda \in P_{+}} g_{\varphi}(\lambda - e_{1}, 0)q^{2(\lambda)\pi_{\lambda}(K^{-4\omega_{1}})} \quad \text{mod} \ c_{0}(\text{SU}_{q}(N)). \]
Unfortunately, this is too much to hope for. Namely, the requirement \(|r_{i,j} - r_{i+1,j}| \to \infty\) is too strong. The estimate (4.2.18) does not hold for a too large set of matrix coefficients, the set of such coefficients is not in \(c_0(\text{SU}_q(2))\). For example in every irreducible representation \(V_\lambda\) there exist GT tableaux \(r\) such that \(A_2 = r_{1,2} = r_{2,1} = r_{2,2}\) (this argument only applies if \(N \geq 3\)). Moreover in (4.2.20) it is not clear why

\[
q^{2((r_{1,1}-r_{2,1})+...+(r_{1,j-1}-r_{2,j-1}))} q^{2(\lambda, \bar{\tau})} \pi_\lambda(K_{-4|\omega_1})
\]

is bounded or tends to 0. We have to be more precise.

Recall the definition of \(b\)-\(\text{lim}\) and the function \(c_{\varphi,\mu}\) introduced in Notation 4.1.14 and respectively Notation 4.1.21.

**Lemma 4.2.17.** Assume \((\lambda_n)_n\) is a sequence in \(P_+\) with \(b\)-\(\text{lim}_n\) \(\lambda_n = x\) for some \(x \in \text{int}(S_+^{N-2})\). If \(\nu \in P\), then

\[
\lim_n \frac{g_\varphi(\lambda_n + \nu, 0)}{g_\varphi(\lambda_n, 0)} = e^{(c_{\varphi,(-x),\nu})}.
\]

**Proof.** By Lemmas 3.2.7 and 4.1.7 and Corollary 4.1.13 we get

\[
g_\mu^{(c)}(\lambda, \nu) = \left( \frac{d_\nu}{d_\lambda} \right)^2 g_\mu^{(c)}(\nu, \lambda)
= \left( \frac{d_\nu}{d_\lambda} \right)^2 \left( \frac{d_\nu}{d_\nu} \right) \sum_{w \in W} \det(w) q^{(\lambda+\rho)-w(\nu+\rho), 2\rho} g_\mu^{(t)}(w(\nu + \rho) - \rho, \lambda)
= \frac{d_\nu}{d_\lambda} \sum_{w \in W} \det(w) q^{(\lambda+\rho)-w(\nu+\rho), -2\rho} g_\mu^{(t)}(\lambda, w(\nu + \rho) - \rho).
\]

Using the asymptotics of Proposition 4.1.20 and following a similar reasoning as in the proof of Lemma 4.1.24 gives

\[
\lim_n \frac{g_\varphi^{(c)}(\lambda_n + \nu, 0)}{g_\varphi^{(c)}(\lambda_n, 0)} = \lim_n \frac{d_{\lambda_n}}{d_{\lambda_n+\nu}} \sum_{w \in W} \det(w) q^{(w(\rho)-w(\nu+\rho), 2\rho)} g_\varphi^{(t)}(\lambda_n + \nu, w(\rho) - \rho)
= \lim_n q^{(\nu, 2\rho)} \sum_{w \in W} \det(w) q^{(w(\rho)-w(\nu+\rho), 2\rho)} q^{(\lambda_n+\nu(\rho)+(\lambda_n+\nu(\rho)), 2\rho)} g_\varphi^{(t)}(\lambda_n, w(\rho) - \rho)
= \lim_n q^{(\nu, 2\rho)} \frac{\sum_{w \in W} \det(w) q^{(w(\rho)-w(\nu+\rho), 2\rho)}} {\sum_{w \in W} \det(w) q^{(w(\rho)-w(\nu+\rho), 2\rho)}} \frac{\pi_\lambda(K_{-4|\omega_1})}{\pi_\lambda(K_{-4|\omega_1})}
= e^{(c_{\varphi,(-x),\nu})} \frac{\sum_{w \in W} \det(w) q^{(w(\rho)-w(\nu+\rho), 2\rho)}} {\sum_{w \in W} \det(w) q^{(w(\rho)-w(\nu+\rho), 2\rho)}} e^{(c_{\varphi,(-x),\nu})}
= e^{(c_{\varphi,(-x),\nu})}.
\]

Note that the requirement \(x \in \text{int}(S_+^{N-2})\) precisely insures that the denominator above is
Indeed, by Lemmas 4.1.22 and 4.1.23,
\[ \sum_{w \in W} \det(w)q^{(w(\rho)-(\lambda+\rho),2\rho)}e^{(c_{\varphi}(-x),\rho-w(\rho))} = q^{-(\lambda+\rho,2\rho)}e^{(c_{\varphi}(-x),\rho)} \sum_{w \in W} \det(w)e^{(2\log(q)\rho-Tc_{\varphi}(-x),w(\rho))} \neq 0, \]
which completes the proof.

**Lemma 4.2.18.** Assume \( x \in \text{int}(S^{N-2}_+) \) and \((\lambda_n)_n\) is a sequence in \( P_+ \) with \( b\lim_n \lambda_n = x \). Suppose that \((n\mathbf{r})_n\) is a sequence of GT tableaux with \((n\mathbf{r})_1 = \lambda_n\) and \((n\mathbf{r})^{(j)}\) is a GT tableau for every \( n \). Write \( n\mathbf{r} = (n^r_{i,j})_{i,j} \) and \( \lambda_n = n\lambda_1e_1 + \ldots + n\lambda_{N-1}e_{N-1}, n\lambda_N = 0 \). Then as \( n \to \infty \)
\[
C_q(1, n\mathbf{r}, (n\mathbf{r})^{(j)})^2 = \begin{cases} 
\prod_{i=1}^{N-1} q^{2(n_{r_2,i}-n_{\lambda_i+1})} (1 + o(1)) & \text{if } j = 1; \\
q^{2(j-1)} \left( \prod_{i=j}^{N-1} q^{2(n_{r_2,i}-n_{\lambda_i+1})} - \prod_{i=j-1}^{N-1} q^{2(n_{r_2,i}-n_{\lambda_i+1})} \right) (1 + o(1)) & \text{if } j = 2, \ldots, N - 1; \\
q^{2(N-1)} \left( 1 - q^{2(n_{r_2,N-1})} \right) (1 + o(1)) & \text{if } j = N. 
\end{cases}
\]
and
\[
C_{\frac{1}{\pi}}(1, n\mathbf{r}, (n\mathbf{r})^{(j)})^2 = \begin{cases} 
(1 - q^{2(n_{\lambda_1-n_{r_2,1})}}) (1 + o(1)) & \text{if } j = 1, \\
 (\prod_{i=1}^{N-1} q^{2(n_{\lambda_i}-n_{r_2,i})} - q^2 \prod_{i=1}^{j} q^{2(n_{\lambda_i}-n_{r_2,i})}) (1 + o(1)) & \text{if } j = 2, \ldots, N - 1; \\
(\prod_{i=1}^{N-1} q^{2(n_{\lambda_i}-n_{r_2,i})} (1 + o(1)) & \text{if } j = N. 
\end{cases}
\]

Note that the expression of this lemma agrees with the statement of [CP08, Eq. (4.12)]. Indeed, since \( \mathbf{r}^{(j)} \) is assumed to be a GT tableau, \( r_{2,j-1} - \lambda_j \geq 1 \), thus \( 0 < q^{2(N-1)}(1-q^2) \leq q^{2(j-1)}(1-q^{2(r_{2,j-1}-\lambda_j)}) \leq 1. \)

**Proof.** Since \( x \in \text{int}(S^{N-2}_+) \), it follows that \( \lim_n |n\lambda_i - n\lambda_j| = \infty \) for all \( i \neq j \). The notation with the subscript on the left of a variable is awkward, so in the proof we remove this subscript \( n \). All asymptotics should be understood as \( n \to \infty \). We assume for the moment that \( 2 \leq j \leq N - 1 \). A computation shows that as \( n \to \infty \) the following asymptotics hold
\[
C_q(1, \mathbf{r}, \mathbf{r}^{(j)})^2 = q^{(-1+j+ \sum_{i=1}^{N-1} r_{2,i} - \sum_{i=1, i \neq j}^{N} r_{1,i})} \prod_{i=1}^{N-1} \prod_{i=1, i \neq j}^{N} [r_{2,i} - r_{1,i} - i + j - 1] \]
\[
= q^{(-1+j+ \sum_{i=1}^{N-1} r_{2,i} - \sum_{i=1, i \neq j}^{N} \lambda_i)} \prod_{i=1}^{N-1} \prod_{i=1, i \neq j}^{N} [\lambda_i - \lambda_j - i + j - 1] q \]
\[ \begin{aligned}
&= q \left( -1 + j + \sum_{i=1}^{N-1} r_{2,i} - \sum_{i=1, i \neq j}^N \lambda_i \right) q^{(\lambda_1 - \lambda_j - 1 + j) + \ldots + (\lambda_{j-1} - \lambda_1 + 1) + (\lambda_{j+1} - \lambda_1 + 1) + \ldots + (\lambda_N - \lambda_N + N - j)} (1 + o(1)) \\
&\times \prod_{i=1}^{N-1} \left( q^{r_{2,i} - \lambda_i - i + j - 1} - q^{-r_{2,i} + \lambda_i + i + j + 1} \right) (1 + o(1)) \\
&= (-1)^{j-1} q \left( -1 + j + \sum_{i=1}^{N-1} (r_{2,i} - \lambda_i) + \sum_{i=1, i \neq j}^N (\lambda_i - \lambda_j - i + j) + \sum_{i=1}^{N-1} (\lambda_i - \lambda_{i+1} + i + 1 - j) \right) \prod_{i=1}^{N-1} \left( q^{r_{2,i} - \lambda_i - i + j - 1} - q^{-r_{2,i} + \lambda_i + i + j + 1} \right) (1 + o(1)) \\
&= (-1)^{j-1} q^{2(j-1)} \prod_{i=1}^{j-1} \left( q^{2(r_{2,i} - \lambda_i - i + j - 1)} - 1 \right) \prod_{i=j}^{N-1} \left( q^{2(r_{2,i} - \lambda_{i+1})} - q^{2(\lambda_j - \lambda_{i+1} + i + j + 1)} \right) (1 + o(1)) \\
&= (-1)^{j-1} q^{2(j-1)} (-1)^{-2} \prod_{i=j}^{N-1} \left( q^{2(r_{2,i} - \lambda_i)} - 1 \right) \prod_{i=j}^{N-1} \left( q^{2(r_{2,i} - \lambda_{i+1})} \right) (1 + o(1)) \\
&= q^{2(j-1)} \left( \prod_{i=j}^{N-1} q^{2(r_{2,i} - \lambda_{i+1})} - \prod_{i=j-1}^{N-1} q^{2(r_{2,i} - \lambda_{i+1})} \right) (1 + o(1)).
\end{aligned} \]

Equality (4.2.21) holds because \(|\lambda_i - \lambda_j| \to \infty\) so we can use the asymptotics of the \(q\)-analogues of Lemma 1.2.3. For (4.2.23) we observe that \(r_{2,i-1} - \lambda_j - i + j - 1 \to \infty\) for \(i = 1, \ldots, j - 2\) and that \(\lambda_j - \lambda_i + i - j + 1 \to \infty\) for \(i = j, \ldots, N\) as \(n \to \infty\). The case \(j = 1\) can be obtained from the above computation. If we substitute \(j = 1\) above, in Equation (4.2.22) the sums of the form \(\sum_{i=1}^{j-1} 1\) vanish. Continuing as in the general case gives the desired result. For \(j = N\), the sums \(\sum_{i=1}^{N-1} 1\) vanish. Proceeding in an analogous way gives the result. We leave the details to the reader.

Similarly, the other asymptotics can be calculated in the following way. Recall that \([n]_q = [n]_q\). Then
\[ C_j(1, r, r^{(j)})^2 = q \left( -1 + j + \sum_{i=1}^{N-1} r_{2,i} - \sum_{i=1, i \neq j}^N r_{1,i} \right) \prod_{i=1}^{N-1} [r_{2,i} - r_{1,j} - i + j - 1]_q \prod_{i=1}^{N} [r_{1,i} - r_{1,j} - i + j]_q \]
\[ = q \left( -1 + j + \sum_{i=1}^{N-1} r_{2,i} + \sum_{i=1, i \neq j}^N \lambda_i \right) \prod_{i=1}^{N-1} [r_{2,i} - \lambda_j - i + j - 1]_q \prod_{i=1}^{N} [\lambda_i - \lambda_j - i + j]_q \]
4.2. PARTIAL RESULTS ON THE MARTIN BOUNDARY OF $\text{SU}_q(N)$

\[ = q \left( 1 - \frac{N-1}{1} \sum_{i=1}^{N} r_{2i} + \frac{N}{i=1, i \neq j} \lambda_i \right) q^{(\lambda_1 - \lambda_1 - j + 1) + (\lambda_1 - \lambda_{j+1} + 1) + \cdots + (\lambda_j - \lambda_N + N - j)} (-1)^{j-1} \]

\[ \times \prod_{i=1}^{N-1} \left( q^{r_{2i} - \lambda_j - i + j - 1} - q^{-r_{2i} + \lambda_j + i + j + 1} \right) (1 + o(1)) \]

\[ = (-1)^{j-1} q^{1 - \frac{N-1}{1} \sum_{i=1}^{N} (\lambda_1 - \lambda_i) + \frac{N}{i=1, i \neq j} \sum_{i=1}^{N} (\lambda_1 - \lambda_i) + \frac{N}{i=1} (\lambda_j - \lambda_{i+1} + 1 - j)} \]

\[ \times \prod_{i=1}^{N-1} \left( q^{r_{2i} - \lambda_j - i + j - 1} - q^{-r_{2i} + \lambda_j + i + j + 1} \right) (1 + o(1)) \]

\[ = (-1)^{j-1} q^{\sum_{i=1}^{j-1} (2\lambda_1 - \lambda_i - 2i + j - 1) + \sum_{i=1}^{N-1} (\lambda_j - \lambda_{i+1} + 1 - j)} \]

\[ \times \prod_{i=1}^{N-1} \left( q^{r_{2i} - \lambda_j - i + j - 1} - q^{-r_{2i} + \lambda_j + i + j + 1} \right) (1 + o(1)) \]

\[ = (-1)^{j-1} \prod_{i=1}^{j-1} q^{2(\lambda_1 - \lambda_j - i + j - 1)} \prod_{i=j}^{N-1} (1 - q^{2(\lambda_j - \lambda_{i+1} + i - j + 1)}) (1 + o(1)) \]

\[ = (-1)^{j-1} (-1)^{j-1} \prod_{i=1}^{j-1} q^{2(\lambda_1 - \lambda_{i+1})} (1 - q^{2(\lambda_j - \lambda_{i+1} + j - 1)}) (1 + o(1)) \]

\[ = \left( \prod_{i=1}^{j-1} q^{2(\lambda_1 - \lambda_{i+1})} - q^2 \prod_{i=1}^{j} q^{2(\lambda_1 - \lambda_{i+1})} \right) (1 + o(1)). \]

Again the cases $j = 1$, $N$ can be derived from this one. □

**Proposition 4.2.19.** Denote $a_0(r) := 0$, $a_N(r) := 1$ and $a_j(r) := \prod_{i=j}^{N-1} q^{2(\lambda_i - \lambda_i + 1)}$ for $j = 1, \ldots, N - 1$. Assume that $(\lambda_n)_n$ is a sequence in $P_+$ such that $\lim_{n} \lambda_n = x$ for some $x \in \text{int}(S_{N+2}^\infty)$, then

\[ \sum_{n=1}^{\infty} K_{\varphi}(m_{\omega_{N-1} \omega_{N-1}}) I_{\lambda_n} \]

\[ = [N]_q q^{N-1} \left( e^{(c_{\varphi}(2\tilde{e}), x_N)} - q^2 e^{(c_{\varphi}(2\tilde{e}), x_{N-1})} \right) \sum_{n=1}^{\infty} q^{2(\lambda_n \tilde{\omega}_1)} \pi_{\lambda_n} (K - 4\omega_{N-1}) \]

\[ + \sum_{j=2}^{N-1} \left( e^{(c_{\varphi}(2\tilde{e}), x_{N+1-j})} - q^2 e^{(c_{\varphi}(2\tilde{e}), x_{N+1-j})} \right) \sum_{n=1}^{\infty} \sum_{r \in G, r_{i+1} = \lambda_n} a_j(r) m_{\lambda_n}^2 \]

\[ + q e^{(c_{\varphi}(2\tilde{e}), x_1)} \sum_{n=1}^{\infty} I_{\lambda_n} \right) \mod \bigoplus_{n=1}^{\infty} B(\mathcal{H}_{\lambda_n}). \tag{4.2.24} \]

**Proof.** We start by computing $\lim_{n} K_{\varphi}(\mathcal{F}(u_{N,N}^{\omega})) I_{\lambda_n}$. Observe that

\[ a_k(r_j) = \begin{cases} a_k(r) & \text{if } k \geq j; \\ q^2 a_k(r) & \text{if } k < j. \end{cases} \]
By (4.2.17) and Lemmas 4.2.15 and 4.2.18 we have

\[
G_{\varphi}(\mathcal{F}(u_{N,N}^{\omega})) = \sum_{\lambda, r} g_{\varphi}(\lambda, 0) \sum_{j=1}^{N} C_q(1, r, r(j))^2 \frac{d\lambda}{d\lambda + \epsilon_j} q^{(e_N, 2\rho)} m_{J(r(j)), J(r(j))}^{\lambda + \epsilon_j}
\]

\[
= \sum_{\lambda, r} \sum_{j=1}^{N} g_{\varphi}(\lambda, 0) q^{(e_N, 2\rho)} q^{2(j-1)} (a_j(r) - a_{j-1}(r))(1 + o(1)) m_{J(r), J(r)}^{\lambda + \epsilon_j}
\]

\[
= \sum_{\lambda, r} \sum_{j=1}^{N} g_{\varphi}(\lambda, 0) (a_j(r) - q^2 a_{j-1}(r))(1 + o(1)) m_{J(r), J(r)}^{\lambda + \epsilon_j}
\]

Observe that \( e_j = -e_{N+1-j} \). The asymptotics of the Green kernel on the center (cf. Lemma 4.2.17) yield

\[
\lim_{n \to \infty} K_{\varphi}(\mathcal{F}(u_{N,N}^{\omega})) I_{\lambda_n} = \lim_{n \to \infty} G_{\varphi}(\mathcal{F}(u_{N,N}^{\omega})) G_{\varphi}(I_0)^{-1} I_{\lambda_n}
\]

\[
= \lim_{n \to \infty} \sum_{r} \sum_{j=1}^{N} g_{\varphi}(\lambda_n - e_j, 0) (a_j(r) - q^2 a_{j-1}(r))(1 + o(1)) m_{J(r), J(r)}^{\lambda_n}
\]

\[
= \lim_{n \to \infty} \sum_{r} \sum_{j=1}^{N} e^{(c_{\varphi}(-x), e_N+1-j)} (a_j(r) - q^2 a_{j-1}(r)) m_{J(r), J(r)}^{\lambda_n} \quad (4.2.25)
\]

Now suppose that \( r_1. = \lambda \), then \( r(N) \) is always a GT tableau. On the other hand, assume \( r_1. = \lambda \) and \( r(j) \) is not a GT tableau for some \( 1 \leq j < N \). This can only happen if \( r_{2,j} = \lambda_j \), because in that situation \( (r(j))_{1,j} = \lambda_j - 1 < r_{2,j} \). However, if that is the case, then

\[
a_j(r) = q^{2(r_2,j - \lambda_{j+1})} \prod_{i=j+1}^{N-1} q^{2(r_2,i - \lambda_{i+1})} = q^{2(\lambda_j - \lambda_{j+1})} \prod_{i=j+1}^{N-1} q^{2(r_2,i - \lambda_{i+1})} = o(1) \quad \text{as } \lambda \to \infty.
\]

and similarly for this \( r \) one also has \( a_{j-1}(r) = o(1) \). Hence the terms \( r \) for which \( r(j) \) is not GT can be included in (4.2.25). Moreover from Lemma 4.2.9 we get \( d_{\omega_1} = [N]_q \) and thus by the same lemma \( \mathcal{F}(u_{N,N}^{\omega}) = [N]_q^{-1} q^{-N+1} m_{-e_N, -e_N}^{\omega_N} \). Therefore

\[
\lim_{n \to \infty} [N]_q^{-1} q^{-N+1} K_{\varphi}(m_{-e_N, -e_N}^{\omega_N}) I_{\lambda_n}
\]

\[
= \lim_{n \to \infty} \sum_{r} \sum_{j=1}^{N} e^{(c_{\varphi}(-x), e_N+1-j)} (a_j(r) - q^2 a_{j-1}(r)) m_{J(r), J(r)}^{\lambda_n} \quad (4.2.26)
\]
4.2. PARTIAL RESULTS ON THE MARTIN BOUNDARY OF SU_q(N)

Gathering the terms \(a_j(r)\) in (4.2.26) gives

\[
\sum_{n=1}^{\infty} K_\varphi(m^{\omega_N, \ldots, \omega_{N-1}}_{\omega_{N-1}, \omega_{N-1}}) I_\lambda_n
\]

\[
= [N] q^{N-1} \sum_{n=1}^{N-1} \left( e^{(c_\varphi(-\bar{x}), c_\varphi x)} - q^2 e^{(c_\varphi(-\bar{x}), c_\varphi x)} \right) a_1(r) m^{\lambda, \ldots, \lambda}_{J(r), J(r)}
\]

\[
+ \sum_{j=2}^{N-1} \left( e^{(c_\varphi(-\bar{x}), c_\varphi x)} - q^2 e^{(c_\varphi(-\bar{x}), c_\varphi x)} \right) \sum_{r \text{ is GT}} a_j(r) m^{\lambda, \ldots, \lambda}_{J(r), J(r)}
\]

\[
+ e^{(c_\varphi(-\bar{x}), c_1)} \sum_{r \text{ is GT}} a_N(r) m^{\lambda, \ldots, \lambda}_{J(r), J(r)} \quad \text{mod } c_0 \bigoplus_{n=1}^{\infty} B(H_\lambda_n).
\]

By Lemma 4.2.8 we have

\[
(\bar{\omega}_1, \text{wt}(J(r))) + (\bar{\lambda}, \bar{\omega}_1) = -(\omega_{N-1}, \text{wt}(r)) + (\lambda, \omega_1)
\]

\[
= \mu_{N-1}(r) - \frac{N-1}{N} \mu_N(r) + r_{1,1} - \frac{1}{N} \mu_N(r),
\]

whence

\[
a_1(r) m^{\lambda, \ldots, \lambda}_{J(r), J(r)} = q^{2(r_{2,1} - \lambda_2) + \ldots + 2(r_{2,N-1} - \lambda_N)} m^{\lambda, \ldots, \lambda}_{J(r), J(r)}
\]

\[
= q^{2(-\mu_N(r) + \mu_{N-1}(r) + r_{1,1})} m^{\lambda, \ldots, \lambda}_{J(r), J(r)}
\]

\[
= q^{2(\bar{\lambda}, \bar{\omega}_1)} m^{\lambda, \ldots, \lambda}_{J(r), J(r)}.
\]

Moreover \(a_N(r) = 1\) for all \(r\), so the term with \(a_N(r)\) acts as a multiple of the identity, which concludes the proof. \(\Box\)

**Remark 4.2.20.** For \(j \neq 1, N\) the functions \(r \mapsto a_j(r)\) are not constant on the weight spaces and are neither in \(c_0 \bigoplus_{n} B(H_\lambda_n)\). Indeed, suppose \(\lambda \in P_+\). Consider two GT tableaux \(r\) and \(r'\) with first two rows given by

\[
\begin{pmatrix}
\lambda_1 & \ldots & \lambda_{j-1} & \lambda_j & \ldots & \lambda_{N-1} & \lambda_N \\
\lambda_1 - 1 & \ldots & \lambda_{j-1} - 1 & \lambda_j - 2 & \ldots & \lambda_{N-1} - 1
\end{pmatrix}
\]

and respectively

\[
\begin{pmatrix}
\lambda_1 & \ldots & \lambda_{j-1} & \lambda_j & \ldots & \lambda_{N-1} & \lambda_N \\
\lambda_1 - 1 & \ldots & \lambda_{j-1} - 2 & \lambda_j - 1 & \ldots & \lambda_{N-1} - 1
\end{pmatrix}
\]

and identical rows 3, \ldots, \(N\). Then \(\mu_i(r) = \mu_i(r')\) for all \(i\), so \(\text{wt}(r) = \text{wt}(r')\). But \(qa_j(r) = a_j(r')\), so \(a_j(r) \neq a_j(r')\). So \(a_j\) is not constant on weight spaces. Examples for which this difference becomes arbitrarily large can be made if \(\lambda\) is taken large enough. To be precise, given \(c > 0\) one can find \(\lambda, r\) and \(r'\) with \(r_{1,1} = r_{1,1} = \lambda\), \(\text{wt}(r) = \text{wt}(r')\) and \(a_j(r) > ca_j(r')\).

If we consider the sequence of highest weight vectors \((r_\lambda)_n = (r_{\lambda_n})_n\) (cf. Lemma 4.2.4)
Proof. To prove this expression we use \( (4.2.4) \) and we exploit the calculations we made but by assumption \( x \in \text{int}(S_+^{N-2}) \), so \( (x, \alpha_j) > 0 \) for all \( j \).

Proposition 4.2.21. Denote \( b_0(r) := 1, b_0(N) := 0 \) and \( b_j(r) := \prod_{i=1}^{j} q^{2(\lambda_i - r_{i+1})} \) for \( j = 1, \ldots, N - 1 \). Assume that \( (\lambda_n) \) is a sequence in \( P_+ \) such that \( \lim_{n} \lambda_n = x \) for some \( x \in \text{int}(S_+^{N-2}) \). Then

\[
\sum_{n=1}^{\infty} K_{\varphi} \left( m_{\epsilon_{n}, e_N}^{\epsilon_1} \right) I_{\lambda_n} = [N] q^{-(N-1)} \sum_{n=1}^{\infty} \left( (e^{(\epsilon_{\varphi}(x), -\epsilon_n)} q^{-2} - e^{(\epsilon_{\varphi}(x), -\epsilon_{n-1})}) q^{2(\lambda_n, \omega_{N-1})} \pi_{\lambda_n}(K_{-\omega_{N-1}}) \right)
\]

\[
+ \sum_{j=1}^{N-2} \sum_{r} \left( (e^{(\epsilon_{\varphi}(x), -e_{j+1})} q^{-2} - e^{(\epsilon_{\varphi}(x), -e_j)}) q^{2(N-j)} b_j(r) m_{r,r}^{\lambda_n} \right)
\]

\[
+ e^{(\epsilon_{\varphi}(x), -e_1)} q^{2(N-1)} I_{\lambda_n} \quad \text{mod } c_0^{-\infty} \bigoplus_{n=1}^{\infty} B(\mathcal{H}_{\lambda_n}).
\]

Proof. To prove this expression we use \( (4.2.4) \) and we exploit the calculations we made to prove Proposition 4.2.19. We get by Lemmas 4.2.9 and 4.2.14

\[
G_{\varphi}(\mathcal{F}(u_{\omega_{N-1}}^{\omega_{N-1}, -\omega_{N-1}})) = Q_{u_{\omega_{N-1}}^{\omega_{N-1}, -\omega_{N-1}}} G_{\varphi}(I_0)
\]

\[
= Q_{u_{\epsilon_1}^{\epsilon_1}} \left( \sum_{\lambda, r} g_{\varphi}(\lambda, 0) d_{\lambda} q^{\langle w(t), 2\rho \rangle} \mathcal{F}(u_{J(r)}^{\tilde{\lambda}}) \right)
\]

\[
= \sum_{\lambda, r} g_{\varphi}(\lambda, 0) d_{\lambda} q^{\langle w(t), 2\rho \rangle} \mathcal{F}(u_{J(r)}^{\tilde{\lambda}}, J_{(r, \epsilon_N)} u_{J(r)}^{\tilde{\lambda}})
\]
4.2. PARTIAL RESULTS ON THE MARTIN BOUNDARY OF $SU_q(N)$

\[
\sum_{\lambda, r} g_{\nu}(\lambda, 0) d_{\lambda} g^{(\text{wt}(r), 2\rho)} F \left( \sum_{\mu, s, t} \langle J(r_{e_N}) \otimes J(r), J(s) \rangle \langle J(r_{e_N}) \otimes J(r), J(t) \rangle u^\mu_{J(s), J(t)} \right) \\
= \sum_{\lambda, r} g_{\nu}(\lambda, 0) d_{\lambda} g^{(\text{wt}(r), 2\rho)} \sum_{\mu, s, t} \langle r \otimes r_{e_N}, s \rangle \langle r \otimes r_{e_N}, t \rangle F (u^\mu_{J(s), J(t)}) \\
= \sum_{\lambda, r} g_{\nu}(\lambda, 0) d_{\lambda} g^{(\text{wt}(r), 2\rho)} \sum_{\mu, s, t} C_2^1(1, r, s) C_2^1(1, r, t) d_\mu^{-1} q^{-\langle \text{wt}(s) + \text{wt}(t), \rho \rangle} m^\mu_{s, t} \\
= \sum_{\lambda, r} \sum_{j=1}^{N} g_{\nu}(\lambda, 0) \frac{d_{\lambda}}{d_{\lambda + e_j}} g^{(\text{wt}(r), 2\rho)} C_2^1(1, r, r^{(j)}) 2q^{(\text{wt}(r) + e_N, -2\rho)} m^{\lambda + e_j}_{r^{(j)}, r^{(j)}}, 
\]  

(4.2.28)

From the definition it follows that

\[
\begin{align*}
& b_k(r^{(j)}) = \begin{cases} 
q^{-2}b_k(r) & \text{if } j \leq k; \\
 b_k(r) & \text{if } j > k.
\end{cases}
\end{align*}
\]

Use the asymptotics of Lemma 4.2.18 to find the following estimates

\[
(4.2.28) = \sum_{\lambda, r} \sum_{j=1}^{N} g_{\nu}(\lambda, 0) q^{(e_j, 2\rho)} q^{(\text{wt}(r) - \text{wt}(r) - e_N, 2\rho)} (b_{j-1}(r) - q^2b_j(r))(1 + o(1)) m^{\lambda + e_j}_{r^{(j)}, r^{(j)}} \\
= \sum_{j=1}^{N} \sum_{\lambda, r} \sum_{r^{(j)} \text{ is GT}} g_{\nu}(\lambda - e_j, 0) q^{(e_j - e_N, 2\rho)} (b_{j-1}(r^{(j)}) - q^2b_j(r^{(j)}))(1 + o(1)) m^\lambda_{r, r} \\
= \sum_{j=1}^{N} \sum_{\lambda, r} \sum_{r^{(j)} \text{ is GT}} g_{\nu}(\lambda - e_j, 0) q^{2(N-j)} (b_{j-1}(r) - b_j(r))(1 + o(1)) m^\lambda_{r, r}. 
\]

(4.2.29)

As before the terms $r$ for which $r^{(j)}$ is not GT are $o(1)$ as $\lambda \to \infty$, so these can be included in the sums. Moreover by Lemma 4.2.8

\[
\begin{align*}
b_{N-1}(r) &= q^{2(\lambda_1 + \cdots + \lambda_{N-1}) - 2(r_{2,1} + \cdots + r_{2, N-1})} = q^{2(\mu_N(r) - N^2_{\omega_{N-1}} \mu_N(r) - (\mu_{N-1}(r) - N^1_{\omega_{N-1}} \mu_N(r))} \\
&= q^{(\lambda, 2\omega_{N-1})} q^{\frac{1}{2}(\text{wt}(r), -4\omega_{N-1})}.
\end{align*}
\]

Thus (4.2.29) equals

\[
\sum_{\lambda, r} g_{\nu}(\lambda - e_1, 0) q^{2(N-1)} (1 + o(1)) b_0(r) m^\lambda_{r, r} + \sum_{j=1}^{N-1} \sum_{\lambda, r} (g_{\nu}(\lambda - e_{j+1}, 0) q^{2(N-j)-2} - g_{\nu}(\lambda - e_j, 0) q^{2(N-j)}) b_j(r)(1 + o(1)) m^\lambda_{r, r}
\]

= \sum_\lambda g_\nu(\lambda - e_1, 0)q^{2(N-1)}(1 + o(1))I_\lambda \\
+ \sum_{j=1}^{N-2} \sum_{\lambda, r} \left( g_\nu(\lambda - e_{j+1}, 0)q^{2(N-j)-2} - g_\nu(\lambda - e_j, 0)q^{2(N-j)} \right) b_j(r)(1 + o(1))m_{\lambda r}^\lambda \\
+ \sum_\lambda \left( g_\nu(\lambda - e_N, 0)q^{-2} - g_\nu(\lambda - e_{N-1}, 0) \right)q^{2(\lambda, \omega_{N-1})}(1 + o(1))\pi_\lambda(K_{-4\omega_{N-1}}).

Observe that by Lemmas 4.2.7 and 4.2.9

\mathcal{F}(\nu^{\omega_{N-1}}_{\omega_{N-1}}) = d^{-1}_{\omega_{N-1}} q^{(\omega_{N-1}, 2\rho)} m_{j(r_{\omega_{N-1}}), j(r_{\omega_{N-1}})} = [N]^{-1}_q q^{N-1} m_{e_1}^{\mu_1, \mu_N}.

Now let \((\lambda_n)\subset P_+\) be a sequence with \(b^\lim_n \lambda_n = x \in \text{int}(S_+^{N-2})\). By lemma 4.2.17

\lim_n [N]^{-1}_q q^{N-1} K_\nu(\mu_1, \mu_N) I_{\lambda_n} = \lim_n K_\nu(\mathcal{F}(\nu^{\omega_{N-1}}_{\omega_{N-1}})) I_{\lambda_n}

= \lim_n \frac{g_\nu(\lambda_n - e_1, 0)}{g_\nu(\lambda_n, 0)}q^{2(N-1)}(1 + o(1))I_{\lambda_n}

+ \lim_n \sum_{j=1}^{N-2} \sum_{r} \left( g_\nu(\lambda_n - e_{j+1}, 0)q^{2(N-j)-2} - g_\nu(\lambda_n - e_j, 0)q^{2(N-j)} \right) b_j(r)(1 + o(1))m_{\lambda r}^\lambda

+ \lim_n \left( g_\nu(\lambda_n - e_N, 0)q^{-2} - g_\nu(\lambda_n - e_{N-1}, 0) \right)q^{2(\lambda_n, \omega_{N-1})}(1 + o(1))\pi_\lambda(K_{-4\omega_{N-1}})

= \lim_n e^{(\nu, -e_1)} q^{2(N-1)} I_{\lambda_n}

+ \lim_n \sum_{j=1}^{N-2} \left( e^{(\nu, -e_{j+1})} q^{-2} - e^{(\nu, -e_j)} \right)q^{2(N-j)} b_j(r)m_{\lambda r}^\lambda

+ \lim_n \left( e^{(\nu, -e_N)} q^{-2} - e^{(\nu, -e_{N-1})} \right)q^{2(\lambda_n, \omega_{N-1})}\pi_\lambda(K_{-4\omega_{N-1}}),

as desired. \(\Box\)

To this expression apply the same remarks as before. So \(\sum_n \sum_r q^{N+1-2j} b_j(r)m_{\lambda r}^\lambda\) is not in \(c_{0\oplus P_+} B(H_{\lambda_n})\) and the coefficient \(e^{(\nu, -e_{j+1})} q^{-2} - e^{(\nu, -e_j)} \) vanishes if and only if \(x\) is in the wall of \(S_+^{N-2}\) corresponding to the root \(\alpha_j = e_j - e_{j-1}\).

4.2.4 Commutation relations in \(M(SU_3^q), \mu)\)

In theory one can compute the whole Martin boundary from the expression of Proposition 4.2.19 using equivariance and more Clebsch–Gordan coefficients. Namely the matrix coefficients of the fundamental representation (or of the dual of the fundamental representation) generate the matrix coefficients of all other irreducible representations. More precisely every matrix coefficient of any irreducible representation is a polynomial of matrix coefficients of the standard representation ([KS97, Prop. 7.22]). The other matrix coefficients of the fundamental representation can be computed using equivariance.

However, in practice this approach becomes very computational and we did not find a suitable way too actually perform such computations. The Clebsch–Gordan coefficients get to complicated and the expressions become too big to be handled. To get an idea of
4.2. PARTIAL RESULTS ON THE MARTIN BOUNDARY OF SU\(_q(N)\)

the algebra one obtains we compute some commutation relations.

**Definition 4.2.22.** Let \((H, \Delta, S, \varepsilon)\) be a Hopf algebra. The *right adjoint action* is defined as

\[
X \triangleleft Y := S(Y_1)X Y_2, \quad (X, Y \in H).
\]

In particular as the quantized universal enveloping algebra \(\hat{\mathcal{U}}_q(g)\) satisfies the axioms of a Hopf algebra we obtain a right adjoint action of \(\hat{\mathcal{U}}_q(g)\). Similarly define

\[
x \triangleleft Y := \pi_s(\hat{S}(Y_1)) x \pi_s(Y_2), \quad (x \in B(H_s), Y \in \hat{\mathcal{U}}_q(g)),
\]

which can be extended to \(l^\infty(\hat{G}_q)\). We call this again the right adjoint action of \(\hat{\mathcal{U}}_q(g)\).

The right adjoint action \(\triangleleft\) is obtained from the left adjoint action (cf. Notation 1.4.19) of \(C(G_q)\) in the following way.

**Lemma 4.2.23.** If \(x \in l^\infty(\hat{G}_q)\) and \(Y \in \hat{\mathcal{U}}_q(g)\), then

\[
x \triangleleft Y = (Y \otimes \iota) \alpha_l(x).
\]

**Proof.** From the definition of the left adjoint action and the decomposition of the multiplicative unitary (1.4.10) we get

\[
(Y \otimes \iota) \alpha_l(x) = (Y \otimes \iota) W^*(1 \otimes x) W
\]

\[
= (Y \otimes \iota) \sum_{\lambda, \mu \in P_+} \sum_{i,j,k,l} (u^\lambda_{ij} \otimes m^\lambda_{kl})^*(1 \otimes x)(u^\mu_{kl} \otimes m^\mu_{kl})
\]

\[
= (Y \otimes \iota) \sum_{\lambda, \mu \in P_+} \sum_{i,j,k,l} (S(u^\lambda_{ij}) \otimes m^\lambda_{ij})(1 \otimes x)(u^\mu_{kl} \otimes m^\mu_{kl})
\]

\[
= \sum_{\lambda \in P_+} \sum_{i,j,k,l} Y(S(u^\lambda_{ij})u^\lambda_{kl})(m^\lambda_{ij}x m^\lambda_{kl})
\]

\[
= \hat{S}(Y_1) x Y_2
\]

\[
= x \triangleleft Y,
\]

as desired. ☐

**Corollary 4.2.24.** \(P_\varphi(x \triangleleft Y) = P_\varphi(x) \triangleright Y\) and \(K_\varphi(x \triangleleft Y) = K_\varphi(x) \triangleright Y\). In particular the Poisson boundary and Martin boundary are closed under the right adjoint action \(\triangleleft\) of \(\hat{\mathcal{U}}_q(g)\).

**Proof.** Combine the preceding lemma with Lemma 3.2.14 and Proposition 3.2.15 to obtain

\[
P_\varphi(x \triangleleft Y) = P_\varphi((Y \otimes \iota) \alpha_l(x)) = (Y \otimes \iota)(\iota \otimes P_\varphi)(\alpha_l(x)) = (Y \otimes \iota) \alpha_l(P_\varphi(x))
\]

\[
P_\varphi(x) \triangleleft Y;
\]

\[
K_\varphi(x \triangleleft Y) = K_\varphi((Y \otimes \iota) \alpha_l(x)) = (Y \otimes \iota)(\iota \otimes K_\varphi)(\alpha_l(x)) = (Y \otimes \iota) \alpha_l(K_\varphi(x))
\]

\[
K_\varphi(x) \triangleleft Y;
\]

which give the result. ☐
Definition 4.2.25 ([DCN15, Def. 2.7]). Let g be a semisimple complex Lie algebra and \( S \subset \Delta_+ \) be a subset of the positive roots. Put
\[
\varepsilon_i := \begin{cases} 
0 & \text{if } \bar{\alpha}_i \notin S; \\
1 & \text{if } \bar{\alpha}_i \in S.
\end{cases}
\]

Define \( \bar{U}_q(g, S) \) to be the universal algebra with generators \( E_i, F_i, K_i, K_i^{-1}, i = 1, \ldots, r \) subject to the relations (1.6.1), (1.6.2), (1.6.4), (1.6.5) and
\[
[E_i, F_j] = \delta_{ij} \frac{\varepsilon_i K_i^2 - K_i^{-2}}{q_i - q_i^{-1}}.
\]

Note that \( \bar{U}_q(g, \Delta_+) = \bar{U}_q(g) \). Moreover \( \bar{U}_q(g, S) \) becomes a Hopf \(*\)-algebra when given the structure defined by (1.6.6)–(1.6.9).

Lemma 4.2.26. The adjoint action satisfies the following relations in \( \bar{U}_q(\mathfrak{su}_N, S) \):

(i) \( 1 \triangleleft Y = \varepsilon(Y)1 \);

(ii) \( K_\omega \triangleleft X_i^\pm = q^{\pm 1}(q^{\pm(\alpha_i, \omega)/2} - 1)X_i^\pm K_\omega, \) for \( \omega \in P \);

(iii) \( x \triangleleft Y = (x^* \triangleleft (S^{-1}(Y^*))^* \).

Proof. The first is immediate from the defining property of the antipode (1.4.6). The second follows from the commutation relations in \( \bar{U}_q(\mathfrak{su}_N) \). Indeed,
\[
K_\omega \triangleleft X_i^\pm = S(X_i^\pm)K_\omega K_\alpha + S(K_\alpha^{-1})K_\omega X_i^\pm = -q^{\pm 1}X_i^\pm K_\omega + K_\alpha + K_\omega X_i^\pm.
\]

The last identity can be computed via
\[
(x \triangleleft Y)^* = (S(Y(1))xY(2))^* = S^{-1}(Y(2))x^*S^{-1}(Y(1)) = S(S^{-1}(Y^*)(1))x^*S^{-1}(Y^*)(2) = x^* \triangleleft (S^{-1}(Y^*)),
\]
as desired. \( \qed \)

To compute some commutation relations we restrict ourselves to the case \( N = 3 \), this is already quite involved. We work in the following setting. Let \( \lambda \in P_+ \), write it as \( \lambda = 1\lambda_1 e_1 + 1\lambda_2 e_2 \) and assume that \( 1\lambda_1 > 1\lambda_2 > 0 \). Consider the sequence \((\lambda_n)_n \) where \( \lambda_n := n\lambda \). Then \( b^-\lim_n \lambda_n = b^-\lim_n n\lambda = \frac{\lambda}{|\lambda|} =: x \in \text{int}(S_+^1) \).

Using the notation of Proposition 4.2.19 we introduce the following element in the algebra \( \mathcal{H}_n \)
\[
l_{-4\omega_2} := [3]q^2 \left( e^{(\epsilon \varphi(-2), e_1)} - q^2 e^{(\epsilon \varphi(-2), e_2)} \right) \sum_{n=1}^\infty q^{2n(\bar{\lambda}, \omega_1)} n! \pi_n \lambda(K_{-4\omega_2})
\]
\[
+ [3]q^2 \left( e^{(\epsilon \varphi(-2), e_2)} - q^2 e^{(\epsilon \varphi(-2), e_1)} \right) \sum_{n=1}^\infty \sum_{r \in \mathcal{G}(\lambda)} q^{2n} m_n \pi_n \lambda(J(r), J(r)).
\]
Here $r_{2,2}$ indicates the entry of the GT tableau $r = (r_{i,j})_{i,j}$. Note that

$$\sum_{n=1}^{\infty} K_{-\varphi} (c_{-\omega_2, \omega_2}) I_{n\lambda} = l_{-4\omega_2} + [3]_q q^2 e^{(r_{-\bar{x}}, e_2)} \sum_{n=1}^{\infty} I_{n\lambda} \mod c_0 \bigoplus_n B(\mathcal{H}_{n\lambda}).$$

Motivated by the case of $SU_q(2)$ in which the Martin boundary is the quantum homogeneous sphere of Podleś (see Subsection 4.4.1), we expect that $M(SU_q(N), \mu)$ will be a field of quantum flag manifolds over the space $M(Irr(SU_q(N)), \mu)$, the Martin boundary of the classical random walk on the set of irreducible representations. For each subset of the positive roots $S \subset \Delta_+$ De Commer and Neshveyev [DCN15] defined the degeneration $\hat{U}_q(g, S)$ of $U_q(g)$. They showed that these algebras $\hat{U}_q(g, S)$ are isomorphic to quantum flag manifolds. Comparing this again with the computations of the Martin kernel of the matrix unit corresponding to the fundamental representation of $SU_q(N)$ (see Lemmas 4.2.27 and 4.2.28 below).

**Lemma 4.2.27.** In the notation above the operator $l_{-4\omega_2}$ is self-adjoint and satisfies the following relations in the algebra $\mathbb{C}^\infty \bigoplus_n B(\mathcal{H}_{n\lambda})/ c_0 \bigoplus_n B(\mathcal{H}_{n\lambda})$:

$$\begin{align*}
(l_{-4\omega_2} \vartriangleleft E_2) l_{-4\omega_2} - q^2 l_{-4\omega_2} l_{-4\omega_2} (l_{-4\omega_2} \vartriangleleft E_2) &= C_x \sum_{n=1}^{\infty} \sum_{r_1, r_2, \lambda} f(r) q^{-r_1, 2 + r_2, 1 + 2r_2, 2 + r_3, 1} m_{J(r_1, 1), J(r)}^{n\lambda}, \quad (4.2.30) \\
(l_{-4\omega_2} \vartriangleleft F_2) l_{-4\omega_2} - q^2 l_{-4\omega_2} l_{-4\omega_2} (l_{-4\omega_2} \vartriangleleft F_2) &= C_x q^{-3} \sum_{n=1}^{\infty} \sum_{r_1, r_2, \lambda} f(r) q^{-r_1, 2 + r_2, 1 + 2r_2, 2 + r_3, 1} m_{J(r_1, 1), J(r_2, 1)}^{n\lambda}, \quad (4.2.31)
\end{align*}$$

where

$$C_x := -([3]_q)^2 q^5 (e^{(r_{-\bar{x}}, e_3)} - q^2 e^{(r_{-\bar{x}}, e_2)}) (e^{(r_{-\bar{x}}, e_2)} - q^2 e^{(r_{-\bar{x}}, e_1)}) (q^{-1} - q)^2$$

is a constant depending on $x$ and $f$ is a function with the property $a < f(r) < b$ for all GT tableaux $r$ such that $r_{2,1}$ is GT. These constants $0 < a < b < \infty$ are independent of $r$.

**Lemma 4.2.26** gives us the identity in $\hat{U}_q(\mathfrak{su}_3)$

$$K_{-4\omega_i} \triangleleft X_i^\pm K_{-4\omega_i} = q^{\pm 1} (q^2 - 1) X_i^\pm K_{-4\omega_i + \alpha_i} K_{-4\omega_i} = q^{\pm 1} (q^2 - 1) q^{(\alpha_i, -4\omega_i)/2} K_{-4\omega_i} X_i^\pm K_{-4\omega_i + \alpha_i} = q^{\pm 2} K_{-4\omega_i} (K_{-4\omega_i} \triangleleft X_i^\pm).$$

From Lemma 4.2.27 it follows that $l_{-4\omega_2}$ does not behave the same as $K_{-4\omega_2}$. Indeed, (4.2.30) is nonzero modulo $c_0 \bigoplus_n B(\mathcal{H}_{n\lambda})$. Namely, consider the sequence of GT tableaux
(r_n)_n$, where
\[ r_n := \begin{pmatrix} n\lambda_1 & n\lambda_2 & 0 \\ n\lambda_2 + 1 & 0 \\ 0 & 0 \end{pmatrix}, \]
then \( f(r_n)q^{-(r_n)_{1,2}+(r_n)_{2,1}+2(r_n)_{2,2}+(r_n)_{3,1}} \geq aq \) for all \( n \), where the constant \( a \) is as in the statement of the lemma.

**Proof of Lemma 4.2.27.** Clearly \( l_{-\omega} \) is self-adjoint as \( K_\omega \) is self-adjoint for any \( \omega \) and \( m_{\xi\eta} \) is self-adjoint for any vector \( \xi \) by (1.4.4).

Equation (4.2.31) can be obtained from (4.2.30) by means of Lemma 4.2.26 part (iii).

Identity (4.2.30) requires a long computation. First
\[ m_{\xi,\eta} \triangleq X = S(X_{(1)})m_{\xi,\eta}X_{(2)} = m_{S(X_{(1)})\xi,\eta}X_{(2)} \]
Thus in particular
\[ m_{\tilde{\lambda}}^3 j(k_0),J(s) \triangleq X_1^\pm = m_{\tilde{\lambda}}^3 S(X_1^\pm)j(k_0),J(s) + m_{\tilde{\lambda}}^3 S(K_0^{-1})j(k_0),(X_1^\pm)^*j(s) \]
\[ = m_{\tilde{\lambda}}^3 j(q^{1/2}X_1^\pm,\lambda)^{-1} + m_{\tilde{\lambda}}^3 j(q^{-1/2}X_1^\pm,\lambda) \]
\[ = q^{-1/2}(wt(s),\alpha_1)\pm 1 m_{\tilde{\lambda}}^3 j(X_1^\pm,\lambda) - q^{-1/2}(wt(r),\alpha_1) m_{\tilde{\lambda}}^3 j(r),J(X_1^\pm) \]

In combination with Lemma 4.2.2 this gives
\[
\sum_r q^{2r_2,2}(m_{\tilde{\lambda}}^3 j(k_0),J(r) \triangleq E_2)(J(s)) \\
= \sum_r q^{2r_2,2}q^{-1/2}(wt(r),\alpha_2) (q m_{\tilde{\lambda}}^3 j(F_2^r),J(r)(J(s)) - m_{\tilde{\lambda}}^3 j(r),J(E_2)(J(s))) \\
= \sum_r q^{2r_2,2}q^{-1/2}(wt(r),\alpha_2) (q(j(r),J(s))J(A_{2,1}(r_{2,1})r_{2,1} + A_{2,2}(r_{2,2})r_{2,2}) \\
- \langle J(E_2)(r),J(s) \rangle \rangle \\
= q^{2s_2,2}q^{-1/2}(wt(s),\alpha_2) (A_{2,1}(s_{2,1})J(s_{2,1}) + A_{2,2}(s_{2,2})J(s_{2,2})) \\
- \sum_r q^{2r_2,2}q^{-1/2}(wt(r),\alpha_2) (F_2^s, r) \langle J(r) \rangle \\
= q^{2s_2,2}q^{-1/2}(wt(s),\alpha_2) (A_{2,1}(s_{2,1})J(s_{2,1}) + A_{2,2}(s_{2,2})J(s_{2,2})) \\
- \sum_r q^{2r_2,2}q^{-1/2}(wt(s),\alpha_2) (A_{2,1}(s_{2,1})(s_{2,1}, r) \langle J(r) \rangle + A_{2,2}(s_{2,2})(s_{2,2}, r) \langle J(r) \rangle) \\
= q^{2s_2,2}q^{-1/2}(wt(s),\alpha_2) (A_{2,1}(s_{2,1})J(s_{2,1}) + A_{2,2}(s_{2,2})J(s_{2,2})) \\
- q^{-1/2}(wt(s),\alpha_2) (q^{2(s_{2,1})2}A_{2,1}(s_{2,1})J(s_{2,2}) + q^{2(s_{2,2})2}A_{2,2}(s_{2,2})J(s_{2,2})) \\
= q^{2s_2,2}q^{-1/2}(wt(s),\alpha_2) (1 - q^{-2})A_{2,2}(s_{2,2})J(s_{2,2}) \\
= (q - q^{-1}) \sum_r q^{2r_2,2}q^{-1/2}(wt(r),\alpha_2) A_{2,2}(r_{2,2})m_{\tilde{\lambda}}^3 j(r_{2,2},J(r))(J(s)).
\]
The notation can be a bit ambiguous, but recall the difference between $r_{2,2}$ and $r_{2,2}$; see Lemma 4.2.5 and Definition 4.2.1. Hence it follows

$$\sum_r q^{2r_{2,2}}(m_{J(r),J(r)}^{n,λ}) \ll E_2 = (q - q^{-1}) \sum_r q^{2r_{2,2}}q^{-\frac{1}{2}(\text{wt}(r),\alpha_2)}A_{2,2}(r_{2,2})m_{J(r),J(r)}^{n,λ}.$$ 

To shorten the notation, let

$$C_3 := [3]q^2(\epsilon(\varphi(-x),e_3) - q^2\epsilon(\varphi(-x),e_2)),$$

$$C_2 := [3]q^2(\epsilon(\varphi(-x),e_2) - q^2\epsilon(\varphi(-x),e_1)),$$

$$C_1 := [3]q^2\epsilon(\varphi(-x),e_1).$$

We get

$$l_{-4\omega_2} \ll E_2 = C_3 \sum_{n=1}^{\infty} q^{2n(\lambda,\omega_2)}\pi_{n,\lambda}(K_{-4\omega_2}) \ll E_2 + C_2 \sum_{n=1}^{\infty} \sum_r q^{2r_{2,2}}(m_{J(r),J(r)}^{n,λ}) \ll E_2$$

$$+ C_2(q - q^{-1}) \sum_{n=1}^{\infty} \sum_{r, \lambda \neq \lambda} q^{2r_{2,2}}q^{-\frac{1}{2}(\text{wt}(r),\alpha_2)}A_{2,2}(r_{2,2})m_{J(r),J(r)}^{n,λ}.$$ 

We compute $(l_{-4\omega_2} \ll E_2)l_{-4\omega_2} = (q - q^{-1})\sum_{n=1}^{\infty}(I_n + II_n + III_n + IV_n)$, where

$$I_n = (C_3)^2 q^{2n(\lambda,\omega_2)}q^{2n(\lambda,\omega_2)}\pi_{n,\lambda}(E_2 K_{-4\omega_2+\alpha_2})\pi_{n,\lambda}(K_{-4\omega_2})$$

$$= (C_3)^2 q^{4n(\lambda,\omega_2)}\pi_{n,\lambda}(E_2 K_{-8\omega_2+\alpha_2});$$

$$II_n = C_2C_3 \sum_{r, \lambda \neq \lambda} q^{2n(\lambda,\omega_2)}q^{r_{2,2}0}\pi_{n,\lambda}(E_2 K_{-4\omega_2+\alpha_2})m_{J(r),J(r)}^{n,λ}$$

$$= C_2C_3 \sum_{r, \lambda \neq \lambda} q^{2n(\lambda,\omega_2)}q^{r_{2,2}}q^{-\frac{1}{2}(\text{wt}(r),-4\omega_2+\alpha_2)}m_{E_2 J(r),J(r)}^{n,λ};$$

$$III_n = -C_2C_3 \sum_{r, \lambda \neq \lambda} q^{2r_{2,2}}q^{-\frac{1}{2}(\text{wt}(r),\alpha_2)}A_{2,2}(r_{2,2})m_{J(r),J(r)}^{n,λ}q^{2n(\lambda,\omega_2)}\pi_{n,\lambda}(K_{-4\omega_2})$$

$$= -C_2C_3 \sum_{r, \lambda \neq \lambda} q^{2r_{2,2}}q^{-\frac{1}{2}(\text{wt}(r),-4\omega_2+\alpha_2)}q^{2n(\lambda,\omega_2)}A_{2,2}(r_{2,2})m_{J(r),J(r)}^{n,λ};$$
Similarly we compute \( I_{n} = \sum_{r_{1}, s_{1} = n_{\lambda}}^{n_{\lambda}} q^{2r_{2,2} q^{r_{1},a_{2}} A_{2,2}(r_{2,2}) m_{J(r_{2,2}), J(r)}} \sum_{s_{1}, s = n_{\lambda}} q^{2s_{2,2} n_{\lambda}} m_{J(s), J(s)} \)

\[ \sum_{r_{1}, s = n_{\lambda}}^{n_{\lambda}} q^{2r_{2,2} q^{r_{1},a_{2}} A_{2,2}(r_{2,2}) m_{J(r_{2,2}), J(r)}} \sum_{s_{1}, s = n_{\lambda}} q^{2s_{2,2} n_{\lambda}} m_{J(s), J(s)} \]

\[ \sum_{r_{1}, s = n_{\lambda}}^{n_{\lambda}} q^{2r_{2,2} q^{r_{1},a_{2}} A_{2,2}(r_{2,2}) m_{J(r_{2,2}), J(r)}} \sum_{s_{1}, s = n_{\lambda}} q^{2s_{2,2} n_{\lambda}} m_{J(s), J(s)} \]

\[ \sum_{r_{1}, s = n_{\lambda}}^{n_{\lambda}} q^{2r_{2,2} q^{r_{1},a_{2}} A_{2,2}(r_{2,2}) m_{J(r_{2,2}), J(r)}} \sum_{s_{1}, s = n_{\lambda}} q^{2s_{2,2} n_{\lambda}} m_{J(s), J(s)} \]

Similarly we compute \( I_{n} = \sum_{n=1}^{\infty} (V_{n} + V_{I_{n}} + V_{I_{I_{n}}} + V_{I_{V_{I_{n}}}}) \), where

\[ V_{n} = (C_{3})^{2} q^{2n(\lambda_{\omega_{2}})} q^{2n(\lambda_{\omega_{2}})} \pi_{n_{\lambda}}(K_{-4\omega_{2}}) \pi_{n_{\lambda}}(E_{2} K_{-4\omega_{2} + \omega_{2}}) \]

\[ = (C_{3})^{2} q^{2n(\lambda_{\omega_{2}})} q^{2n(\lambda_{\omega_{2}})} \pi_{n_{\lambda}}(E_{2} K_{-8\omega_{2} + \omega_{2}}) \]

\[ VI_{n} = C_{2} C_{3} \sum_{r_{1}, r_{2}, r_{3}, s_{1} = n_{\lambda}}^{n_{\lambda}} q^{2r_{2,2} q^{2n(\lambda_{\omega_{2}})} m_{J(r), J(r)} \pi_{n_{\lambda}}(E_{2} K_{-4\omega_{2} + \omega_{2}})} \]

\[ = C_{2} C_{3} \sum_{r_{1}, r_{2}, r_{3}, s_{1} = n_{\lambda}}^{n_{\lambda}} q^{2n(\lambda_{\omega_{2}})} q^{2r_{2,2} q^{r_{1},a_{2}} A_{2,2}(r_{2,2}) m_{J(r_{2,2}), J(r)}} \]

\[ = C_{2} C_{3} q \sum_{r_{1}, r_{2}, r_{3}, s_{1} = n_{\lambda}}^{n_{\lambda}} q^{2n(\lambda_{\omega_{2}})} q^{2r_{2,2} q^{r_{1},a_{2}} A_{2,2}(r_{2,2}) m_{J(r_{2,2}), J(r)}} \]

\[ VIII_{n} = -C_{2} C_{3} \sum_{r_{1}, r_{2}, r_{3}, s_{1} = n_{\lambda}}^{n_{\lambda}} q^{2n(\lambda_{\omega_{2}})} q^{2r_{2,2} q^{r_{1},a_{2}} A_{2,2}(r_{2,2}) m_{J(r_{2,2}), J(r)}} \]

\[ = -C_{2} C_{3} \sum_{r_{1}, r_{2}, r_{3}, s_{1} = n_{\lambda}}^{n_{\lambda}} q^{2n(\lambda_{\omega_{2}})} q^{2r_{2,2} q^{r_{1},a_{2}} A_{2,2}(r_{2,2}) m_{J(r_{2,2}), J(r)}} \]

\[ = -C_{2} C_{3} \sum_{r_{1}, r_{2}, r_{3}, s_{1} = n_{\lambda}}^{n_{\lambda}} q^{2n(\lambda_{\omega_{2}})} q^{2r_{2,2} q^{r_{1},a_{2}} A_{2,2}(r_{2,2}) m_{J(r_{2,2}), J(r)}} \]

It follows that \( I_{n} = q^{2} V_{n}, III_{n} = q^{2} VII_{n} \) and \( IV_{n} = q^{2} VIII_{n} \). To complete the proof it
remains to compute $\Pi_n - q^2 \text{VI}_n$. We have

$$\sum_{r_1, = \lambda \atop r \text{ is GT}} q^{2r_2, q} q^{-\frac{1}{2}(\omega(t), -4\omega_2 + \alpha_2)} \left(q^3 m_{J(r)}^{\lambda_1} - m_{E_2J(r), J(r)}^{\lambda_1}\right)(J(s))$$

$$= \sum_{r_1, = \lambda \atop r \text{ is GT}} q^{2r_2, q} q^{-\frac{1}{2}(\omega(t), -4\omega_2 + \alpha_2)} \left(q^3 (J(E_2r), J(s))J(r) - (J(r), J(s))\pi_{\lambda}(E_2J(r))\right)$$

$$= \sum_{r_1, = \lambda \atop r \text{ is GT}} q^{2r_2, q} q^{-\frac{1}{2}(\omega(t), -4\omega_2 + \alpha_2)} \left(-q^3 (\langle s, A_{2,1}(r)\rangle r^{2,1} + A_{2,2}(r)r^{2,2})\right)J(r)$$

$$+ \langle s, r\rangle \left(A_{2,1}(r_2, 1)J(r_2, 1) + A_{2,2}(r_2, 2)J(r_2, 2)\right)$$

$$= -q^{2s_2, q} q^{-\frac{1}{2}(\omega(s), -4\omega_2 + \alpha_2)} q^3 A_{2,1}(s_2, 1)J(s_2, 1)$$

$$- q^{2s_2, q} q^{-\frac{1}{2}(\omega(s), -4\omega_2 + \alpha_2)} q^3 A_{2,2}(s_2, 2)J(s_2, 2)$$

$$+ q^{2s_2, q} q^{-\frac{1}{2}(\omega(s), -4\omega_2 + \alpha_2)} \left(A_{2,1}(s_2, 1)J(s_2, 1) + A_{2,2}(s_2, 2)J(s_2, 2)\right)$$

$$= q^{2s_2, q} q^{-\frac{1}{2}(\omega(s), -4\omega_2 + \alpha_2)} (1 - q^2) A_{2,1}(s_2, 1)J(s_2, 1)$$

$$= (1 - q^2) \sum_{r_1, = \lambda \atop r, r_2, 2, 1 \text{ are GT}} q^{2r_2, q} q^{-\frac{1}{2}(\omega(t), -4\omega_2 + \alpha_2)} A_{2,1}(r_2, 1)m_{J(r_2, 1), J(r)}^{\lambda_1}(J(s)).$$

Observe that by Lemma 1.2.3 there exists $0 < c_1 < c_2 < \infty$ and a function $f$ such that the constants $A_{2,1}(r)$ are of the form

$$A_{2,1}(r_2, 1) = \left(f(r)^2 q^{(-r_1, 1 + r_2, 2, 1) + (-r_1, 1 - r_2, 1) + (r_3, 1 - r_2, 1) + (r_2, 1 - r_2, 2) + (r_1, 2 - r_2, 2, 1)}\right)^{1/2}$$

$$= f(r) q^{\frac{1}{2}(-r_1, 1 + r_2, 1 - 2r_2, 2, 1)}$$

with the property that $c_1 < f(r) < c_2$ for all $r$ for which $r_2, 2, 1$ is GT. This implies that

$$(q^{-1} - q)\sum_{n=1}^{\infty} q^n \text{VI}_n - \Pi_n$$

$$= C_2C_3(q^{-1} - q)(1 - q^2) \sum_{n=1}^{\infty} \sum_{r_1, = \lambda \atop r, r_2, 2, 1 \text{ are GT}} q^{2n(\lambda, \omega_2)} q^{2r_2, q} q^{-\frac{1}{2}(\omega(t), -4\omega_2 + \alpha_2)} A_{2,1}(r_2, 1)m_{J(r_2, 1), J(r)}^{\lambda_1}$$

$$= C_2C_3 q^{-1} - q^2 \sum_{n=1}^{\infty} \sum_{r_1, = \lambda \atop r, r_2, 1 \text{ are GT}} q^{2n(\lambda_1 - \frac{1}{2}(\lambda_1 + \lambda_2))} q^{2r_2, q} q^{2(\mu_2(r) - \frac{1}{2} \mu_3(r)) - \mu_2(r) + \frac{1}{2} \mu_3(r) + \frac{1}{2} \mu_1(r)}$$

$$\times f(r) q^{\frac{1}{2}(-r_1, 1 + r_2, 1 - 2r_2, 2, 1)} m_{J(r_2, 1), J(r)}^{\lambda_1}$$

$$= C_2C_3 q^{-1} - q^2 \sum_{n=1}^{\infty} \sum_{r_1, = \lambda \atop r, r_2, 1 \text{ are GT}} f(r) q^{-(r_1, 2 + r_2, 1 + 2r_2, 2, 1 + r_2, 2, 1)} m_{J(r_2, 1), J(r)}^{\lambda_1},$$

as required.
The following relation holds in $\bar{U}_q(\mathfrak{su}_N, S)$

\[(q^2 - q)^{-1}(q^2 - 1)^{-1}(q^{-1}(K_{-4\omega_i} \triangleleft E_i)(K_{-4\omega_i} \triangleleft F_i) - q(K_{-4\omega_i} \triangleleft F_i)(K_{-4\omega_i} \triangleleft E_i)) = \begin{cases} 
\frac{1}{q-q^{-1}}(K_{-8\omega_i+4\alpha_i} - K_{-8\omega_i}), & \text{if } S = \Delta_i; \\
\frac{-1}{q-q^{-1}} K_{-8\omega_i}, & \text{if } S = \emptyset. 
\end{cases} \quad (4.2.33)\]

A similar but even longer computation than Lemma 4.2.27 yields the following result. We omit the proof.

**Lemma 4.2.28.** In the notation introduced above the operator $l_{-4\omega_2}$ satisfies the following relation in the algebra $\mathfrak{l}^\infty_0 \bigoplus_n B(\mathcal{H}_n, \lambda) / c_0^0 \bigoplus_n B(\mathcal{H}_n, \lambda)$.

\[(q^2 - q)^{-1}(q^2 - 1)^{-1}(q^{-1}(l_{-4\omega_i} \triangleleft E_i)(l_{-4\omega_i} \triangleleft F_i) - q(l_{-4\omega_i} \triangleleft F_i)(l_{-4\omega_i} \triangleleft E_i)) = \sum_{n=1}^{\infty} \left( (C_3)^2(q - q^{-1})^{-1} q^{4(n\lambda, \omega_2)} \pi_{n\lambda}(-K_{-8\omega_2}) + C_2 C_3 q^{2(n\lambda, \omega_2)} \right) \times \left( \sum_{r_{1,1} = n\lambda} q^{\frac{1}{2}(\text{wt}(r), 4\omega_2 - 2\omega_2)} q^{2r_{2,2}} \left( (1 + q)(A_{2,2}(r))^{-2} - (1 + q^{-1})(A_{2,2}(r_2, 2))^{-2} \right) m_{J(r), J(r)}^{n\lambda} + \sum_{r_{1,1} = n\lambda} q^{\frac{1}{2}(\text{wt}(r), 5\omega_2 - 2\omega_2)} q^{2r_{2,2}} \left( (A_{2,2}(r_2, 2))^{2} - (A_{2,2}(r_2, 2))^{-2} \right) m_{J(r), J(r)}^{n\lambda} \right)
\]

where $C_2$ and $C_3$ are as in (4.2.32).

Because of Equation (4.2.33) above we want that modulo $c_0^0 \bigoplus_n B(\mathcal{H}_n, \lambda)$ this big expression equals $\frac{-1}{q-q^{-1}} I^2_{-4\omega_2}$, but again this does not hold. Consider for instance the term

\[q^{2(n\lambda, \omega_2)} q^{\frac{1}{2}(\text{wt}(r), 4\omega_2 - 2\omega_2)} q^{2r_{2,2}} (A_{2,2}(r) A_{2,1}(r_2, 2) - q A_{2,2}(r_2, 1) A_{2,1}(r_2, 1))
\]

\[= q^{2(n\lambda, \omega_2)} q^{\frac{1}{2}(\text{wt}(r), 4\omega_2 - 2\omega_2)} q^{2r_{2,2}}
\]

\[\times \left( \left[ q_{1,1} - r_{2,2} + 1 \right] q_{1,1} - r_{2,2} q \left[ r_{1,3} - r_{2,2} - 1 \right] q \left[ r_{3,1} - r_{2,2} \right] q \right)^{\frac{1}{2}}
\]

\[\times \left[ q_{1,1} - r_{2,1} + 1 \right] q_{1,1} - r_{2,1} q \left[ r_{1,3} - r_{2,1} - 1 \right] q \left[ r_{3,1} - r_{2,1} \right] q \right)^{\frac{1}{2}}
\]

\[\times - q \left( q_{1,1} - r_{2,2} + 1 \right] q_{1,1} - r_{2,2} q \left[ r_{1,3} - r_{2,2} - 1 \right] q \left[ r_{3,1} - r_{2,2} \right] q \right)^{\frac{1}{2}}
\]

\[\times \left[ q_{1,1} - r_{2,1} + 1 \right] q_{1,1} - r_{2,1} q \left[ r_{1,3} - r_{2,1} - 1 \right] q \left[ r_{3,1} - r_{2,1} \right] q \right)^{\frac{1}{2}}
\]

\[= q^{r_{1,1} - r_{1,2} + 2r_{2,2} + r_{3,1}} f(r) (1 - q) \left( q^{-r_{1,1} + r_{2,2} - 1} q^{-r_{1,2} + r_{2,2}} q^{-r_{2,2} + r_{1,3} - 1} q^{-r_{3,1} + r_{2,2}} q^{-r_{1,1} + r_{2,2}} \right)^{\frac{1}{2}}
\]

\[\times q^{-r_{2,1} + r_{1,2}} q^{-r_{2,1} + r_{1,3} - 1} q^{-r_{2,1} + r_{3,1}} q^{-r_{2,1} - r_{2,2}} q^{-r_{2,1} - r_{2,2} - 1} q^{-r_{2,1} - r_{2,2} - 1} \]
4.2. PARTIAL RESULTS ON THE MARTIN BOUNDARY OF SU_q(N)

\[ C_2C_3q^{2(\lambda,\omega_2)} \sum_{n=1}^{\infty} \sum_{\lambda_1,=n\lambda} q^{\frac{1}{2}(\text{wt}(r),4\omega_2-2\omega_2)} q^{r_2,2} (A_{2,2}(r)A_{2,1}(r^2_{2,1}) - qA_{2,2}(r_{2,1})A_{2,1}(r_{2,1})) \times m_{J(r^2_{2,1}),J(r)}^{n\lambda} \]

is not in \( c_0 \bigoplus_n B(H_{\lambda_n}) \). Observe that \( \frac{1}{q-q^{-1}} l^2_{-4\omega_2} \) does not have terms of the form \( m_{J(r^2_{2,1}),J(r)}^{n\lambda} \). Indeed, from Proposition 4.2.19 it follows that modulo \( c_0 \bigoplus_n B(H_{\lambda_n}) \) the operator \( \pi_{n\lambda}(l_{-4\omega_2}) \) is a diagonal matrix with respect to the basis \( \{J(r) : r_1, = n\lambda\} \). Thus (4.2.34) cannot be equal to \( \frac{1}{q-q^{-1}} l^2_{-4\omega_2} \).

**Conclusion 4.2.29.** These phenomena which occur for \( SU_q(3) \) of course also occur for \( SU_q(N) \) for \( N \geq 3 \). At this moment it is not clear what kind of \( C^* \)-algebra we can expect to obtain as the Martin boundary.

The quantum random walk restricted to the center \( Z(l^\infty(SU_q(N))) \) defines a classical random walk on \( P_+ = \text{Irr}(SU_q(N)) \). The Martin boundary of this classical random walk \( M(P_+, \mu) \) can be embedded into the center \( Z(M(SU_q(N), \mu)) \) (see (3.2.4)). So the full Martin boundary forms a \( C(M(P_+, \mu)) \)-algebra. Recall, if \( X \) is a compact space, a \( C(X) \)-algebra is a special case of a \( C_0(Y) \)-algebra introduced by Kasparov [Kas88] for locally compact spaces \( Y \). Since the Martin boundary \( M(P_+, \mu) \) contains contains the space \( \text{int}(S^N_{\perp}) \) (cf. Theorem 4.1.27) we tried to compute the fibers over points \( x \in \text{int}(S^N_{\perp}) \) to get some information on the \( C^* \)-algebra \( M(SU_q(N), \mu) \). In order to do this we needed to compute the \( C^* \)-subalgebra in \( l^\infty(\bigoplus_n B(H_{\lambda_n})) \) generated by \( \bigoplus_n K_\varphi(x)I_{\lambda_n} \) and \( c_0 \bigoplus_n B(H_{\lambda_n}) \) for sequences \( (\lambda_n)_n \) with \( b\text{-lim} \lambda_n = x \). A priori we hoped that these fibers were isomorphic to quantum flag manifolds via the degenerated quantum enveloping algebras and the isomorphism described in [DCN15, Thm. 2.8]. Then \( \pi_{\lambda_n}(K_\varphi(m^\omega_{\gamma_{\omega,\omega\omega}},\gamma_{\omega\omega})) \) should correspond to \( c\pi_{\lambda_n}(K_{-4\omega}) + dI_{\lambda_n} \) for some \( c, d \in \mathbb{C} \). However, this turns out not to be true, as the commutation relations in Lemmas 4.2.27 and 4.2.28 point out.

The reason why such a correspondence fails is the appearance of the intermediate terms in (4.2.24). Without these intermediate terms the obstructions vanish, so in fact these elements \( K_\varphi(m^\omega_{\gamma_{\omega,\omega\omega}},\gamma_{\omega\omega}) \) are very similar to \( cK_{-4\omega} + d\text{Id} \). Note that these lemmas do not tell us that an isomorphism between the fibers and the quantum flag manifolds does not exist at all, they only state that \( \pi_{\lambda_n}(K_\varphi(m^\omega_{\gamma_{\omega,\omega\omega}},\gamma_{\omega\omega})) \) cannot correspond to \( c\pi_{\lambda_n}(K_{-4\omega}) + dI_{\lambda_n} \).

At this point the current approach seems not very fruitful to compute the Martin boundary since we no longer have a canonical algebra to look for, while at the same time further computations become increasingly more complicated. One other approach to get some partial understanding of the algebra can be to find an appropriate ideal \( I \) in \( M(SU_q(N), \mu) \) such that \( M(SU_q(N), \mu)/I \) is indeed isomorphic to a quantum flag manifold. The idea is that such an ideal absorbs all the terms that one wants to get rid of, i.e. all the terms...
that obstruct the correspondence described above. To look for such an ideal is again not an easy task. Indeed, suppose that $J$ is an ideal in $l^\infty(SU_q(N))$, then $J \cap B(H_\lambda)$ is an ideal in $B(H_\lambda)$ so it equals either $\{0\}$ or $B(H_\lambda)$. Hence $J$ is a sum of full matrix algebras $l^\infty(\bigoplus_{s \in S} B(H_\lambda))$ for some subset $S \subset P_+$, but that is not the kind of ideal we are after. So one really needs an ideal in $M(SU_q(N), \mu)$ and not an ideal which comes from $l^\infty(SU_q(N))$. Unfortunately, to define such an ideal one needs more information about $M(SU_q(N), \mu)$ which is exactly what we wanted in the first place, so one ends up reasoning in a kind of circle.

4.3 Convergence to the boundary for quantum groups

Discrete Markov chains are known to converge to the boundary (cf. Theorem 3.1.12). For random walks on discrete quantum groups this is still an open problem. In this section we prove some results regarding the convergence of paths in the quantum setting.

**Definition 4.3.1.** Recall maps $j_n$ defined in Definition 3.2.12. An element $x$ is called $\varphi$-regular if $s^* \lim_n j_n(x)$ exists in $\bigotimes_{-\infty}^{-1}(l^\infty(\hat{G}), \varphi)$. Let $R_\varphi := \{x \in l^\infty(\hat{G}) : x$ is $\varphi$-regular$\}$. For $x \in R_\varphi$ define $j_\infty(x) := s^* \lim_n j_n(x)$. Write $\varphi^\infty := \cdots \otimes \varphi \otimes \varphi$ and observe that $\varphi^\infty \circ j_n = \varphi^n$.

**Proposition 4.3.2** (Izumi [Izu02] and Neshveyev–Tuset [NT03]). Let $(l^\infty(\hat{G}), \hat{\Delta})$ be a discrete quantum group and $\mu$ be a probability measure on $\text{Irr}(G)$. Assume that the random walk defined by $\varphi = \varphi_\mu$ is generating and transient. The following holds:

1. $R_\varphi$ is a $C^*$-subalgebra of $l^\infty(\hat{G})$ that contains $c_0(\hat{G})$ and $H^\infty(\hat{G}, \varphi)$.

Write $\theta := j_\infty|_{H^\infty(\hat{G}, \varphi)} : H^\infty(\hat{G}, \varphi) \to \bigotimes_{-\infty}^{-1}(l^\infty(\hat{G}), \varphi)$. Then

2. $j_\infty : R_\varphi \to \bigotimes_{-\infty}^{-1}(l^\infty(\hat{G}), \varphi)$ is a $*$-homomorphism onto $\theta(H^\infty(\hat{G}, \varphi))$;

3. $c_0(\hat{G}) \subset \ker(\theta)$;

4. $\theta : H^\infty(\hat{G}, \varphi) \to \theta(H^\infty(\hat{G}, \varphi))$ is a $*$-isomorphism;

5. $\theta^{-1} \circ j_\infty(x) = s^* \lim_n \varphi^n(x)$, for any $x \in R_\varphi$. In particular $\varepsilon \circ \theta^{-1} \circ j_\infty = \lim_n \varphi^n$.

Denote $\theta_0 := \theta^{-1} \circ j_\infty : R_\varphi \to H^\infty(\hat{G}, \varphi)$. This is a $*$-homomorphism by (ii) and (iv).

No proof of this result is given in [NT03] and only some parts are in [Izu02, Thm. 3.6], so we give a proof for completeness. Note that part (iv) does not follow from part (ii) since the Poisson boundary is equipped with different product than $l^\infty(\hat{G})$ (cf. Definition 3.2.11).

**Proof of Proposition 4.3.2.** (i) Clearly $R_\varphi$ is a linear space which is closed under the

---

$^2j_\infty$ should not be confused with $j_\infty$ defined previously in (3.2.3).
involution $\ast$. Also since each $j_n$ is a $\ast$-homomorphism, it is norm-decreasing, i.e. $\|j_n(x)\| \leq \|x\|$. Therefore if $x, y \in R_\varphi$ and $\xi \in \mathcal{H}_\varphi$,

$$
\|(j_n(xy) - j_\infty(xy))\xi\| = \|(j_n(x)j_n(y) - j_\infty(x)j_\infty(y))\xi\|
$$

$$
\leq \|(j_n(x)j_n(y) - j_\infty(x)j_\infty(y))\xi\| + \|(j_n(x)j_\infty(y) - j_\infty(x)j_\infty(y))\xi\|
$$

$$
\leq \sup_m \|j_n(x)\|\|(j_n(y) - j_\infty(y))\xi\| + \|(j_n(x) - j_\infty(x))j_\infty(y)\xi\|
$$

which tends to 0 as $n \to \infty$, thus $xy \in R_\varphi$ and $j_\infty(xy) = j_\infty(x)j_\infty(y)$. To show that $R_\varphi$ is closed in norm, let $(x_n)_n \subset R_\varphi$ be a bounded sequence with $\lim_n \|x_n - x\| = 0$. Let $\varepsilon > 0$ and $\xi \in \mathcal{H}_\varphi$. Find $m$ such that $\|x - x_m\| < \varepsilon$. Then there exists $N \in \mathbb{N}$ such that for all $n, n' \geq N$ it holds $\|(j_n(x_m) - j_{n'}(x_m))\xi\| \leq \varepsilon$. This gives for $n, n' \geq N$

$$
\|(j_n(x) - j_{n'}(x))\xi\|
$$

$$
\leq \|(j_n(x) - j_n(x_m))\xi\| + \|(j_n(x_m) - j_{n'}(x_m))\xi\| + \|(j_{n'}(x_m) - j_{n'}(x))\xi\|
$$

$$
\leq \|x - x_m\|\|\xi\| + \varepsilon + \|x_m - x\|\|\xi\| \leq \varepsilon + 2\|\xi\|\varepsilon.
$$

(4.3.1)

Thus $R_\varphi$ is closed in norm and hence a $C^*$-algebra.

We claim that $s^\ast$-lim$_n j_n(x) = 0$ for every $x \in c_0(\hat{G})$. This would show that $c_0(\hat{G}) \subset R_\varphi$ and $c_0(\hat{G}) \subset \ker(j_\infty)$. First consider $I_s \in B(H_s)$, then from (3.2.7) we see

$$
\varphi^\infty_j(I_s) = \hat{\varphi}(P^n_{\varphi^\mu}(I_s)) = \sum_{t \in \text{Irr}(G)} \hat{\varphi}(P^n_{\varphi^\mu}(I_s)I_t) = \sum_{t \in \text{Irr}(G)} \hat{\varphi}(I_t)p^n_{\mu}(t, s) = p^n_{\mu}(0, s).
$$

By assumption $P^\mu_{\mu}$ is transient, thus $\sum_n p^n_{\mu}(0, s) = g_\mu(0, s) < \infty$ and hence $\lim_n p^n_{\mu}(0, s) = 0$. If $x \in c_{00}(\hat{G})$ is finitely supported, then $x^\ast x$ is dominated by a linear combination $\sum_{i=1}^m c_s I_s$, for some $s_i \in \text{Irr}(G)$. Therefore

$$
\lim_n \varphi^\infty_j(j_n(x)^\ast j_n(x)) \leq \sum_i c_{s_i} \lim_n p^n_{\mu}(0, s_i) = 0.
$$

Since $R_\varphi$ is a $C^*$-algebra and $c_{00}(\hat{G})$ is dense in $c_0(\hat{G})$ the claim follows.

The statement that $H^\infty(\hat{G}, P_\varphi) \subset R_\varphi$ is contained in [Izu02, Thm. 3.6]. Izumi only deals with finitely supported measures $\mu$, but the proof also applies to our case. Again we give a proof of this result.

Recall the conditional expectations $E_n$ and maps $E'_n$ introduced in Lemma 1.3.7. By coassociativity one has for $m > n$ and any $x \in l^\infty(\hat{G})$

$$
E'_n(j_m(x)) = (\cdots \otimes \varphi \otimes \varphi \otimes t^{\otimes n})(\cdots \otimes 1 \otimes 1 \otimes \hat{\Delta}^{m-1}(x))
$$

$$
= (\varphi \otimes \cdots \otimes \varphi \otimes t^{\otimes n})((t^{\otimes m-n} \otimes \hat{\Delta}^{m-n}(x)) = \hat{\Delta}^{m-n}(P^m_{\varphi}(x))
$$

and thus

$$
E_n(j_m(x)) = j_n(P^m_{\varphi}(x)).
$$

(4.3.2)

Therefore if $h \in H^\infty(\hat{G}, P_\varphi)$ is a harmonic element, $E_n(j_{n+1}(h)) = j_n(h)$. In which case
the noncommutative Martingale convergence theorem (cf. Lemma 1.3.7) implies that there exists a unique element \( y \in \bigotimes_{-\infty}^{1}(l^\infty(\hat{G}), \varphi) \) such that \( E_n(y) = j_n(h) \) for all \( n \) and that the sequence \( (E_n(y))_n \) converges to \( y \) in strong* topology. In other words \( s^* \lim_n j_n(h) = y \), thus \( H^\infty(\hat{G}, P_\varphi) \subset R_\varphi \), concluding part (i).

We now deal with (iv). Clearly \( \theta \) is surjective on its image, linear and preserves the involution. For an element \( x \in R_\varphi \) the limit \( s^* \lim_n P^n_\varphi(x) \) exists by (4.3.2) and is harmonic. Therefore by part (i) it is again in \( R_\varphi \). Thus if \( h_1, h_2 \in H^\infty(\hat{G}, P_\varphi) \) are harmonic, by the noncommutative martingale convergence theorem and (4.3.2)

\[
j_\infty(h_1)j_\infty(h_2) = j_\infty(h_1h_2)
\]

\[
= s^* \lim_n E_nj_\infty(h_1h_2)
\]

\[
= s^* \lim_n s^* \lim_m E_nj_m(h_1h_2)
\]

\[
= s^* \lim_n s^* \lim_m j_n(P^{m-n}(h_1h_2))
\]

\[
= s^* \lim_n j_n(h_1 \cdot h_2)
\]

\[
= j_\infty(h_1 \cdot h_2),
\]

thus \( j_\infty : H^\infty(\hat{G}, P_\varphi) \to \bigotimes_{-\infty}^{1}(l^\infty(\hat{G}), \varphi) \) is a *-homomorphism. It remains to show that \( j_\infty \) is isometric. Since \( j_\infty \) is a *-homomorphism it is norm-decreasing, so \( \|h\| \geq \|j_\infty(h)\| \).

It thus suffices to prove the reverse inequality. Denote again \( U := \bigoplus_{s \in \text{supp} \mu} U_s \). Since the measure \( \mu \) is assumed to be generating we obtain

\[
\|h\| = \sup\{\|\pi_s(h)\| : s \in \text{Irr}(G)\}
\]

\[
= \sup\{\|\pi_s(h)\| : s \in \text{supp}(\mu^s), n \in \mathbb{N}\}
\]

\[
= \sup\{\|\pi_{U_s}^{\otimes n}(\hat{\Delta}^n(h))\| : n \in \mathbb{N}\}
\]

\[
= \sup\{\|j_n(h)\| : n \in \mathbb{N}\}.
\]

From (4.3.2) we conclude \( j_n(h) = E_n(j_\infty(h)) \). Since \( E_n \) is a conditional expectation we get

\[
\|h\| = \sup_{n \in \mathbb{N}} \|j_n(h)\| = \sup_{n \in \mathbb{N}} \|E_n(j_\infty(h))\| \leq \|j_\infty(h)\|.
\]

Hence \( j_\infty \) is isometric and thus a *-isomorphism onto its image, which proves (iv).

We prove (ii) and (v) simultaneously. Clearly \( j_\infty \) is a *-homomorphism. By (i) it is immediate that \( \theta(H^\infty(\hat{G}, P_\varphi)) \subset j_\infty(R_\varphi) \). Let \( x \in R_\varphi \). By the noncommutative martingale convergence theorem and (4.3.2) we get

\[
j_\infty(x) = s^* \lim_n E_nj_\infty(x)
\]

\[
= s^* \lim_n s^* \lim_m E_nj_m(x)
\]

\[
= s^* \lim_n s^* \lim_m j_n(P^{m-n}(x))
\]

\[
= s^* \lim_n s^* \lim_m j_n(P^m(x))
\]

\[
= j_\infty(s^* \lim_m P^m(\varphi(x))). \tag{4.3.3}
\]
4.3. CONVERGENCE TO THE BOUNDARY FOR QUANTUM GROUPS

Since $s^*\lim_n P^n_\varphi(x)$ is harmonic, we obtain $j_\infty(R_\varphi) \subset j_\infty(H^\infty(\hat{G}, P_\varphi))$ which establishes (ii). Moreover (4.3.3) can be written as $\theta^{-1} \circ j_\infty(x) = s^*\lim_m P^m_\varphi(x)$. This gives

$$\hat{\varepsilon} \circ \theta^{-1} \circ j_\infty(x) = \lim_m \hat{\varepsilon} \circ P^m_\varphi(x) = \lim_m \hat{\varepsilon}(\varphi^m \otimes t)\hat{\Delta}(x) = \lim_m \varphi^m(x),$$

which finishes the proof.

Conjecture 4.3.3 ([NT03]). Let $\mu$ be a transient and generating probability measure on $\text{Irr}(G)$. The following holds for the random walk defined by $\varphi = \varphi_\mu$ on $(\hat{G}, \hat{\Delta})$:

(i) $K_\varphi(x) \in R_\varphi$, for every $x \in c_{00}(\hat{G})$;

(ii) If $\nu = \lim_n \varphi^n|_{R_\varphi} = \hat{\varepsilon}\theta$, then $\hat{\psi}(xh) = \nu(K_\varphi(x)h)$ for every $x \in c_{00}(\hat{G})$ and any harmonic element $h \in H^\infty(\hat{G}, P_\varphi)$.

There is no proof known for this conjecture. We therefore say that a random walk on a discrete quantum group $(\hat{G}, \hat{\Delta})$ defined by a probability measure $\mu$ converges to the boundary if the statements of the conjecture hold for $(\hat{G}, \mu)$. This notion of convergence to the boundary is compatible with the classical case, see Proposition 4.3.5 below.

The following construction is very similar to Lemma 1.3.12. The state $\nu|_{M(\hat{G}, \varphi)}$ is a $\sigma_\hat{\psi}$-KMS state representing the unit $1 \in H^\infty(\hat{G}, \varphi)$ (see [NT04, Thm. 3.10]). So in particular $\sigma_t^\varphi = \rho_t^\hat{\psi}$ for all $t$. Define the map $K^*_\varphi: \pi_\nu(\hat{M}(\hat{G}, \varphi))'' \rightarrow l^\infty(\hat{G})$ by the identity

$$(a, K^*_\varphi(x))_\nu = (K^*_\varphi(a), x)_\hat{\psi}, \quad (a \in \pi_\nu(\hat{M}(\hat{G}, \varphi))'', x \in c_{00}(\hat{G})), \quad (4.3.4)$$

where $(\cdot, \cdot)_\nu$ and $(\cdot, \cdot)_\hat{\psi}$ are as in Lemma 1.3.11. Note that $\hat{\psi}$ is unbounded on $l^\infty(\hat{G})$, but it is well defined on $c_{00}(\hat{G})$. Identity (4.3.4) determines $K^*_\varphi$ uniquely because we can take for $x$ all matrix units $m_{ij}^\varphi$. In [NT04, Lem. 3.9] it is shown that $\text{Im}(K^*_\varphi) \subset H^\infty(\hat{G}, P_\varphi)$.

Theorem 4.3.4 ([NT03, Thm. 6.2]). If Conjecture 4.3.3 holds for a state $\varphi$ on $(l^\infty(\hat{G}), \hat{\Delta})$, then this random walk has the following properties:

(i) for any positive harmonic element $h \in H^\infty(\hat{G}, P_\varphi)$ the positive linear functional $(h, \cdot)_\nu$ represents $h$, meaning $(h, x)_\hat{\psi} = (h, K^*_\varphi(x))_\nu$ for all $x \in c_{00}(\hat{G})$;

(ii) the map $K^*_\varphi|_{M(\hat{G}, \varphi)}: M(\hat{G}, \varphi) \rightarrow H^\infty(\hat{G}, P_\varphi)$ equals the map $\theta_0|_{M(\hat{G}, \varphi)}$. It induces an isomorphism $\pi_\nu(M(\hat{G}, \varphi))'' \cong H^\infty(\hat{G}, P_\varphi)$ which respects the action of the compact quantum group $G$ and the dual discrete quantum group $\hat{G}$.

The proof of this theorem can be found later in this section, see page 134.

Proposition 4.3.5. Given a probability measure $\mu$ on a discrete group $\Gamma$. Define the compact quantum group $G = (C(G), \Delta) = (C^*_\Gamma(\Gamma), \Delta)$ as in Example 1.4.2. Suppose $\mu$ is generating and transient. Let $P_\mu$ be the Markov kernel of the random walk on $l^\infty(\hat{G})$ as described in Example 3.2.5. Then this quantum random walk converges to the boundary in the sense of Conjecture 4.3.3.
Proof. We adopt the notation of Examples 1.4.14 and 3.2.5. So \( G = (C(G), \Delta) = (C^*(\Gamma), \Delta) \) is the compact quantum group. The discrete dual equals \( l^\infty(\hat{G}) = l^\infty(\Gamma) \). In the proof we pass back and forth between the classical and quantum picture. To distinguish between these two pictures we write \( \Gamma \) and lowercase letters \( p \) and \( k \) to indicate the classical random walk on \( \Gamma \) and we use \( \hat{G} \) and capital letters \( P \) and \( K \) for the quantum random walk on \( \hat{G} \).

Consider the path space of the classical random walk \( \Omega := \Gamma^\mathbb{N} \) with associated coordinate maps \( X_n : \Omega \to \Gamma \). Since \( \mu \) is generating and transient the Martin compactification \( \hat{M}(\Gamma, p_\mu) \) of the classical random walk is well-defined. By convergence to the boundary in the classical case (Theorem 3.1.12) there exists \( X_\infty : \Omega \to M(\Gamma, p_\mu) \) such that \( \lim_n X_n = X_\infty \) \( \mathbb{P}_\gamma \)-a.e. for all \( \gamma \in \Gamma \). Hence by Lemma 3.1.14 for every continuous function \( f : \hat{M}(\Gamma, p_\mu) \to \mathbb{R} \) it holds that \( \lim_n f \circ X_n = f \circ X_\infty \) \( \mathbb{P}_\gamma \)-a.e. for every \( \gamma \in \Gamma \). Since almost everywhere convergence is stronger than convergence in mean, Lemma 1.3.2 gives us that \( s^- \lim_n f \circ X_n = f \circ X_\infty \) in the von Neumann algebra \( L^\infty(\hat{M}(\Gamma, p_\mu), \mathbb{P}_e) \), where \( e \in \Gamma \) is the identity element. Now invoke the correspondence \( \kappa \) (see Equation (3.2.4)) to obtain that \( \lim_n j_n(\kappa(f)) = \lim_n f^\infty(f \circ X_n) \) exists in \( s^- \)-topology in \( \bigotimes_{n=1}^{\infty} L^\infty(\hat{G}, \varphi_\mu) \). The algebra \( L^\infty(\Gamma) \) is commutative, so \( \kappa : C(\hat{M}(\text{Irr}(G), \mu)) \to \hat{M}(\hat{G}, p_\mu) \) is an isomorphism. It follows that \( s^- \lim_n j_n(K_\mu(x)) \) exists for all \( x \in c_{00}(\hat{G}) \).

To prove the representation of harmonic elements we first need to find the functional \( \nu \) that represents the unit. We take for our reference point to define the Martin kernel \( x_0 = e \), the unit of \( \Gamma \). Each irreducible representation of \( G \) is one-dimensional, therefore \( d_s = 1 \) for all \( s \in \text{Irr}(G) \cong \Gamma \). This implies that the map \( \kappa \) satisfies

\[
\kappa\left( \sum_{\gamma \in \Gamma} k_\mu(\gamma, \cdot) f(\gamma) \right) = K_\mu(\kappa(f)).
\]

In addition we have\(^3\)

\[
\hat{\psi}(\kappa(f)) = \sum_{\gamma \in \text{Irr}(G)} d_s \text{Tr}(\kappa(f) \rho_s^{-1}) = \sum_{\gamma \in \Gamma} f(\gamma).
\]

Let \( x \in c_{00}(\hat{G}) \) and write \( x = \kappa(f) \) for some \( f \in c_{00}(\Gamma) \). From (3.1.3) and the above observations it follows that

\[
\hat{\psi}(x) = \sum_{\gamma \in \Gamma} f(\gamma) = \sum_{\gamma \in \Gamma} f(\gamma) \int_{M(\Gamma, p_\mu)} k_\mu(\gamma, \cdot) \, d\nu_e = \int_{M(\Gamma, p_\mu)} \sum_{\gamma \in \Gamma} f(\gamma) k_\mu(\gamma, \cdot) \, d\nu_e = \int_{M(\Gamma, p_\mu)} \kappa^{-1}(K_\mu(\kappa(f))) \, d\nu_e = \int_{M(\Gamma, p_\mu)} \kappa^{-1}(K_\mu(x)) \, d\nu_e.
\]

If \( \nu \) represents the unit, then it fulfils the defining equation \( \hat{\psi}(x) = \nu(K_\mu(x)) \) for all \( x \in c_{00}(\hat{G}) \). We see that \( \nu = \int_{M(\Gamma, p_\mu)} \kappa^{-1}(\cdot) \, d\nu_e \).

Now let \( y \in H^\infty(\hat{G}, \mu) \) be a harmonic element, then \( h := \kappa^{-1}(y) \in l^\infty(\Gamma) \) is a bounded

\(^3\)this is no surprise, because the Haar measure of a discrete group is a multiple of the counting measure.
4.3. CONVERGENCE TO THE BOUNDARY FOR QUANTUM GROUPS

The morphism

\[ \psi(xy) = \sum_{\gamma \in \Gamma} f(\gamma)h(\gamma) = \sum_{\gamma \in \Gamma} f(\gamma) \int_{\Gamma} k_{\mu}(\gamma, \cdot) h \, d\nu_e \]

\[ = \int_{\Gamma} (\sum_{\gamma \in \Gamma} k_{\mu}(\gamma, \cdot) f(\gamma)) h \, d\nu_e \]

\[ = \int_{\Gamma} \kappa^{-1}(K_{\hat{\mu}}(f)) h \, d\nu_e \]

\[ = \int_{\Gamma} \kappa^{-1}(K_{\hat{\mu}}(x)y) d\nu_e \]

\[ = \nu(K_{\hat{\mu}}(x)y), \]

as desired. \( \blacksquare \)

Convergence to the boundary describes the asymptotic behaviour of the paths in the random walk, but by Theorem 4.3.4 it also gives a natural representation of harmonic elements. Let us take a closer look at this second part of the conjecture.

**Lemma 4.3.6.** The morphism \( j_\infty: R_\varphi \rightarrow \bigotimes_{-\infty}^{-1}(l_\infty(\hat{G}), \varphi) \) can be extended to a s\(^*\)-continuous map \( j_\infty: \pi_\nu(R_\varphi)'' \rightarrow \theta(H_\infty(\hat{G}, \varphi)) \).

This extension is denoted by \( j_\infty \). Similarly write \( \theta_0 := \theta_{-1} \circ j_\infty : \pi_\nu(R_\varphi)'' \rightarrow H_\infty(\hat{G}, \varphi) \).

**Proof of Lemma 4.3.6.** Recall that \( \varphi_\infty \) is a faithful state on the von Neumann algebra \( \bigotimes_{-\infty}^{-1}(l_\infty(\hat{G}), \varphi) \). Consider the GNS representation \( (\pi_\varphi, H_\varphi, \xi_\varphi) \) of \( \theta(H_\infty(\hat{G}, \varphi)) \) defined by \( \varphi_\infty \). Since \( j_\infty: R_\varphi \rightarrow \theta(H_\infty(\hat{G}, \varphi)) \) is surjective and

\[ \nu = \lim_n \varphi^n = \lim_n (\varphi \otimes \cdots \otimes \varphi) \hat{\Delta}_n^{-1} = \varphi_\infty \circ j_\infty \]

it follows that \( \pi_\nu(R_\varphi) = \pi_\varphi(\theta(H_\infty(\hat{G}, \varphi))) \) inside \( B(H_\nu) = B(H_\varphi) \). Hence \( j_\infty \) can be extended. \( \blacksquare \)

**Lemma 4.3.7.** For \( x, y \in R_\varphi \) the following holds:

\[ \theta_0(\theta_0(x)) = \theta_0(x) \quad \text{and} \quad \theta_0(xy) = \theta_0(x) \cdot \theta_0(y), \]

where \( \cdot \) is the product in \( H_\infty(\hat{G}, P_\varphi) \). If in addition \( \hat{M}(\hat{G}, \varphi) \subset R_\varphi \), then

\[ \nu(K_{\hat{\varphi}}(x)\theta_0(a)) = \nu(K_{\hat{\varphi}}(x)a), \quad \text{for } a \in \pi_\nu(\hat{M}(\hat{G}, \varphi))'' \text{ and } x \in c_{00}(\hat{G}). \quad (4.3.6) \]

**Proof.** Let \( x \) be \( \varphi \)-regular. By Proposition 4.3.2 it follows \( \theta_0(x) \in H_\infty(\hat{G}, P_\varphi) \). Thus
\[ P_\varphi^m(\theta_0(x)) = \theta_0(x), \quad \text{for every } m \geq 0. \]

We also know from Proposition 4.3.2 that \( \theta_0 = s^*\text{-}\lim_n P_\varphi^n \). Hence
\[ \theta_0 \circ \theta_0(x) = s^*\text{-}\lim_n P_\varphi^n(\theta_0(x)) = s^*\text{-}\lim \theta_0(x) = \theta_0(x), \]
which proves the first identity. The second one follows from Proposition 4.3.2 part (iv).

To prove the last statement, observe that if \( a \in \pi_\nu(\hat{M}(\hat{G}, \varphi))'' \), then \( \theta_0(a) \) is \( P_\varphi \)-harmonic and \( \nu = \hat{\epsilon} \circ \theta^{-1} \circ j_\infty = \epsilon \circ \theta_0 \). This gives us by means of Lemma 4.3.6
\[
\nu(K_{\hat{\varphi}}(x)\theta_0(a)) = \hat{\epsilon} \circ \theta_0(K_{\hat{\varphi}}(x)\theta_0(a)) \\
= \hat{\epsilon}(\theta_0(K_{\hat{\varphi}}(x)) \cdot (\theta_0(\theta_0(a)))) \\
= \hat{\epsilon}(\theta_0(K_{\hat{\varphi}}(x)) \cdot (\theta_0(a))) \\
= \hat{\epsilon} \circ \theta_0(K_{\hat{\varphi}}(x)a) \\
= \nu(K_{\hat{\varphi}}(x)a), \quad (4.3.7)
\]
which proves the result.

**Proof of Theorem 4.3.4.** (i) This can be obtained by rewriting the first statement of convergence to the boundary. Indeed, using Lemma 3.2.17 we get
\[
(h, K_{\hat{\varphi}}(x))_\nu = \nu(K_{\hat{\varphi}}(x)\sigma_\nu^{\varphi}(h^*)) = \nu(K_{\hat{\varphi}}(\sigma_\nu^{\varphi}(x))h^*) = \hat{\psi}(\sigma_\nu^{\varphi}(x)h^*) = (h, x)_{\hat{\varphi}}.
\]

(ii) Let \( a \in \pi_\nu(\hat{M}(\hat{G}, \varphi))'' \) and \( x \in c_{00}(\hat{G}) \). Then convergence to the boundary and Lemma 4.3.7 yield
\[
\hat{\psi}(x_\theta_0(a)) = \nu(K_{\hat{\varphi}}(x)\theta_0(a)) = \nu(K_{\hat{\varphi}}(x)a).
\]
Therefore for every \( x \in c_{00}(\hat{G}) \)
\[
(\theta_0(a), x)_\varphi = \hat{\psi}(\sigma_\varphi^{\hat{\varphi}}(x)\theta_0(a^*)) = \nu(K_{\hat{\varphi}}(\sigma_\varphi^{\hat{\varphi}}(x))a^*) = \nu(K_{\hat{\varphi}}(x)\sigma_{-1\varphi}^{\hat{\varphi}}(a^*)) \\
= (a, K_{\hat{\varphi}}(x))_\nu = (K_{\hat{\varphi}}^*(a), x)_\varphi.
\]
Hence \( K_{\hat{\varphi}}^* = \theta_0 \) on \( \pi_\nu(\hat{M}(\hat{G}, \varphi))'' \). Now \( \theta_0 = s^*\text{-}\lim_n P_\varphi^n \) and \( P_\varphi \) is \( \alpha_1 \) and \( \Delta \)-equivariant, hence so is \( \theta_0 \). From Proposition 4.3.2 we know that \( \theta_0 \) is a \(*\)-homomorphism and the proof of Lemma 4.3.6 shows that \( \theta_0: \pi_\nu(\hat{M}(\hat{G}, \varphi))'' \rightarrow H^\infty(\hat{G}, \mu) \) is injective. It remains to show that the map is surjective. Assume that \( h \in H^\infty(\hat{G}, \mu)_+ \) and \( \|h\| \leq 1 \).

By part (i) \( h \) is represented by the functional \( (h, \cdot)_\nu \). As \( 0 \leq h \leq 1 \) the functional satisfies \( 0 \leq (h, \cdot)_\nu \leq \nu \) and thus by the Radon–Nikodym theorem for bounded linear functionals (cf. [Bla06, II.6.4.6]), there exists a unique \( a' \in \pi_\nu(\hat{M}(\hat{G}, \varphi))' \) such that \( (h, \cdot)_\nu = \nu(a' \cdot) \).

Put \( a := J a' J \in \pi_\nu(\hat{M}(\hat{G}, \varphi))'' \). Note that \( J a' J \xi_\nu = J a' \xi_\nu = \Delta_\nu^{\frac{1}{2}}(a^*)n_\nu = \sigma_{\varphi^{\frac{1}{2}}}(a^*)n_\nu \), where \( \xi_\nu \) is the GNS vector, \( \Delta_\nu \) the modular group and \( J \) the modular conjugation. We get
\[
(h, x)_\varphi = (h, K_{\hat{\varphi}}(x))_\nu = \nu(K_{\hat{\varphi}}(x)a') = (\sigma_{\varphi^{\frac{1}{2}}}(a^*), K_{\hat{\varphi}}(x))_\nu = (a, K_{\hat{\varphi}}(x))_\nu = (K_{\hat{\varphi}}^*(a), x)_\varphi
\]
and thus \( h = K^\ast_{\tilde{\varphi}}(\sigma_{-\frac{x}{2}}(a^*)) \). Since any harmonic element can be written as a linear combination of positive harmonic elements of norm \( \leq 1 \) the proof is complete.

**Proposition 4.3.8.** Suppose that \( \tilde{M}(\hat{G}, \varphi) \subset R_{\varphi} \). Then the following two conditions are equivalent:

(i) \( K^\ast_{\hat{\varphi}} = \theta_0|_{\tilde{M}(\hat{G}, \varphi)} \) and \( j_\infty: \pi_\nu(\tilde{M}(\hat{G}, \varphi))^\prime\prime \to \theta(H^\infty(\hat{G}, P_\varphi)) \) is surjective;

(ii) part (ii) of Conjecture 4.3.3 holds.

**Proof.** Suppose that \( K^\ast_{\hat{\varphi}} = \theta_0|_{\tilde{M}(\hat{G}, \varphi)} \) and \( j_\infty: \pi_\nu(\tilde{M}(\hat{G}, \varphi))^\prime\prime \to \theta(H^\infty(\hat{G}, P_\varphi)) \) is surjective. Let \( h \in H^\infty(\hat{G}, P_\varphi) \). By assumption there exists an \( a \in \pi_\nu(\tilde{M}(\hat{G}, \varphi))^\prime\prime \) such that \( K^\ast_{\hat{\varphi}}(a) = \theta^{-1}(j_\infty(a)) = h \). Lemma 4.3.7, the definition of \( K^\ast_{\hat{\varphi}} \) in terms of the inner products and identity (ii) of Lemma 3.2.17 give

\[
\nu(K_{\hat{\varphi}}(x)h) = \nu(K_{\hat{\varphi}}(x)\theta_0(a)) = \nu(K_{\hat{\varphi}}(x)a) \\
= \nu\left(K_{\hat{\varphi}}(\sigma_{-\frac{x}{2}}(x))\sigma_{-\frac{x}{2}}(a^*)\right) \\
= (a^*, K_{\hat{\varphi}}(\sigma_{-\frac{x}{2}}(x)))_\nu \\
= (K^\ast_{\hat{\varphi}}(a^*), \sigma_{-\frac{x}{2}}(x))_{\hat{\varphi}} \\
= (x\sigma_{-\frac{x}{2}}(x)a)_{\hat{\varphi}} (K^\ast_{\hat{\varphi}}(a^*)) \\
= (xK^\ast_{\hat{\varphi}}(a)) = \hat{\psi}(xh),
\]

which is exactly the second condition.

To prove the converse implication, note that Theorem 4.3.4 implies that \( j_\infty = \theta \circ K^\ast_{\hat{\varphi}} \) and that \( K^\ast_{\hat{\varphi}} \) is an isomorphism. Hence \( j_\infty \) is surjective.

**Corollary 4.3.9.** Suppose that \( \tilde{M}(\hat{G}, \varphi) \subset R_{\varphi} \). If \( K_{\hat{\varphi}}: c_0(\hat{G}) \to M(\hat{G}, \varphi) \) has dense range and \( K^\ast_{\hat{\varphi}}: \pi_\nu(M(\hat{G}, \varphi))^\prime\prime \to H^\infty(\hat{G}, P_\varphi) \) intertwines the right coactions of \((c_0(\hat{G}), \hat{\Delta})\), then part (ii) of the conjecture holds.

**Proof.** From [NT04, Prop. 3.12] it follows that \( K^\ast_{\hat{\varphi}}: \pi_\nu(M(\hat{G}, \varphi))^\prime\prime \to H^\infty(\hat{G}, P_\varphi) \) is an isomorphism, while [NT04, Prop. 3.11] implies that \( K^\ast_{\hat{\varphi}} = (\nu \otimes \iota)\hat{\Delta} = \theta_0 \) which is thus surjective. Now apply Proposition 4.3.8.

**Remark 4.3.10.** As observed in Example 1.3.2 strong* convergence corresponds in the classical case to convergence in square mean. For classical random walks, it is known that the convergence to the boundary has a better behaviour than only convergence in mean, namely convergence almost everywhere. There exist noncommutative versions of almost everywhere convergence, the so-called “almost uniform convergence” (see [Lan76, Jaj85, Sau91]). There are several slightly different versions of this almost uniform convergence. The thing they have in common is that they are based on Egoroff’s theorem (see e.g. [Bog07, Thm. 2.2.1]), which gives an equivalent formulation of almost every convergence in terms of uniform convergence on subsets of the underlying measurable space. This second formulation can be generalised to von Neumann algebras by using projections
which play the role of indicator functions of sets. So a natural thing to do is to formulate and try to prove the conjecture of convergence to the boundary in terms of this almost uniform convergence, so that one has a proper analogue of convergence to the boundary in the noncommutative world.

The downside of almost uniform convergence is that it is very hard to work with. When we tried to prove results analogous to Proposition 4.3.2 or tried to prove Conjecture 4.3.3, we could not get any further than a reformulation of the classical case. For example one can prove that central elements in $K_{\hat{\mu}}(I_s)$ for some $s \in \operatorname{Irr}(G)$ are regular with respect to almost uniform convergence, because one can define a classical random walk on the center which is known to converge to the boundary almost everywhere. However, to prove almost uniform regularity of $K_{\hat{\mu}}(x)$ for general elements $x \in c_0(\hat{G})$ is even harder than to prove strong$^*$ regularity.

## 4.4 Convergence to the boundary for $SU_q(2)$

If $G$ is a compact group, then the quantum group defined by $G$ converges to the boundary as proved in Proposition 4.3.5. To obtain a “true” quantum example we consider random walks on $SU_q(2)$. We shall show that if a probability measure $\mu$ on $\operatorname{Irr}(SU_q(2))$ is nice enough, then the random walk on $SU_q(2)$ defined by $\mu$ converges to the boundary.

### 4.4.1 $SU_q(2)$ and its Martin boundary

To establish convergence to the boundary for $SU_q(2)$ we need knowledge of the Martin boundary of random walks on $SU_q(2)$. These boundaries have been computed in [NT04], here we review the results we need.

Fix $q \in (0, 1)$. The C*-algebra $C(SU_q(2))$ is the universal unital C*-algebra generated by $\alpha$ and $\gamma$ such that the matrix

$$
\begin{pmatrix}
\alpha^* & \gamma \\
-q\gamma^* & \alpha
\end{pmatrix}
$$

is unitary. The comultiplication $\Delta$ is defined as

$$
\Delta(\alpha) := \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) := \gamma \otimes \alpha + \alpha^* \otimes \gamma.
$$

The Hopf $*$-algebra of matrix coefficients $C[SU_q(2)]$ is the $*$-algebra generated by $\alpha$ and $\gamma$. The irreducible representations can be identified as $\operatorname{Irr}(SU_q(2)) \cong \frac{1}{2} \mathbb{Z}_+$, and the discrete dual equals

$$
l^\infty(SU_q(2)) = l^\infty(\bigoplus_{s \in \frac{1}{2} \mathbb{Z}_+} B(H_s)),
$$

where $H_s$ is a $2s+1$-dimensional Hilbert space. This is enough information to be able to state the main theorem of this section.

**Theorem 4.4.1.** Assume that $\mu$ is a probability measure on $\frac{1}{2} \mathbb{Z}_+$ which satisfies the following two conditions:
4.4. CONVERGENCE TO THE BOUNDARY FOR SU_q(2)

(i) there exists $s \in \frac{1}{2}\mathbb{Z}_+ \setminus \mathbb{Z}_+$ with $s \in \text{supp}(\mu)$;

(ii) $\sum_{t \in \frac{1}{2}\mathbb{Z}_+} \mu(t)(1 + q^2)^{2t} < \infty$,

then the random walk on $\widehat{SU_q(2)}$ defined by $\mu$ converges to the boundary.

**Proof.** We postpone the proof of the regularity of the elements $\hat{K}_\mu(x)$ for $x \in \mathcal{C}_0(\hat{SU_q(2)})$ to the next subsection. For now assume that the first part of the conjecture holds.

The space $K_{\hat{\psi}}(\mathcal{C}_0(\hat{SU_q(2)})) \subset \mathcal{M}(\hat{G},\mu)$ is dense by [NT04, Thm. 4.10]. Moreover by [NT04, Cor. 4.13, Prop. 3.11] the mapping $K^*_{\hat{\psi}_\mu}$ intertwines the right coactions of $(\mathcal{C}_0(\hat{G}),\hat{\Delta})$. Now Corollary 4.3.9 gives the second part of the conjecture of convergence to the boundary.

To prove the regularity of elements in the Martin boundary, we need some knowledge of $\hat{\mathcal{M}}(\hat{SU_q(2)},\mu)$. Here we give a short recap of known results. In this section we follow the conventions of [NT04]. The Hopf algebra they constructed from $su_2$ is slightly different from the one we defined in Section 1.6.2. The following gives the precise relation.

Consider the two Hopf $*$-algebras $\hat{U}_q(su_2)$ and $U_q(su_2)$ given by the following generators and relations.

<table>
<thead>
<tr>
<th>$\hat{U}_q(su_2)$ as in §1.6.2</th>
<th>$U_q(su_2)$ as in [NT04]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$, $E$, $F$</td>
<td>$k$, $e$, $f$</td>
</tr>
<tr>
<td>$KK^{-1} = K^{-1}K = 1$</td>
<td>$kk^{-1} = k^{-1}k = 1$</td>
</tr>
<tr>
<td>$KE = qEK$</td>
<td>$ke = qek$</td>
</tr>
<tr>
<td>$KF = q^{-1}FK$</td>
<td>$kf = q^{-1}fk$</td>
</tr>
<tr>
<td>$[E, F] = \frac{K^2 - K^{-2}}{q-q^{-1}}$</td>
<td>$[e, f] = \frac{k^2 - k^{-2}}{q-q^{-1}}$</td>
</tr>
<tr>
<td>$\hat{\Delta}(E) = E \otimes K + K^{-1} \otimes E$</td>
<td>$\hat{\Delta}(e) = e \otimes k^{-1} + k \otimes e$</td>
</tr>
<tr>
<td>$\hat{\Delta}(F) = F \otimes K + K^{-1} \otimes F$</td>
<td>$\hat{\Delta}(f) = f \otimes k^{-1} + k \otimes f$</td>
</tr>
<tr>
<td>$\hat{\Delta}(K) = K \otimes K$</td>
<td>$\hat{\Delta}(k) = k \otimes k$</td>
</tr>
<tr>
<td>$\hat{S}(E) = -qE$</td>
<td>$\hat{S}(e) = -q^{-1}e$</td>
</tr>
<tr>
<td>$\hat{S}(F) = -q^{-1}F$</td>
<td>$\hat{S}(f) = -qf$</td>
</tr>
<tr>
<td>$\hat{S}(K) = K^{-1}$</td>
<td>$\hat{S}(k) = k^{-1}$</td>
</tr>
<tr>
<td>$\hat{\epsilon}(K) = 1$</td>
<td>$\hat{\epsilon}(k) = 1$</td>
</tr>
<tr>
<td>$\hat{\epsilon}(E) = \hat{\epsilon}(F) = 0$</td>
<td>$\hat{\epsilon}(e) = \hat{\epsilon}(f) = 0$</td>
</tr>
<tr>
<td>$E^* = F$</td>
<td>$e^* = f$</td>
</tr>
<tr>
<td>$F^* = E$</td>
<td>$f^* = e$</td>
</tr>
<tr>
<td>$K^* = K$</td>
<td>$k^* = k$</td>
</tr>
</tbody>
</table>

It is straightforward to verify that the map $\psi: \hat{U}_q(su_2) \to U_q(su_2)$ given on the gener-
Ators by
\[ \psi(E) := f, \quad \psi(F) := e, \quad \psi(K) := k^{-1} \]
is a \text{-isomorphism.}

These Hopf \text{-algebras} have irreducible representations. The admissible representations can be described in the following way. For each \( s \in \frac{1}{2}\mathbb{Z}^+ \) let \( \{ \tilde{e}^s_j \}_{j = -s} \) and \( \{ \xi^s_j \}_{j = -s} \) be orthonormal bases of a \( 2s + 1 \)-dimensional Hilbert space. There exist representations of the above Hopf \text{-algebras} given by (cf. [KS97, §3.2.3], [NT04, §4.1])

\[
\begin{align*}
\pi_s(K) \tilde{e}^s_j &= q^j \tilde{e}^s_j, \\
\pi_s(F) \tilde{e}^s_j &= ([s + j]_q[s + j + 1]_q)^{1/2} \tilde{e}^s_{j-1}, \\
\pi_s(k) \xi^s_j &= q^{-j} \xi^s_j, \\
\pi_s(F) \xi^s_j &= ([s + j]_q[s + j + 1]_q)^{1/2} \xi^s_{j-1}
\end{align*}
\]

For \( s \in \frac{1}{2}\mathbb{Z}^+ \) we denote these representations by respectively \( \hat{U}_s \) and \( U_s \) on the Hilbert spaces \( \hat{\mathcal{H}}_s \) and \( \mathcal{H}_s \). Up to equivalence, these are all irreducible unitary representations. It can be shown that both \( U_q(\mathfrak{su}_2) \) and \( \hat{U}_q(\mathfrak{su}_2) \) have a faithful representation on \( \mathcal{U}(\mathfrak{su}_q(2)) \).

Define \( \tilde{\psi}: \hat{\mathcal{H}}_s \rightarrow \mathcal{H}_s \) by \( \tilde{\psi}(\tilde{e}^s_j) := \xi^s_j \). It immediately follows from the above relations that
\[
\pi_s(\tilde{\psi}(X)) = \tilde{\psi}(\pi_s(X) \tilde{e}^s_j),
\]
whenever \( X \) is one of the generators \( E, F \) or \( K \).

Any finite dimensional unitary representation is completely reducible. The fusion rules of representations of \( U_q(\mathfrak{su}_2) \) are the same as for the Lie algebra \( \mathfrak{su}_2 \). These are explicitly known:
\[
U_t \times U_s \cong U_{t+s} \oplus U_{t+s-1} \oplus \ldots \oplus U_{|t-s|}.
\]
(4.4.1)

In particular \( U_{\frac{1}{2}} \times U_{\frac{1}{2}} \cong U_{1} \oplus U_{-1} \). As discussed in §4.2.2 there exist two orthonormal bases of \( \hat{\mathcal{H}}_\frac{1}{2} \otimes \mathcal{H}_t \). Namely \( \{ \xi^\frac{1}{2}_j \otimes \xi^t_i \}_{i,j} \) and \( \{ \xi^{-\frac{1}{2}}_j \otimes \{ \xi^{t+\frac{1}{2}}_j \}_{j} \}_{i,j} \) and similarly for \( \{ \tilde{e}^s_j \}_{j} \). These two bases can be expressed in terms of each other by means of the Clebsch–Gordan coefficients. We have
\[
\langle \xi^s_j \otimes \xi^t_i, \xi^s_m \rangle = \langle \tilde{\psi}(\tilde{e}^s_j) \otimes \tilde{\psi}(\tilde{e}^t_i), \tilde{\psi}(\tilde{e}^s_m) \rangle = \langle \tilde{e}^s_j \otimes \tilde{e}^t_i, \tilde{e}^s_m \rangle.
\]

Denote \( C_q(s, t; r; j, k, m) := \langle \xi^s_j \otimes \xi^t_k, \xi^m_r \rangle \). For \( SU_q(2) \) the formulas (4.2.11) simplify as follows ([KS97, Eq. (3.4.68), (3.4.69)])

\[
\begin{align*}
C_q \left( \frac{1}{2}, s, s + \frac{1}{2}; \pm \frac{1}{2}, j, j \pm \frac{1}{2} \right) &= q^{\frac{1}{2}(\mp s + j)} \left( \frac{[s + j + 1]_q}{[2s + 1]_q} \right)^{\frac{1}{2}}, \\
C_q \left( \frac{1}{2}, s, \pm \frac{1}{2}; \pm \frac{1}{2}, j, j \pm \frac{1}{2} \right) &= q^{\frac{1}{2}(\pm s + j \mp 1)} \left( \frac{[s + j]_q}{[2s + 1]_q} \right)^{\frac{1}{2}}.
\end{align*}
\]

To make the connection with the results in §4.2.1 and §4.2.2 more apparent, observe that

\[
\text{In [KS97, Eq. (6.3.68)] a factor } +1/2 \text{ should be replaced by } +1, \text{ this is done in our formulas.}
the basis vector \( \xi_j^s \) is represented by the GT tableau

\[
r_j^s := \begin{pmatrix} 2s \\ s+j \end{pmatrix}.
\]

Indeed, \( r_j^s \in V_{2s+1} \) and \( \text{wt}(r_j^s) = (2\mu_1(r_j^s) - \mu_2(r_j^s))e_1 = (2(s+j) - 2s)e_1 = 2je_1 \).

Observe that the fusion rules (4.4.1) imply that if \( s \in \frac{1}{2}\mathbb{Z}_+ \setminus \mathbb{Z}_+ \), then for any \( t \in \frac{1}{2}\mathbb{Z}_+ \) there exists \( n \in \mathbb{N} \) such that \( U_t \) is a subrepresentation of \( U_s^\otimes n \). So \( s \) generates all irreducible representations. However, if \( s \in \mathbb{Z}_+ \setminus \frac{1}{2}\mathbb{Z}_+ \) and \( t \in \frac{1}{2}\mathbb{Z}_+ \), then \( U_t \) is a subrepresentation of \( U_s^\otimes n \) for some \( n \in \mathbb{N} \) if and only if \( t \in \mathbb{Z}_+ \). So a probability measure on \( \text{Irr}(SU_q(2)) \) is generating if and only if \( \text{supp}(\mu) \cap (\frac{1}{2}\mathbb{Z}_+ \setminus \mathbb{Z}_+) \neq \emptyset \). Note that \( \rho = k^{-2} \) (see §1.6.2).

In the same spirit as Lemma 1.4.8 we have the following (stronger) estimate for \( SU_q(2) \):

\[
\sum_{t \in \frac{1}{2}\mathbb{Z}_+} m_t^{\otimes n} \leq 2^n. \tag{4.4.2}
\]

Indeed, use (4.4.1) to obtain

\[
U_{t/2} \otimes U_t \cong \begin{cases} U_{t-1/2} \oplus U_{t+1/2}, & \text{if } t \geq 1/2; \\ U_{1/2}, & \text{if } t = 0. \end{cases} \tag{4.4.3}
\]

Hence by induction

\[
\sum_{r \in \frac{1}{2}\mathbb{Z}_+} m_r^{\otimes n} \leq \sum_{r \in \frac{1}{2}\mathbb{Z}_+} \sum_{t \in \frac{1}{2}\mathbb{Z}_+} m_t^{\otimes n} \leq 2^n 2 = 2^{n+1},
\]

as desired.

**Notation 4.4.2.** In accordance with [NT04] define \( \lambda, \tilde{\lambda} \in \prod_s B(H_s) \) by

\[
\pi_s(\lambda) := \frac{q(q^{2s+1} + q^{-2s-1})}{(q - q^{-1})} I_s, \quad \pi_s(\tilde{\lambda}) := q^{-2s} I_s.
\]

Define \( \chi_{-1} := -qfk, \chi_0 := \frac{e^{t-e^t} t^e}{ \sqrt{2q} } \) and \( \chi_1 := qek \). Denote \( X_j := \lambda^{-1} X_j \) for \( j = -1, 0, 1 \).

Moreover, write \( \tilde{X}_{-1} := \tilde{\lambda}^{-1} f_k, \tilde{X}_0 := \tilde{\lambda}^{-1} k^2 \) and \( \tilde{X}_1 := \tilde{\lambda}^{-1} e_k \). It can be shown that \( X_j, \tilde{X}_j \in l^\infty(SU_q(2)) \). Let \( \Psi \) be the unital C*-algebra in \( l^\infty(SU_q(2)) \) generated by \( c_0(SU_q(2)) \) and the elements \( X_i, \tilde{X}_i, i = -1, 0, 1 \). The quantum homogeneous sphere of Podleś [Pod87] is the C*-algebra \( C(S^2_{q,0}) := \Psi / c_0(SU_q(2)) \).

**Lemma 4.4.3.** The elements \( \tilde{X}_i \) and \( X_i \) satisfy the following:

\[
\mathbb{C}X_i + c_0(SU_q(2)) = \mathbb{C}\tilde{X}_i + c_0(SU_q(2)), \quad (\text{for } i = -1, 1);
\]

\[
\mathbb{C}X_0 + c_0(SU_q(2)) \subset \mathbb{C}1 + \mathbb{C}\tilde{X}_0 + c_0(SU_q(2));
\]

\[
\mathbb{C}\tilde{X}_0 + c_0(SU_q(2)) \subset \mathbb{C}1 + \mathbb{C}X_0 + c_0(SU_q(2)).
\]
Here $CX := \{ tX : t \in \mathbb{C} \}$ denotes the complex linear span. In particular $\Psi$ equals the unital $C^*$-subalgebra generated by $\hat{X}_i$, $i = -1, 0, 1$ and $c_0(SU_q(2))$.

**Proof.** We have

\[
\pi_s(\hat{\lambda}^{-1}\lambda) = q^{2s}q(q^{2s+1} + q^{-2s-1})I_s = \frac{1}{(q-q^{-1})\sqrt{2}}I_s + \frac{q^{4s+2}}{(q-q^{-1})\sqrt{2}}I_s;
\]

\[
\pi_s(\hat{\lambda}^{-1}\lambda) = q^{-2s}\frac{(q-q^{-1})\sqrt{2}}{q(q^{2s+1} + q^{-2s-1})}I_s = (q-q^{-1})\sqrt{2}I_s + (q-q^{-1})\sqrt{2}\frac{-q^{4s+2}}{q^{4s+2}+1}I_s.
\]

Write $c := (q-q^{-1})\sqrt{2}$. It follows that $\hat{\lambda}^{-1}\lambda = c^{-1}1 + a$ and $\hat{\lambda}\lambda^{-1} = c1 + a'$ for some $a, a' \in c_0(SU_q(2))$. Therefore if $j = -1, 1$

\[
X_j = \lambda^{-1}\chi_j = (\lambda^{-1}\hat{\lambda}^{-1}\chi_j = (c1 + a')\hat{\lambda}^{-1}\chi_j = jqc\hat{X}_j + jqa'\hat{X}_j \in CX_j + c_0(SU_q(2));
\]

\[
\hat{X}_j = \hat{\lambda}^{-1}j\lambda^{-1}\chi_j = (\hat{\lambda}^{-1}\lambda^{-1})j\lambda^{-1}\chi_j = (c^{-1}1 + a)\lambda^{-1}j\lambda^{-1}\chi_j
\]

\[= c^{-1}j\lambda^{-1}\chi_j + jqa\chi_j \in CX_j + c_0(SU_q(2)).\]

Hence for $j = -1, 1$ the statement follows. For $j = 0$ observe first that

\[
\pi_s(\chi_0)\xi_i^s = \pi_s\left(\frac{ef - q^2fe}{\sqrt{2}}\right)\xi_i^s = \frac{1}{\sqrt{2}}((s-i)q[s+i+1]q - q^2[s+i]q[s-i+1]q)\xi_i^s
\]

\[= \frac{1}{\sqrt{2}}(q-q^{-1})^2((q^{2s+1} + q^{-2s-1} - q^{2i+1} - q^{-2i-1}) - q^2(q^{2s+1} + q^{-2s-1} - q^{2i+1} - q^{-2i-1}))\xi_i^s
\]

\[= \frac{1}{\sqrt{2}}\frac{1-q^2}{(q-q^{-1})^2}(q^{2s+1} + q^{-2s-1}) + \frac{1-q^4}{(q-q^{-1})^2}q^{-2i-1}\xi_i^s
\]

\[= \pi_s(\lambda)\xi_i^s + \frac{q(q^2 - q^4)\sqrt{2}}{(q^{-1} + q)(q-q^{-1})^2}q^{-2i}\xi_i^s
\]

\[= \pi_s(-\lambda + \frac{q\sqrt{2}\xi_i^s}{q-q^{-1}k^2}).
\]

Thus $\chi_0 = -\lambda + \frac{q\sqrt{2}\xi_i^s}{q-q^{-1}k^2}$. Hence

\[
X_0 = -1 + \lambda^{-1}q\sqrt{2}q^{-1}k^2 = -1 + \frac{q\sqrt{2}q^{-1}}{q-q^{-1}}k^2 = -1 + \frac{q\sqrt{2}}{q-q^{-1}}(c1 + a')\hat{X}_0;
\]

\[
\hat{X}_0 = \hat{\lambda}^{-1}k^2 = \left(\frac{q-q^{-1}}{q\sqrt{2}}\right)\lambda^{-1}q\sqrt{2}q^{-1}k^2
\]

\[= \left(\frac{q-q^{-1}}{q\sqrt{2}}(c^{-1}1 + a)\right)\lambda^{-1}q\sqrt{2}q^{-1}k^2.
\]
\[ q - q^{-1} \frac{1}{q^2} c^{-1} + q - q^{-1} \frac{1}{q^2} c^{-1} \lambda^{-1} \left( -\lambda + \frac{q - q^{-1}}{q^2} k^2 \right) + a \lambda^{-1} k^2 \]

\[ = q - q^{-1} \frac{1}{q^2} c^{-1} + q - q^{-1} \frac{1}{q^2} \lambda^{-1} X_0 + a \lambda^{-1} k^2, \]

which proves the statement for \( j = 0 \).

**Theorem 4.4.4 ([NT04, Thm. 4.1, 4.10]).** Assume that \( \mu \) is a generating and transient probability measure on \( \frac{1}{2} \mathbb{Z}_+ \) with finite mean, that is \( \sum_{s \in \frac{1}{2} \mathbb{Z}_+} \mu(s) s < \infty \). Then the Martin compactification \( \hat{M}(SU_q(2), \mu) \) of \( SU_q(2) \) with respect to \( \mu \) equals \( \Psi \) and thus the Martin boundary is isomorphic to \( C(S_{q,0}^2) \).

### 4.4.2 Regularity

Let \( \mu = \sum_{i \in \frac{1}{2} \mathbb{Z}_+} \mu_i \delta_i \) be a probability measure on \( \frac{1}{2} \mathbb{Z}_+ \). We form the infinite tensor product with respect to the state \( \varphi_\mu \) (see Definition 1.3.6)

\[
\bigotimes_{n=-\infty}^{-1} (\mathbb{L}^\infty(SU_q(2)), \varphi_\mu).
\]

(4.4.4)

Recall that the state \( \varphi_\mu^\infty \) given by

\[
\varphi_\mu^\infty \left( \cdots \otimes 1 \otimes 1 \otimes x_n \otimes \cdots \otimes x_1 \right) := \varphi_\mu(x_n) \cdots \varphi_\mu(x_1)
\]

is faithful on the infinite tensor product (4.4.4).

**Proposition 4.4.5.** There exists a constant \( C > 0 \) such that the elements \( \hat{X}_i \) satisfy the estimate

\[
\|1 \frac{1}{2} \otimes \pi_s(\hat{X}_i) - (\pi_\frac{1}{2} \otimes \pi_s) \hat{\Delta}(\hat{X}_i)\|_{\varphi_{\frac{1}{2}} \otimes \varphi_s}^2 \leq C d_s^{-1}.
\]

for every \( s \in \frac{1}{2} \mathbb{Z}_+ \).

**Proof.** To shorten the notation slightly, write

\[
a_{\pm, i}^{\frac{1}{2}} := C_q \left( \frac{1}{2}, s, \frac{1}{2} i, i \pm \frac{1}{2} \right); \quad a_{\pm, j}^{-\frac{1}{2}} := C_q \left( \frac{1}{2}, s, -\frac{1}{2} i, i \pm \frac{1}{2} \right),
\]

(4.4.5)

so that \( \xi_{\pm, i}^{\frac{1}{2}} \otimes \xi_{\pm, j}^{\frac{1}{2}} = a_{\pm, i}^{\frac{1}{2}} \xi_{\pm, j}^{\frac{1}{2}} + a_{\pm, j}^{-\frac{1}{2}} \xi_{\pm, i}^{-\frac{1}{2}} \). Recall that \( \langle \xi_{i, i}^{\frac{1}{2}}, \xi_{i, j}^{\frac{1}{2}} \rangle = \delta_{i, i} \delta_{i, j} \). We compute

\[
d_{\pm} d_{\varphi_{\frac{1}{2}}} (\varphi_\frac{1}{2} \otimes \varphi_s, (1 \frac{1}{2} \otimes \pi_s(\hat{X}_0))^*(\pi_\frac{1}{2} \otimes \pi_s)(\hat{\Delta}(\hat{X}_0)))
\]

\[
= (\Tr_\frac{1}{2} \otimes \Tr_s) (q^{2s} (1 \frac{1}{2} \otimes \pi_s(k^2))(\pi_\frac{1}{2} \otimes \pi_s)(\hat{\Delta}(\hat{X}_0))^2)(\pi_\frac{1}{2} \otimes \pi_s)(\hat{\Delta}(\hat{X}_0))^2(k^2 \otimes k^2)(\rho^{-1} \otimes \rho^{-1}))
\]

\[
= q^{2s} (\Tr_\frac{1}{2} \otimes \Tr_s) (k^2 \otimes k^2)(\pi_\frac{1}{2} \otimes \pi_s)(\hat{\Delta}(\hat{X}_0))^2(k^4 \otimes k^4))
\]

\[
= q^{2s} \sum_{j \in \{1, \cdots, 2s\}} \langle (1 \frac{1}{2} \otimes \pi_s(k^2))(\pi_\frac{1}{2} \otimes \pi_s)(\hat{\Delta}(\hat{X}_0))^2(k^4 \otimes k^4))\xi_{j, i}^{\frac{1}{2}} \otimes \xi_{j, i}^{\frac{1}{2}}, \xi_{j, i}^{\frac{1}{2}} \otimes \xi_{j, i}^{\frac{1}{2}} \rangle.
\]
\[ q^{2s} \sum_{\pm,i} q^{-6i+2} \left\langle (\pi_{\frac{1}{2}} \otimes \pi_s)(\tilde{\Delta}(\tilde{\lambda}^{-1}))\xi_{\pm}^{i+\frac{1}{2}} \otimes \xi_{\mp}^{i} \right\rangle \]

\[ = q^{2s} \sum_{\pm,i} q^{-6i+2} \left\langle (\pi_{\frac{1}{2}} \otimes \pi_s)(\tilde{\Delta}(\tilde{\lambda}^{-1}))\xi_{\pm}^{i+\frac{1}{2}} \otimes \xi_{\mp}^{i} \right\rangle \]

\[ = q^{2s} \sum_{\pm,i} q^{-6i+2} \left\langle a_{\pm,i}^{1/2} \pi_{s+1/2} (\tilde{\lambda}^{-1})\xi_{\pm}^{i+\frac{1}{2}} + a_{\pm,i}^{-1/2} \pi_{s-1/2} (\tilde{\lambda}^{-1})\xi_{\pm}^{i+\frac{1}{2}} \right\rangle \]

\[ = q^{2s} \sum_{\pm,i} q^{-6i+2} \left\langle 2^{s+1} a_{\pm,i}^{1/2} \xi_{\pm}^{i+\frac{1}{2}} + q^{2s-1} a_{\pm,i}^{-1} \xi_{\pm}^{i+\frac{1}{2}} \right\rangle \]

\[ = q^{4s} \sum_{\pm,i} q^{-6i+2} \left\langle (a_{\pm,i}^{1/2})^2 + q^{-1} (a_{\pm,i}^{-1})^2 \right\rangle \]

\[ = q^{4s} \sum_{i \in \{-s,...,s-1,s\}} q^{-6i+2} \left( q^2 q^{(s+i-1)}_{\frac{1}{2},s+i} + q^{-2} q^{(s+i+1)}_{\frac{1}{2},s+i} \right). \tag{4.4.6} \]

It is a standard result that for a positive linear functional \( \omega \) it holds that \( \omega(a^*) = \overline{\omega(a)} \). Clearly \( \varphi_s \) is positive, hence

\[ \varphi_s(ab) = \overline{\varphi_s(b^*a^*)}. \tag{4.4.7} \]

Since \( \tilde{X}_0 \) is self-adjoint and (4.4.6) is real-valued we see

\[ d_s d_s (\varphi_\frac{1}{2} \otimes \varphi_s)((\pi_{\frac{1}{2}} \otimes \pi_s)(\tilde{\Delta}(\tilde{\lambda}^{-1}))^*(\pi_{\frac{1}{2}} \otimes \pi_s)(\tilde{\Delta}(\tilde{\lambda}^{-1}))) \]

\[ = q^{4s} \sum_{i \in \{-s,...,s-1,s\}} q^{-6i+2} \left( q^2 q^{(s+i-1)}_{\frac{1}{2},s+i} + q^{-2} q^{(s+i+1)}_{\frac{1}{2},s+i} \right). \]

Similarly using that \( \tilde{\lambda} \) is central gives

\[ d_s d_s (\varphi_\frac{1}{2} \otimes \varphi_s)((\pi_{\frac{1}{2}} \otimes \pi_s)(\tilde{\Delta}(\tilde{\lambda}^{-2})(k^{-1} \otimes k^{-1}))) \]

\[ = q^{4s} \sum_{i \in \{-s,...,s-1,s\}} q^{-6i+2} \left( q^2 q^{(s+i-1)}_{\frac{1}{2},s+i} + q^{-2} q^{(s+i+1)}_{\frac{1}{2},s+i} \right). \]
Combining these four calculations yields

\[ 1 = \langle \xi_{\pm \frac{1}{2}}^2 \otimes \xi_s, \xi_{\pm \frac{1}{2}}^2 \otimes \xi_s \rangle = \langle a_{\pm \frac{3}{2}} \xi_{\pm \frac{1}{2}}^2 + a_{\pm \frac{3}{2}} \xi_{\pm \frac{1}{2}}^2, a_{\pm \frac{3}{2}} \xi_{\pm \frac{1}{2}}^2 + a_{\pm \frac{3}{2}} \xi_{\pm \frac{1}{2}}^2 \rangle = (a_{\pm \frac{1}{2}}^2 + (a_{\pm \frac{1}{2}}^2)^2 \]

gives

\[
d_s^2(\varphi \otimes \varphi) ((1 \otimes \pi_s(\tilde{X} \tilde{X}))^r(1 \otimes \pi_s(\tilde{X} \tilde{X})))
= (\text{Tr} \otimes \text{Tr}) ((\pi \otimes \pi) ((1 \otimes (\tilde{X}^2 k^4))(\rho^{-1} \otimes \rho^{-1})))
= q^{4s} \sum_{\pm, i} \langle (\pi \otimes \pi) (k^4) \xi_{\pm \frac{1}{2}} \otimes \xi_s, \xi_{\pm \frac{1}{2}} \otimes \xi_s \rangle
= q^{4s} \sum_{\pm, i} q^{-6i(\mp 1)}
= q^{4s} \sum_{\pm, i} q^{-6i(\mp 1)} \left( q^{(\mp 2i + 1)} \frac{[s \mp i + 1]_q}{[2s + 1]_q} + q^{(\pm s + i \pm 1)} \frac{[s \mp i]_q}{[2s + 1]_q} \right).
\]

Combining these four calculations yields

\[
(\varphi \otimes \varphi) ((1 \otimes \pi_s(\tilde{X} \tilde{X})) - (\pi \otimes \pi_s(\tilde{X} \tilde{X}))^r(1 \otimes \pi_s(\tilde{X} \tilde{X})))
= (\varphi \otimes \varphi) ((1 \otimes \pi_s(\tilde{X} \tilde{X}))^r(1 \otimes \pi_s(\tilde{X} \tilde{X})))
+ (\varphi \otimes \varphi) ((\pi \otimes \pi) (\Delta(\tilde{X} \tilde{X}))^r(\pi \otimes \pi) (\Delta(\tilde{X} \tilde{X})))
- 2(\varphi \otimes \varphi) ((1 \otimes \pi_s(\tilde{X} \tilde{X}))^r(\pi \otimes \pi) (\Delta(\tilde{X} \tilde{X})))
= (d_s^2)^{-1} [2s + 1] q^{-1} q^{4s} \left( \sum_{\pm, i} q^{(\pm 6i \mp 1)} \left( (1 + q^{2i} q^2 - 2q^{1 i} q) q^{(\mp 2i + 1)} [s \mp i + 1]_q \right.ight.
+ (1 + q^{2i} q^2 - 2q^{1 i} q) q^{(\pm s + i \pm 1)} [s \mp i]_q \bigg)
= d_s^{-1} d_s^{-2} q^{4s} \left( \sum_{i = -s}^s q^{(-6i - 1)} \left( (1 + 1 - 2) q^{i \mp s} [s + i + 1]_q + (1 + q^{-4} - 2q^{-2}) q^{s + i \mp 1} [s - i]_q \right)
+ \sum_{i = -s}^s q^{(-6i + 1)} \left( (1 + q^4 - 2q^2) q^{s \mp i} [s - i + 1]_q + (1 + 1 - 2) q^{s \mp i \mp 1} [s + i]_q \right) \bigg)
= d_s^{-1} d_s^{-2} q^{4s} \sum_{i = -s}^s q^{-6i} \left( q^{-1}(1 - q^{-2}) q^{s \mp i \mp 1} [s - i]_q \right) + \left( q(1 - q^2) q^{s \mp i} [s - i + 1]_q \right).
\]
which proves the result for $\tilde{X}_0$. In fact this estimate is stronger than stated in the proposition, but we will not need that.

We deal with $\tilde{X}_1$ in an analogous way. However, the estimates become slightly more involved, because the $\xi_i^s$ are no longer eigenvectors for $\pi_s(e)$ and $\pi_s(f)$. The calculations for $\tilde{X}_{-1}$ are similar and are omitted. First we calculate

$$d_s\varphi_s(\pi_s(\tilde{X}_1)) = \text{Tr}_s(\pi_s(\tilde{X}_1) (\tilde{X}_1)_{\pi_s(\rho^-1)})$$

$$= \text{Tr}_s(\pi_s(\tilde{X}_1) (\tilde{X}_1)_{\pi_s(\rho^-1)})$$

$$= q^{4s} \sum_{i=-s}^s \langle \pi_s(f) \xi_i^s, \xi_i^s \rangle$$

$$= q^{4s} \sum_{i=-s}^s q^{-4i} ([s + i]_q [s - i + 1]_q) \frac{1}{2}([s - (i - 1)]_q [s + (i - 1) + 1]_q) \frac{1}{2} \langle \xi_i^s, \xi_i^s \rangle$$

$$= q^{4s} \sum_{i=-s}^s q^{-4i} [s + i]_q [s - i + 1]_q$$

$$= \frac{q^{4s}}{(q - q^{-1})^2} \sum_{i=-s}^s q^{-4i} (q^{s+i} - q^{-s-i})(q^{s-i+1} - q^{-s+i-1})$$

$$= \frac{q^{4s}}{(q - q^{-1})^2} \sum_{i=-s}^s q^{-4i} (q^{2s+1} + q^{-2s-1} - q^{2i-1} - q^{-2i+1})$$

$$= \frac{q^{4s}}{(q - q^{-1})^2} \left( (q^{2s+1} + q^{-2s-1}) \sum_{i=-s}^s q^{-4i} - q^{-1} \sum_{i=-s}^s q^{-2i} - q \sum_{i=-s}^s q^{-6i} \right)$$

$$= \frac{q^{4s}}{(q - q^{-1})^2} \left( (q^{2s+1} + q^{-2s-1}) [2s + 1]_q^2 - q^{-1}[2s + 1]_q - q[2s + 1]_q^3 \right).$$
This gives

\[
d_{\frac{1}{2}} d_s(\varphi_{\frac{1}{2}} \otimes \varphi_s)((1_{\frac{1}{2}} \otimes \pi_s(\tilde{X}_1))^* (1_{\frac{1}{2}} \otimes \pi_s(\tilde{X}_1))) = q^4 s \frac{q + q^{-1}}{(q - q^{-1})^2} \left( (q^{2s+1} + q^{-2s-1})[2s + 1]_q^2 - q^{-1}[2s + 1]_q - q[2s + 1]_q^3 \right).
\]

Using (4.4.8) for \( s \pm \frac{1}{2} \) instead of \( s \) we also get

\[
d_{\frac{1}{2}} d_s(\varphi_{\frac{1}{2}} \otimes \varphi_s)((\pi_{\frac{1}{2}} \otimes \pi_s)(\hat{\Delta}(\tilde{\lambda}^{-1}ek))\hat{\Delta}(\tilde{\lambda}^{-1}ek))) = (\text{Tr}_{\frac{1}{2}} \otimes \text{Tr}_s)((\pi_{\frac{1}{2}} \otimes \pi_s)(\hat{\Delta}(kf\tilde{\lambda}^{-1}ek)(k^2 \otimes k^2)) = (\text{Tr}_{\frac{1}{2}} \otimes \text{Tr}_s)((\pi_{\frac{1}{2}} \otimes \pi_s)(\hat{\Delta}(\tilde{\lambda}^{-2}fek^4))) = \text{Tr}_{s-\frac{1}{2}}(\pi_{s-\frac{1}{2}}(\tilde{\lambda}^{-2}fek^4)) + \text{Tr}_{s+\frac{1}{2}}(\pi_{s+\frac{1}{2}}(\tilde{\lambda}^{-2}fek^4)) = q^{4s-2} \frac{q^2 + q^{-2s}}{(q - q^{-1})^2} \left( (2s + 2)_q^2 - q^{-1}[2s + 2]_q - q[2s + 2]_q^3 \right)
\]

Recall the abbreviations of the Clebsch–Gordan coefficients \( a_{\pm,i}^{\pm} \) and \( a_{\pm,i}^{-} \) introduced in (4.4.5) above. We obtain

\[
d_{\frac{1}{2}} d_s(\varphi_{\frac{1}{2}} \otimes \varphi_s)((1_{\frac{1}{2}} \otimes \pi_s(\tilde{X}_1))^* (1_{\frac{1}{2}} \otimes \pi_s)(\hat{\Delta}(\tilde{X}_1))) = (\text{Tr}_{\frac{1}{2}} \otimes \text{Tr}_s)((1_{\frac{1}{2}} \otimes \pi_s(\tilde{\lambda}^{-1}ek))^* (\pi_{\frac{1}{2}} \otimes \pi_s)(\hat{\Delta}(\tilde{\lambda}^{-1}ek))(k^2 \otimes k^2)) = (\text{Tr}_{\frac{1}{2}} \otimes \text{Tr}_s)(q^{2s}(\pi_{\frac{1}{2}} \otimes \pi_s)((1 \otimes efk)\hat{\Delta}(\tilde{\lambda}^{-1}e)(k^3 \otimes k^3))) = q^{2s} \sum_{i \in \{-s,-s+1,...,s-1,s\}} \langle (\pi_{\frac{1}{2}} \otimes \pi_s)((1 \otimes efk)\hat{\Delta}(\tilde{\lambda}^{-1}e)(k^3 \otimes k^3))\xi^{\frac{i}{s+1}} \otimes \xi^s, \xi^{\frac{i}{s+1}} \otimes \xi^s \rangle
\]

\[
= q^{2s} \sum_{\pm,i} q^{\frac{3}{2}} q^{-3i} \langle (\pi_{\frac{1}{2}} \otimes \pi_s)(\hat{\Delta}(\tilde{\lambda}^{-1}e))\xi^{\frac{i}{s+1}} \otimes \xi^s, (\pi_{\frac{1}{2}} \otimes \pi_s)(1 \otimes efk)\xi^{\frac{i}{s+1}} \otimes \xi^s \rangle
\]

\[
= q^{2s} \sum_{\pm,i} q^{\frac{3}{2}} q^{-3i} \langle [s + i]_q[s - i + 1]_q \xi^{\frac{i}{s+1}} (\pi_{\frac{1}{2}} \otimes \pi_s)(\hat{\Delta}(\tilde{\lambda}^{-1}e))\xi^{\frac{i}{s+1}} \otimes \xi^s, \xi^{\frac{i}{s+1}} \otimes \xi^s \rangle
\]

\[
= q^{2s} \sum_{\pm,i} q^{\frac{3}{2}} q^{-3i} \langle [s + i]_q[s - i + 1]_q \xi^{\frac{i}{s+1}} \rho_{\pm,i} \xi^{\frac{s+i}{s+1}} a_{\pm,i}^{\pm} \xi^{\frac{s-i}{s+1}} \rho_{\pm,i} a_{\pm,i}^{-} \rangle
\]

\[
= q^{2s} \sum_{\pm,i} q^{\frac{3}{2}} q^{-3i} \langle [s + i]_q[s - i + 1]_q \xi^{\frac{i}{s+1}} \rho_{\pm,i} \xi^{\frac{s+i}{s+1}} a_{\pm,i}^{\pm} \xi^{\frac{s-i}{s+1}} \rho_{\pm,i} a_{\pm,i}^{-} \rangle
\]
where we used in the third last line that 

$$a_{\pm,i}^\pm = a_{\pm,i}^\pm$$

and in the last line

$$q^m[n]_q + q^{-m}[n]_q = [m+n]_q$$
(4.4.9). Since this expression is real-valued, by (4.4.7) it follows that also

\[
d_1 d_s(\varphi \otimes \varphi_s)((\pi_2 \otimes \pi_s)(\hat{\Delta}(\hat{X}_1))^*(1_2 \otimes \pi_s(\hat{X}_1)))
\]

\[
= q^{4s} \frac{q[2s+3]_q + q^{-1}[2s-1]_q}{[2s+1]_q(q-q^{-1})^2} \left( (q^{2s+1} + q^{-2s-1})[2s+1]_q - q^{-1}[2s+1]_q - q[2s+1]_q \right).
\]

Adding these four terms shows that \( \hat{X}_1 \) satisfies the required estimated. Indeed, using the asymptotic behaviour of the \( q \)-numbers (cf. Lemma 1.2.3) gives

\[
d_1 d_s(\varphi \otimes \varphi_s)((1_2 \otimes \pi_s(\hat{X}_1) - (\pi_2 \otimes \pi_s)\Delta(\hat{X}_1))^*(1_2 \otimes \pi_s(\hat{X}_1) - (\pi_2 \otimes \pi_s)\Delta(\hat{X}_1)))
\]

\[
= q^{4s} \frac{q + q^{-1}}{(q-q^{-1})^2} \left( (q^{2s+1} + q^{-2s-1})[2s+1]_q - q^{-1}[2s+1]_q - q[2s+1]_q \right)
\]

\[
+ \frac{q^{4s-2}}{(q-q^{-1})^2} \left( (q^{2s} + q^{-2s})[2s]_q - q^{-1}[2s]_q - q[2s]_q \right)
\]

\[
+ \frac{q^{4s+2}}{(q-q^{-1})^2} \left( (q^{2s+2} + q^{-2s-2})[2s+2]_q - q^{-1}[2s+2]_q - q[2s+2]_q \right)
\]

\[
- 2q^{4s} \frac{q[2s+3]_q + q^{-1}[2s-1]_q}{[2s+1]_q(q-q^{-1})^2}
\]

\[
\times \left( (q^{2s+1} + q^{-2s-1})[2s+1]_q - q^{-1}[2s+1]_q - q[2s+1]_q \right)
\]

\[
= \frac{q^{4s}}{(q-q^{-1})^2} \left( (q + q^{-1})(O(q^{-4s}) + q^{-2s-1} - q^{-6s-3} q^{-q-2s} - O(q^{-2s}) + O(q^{6s}) - q^{-6s-3} - q^{q-2s}) \right)
\]

\[
+ q^{-2}(O(q^{-4s}) + q^{-2s-2} q^{-q-2s} - O(q^{-4s}) - O(q^{6s}) - q^{-6s-6} - q^{q-2s})
\]

\[
+ q^{2}(O(q^{-4s}) + q^{-2s-2} q^{-q-2s} - O(q^{-4s}) - O(q^{6s}) - q^{q-2s} - q^{q-2s})
\]

\[
- 2(q^{-2}(1 + O(q^{-4s})) + q^{-1} q^2 (1 + O(q^{-4s})))
\]

\[
\times \left( (O(q^{-4s}) + q^{-2s-1} q^{-q-2s} + O(q^{-4s}) + O(q^{6s}) - q^{-6s-3} - q^{q-2s}) \right)
\]

\[
= \frac{q^{4s}}{(q-q^{-1})^2} \left( (q + q^{-1}) \left( \frac{q^{-6s-3}}{q^2-q^{-2}} + \frac{q^{-6s-2}}{q^3-q^{-3}} - \frac{q^{-6s-2}}{q^2-q^{-2}} + \frac{q^{-6s-1}}{q^3-q^{-3}} - \frac{q^{-6s-4}}{q^2-q^{-2}} \right) \right)
\]

\[
+ \frac{q^{-6s-3}}{q^3-q^{-3}} + 2(q + q^{-1})(1 + O(q^{-4s})) \left( \frac{q^{-6s-3}}{q^2-q^{-2}} - \frac{q^{-6s-2}}{q^3-q^{-3}} + O(q^{-4s}) \right)
\]

\[
= \frac{q^{4s}}{(q-q^{-1})^2} \left( O(q^{-2s}) + O(q^{-4s}) \right)
\]

\[
= O(1),
\]

which proves the claim for \( \hat{X}_1 \).

\( \blacksquare \)

Lemma 4.4.6. Let \( C > 0, x \in l^\infty(\text{SU}_q(2)) \) and \( \mu = \sum_{t \in 1/2\mathbb{Z}_+} \mu_t \delta_t \) be a probability measure on \( 1/2\mathbb{Z}_+ \). Denote \( U = \bigoplus_{t \in \text{supp}(\mu)} U_t \). Assume that the element \( x \) satisfies the estimate

\[
\|1_{1/2} \otimes \pi_s(x) - (\pi_1 \otimes \pi_s)\hat{\Delta}(x)\|_{\varphi_2 \otimes \varphi_s}^2 \leq C d_s^{-1}
\]

(4.4.10)
for every \( s \in \frac{1}{2} \mathbb{Z}_+ \), then the inequality

\[
\|1 \sigma \pi_s(x) - (\pi U \sigma \pi_s) \hat{\Delta}(x)\|_{\varphi_{\mu} \otimes \varphi_s}^2 \leq C d_s^{-1} \frac{[2]^k}{\sqrt{[2]^k - 2}} \sum_{t \in \frac{1}{2} \mathbb{Z}_+} \mu_t \frac{(d_1)^{2t}}{d_t}
\]

holds for every \( s \in \frac{1}{2} \mathbb{Z}_+ \).

**Proof.** Let \( x \in t^\infty(\mathbb{S}_U(2)) \) satisfying (4.4.10). We start with the following estimate

\[
\begin{align*}
&\left\| (1_{\frac{1}{2}} \otimes \cdots \otimes 1_{\frac{1}{2}} \otimes (\pi \otimes \pi_s) \hat{\Delta}^k(x) \right\|_{\varphi_{\frac{1}{2}} \otimes \varphi_{\frac{1}{2}} \otimes \varphi_s}^2 \\
&\quad - (1_{\frac{1}{2}} \otimes \cdots \otimes 1_{\frac{1}{2}} \otimes (\pi \otimes \pi_s) \hat{\Delta}^{k-1}(x)) \left\|_{\varphi_{\frac{1}{2}} \otimes \varphi_{\frac{1}{2}} \otimes \varphi_s}^2 \\
&= \left\| (\pi \otimes \pi_s) \left( (\hat{\Delta}^k - 1_{\frac{1}{2}} \otimes \hat{\Delta}^{k-1})(x) - (1_{\frac{1}{2}} \otimes \hat{\Delta}^{k-1})(x) \right) \right\|_{\varphi_{\frac{1}{2}} \otimes \varphi_{\frac{1}{2}} \otimes \varphi_s}^2 \\
&= (\varphi_{\frac{1}{2}} \otimes \varphi_{\frac{1}{2}} \otimes \varphi_s)(\pi \otimes \pi_s) \\
&\quad \cdot \left( (\hat{\Delta}^k - 1_{\frac{1}{2}} \otimes \hat{\Delta}^{k-1})(x) - (1_{\frac{1}{2}} \otimes \hat{\Delta}^{k-1})(x) \right) - (\hat{\Delta}^k - 1_{\frac{1}{2}} \otimes \hat{\Delta}^{k-1})(x) - (1_{\frac{1}{2}} \otimes \hat{\Delta}^{k-1})(x) \right) \\
&= \sum_{r \in \frac{1}{2} \mathbb{Z}_+} m_{\frac{1}{2}} \hat{\Delta}^{k-1} d_s^{-1} \sum_{t \in \frac{1}{2} \mathbb{Z}_+} \frac{d_r}{d_1} C d_r^{-1} \\
&= d_{\frac{1}{2}}^{(k-1)} d_s^{-1} C \sum_{r \in \frac{1}{2} \mathbb{Z}_+} m_{\frac{1}{2}} \hat{\Delta}^{k-1} \\
&\leq d_{\frac{1}{2}}^{(k-1)} d_s^{-1} C 2^{k-1} \\
&= C \left( \frac{2}{d_{\frac{1}{2}}} \right)^{k-1} d_s^{-1}.
\end{align*}
\]

Here we used (4.4.2) to obtain a bound on the sum of the multiplicities. Now let \( t \in \frac{1}{2} \mathbb{Z}_+ \). Then the representation \( \pi_t \) embeds into \((\pi_{\frac{1}{2}})^{2t}\) with multiplicity 1. Therefore for any positive element \( y \) it holds

\[
\frac{d_t}{(d_1)^{2t}} \varphi_t(y) \leq \sum_{r \in \frac{1}{2} \mathbb{Z}_+} m_{\frac{1}{2}} \otimes 2t \frac{d_r}{(d_1)^{2t}} \varphi_r(y) = (\varphi_{\frac{1}{2}})^{2t}(y).
\]

Combining this with the previous estimate we obtain

\[
\begin{align*}
&\left\|1_{\frac{1}{2}} \otimes \pi_s(x) - (\pi U \otimes \pi_s) \hat{\Delta}(x)\|_{\varphi_{\frac{1}{2}} \otimes \varphi_s}^2 \\
&\quad \leq \frac{(d_1)^{2t}}{d_t} \left\|1_{\frac{1}{2}} \otimes \cdots \otimes 1_{\frac{1}{2}} \otimes \pi_s(x) - (\pi \otimes \pi_s) \hat{\Delta}^{2t-1} \otimes \hat{\Delta}(x)\right\|_{\varphi_{\frac{1}{2}} \otimes \varphi_{\frac{1}{2}} \otimes \varphi_s}^2.
\end{align*}
\]
Since

\[ \| 1_U \otimes \pi_s(x) - (\pi_U \otimes \pi_s)\hat{\Delta}(x) \|_{\varphi_1 \otimes \cdots \otimes \varphi_1 \otimes \varphi_s}^2 = \sum_{t} \mu_t \| 1_U \otimes \pi_s(x) - (\pi_U \otimes \pi_s)\hat{\Delta}(x) \|_{\varphi_1 \otimes \cdots \otimes \varphi_1 \otimes \varphi_s}^2, \]

the proof is complete.

\[ \square \]

**Corollary 4.4.7.** Let \( C > 0, x \in l^{\infty}(\text{SU}_q(2)) \) and \( \mu = \sum_{t \in \frac{1}{2}\mathbb{Z}_+} \mu_t \delta_t \) be a probability measure on \( \frac{1}{2}\mathbb{Z}_+ \). Assume that \( \sum_{t} \mu_t (1 + q^2)^{2t} < \infty \) and that the element \( x \) satisfies estimate (4.4.10) for every \( s \in \frac{1}{2}\mathbb{Z}_+ \), then there exists a constant \( C' \) independent of \( n \) such that

\[ \| j_n(x) - j_{n+1}(x) \|_{\varphi_{\mu}^{\otimes n+1}}^2 \leq C' \left( \sum_{r \in \frac{1}{2}\mathbb{Z}_+} \mu(r) \frac{\dim(U_r)}{d_r} \right)^n \]

In particular if \( \text{supp}(\mu) \neq \{0\} \), then

\[ \sum_{n=0}^{\infty} \| j_n(x) - j_{n+1}(x) \|_{\varphi_{\mu}^{\otimes n+1}}^2 < \infty. \]

**Proof.** For \( n \geq 1 \) it holds that \( \frac{q^{-1}q^n}{q^n - q^{-1}} q^n \leq \frac{1}{|n_q|} \leq q^n(q - q^{-1}) \). Also \( \sum_{t} \mu_t [2]_{q^2} q^{2t} = \sum_{t} \mu_t (q^2 + 1)^{2t} \). So \( \sum_{t} \mu_t \frac{(d_{\frac{1}{2}})^{2t}}{d_{\frac{1}{2}}} < \infty \) if and only if \( \sum_{t} \mu_t (1 + q^2)^{2t} < \infty \). Use Lemma 4.4.6 and the decomposition of \( \varphi_\mu^n \) using the constants \( c_{n,s}(\mu) \) (see (3.2.2)) to obtain the following chain of inequalities.

\[ \| j_n(x) - j_{n+1}(x) \|_{\varphi_{\mu}^{\otimes n+1}}^2 = \| 1 \otimes \hat{\Delta}^{n-1}(x) - \hat{\Delta}^n(x) \|_{\varphi_{\mu}^{\otimes n+1}}^2 \]

\[ = \| 1 \otimes x - \hat{\Delta}(x) \|_{\varphi_\mu^{\otimes (\varphi_{\mu})}}^2 \]

\[ = \sum_{s \in \frac{1}{2}\mathbb{Z}_+} c_{n,s}(\mu) \| 1 \otimes x - \hat{\Delta}(x) \|_{\varphi_{\mu}^{\otimes \varphi_s}}^2 \]
\[
\leq \sum_{s \in \frac{1}{2}\mathbb{Z}_+} c_{n,s}(\mu) C d_s^{-1} \frac{[2]_q}{(\sqrt{[2]_q} - \sqrt{2})^2} \left( \sum_{t \in \frac{1}{2}\mathbb{Z}_+} \mu_t \frac{(d_\frac{1}{2})^{2t}}{d_t} \right) \\
\leq C \frac{[2]_q}{(\sqrt{[2]_q} - \sqrt{2})^2} \left( \sum_{t \in \frac{1}{2}\mathbb{Z}_+} \mu_t \frac{(d_\frac{1}{2})^{2t}}{d_t} \right) \left( \sum_{r \in \frac{1}{2}\mathbb{Z}_+} \mu(r) \frac{\dim(U_r)}{d_r} \right)^{n}.
\]

This proves the first part of the corollary. Write \( d := \sum_r \mu(r) \frac{\dim(U_r)}{d_r} \). If \( \text{supp}(\mu) \neq \{0\} \) the sum satisfies \( 0 < d < 1 \) and thus \( \sum_{n=0}^{\infty} d^n = \frac{1}{1-d} < \infty \).

Now it is a matter of putting everything together to prove the regularity for random walks on \( \hat{\text{SU}}_q(2) \).

**Proof of the first part of Theorem 4.4.1.** Proposition 4.4.5 and Corollary 4.4.7 imply that for elements \( x = \tilde{X}_i, i = -1, 0, 1 \) the following holds

\[
\sum_{n=0}^{\infty} \left( \cdots \otimes \varphi_\mu \otimes \varphi_\mu \right) \left( (j_n(x) - j_{n+1}(x))^* (j_n(x) - j_{n+1}(x)) \right) < \infty.
\]

Hence \( s\lim_n j_n(x) \) exists by means of Lemma 1.3.1. As \( X_0^* = X_0 \) and \( \tilde{X}_{-1}^* = q\tilde{X}_1 \), we in fact have that \( s^*\lim_n j_n(x) \) exists. In other words, the elements \( \tilde{X}_j \) are regular. By Lemma 4.4.3 and Theorem 4.4.4 the Martin boundary \( \tilde{M}(\hat{\text{SU}}_q(2), \mu) \) is generated as a \( C^* \)-algebra by \( c_0(\hat{\text{SU}}_q(2)) \) and \( \tilde{X}_i, i = -1, 0, 1 \). Proposition 4.3.2 asserts that all elements in \( c_0(\hat{\text{SU}}_q(2)) \) are \( \varphi_\mu \)-regular and that the set of regular elements forms a \( C^* \)-algebra. Hence all elements in \( \tilde{M}(\hat{\text{SU}}_q(2), \mu) \) are regular.

**Remark 4.4.8.** This proof of regularity of “functions” on the Martin compactification is fundamentally different from the proof in the classical case. There is no need to use some sort of “noncommutative stopping times” (it is even not clear what such objects should be). Due to the fact that \( 0 < q < 1 \), there is exponentially fast convergence to the boundary if the measure \( \mu \) is nice enough, this is much stronger convergence than classically. We emphasize that this proof does not work in the classical case, so not for \( q = 1 \).
Chapter 5

A categorical approach to the Martin boundary

In this chapter we investigate to what extent the results of the previous chapters can be generalised to C*-tensor categories. Random walks on C*-tensor categories have been defined by Neshveyev and Yamashita in [NY14c], moreover they introduced the Poisson boundary for such random walks and used it in subsequent papers for a classification of quantum groups. Here we build on their categorical viewpoint. An intermediate step in this process of generalisation to categories is the paper [DRVV10] in which the authors establish a method to compute the Poisson and Martin boundary of a random walk on a quantum group from the respective boundary of a monoidally equivalent one. Based on these results we show in the first section that the property of convergence to the boundary is invariant under monoidal equivalence. In the remaining sections we turn our attention to giving a purely categorical approach to the Martin boundary and to convergence to the boundary.

5.1 Boundary convergence and monoidal equivalence

We show that the property of boundary convergence is stable under monoidal equivalence. In more detail, if we have a random walk on a discrete quantum group and a second discrete quantum group monoidally equivalent to the first one, then there is a natural way to define a random walk on the second quantum group and the random walk on the first quantum group converges to the boundary if and only if the random walk on the second does.

We mainly follow the approach of [DRVV10] whenever dealing with monoidal equivalence. However, we are forced to slightly modify their notation due to different conventions. Monoidal equivalence has been introduced in [BDRV06]. We start by stating their results.

Definition 5.1.1 ([BDRV06, Def. 3.1]). Two compact quantum groups $G_i := (C(G_i), \Delta_i)$ ($i = 1, 2$) are said to be monoidally equivalent if the representation categories $\text{Rep}(G_1)$ and $\text{Rep}(G_2)$ are unitarily monoidally equivalent as C*-tensor categories.
The following theorem is proven in [BDRV06] the formulation is from [DRVV10, Thm. 4.2]. Since this result will play a fundamental role and we use different conventions we recall it here for convenience.

**Theorem 5.1.2** ([BDRV06]). Given two monoidally equivalent compact quantum groups \( G_1 \) and \( G_2 \) together with a unitary monoidal equivalence \( \kappa : \text{Rep}(G_1) \to \text{Rep}(G_2) \), the following holds:

(i) there exists a (up to isomorphism) unique unital \(*\)-algebra \( \mathcal{B} \) equipped with a faithful state \( \omega \) and unitary elements \( X^s \in \mathcal{B} \otimes B(\mathcal{H}_s, \mathcal{H}_{\kappa(s)}) \) satisfying

- \( X^s_{12}X^t_{13}(\iota_B \otimes S) = (\iota_B \otimes \kappa(S))X^r \), whenever \( S \in \text{Hom}(r, s \otimes t) \);
- the matrix coefficients of \( \{X^s\} \) form a basis in \( \mathcal{B} \) as a vector space;
- \( (\omega \otimes \iota)(X^s) = 0 \) if \( s \neq 0 \);

(ii) there exist unique commuting ergodic actions \( \delta_1 : \mathcal{B} \to \mathcal{B} \otimes_{\text{alg}} \mathbb{C}[G_1] \) and \( \delta_2 : \mathcal{B} \to \mathbb{C}[G_2] \otimes_{\text{alg}} \mathcal{B} \) satisfying for every \( s \in \text{Irr}(G_1) \)

\[
(\delta_1 \otimes \iota)(X^s) = (X^s_{13})(U_s)_{23} \in \mathcal{B} \otimes \mathbb{C}[G_1] \otimes B(\mathcal{H}_s, \mathcal{H}_{\kappa(s)});
(\delta_2 \otimes \iota)(X^s) = (U_{\kappa(s})_{13}(X^s_{23}) \in \mathcal{C}[G_2] \otimes \mathcal{B} \otimes B(\mathcal{H}_s, \mathcal{H}_{\kappa(s)});
\]

(iii) the state \( \omega \) is invariant under \( \delta_1 \) and \( \delta_2 \) and is given by \( \omega(b)1_B = (\iota \otimes h)\delta_1(b) \).

**Proof.** See [BDRV06, Thm. 3.9 and Prop. 3.13]. Since we interchanged some of the legs of the tensor product, let us check that \( \delta_1 \) and \( \delta_2 \) indeed define actions. We are in the realm of Hopf algebras so there is no topology involved. We have

\[
(\delta_1 \otimes \iota \otimes \iota)(\delta_1 \otimes \iota)(X^s) = (\delta_1 \otimes \iota \otimes \iota)(X^s_{13}U^s_{23}) = X^s_{14}U^s_{24}U^s_{34} = (\iota \otimes \Delta \otimes \iota)(X^s_{13}U^s_{23}) = (\iota \otimes \Delta \otimes \iota)(\delta_1 \otimes \iota)(X^s)
\]

and similarly

\[
(\iota \otimes \delta_2 \otimes \iota)(\delta_2 \otimes \iota)(X^s) = (\iota \otimes \delta_2 \otimes \iota)(U^s_{13}X^s_{23}) = U^s_{14}U^s_{24}X^s_{34} = (\Delta \otimes \iota \otimes \iota)(U^s_{13}X^s_{23}) = (\Delta \otimes \iota \otimes \iota)(\delta_2 \otimes \iota)(X^s).
\]

Thus \( \delta_1 \) and \( \delta_2 \) are actions. \( \Box \)

This algebra \( \mathcal{B} \) is called the link algebra of \( G_1 \) and \( G_2 \) under the monoidal equivalence \( \kappa \). The algebra \( \mathcal{B} \) can be constructed explicitly, however such a definition is irrelevant for our purposes. We will only work with the properties described above. Using the faithful state \( \omega \) this can all be extended to the von Neumann framework. Let \( B \) be the von Neumann algebra generated by \( \mathcal{B} \) in the GNS representation of \( \omega \). The actions \( \delta_1 \) and \( \delta_2 \) have unitary implementations and can therefore be extended to \( \delta_1 : B \to B \otimes L^\infty(G_1) \) and \( \delta_2 : B \to L^\infty(G_2) \otimes B \) (see [DRVV10, Rem. 3.12]).
**Notation 5.1.3.** In this section $X$ denotes the following element\(^1\)

$$X := \prod_{s \in \text{Irr}(G)} X^s \in \mathbb{1}_{\text{rep}} \bigoplus_{s \in \text{Irr}(G)} \mathcal{B}(\mathcal{H}_s, \mathcal{H}_{\kappa(s)}) \bar{\otimes} B.$$  

Use this to define the collection of maps

$$k_n : \bigotimes_{i = -n}^{-1} l^\infty(\hat{G}_2) \to B \bar{\otimes} \left( \bigotimes_{i = -n}^{-1} l^\infty(\hat{G}_1) \right), \quad (n \geq 1),$$

$$x \mapsto (X_{1,n+1}^* \cdots X_{1,2}^*)(1_B \otimes x)(X_{1,2} \cdots X_{1,n+1});$$

$$k_0 : \mathbb{C} \to B, \quad z \mapsto z1_B.$$  

Observe that if $W^{(i)}$ is the multiplicative unitary of $G_i$, then $(\delta_1 \otimes \iota)(X) = X_{13}W^{(1)}_{23}$ and $(\iota \otimes \delta_2)(X) = W^{(2)}_{13}X_{23}$ when viewed in the multiplier algebra.

Proposition 5.1.5 below will play a crucial role to transport the convergence properties of one quantum group to a monoidal equivalent one. The result shows that all the algebraic operations can be transferred from $G_1$ to $G_2$ by means of the link algebra $B$. The construction performed in the proof is motivated by and can be proved in a spirit similar to [DRVV10, §7 and §8]. It extends their results to higher tensor powers. When defining random walks De Rijdt and Vander Vennet work with states of the form $\psi_s := d^{-1} \text{Tr}(\pi_s(\rho))$ and slice in the right leg, while we work with $\varphi_s := d^{-1} \text{Tr}(\pi_s(\rho^{-1}))$ slicing in the left leg. Moreover they interchange the roles of $G_1$ and $G_2$ in the monoidal equivalence, while we do not. Therefore our maps defining the isomorphisms have a slightly different form than theirs.

For the remainder of this section we assume that $G_1$ and $G_2$ are two monoidally equivalent compact quantum groups with a unitary monoidal equivalence $\kappa : \text{Rep}(G_1) \to \text{Rep}(G_2)$. We will freely use the results of Theorem 5.1.2 and the remarks following that theorem.

**Notation 5.1.4.** If $\alpha : N \to L^\infty(G) \bar{\otimes} N$ is a left action of the compact quantum group $G = G_1$ on a von Neumann algebra $N$, denote

$$B \boxtimes^\alpha N := \{x \in B \bar{\otimes} N : (\delta_1 \otimes \iota)(x) = (\iota \otimes \alpha)(x)\}.$$  

In the algebraic setting, if $\mathcal{N}$ is a unital $*$-algebra and $\alpha : \mathcal{N} \to \mathbb{C}[G] \otimes_{\text{alg}} \mathcal{N}$ an action, let

$$B \boxtimes^\alpha_{\text{alg}} \mathcal{N} := \{x \in B \otimes_{\text{alg}} \mathcal{N} : (\delta_1 \otimes \iota)(x) = (\iota \otimes \alpha)(x)\}.$$  

The $C^*$-algebraic case is more involved due to technicalities with the range. We formulate it in general, but we will only need it if the $C^*$-algebra under consideration equals the Martin boundary or Martin compactification. Suppose $D$ is a $C^*$-algebra with an action $\alpha : D \to C(G) \otimes D$. Define for $s \in \text{Irr}(G)$ the spectral subspaces

$$K_s := \{x \in D \otimes \mathcal{H}_s : (\alpha \otimes \iota)(x) = x_{13}U_{23}^s\}.$$  

---

\(^1\)Later the symbol $X$ will again be used to indicate objects in categories.
Denote
\[ \mathcal{D}_s := \text{span}\{ x(1 \otimes \xi) : x \in K_s, \xi \in \mathcal{H}_s \}, \quad \mathcal{D} := \text{span}\{ \mathcal{D}_s : s \in \text{Irr}(G) \}. \]

It follows that \( \alpha : \mathcal{D}_s \to \mathbb{C}[G]_s \otimes_{\text{alg}} \mathcal{D}_s \) is a Hopf algebra coaction. An action \( \alpha \) is called reduced if the conditional expectation onto the fixed point algebra \( (\iota \otimes \hat{h})\alpha : D \to D^\alpha \) is faithful. For a reduced action \( \alpha : D \to C(G) \otimes D \) define \( B \mathbb{C}_{\text{red}}^\alpha \) to be the norm closure of \( B \mathbb{C}_{\text{alg}}^\alpha \mathcal{D} \) in \( B \otimes D \).

**Proposition 5.1.5.** Let \( \alpha_1 \) be the adjoint action as defined in Notation 1.4.19. Using the notation introduced above, the following holds for \( 0 \leq m \leq n - 1 \):

(i) \( k_n : \bigotimes_{-n}^- l^\infty(\hat{G}_2) \to B \mathbb{C}_{\alpha_1} = (\bigotimes_{-n}^- l^\infty(\hat{G}_1)) \) is a \( \ast \)-isomorphism;

(ii) \( k_{n+1} \circ (\iota \otimes m \otimes \hat{\Delta}_2 \otimes l^\infty_{m-1}) = (\iota_B \otimes l^\otimes m \otimes \hat{\Delta}_1 \otimes l^\otimes_{m-1}) \circ k_n; \)

(iii) \( k_{n-1} \circ (\iota \otimes m \otimes \varphi_s \otimes l^\otimes_{m-1}) = (\iota_B \otimes l^\otimes m \otimes \varphi_s \otimes l^\otimes_{m-1}) \circ k_n; \)

(iv) \( k_{n+1}(1 \otimes x) = (k_n(x))_1 \otimes 1_{C(G)} \otimes (k_n(x))_2, \ldots, n = (\iota_B \otimes 1_{C(G)} \otimes l^\otimes)(k_n(x)), \) if \( x \in \bigotimes_{-n}^- l^\infty(\hat{G}_2); \)

**Proof.** For simplicity we write \( W = W^{(1)} \) for the multiplicative unitary of \( G_1 \) whenever no confusion arises and denote \( k := n - m - 1 \).

(i) Clearly \( k_n \) is an injective \( \ast \)-homomorphism, because \( X \) is unitary. The map \( k_n \) has the correct range. Indeed, first note \((\delta_1 \otimes l^\otimes)(X_{1,j}) = X_{1,j+1}W_{2,j+1}, \) so that

\[
(\delta_1 \otimes l^\otimes)(k_n(x)) = (\delta_1 \otimes l^\otimes)((X_{1,n+1} \cdots X_{1,2}) (1_B \otimes x) (X_{1,2} \cdots X_{1,n+1}))
\]

\[
= ((X_{1,n+2}W_{2,n+2})^* \cdots (X_{1,3}W_{2,3})^*(1_B \otimes 1_{C(G)} \otimes x)((X_{1,3}W_{2,3}) \cdots (X_{1,n+2}W_{2,n+2}))
\]

\[
= (W_{2,n+2}^* \cdots W_{2,3})^*(X_{1,n+2} \cdots X_{1,3}) (1_B \otimes 1_{C(G)} \otimes x)(X_{1,3} \cdots X_{1,n+2} (W_{2,3} \cdots W_{2,n+2})
\]

\[
= (\iota \otimes \alpha_1)(X_{1,n+1} \cdots X_{1,2}) (1_B \otimes x) (X_{1,2} \cdots X_{1,n+1})
\]

\[
= (\iota \otimes \alpha_1)k_n(x).
\]

For surjectivity let \( x \in B \mathbb{C}^\otimes \bigotimes_{-n}^- l^\infty(\hat{G}_1) \), then

\[
(\delta_1 \otimes l^\otimes)(X_{1,2} \cdots X_{1,n+1})x(X_{1,n+1}^* \cdots X_{1,2}^*)
\]

\[
= (X_{1,3}W_{2,3} \cdots X_{1,n+2}W_{2,n+2})((\delta_1 \otimes \iota)(x))(W_{2,n+2}^*X_{1,n+2}^* \cdots W_{2,3}^*X_{1,3})
\]

\[
= (X_{1,3} \cdots X_{1,n+2})(W_{2,3} \cdots W_{2,n+2})((\iota \otimes \alpha)(x))(W_{2,n+2}^* \cdots W_{2,3}^*)(X_{1,n+2}^* \cdots X_{1,3}^*)
\]

\[
= (X_{1,3} \cdots X_{1,n+2})(W_{2,3} \cdots W_{2,n+2})(W_{2,n+2}^* \cdots W_{2,3}^*)x_{1,3,4, \ldots, n+2} (W_{2,3} \cdots W_{2,n+2})
\]

\[
\times (W_{2,n+2}^* \cdots W_{2,3}^*)(X_{1,n+2}^* \cdots X_{1,3}^*)
\]

\[
= (X_{1,3} \cdots X_{1,n+2})x_{1,3,4, \ldots, n+2} (X_{1,n+2}^* \cdots X_{1,3}^*).
\]

So by ergodicity of \( \delta_1 \) the element \( X_{1,2} \cdots X_{1,n+1})x(X_{1,n+1}^* \cdots X_{1,2}^*) \) is of the form \( 1_B \otimes y \) for some \( y \in \bigotimes_{-n}^- l^\infty(\hat{G}_2) \), hence

\[
x = (X_{1,n+1}^* \cdots X_{1,2}^*)(1_B \otimes y)(X_{1,2} \cdots X_{1,n+1}) = k_n(y).
\]
So $k_n$ is surjective and thus a $*$-isomorphism. Note that
\[ y = (\omega \otimes \iota \otimes \kappa)(1_B \otimes y) = (\omega \otimes \iota \otimes \kappa)((X_{1,2} \cdots X_{n+1})x(X_{1,2}^* \cdots X_{n+1}^*)) \]

(ii) We first prove that $(\iota_B \otimes \hat{\Delta}_1)k_1(x) = k_2\hat{\Delta}_2(x)$ for $x \in l^\infty(\hat{G}_2)$. This is essentially already shown in [DRVV10, Prop. 8.3]. Assume $T \in \text{Hom}(r, s \otimes t)$ and $x \in l^\infty(\hat{G}_2)$, then by definition of $\hat{\Delta}_1$
\[
((\iota_B \otimes \pi_s \otimes \pi_t)(\iota_B \otimes \hat{\Delta}_1)k_1(x))(1 \otimes T) = ((\iota_B \otimes (\pi_s \otimes \pi_t)\hat{\Delta}_1)(X_{12}(1_B \otimes x)X_{12}))(1 \otimes T) \\
= (1 \otimes T)((\iota_B \otimes \pi_r)(X_{12}(1_B \otimes x)X_{12})) \\
= (1 \otimes T)(X_{12}^*)(1_B \otimes \pi_{\kappa(r)}(x))X_{12}^*.
\]

Using the properties of $X$ and the definition of $\hat{\Delta}_2$ this equals
\[
(5.1.1) = (X_{13})^*(X_{12}^*)(1_B \otimes \kappa(T))(1_B \otimes \pi_{\kappa(r)}(x))X_{12}^* \\
= (X_{13})^*(X_{12}^*)(1_B \otimes (\pi_{\kappa(s)} \otimes \pi_{\kappa(t)})\hat{\Delta}_2(x)\kappa(T))X_{12}^* \\
= (X_{13})^*(X_{12}^*)(1_B \otimes (\pi_{\kappa(s)} \otimes \pi_{\kappa(t)})\hat{\Delta}_2(x))X_{12}^*X_{13}(1 \otimes T) \\
= ((\iota_B \otimes \pi_s \otimes \pi_t)k_2\hat{\Delta}_2(x))(1 \otimes T),
\]
which proves the claim. Using the definition of $\hat{\Delta}$ in terms of the multiplicative unitary yields
\[
W_{23}^{(1)}X_{12}^*(1_B \otimes x \otimes 1)X_{12}(W_{23}^{(1)})^* = (\iota_B \otimes \hat{\Delta}_1)k_1(x) = k_2\hat{\Delta}_2(x) \\
= X_{13}^*W_{23}^{(2)}X_{12}^*(1_B \otimes x \otimes 1)(W_{23}^{(2)})^*X_{12}X_{13}.
\]

This can now be used to prove (ii) as follows
\[
(\iota_B \otimes \iota \otimes \kappa)(k_n(x)) \\
= (\iota_B \otimes \iota \otimes \hat{\Delta}_1 \otimes \iota \otimes k)((X_{1,n+1}^* \cdots X_{1,2}^*)(1_B \otimes x)(X_{1,2} \cdots X_{1,n+1}) \\
= W_{m+2,m+3}^{(1)}X_{1,n+2}^* \cdots X_{1,m+4}X_{1,m+2}^* \cdots X_{1,2}^*(1_B \otimes x_{1,\ldots,m+1,m+3,\ldots,n+1}) \\
\times (X_{1,2} \cdots X_{1,m+2}X_{1,m+4} \cdots X_{1,n+2})(W_{m+2,m+3}^{(1)})^* \\
= (X_{1,n+2}^* \cdots X_{1,2}^*)W_{m+2,m+3}^{(2)}(1_B \otimes x_{1,\ldots,m+1,m+3,\ldots,n+1})(W_{m+2,m+3}^{(2)})^*(X_{1,2} \cdots X_{1,n+2}) \\
= k_{n+1}((\iota \otimes \hat{\Delta}_2 \otimes \iota \otimes k)(x)).
\]

(iii) The following orthogonality relations hold in $B$ (cf. [BDRV06, Pf. of Thm. 3.9])
\[
(\omega \otimes \iota)(X^*(1_B \otimes \xi_1\bar{\eta}_1)(X^*)^* = \delta_{s,t}\frac{\langle \eta_1, \rho \xi_1 \rangle_{1,\kappa(s)}}{d_s}1_{\kappa(s)}, \quad (\eta_1 \in \mathcal{H}_t, \xi_1 \in \mathcal{H}_s); \\
(\omega \otimes \iota)((X^*)^*(1_B \otimes \xi_2\bar{\eta}_2)X^*) = \delta_{s,t}\frac{\langle \eta_2, \rho^{-1} \xi_2 \rangle_{1,s}}{d_s}1_s, \quad (\eta_2 \in \mathcal{H}_{\kappa(t)}, \xi_2 \in \mathcal{H}_{\kappa(s)}).
\]
This implies that
\[
(\omega \otimes \psi)(X^*(1_B \otimes m^2_{\xi_1,\eta_1})(X^i)^*) = \delta_{s,t} \psi_s(m^2_{\xi_1,\eta_1}1_{\kappa(s)}), \quad (\eta_1 \in \mathcal{H}_1, \xi_1 \in \mathcal{H}_s);
\]
\[
(\omega \otimes \varphi)((X^*)^*(1_B \otimes m^{s(s)}_{\xi_2,\eta_2})X^i) = \delta_{s,t} \varphi_s(m^{s(s)}_{\xi_2,\eta_2}1_s), \quad (\eta_2 \in \mathcal{H}_{\kappa(t)}, \xi_2 \in \mathcal{H}_{\kappa(s)}),
\]
here \(\psi\) is as in Remark 3.2.4. We obtain formulas similar to (3.2.1).

Use (3.2.1) and the definition of \(\delta\) to find
\[
(t_B \otimes t^\otimes \otimes \varphi \otimes t^\otimes \otimes \kappa)k_n(x)
\]
\[
= (t_B \otimes t^\otimes \otimes \varphi \otimes t^\otimes \otimes \kappa)((X^*_{1,n+1} \cdots X^*_{1,2})(1_B \otimes x)(X_1,2 \cdots X_{1,n+1})
\]
\[
= (t_B \otimes t^\otimes \otimes \otimes \varphi \otimes t^\otimes \otimes \kappa)(W_{m+2,m+3}X^*_{1,n+2} \cdots X^*_{1,m+3}X^*_{1,m+1} \cdots X^*_{1,2})
\]
\[
\times (1_B \otimes x_{1,1,\cdots,m} \otimes \delta_{\mathcal{G}_2} \otimes x_{m+1,\cdots,n})(X_1,2 \cdots X_{1,m+1}X_{1,m+3} \cdots X_{1,n+2})W_{m+2,m+3}
\]
\[
= (t_B \otimes t^\otimes \otimes \otimes \varphi \otimes t^\otimes \otimes \kappa)((X^*_{1,n+1} \cdots X^*_{1,2})(1_B \otimes x)(X_1,2 \cdots X_{1,n+1})
\]
\[
(5.1.3)
\]
Here \((\delta \otimes \otimes \otimes \otimes \varphi \otimes t^\otimes \otimes \otimes \kappa)\) acting in the legs 1, \(m+2\) and \(m+3\), meaning
\[
(\delta \otimes \otimes \otimes \otimes \otimes \varphi \otimes t^\otimes \otimes \otimes \otimes \kappa)_{1,m+2,m+3}(X_{1,i}) = \begin{cases} X_{1,i} & \text{if } i \neq m+2; \\ X_{1,m+3}W_{m+2,m+3} & \text{if } i = m+2. \end{cases}
\]
Since \(\omega = (\otimes \otimes \delta)\), identity (5.1.2) gives
\[
(5.1.3) = (t_B \otimes t^\otimes \otimes \otimes \varphi \otimes t^\otimes \otimes \otimes \kappa)((X^*_{1,n+1} \cdots X^*_{1,m+2}X^*_{1,m+1} \cdots X^*_{1,2})(1_B \otimes x)
\]
\[
\times (X_1,2 \cdots X_{1,m+1}X_{1,m+2} \cdots X_{1,n+1})
\]
\[
= (X^*_{1,n} \cdots X^*_{1,2})(1_B \otimes ((t^\otimes \otimes \otimes \otimes \varphi \otimes t^\otimes \otimes \otimes \otimes \kappa)(x))(X_1,2 \cdots X_{1,n})
\]
\[
= k_n((t^\otimes \otimes \otimes \otimes \varphi \otimes t^\otimes \otimes \otimes \otimes \kappa)(x)).
\]

(iv) Clearly
\[
k_{n+1}(1 \otimes x) = (X^*_{1,n+1} \cdots X^*_{1,2})(1_B \otimes 1_{\otimes \otimes \otimes \otimes \kappa} \otimes x)(X_1,2 \cdots X_{1,n+2})
\]
\[
= (X^*_{1,n+2} \cdots X^*_{1,2})(1_B \otimes 1_{\otimes \otimes \otimes \otimes \kappa} \otimes x)(X_1,2 \cdots X_{1,n+2})
\]
\[
= (X^*_{1,n+2} \cdots X^*_{1,2})(1_B \otimes 1_{\otimes \otimes \otimes \otimes \kappa} \otimes x)(X_1,2 \cdots X_{1,n+2})
\]
\[
= k_n(x) \otimes 1_{\otimes \otimes \otimes \otimes \kappa} = k_n(x)_{2,\cdots,n+1} = (t_B \otimes 1_{\otimes \otimes \otimes \otimes \kappa})(k_n(x)),
\]
as desired.

If \(\mu\) is a probability measure on \(\text{Irr}(G_1)\), one can directly verify that the function
Thus pushforward measure (see for example [Bog07, §3.6]). It is clear that \( \kappa(s) = \kappa(t) \) and thus \( \kappa_s(\bar{\mu}) = \kappa_s(\mu) \). Observe that \( \kappa_s(\mu)(\kappa(t)) = \mu(t) \), so \( \kappa_s(\mu) = \sum_s \mu(s)\delta_{\kappa(s)} \).

The following corollary is an easy consequence of the preceding computations.

**Corollary 5.1.6.** Maintaining the same notation as above, the following identities hold for \( 0 \leq m \leq n - 1 \):

1. \( k_{n-1} \circ (t^m \otimes \varphi_{\kappa_s(\mu)} \otimes t^{n-m-1}) = (t_B \otimes t^m \otimes \varphi_{\mu} \otimes t^{n-m-1}) \circ k_n \);
2. \( (\varphi_{\kappa_s(\mu)} \otimes \cdots \otimes \varphi_{\kappa_s(\mu)}) = (\omega \otimes \varphi_{\mu} \otimes \cdots \otimes \varphi_{\mu}) \circ k_n \);
3. \( k_n \circ (t^m \otimes P_{\kappa_s(\mu)} \otimes t^{n-m-1}) = (t_B \otimes t^m \otimes P_{\mu} \otimes t^{n-m-1}) \circ k_n \);
4. \( k_1 : H^\infty(\hat{G}_2, \kappa_s(\mu)) \to B \otimes H^\infty(\hat{G}_1, \mu) \) is a \(*)\)-isomorphism;
5. \( k_n \circ (t^m \otimes K_{\kappa_s(\mu)} \otimes t^{n-m-1}) = (t_B \otimes t^m \otimes K_{\mu} \otimes t^{n-m-1}) \circ k_n \);
6. \( k_1 : \hat{M}(\hat{G}_2, \kappa_s(\mu)) \to B \otimes H(\hat{G}_1, \mu) \) and \( M(\hat{G}_2, \kappa_s(\mu)) \to B \otimes \hat{M}(\hat{G}_1, \mu) \) are \(*\)-isomorphisms.

**Proof.** Write again \( k := n - m - 1 \).

1. As \( \kappa_s(\mu) = \sum_s \mu(s)\delta_{\kappa(s)} \), it holds that
   \[
   k_{n-1}(t^m \otimes \varphi_{\kappa_s(\mu)} \otimes t^k) = \sum_{s \in \text{Irr}(G_1)} k_{n-1}(t^m \otimes \mu(s)\varphi_{\kappa(s)} \otimes t^k) = \sum_{s \in \text{Irr}(G_1)} \mu(s)(t_B \otimes t^m \otimes \varphi_s \otimes t^k)k_n = (t_B \otimes t^m \otimes \varphi_{\mu} \otimes t^k)k_n.
   \]
2. This follows by repeatedly applying (i) and observing that \( \omega(k_0(z)) = \omega(z1_B) = z \) for any complex number \( z \).
3. By definition \( P_{\mu} = (\varphi_{\mu} \otimes \iota)\hat{\Delta} \). Apply part (i) of this corollary and (ii) of the above proposition.
4. From (iii) it follows that \( x \in l^\infty(\hat{G}_2) \) is \( \kappa_s(\mu) \)-harmonic if and only if \( k_1(x) \) is \( (\iota \otimes P_{\mu}) \)-invariant. Recall that \( x \cdot y = s^*\lim_j P^2_{\nu}(xy) = s^*\lim_j P_{\bar{\nu}s}(xy) \), so (iii) implies that \( k_1 \) is a \(*\)-homomorphism.
5. This is immediate from (iii), since \( k_1(I_0) = I_0 \).
6. The actions \( \alpha_t \) are reduced ([DRVV10, Rem. 6.3]), so the tensor products are well-defined. See [DRVV10, Thm. 10.1] for the proof that \( k_1 \) are isomorphisms.

**Theorem 5.1.7.** Suppose that \( G_1 \) and \( G_2 \) are two monoidally equivalent quantum groups with a unitary monoidal equivalence \( \kappa : \text{Rep}(G_1) \to \text{Rep}(G_2) \) and \( \mu \) is a probability measure on \( \text{Irr}(G_1) \). If the random walk on \( \hat{G}_1 \) defined by \( \mu \) converges to the boundary, then so does the random walk defined by \( \kappa_s(\mu) \) on \( \hat{G}_2 \).
Proof. Let $x \in l^\infty(\hat{G}_2)$. Assume $n > m$ and write $k := n - m$. Use Corollary 5.1.6 to compute

\[
(\cdots \otimes \varphi_{\kappa_\mu}(x) \otimes \varphi_{\kappa_\mu}(y))(j_n(x) - j_m(x))(j_n(x) - j_m(x))
= (\varphi_{\kappa_\mu}(x) \otimes \cdots \otimes \varphi_{\kappa_\mu}(y))(\Delta^{n-1}(x) - 1^{\otimes k} \otimes \Delta^{m-1}(x))\ast (\Delta^{n-1}(x) - 1^{\otimes k} \otimes \Delta^{m-1}(x))
= (\omega \otimes \varphi_\mu \otimes \cdots \otimes \varphi_\mu) \otimes k_n(\Delta^{n-1}(x) - 1^{\otimes k} \otimes \Delta^{m-1}(x))\ast (\Delta^{n-1}(x) - 1^{\otimes k} \otimes \Delta^{m-1}(x))
\times (\Delta^{n-1}(x) - 1^{\otimes k} \otimes \Delta^{m-1}(x))
= (\omega \otimes \varphi_\mu \otimes \cdots \otimes \varphi_\mu)((t_B \otimes \Delta^{n-1})k_1(x) - (t_B \otimes (1^{\otimes k} \otimes \Delta^{m-1}))k_1(x))\ast (t_B \otimes j_n)k_1(x) - (t_B \otimes j_m)k_1(x))
\times (t_B \otimes j_n)k_1(x) - (t_B \otimes j_m)k_1(x)).
\] (5.1.4)

The same equalities also hold if we replace $x$ by $x^\ast$. Since $\omega$ is faithful, this computation and Lemma 1.3.1 show that $s^\ast\lim_m j_n(x)$ exists in $\bigotimes_{-\infty}^{-1}(l^\infty(\hat{G}_2), \varphi_{\kappa_\mu})$ if and only if $s^\ast\lim_m (t_B \otimes j_n)k_1(x)$ exists in $B \boxtimes_{a_{\text{alg}}}^\ast (\bigotimes_{-\infty}^{-1}(l^\infty(\hat{G}_1), \varphi_\mu))$. Combining this with Corollary 5.1.6 part (vi), we see that to prove that $\hat{M}(\hat{G}_2, \kappa_\mu) \subset R_{\kappa_\mu}$ it suffices to argue that $s^\ast\lim_m (t_B \otimes j_n)y$ exists for all $y \in B \boxtimes_{a_{\text{alg}}}^\ast \hat{M}(\hat{G}_1, \mu)$.

Recall that by definition $B \boxtimes_{a_{\text{alg}}}^\ast \hat{M}(\hat{G}_1, \mu)$ is the norm closure of $B \boxtimes_{a_{\text{alg}}}^\ast \hat{M}(\hat{G}_1, \mu)$ in $B \boxtimes l^\infty(\hat{G}_1)$ (see Notation 5.1.4). Suppose $y \in B \boxtimes_{a_{\text{alg}}}^\ast \hat{M}(\hat{G}_1, \mu)$, say $y = \sum_{i=1}^n b_i \otimes y_i$. Since $G_1$ satisfies convergence to the boundary it follows that $s^\ast\lim_m j_m(y_i)$ exists for $i = 1, \ldots, n$. Because it is a finite sum we get

\[
\sum_{i=1}^n b_i \otimes (s^\ast\lim_m j_m(y_i)) = s^\ast\lim_m \left( \sum_{i=1}^n (t_B \otimes j_m)(b_i \otimes y_i) \right) = s^\ast\lim_m (t_B \otimes j_m)(y)
\]

and thus $s^\ast\lim_m (t_B \otimes j_m)(y)$ exists. Now let $y \in B \boxtimes_{a_{\text{alg}}}^\ast \hat{M}(\hat{G}_1, \mu)$. Then there exists a bounded net $(y_i)_i \subset B \boxtimes_{a_{\text{alg}}}^\ast \hat{M}(\hat{G}_1, \mu)$ with $\|y_i - y\| \to 0$. Since the maps $t_B \otimes j_n$ are $*$-homomorphisms, they are norm-decreasing. Hence a calculation similar to (4.3.1) shows that $s^\ast\lim_n (t_B \otimes j_n)(y)$ exists, which completes the argument that $\hat{M}(\hat{G}_2, \kappa_\mu) \subset R_{\kappa_\mu}$.

Now the last part, let $x \in c_{00}(\hat{G}_2)$ and $h \in H^\infty(\hat{G}_2, \kappa_\mu)$. Then

\[
\nu_{G_2}(K_{\kappa_\mu}(x))h = \lim_n(\varphi_{\kappa_\mu}(x) \otimes \cdots \otimes \varphi_{\kappa_\mu})(\hat{\Delta}^n(K_{\kappa_\mu}(x))h)
= \lim_n(\omega \otimes \varphi_\mu \otimes \cdots \otimes \varphi_\mu)(k_n(\hat{\Delta}^n(K_{\kappa_\mu}(x))h))
= \lim_n(\omega \otimes \varphi_\mu \otimes \cdots \otimes \varphi_\mu)((t_B \otimes \hat{\Delta}^n)(k_1(K_{\kappa_\mu}(x))h))
= (\omega \otimes \nu_{G_1})(k_1(K_{\kappa_\mu}(x))h)).
\] (5.1.5)

Note that (5.1.5) exists. Indeed, we established before that $\hat{M}(\hat{G}_2, \kappa_\mu) \subset R_{\kappa_\mu}$ and we already know (see Proposition 4.3.2) that $H^\infty(\hat{G}_2, \kappa_\mu) \subset R_{\kappa_\mu}$ and that $R_{\kappa_\mu}$ is an algebra. Therefore it follows that $K_{\kappa_\mu}(x)h \in R_{\kappa_\mu}$. From the conclusion following (5.1.4) we obtain that $(\omega \otimes \nu_{G_1})(k_1(K_{\kappa_\mu}(x)))$ exists. Clearly $k_1(x) = \sum_{i=1}^n a_i \otimes x_i \in
5.1. BOUNDARY CONVERGENCE AND MONOIDAL EQUIVALENCE

Recall the spectral subspaces $\mathcal{H}^\infty(\hat{G}_2, \kappa_*(\mu))$ and $\mathcal{H}^\infty(\hat{G}_2, \kappa_*(\mu)) := \text{span}\{\mathcal{H}^\infty(\hat{G}_2, \kappa_*(\mu)) : s \in \text{Irr}(G_2)\}$ (cf. Notation 5.1.4). Assume for the moment that $h \in \mathcal{H}^\infty(\hat{G}_2, \kappa_*(\mu))$. By Corollary 5.1.6 the element $k_1(h)$ can be decomposed as a finite sum $k_1(h) = \sum_{i=1}^m b_i \otimes h_i \in \mathcal{B} \boxtimes_{\text{alg}} \mathcal{H}^\infty(\hat{G}_1, \mu)$. Use the convergence to the boundary of $G_1$ to obtain

\begin{equation}
(5.1.5) = (\omega \otimes \nu_{G_1})((\iota \otimes K_{\mu})(k_1(x))k_1(h)) = \sum_{i=1}^n \sum_{j=1}^m (\omega \otimes \nu_{G_1})(a_{ij} b_j \otimes K_{\mu}(x_i)h_j) = \sum_{i=1}^n \sum_{j=1}^m (\omega \otimes \hat{\psi}_{G_1})(a_{ij} b_j \otimes x_i h_j) = (\omega \otimes \hat{\psi}_{G_1})(k_1(x)k_1(h)).
\end{equation}

Since $x \in c_00(\hat{G}_2)$, the definition of $\hat{\psi}$ implies that this can be written as

$$
\nu_{G_2}(K_{\kappa_*(\mu)}(x)h) = \sum_{s \in \text{Irr}(G_1)} d^2_s(\omega \otimes \varphi_s)(k_1(x)h) = \sum_{s \in \text{Irr}(G_1)} d^2_{\kappa_*(s)} \varphi_{\kappa(s)}(x h) = \hat{\psi}_{G_2}(x h).
$$

On the von Neumann algebraic level, the span of spectral subspaces $\mathcal{H}^\infty(\hat{G}_2, \kappa_*(\mu))$ is weakly dense in $H^\infty(\hat{G}_2, \kappa_*(\mu))$ ([Vae01, Thm. 2.6]). So if $h \in H^\infty(\hat{G}_2, \kappa_*(\mu))$ is an arbitrary harmonic element, approximate $h = w\text{-lim}_i h_i$ by elements $h_i \in \mathcal{H}^\infty(\hat{G}_2, \kappa_*(\mu))$. Then

$$
\nu_{G_2}(K_{\kappa_*(\mu)}(x)h) = \nu_{G_2}(K_{\kappa_*(\mu)}(x \text{ w-lim}_i h_i)) = \lim_i \nu_{G_2}(K_{\kappa_*(\mu)}(x)h_i) = \lim_i \hat{\psi}_{G_2}(x h_i) = \hat{\psi}_{G_2}(x \text{ w-lim}_i h_i) = \hat{\psi}_{G_2}(x h)
$$

and thus the random walk on $\hat{G}_2$ defined by $\kappa_*(\mu)$ converges to the boundary. \hfill \Box

**Example 5.1.8.** In [VDW96] the free orthogonal quantum groups were introduced. Let $F \in \text{GL}_n(\mathbb{C})$ be a matrix with $FF = \pm 1$, where $\bar{F}$ is obtained from $F$ by taking the complex conjugate in every entry. Define $A_0(F)$ to be the universal $C^*$-algebra generated by elements $u_{ij}$, $1 \leq i, j \leq n$ satisfying:

$$
U = (u_{ij})_{i,j} \text{ is unitary, } U = FU^c F^{-1},
$$

where $(U^c)_{ij} = u_{ji}^*$. The comultiplication is given by $\Delta(u_{ij}) := \sum_{k=1}^n u_{ik} \otimes u_{kj}$. Note that for

$$
F_q = \begin{pmatrix} 0 & -\sqrt{q} \\ \sqrt{q} & 0 \end{pmatrix}
$$

and $F_q^n = F_{q^n}$.
it holds \( A_0(F_q) = SU_q(2) \). Also \((A_0(F_1), \Delta_1)\) is monoidally equivalent to \((A_0(F_2), \Delta_2)\) if and only if \( \text{sgn}(F_1 \bar{F}_1) = \text{sgn}(F_2 \bar{F}_2) \) and \( \text{Tr}(F_1^* F_1) = \text{Tr}(F_2^* F_2) \) (see [BDRV06, Thm. 5.3, Cor. 5.4]). It follows that every free orthogonal quantum group and only if \( \text{sgn}(F) \), then by Theorem 4.4.1 the random walk defined by \( \kappa \) and \( \hat{\kappa} \), boundary. Hence Theorem 5.1.7 shows that the random walk on \( A_0(F) \) be a free orthogonal quantum group monoidally equivalent to \( SU_q(2) \) for some \( q \in [0,1] \setminus \{0\} \).

Let \( A_0(F) \) be a free orthogonal quantum group monoidally equivalent to \( SU_q(2) \) for some \( q \in (0,1) \) (thus \( F \bar{F} = -1 \) and \( \text{Tr}(F^* F) > 2 \)), so \( q \) satisfies \( q + q^{-1} = \text{Tr}(F^* F) \). Suppose we have an explicit identification \( \kappa: \text{Irr}(A_0(F)) \rightarrow \frac{1}{2} \mathbb{Z}_+ \cong \text{Irr}(SU_q(2)) \) (see also [Ban96, Thm. 1]) and assume that a generating probability measure \( \mu \) on \( \text{Irr}(A_0(F)) \) satisfies the condition

\[
\sum_{s \in \text{Irr}(A_0(F))} \mu(s)(1 + q^2)^{2\kappa(s)} < \infty,
\]

then by Theorem 4.4.1 the random walk defined by \( \kappa_{\mu}(\mu) \) on \( SU_q(2) \) converges to the boundary. Hence Theorem 5.1.7 shows that the random walk on \( A_0(F) \) defined by \( \mu \) converges to the boundary.

### 5.2 Random walks on \( C^* \)-tensor categories

Neshveyev and Yamashita found a way to define random walks on \( C^* \)-tensor categories [NY14c]. For such random walks they defined a Poisson boundary and they used it to prove their characterisation of quantum groups with the same representation theory as \( SU(N) \). Motivated by their work we will construct a Martin boundary and formulate convergence to the boundary for such random walks. This section will form the starting point of this theory. We introduce the analogue of algebra of functions on the path space and the necessary functors on these spaces. It is motivated by [NY14c, §3.1].

**Notation 5.2.1.** Let \( C \) be a strict \( C^* \)-tensor category with simple unit \( \mathbf{1} \). For \( n \geq 1 \) and an object \( U \in \text{Ob}(C) \) define the \( n \)-ary functor

\[
(i^{\otimes n} \otimes U): C \times \cdots \times C \rightarrow C,
\]

\[
(i^{\otimes n} \otimes U)(X_1, \ldots, X_n) := (X_1 \otimes \cdots \otimes X_n \otimes U),
\]

\[
(i^{\otimes n} \otimes U)(f_1, \ldots, f_n) := f_1 \otimes \cdots \otimes f_n \otimes U.
\]

Clearly \( i^{\otimes n} \otimes U \) is unitary. For two objects \( U, V \in \text{Ob}(C) \) consider the space of natural transformations \( \text{Nat}(i^{\otimes n} \otimes U, i^{\otimes n} \otimes V) \). Since every object can be decomposed in simple ones and \( i^{\otimes n} \otimes U \) respects direct sums, a natural transformation \( \eta: i^{\otimes n} \otimes U \rightarrow i^{\otimes n} \otimes V \) is completely determined by its action on simple objects \( U_s \). Thus

\[
\text{Nat}(i^{\otimes n} \otimes U, i^{\otimes n} \otimes V) \cong \prod_{s_1, \ldots, s_n \in \text{Irr}(C)} \text{Hom}_C(U_{s_1} \otimes \cdots \otimes U_{s_n} \otimes U, U_{s_1} \otimes \cdots \otimes U_{s_n} \otimes V). \tag{5.2.1}
\]

A natural transformation \( \eta \in \text{Nat}(i^{\otimes n} \otimes U, i^{\otimes n} \otimes V) \) is \((\text{uniformly}) \text{ bounded} \) if

\[
\|\eta\|_\infty := \sup\{\|\eta_{s_1,\ldots,s_n}\| : s_1, \ldots, s_n \in \text{Irr}(C)\} < \infty.
\]

Denote by \( \text{Nat}_b(i^{\otimes n} \otimes U, i^{\otimes n} \otimes V) \) the uniformly bounded natural transformations.
Define $\mathcal{C}_{-n}$ as the C*-category which is the subobject completion of the category with objects $\text{Ob}(\mathcal{C})$ and morphisms

$$\text{Hom}_{\mathcal{C}_{-n}}(U, V) := \text{Nat}_b(\iota^\otimes n \otimes U, \iota^\otimes n \otimes V), \quad (U, V \in \text{Ob} (\mathcal{C})).$$

So a morphism $\eta = (\eta_{X_1, \ldots, X_n})_{X_1, \ldots, X_n} \in \text{Hom}_{\mathcal{C}_{-n}}(U, V)$ consists of a bounded collection of morphisms

$$\eta_{X_1, \ldots, X_n} : X_1 \otimes \cdots \otimes X_n \otimes U \to X_1 \otimes \cdots \otimes X_n \otimes V$$
natural in $X_1, \ldots, X_n \in \text{Ob} (\mathcal{C})$. In $\mathcal{C}_{-n}$ the multiplication of morphisms is given by composition and the involution is coming from the $*$ on $\mathcal{C}$. The C*-norm on $\text{Hom}_{\mathcal{C}_{-n}}(U, V)$ is given by $\|\eta\|_{\infty} := \sup \{\|\eta_{s_1, \ldots, s_n}\| : s_1, \ldots, s_n \in \text{Irr}(\mathcal{C})\}$ for $\eta \in \text{Nat}_b(\iota^\otimes n \otimes U, \iota^\otimes n \otimes V)$. 

Observe that there exists a canonical functor $\mathcal{E} : \mathcal{C} \to \mathcal{C}_{-n}$ given on objects by $U \mapsto U$ and on morphisms

$$\mathcal{E}(T) := \iota^\otimes n \otimes T, \quad \text{where } (\iota^\otimes n \otimes T)_{X_1, \ldots, X_n} := \iota_{X_1} \otimes \cdots \otimes \iota_{X_n} \otimes T. \quad (5.2.2)$$

Obviously $\mathcal{E}$ is a unitary functor.

The category $\mathcal{C}_{-n}$ should be thought of as the space of functions on paths of length $n$ (see also Corollary 5.5.9)). In particular for $n = 1$ we obtain the functions on the space.

**Notation 5.2.2.** For $n = 1$ there is more structure present. Define the tensor product of objects in $\mathcal{C}_{-1}$ to be inherited from $\mathcal{C}$. For natural transformations define

$$(\nu \otimes \eta) := (\nu \otimes \iota_X)(\iota_U \otimes \eta), \quad (\nu \in \text{Hom}_{\mathcal{C}_{-1}}(U, V), \eta \in \text{Hom}_{\mathcal{C}_{-1}}(W, X)), \quad (5.2.3)$$

where

$$(\nu \otimes \iota_X)_Y := \nu_Y \otimes \iota_X : Y \otimes U \otimes X \to Y \otimes V \otimes X; \quad (5.2.4)$$

$$((\iota_U \otimes \eta)_Y := \eta_Y \otimes U : Y \otimes U \otimes W \to Y \otimes U \otimes X. \quad (5.2.5)$$

Extend this to the subobject completion. With this tensor structure $\mathcal{C}_{-1}$ becomes a C*-tensor category. For $\eta \in \text{Nat}(\iota \otimes U, \iota \otimes V)$ write $\text{supp}(\eta) := \{s \in \text{Irr}(\mathcal{C}) : \eta_s \neq 0\}$ for the support of $\eta$. A natural transformation $\eta \in \text{Nat}(\iota \otimes V, \iota \otimes W)$ is called a

1. **natural transformation vanishing at infinity** if $\{\|\eta_s\|\}_{s \in \text{Irr}(\mathcal{C})} \in c_0(\text{Irr}(\mathcal{C}));$

2. **compactly or finitely supported natural transformation** if $|\text{supp}(\eta)| < \infty$.

These classes will be denoted by $\text{Nat}_0(\iota \otimes V, \iota \otimes W)$ and respectively $\text{Nat}_{00}(\iota \otimes V, \iota \otimes W)$. By the same argument as for (5.2.1) we get

$$\text{Nat}_0(\iota \otimes V, \iota \otimes W) \cong \bigoplus_{s \in \text{Irr}(\mathcal{C})} \text{Hom}_{\mathcal{C}}(U_s \otimes V, U_s \otimes W);$$

$$\text{Nat}_{00}(\iota \otimes V, \iota \otimes W) \cong \bigoplus_{s \in \text{Irr}(\mathcal{C})} \text{Hom}_{\mathcal{C}}(U_s \otimes V, U_s \otimes W).$$
Observe that $M(\text{Nat}_0(\iota \otimes V, \iota \otimes V)) = \text{Nat}_b(\iota \otimes V, \iota \otimes V)$, where $M$ indicates the multiplier algebra.

**Notation 5.2.3.** Define for $0 \leq m < n$ functors $(\iota \otimes \cdot): \mathcal{C}_{-n} \to \mathcal{C}_{-(n+1)}$ and $\iota^m \otimes \hat{\Delta} \otimes \iota^{\otimes n-m-1}: \mathcal{C}_{-n} \to \mathcal{C}_{-(n+1)}$. On objects $U \in \text{Ob}(\mathcal{C}) \subset \text{Ob}(\mathcal{C}_{-n})$ they are given by the identity while on morphisms $\eta \in \text{Hom}_{\mathcal{C}_{-n}}(U, V)$ they are defined by

\[
(t \otimes \eta)x_{1},...,x_{n+1} := t_{X} \otimes \eta_{x_{2},...,x_{n+1}}; \tag{5.2.6}
\]

\[
((\iota^m \otimes \hat{\Delta} \otimes \iota^{\otimes n-m-1})(\eta))x_{1},...,x_{n+1} := \eta_{x_{1},...,x_{m},x_{m+1}\otimes x_{m+2},x_{m+3},...,x_{n+1}}. \tag{5.2.7}
\]

Extend these functors to the subobject completions (see Remark 1.5.11). Given objects $U, V, X \in \text{Ob}(\mathcal{C})$ pick a standard solution $(R_X, \overline{R}_X)$ of the conjugate equations for $X$. Define the partial trace

\[
\text{tr}_X \otimes \iota_V: \text{Hom}_\mathcal{C}(X \otimes U, X \otimes V) \to \text{Hom}_\mathcal{C}(U, V);
\]

\[
(\text{tr}_X \otimes \iota_V)(T) := d_X^{-1}(R_X^s \otimes \iota_V)(t_X \otimes T)(R_X \otimes \iota_U).
\]

Notice that $\text{Hom}_\mathcal{C}(X \otimes U, X \otimes V)$ is not a tensor product, so this notation might seem a bit misleading at first. Use these partial traces to define for each $X \in \text{Ob}(\mathcal{C})$ and $n \geq 2$ the collection of completely positive linear maps

\[
\text{tr}_X \otimes \iota^{\otimes n-1}: \text{Hom}_{\mathcal{C}_{-n}}(U, V) \to \text{Hom}_{\mathcal{C}_{-(n-1)}}(U, V);
\]

\[
((\text{tr}_X \otimes \iota^{\otimes n-1})(\eta))X_{1},...,X_{n-1} := (\text{tr}_X \otimes \iota^{\otimes n-1} \otimes \iota_V)(\eta_{X_{1},...,X_{n-1}}) \tag{5.2.8}
\]

\[
= d_X^{-1}(R_X^s \otimes \iota^{\otimes n-1} \otimes \iota_V)(t_X \otimes \eta_{X_{1},...,X_{n-1}})(R_X \otimes \iota^{\otimes n-1} \otimes \iota_U).
\]

Note that $(\text{tr}_X \otimes \iota^{\otimes n-1})$ does not define a functor, because it does not preserve composition of morphisms. Similarly if $n = 1$ and $\eta \in \mathcal{C}_{-1}(U, V)$ denote

\[
\text{tr}_X(\eta) := (\text{tr}_X \otimes \iota_V)(\eta_X) \in \text{Hom}_\mathcal{C}(U, V) \tag{5.2.9}
\]

The same argument as in Remark 1.5.11 works to extend $(\text{tr}_X \otimes \iota^{\otimes n-1})$ to the subobject completion. The lemma below shows that (5.2.6)–(5.2.9) are well-defined.

Since $\mathcal{C}$ is assumed to be strict, the functor $\hat{\Delta}$ is coassociative. Define inductively $\hat{\Delta}^n := (\hat{\Delta} \otimes t^{\otimes n-2})\hat{\Delta}^{n-1}$. By linearity define for a probability measure $\mu$ on $\text{Irr}(\mathcal{C})$

\[
\text{tr}_\mu \otimes t^{\otimes n-1} := \sum_{s \in \text{Irr}(\mathcal{C})} \mu(s)(\text{tr}_{U_s} \otimes t^{\otimes n-1}).
\]

Note that simplicity of the unit $\mathbb{1}$ ensures that the traces $\text{tr}_\mu$ take values in $\mathbb{C} \cong \text{End}_\mathcal{C}(\mathbb{1})$.

**Lemma 5.2.4.** If $\eta \in \text{Hom}_{\mathcal{C}_{-n}}(U, V)$, then $(\iota \otimes \eta)$ and $\iota^m \otimes \hat{\Delta} \otimes \iota^{\otimes n-m-1}(\eta)$ are in $\text{Hom}_{\mathcal{C}_{-(n+1)}}(U, V)$ and $(\text{tr}_\mu \otimes t^{\otimes n-1})(\eta) \in \text{Hom}_{\mathcal{C}_{-(n-1)}}(U, V)$. Moreover if $\mu$ is a probability measure on $\text{Irr}(\mathcal{C})$, the series

\[
\sum_{s \in \text{Irr}(\mathcal{C})} \mu(s)((\text{tr}_{U_s} \otimes t^{\otimes n-1})(\eta))x_{1},...,x_{n-1}
\]
converges in norm in $\text{Hom}_C(X_1 \otimes \cdots \otimes X_{n-1} \otimes U, X_1 \otimes \cdots \otimes X_{n-1} \otimes V)$.

**Proof.** The proof is straightforward. Clearly $(t \otimes \cdot)$ and $t^\otimes m \otimes \hat{\Delta} \otimes t^\otimes n-m-1$ preserve compositions. Let $f_i : X_i \to Y_i$, for some objects $X_i, Y_i \in \text{Ob}(C)$. Then

$$(t \otimes \eta)_{Y_1, \ldots, Y_{n+1}}(f_1 \otimes \cdots \otimes f_{n+1} \otimes t_U) = (f_1 \otimes \cdots \otimes f_{n+1} \otimes t_V) \eta_{X_1, \ldots, X_{n+1}}$$

Thus all three define natural isomorphisms. Clearly they are all bounded by $(\|\eta\|_\infty)$. So the series converges and the proof is complete.

Write $k := n - m - 1$. Clearly $f_{m+1} \otimes f_{m+2} : X_{m+1} \otimes X_{m+2} \to Y_{m+1} \otimes Y_{m+2}$, so that

$$((t^\otimes m \otimes \hat{\Delta} \otimes t^\otimes k)(\eta))_{Y_1, \ldots, Y_{n+1}}(f_1 \otimes \cdots \otimes f_{n-1} \otimes t_U)$$

To check the last one, by linearity it suffices to deal with $\text{tr}_X$ instead of $\text{tr}_\mu$. We have

$$(\text{tr}_X \otimes t^\otimes n-1(\eta))_{Y_1, \ldots, Y_{n-1}}(f_1 \otimes \cdots \otimes f_{n-1} \otimes t_U)$$

Thus all three define natural isomorphisms. Clearly they are all bounded by $\|\eta\|_\infty$. For $\text{tr}_\mu$ it is also easy to verify that

$$\left\| \sum_s \mu(s)(\text{tr}_{U_s} \otimes t^\otimes n-1)(\eta_{X_1, \ldots, X_{n-1}}) \right\| \leq \|\eta\|_\infty,$$

so the series converges and the proof is complete.

**Definition 5.2.5** ([NY14c]). Suppose that $\mu$ is a probability measure on $\text{Irr}(C)$. Define the collection of positive linear maps

$$P_\mu : \text{Hom}_{C_{-1}}(U, V) \to \text{Hom}_{C_{-1}}(U, V), \quad P_\mu := (\text{tr}_\mu \otimes t)\hat{\Delta}.$$ 

$P_\mu$ is called the **Markov operator** of $\mu$. $P_\mu$ is well-defined due to Lemma 5.2.4 above. Explicitly we have

$$P_\mu(\eta)_X = \sum_{s \in \text{Irr}(C)} \mu(s)(\text{tr}_{U_s} \otimes t_X \otimes t_V)(\eta_{U_s \otimes X}). \quad (\eta \in \text{Nat}_b(t \otimes U, t \otimes V) \subset \text{Hom}_{C_{-1}}(U, V)).$$

As in the quantum case we need a classical discrete Markov chain on the set of irreducible objects. To define this random walk we need a couple of central natural transform-
For the operators

\[ \kappa^{s,t} \in \text{Hom}_{\mathcal{C}_{-1}}(V,V) \cong l^\infty_{\iota} \bigoplus_{s \in \text{Irr}(\mathcal{C})} \text{End}_\mathcal{C}(U_s \otimes V), \]

\[ \kappa^{s,t}_s := \begin{cases} t_{U_t \otimes V} & \text{if } s = t; \\ 0 & \text{if } s \neq t. \end{cases} \]

From (5.2.1) we see \( \text{Nat}_0(\iota \otimes \mathbb{1}, \iota \otimes \mathbb{1}) \cong l^\infty_{\iota} \bigoplus_s \text{Hom}_\mathcal{C}(U_s, U_s) = l^\infty_{\iota} \bigoplus_s \mathbb{C} \iota_s. \) Therefore the matrix \( \{p_\mu(s,t)\}_{s,t \in \text{Irr}(\mathcal{C})} \) determined by the identity

\[ p_\mu(\kappa^{s,t}) = p_\mu(s,t) \kappa^{s,t} \tag{5.2.10} \]

is well-defined. It describes a Markov kernel on \( \text{Irr}(\mathcal{C}). \) Indeed, if \( \mu \) is a probability measure, then \( \|P_\mu\| \leq 1. \) Clearly \( P_\mu \) is positive, thus \( p_\mu(s,t) \in [0,1] \) for all \( s,t. \) Furthermore if we denote by \( I \) the natural transformation which is the identity on all objects, then

\[ \sum_t P_r(\kappa^{s,t})_s = P_r(I)_s = \iota_s \text{ for all } r,s. \]

Therefore \( \sum_t P_\mu(s,t) \iota_s = \sum_t P_\mu(\kappa^{s,t})_s = \iota_s \) and hence \( \sum_t P_\mu(s,t) = 1. \)

Define the map \( \vee: P_s \mapsto \bar{P}_s := P_s \) and extend it to operators \( P_U \) and \( P_{\mu} \) by antilinearity.

Analogous to Lemma 3.2.7 we have the following.

**Lemma 5.2.7.** For the operators \( P_U, P_\mu \) defined above the following properties hold:

\( P_U \circ P_V = P_{V \otimes U}; \)

\( \bar{P}_\mu = P_{\bar{\mu}} \) and \( \bar{P}_U = P_{\bar{U}}; \)

\( \tilde{P}_U \circ P_V = P_{U \otimes V}; \)

\( p_\mu(s,t) = \sum_r \mu(r) m_{r,s}^{t} \frac{d_s}{d_s}; \)

\( p_\mu^n(s,t) = \left( \frac{d_s}{d_s} \right)^2 p_\mu^n(t,s); \)

\( P_\mu^n(\kappa^{s,t}) = p_\mu^n(s,t) \kappa^{s,t}. \)

**Proof.** (i) Let \( \eta \in Z(\text{Hom}_{\mathcal{C}_{-1}}(Z, Z)), \) then

\[ (P_U(P_V(\eta)))_X = (\text{tr}_{U \otimes X \otimes Z})(P_V(\eta)_{U \otimes X}) = (\text{tr}_{U \otimes X \otimes Z})(\text{tr}_{V \otimes U \otimes X \otimes Z})(\eta_{V \otimes U \otimes X}) = (\text{tr}_{U \otimes V \otimes X \otimes Z})(\eta_{V \otimes U \otimes X}) = P_{V \otimes U}(\eta)_X. \]

(ii) By linearity \( \bar{P}_\mu = (\sum_s \mu(s)P_s)^\vee = \sum_s \mu(s)P_s = \sum_t \mu(t)P_t = P_{\bar{\mu}}. \) The other one is similar, decompose \( U \) into a sum of simple objects.

(iii) This follows from (i), (ii) and the fact that \( U \otimes V \cong \bar{V} \otimes \bar{U}. \)
(iv) Suppose for the moment that \( \mu \) is a Dirac measure concentrated on \( r \in \text{Irr}(C) \), so \( P_\mu = P_r \). Then

\[
p_\mu(s, t) = \text{tr}_{U_s}(p_\mu(s, t) \kappa_s^{s,1}) = \text{tr}_{U_s}((P_\mu(\kappa^{t,1}) \kappa_s^{s,1})_s) = \text{tr}_{U_s}(P_r(\kappa^{t,1})_s) = \text{tr}_{U_s}((\text{tr}_{U_r} \otimes \iota)(\kappa^{t,1}_{U_r \otimes U_s})) = (\text{tr}_{U_r} \otimes \text{tr}_{U_s})(\kappa^{t,1}_{U_r \otimes U_s}) = m_{r,s}^t \frac{d_t}{d_r d_s}.
\]

The general case follows now by linearity in \( \mu \).

(v) This is proved by induction. For \( n = 1 \), note that Frobenius reciprocity implies that

\[
m_{r,s}^t = m_{s,t}^r.
\]

Furthermore the intrinsic dimension satisfies \( d_r = d_r \). Therefore using (iv) we obtain

\[
p_\mu(s, t) = \sum_r \bar{\mu}(r)m_{r,s}^t \frac{d_t}{d_r d_s} = \sum_r \mu(r)m_{r,s}^t \frac{d_t}{d_r d_s} = \sum_r \mu(r)m_{r,t}^s \frac{d_s}{d_r d_t} \left( \frac{d_t}{d_s} \right)^2
\]

\[
= \left( \frac{d_t}{d_s} \right)^2 p_\mu(t, s).
\]

For the induction step, assume that the result is true for \( 1, 2, \ldots, n \). Then

\[
p^{n+1}_\mu(s, t) = \sum_r p^n_\mu(s, r)p_\mu(r, t) = \sum_r \left( \frac{d_r}{d_s} \right)^2 p^n_\mu(r, s) \left( \frac{d_t}{d_r} \right)^2 p_\mu(t, r)
\]

\[
= \left( \frac{d_t}{d_s} \right)^2 \sum_r p^n_\mu(r, s)p_\mu(t, r) = \left( \frac{d_t}{d_s} \right)^2 p^{n+1}_\mu(t, s).
\]

(vi) Apply induction to \( n \). The case \( n = 1 \) is simply the definition. For the induction step, assume that the result is true for \( 1, 2, \ldots, n \). It suffices to check that \( \text{tr}_{s}(P^{n+1}_\mu(\kappa^{t,1}))_s = p^{n+1}_\mu(s, t) \). Decompose \( U_r \otimes U_s \) into irreducibles, apply the induction hypothesis and use assertion (iv) to obtain

\[
\text{tr}_{U_s}(P^{n+1}_\mu(\kappa^{t,1}))_s = \text{tr}_{U_s} \left( \sum_r \mu(r)(\text{tr}_{U_r} \otimes \iota_{U_s})(P^n_\mu(\kappa^{t,1})_{U_r \otimes U_s}) \right)
\]

\[
= \sum_r \mu(r) \text{tr}_{U_r \otimes U_s} \left( P^n_\mu(\kappa^{t,1})_{U_r \otimes U_s} \right)
\]

\[
= \sum_r \mu(r) \sum_q m^q_{r,s} \frac{d_q}{d_r d_s} \text{tr}_{U_q} (P^n_\mu(\kappa^{t,1})_q)
\]

\[
= \sum_q \left( \sum_r \mu(r)m^q_{r,s} \frac{d_q}{d_r d_s} \right) p^n_\mu(q, t)
\]

\[
= \sum_q p_\mu(s, q)p^n_\mu(q, t)
\]

\[
= p^{n+1}_\mu(s, t),
\]

which concludes the proof. \( \qed \)
5.3 The categorical Martin boundary

Even though this section is called “The categorical Martin boundary”, we will start by presenting the Poisson boundary, as that one is easier to define and we will need it later in the thesis. After stating the main properties we move our attention to the Martin boundary for random walks on $C^*$-tensor categories. We need to prove some additional results before we can state the definition.

**Definition 5.3.1.** Let $\mu$ be a probability measure on $\Irr(\mathcal{C})$. A natural transformation $\eta \in \Nat_b(\iota \otimes U, \iota \otimes V)$ is called $\mu$-harmonic if $P_\mu(\eta) = \eta$.

**Lemma 5.3.2** ([NY14c, Prop. 2.1]). Let $\eta \in \Nat_b(\iota \otimes U, \iota \otimes V)$ and $\nu \in \Nat_b(\iota \otimes V, \iota \otimes W)$. The limit

$$(\nu \cdot \eta)_X = \lim_{n \to \infty} P_\mu^n(\nu \eta)_X$$

exists for all objects $X \in \Ob(\mathcal{C})$ and defines a bounded $P_\mu$-harmonic natural transformation $\iota \otimes U \to \iota \otimes W$. Moreover the product $\cdot$ is associative.

If $T \in \Hom_\mathcal{C}(X, Y)$, then $\mathcal{E}(T) = (\iota_U \otimes T)_U \in \Hom_{\mathcal{C}_{-1}}(X, Y)$ (see (5.2.2)) is a natural transformation. The tensor product of natural transformations (5.2.3) gives $\kappa^{t,V} \otimes T = \kappa^{t,V} \otimes \mathcal{E}(T)$ which is a natural transformation given by

$$(\kappa^{t,V} \otimes T)_U := \kappa^{t,V}_U \otimes T \in \Hom(U \otimes V \otimes X, U \otimes V \otimes Y).$$

Note that

$$P_s(\kappa^{t,\iota}_U \otimes T)_U = (\text{tr}_s \otimes \iota_U \otimes t_W)((\kappa^{t,\iota}_U \otimes T)_{U_s \otimes U}) = (\text{tr}_s \otimes \iota_U)((\kappa^{t,\iota}_U)_{U_s \otimes U} \otimes T)$$

and thus

$$P_\mu(\kappa^{t,\iota}_U \otimes T) = P_\mu(\kappa^{t,\iota}_U) \otimes T. \quad (5.3.1)$$

In particular as $\kappa^{t,V} = \kappa^{t,\iota} \otimes \iota_V$, it follows that

$$P_\mu(\kappa^{t,V}_U) = P_\mu(\kappa^{t,\iota}_U) \otimes \iota_V. \quad (5.3.2)$$

**Definition 5.3.3** ([NY14c]). The categorical Poisson boundary of $\mathcal{C}$ with respect to a probability measure $\mu$ on $\Irr(\mathcal{C})$ consists of a pair $(\mathcal{P}(\mathcal{C}, \mu), \mathcal{E})$, where $\mathcal{P}(\mathcal{C}, \mu)$ is the $C^*$-tensor category which is the subobject completion of $\mathcal{C}$ with morphism sets given by

$$\Hom_{\mathcal{P}(\mathcal{C}, \mu)}(U, V) := \{\eta \in \Nat_b(\iota \otimes U, \iota \otimes V) : \eta \text{ is } \mu\text{-harmonic}\}.$$
5.3. THE CATEGORICAL MARTIN BOUNDARY

If $\nu$ and $\eta$ are $\mu$-harmonic, then so are $(\nu \otimes \iota_X)$ and $(\iota_U \otimes \eta)$. Indeed, if $\eta \in \text{Nat}_b(\iota \otimes U, \iota \otimes V)$ and $\nu \in \text{Nat}_b(\iota \otimes X, \iota \otimes Y)$, then for any $s \in \text{Irr}(C)$ and $W \in \text{Ob}(C)$

\[
(P_s(\eta) \otimes \iota_Y)_W = P_s(\eta)_W \otimes \iota_Y = ((\text{tr}_s \otimes \iota_W \otimes \iota_Y)(\eta_{U \otimes W})) \otimes \iota_Y
= (\text{tr}_s \otimes \iota_W \otimes \iota_Y)((\eta \otimes \iota_Y)_{U \otimes W}) = P_s(\eta \otimes \iota_Y)_W; \tag{5.3.3}
\]

\[
(\iota_U \otimes P_s(\nu))_W = P_s(\nu)_{W \otimes U} = (\text{tr}_s \otimes \iota_W \otimes \iota_Y)(\nu_{U \otimes W \otimes U})
= (\text{tr}_s \otimes \iota_W \otimes \iota_Y)((\iota_U \otimes \nu)_{U \otimes W}) = P_s(\iota_U \otimes \nu)_W \tag{5.3.4}
\]

and the claim follows by linearity in $\mu$. So the tensor product is well-defined. Also $(\iota_U \otimes T)_U$ is harmonic for every $T \in \text{Hom}_C(X, Y)$, so the functor $E$ is well-defined. Note that in general the unit object $1$ need no longer be simple in $\mathcal{P}(C, \mu)$.

**Lemma 5.3.4.** Let $\mu$ be a probability measure on $\text{Irr}(C)$. The following are equivalent:

(i) The random walk $\{P_{\mu}(s, t)\}_{s, t \in \text{Irr}(C)}$ is transient;

(ii) $\|\sum_{n \in \mathbb{N}} P^{\mu}_n(\kappa^{s,V})\|_\infty < \infty$, for all $s \in \text{Irr}(C)$ and $V \in \text{Ob}(C)$;

(iii) $\|\sum_{n \in \mathbb{N}} P^{\mu}_n(\kappa^{s,1})\|_\infty < \infty$, for all $s \in \text{Irr}(C)$.

Moreover if these statements are satisfied, for every $s \in \text{Irr}(C)$ the quantity $\sum_{n \in \mathbb{N}} P^{\mu}_n(t, s)$ is uniformly bounded in $t$.

**Proof.** This is a classical result in disguise. Indeed, define $\eta := \sum_{n=0}^{\infty} P^{\mu}_n(\kappa^{s,1})$, then

$$
\|\eta\|_\infty = \sup_{t \in \text{Irr}(C)} \|\eta_t\| = \sup_{t \in \text{Irr}(C)} \sum_{n=0}^{\infty} P^{\mu}_n(t, s).
$$

The maximum principle (see for instance [Rev84, §2.1]) implies equivalence of (i) and (ii). Obviously (ii) and (iii) are equivalent. $\blacklozenge$

Recall that $0 \in \text{Irr}(C)$ labels the unit object, so $\kappa^{0,1}$ indicates the natural transformation which is the identity on $1$ and zero on $U_s$ for $s \neq 0$.

**Lemma 5.3.5.** Let $\mu$ be a probability measure on $\text{Irr}(C)$. The following are equivalent:

(i) the random walk $\{P_{\mu}(s, t)\}_{s, t}$ on $\text{Irr}(C)$ is irreducible;

(ii) for all $s, t \in \text{Irr}(C)$ and $V \in \text{Ob}(C)$ there exists an $n \in \mathbb{N}$ such that $P^{\mu}_n(\kappa^{s,V})_s \in \text{End}_C(U_t \otimes V)$ is invertible;

(iii) for all $s \in \text{Irr}(C)$ there exists an $n \in \mathbb{N}$ such that $P^{\mu}_n(\kappa^{0,1})_s \in \text{End}_C(U_t)$ is invertible;

(iv) for all $t \in \text{Irr}(C)$ and $V \in \text{Ob}(C)$, $\bigcup_{n=1}^{\infty} \text{supp}(P^{\mu}_n(\kappa^{0,V})) = \text{Irr}(C)$;

(v) $\bigcup_{n=1}^{\infty} \text{supp}(P^{\mu}_n(\kappa^{0,1})) = \text{Irr}(C)$.

The number $n$ of (ii) can be chosen independently of $V$. 
Proof. Suppose (i) holds. Let \( s,t \in \text{Irr}(C) \), let \( n \) be such that \( p^m_\mu(s,t) > 0 \). By Lemma 5.2.7 we have \( P^m_\mu(\kappa_{t,s}) = p^m_\mu(s,t)\mu_U \). Thus by (5.3.2) it follows that \( P^m_\mu(\kappa_{t,V}) = p^m_\mu(s,t)\mu_U \otimes V \), which is invertible because \( p^m_\mu(s,t) > 0 \). Thus (ii) holds.

(iii) is just a special case of (ii) and clearly (ii) implies (iv), because invertible operators are nonzero. For the same reason (iii) implies (v). Furthermore (v) is just a special case of (iv).

It remains to prove the implication (v) to (i). Let \( s \in \text{Irr}(C) \), by assumption there exists \( m \geq 1 \) such that \( s \in \supp(P^m_\mu(\kappa_{t,0})) \). From the defining equation (5.2.10) we see that \( p^m_\mu(s,0) > 0 \). Note that Lemma 5.2.7 implies that for any probability measure \( \nu \) on \( \text{Irr}(C) \) it holds that \( p_\nu(t,0) = p_\nu(t,0) \). So for any \( t \in \text{Irr}(C) \) there exists an \( n \geq 1 \) such that \( p^n_\mu(t,0) > 0 \). Let \( s,t \in \text{Irr}(C) \) and select the corresponding \( m \) and \( n \). Again by Lemma 5.2.7 we obtain

\[
p^{m+n}_\mu(s,t) = \sum_{r \in \text{Irr}(C)} p^m_\mu(s,r)p^n_\mu(r,t) \geq p^m_\mu(s,0)p^n_\mu(t,0) = \left( \frac{d_t}{d_0} \right)^m p^m_\mu(s,0)p^n_\mu(t,0) > 0
\]

and thus \( p_\mu \) is irreducible. \( \Box \)

**Definition 5.3.6.** If the operator \( P_\mu \) satisfies one (and therefore all) conditions of Lemma 5.3.4 we call \( \mu \) or \( P_\mu \) transient. Similarly, if \( P_\mu \) satisfies the equivalent conditions of Lemma 5.3.5, then \( P_\mu \) is called generating. If \( P_\mu \) is transient, we define the Green kernel \( G_\mu := \sum_{n=0}^\infty P^n_\mu \).

**Remark 5.3.7.** If \( C = \text{Rep}(G) \) for a compact quantum group \( G \). Then \( \kappa_{0,1} \) corresponds to \( I_* \). So a probability measure \( \mu \) on \( \text{Irr}(G) \) is generating respectively transient in the sense of quantum groups (cf. Definition 3.2.8) if and only if \( \mu \) is generating respectively transient in the sense of categories (cf. Definition 5.3.6).

From item (v) of Lemma 5.2.7 we immediately conclude the following.

**Corollary 5.3.8.** Let \( \mu \) be a probability measure on \( \text{Irr}(C) \). Then

(i) \( \mu \) is generating if and only if \( \bar{\mu} \) is generating;

(ii) \( \mu \) is transient if and only if \( \bar{\mu} \) is transient.

**Corollary 5.3.9.** If \( \eta \in \text{Nat}_{00}(t \otimes V,t \otimes V) \subset \text{Hom}_{\mathcal{C},-1}(V,V) \) and \( \mu \) is transient, then \( \|G_\mu(\eta)\|_\infty < \infty \) and thus \( G_\mu(\eta)_U \) is well-defined for all \( U \in \text{Ob}(C) \). If in addition \( \mu \) is generating, then \( (G_\mu(\kappa_{0,0}^t))_U \) is invertible for all \( U,V \in \text{Ob}(C) \).

**Proof.** The categorical traces \( tr_U \) are completely positive maps. Therefore also the Markov operators \( P_\mu \) are unital completely positive. In particular by taking linear combinations it follows that the Green kernel \( G_\mu^t : \text{Nat}_{00}(t \otimes V,t \otimes V) \to \text{Nat}(t \otimes V, t \otimes V) \) is positive. Put \( C := \|\eta\|_\infty \). Then \( |\eta_n| \leq C \sum_{t \in \supp(\eta)} \kappa_{t,s}^V \), for every \( s \in \text{Irr}(C) \). So

\[
\|G_\mu(\eta)\|_\infty \leq \|G_\mu(\bar{\eta})\|_\infty \leq \left\|G_\mu \left( C \sum_{t \in \supp(\eta)} \kappa_{t,s}^V \right) \right\|_\infty \leq C \sum_{t \in \supp(\eta)} \|G_\mu(\kappa_{0,0}^t)\|_\infty,
\]
which is finite by Lemma 5.3.4, because $\eta$ is finitely supported. The support $\text{supp}(\kappa^{0, V}) = \{0\}$, so $G_\mu(\kappa^{0, V})_U$ converges in norm for all $U$. Use the generating property of $\mu$ to find for each $t \in \text{Irr}(\mathcal{C})$ an $n_t \in \mathbb{N}$, such that $P^{n_t}_\mu(\kappa^{0, V})_t$ is invertible. Because $\kappa^{0, V}$ is positive and $P^{n_t}_\mu$ is a positive operator, $P^{n_t}_\mu(\kappa^{0, V})_t$ is strictly positive (meaning there exists $c > 0$ such that $P^{n_t}_\mu(\kappa^{0, V})_t > c I_{t \oplus V}$). Let $U \in \text{Ob}(\mathcal{C})$ and decompose $U = \bigoplus_t m_U^t U_t$. Denote $I := \{t \in \text{Irr}(\mathcal{C}) : m_U^t \neq 0\}$ and $N := \max\{n_t : t \in I\}$. Then for each $t \in I$ the inequality $\sum_{n=0}^N P^{n_t}_\mu(\kappa^{0, V})_t \geq P^{n_t}_\mu(\kappa^{0, V})_t$ implies that $\sum_{n=0}^N P^{n_t}_\mu(\kappa^{0, V})_t$ is strictly positive and thus invertible. Now

$$G_\mu(\kappa^{0, V})_U \geq \bigoplus_{t \in I} m_U^t \sum_{n=0}^N P^{n_t}_\mu(\kappa^{0, V})_t$$

shows that $G_\mu(\kappa^{0, V})_U$ is strictly positive, thus invertible.

It is often easier to work with $\text{Nat}_b(\iota \otimes V, \iota \otimes V)$ than with $\text{Nat}_b(\iota \otimes V, \iota \otimes W)$ for $V \neq W$. Fortunately most of the reducing cases can be done by taking the direct sum of $V$ and $W$. We make it precise.

**Notation 5.3.10.** Let $p_V \in \text{Hom}_\mathcal{C}(V \oplus W, V)$ and $p_W \in \text{Hom}_\mathcal{C}(V \oplus W, W)$ be the projections on $V$ respectively $W$. To be precise $p_V p_V^* = \iota_V$ and $p_V^* p_V = \iota_W$ where $\iota_W$ is a projection in $\text{End}_\mathcal{C}(V \oplus W)$, similarly for $p_W$. Then $q_V + q_W = \iota_{V \oplus W}$. Suppose $\eta \in \text{Nat}_b(\iota \otimes V, \iota \otimes W)$, the extension of $\eta$ is defined as $\eta^c := (\eta_U^c)_U$, where $\eta_U^c$ equals the composition

$$U \otimes (V \oplus W) \xrightarrow{\iota \otimes p_V} U \otimes V \xrightarrow{\eta_U} U \otimes W \xrightarrow{\iota \otimes p_W} U \otimes (V \oplus W).$$

Similarly for $\rho \in \text{Nat}_b(\iota \otimes (V \oplus W), \iota \otimes (V \oplus W))$, the restriction of $\eta$ is defined as $\eta^c := (\rho_U^c)_U$, where $\rho_U^c$ equals the composition

$$U \otimes V \xrightarrow{\iota \otimes p_V^*} U \otimes (V \oplus W) \xrightarrow{\rho_U} U \otimes (V \oplus W) \xrightarrow{\iota \otimes p_W} U \otimes W.$$

**Lemma 5.3.11.** Suppose that $\eta \in \text{Nat}_b(\iota \otimes (V \oplus W), \iota \otimes (V \oplus W))$ and $\mu$ is a probability measure on $\text{Irr}(\mathcal{C})$. The following holds:

(i) $\eta^c \in \text{Nat}_b(\iota \otimes (V \oplus W), \iota \otimes (V \oplus W))$, if $\eta$ is a natural transformation vanishing at infinity, then $\eta^c$ vanishes at infinity as well;

(ii) $\rho^c \in \text{Nat}_b(\iota \otimes V, \iota \otimes W)$, if $\rho$ is a natural transformation vanishing at infinity, then $\rho^c$ vanishes at infinity as well;

(iii) $(\eta^c)^c = \eta$;

(iv) $P_\mu(\eta^c) = P_\mu(\eta)^c$;

(v) $P_\mu(\rho^c) = P_\mu(\rho)^c$.

If in addition $\mu$ is transient, $|\text{supp}(\eta)| < \infty$ and $|\text{supp}(\rho)| < \infty$, then
(vi) \( G_\mu(\eta^e) = G_\mu(\eta)^e \);

(vii) \( G_\mu(\rho^e) = G_\mu(\rho)^e \).

**Proof.** We will not prove (ii) and (v), because they are very similar to (i) and (iv). For (i) suppose \( T \in \text{Hom}_C(U, X) \). Since \( \eta \) is a natural transformation, the diagram

\[
\begin{array}{ccc}
U \otimes (V \oplus W) & \xrightarrow{\iota \otimes p_V} & X \otimes V \\
& \downarrow{\eta_V} & \downarrow{\eta_X} \\
U \otimes W & \xrightarrow{T \otimes \iota} & X \otimes (V \oplus W)
\end{array}
\]

commutes. Because \( T \otimes \iota \) and the projections \( \iota \otimes p_V \) and \( \iota \otimes p_W \) act in different legs, the diagram

\[
\begin{array}{ccc}
U \otimes (V \oplus W) & \xrightarrow{T \otimes (\iota \otimes \iota)} & X \otimes (V \oplus W) \\
& \downarrow{\iota \otimes p_V} & \downarrow{\iota \otimes p_W} \\
U \otimes V & \xrightarrow{\iota \otimes \iota} & U \otimes (V \oplus W) & \xrightarrow{T \otimes (\iota \otimes \iota)} & X \otimes (V \oplus W)
\end{array}
\]

commutes. Hence \( \eta^e \) is a natural transformation. Since \( \|\eta^e_U\| = \|(\iota \otimes p_W^* \eta_U \iota \otimes p_V)\| \leq \|\eta_U\| \), the natural transformation \( \eta^e \) is uniformly bounded and it is vanishing at infinity if \( \eta \) is vanishing at infinity.

Since \( p_V p_V^* = \iota_V \) and \( p_W p_W^* = \iota_W \) it is immediate that

\[
(\eta^e)^* = (p_W \otimes \iota)(p_W^* \otimes \iota)\eta_U(p_V \otimes \iota)(p_V^* \otimes \iota) = \eta_U
\]

and thus (iii) holds.

To prove (iv), by linearity it is sufficient to prove the statement for \( P_1 \) for some simple object \( U_1 \). We have

\[
P_1(\eta^e)_U = (\text{tr}_{U_1} \otimes \eta_{(1,1)}) (\eta^e_{U_1 \otimes U})
= (\text{tr}_{U_1} \otimes \eta_{(1,1)}) \left((\iota_{U_1} \otimes \eta_U \iota_U \otimes p_V)\eta_{U_1 \otimes U}(\eta_{U_1 \otimes U})\right)
= (\iota_U \otimes p_V^*) \left((\text{tr}_{U_1} \otimes \eta_{(1,1)})(\eta_{U_1 \otimes U})(\eta_{U_1 \otimes U})\right)(\iota_U \otimes p_V)
= (\mu^e_{U_1})(\eta)_U.
\]

The assertions (vi) and (vii) are an immediate consequence of (iv) and (v).  \( \Box \)

**Definition 5.3.12.** Suppose that \( \mu \) is a generating and transient probability measure on \( \text{Irr}(C) \). The **Martin kernel** for \( P_\mu \) is the operator given by

\[
K_{\bar{\mu}}: \text{Nat}_0(\iota \otimes V, \iota \otimes W) \to \text{Hom}_{C^{-1}}(V, W), \quad K_{\bar{\mu}}(\eta)_U := G_{\bar{\mu}}(\eta)_U (G_{\bar{\mu}}(\kappa_{0,V})_U)^{-1}.
\]

By Corollary 5.3.8 \( \bar{\mu} \) is generating and transient, so Lemmas 5.3.4 and 5.3.5 imply that the operators \( G_{\bar{\mu}}(\eta)_U \) and \( (G_{\bar{\mu}}(\kappa_{0,V})_U)^{-1} \) are well-defined. Hence the Martin kernel \( K_{\bar{\mu}} \).
makes sense.

Note that (5.3.2) implies by linearity that $G_\mu(\kappa^t V) = G_\mu(\kappa^t) \otimes \iota_V$. Hence as $P_\tau$ and thus $G_\bar{\mu}$ preserves the center, for any $\nu \in \text{Nat}_0(\iota \otimes V, \iota \otimes W)$ it holds that

$$G_\bar{\mu}(\kappa^{0,W} V) \nu_U = (G_\bar{\mu}(\kappa^{0,1}) \otimes \iota_W) U \nu_U = \nu_U (G_\bar{\mu}(\kappa^{0,1}) \otimes \iota_V) U = \nu_U G_\bar{\mu}(\kappa^{0,V}) U.$$ 

Therefore we obtain

$$G_\bar{\mu}(\eta) U (G_\bar{\mu}(\kappa^{0,V}) U)^{-1} = (G_\bar{\mu}(\kappa^{0,W}) U)^{-1} G_\bar{\mu}(\eta) U.$$ 

This means that the appropriate inverse can be placed on either side of $G_\bar{\mu}(\eta)$ when defining the Martin kernel.

**Lemma 5.3.13.** Suppose that $\mu$ is a generating and transient probability measure on $\text{Irr}(\mathcal{C})$, then

\[
\begin{align*}
K_\mu(\eta^c) &= K_\mu(\eta)^c \quad \text{for } \eta \in \text{Nat}_0(\iota \otimes V, \iota \otimes W); \\
K_\mu(\eta^c) &= K_\mu(\eta)^c \quad \text{for } \eta \in \text{Nat}_0(\iota \otimes (V \oplus W), \iota \otimes (V \oplus W)); \\
K_\mu(\kappa^{0,1} \otimes T) &= \iota \otimes T \quad \text{for } T \in \text{Hom}_c(V, W).
\end{align*}
\]

**Proof.** We know $G_\mu(\kappa^t V) = G_\mu(\kappa^t) \otimes \iota_V$. Combine this with Lemma 5.3.11 to obtain

\[
\begin{align*}
K_\mu(\eta^c) &= (\iota \otimes p^*_W) G_\mu(\eta)(G_\mu(\kappa^{0,1} \otimes \iota_V))^{-1}(\iota \otimes p_V) \\
&= (\iota \otimes p^*_W) G_\mu(\eta)(\iota \otimes p_V)(G_\mu(\kappa^{0,1}))^{-1} \otimes \iota_{V \oplus W} \\
&= G_\mu(\eta)^c(G_\mu(\kappa^{0,1}))^{-1} \otimes \iota_{V \oplus W} \\
&= G_\mu(\eta^c)(G_\mu(\kappa^{0,1}))^{-1} \otimes \iota_{V \oplus W} = K_\mu(\eta^c).
\end{align*}
\]

The second identity can be proved similarly. Observe that by linearity (5.3.1) implies that

$$G_\mu(\kappa^{0,1} \otimes T) = G_\mu(\kappa^{0,1}) \otimes T.$$ 

Therefore

$$K_\mu(\kappa^{0,1} \otimes T) = G_\mu(\kappa^{0,1} \otimes T) G_\mu(\kappa^{0,V})^{-1} = (G_\mu(\kappa^{0,1}) \otimes \iota_V)^{-1} = \iota \otimes T,$$

which proves the lemma. \(\Box\)

**Definition 5.3.14.** Given a strict $C^*$-tensor category $\mathcal{C}$ with simple unit $1$ and a generating and transient probability measure $\mu$ on $\text{Irr}(\mathcal{C})$, let $\mathcal{M}'$ be the smallest $C^*$-subcategory of $\mathcal{C}_{-1}$ containing the objects $\text{Ob}(\mathcal{C})$ and morphism sets $\text{Nat}_0(\iota \otimes U, \iota \otimes V)$ and $K_\bar{\mu}(\text{Nat}_0(\iota \otimes U, \iota \otimes V))$. The composition of morphisms in $\mathcal{M}'$ is given by the composition of natural transformations.

The **Martin compactification** of $\mathcal{C}$ with respect to $\mu$ consists of the pair $(\mathcal{M}(\mathcal{C}, \mu), \mathcal{E})$, where $\mathcal{M}(\mathcal{C}, \mu)$ is direct sum and subobject completion of $\mathcal{M}'$ and $\mathcal{E} : \mathcal{C} \rightarrow \mathcal{M}(\mathcal{C}, \mu)$ is the restriction of the canonical functor $\mathcal{E} : \mathcal{C} \rightarrow \mathcal{C}_{-1}$ introduced in Notation 5.2.1 above. $\mathcal{E}$ is
well-defined due to the preceding lemma. \( \tilde{\mathcal{M}}(\mathcal{C}, \mu) \) is a C*-tensor category, see Lemma 5.3.16 below.

Again note that the unit object \( 1 \) of \( \tilde{\mathcal{M}}(\mathcal{C}, \mu) \) need no longer be simple.

**Remark 5.3.15.** Observe that \( \text{End}_{\tilde{\mathcal{M}}(\mathcal{C}, \mu)}(U) \) equals the C*-algebra generated by the morphisms \( K_\mu(\text{Nat}_0(\iota \otimes U, \iota \otimes U)) \) and \( \text{Nat}_0(\iota \otimes U, \iota \otimes U) \). Indeed, it suffices to show that if \( V \in \text{Ob}(\mathcal{C}) \) is another object and \( \eta \) is an element of the C*-algebra generated by \( \text{Nat}_0(\iota \otimes (U \oplus V), \iota \otimes (U \oplus V)) \) and \( K_\mu(\text{Nat}_0(\iota \otimes (U \oplus V), \iota \otimes (U \oplus V))) \), then \( (\iota \otimes p_U)\eta(\iota \otimes p_U^\ast) \) is an element of the C*-algebra generated by \( \text{Nat}_0(\iota \otimes U, \iota \otimes U) \) and \( K_\mu(\text{Nat}_0(\iota \otimes U, \iota \otimes U)) \).

If \( \eta \) is a generator, so an element in \( \text{Nat}_0(\iota \otimes (U \oplus V), \iota \otimes (U \oplus V)) \cup K_\mu(\text{Nat}_0(\iota \otimes (U \oplus V), \iota \otimes (U \oplus V))) \), this can directly be verified by a proof similar to Lemmas 5.3.11 and 5.3.13. But then it immediately holds for all \( \eta \).

Since \( \tilde{\mathcal{M}}(\mathcal{C}, \mu) \) is a C*-category we also have that

\[
\text{Hom}_{\tilde{\mathcal{M}}}(V, W) = (\text{End}_{\tilde{\mathcal{M}}}(V \oplus W))^r,
\]

where \( r \) denotes the restriction.

**Lemma 5.3.16.** The category \( \tilde{\mathcal{M}}(\mathcal{C}, \mu) \) forms a C*-tensor category.

**Proof.** Since \( \mathcal{C}_{-1} \) is a C*-tensor category we only need to show that \( \tilde{\mathcal{M}}(\mathcal{C}, \mu) \) is closed under tensor products. Suppose that \( \eta \in \text{Hom}_{\tilde{\mathcal{M}}(\mathcal{C}, \mu)}(U, V) \) and \( \nu \in \text{Hom}_{\tilde{\mathcal{M}}(\mathcal{C}, \mu)}(X, Y) \), then \( \eta \otimes \nu \in \text{Hom}_{\tilde{\mathcal{M}}(\mathcal{C}, \mu)}(U \otimes X, V \otimes Y) \). Recall the tensor product in \( \mathcal{C}_{-1} \) (5.2.3). It therefore suffices to show that

\[
\begin{align*}
\text{(a)} \quad & (\eta \otimes \iota_Y) \in \text{Nat}_0(\iota \otimes U \otimes Y, \iota \otimes V \otimes Y) \text{ if } \eta \in \text{Nat}_0(\iota \otimes U, \iota \otimes V); \\
\text{(b)} \quad & (\iota_U \otimes \nu) \in \text{Nat}_0(\iota \otimes U \otimes X, \iota \otimes U \otimes Y) \text{ if } \nu \in \text{Nat}_0(\iota \otimes X, \iota \otimes Y); \\
\text{(c)} \quad & K_\mu(\eta) \otimes \iota_Y = K_\mu(\eta \otimes \iota_Y) \text{ if } \eta \in \text{Nat}_0(\iota \otimes U, \iota \otimes V); \\
\text{(d)} \quad & \iota_U \otimes K_\mu(\nu) = K_\mu(\iota_U \otimes \nu) \text{ if } \nu \in \text{Nat}_0(\iota \otimes X, \iota \otimes Y).
\end{align*}
\]

(a) is trivial, because \( (\eta \otimes \iota_Y)_U = \eta_U \otimes \iota_Y \), thus \( \text{supp}(\eta \otimes \iota_Y) = \text{supp}(\eta) \) and \( \| (\eta \otimes \iota_Y)_W \| = \| \eta_W \| \).

For (b), suppose that \( \eta \) has finite support. Recall \( (\iota_U \otimes \nu)_W = \eta_W \otimes U \), so by Frobenius reciprocity

\[
| \text{supp}(\iota_U \otimes \nu) | = | \{ s \in \text{Irr}(\mathcal{C}) : \eta_{U,s} \otimes U \neq 0 \} |
= | \{ s \in \text{Irr}(\mathcal{C}) : \exists t \in \text{supp}(\nu), \eta_U^{t,s_U} \otimes U \neq 0 \} |
= | \{ s \in \text{Irr}(\mathcal{C}) : \exists t \in \text{supp}(\nu), \eta_U^{t,s} \otimes U \neq 0 \} |
\leq \sum_{t \in \text{supp}(\nu)} | \{ s \in \text{Irr}(\mathcal{C}) : \eta_U^{t,s} \otimes U \neq 0 \} | < \infty.
\]
Hence $\iota_U \otimes \nu \in \text{Nat}_0(\iota \otimes (U \otimes X), \iota \otimes (U \otimes Y))$. Since $\text{Nat}_0(\iota \otimes (U \otimes X), \iota \otimes (U \otimes Y))$ is the norm-closure of $\text{Nat}_0(\iota \otimes (U \otimes X), \iota \otimes (U \otimes Y))$ the claim follows.

(c) Use (5.3.3) to obtain

$$K_\mu(\eta) \otimes \iota_Y = (G_\mu(\eta)G_\mu(\kappa^{0,U})^{-1}) \otimes \iota_Y = (G_\mu(\iota \otimes \iota_Y))G_\mu(\kappa^{0,U})^{-1} \otimes \iota_Y = K_\mu(\eta \otimes \iota_Y).$$

(d) Similarly use (5.3.4) to obtain

$$\iota_U \otimes K_\mu(\nu) = \iota_U \otimes G_\mu(\nu)G_\mu(\kappa^{0,X})^{-1} = G_\mu(\iota_U \otimes \nu)G_\mu(\nu \otimes \kappa^{0,X})^{-1} = K_\mu(\iota_U \otimes \nu),$$

which completes the proof.

\begin{lemma}
Define $\mathcal{M}'$ to be the category with $\text{Ob}(\mathcal{M}') := \text{Ob}(\tilde{\mathcal{M}}') = \text{Ob}(\mathcal{C})$ and

$$\text{Hom}_{\mathcal{M}'}(V, W) := \{[\eta] : \eta \in \text{Hom}_{\tilde{\mathcal{M}}'}(V, W)\},$$

where $[\eta]$ denotes the equivalence class of $\eta$ in $\text{Nat}_0(\iota \otimes V, \iota \otimes W)/\text{Nat}_0(\iota \otimes V, \iota \otimes W)$. Let $\mathcal{M}(\mathcal{C}, \mu)$ be the subobject and direct sum completion of $\mathcal{M}'$. Then

(i) $\mathcal{M}'$ is a category;

(ii) $[\cdot] : \tilde{\mathcal{M}}' \to \mathcal{M}'$ is a functor and defines a tensor and $*$-structure on $\mathcal{M}'$ by $[\nu \otimes [\eta]] := [\nu \otimes \eta]$ and $[\eta]^* := [\eta^*]$;

(iii) $[\cdot]$ extends to a full unitary tensor functor $\tilde{\mathcal{M}}(\mathcal{C}, \mu) \to \mathcal{M}(\mathcal{C}, \mu)$.

\end{lemma}

\begin{proof}
(i) Since $\|(\eta \nu)\| \leq \|\eta\| \|\nu\|$, the morphism $[\eta \nu] = 0$ in $\text{Hom}_{\tilde{\mathcal{M}}'}(U, W)$ if $\eta \in \text{Hom}_{\tilde{\mathcal{M}}'}(V, W) \cap \text{Nat}_0(\iota \otimes V, \iota \otimes W)$ or $\nu \in \text{Hom}_{\tilde{\mathcal{M}}'}(U, V) \cap \text{Nat}_0(\iota \otimes U, \iota \otimes V)$. Thus the equivalence relation is well-defined and $\mathcal{M}'$ becomes a category.

(ii) Clearly $[\cdot]$ defines a functor. It suffices to check that the maps $[\nu \otimes [\eta]] \mapsto [\nu \otimes \eta]$ and $[\eta]^* \mapsto [\eta^*]$ are well-defined on the quotient. Obviously $\eta^* = (\eta^*)_X \in \text{Nat}_0(\iota \otimes W, \iota \otimes V)$ if $\eta \in \text{Nat}_0(\iota \otimes V, \iota \otimes W)$. To show that $[\nu \otimes \eta]$ is well-defined, by the proof of (i) it suffices to show that $\eta \otimes \iota_X \in \text{Nat}_0(\iota \otimes (V \otimes X), \iota \otimes (W \otimes X))$ and $\iota_U \otimes \eta \in \text{Nat}_0(\iota \otimes (U \otimes V), \iota \otimes (U \otimes W))$ for any $\eta \in \text{Hom}_{\tilde{\mathcal{M}}'}(V, W) \cap \text{Nat}_0(\iota \otimes V, \iota \otimes W)$, which is already established in the proof of Lemma 5.3.16.

(iii) It is immediate that $[\cdot] : \text{Hom}_{\tilde{\mathcal{M}}'}(V, W) \to \text{Hom}_{\mathcal{M}'}(V, W)$ is surjective. Thus $[\cdot]$ is full. Since any functor between two categories can be extended to the subobject and direct sum completions, also $[\cdot]$ can be extended to a full functor.

\end{proof}

\begin{definition}
The Martin boundary of $\mathcal{C}$ with respect to $\mu$ is the pair $(\mathcal{M}(\mathcal{C}, \mu), \mathcal{E})$ where $\mathcal{M}(\mathcal{C}, \mu)$ is the $C^*$-tensor category as defined in the previous lemma and $\mathcal{E} : \mathcal{C} \to \mathcal{M}(\mathcal{C}, \mu)$ is the unitary tensor functor which is the composition of the functor $\mathcal{C} \to \tilde{\mathcal{M}}(\mathcal{C}, \mu)$ with $[\cdot]$.

\end{definition}
5.4 Categorical convergence to the boundary

It is also possible to put convergence to the boundary in a categorical framework. In this section we work towards the definition.

**Notation 5.4.1.** Let $\mu$ be a probability measure on $\text{Irr}(\mathcal{C})$. For $n > m$ define recursively the maps $\text{Hom}_{\mathcal{C}_n}(U, V) \to \text{Hom}_{\mathcal{C}_m}(U, V)$ by

$$
(\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu \otimes t^m)(\eta) := (\text{tr}_\mu \otimes t^m)((\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu \otimes t^{m+1})(\eta)), \quad (\eta \in \text{Hom}_{\mathcal{C}_n}(U, V)),
$$

where the base case $m = n - 1$ is defined by (5.2.8). Denote

$$
(\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu \otimes \text{tr}_U)(\eta) := \text{tr}_U((\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)(\eta)), \quad (\eta \in \text{Hom}_{\mathcal{C}_n}(U, U)).
$$

Write

$$
||\eta||_{\mu \otimes n} := ((\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu \otimes \text{tr}_U)(\eta^* \eta))^{\frac{1}{2}}.
$$

(5.4.1)

Since $\text{tr}_\mu$ is a positive linear functional, $|| \cdot ||_{\mu \otimes n}$ defines a seminorm on $\text{Hom}_{\mathcal{C}_n}(U, V)$. As $\eta^* \in \text{Hom}_{\mathcal{C}_n}(V, U)$ Equation (5.4.1) also defines

$$
||\eta^*||_{\mu \otimes n} = ((\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu \otimes \text{tr}_V)(\eta \eta^*))^{\frac{1}{2}}.
$$

**Definition 5.4.2.** A natural transformation $\eta \in \text{Hom}_{\mathcal{C}_1}(U, V)$ is called $\mu$-regular if the following condition holds: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$

$$
||\hat{\Delta}^{n-1}(\eta) - \hat{\Delta}^{m-1}(\eta)||_{\mu \otimes n}^2 < \varepsilon,
$$

(5.4.2)

$$
||\hat{\Delta}^{n-1}(\eta) - \hat{\Delta}^{m-1}(\eta)^*||_{\mu \otimes n}^2 < \varepsilon.
$$

(5.4.3)

Denote the set of $\mu$-regular elements in $\text{Hom}_{\mathcal{C}_1}(U, V)$ by $R_{\mathcal{C}, \mu}(U, V)$ or simply $R_{\mu}(U, V)$. These morphisms can be collected into a category. Denote $R_{\mathcal{C}, \mu}$ for the subcategory of $\mathcal{C}_1$ with $\text{Hom}_{R_{\mathcal{C}, \mu}}(U, V) := R_{\mathcal{C}, \mu}(U, V)$, this is indeed a subcategory, see Proposition 5.4.4.

For $U = V$ regularity can be formulated in an equivalent way. First, we define the path spaces of paths of infinite length (cf. [NY14c, §3.1]). For this consider the von Neumann algebras $\text{Hom}_{\mathcal{C}_n}(U, U)$. Fix $\mu$ and define a conditional expectation

$$
E_{n+1,n} : \text{Hom}_{\mathcal{C}_{n+1}}(U, U) \to \text{Hom}_{\mathcal{C}_n}(U, U), \quad E_{n+1,n}(\eta) := (\text{tr}_\mu \otimes \ell^{\otimes n})(\eta).
$$

(5.4.4)

Clearly $E_{n+1,n}(\ell \otimes \eta) = \eta$, so the embedding $\text{Hom}_{\mathcal{C}_n}(U, U) \hookrightarrow \text{Hom}_{\mathcal{C}_{n+1}}(U, U)$ is preserved by $E_{n+1,n}$. For $n > m$ define $E_{n,m} := E_{m+1,m} \circ \cdots \circ E_{n,n-1}$. Define a state $\varphi_U^{(n)} := \text{tr}_U \circ E_{n,0}$ on $\text{Hom}_{\mathcal{C}_n}(U, U)$. This collection $(\varphi_U^{(n)})_n$ gives a state $\varphi_U^{(\infty)}$ on the union $\bigcup_n \text{Hom}_{\mathcal{C}_n}(U, U)$. Define the von Neumann algebra $\text{Hom}_{\mathcal{C}_n}(U, U)$ as the completion of $\bigcup_n \text{Hom}_{\mathcal{C}_n}(U, U)$ in the GNS representation defined by the state $\varphi_U^{(\infty)}$. Denote the
composition

\[ j_n: \text{Hom}_{\mathcal{C}-1}(U,U) \to \text{Hom}_{\mathcal{C}-n}(U,U) \to \text{Hom}_{\mathcal{C}-\infty}(U,U), \]

\[ \eta \mapsto \hat{\Delta}^{n-1}(\eta) \mapsto \cdots \otimes \iota \otimes \hat{\Delta}^{n-1}(\eta). \]

It follows that a morphism \( \eta \in \text{Hom}_{\mathcal{C}-1}(U,U) \) is \( \mu \)-regular if and only if \( s^*\text{-}\lim_n j_n(\eta) \) exists in the von Neumann algebra \( \text{Hom}_{\mathcal{C}-\infty}(U,U) \). We will slightly extend the results of Lemma 5.3.11.

**Lemma 5.4.3.** Let \( \eta \in \text{Hom}_{\mathcal{C}-n}(U,V) \) and \( \nu \in \text{Hom}_{\mathcal{C}-n}(U \oplus V, U \oplus V) \) the restriction and extension operations satisfy the following properties:

(i) \( (\iota \otimes \hat{\Delta} \otimes \iota \otimes \eta^{\otimes n-1})(\eta^\varepsilon) = ((\iota \otimes \hat{\Delta} \otimes \iota \otimes \eta^{\otimes n-1})(\eta))^\varepsilon; \)

(ii) \( (\iota \otimes \nu^{\otimes n-1})(\nu^\varepsilon) = ((\iota \otimes \hat{\Delta} \otimes \iota \otimes \eta^{\otimes n-1})(\nu))^\varepsilon; \)

(iii) \( \iota \otimes \eta^\varepsilon = (\iota \otimes \eta)^\varepsilon; \)

(iv) \( \iota \otimes \nu^\varepsilon = (\iota \otimes \nu)^\varepsilon; \)

(v) \( \|\eta^\varepsilon\|_{\mu^{\otimes n}} = \frac{d_\mu}{d_\nu + d_V} \|\eta\|_{\mu^{\otimes n}}; \)

(vi) \( \|\nu^\varepsilon\|_{\mu^{\otimes n}} \leq \frac{d_\mu}{d_\nu + d_V} \|\nu\|_{\mu^{\otimes n}}. \)

Moreover, if \( \eta \in \text{Hom}_{\mathcal{C}-1}(U,V) \) and \( \nu \in \text{Hom}_{\mathcal{C}-1}(U \oplus V, U \oplus V) \), then

(vii) \( \text{if } \eta \in R_\mu(U,V), \text{ then } \eta^\varepsilon \in R_\mu(U \oplus V, U \oplus V); \)

(viii) \( \text{if } \nu \in R_\mu(U \oplus V, U \oplus V), \text{ then } \nu^\varepsilon \in R_\mu(U,V). \)

**Proof.** To prove (i)–(vi) we restrict ourselves to \( n = 1 \), the case \( n > 1 \) is similar but only more complicated in notation. We have

\[ \hat{\Delta}(\eta^\varepsilon)_{X,Y} = \eta^\varepsilon_{X \otimes Y} = (\iota \otimes p_U^* \eta)_{X \otimes Y}(\iota \otimes p_V)(\hat{\Delta}(\eta))_{X,Y}(\iota \otimes p_V) = (\hat{\Delta}(\eta))_{X,Y}^\varepsilon; \]

\[ \hat{\Delta}(\nu^\varepsilon)_{X,Y} = \nu^\varepsilon_{X \otimes Y} = (\iota \otimes p_V \nu)_{X \otimes Y}(\iota \otimes p_U^*)(\hat{\Delta}(\nu))_{X,Y}(\iota \otimes p_U) = (\hat{\Delta}(\nu))_{X,Y}^\varepsilon; \]

\[ (\iota \otimes \eta^\varepsilon)_{X,Y} = \iota_X \otimes \eta^\varepsilon_{X \otimes Y} = \iota_X \otimes ((\iota \otimes p_U^* \eta_{X \otimes Y})(\iota \otimes p_V)) = (\iota_{X \otimes Y} \otimes p_U^*)(\iota_X \otimes \eta_{X \otimes Y})(\iota_{X \otimes Y} \otimes p_V) \]

\[ = (\iota \otimes \eta)^\varepsilon_{X,Y}; \]

\[ (\iota \otimes \nu^\varepsilon)_{X,Y} = \iota_X \otimes \nu^\varepsilon_{X \otimes Y} = \iota_X \otimes ((\iota \otimes p_V \nu_{X \otimes Y})(\iota \otimes p_U)) = (\iota_{X \otimes Y} \otimes p_V)(\iota_X \otimes \nu_{X \otimes Y})(\iota_{X \otimes Y} \otimes p_U) \]

\[ = (\iota \otimes \nu)^\varepsilon_{X,Y}. \]

Note that

\[ \text{tr}_{U \oplus V} = \frac{1}{d_{U \oplus V}} \text{Tr}_{U \oplus V} = \frac{1}{d_U + d_V} \text{Tr}_U \oplus \text{Tr}_V = \frac{d_U}{d_U + d_V} \text{tr}_U \oplus \frac{d_V}{d_U + d_V} \text{tr}_V. \]
Therefore
\[
\|\eta^r\|_\mu^2 = (\text{tr}_\mu \otimes \text{tr}_{U \oplus V})(\eta^r)^* \eta^r
\]
\[
= \left( \text{tr}_\mu \otimes \left( \frac{d_U}{d_U + d_V} \text{tr}_U \oplus \frac{d_V}{d_U + d_V} \text{tr}_V \right) \right)( (\iota \otimes p_U^*)\eta^r (\iota \otimes p_V) (\iota \otimes p_U^*)\eta (\iota \otimes p_V) )
\]
\[
= \left( \text{tr}_\mu \otimes \left( \frac{d_U}{d_U + d_V} \text{tr}_U \oplus \frac{d_V}{d_U + d_V} \text{tr}_V \right) \right)( (\iota \otimes p_U^*)\eta^r \eta (\iota \otimes p_V) )
\]
\[
= \left( \frac{d_U}{d_U + d_V} \text{tr}_\mu \otimes \text{tr}_{U \oplus V} \right)(\eta^* \eta) = \frac{d_U}{d_U + d_V} \|\eta\|_\mu^2,
\]
\[
\|\nu^r\|_\mu^2 = (\text{tr}_\mu \otimes \text{tr}_{U \oplus V})((\nu^r)^* \nu^r) = (\text{tr}_\mu \otimes \text{tr}_U)((\iota \otimes p_U)\nu^r (\iota \otimes p_U^*)\nu (\iota \otimes p_U^*))
\]
\[
\leq (\text{tr}_\mu \otimes \text{tr}_U)((\iota \otimes p_U)\nu^r \nu (\iota \otimes p_U^*))
\]
\[
\leq \frac{d_U}{d_U + d_V} (\text{tr}_\mu \otimes (\text{tr}_{U \oplus V}))( (\iota \otimes p_U)\nu^r (\iota \otimes p_U^*) \oplus (\nu \otimes p_U^*)\nu^r (\iota \otimes p_U^*))
\]
\[
= \frac{d_U + d_V}{d_U} (\text{tr}_\mu \otimes \text{tr}_{U \oplus V})(\nu^* \nu) = \frac{d_U + d_V}{d_U} \|\nu\|_\mu^2,
\]
where in (vi) we used that \(\iota_{U \oplus V} \geq p_U^* p_V\) so that \(\nu^r \nu = \nu^r (\iota \otimes \iota_{U \oplus V}) \nu \geq \nu^r (\iota \otimes p_U^*) (\iota \otimes p_V) \nu\).

Statements (vii) and (viii) can now be proved by combining all the previous identities. Namely,
\[
\|\hat{\Delta}^{n-1}(\eta^r) - \iota^{\otimes n-m} \otimes \hat{\Delta}^{m-1}(\eta^r)\|_{\mu \otimes n}^2 = \| (\hat{\Delta}^{n-1}(\eta) - \iota^{\otimes n-m} \otimes \hat{\Delta}^{m-1}(\eta))^r\|_{\mu \otimes n}^2
\]
\[
= \frac{d_U}{d_U + d_V} \|\hat{\Delta}^{n-1}(\eta) - \iota^{\otimes n-m} \otimes \hat{\Delta}^{m-1}(\eta)\|_{\mu \otimes n}^2,
\]
\[
\|\hat{\Delta}^{n-1}(\nu^r) - \iota^{\otimes n-m} \otimes \hat{\Delta}^{m-1}(\nu^r)\|_{\mu \otimes n}^2 = \| (\hat{\Delta}^{n-1}(\nu) - \iota^{\otimes n-m} \otimes \hat{\Delta}^{m-1}(\nu))^r\|_{\mu \otimes n}^2
\]
\[
\leq \frac{d_U + d_V}{d_U} \|\hat{\Delta}^{n-1}(\nu) - \iota^{\otimes n-m} \otimes \hat{\Delta}^{m-1}(\nu)\|_{\mu \otimes n}^2,
\]
as desired. \(\blacksquare\)

Since \((\eta^r)^r = \eta\) this lemma implies that a morphism \(\eta \in \text{Hom}_{C_{-1}}(U, V)\) is \(\mu\)-regular if and only if \(s^* \lim_n \gamma_n(\eta^r)\) exists in the von Neumann algebra \(\text{Hom}_{C_{-\infty}}(U \oplus V, U \oplus V)\).

**Proposition 5.4.4.** The following holds:

(i) \(R(C, \mu)\) forms a \(C^*\)-tensor subcategory of \(C_{-1}\);

(ii) if \(\eta\) is \(\mu\)-harmonic, then \(\eta\) is \(\mu\)-regular;

(iii) if \(\eta \in \text{Hom}_{R(C, \mu)}(U, V)\), then \(\lim_n (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)(\hat{\Delta}^{n-1}(\eta)) \in \text{Hom}_C(U, V)\) exists in norm.

We write
\[
\text{tr}_\mu^{\infty}(\eta) := \lim_n (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)(\hat{\Delta}^{n-1}(\eta)) \in \text{Hom}_C(U, V).
\] (5.4.5)

Note that we cannot say that the Poisson boundary \(P(C, \mu)\) is a subcategory of \(R(C, \mu)\) since the product in \(P(C, \mu)\) is different.
Proof of Proposition 5.4.4. (i) From the definition of $\mu$-regularity it is immediate that $\eta^* \in R_\mu(V,U)$ whenever $\eta \in R_\mu(U,V)$. Moreover it is clear that $R_\mu(U,V)$ is a linear space. $R_\mu(U,V)$ has norm $\|\cdot\|_\infty$ which is inherited from $C$ and thus has the properties of a norm in a C*-category. For $R(C,\mu)$ to be a C*-category it thus remains to show that $R_\mu(U,V)$ is closed in norm $\|\cdot\|_\infty$ and if $\eta$ and $\nu$ are $\mu$-regular, then so is their composition. First observe that $\text{tr}_U(x^*x) \leq \|x\|^2 \text{ for any } x \in \text{Hom}_C(U,V)$. So that $\|\eta\|_{\mu^\otimes n} \leq \|\eta\|_\infty$ whenever $\eta \in \text{Hom}_{C_n}(U,V)$. Suppose that $(\nu_n)_n \subset R_\mu(U,V)$ and $\nu \in \text{Hom}_{C_1}(U,V)$ such that $\|\nu_n - \nu\|_\infty \to 0$. Let $\varepsilon > 0$ and find $k$ such that $\|\nu_k - \nu\|_\infty < \varepsilon$. Pick $N$ such that (5.4.2) and (5.4.3) hold for $\eta = \nu_k$ and all $n > m \geq N$. For such $n > m \geq N$ we obtain

$$
\|\hat{\Delta}^{n-1}(\nu) - \iota^{\otimes(n-m)} \otimes \hat{\Delta}^{m-1}(\nu)\|_{\mu^\otimes n} \\
\leq \|\hat{\Delta}^{n-1}(\nu) - \hat{\Delta}^{n-1}(\nu_k)\|_{\mu^\otimes n} + \|\hat{\Delta}^{n-1}(\nu_k) - \iota^{\otimes(n-m)} \otimes \hat{\Delta}^{m-1}(\nu_k)\|_{\mu^\otimes n} \\
+ \|\iota^{\otimes(n-m)} \otimes \hat{\Delta}^{m-1}(\nu_k) - \iota^{\otimes(n-m)} \otimes \hat{\Delta}^{m-1}(\nu)\|_{\mu^\otimes n} \\
\leq \|\hat{\Delta}^{n-1}(\nu) - \hat{\Delta}^{n-1}(\nu_k)\|_{\mu^\otimes n} + \|\iota^{\otimes(n-m)} \otimes \hat{\Delta}^{m-1}(\nu_k) - \nu\|_{\mu^\otimes n} \\
\leq \|\nu - \nu_k\|_\infty + \varepsilon + \|\nu - \nu_k\|_{\mu^\otimes n} \\
\leq \|\nu - \nu_k\|_\infty + \varepsilon + \|\nu_k - \nu\|_\infty \leq 3\varepsilon.
$$

In the last line we used again that $*$-homomorphisms of C*-algebras are norm-decreasing. A similar argument applies to $\|\hat{\Delta}^{n-1}(\nu^*) - \iota^{\otimes(n-m)} \otimes \hat{\Delta}^{m-1}(\nu^*)\|_{\mu^\otimes n}$, so that $\nu \in R_\mu(U,V)$ and thus $R_\mu(U,V)$ is norm-closed.

For multiplicativity we use the same estimate as in the proof of Lemma 5.4.3, namely that $x^*x \leq \|x\|^2 1$, so that $y^*x^*xy \leq \|x\|^2 y^*y$ and thus

$$
\|xy\|_U^2 = \text{tr}_U((xy)^*(xy)) \leq \|x\|^2 \text{tr}_U(y^*y) = \|x\|^2 \|y\|_U^2.
$$

Let $\nu \in R_\mu(U,V)$, $\eta \in R_\mu(V,W)$ and $\varepsilon > 0$. Then for $n$ and $m$ large enough

$$
\|\hat{\Delta}^{n-1}(\eta\nu) - \iota^{\otimes(n-m)} \otimes \hat{\Delta}^{m-1}(\eta\nu)\|_{\mu^\otimes n} \\
\leq \|\hat{\Delta}^{n-1}(\eta)\|_{\mu^\otimes n} \|\hat{\Delta}^{n-1}(\nu)\|_{\mu^\otimes n} \\
+ \|\hat{\Delta}^{n-1}(\eta)\|_{\mu^\otimes n} \|\hat{\Delta}^{m-1}(\nu)\|_{\mu^\otimes n} \\
\leq \|\hat{\Delta}^{n-1}(\eta)\|_\infty \|\hat{\Delta}^{n-1}(\nu)\|_\infty \|\hat{\Delta}^{n-1}(\eta)\|_{\mu^\otimes n} \\
+ \|\hat{\Delta}^{m-1}(\nu)\|_{\mu^\otimes n} \|\hat{\Delta}^{m-1}(\eta)\|_{\mu^\otimes n} \\
\leq \|\eta\|_\infty \varepsilon + \|\nu\|_\infty \varepsilon.
$$

Thus $\eta\nu \in R_\mu(U,W)$ and $R(C,\mu)$ is a C*-category.

To show that it $R_\mu(C,\mu)$ admits a tensor structure we must verify that

$$(\eta \otimes \nu) \in \text{Hom}_{R(C,\mu)}(U \otimes X, V \otimes Y), \quad \text{if } \eta \in \text{Hom}_{R(C,\mu)}(U, V), \nu \in \text{Hom}_{R(C,\mu)}(X, Y).$$

Here $(\eta \otimes \nu)$ is defined by Identity (5.2.3). We have already shown that $R_\mu$ is closed under composition, so it suffices to show that the natural transformations $\eta \otimes \iota_Y$ and $\iota_U \otimes \nu$ are
Similarly it can be shown that 

\[ (\Delta^{n-1} - e^{m} \Delta^{m-1})(\eta \otimes \tau_{Y}) \] 

for some simple \( t \in \text{supp}(\mu^{k}) \) for some \( k \geq 1 \). Then

\[ (\Delta^{n-1} - e^{m} \Delta^{m-1})(\eta \otimes \tau_{Y}) \leq d_{t}^{-1} \frac{d_{s_{1}} \cdots d_{s_{k}}}{d_{s_{1}} \cdots d_{s_{k}}} (\Delta^{n-1} - e^{m} \Delta^{m-1})(\eta \otimes \tau_{Y}) \Delta^{k-1}. \]

Observe that \((\eta \otimes \nu)_{X} = \nu_{X \otimes U} = \Delta^{n-1} - e^{m} \Delta^{m-1} \), where on the spot one has to put an \( n \)-tuple of objects. We still assume that \( U = U_{t} \) for some simple object \( U_{t} \). We obtain

\[ (\Delta^{n-1} - e^{m} \Delta^{m-1})(\eta \otimes \tau_{Y}) \leq d_{t}^{-1} \frac{d_{s_{1}} \cdots d_{s_{k}}}{d_{s_{1}} \cdots d_{s_{k}}} (\Delta^{n-1} - e^{m} \Delta^{m-1})(\eta \otimes \tau_{Y}) \Delta^{k-1}. \]

Similarly it can be shown that

\[ \| \Delta^{n-1} - e^{m} \Delta^{m-1} \|_{\mu^{\otimes n}}^{2} \rightarrow 0, \quad \text{as } m, n \rightarrow \infty. \]
If $U$ is not simple, then decomposing $U$ into simple objects and taking direct sums gives the result.

(ii) This is similar to [NY14c, §3.1]. Let $\eta \in \text{Hom}_{C^{-1}}(U,V)$ be $\mu$-harmonic. By Lemmas 5.4.3 and 5.3.11 we may assume $U = V$, since if $U \neq V$ we can consider the extension $\eta^*$ to $U \oplus V$. Again we apply the noncommutative martingale convergence theorem. Consider the sequence $(j_n(\eta))_{n=1}^\infty \subset \text{Hom}_{C^{-\infty}}(U,U)$. Recall the conditional expectations $E_{n,m}$ defined in (5.4.4). It holds that for $n \geq m$

\[
E_{n,m}(j_n(\eta)) = (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu \otimes \iota^{\otimes m}) \hat{\Delta}^{n-1}(\eta)
= (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu \otimes \iota^{\otimes m})(\iota^{\otimes n-m} \otimes \hat{\Delta}^{m-1}) \hat{\Delta}^{n-m}(\eta))
= \hat{\Delta}^{m-1}(P^{n-m}(\eta)) = \hat{\Delta}^{m-1}(\eta).
\]

Write $\hat{E}_{n,m}$ for the composition of the conditional expectation $E_{n,m}$ with the embedding $\text{Hom}_{C^{-\infty}}(U,U) \hookrightarrow \text{Hom}_{C^{-\infty}}(U,U)$. It follows from the above computation that $\hat{E}_{n,m}(j_n(\eta)) = j_m(\eta)$. Consider $\text{Hom}_{C^{-\infty}}(U,U)$ as a subalgebra of $\text{Hom}_{C^{-\infty}}(U,U)$ and denote

$\hat{E}_m : \text{Hom}_{C^{-\infty}}(U,U) \to \text{Hom}_{C^{-\infty}}(U,U) \subset \text{Hom}_{C^{-\infty}}(U,U)$

for the conditional expectation defined by $\{\hat{E}_m\}_{n \geq m}$. As we are dealing with von Neumann algebras the noncommutative martingale convergence theorem (cf. Lemma 1.3.7) shows that there exists $\tilde{\eta} \in \text{Hom}_{C^{-\infty}}(U,U)$ such that $j_n(\eta) = \hat{E}_n(\tilde{\eta})$ and $(j_n(\eta))_{n=1}^\infty$ converges in strong* topology to $\tilde{\eta}$. So $\eta$ is $\mu$-regular.

(iii) Let $\eta \in R_\mu(U,V)$. Using Lemma 5.4.3 we get

\[
\lim_n (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)(\hat{\Delta}^{n-1}(\eta)) = \lim_n (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)((\hat{\Delta}^{n-1}(\eta^e))^r)
= \lim_n (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)((\iota^{\otimes n} \otimes \text{tr}_\nu)(\hat{\Delta}^{n-1}(\eta^e)))(\iota^{\otimes n} \otimes p_U^*)
= \lim_n \text{tr}_\nu(\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)(\hat{\Delta}^{n-1}(\eta^e))p_U^*
= \text{tr}_\nu(\lim_n (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)(j_n(\eta^e)))p_U^*
= \text{tr}_\nu(\lim_n (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)(\eta^e))p_U^*.
\]

From the observation following the proof of Lemma 5.4.3 it follows that $s^* - \lim_n j_n(\eta^e)$ exists in $\text{Hom}_{C^{-\infty}}(U \oplus V, U \oplus V)$. As $\text{Hom}_{C}(U \oplus V, U \oplus V)$ is a finite dimensional $C^*$-algebra the limit (5.4.6) exists in norm.

**Definition 5.4.5.** Let $C$ be a strict $C^*$-tensor category with simple unit $\mathbb{1}$ and $\mu$ be a probability measure on the set of irreducible objects $\text{Irr}(C)$. The random walk on $C$ with Markov kernel $P_\mu$ converges to the boundary if the following two conditions hold:

(i) $K_\mu(\eta) \in R_\mu(U,V)$ for every $\eta \in \text{Nat}_{00}(\iota \otimes U, \iota \otimes V) \subset \text{Hom}_{C^{-}}(U,V)$;

(ii) for every $\eta \in \text{Nat}_{00}(\iota \otimes U, \iota \otimes V)$ and $\nu \in \text{Nat}_{b}(\iota \otimes X, \iota \otimes Y)$ with $P_\mu(\nu) = \nu$ it holds

\[
\sum_{s \in \text{Irr}(C)} d_s^2 \text{tr}_s((\eta \otimes \nu)_{U_s}) = \text{tr}_\mu^\infty(K_\mu(\eta) \otimes \nu).
\]
Here the tensor product of natural transformations is defined by (5.2.3) and the functionals \( \text{tr}_s \) and \( \text{tr}_\infty^\mu \) are defined by (5.2.9) and respectively (5.4.5). Note that both sides of (5.4.7) are in \( \text{Hom}_{\mathcal{C}}(U \otimes X, V \otimes Y) \).

Observe that requirement (ii) of this definition makes sense due to Proposition 5.4.4.

5.5 Correspondence with quantum groups

To make sure that the definition of a Martin boundary and the definition of convergence to the boundary for random walks on \( \mathcal{C}^* \)-tensor categories are sensible, we need to check that they correspond in some way to theory that already exists for random walks on discrete quantum groups. So for a compact group \( G \) one should be able to reconstruct the Martin boundary \( M(\hat{G}, \mu) \) from \( M(\text{Rep}(G), \mu) \) and vice versa. Similarly a random walk \( (\hat{G}, \mu) \) should converge to the boundary if and only if \( (\text{Rep}(G), \mu) \) converges to the boundary.

5.5.1 Duality between \( G \)-C*-algebras and categories

This subsection again contains preliminary material. We review the results of [DCY13], [Nes14] and [NY14a] which we need to prove a correspondence between the categorical picture and the quantum group picture in the next subsection.

**Definition 5.5.1.** Let \( B \) be a \( G \)-C*-algebra, with left action \( \alpha : B \to C(G) \otimes B \). The regular subalgebra of \( B \) is denoted by

\[ B := \{ x \in B : \alpha(x) \in \mathbb{C}[G] \otimes_{\text{alg}} B \}. \]

It is a dense \( * \)-subalgebra of \( B \) ([DCY13, Lem. 4.3]).

Since \( \Delta(\mathbb{C}[G]) \subset \mathbb{C}[G] \otimes_{\text{alg}} \mathbb{C}[G] \) we see that for \( x \in B \)

\[(\iota \otimes \alpha)x = (\Delta \otimes \iota)x \in \mathbb{C}[G] \otimes_{\text{alg}} \mathbb{C}[G] \otimes_{\text{alg}} B. \]

So that \( \alpha(x) \in \mathbb{C}[G] \otimes_{\text{alg}} B \) and thus \( \alpha \) defines a Hopf algebra coaction of \( \mathbb{C}[G] \) on \( B \).

**Definition 5.5.2.** If \( \mathcal{D} \) is a C*-category, define the category \( \text{End}(\mathcal{D}) \) with objects given by functors \( \mathcal{D} \to \mathcal{D} \) and morphisms \( \text{Hom}_{\text{End}(\mathcal{D})}(F, G) := \text{Nat}_{\mathcal{D}}(F, G) \). \( \mathcal{D} \) is called a \( \text{Rep}(G) \)-module category if \( \mathcal{D} \) comes equipped with a unitary tensor functor \( \text{Rep}(G) \to \text{End}(\mathcal{D}) \). If \( U \in \text{Ob}(\text{Rep}(G)) \) the induced functor in \( \text{End}(\mathcal{D}) \) is denoted \( X \mapsto X \times U \). An object \( X \in \text{Ob}(\mathcal{D}) \) is generating if for any object \( Y \in \text{Ob}(\mathcal{D}) \) there exists \( U \in \text{Rep}(G) \) such that \( Y \) is a subobject of \( X \times U \).

**Theorem 5.5.3** ([DCY13, Thm. 6.4], [Nes14, Thm. 3.3]). Let \( G \) be a reduced compact quantum group. The following two categories are equivalent:

(i) the category of unital \( G \)-C*-algebras with unital \( G \)-equivariant \( * \)-homomorphisms as morphisms;
5.5. CORRESPONDENCE WITH QUANTUM GROUPS

(ii) the category of pairs \( (\mathcal{D}, X) \), where \( \mathcal{D} \) is a \( \text{Rep}(G) \)-module \( C^* \)-category and \( X \) is a generating object in \( \mathcal{D} \) with morphisms given by equivalence classes of unitary \( \text{Rep}(G) \)-module functors respecting the designated generating objects.

The formulation of this result is taken from [NY14a, Thm. 1.1]. For later use we explicitly describe the correspondence. Given a \( G \)-\( C^* \)-algebra \( B \). Let \( \mathcal{D}' \) be the \( C^* \)-category with objects \( \text{Ob}(\text{Rep}(G)) \) but morphism sets

\[
\text{Hom}_{\mathcal{D}'}(U, V) := \{ T \in B \otimes B(\mathcal{H}_U, \mathcal{H}_V) : V_{13}^*(\alpha \otimes \iota)(T)U_{13} = 1 \otimes T \}.
\]

Let \( \mathcal{D}_B \) be the subobject and direct sum completion of \( \mathcal{D}' \). Then \( \mathcal{D}_B \) is the category corresponding to \( B \). The generating object is given by unit object \( 1 \in \text{Ob}(\text{Rep}(G)) \subset \text{Ob}(\mathcal{D}_B) \). If \( U \in \text{Ob}(\text{Rep}(G)) \), then \( \iota \otimes U \) defines a functor on \( \text{Rep}(G) \). This functor \( \iota \otimes U \) can be extended to the completion \( \mathcal{D}_B \) the extension is again denoted by \( \iota \otimes U \). The unitary tensor functor \( \text{Rep}(G) \rightarrow \text{End}(\mathcal{D}_B) \) is given by \( U \mapsto \iota \otimes U \).

Conversely, let \( \mathcal{D} \) be a \( \text{Rep}(G) \)-module category with generating object \( X \). We may assume that \( \mathcal{D} \) is equivalent to an idempotent completion of \( \text{Rep}(G) \) with some larger morphism sets and that \( X = 1 \in \text{Rep}(G) \). Indeed, let \( \mathcal{D}' \) be the category with \( \text{Hom}_{\mathcal{D}'}(U, V) := \text{Hom}_D(X \times U, X \times V) \) and take the idempotent completion. Define

\[
\mathcal{B} := \bigoplus_{s \in \text{Irr}(G)} (\mathcal{H}_s \otimes \text{Hom}(1, U_s)); \quad \hat{\mathcal{B}} := \bigoplus_{U \in \text{Rep}(G)} (\mathcal{H}_U \otimes \text{Hom}(1, U)).
\]

For every \( U \in \text{Rep}(G) \) fix isometries \( w_i : \mathcal{H}_{s_i} \rightarrow \mathcal{H}_U \) to obtain a decomposition of \( U \) in irreducibles. Define

\[
\pi : \hat{\mathcal{B}} \rightarrow \mathcal{B}, \quad \pi(\tilde{\xi} \otimes T) := \sum_i w_i^* \tilde{\xi} \otimes w_i^* T, \quad \text{for } \tilde{\xi} \otimes T \in \mathcal{H}_U \otimes \text{Hom}(1, U).
\]

Then \( \pi \) is independent of the choices of \( w_i \). The space \( \hat{\mathcal{B}} \) becomes an associative algebra with product \( \cdot \) given by

\[
(\tilde{\xi} \otimes T) \cdot (\tilde{\zeta} \otimes S) := (\xi \otimes \zeta) \otimes ((T \otimes \iota)S), \quad (5.5.1)
\]

for \( \tilde{\xi} \otimes T \in \mathcal{H}_U \otimes \text{Hom}(1, U) \) and \( \tilde{\eta} \otimes S \in \mathcal{H}_V \otimes \text{Hom}(1, V) \). We get a product on \( \mathcal{B} \) by \( \pi(x)\pi(y) := \pi(x \cdot y) \) for \( x, y \in \hat{\mathcal{B}} \). Define an antilinear map \( \cdot \) on \( \hat{\mathcal{B}} \)

\[
(\tilde{\xi} \otimes T)^* := \rho^{-1/2} \tilde{\xi} \otimes (T^* \otimes \iota)\tilde{R}_U, \quad \text{for } \tilde{\xi} \otimes T \in \mathcal{H}_U \otimes \text{Hom}(1, U). \quad (5.5.2)
\]

This map is not an involution on \( \hat{\mathcal{B}} \), but defines an involution on \( \mathcal{B} \) by \( \pi(x)^* := \pi(x^*) \) whenever \( x \in \hat{\mathcal{B}} \). Note that \( \rho^{-1/2} \tilde{\xi} = (\iota \otimes \tilde{\xi})\tilde{R}_U(1) \). There exists a left action of \( \mathbb{C}[G] \) on \( \mathcal{B} \) defined in the following way. Let \( \{ \xi_i \} \) be an orthonormal basis of \( \mathcal{H}_u \) and \( u_{ij} \) the matrix coefficients of the representation \( U \) with respect to this basis. Define

\[
\alpha(\pi(\tilde{\xi} \otimes T)) := \sum_j u_{ij} \otimes \pi(\xi_j \otimes T).
\]
It can be shown that there exists a unique completion of $\mathcal{B}$ turning it into a $C^*$-algebra $B$ such that $\alpha$ extends to a left $G$-action on $B$, see [DCY13, §4]. This is the $G$-$C^*$-algebra $B$ corresponding to the category $\mathcal{D}$.

Suppose we start with a unital $G$-$C^*$-algebra $B$. We form $\mathcal{D}_B$ and let $B'$ be the algebra corresponding to $\mathcal{D}_B$. Then there is a $*$-isomorphism [NY14a, §2.5]

$$\lambda: B' \to B, \quad \pi(\xi \otimes T) \mapsto (\iota \otimes \xi)T. \quad (5.5.3)$$

In case there is more structure present, there is again an equivalence of categories.

**Definition 5.5.4.** Let $B$ be a unital $C^*$-algebra. Assume that there exists a continuous left action $\alpha: B \to C(G) \otimes B$ of $G$ on $B$ and a continuous right action $\beta: B \to M(B \otimes c_0(\hat{G}))$ of the dual discrete quantum group $\hat{G}$ on $B$. Define the left $\mathbb{C}[G]$-module algebra structure

$$\triangleright: \mathbb{C}[G] \otimes B \to B, \quad x \triangleright a := (\iota \otimes x)\beta(a), \quad \text{for } x \in \mathbb{C}[G] \text{ and } a \in B.$$ 

Here $c_0(\hat{G})$ is identified with a subalgebra of $\mathbb{C}[G]^*$ as described by the isomorphism (1.4.7). Consider the regular subalgebra $\mathcal{B} \subset B$. Let $S$ be the antipode of the Hopf algebra $(\mathbb{C}[G], \Delta)$. The algebra $B$ is called a Yetter–Drinfeld $G$-$C^*$-algebra if

$$\alpha(x \triangleright a) = x_{(1)}a_{(1)}S(x_{(3)}) \otimes (x_{(2)} \triangleright a_{(2)}), \quad \text{for all } x \in \mathbb{C}[G] \text{ and } a \in \mathcal{B}. \quad (5.5.4)$$

Here Sweedler’s sumless notation is used, so $\Delta(x) = x_{(1)} \otimes x_{(2)}$ and $\alpha(a) = a_{(1)} \otimes a_{(2)}$. A Yetter–Drinfeld $G$-$C^*$-algebra $B$ is called braided-commutative whenever

$$ab = b_{(2)}(S^{-1}(b_{(1)}) \triangleright a), \quad \text{for all } a, b \in \mathcal{B}. \quad (5.5.5)$$

**Theorem 5.5.5** ([NY14a, Thm. 2.1]). Let $G$ be a reduced compact quantum group. The following two categories are equivalent:

(i) the category of unital braided-commutative Yetter–Drinfeld $G$-$C^*$-algebras with unital $G$- and $G$-equivariant $*$-homomorphisms as morphisms;

(ii) the category of pairs $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is a $C^*$-tensor category and $\mathcal{E}: \text{Rep}(G) \to \mathcal{C}$ is a unitary tensor functor such that $\mathcal{C}$ is generated by the image of $\mathcal{E}$. The morphisms $(\mathcal{C}, \mathcal{E}) \to (\mathcal{C}', \mathcal{E}')$ in this category are given by the set of equivalence classes of pairs $(\mathcal{F}, \eta)$ where $\mathcal{F}: \mathcal{C} \to \mathcal{C}'$ is a unitary tensor functor and $\eta: \mathcal{F}\mathcal{E} \to \mathcal{E}'$ is a natural unitary monoidal isomorphism.

The correspondence between these two categories is the same as the correspondence given by Theorem 5.5.3, but one needs to account for the extra structure present. So given a $C^*$-tensor category, one needs to define the $\mathbb{C}[G]$-module structure on the algebra. Denote $\mathbb{C}[[G]] := \bigoplus_U (\hat{H}_U \otimes H_U)$ and let

$$\pi_G: \mathbb{C}[[G]] \to \mathbb{C}[G], \quad \pi_G(\xi \otimes \zeta) := (\iota \otimes (\xi, \zeta))(U) \quad \text{for } \xi \otimes \zeta \in \hat{H}_U \otimes H_U.$$
5.5. CORRESPONDENCE WITH QUANTUM GROUPS

Define \( \triangleright : \widehat{\mathbb{C}[G]} \otimes \hat{\mathcal{B}} \to \hat{\mathcal{B}} \) by letting for \( \xi \otimes \zeta \in \hat{\mathcal{H}}_U \otimes \mathcal{H}_U \) and \( \eta \otimes T \in \hat{\mathcal{H}}_V \otimes \text{Hom}_C(1, V) \):

\[
(\xi \otimes \zeta) \triangleright (\eta \otimes T) := (\xi \otimes \eta \otimes \bar{\rho}^{-1/2} \zeta) \otimes (\iota \otimes T \otimes \iota) \bar{R}_U \in \hat{\mathcal{H}}_{U \times V \times U} \otimes \text{Hom}_C(1, U \times V \times U).
\]

Identify \( \mathbb{C}[G] \) with \( \bigoplus_s (\mathcal{H}_s \otimes \mathcal{H}_s) \subset \widehat{\mathbb{C}[G]} \), where \( \bar{\xi} \otimes \bar{\zeta} \in \mathcal{H}_s \otimes \mathcal{H}_s \) corresponds to the matrix coefficient \( (\iota \otimes (\xi, \zeta))(U) \) of \( U \). We define \( \triangleright : \mathbb{C}[G] \otimes \mathcal{B} \to \mathcal{B} \) by

\[
u \triangleright x := \pi(\nu \triangleright x), \quad \text{for } \nu \in \mathbb{C}[G] \text{ and } x \in \mathcal{B}.
\]

This action gives \( B_C \) the braided-commutative Yetter–Drinfeld structure.

For the converse direction we need to describe the tensor product. Let \( B \) be a unital braided-commutative Yetter–Drinfeld \( G \)-\( C^* \)-algebra. Using the construction above we obtain a \( \text{Rep}(G) \)-module \( C^* \)-category \( \mathcal{C}_B \). The tensor product can be described as follows: if \( U, V \in \text{Ob}(\text{Rep}(G)) \subset \text{Ob}(\mathcal{C}_B) \), the tensor product \( U \otimes V \) in \( \mathcal{C}_B \) is defined as in \( \text{Rep}(G) \).

For morphisms \( S \in \text{Hom}_C(U, V) \) and \( T = \sum_i b_i \otimes T_i \in \text{Hom}_C(W, X) \), define

\[
u_U \otimes T := \sum_{i,k,l} (u_{kl} \triangleright b_i) \otimes m_{kl} \otimes T_i \in \text{Hom}_C(U \times W, U \times X);
\]

\[
u \otimes T := \text{Hom}(S \otimes T, (\iota_U \otimes T)), \quad \text{which equals } (\iota_V \otimes T)(S \otimes \iota_W). \quad \text{The functor } \mathcal{E}_B : \text{Rep}(G) \to \mathcal{C}_B \text{ is given as the identity map on objects and } T \mapsto 1_B \otimes T \text{ whenever } T \in \text{Hom}_{\text{Rep}(G)}(U, V).
\]

Example 5.5.6 ([NY14a, §3.3]). We consider the example of \( \ell^\infty(\hat{G}) \). We will use the results of this specific example later on. \( \ell^\infty(\hat{G}) \) comes equipped with a left \( G \)-action given by the adjoint action \( \alpha_l \) and with a right \( \hat{G} \)-action given by the comultiplication \( \hat{\Delta} \). The adjoint action is an action of von Neumann algebras, so it is not continuous in norm.

To define a \( C^* \)-algebraic action of \( C(G) \) we need a smaller algebra. Consider the regular algebra

\[
\ell^\infty_{\text{alg}}(\hat{G}) := \{ x \in \ell^\infty(\hat{G}) : \alpha_l(x) \in \mathbb{C}[G] \otimes_{\text{alg}} \ell^\infty(\hat{G}) \}
\]

and let \( C_{-1}(\hat{G}) \) be the unique completion of \( \ell^\infty_{\text{alg}}(\hat{G}) \) to which \( \alpha_l \) extends as described above. Then \( C_{-1}(\hat{G}) \) is a unital braided-commutative Yetter–Drinfeld \( G \)-\( C^* \)-algebra with the \( G \)-action and \( \mathbb{C}[G] \)-module structure given by

\[
\alpha_l : C_{-1}(\hat{G}) \to C(G) \otimes C_{-1}(\hat{G}),
\]

\[
u \triangleright x := (\iota \otimes u) \hat{\Delta}(x) \quad \text{for } x \in C_{-1}(\hat{G}), \nu \in \mathbb{C}[G].
\]  \((5.5.6)\)

To show that this \( \mathbb{C}[G] \)-module structure extends from \( \ell^\infty_{\text{alg}}(\hat{G}) \) to \( C_{-1}(\hat{G}) \) it suffices to prove that \( \hat{\Delta}(x) \in \ell^\infty_{\text{alg}}(\hat{G}) \otimes \ell^\infty(\hat{G}) \) whenever \( x \in \ell^\infty_{\text{alg}}(\hat{G}) \). This holds, because if \( u \in \mathbb{C}[G] \), then the Yetter–Drinfeld condition \((5.5.4)\) implies

\[
\alpha_l((\iota \otimes u) \hat{\Delta}(x)) = \alpha_l(u \triangleright x) = u_{(1)} x_{(1)} S(u_{(3)}) \otimes u_{(2)} \triangleright x_{(2)} \in \mathbb{C}[G] \otimes_{\text{alg}} \ell^\infty(G).
\]

Here \( \alpha_l(x) = x_{(1)} \otimes x_{(2)} \) and \( \Delta(u) = u_{(1)} \otimes u_{(2)} \). It can now be shown that the defining

\footnote{The reason why we use this notation \( C_{-1}(\hat{G}) \) will become clear in §5.5.2.}
identities of Definition 5.5.4 are satisfied. The corresponding category is the completion of the category with objects \( \text{Ob}(\text{Rep}(G)) \). The morphisms between \( U \) and \( V \) in this new category are given by the space of natural bounded transformations \( \text{Nat}_s(\iota \otimes U, \iota \otimes V) \).

This category will be described in more detail in the next subsection.

### 5.5.2 The correspondence with discrete quantum groups

We will show that the Martin boundary and Martin compactification of a random walk on a discrete quantum group define braided-commutative Yetter–Drinfeld \( G \)-C*-algebras. Therefore by the duality described in Subsection 5.5.1 these algebras define C*-tensor categories. These categories are shown to be unitarily monoidally equivalent to the previously defined categorical Martin boundary and compactification of \( \text{Rep}(G) \). The results in Section 5.1 above suggest that convergence to the boundary for random walks on discrete quantum groups is a property from the underlying category \( \text{Rep}(G) \) and not from the actual realization via a fiber functor \( \text{Rep}(G) \to \text{Hilb}_q \). We show that this is indeed the case.

**Notation 5.5.7.** Recall the left adjoint action \( \alpha_l \) defined by (1.4.17). Define the space

\[
\Big( \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \Big)_{\text{alg}} := \left\{ x \in \bigotimes_{-n}^{-1} l^\infty(\hat{G}) : \alpha_l(x) \in \mathbb{C}[G] \otimes_{\text{alg}} \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \right\}
\]

and denote \( C_{-n}(\hat{G}) \) for the norm-closure of \( \left( \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \right)_{\text{alg}} \) in \( \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \). The restriction \( \alpha_l : C_{-n}(\hat{G}) \to C(G) \otimes C_{-n}(\hat{G}) \) defines an action of C*-algebras. Indeed, from the observation following Definition 5.5.1 we obtain that \( \alpha_l(x) \in \mathbb{C}[G] \otimes_{\text{alg}} \left( \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \right)_{\text{alg}} \) if \( x \in \left( \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \right)_{\text{alg}} \). Since \( \alpha_l \) is continuous in norm, it extends to a map of the norm closures, again denoted by \( \alpha_l : C_{-n}(\hat{G}) \to C(G) \otimes C_{-n}(\hat{G}) \). As \( (\varepsilon \otimes \iota)\alpha_l = \iota \) on \( \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \) this identity also holds on \( \left( \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \right)_{\text{alg}} \). It follows now from [NT04, Cor. 1.4] that \( \alpha_l \) is a left \( G \)-action on the C*-algebra \( C_{-n}(\hat{G}) \).

We already know (cf. Example 5.5.6) that \( C_{-1}(\hat{G}) \) contains more structure. It is a unital braided-commutative Yetter–Drinfeld \( G \)-C*-algebra with the right action of \( \hat{G} \) given by the comultiplication \( \hat{\Delta} \).

Recall the categorical path spaces of Notation 5.2.1. We specialise to \( \mathcal{C} = \text{Rep}(G) \).

**Lemma 5.5.8.** For \( \mathcal{C} = \text{Rep}(G) \) and \( U,V \) finite dimensional unitary representations of \( G \) it holds that

\[
\text{Hom}_{\text{Rep}(G)-\text{alg}}(U,V) \cong \left\{ T \in \left( \bigotimes_{-n}^{-1} l^\infty(\hat{G}) \right) \otimes B(\mathcal{H}_U,\mathcal{H}_V) : V_{13}^*(\alpha_l \otimes \iota)(T)U_{13} = 1 \otimes T \right\}
\]

The isomorphism is explicitly given by \( \eta(T) \leftrightarrow T \), where

\[
\eta(T)_{x_1,...,x_n} := T|_{\mathcal{H}_{X_1} \otimes \cdots \otimes \mathcal{H}_{X_n} \otimes \mathcal{H}_U}
\]

and \( \mathcal{H}_{X_i} \) is considered as \( \mathcal{H}_{X_i} = \bigoplus_s m^s_{X_i} \mathcal{H}_s \).
Proof. Note that \( T \in (\bigotimes_n^{-1} l^\infty(G)) \otimes B(\mathcal{H}_U, \mathcal{H}_V) \) satisfies
\[
V_{13}^*(\alpha_l \otimes \iota_V)(T)U_{13} = 1 \otimes T
\]
if and only if
\[
(1 \otimes T)(W \times \cdots \times W \times U) = (W \times \cdots \times W \times V)(1 \otimes T),
\]
so if and only if \( T \) intertwines the representations \((W^\times n \times U)\) and \((W^\times n \times V)\). Since any finite dimensional representation embeds in the regular representation coming from the multiplicative unitary \( W \) the above holds if and only if
\[
(1 \otimes T|_{\mathcal{H}_{X_1} \otimes \cdots \otimes \mathcal{H}_{X_n} \otimes \mathcal{H}_U})(X_1 \times \cdots \times X_n \times U) = (X_1 \times \cdots \times X_n \times V)(1 \otimes T|_{\mathcal{H}_{X_1} \otimes \cdots \otimes \mathcal{H}_{X_n} \otimes \mathcal{H}_U})
\]
for all finite dimensional representations \( X_1, \ldots, X_n \). Which states exactly that
\[
T|_{\mathcal{H}_{X_1} \otimes \cdots \otimes \mathcal{H}_{X_n} \otimes \mathcal{H}_U} \in \text{Hom}_{\text{Rep}(G)}(X_1 \times \cdots \times X_n \times U, X_1 \times \cdots \times X_n \times V),
\]
for all \( X_1, \ldots, X_n \in \text{Ob}(\text{Rep}(G)) \). Now if each \( X_i \) and \( Y_i \) are irreducible representations and \( f_i : X_i \to Y_i \) is a morphism in \( \text{Rep}(G) \). Then \( f_i \) is a multiple of the identity if \( X_i \simeq Y_i \) or zero if \( X_i \not\simeq Y_i \). So clearly the following diagram commutes
\[
\begin{array}{ccc}
X_1 \times \cdots \times X_n \times U & \xrightarrow{(\eta \tau)_{X_1,\ldots,X_n}} & Y_1 \times \cdots \times Y_n \times U \\
\downarrow & & \downarrow \\
X_1 \times \cdots \times X_n \times V & \xrightarrow{f_1 \otimes \cdots \otimes f_n \otimes \iota_V} & Y_1 \times \cdots \times Y_n \times V
\end{array}
\]
If \( X_i \) and \( Y_i \) are not necessarily simple then by decomposing the representations into irreducible ones it follows that the above diagram again commutes. Therefore we conclude that \( T \) satisfies (5.5.8) if and only if \( \eta(T) \in \text{Nat}_b(\iota^\otimes n \otimes U, \iota^\otimes n \otimes V) \). Thus \( \eta(T) \in \text{Hom}_{\text{Rep}(G)_{\otimes n}}(U, V) \).

Clearly \( T \mapsto \eta(T) \) is an injective \(*\)-homomorphism. It thus remains to show that it is surjective. For this suppose \( \eta \in \text{Hom}_{\text{Rep}(G)_{\otimes n}}(U, V) \). Then \( \eta_{X_1,\ldots,X_n} \in \text{Hom}_{\text{Rep}(G)}(X_1 \otimes \cdots \otimes X_n \otimes U, X_1 \otimes \cdots \otimes X_n \otimes V) \). If we write \( T = \bigoplus_{s_1,\ldots,s_n} \eta_{U_{s_1,\ldots,U_{s_n}}} \), then it is immediate that \( \eta = \eta(T) \) and the previous computations imply that \( T \) satisfies (5.5.8).

Let \( U \) and \( V \) be unitary representations. Decompose \( V \) as \( V = \sum_{i,j=1}^{\dim V} v_{ij} \otimes m_{ij}^V \in C[G] \otimes B(\mathcal{H}_V) \) and similarly decompose \( U \). Observe that (5.5.8) can be written as
\[
(\alpha_l \otimes \iota)(T) = V_{13}(1 \otimes T)U_{13} = \sum_{i,j=1}^{\dim V} \sum_{k,l=1}^{\dim U} v_{ij} u_{kl}^* \otimes ((m_{ij}^V \otimes 1)T((m_{kl}^U)^* \otimes 1)),
\]
which lies in \( C[G] \otimes_{\text{alg}} (\bigotimes_n^{-1} l^\infty(G)) \otimes B(\mathcal{H}_U, \mathcal{H}_V) \). It follows that if \( T \) satisfies (5.5.8), then \( T \in (\bigotimes_n^{-1} l^\infty(G))_{\text{alg}} \otimes B(\mathcal{H}_U, \mathcal{H}_V) \). From the discussion following Theorem 5.5.3 we can now immediately conclude the following.

Corollary 5.5.9. For a reduced compact quantum group \( G \) the unital \( G\text{-C}^*\)-algebra corres-
Corresponding to the $C^*$-category $\text{Rep}(G)_{-n}$ via the equivalence in Theorem 5.5.3 is isomorphic to $C_{-n}(\hat{G})$. Or equivalently the $C^*$-category corresponding to the unital $G$-$C^*$-algebra $C_{-n}(\hat{G})$ is unitarily equivalent to the category $\text{Rep}(G)_{-n}$ as a right $\text{Rep}(G)$-module category. For $n = 1$ we have unitarily monoidal equivalence of $C^*$-tensor categories.

In a similar spirit as (5.5.9) we can write the tensor product in $\text{Rep}(G)_{-1}$ as follows. Suppose $\eta = \eta(T)$ and $\nu = \eta(S)$ (see (5.5.7)) for some $T = \sum_i x_i \otimes T_i \in l^\infty(\hat{G}) \otimes B(\mathcal{H}_{U_1}, \mathcal{H}_{U_2})$ and $S = \sum_j y_j \otimes S_j \in l^\infty(\hat{G}) \otimes B(\mathcal{H}_{V_1}, \mathcal{H}_{V_2})$. Then (5.2.3) translates to

$$\eta \otimes \nu = \sum_{i,j} x_i y_j^{(1)} \otimes T_i \pi_{U_1}(y_j^{(2)}) \otimes S_j.$$  

(5.5.10)

Lemma 5.5.10. For $\mathcal{C} = \text{Rep}(G)$, the isomorphism (5.5.7) satisfies

1. $(\iota \otimes \eta(T)) = \eta(1 \otimes T);
2. (\iota^{\otimes m} \otimes \hat{\Delta} \otimes \iota^{\otimes n-m-1})(\eta(T)) = \eta((\iota^{\otimes m} \otimes \hat{\Delta} \otimes \iota^{\otimes n-m-1} \otimes \iota_{U})(T));
3. (\text{tr}_\mu \otimes \iota^{\otimes n-1})(\eta(T)) = \eta((\varphi_\mu \otimes \iota^{\otimes n-1} \otimes \iota_{V})(T)).$

If $n = 1$, then in addition we have

4. $P_\mu(\eta(T)) = \eta((P_\mu \otimes \iota)(T));$
5. $K_\mu(\eta(T)) = \eta((K_\mu \otimes \iota)(T)).$

Here the first $P_\mu$ is the Markov operator on $\text{Rep}(G)$, while the one on the right is the Markov operator on $\hat{G}$ and similarly for $K_\mu$.

Proof. Assume that $T \in (\bigotimes_{-1}^{-1} l^\infty(\hat{G})) \otimes B(\mathcal{H}_U, \mathcal{H}_V)$ satisfies (5.5.8). We compute

$$(\iota \otimes \eta(T))_{X_1, \ldots, X_{n+1}} = \iota_{U_{X_1}} \otimes \eta(T)_{X_2, \ldots, X_{n+1}} = \iota_{U_{X_1}} \otimes T |_{\mathcal{H}_{X_2} \otimes \cdots \otimes \mathcal{H}_{X_{n+1}} \otimes \mathcal{H}_U} = \eta(1 \otimes T)_{X_1, \ldots, X_n}.$$  

For the second, let $\hat{X}$ be a finite dimensional representation of $G$ and $\hat{a} \in \text{Hom}(\hat{X}, X_{m+1} \otimes X_{m+2})$. Write $a := \iota^{\otimes m} \otimes \hat{a} \otimes \iota^{\otimes n-m-2}$ and $k := n - m - 1$. Then by definition of $\hat{\Delta}$ and naturality of $\eta(T)$

$$(\iota^{\otimes m} \otimes \hat{\Delta} \otimes \iota^{\otimes k})(\eta(T))_{X_1, \ldots, X_{n+1}} \circ (a \otimes \iota_{U})$$

$= \eta(T)_{X_1, \ldots, X_m, X_{m+1} \otimes X_{m+2} \otimes \cdots \otimes X_{n+1}} \circ (a \otimes \iota_{U})$

$= (a \otimes \iota_{U}) \circ \eta(T)_{X_1, \ldots, X_m, X_{m+1} \otimes \cdots \otimes X_{n+1}}$

$= (a \otimes \iota_{U}) \circ (\pi_{X_1} \otimes \cdots \otimes \pi_{X_m} \otimes \pi_{X_{m+1}} \circ \cdots \otimes \pi_{X_{n+1}} \otimes \pi_U)(T)$

$= (\iota^{\otimes m} \otimes \hat{\Delta} \otimes \iota^{\otimes k} \otimes \iota_{U})((\pi_{X_1} \otimes \cdots \otimes \pi_{X_{n+1}} \otimes \pi_U)(T)) \circ (a \otimes \iota_{U})$

$= \eta((\iota^{\otimes m} \otimes \hat{\Delta} \otimes \iota^{\otimes k} \otimes \iota_{U})(T))_{X_1, \ldots, X_{n+1}} \circ (a \otimes \iota_{U}).$

As this holds for all $\hat{a}$, we get the second identity. To establish the third one it suffices to deal with $\mu = \delta_s$ for some $s \in \text{Irr}(G)$. As $S \in \text{End}_{\text{Rep}(G)}(U_s) \subset B(\mathcal{H}_s)$, it holds that.
tr_s(S) = \varphi_s(S) (see the observation following Definition 1.5.16). We obtain
\[(\text{tr} \otimes \iota^{\otimes n-1})(\eta(T))_{x_1,\ldots,x_{n-1}} = (\varphi_s \otimes \iota^{\otimes n-1} \otimes \iota_V)|_{\mathcal{H}_s \otimes \mathcal{H}_s \otimes \cdots \otimes \mathcal{H}_{x_{n-1}} \otimes \mathcal{H}_U} = \eta((\varphi_s \otimes \iota^{\otimes n-1} \otimes \iota_V)(T))_{x_1,\ldots,x_{n-1}}.\]

Now the identity with \(P_\mu\) is obvious from the (ii) and (iii). To prove the last one, observe that from (iv) it follows that \(\eta((G_\mu \otimes \iota)(T)) = G_\mu(\eta(T))\). Moreover \(\kappa^{0,U} = \eta(I_0 \otimes \iota_U)\).

Therefore
\[K_\mu(\eta(T)) = G_\mu(\eta(T))(G_\mu(\kappa^{0,U}))^{-1} = G_\mu(\eta(T))(G_\mu(\eta(I_0 \otimes \iota_U)))^{-1} = \eta((G_\mu \otimes \iota)(T))(G_\mu(I_0)^{-1} \otimes \iota_U)) = \eta((K_\mu \otimes \iota)(T)),\]
as desired. \(\square\)

We apply the results from §5.5.1 to reconstruct the length \(n\) path spaces \(\bigotimes_{-1}^{-n} l^\infty(G)\).

Define the algebras
\[\mathcal{B}_{-n}(\hat{G}) := \bigoplus_{s \in \text{Irr}(G)} \mathcal{H}_s \otimes \text{Hom}_{\text{Rep}(G)_{-n}}(1, U_s);\]
\[\hat{\mathcal{B}}_{-n}(\hat{G}) := \bigoplus_{U \in \text{Ob}(G)} \mathcal{H}_U \otimes \text{Hom}_{\text{Rep}(G)_{-n}}(1, U)\]
and denote
\[\lambda_n: \mathcal{B}_{-n}(\hat{G}) \to C_{-n}(\hat{G}), \quad \pi(\tilde{\xi} \otimes \eta) \mapsto (\iota \otimes \tilde{\xi})\eta. \quad (5.5.11)\]

Let \(B_{-n}(\hat{G})\) be the unique completion of \(\mathcal{B}_{-n}(\hat{G})\) to a \(G\)-\(C^*\)-algebra such that \(\lambda_n\) extends to a \(G\)-equivariant isomorphism \(\lambda_n: B_{-n}(\hat{G}) \to C_{-n}(\hat{G})\) (cf. (5.5.3)).

Let us consider the case \(n = 0\). Put \(B = \mathbb{C}\), this is a unital braided-commutative Yetter–Drinfeld \(G\)-\(C^*\)-algebra with trivial left and right actions. The associated \(C^*\)-tensor category \(C_B\) equals \(\text{Rep}(G)\). Indeed, for objects \(U, V \in \text{Rep}(G) \subset \text{Ob}(C_B)\)
\[\text{Hom}_{C_B}(U, V) = \{T \in B \otimes B(\mathcal{H}_U, \mathcal{H}_V) : V_{13}^*(\alpha \otimes \iota)(T)U_{13} = 1 \otimes T\} \cong \{S \in B(\mathcal{H}_U, \mathcal{H}_V) : (\iota \otimes S)U = V(\iota \otimes S)\}\]
\[= \text{Hom}_{\text{Rep}(G)}(U, V).\]
So put \(\text{Rep}(G)_0 := \text{Rep}(G)\) and \(\mathcal{B}_0(\hat{G}) := \bigoplus_{s \in \text{Irr}(G)} \mathcal{H}_s \otimes \text{Hom}_{C_B}(1, U_s) \cong \text{End}_{\text{Rep}(G)}(1)\).

Denote the isomorphism
\[\lambda_0: \mathcal{B}_0(\hat{G}) \to \mathbb{C}, \quad \pi(\tilde{\xi} \otimes T) \mapsto (\iota \otimes \tilde{\xi})T. \quad (5.5.12)\]
The algebra \(\mathcal{B}_0(\hat{G})\) is already complete, but to be consistent in notation we write \(\mathcal{B}_0(\hat{G}) = \hat{\mathcal{B}}_0(\hat{G})\).
Lemma 5.5.11. The map $\lambda_1$ restricts to $G$-equivariant $\ast$-isomorphisms

$$
\lambda_1: \bigoplus_t \bar{H}_t \otimes \text{Nat}_0(\iota \otimes I, \iota \otimes U_t) \to c_0(\hat{G}); \tag{5.5.13}
$$

$$
\lambda_1: \bigoplus_t \bar{H}_t \otimes \text{Nat}_0(\iota \otimes I, \iota \otimes U_t) \to c_0(\hat{G}). \tag{5.5.14}
$$

Proof. $B(\mathcal{H}_s)$ is a unital $G$-$C^\ast$-algebra with left action $\alpha_{l,s}$ given by the restriction of the left adjoint action $\alpha_l$, so $\alpha_{l,s}(x) := U_s^*(1 \otimes x)U_s$. Form the $\text{Rep}(G)$-module category $\mathcal{C}_{B(\mathcal{H}_s)}$ as described by Theorem 5.5.3. From this theorem it follows that $\lambda$ is an $\ast$-isomorphism. Since

$$
\text{Hom}_{\mathcal{C}_{B(\mathcal{H}_s)}}(U, V) = \{ T \in B(\mathcal{H}_s) \otimes B(\mathcal{H}_U, \mathcal{H}_V) : U_{13}^* (\alpha_{l,s} \otimes \iota)(T) V_{13} = 1 \otimes T \}
$$

$$
\cong \{ \eta \in \text{Nat}_0(\iota \otimes U, \iota \otimes V) : \text{supp}(\eta) \subset \{ s \} \},
$$

we obtain that the function $\lambda := \bigoplus_s \lambda_s$ acting as

$$
\lambda: \bigoplus_s \bigoplus_t \bar{H}_t \otimes \text{Nat}_0(\iota \otimes I, \iota \otimes U_t) \to \bigoplus_s B(\mathcal{H}_s)
$$

is an isomorphism. But this map equals exactly $\lambda_1$ of (5.5.13). Taking closures gives the second isomorphism. \(\Box\)

If $\mathcal{F}: \text{Rep}(G)_{-n} \to \text{Rep}(G)_{-k}$ is a functor of module categories, we obtain maps

$$
\bar{H}_s \otimes \text{Hom}_{\text{Rep}(G)_{-n}}(1, U_s) \to \bar{H}_s \otimes \text{Hom}_{\text{Rep}(G)_{-k}}(1, U_s), \quad \xi \otimes \eta \mapsto \xi \otimes \mathcal{F}(\eta).
$$

Taking direct sums and passing to the completion gives a $\ast$-morphism of $C^\ast$-algebras (see [DCY13, Prop. 4.5]) we denote it by

$$
(\iota_{\mathcal{R}} \otimes \mathcal{F}): B_{-n}(\hat{G}) \to B_{-k}(\hat{G}). \tag{5.5.15}
$$

Similarly for $\text{tr}_\mu$ we obtain positive maps

$$
\bar{H}_s \otimes \text{Hom}_{\text{Rep}(G)_{-n}}(1, U_s) \to \bar{H}_s \otimes \text{Hom}_{\text{Rep}(G)_{-(n-1)}}(1, U_s), \quad \xi \otimes \eta \mapsto \xi \otimes (\text{tr}_\mu \otimes \iota^{n-1})(\eta).
$$

By taking direct sums and completion these maps extended to positive maps

$$
(\iota_{\mathcal{R}} \otimes \text{tr}_\mu \otimes \iota^{n-1}): B_{-n}(G) \to B_{-(n-1)}(G). \tag{5.5.16}
$$

In fact, using the functor $(\iota \otimes \cdot)$ one can embed $\bar{H}_s \otimes \text{Hom}_{\text{Rep}(G)_{-(n-1)}}(1, U_s) \hookrightarrow \bar{H}_s \otimes \text{Hom}_{\text{Rep}(G)_{-n}}(1, U_s)$ and thus one obtains an embedding $B_{-(n-1)}(G) \hookrightarrow B_{-n}(G)$. The map $(\iota_{\mathcal{R}} \otimes \text{tr}_\mu \otimes \iota^{n-1})$ defines a conditional expectation $B_{-n}(G) \to B_{-(n-1)}(G)$.

Corollary 5.5.12. The following identities hold on $B_{-n}(\hat{G})$ for $n \geq 1$:
5.5. CORRESPONDENCE WITH QUANTUM GROUPS

(i) \( \lambda_{n+1} \circ (\iota_{\hat{R}} \otimes \iota \otimes \cdot) = (1 \otimes \cdot) \circ \lambda_n; \)

(ii) \( \lambda_{n+1} \circ (\iota_{\hat{R}} \otimes \hat{\Delta} \otimes \iota^{n-1}) = (\hat{\Delta} \otimes \iota^{n-1}) \circ \lambda_n; \)

(iii) \( \lambda_{n-1} \circ (\iota_{\hat{R}} \otimes \text{tr}_\mu \otimes \iota^{n-1}) = (\varphi_\mu \otimes \iota^{n-1}) \circ \lambda_n, \)

where the *-morphisms \((\iota_{\hat{R}} \otimes \iota \otimes \cdot), (\iota_{\hat{R}} \otimes \hat{\Delta} \otimes \iota^{n-1})\) are defined by (5.5.15) and \((\iota_{\hat{R}} \otimes \text{tr}_\mu \otimes \iota^{n-1})\) by (5.5.16). In particular it holds that

(iv) \( \lambda_1 \circ (\iota_{\hat{R}} \otimes P_\mu) = P_\mu \circ \lambda_1; \)

(v) \( \lambda_1 \circ (\iota_{\hat{R}} \otimes K_\mu) = K_\mu \circ \lambda_1. \)

**Proof.** This is more a matter of notation than actually something new. The key part is Lemma 5.5.10. Let \( \xi \otimes \eta \in \hat{\mathcal{H}}_s \otimes \text{Hom}_{\text{Rep}(G)}(\mathbb{1}, U_s) \subset B_{-n}(\hat{G}) \subset B_{-n}(\hat{G}). \) By Lemma 5.5.8 we may assume that \( \eta \) is of the form \( \eta(T) \) for some \( T \in (\otimes_{-n}^1 \mathcal{L}^\infty(G)) \otimes B(\mathbb{C}, \mathcal{H}_s). \) Using Lemma 5.5.10 we conclude

\[
\lambda_{n+1}((\iota_{\hat{R}} \otimes \hat{\Delta} \otimes \iota^{n-1})(\xi \otimes \eta(T)))|_{s_1 \otimes \cdots \otimes s_{n+1}} \\
= (\xi^{n+1} \otimes \hat{\xi})(((\hat{\Delta} \otimes \iota^{n-1}) \circ \lambda_n)(\eta(T)))|_{s_1 \otimes \cdots \otimes s_{n+1}} \\
= (\hat{\Delta} \otimes \iota^{n-1})(\eta(\lambda_n(\xi \otimes \eta(T))))|_{s_1 \otimes \cdots \otimes s_{n+1}} \\
= (\hat{\Delta} \otimes \iota^{n-1})(\lambda_n(\xi \otimes \eta(T)))|_{s_1 \otimes \cdots \otimes s_{n+1}}.
\]

Identities (ii) and (iii) can be verified in an analogous way and (iv) follows immediately from the definition of \( P_\mu \) (see Definition 5.2.5).

(v) Consider \( \bar{1} \otimes \kappa^{0,1} \) as an element of

\[ \mathbb{C} \otimes \text{Hom}_{\text{Rep}(G)}(\mathbb{1}, \mathbb{1}) \subset \bigoplus_s \hat{\mathcal{H}}_s \otimes \text{Hom}_{\text{Rep}(G)}(\mathbb{1}, U_s) = B_{-1}(\hat{G}). \]

Then \( \lambda_1(\bar{1} \otimes \kappa^{0,1}) = I_0 \in B(\mathcal{H}_0). \) By linearity in \( \mu \) we see from (iv) that \( \lambda_1 \circ (\iota_{\hat{R}} \otimes G_\mu) = G_\mu \circ \lambda_1. \) Now take \( \bar{\xi} \otimes \eta \in \hat{\mathcal{H}}_s \otimes \text{Nat}_{00}(\iota \otimes \mathbb{1}, \iota \otimes U_s) \) for some \( s. \) We get

\[
K_\mu \circ \lambda_1(\bar{\xi} \otimes \eta) = G_\mu(\lambda_1(\bar{\xi} \otimes \eta))G_\mu(I_0)^{-1} \\
= G_\mu(\lambda_1(\bar{\xi} \otimes \eta))G_\mu(\lambda_1(\bar{1} \otimes \kappa^{0,1}))^{-1} \\
= \lambda_1((\iota_{\hat{R}} \otimes G_\mu)(\bar{\xi} \otimes \eta)) (\lambda_1((\iota_{\hat{R}} \otimes G_\mu)(\bar{1} \otimes \kappa^{0,1})))^{-1} \\
= \lambda_1((\bar{\xi} \otimes G_\mu(\eta)) \cdot (\bar{1} \otimes G_\mu(\kappa^{0,1}))) \\
= \lambda_1(\bar{\xi} \otimes G_\mu(\eta))G_\mu(\kappa^{0,1})^{-1} \\
= \lambda_1(\bar{\xi} \otimes K_\mu(\eta)),
\]

as desired. \( \square \)

Suppose that \( \mu \) is a generating and transient probability measure on \( \text{Irr}(G). \) The action \( \alpha_\iota \) defines adjoint actions on the Martin compactification \( \hat{M}(\hat{G}, \mu) \) and Martin boundary
M(\hat{G}, \mu)$ (cf. Proposition 3.2.15). Denote $M(\hat{G}, \mu)_{\text{alg}} := \hat{M}(\hat{G}, \mu) \cap l^\infty_{\text{alg}}(\hat{G})$. This is a $\ast$-algebra which is norm-dense in $\hat{M}(\hat{G}, \mu)$ (see Subsection 5.5.1). For the Martin boundary we consider

$$M(\hat{G}, \mu)_{\text{alg}} := \hat{M}(\hat{G}, \mu)_{\text{alg}}/(c_0(\hat{G}) \cap l^\infty_{\text{alg}}(\hat{G})) = \{ x \in M(\hat{G}, \mu) : \alpha_l(x) \in \mathbb{C}[\hat{G}] \otimes_{\text{alg}} M(\hat{G}, \mu) \}.$$

Then again $M(\hat{G}, \mu)_{\text{alg}}$ is norm-dense in $M(\hat{G}, \mu)$.

**Lemma 5.5.13.** The $C^*$-algebras $\hat{M}(\hat{G}, \mu)$ and $M(\hat{G}, \mu)$ are unital braided-commutative Yetter–Drinfeld $G$-$C^*$-algebras. The left action of $G$ on $\hat{M}(\hat{G}, \mu)$ is given by the restriction of $\alpha_l$ to $\hat{M}(\hat{G}, \mu)$. The left $\mathbb{C}[G]$-module structure $\triangleright$ is defined by the restriction of (5.5.6) to $\hat{M}(\hat{G}, \mu)$. Both actions factor through $M(\hat{G}, \mu)$.

**Proof.** By definition $\hat{M}(\hat{G}, \mu)$ and $M(\hat{G}, \mu)$ are $C^*$-algebras. They are unital, because $K_{\hat{G}}(I_0) = G_\mu(I_0)G_{\hat{G}}(I_0)^{-1} = 1$. The mappings $\alpha_l$ and $\hat{\Delta}$ define a left $G$-action and respectively a right $\hat{G}$-action on both $\hat{M}(\hat{G}, \mu)$ and $M(\hat{G}, \mu)$ (Proposition 3.2.15). Thus $\hat{M}(\hat{G}, \mu)$ and $M(\hat{G}, \mu)$ are closed under $\alpha_l$ and $\triangleright$. As $C_{-1}(\hat{G})$ is a unital braided-commutative Yetter–Drinfeld $G$-$C^*$-algebra (see Example 5.5.6) and $\hat{M}(\hat{G}, \mu) \subset C_{-1}(\hat{G})$ it follows that the defining identities (5.5.4) and (5.5.5) also hold $\hat{M}(\hat{G}, \mu)$ on $M(\hat{G}, \mu)$.

**Theorem 5.5.14.** Let $G$ be a reduced compact quantum group and $\mu$ a generating and transient probability measure on $\text{Irr}(G)$. Denote by $\hat{B}(\hat{G}, \mu)$ and respectively $B(\hat{G}, \mu)$ the unital braided-commutative Yetter–Drinfeld algebras associated to the categorical Martin compactification $\hat{M}(\text{Rep}(G), \mu)$ and categorical Martin boundary $M(\text{Rep}(G), \mu)$ as in Theorem 5.5.5. Then $\lambda_1 : \hat{B}(\hat{G}, \mu) \to \hat{M}(\hat{G}, \mu)$ is a $\ast$-isomorphism preserving the left $G$-action and right $\hat{G}$-action. Moreover $\lambda_1$ factors through the Martin boundary $\lambda_1 : B(\hat{G}, \mu) \to M(\hat{G}, \mu)$.

Equivalently, the $C^*$-tensor categories $\hat{B}(G, \mu)$ and $B(G, \mu)$ associated to the unital braided-commutative Yetter–Drinfeld $G$-$C^*$-algebras $\hat{M}(\hat{G}, \mu)$ and respectively $M(\hat{G}, \mu)$ are unitarily monoidally equivalent to the categorical Martin compactification $\hat{M}(\text{Rep}(G), \mu)$ and Martin boundary $M(\text{Rep}(G), \mu)$ of the $C^*$-tensor category $\text{Rep}(G)$. These monoidal equivalences preserve the functors of $\text{Rep}(G)$ into the respective categories.

**Proof.** From the construction of the categorical Martin compactification $\hat{M}(\text{Rep}(G), \mu)$ we see that $\text{Hom}_{\hat{M}(\text{Rep}(G), \mu)}(U, V) \subset \text{Hom}_{\text{Rep}(G)}(U, V)$ for any objects $U, V \in \text{Ob}(\text{Rep}(G))$, thus $\hat{B}(\hat{G}, \mu) \subset B_{-1}(\hat{G})$. Moreover, we already know that $\lambda_1 : B_{-1}(\hat{G}) \to C_{-1}(\hat{G})$ is a $G$-$\hat{G}$-equivariant $\ast$-isomorphism. Hence the restriction $\lambda_1 : \hat{B}(\hat{G}, \mu) \to C_{-1}(\hat{G})$ is an injective $\ast$-homomorphism preserving the actions of $G$ and $\hat{G}$. To show that $\lambda_1$ defines an isomorphism for the compactifications, it therefore suffices to show two more things:

(a) $\lambda_1(\hat{B}(G, \mu)) \subset \hat{M}(G, \mu)$;

(b) $\lambda_1 : \hat{B}(G, \mu) \to \hat{M}(G, \mu)$ is surjective.
To prove (a), note that as a vector space

\[
\tilde{B}(\hat{G}, \mu) \cong \bigoplus_{s \in \text{Irr}(G)} \tilde{H}_s \otimes \text{Hom}_{\tilde{M}(\text{Rep}(G), \mu)}(1, U_s)
\]

where \( r \) denotes again the operation of restriction. Now \( \text{Hom}_{\tilde{M}(\text{Rep}(G), \mu)}(1 \oplus U_s, 1 \oplus U_s) \) is generated as a C*-algebra by \( \text{Nat}_0(t \otimes (1 \oplus U_s), t \otimes (1 \oplus U_s)) \) and \( K_\mu(\text{Nat}_0(t \otimes (1 \oplus U_s), t \otimes (1 \oplus U_s))) \). Lemma 5.5.11 shows that if \( T \in \bigoplus_s \tilde{H}_s \otimes \text{Nat}_0(t \otimes (1 \oplus U_s), t \otimes (1 \oplus U_s)) \), then \( \lambda_1(T) \in c_0(\hat{G}) \) and similarly if \( S \in \bigoplus_s \tilde{H}_s \otimes \text{Nat}_0(t \otimes (1 \oplus U_s), t \otimes (1 \oplus U_s)) \), then \( \lambda_1(S) \in c_0(\hat{G}) \). In the latter case, write \( S^r := S_1 \otimes S_2^r \in \bigoplus_s \tilde{H}_s \otimes \text{Nat}_0(t \otimes 1, t \otimes U_s) \) for the restriction applied to the second leg of \( S \). Lemmas 5.3.13 and 5.5.12 imply that

\[
\lambda_1((t_H \otimes K_\mu)(S^r)) = \lambda_1((t_H \otimes K_\mu)(S^r)) = K_\mu(\lambda_1(S^r)),
\]

which is thus an element of \( K_\mu(c_0(\hat{G})) \subset \tilde{M}(\hat{G}, \mu) \). Hence \( \lambda_1 \) maps the generators of \( \tilde{B}(\hat{G}, \mu) \) in \( \tilde{M}(\hat{G}, \mu) \), which proves (a).

To establish surjectivity, we reverse the argument. Clearly every \( x \in c_0(\hat{G}) \) lies in the image of \( \lambda_1 \) (see Lemma 5.5.11). Assume that \( x \in c_0(\hat{G}) \), then by the same lemma there exists \( S \in \bigoplus_s \tilde{H}_s \otimes \text{Nat}_0(t \otimes 1, t \otimes U_s) \) such that \( \lambda_1(S) = x \). Then \( S^e \in \bigoplus_s \tilde{H}_s \otimes \text{Nat}_0(t \otimes (1 \oplus U_s), t \otimes (1 \oplus U_s)) \) and \((t_H \otimes K_\mu)(S^e))^r \in \bigoplus_s \tilde{H}_s \otimes \text{Hom}_{\tilde{M}(\text{Rep}(G), \mu)}(1, U_s) \). Invoking again Lemmas 5.3.13 and 5.5.12 gives us

\[
\lambda_1((t_H \otimes K_\mu)(S^e))^r = \lambda_1((t_H \otimes K_\mu)(S)) = K_\mu(\lambda_1(S)) = K_\mu(x).
\]

Hence \( \lambda_1 \) is surjective.

From the second part of Lemma 5.5.11 we immediately conclude that \( \lambda_1 \) factors through the Martin boundary. \( \Box \)

Since \( R(C, \mu) \) is a C*-tensor category (see Proposition 5.4.4) one could try to reconstruct the C*-algebra \( R_{\varphi_\mu} \) of regular elements from \( R(\text{Rep}(G), \mu) \). There is however one problem, the algebra \( R_{\varphi_\mu} \) is only a C*-subalgebra of \( l^\infty(\hat{G}) \), so one cannot talk about actions in the von Neumann sense. On the other hand it is unknown whether the left \( G \) action is continuous in the C*-sense. But one can consider \( R_{\varphi_\mu} \cap C_{-1}(\hat{G}) \) and show that this algebra admits a braided-commutative Yetter–Drinfeld structure.

**Lemma 5.5.15.** Denote \( \hat{R}(\hat{G}, \mu) := \{ x \in C_{-1}(\hat{G}) : x \text{ is } \mu\text{-regular} \} = C_{-1}(\hat{G}) \cap R_{\varphi_\mu} \). Then \( \hat{R}(\hat{G}, \mu) \) is a unital braided-commutative Yetter–Drinfeld \( G \)-C*-algebra.

**Proof.** By definition \( \hat{R}(\hat{G}, \mu) \subset C_{-1}(\hat{G}) \) and \( \alpha_l \) defines a continuous left action of \( G \) on \( C_{-1}(\hat{G}) \). Using the same argument as in Lemma 5.5.13 it thus suffices to show that \( \hat{R}(\hat{G}, \mu) \) is closed under the actions \( \alpha_l \) and \( \triangleright \).

We deal with \( \alpha_l \) first. Using the pentagon equation it is easy to show that \( \alpha_l(\hat{\Delta}^n(x)) = \).
By (3.2.1) this equals

\[ (5.5.17) = (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)((\Delta^n(x) - 1^{\otimes n-m} \otimes \hat{\Delta}^m(x))^*(\Delta^n(x) - 1^{\otimes n-m} \otimes \hat{\Delta}^m(x))) \]

which tends to 0 as \( m, n \to \infty \). The same argument holds if one replaces \( x \) by \( x^* \) and thus \( \alpha_l(x) \in C(G) \otimes \tilde{R}(\hat{G}, \mu) \).

For \( \triangleright \) we use an argument similar to the proof of Proposition 5.4.4 part (i). Assume that \( x \in \tilde{R}(\hat{G}, \mu) \) and \( t \in \text{Irr}(G) \). Since \( \mu \) is generating, let \( k \geq 0 \) be such that \( t \in \text{supp}(\mu^k) \). Select \( s_1, \ldots, s_k \in \text{supp}(\mu) \) such that \( m^k_{s_1, \ldots, s_k} \geq 1 \), then by Lemma 3.2.9

\[ \varphi_t \leq \frac{d_{s_1} \cdots d_{s_k}}{d_t} m_{s_1, \ldots, s_k}(\varphi_{s_1} \otimes \cdots \otimes \varphi_{s_k}) \Delta^{k-1} \leq d_t^{-1} \frac{d_{s_1} \cdots d_{s_k}}{\mu(s_1) \cdots \mu(s_k)} \varphi_\mu^k. \]

Write \( C := d_t^{-1} \frac{d_{s_1} \cdots d_{s_k}}{\mu(s_1) \cdots \mu(s_k)} \). It follows that

\[
\begin{align*}
&\left(\varphi_\mu \otimes \cdots \otimes \varphi_\mu \otimes \varphi_t\right) \left( ((\Delta^n \otimes t)\Delta(x) - 1^{\otimes n-m} \otimes (\hat{\Delta}^m \otimes t)(\Delta(x)))^* \\
&\times ((\Delta^n \otimes t)\Delta(x) - 1^{\otimes n-m} \otimes (\hat{\Delta}^m \otimes t)(\Delta(x))) \right) \\
&\leq C \left( \varphi_\mu \otimes \cdots \otimes \varphi_\mu \otimes \varphi_\mu \otimes \cdots \otimes \varphi_\mu \right) \Delta^{k-1} \left( ((\Delta^{n+1} \otimes t)\Delta^{k-1}(x) - 1^{\otimes n-m} \otimes \hat{\Delta}^{m+1}(x))^* \\
&\times ((\Delta^{n+1} \otimes t)\Delta^{k-1}(x) - 1^{\otimes n-m} \otimes \hat{\Delta}^{m+1}(x))) \right) \\
&= C \left( \varphi_\mu \otimes \cdots \otimes \varphi_\mu \right) \left( ((\Delta^{n+k} \otimes t)\Delta^{k-1}(x) - 1^{\otimes n-m} \otimes \hat{\Delta}^{m+k}(x))^* \Delta^{k-1}(x) - 1^{\otimes n-m} \otimes \hat{\Delta}^{m+k}(x)) \right),
\end{align*}
\]

which, by regularity of \( x \), tends to 0 as \( m, n \to \infty \). So \( \hat{\Delta}(x) \in \tilde{R}(\hat{G}, \mu) \otimes t^\infty(\hat{G}) \) and thus \( \tilde{R}(\hat{G}, \mu) \) is closed under \( \triangleright \). 

The algebra \( \tilde{R}(\hat{G}, \mu) \) is a braided-commutative Yetter-Drinfeld \( \text{G-}\text{C}^*-\text{algebra} \) and hence induces a \( \text{C}^*-\text{tensor category} \). To prove that it corresponds to the category of regular natural transformations we need the following estimates.

**Lemma 5.5.16.** The following estimates hold for \( \xi \otimes \eta \in \mathcal{H}_s \otimes \text{Hom}_{\text{Rep}(G)}(1, U_s) \):

\[
\begin{align*}
\|\lambda_n(\xi \otimes \eta)\|_{\varphi_\mu^n}^2 &\leq \|\xi\|^2 \|\eta\|_{\varphi_\mu^n}^2, \\
\|\lambda_n(\xi \otimes \eta)^*\|_{\varphi_\mu^n}^2 &\leq d_s^2 \|\xi\|^2 \|\eta^*\|_{\varphi_\mu^n}^2.
\end{align*}
\]
In particular, if $\xi \otimes \eta \in \tilde{\mathcal{H}}_s \otimes \text{Hom}_{\text{Rep}(G)}(1, U_s)$ and $n > m$, then

$$
\|\hat{\Delta}^{n-1}(\lambda_1(\bar{\xi} \otimes \eta)) - 1^{\otimes n-m} \otimes \hat{\Delta}^{m-1}(\lambda_1(\bar{\xi} \otimes \eta))\|_{\varphi_{\mu}}^2 \\
\leq \|\xi\|^2 \|\hat{\Delta}^{n-1}(\eta) - (1^{\otimes n-m} \otimes \hat{\Delta}^{m-1})(\eta)\|_{\varphi_{\mu}}^2; \\
\|(\hat{\Delta}^{n-1}(\lambda_1(\bar{\xi} \otimes \eta)) - 1^{\otimes n-m} \otimes \hat{\Delta}^{m-1}(\lambda_1(\bar{\xi} \otimes \eta)))^\ast\|_{\varphi_{\mu}}^2 \\
\leq d_\xi^2 \|\xi\|^2 \|\hat{\Delta}^{n-1}(\eta) - (1^{\otimes n-m} \otimes \hat{\Delta}^{m-1})(\eta)\|_{\varphi_{\mu}}^2.
$$

Proof. Let $\bar{\xi} \otimes \eta \in \tilde{\mathcal{H}}_s \otimes \text{Hom}_{\text{Rep}(G)}(1, U_s)$, we compute using (5.5.2), (5.5.1) and Corollary 5.5.12 that

$$
\|\lambda_n(\bar{\xi} \otimes \eta)\|_{\varphi_{\mu}}^2 = (\varphi_{\mu} \otimes \cdots \otimes \varphi_{\mu})(\lambda_n(\bar{\xi} \otimes \eta)\ast \lambda_n(\bar{\xi} \otimes \eta)) \\
= (\varphi_{\mu} \otimes \cdots \otimes \varphi_{\mu})(\lambda_n((\bar{\xi} \otimes \eta)\ast(\bar{\xi} \otimes \eta))) \\
= (\varphi_{\mu} \otimes \cdots \otimes \varphi_{\mu})(\lambda_n((\pi((\xi \otimes \eta)\ast)(\xi \otimes \eta)))) \\
= (\varphi_{\mu} \otimes \cdots \otimes \varphi_{\mu})\left(\lambda_n\left(\pi\left((\rho_s^{-1/2}\xi \otimes (\eta\ast)\iota)(1^{\otimes n} \otimes \bar{R}_s)\cdot(\bar{\xi} \otimes \eta)\right)\right)\right) \\
= \lambda_0\left((\iota_{\bar{R}_s} \otimes \text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu)(\pi((\rho_s^{-1/2}\xi \otimes (\eta^\ast)\iota)(1^{\otimes n} \otimes \bar{R}_s)) \otimes \eta)\right) \\
= (\rho_s^{-1/2}\xi \otimes \eta)((\text{tr}_\mu \cdots \otimes \text{tr}_\mu)((\eta\ast)\iota)(1^{\otimes n} \otimes \bar{R}_s)) \otimes \eta). \tag{5.5.18}
$$

Note that since $\eta \in \text{Hom}_{\text{Rep}(G)}(1, U_s) \subset \text{Nat}_{b}(1^{\otimes n} \otimes 1, 1^{\otimes n} \otimes U)$ the tensor product of natural transformations yields

$$
((\eta\ast)\iota)(1^{\otimes n} \otimes \bar{R}_s)) \otimes \eta = (\eta^\ast \otimes \iota_{\bar{R}_s} \otimes \iota_{\bar{R}_s})(1^{\otimes n} \otimes \bar{R}_s \otimes \iota_{\bar{R}_s})\eta.
$$

By means of Lemma 5.5.8 write $\eta = \sum_{i=1}^k x_i \otimes T_i$, where $x_i \in l^\infty(\hat{G})^{\otimes n}$ and $T_i \in B(\mathcal{H}_{1}, \mathcal{H}_{U_s}) = B(\mathbb{C}, \mathcal{H}_s)$. Since $B(\mathbb{C}, \mathcal{H}_s)$ is finite dimensional, we may choose $k = \dim(U_s)$ and $T_i: \mathbb{C} \rightarrow H_s$, $c \mapsto c\zeta_i$, where $\{\zeta_i\}_{i=1}^{\dim(U_s)}$ forms an orthonormal basis in $\mathcal{H}_s$ and $\langle \zeta_i, \xi \rangle = 0$ if $i \neq 1$. Recall the solutions $(R_s, \bar{R}_s)$ of the conjugate equations for $U_s$ (see (1.5.2)). With these choices we obtain

$$
\begin{align*}
(5.5.18) &= \sum_{i,j=1}^{\dim(U_s)} (\rho_s^{-1/2}\xi \otimes \xi)((\varphi_{\mu} \otimes \cdots \otimes \varphi_{\mu} \otimes \iota_{U_s \otimes U_s})(x_i \otimes T_i) \\
&= \sum_{i,j=1}^{\dim(U_s)} (\varphi_{\mu} \otimes \cdots \otimes \varphi_{\mu})(x_i^\ast x_j)(\rho_s^{-1/2}\xi \otimes \xi)((T_i^\ast \otimes \iota_{\bar{R}_s} \otimes \iota_{\bar{R}_s})(\bar{R}_s \otimes \iota_{\bar{R}_s})T_j) \\
&= \sum_{i,j=1}^{\dim(U_s)} (\varphi_{\mu} \otimes \cdots \otimes \varphi_{\mu})(x_i^\ast x_j) \sum_{k=1}^{\dim(U_s)} T_i^\ast(\rho_s^{-1/2}\xi \otimes \xi)((T_i^\ast \otimes \iota_{\bar{R}_s} \otimes \iota_{\bar{R}_s})(\bar{R}_s \otimes \iota_{\bar{R}_s})T_j)
\end{align*}
$$
\[
\dim(U_u) = \sum_{i,j=1}^{\dim(U_u)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i^*x_j) \quad \sum_{k=1}^{\dim(U_u)} T_i^*(\xi_k^*) \xi_k^*(T_j) \\
= \sum_{i,j=1}^{\dim(U_u)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i^*x_j) T_i^*(\xi)(\tilde{\xi}_j).
\]

By the choice of the \( T_i \)'s in the decomposition of \( \eta \), this equals

\[
(5.5.19) = \sum_{i,j=1}^{\dim(U_u)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i^*x_j) ||\xi||^2 \\
\leq ||\xi||^2 \sum_{i=1}^{\dim(U_u)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i^*x_i) \varphi_1(T_i^*T_i) \\
= ||\xi||^2(\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu(\eta^*\eta)) \\
= ||\xi||^2||\eta||^2_{\mu \otimes n}.
\]

The second estimate is similar but slightly trickier. Along the same lines one can show that

\[
||\lambda_\eta(\tilde{\xi} \otimes \eta)^*||^2_{\mu \otimes n} = (\xi \otimes \rho_s^{-1/2}\xi)((\varphi_\mu \otimes \cdots \otimes \varphi_\mu \otimes \tau_{U_s \otimes U_s})(x_i \otimes T_i \otimes \tau_s)) \\
= (\xi \otimes \rho_s^{-1/2}\xi)((\varphi_\mu \otimes \cdots \otimes \varphi_\mu \otimes \tau_{U_s \otimes U_s})(x_i \otimes T_i \otimes \tau_s)) \\
= (\xi \otimes \rho_s^{-1/2}\xi)(\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu(\eta \otimes \tau_s)(\eta^* \otimes \tau_s)(\tilde{\xi} \otimes \tilde{\rho}_s)).
\]

As before decompose \( \eta \) by means of Lemma 5.5.8 as \( \eta = \sum_{i=1}^{\dim(U_u)} x_i \otimes T_i \), but this time let \( T_i : \mathbb{C} \to \mathcal{H}_s \), \( c \mapsto c \xi_i^* \), where \( \{\xi_i^*\}_i \) are the eigenvectors of \( \rho_s \). Assume for the moment that \( \xi = \xi_i^* \) for some \( l \in \{1, \ldots, \dim(U_u)\} \). Then in this case

\[
(5.5.20) = \sum_{i,j=1}^{\dim(U_u)} (\xi_i^* \otimes \rho_s^{-1/2}\xi_i^*)(((\varphi_\mu \otimes \cdots \otimes \varphi_\mu \otimes \tau_{U_s \otimes U_s})(x_i \otimes T_i \otimes \tau_s)) \\
= \sum_{i,j=1}^{\dim(U_u)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i^*x_j) \sum_{k=1}^{\dim(U_u)} (\xi_i \otimes \rho_s^{-1/2}\xi_i)(T_i^*T_j^*(\rho_s^{1/2}\xi_k^*) \otimes \tilde{\xi}_k^*) \\
= \sum_{i,j=1}^{\dim(U_u)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i^*x_j) \sum_{k=1}^{\dim(U_u)} (\xi_i \otimes \rho_s^{-1/2}\xi_i)(T_j^*(\tilde{\xi}_k^*) \otimes \tilde{\xi}_k^*) \\
= \sum_{i,j=1}^{\dim(U_u)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i^*x_j) \sum_{k=1}^{\dim(U_u)} (\xi_i^* \otimes \xi_i^*)(\xi_j^* \otimes \rho_s^{1/2}\xi_k^*) \otimes (\tilde{\xi}_k^* \otimes \rho_s^{-1/2}\xi_i^*) \\
= (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i^*x_i^*).
\]
On the other hand
\[ \|\xi\|^2\|\eta\|^2_{\mu \otimes n} = (\text{tr}_\mu \otimes \cdots \otimes \text{tr}_\mu \otimes \text{tr}_{U_s})(\eta\eta^*) \]
\[ = \sum_{i,j=1}^{\dim(U_s)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu \otimes \varphi_s)(x_i x_j^* \otimes T_i T_j^*) \]
\[ = \sum_{i,j=1}^{\dim(U_s)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i x_j^*) \sum_{k=1}^{\dim(U_s)} \langle \xi_k^s, \xi_k^s | \rho_s^{-1} \xi_k^s \rangle) \]
\[ = \sum_{i=1}^{\dim(U_s)} (\varphi_\mu \otimes \cdots \otimes \varphi_\mu)(x_i x_i^*) (\rho_s^{-1} \xi_k^s). \tag{5.5.22} \]

Note that \( d_s = \sum_{j=1}^{\dim(U_s)} (\rho_s)_{jj} \geq (\rho_s)_{ii} \) for any \( i \). Therefore comparing (5.5.21) and (5.5.22) gives
\[ \|\lambda_n(\xi \otimes \eta)^*\|_{\nu \otimes n}^2 \leq d_s \|\xi\|^2\|\eta\|^2_{\mu \otimes n}. \tag{5.5.23} \]

Now we deal with general vectors \( \xi \in \mathcal{H}_s \) as follows. Write \( \xi = \sum_i c_i \xi_i^s \), then \( \|\xi\|^2 = \sum_i |c_i|^2 \). By (5.5.23), Cauchy–Schwarz and Jensen’s inequality we get
\[ \|\lambda_n(\tilde{\xi} \otimes \eta)^*\|_{\nu \otimes n}^2 \leq \left( \sum_{i=1}^{\dim(U_s)} |c_i|\|\lambda_n(\xi_i^s \otimes \eta)^*\|_{\nu \otimes n} \right)^2 \]
\[ \leq d_s \left( \sum_{i=1}^{\dim(U_s)} |c_i| \right)^2 \|\eta\|^2_{\mu \otimes n} \]
\[ \leq d_s \dim(U_s) \sum_{i=1}^{\dim(U_s)} |c_i|^2 \|\eta\|^2_{\mu \otimes n} \]
\[ \leq d_s^2 \|\xi\|^2\|\eta\|^2_{\mu \otimes n}, \]
which completes the second inequality.

The third and fourth inequalities now have become easy. Indeed, by Corollary 5.5.12 and the first estimate we get for \( \tilde{\xi} \otimes \eta \in \mathcal{H}_s \otimes \text{Hom}_{\text{Rep}(G)}(1, U_s) \) and \( n > m \) that
\[ \|\hat{\Delta}^{n-1}(\lambda_1(\tilde{\xi} \otimes \eta)) - 1 \otimes_{n-m} \hat{\Delta}^{m-1}(\lambda_1(\tilde{\xi} \otimes \eta))\|_{\nu \otimes n}^2 \]
\[ = \|\lambda_n(\tilde{\xi} \otimes \Delta^{n-1}(\eta)) - 1 \otimes_{n-m} \lambda_m(\tilde{\xi} \otimes \Delta^{m-1}(\eta))\|_{\nu \otimes n}^2 \]
\[ = \|\lambda_n(\tilde{\xi} \otimes \Delta^{n-1}(\eta)) - \lambda_n(\tilde{\xi} \otimes \lambda \otimes_{n-m} \Delta^{m-1}(\eta))\|_{\nu \otimes n}^2 \]
\[ = \|\lambda_n(\tilde{\xi} \otimes (\Delta^{n-1}(\eta) - \lambda \otimes_{n-m} \Delta^{m-1}(\eta)))\|_{\nu \otimes n}^2 \]
\[ \leq \|\xi\|^2\|\hat{\Delta}^{n-1}(\eta) - (\lambda \otimes_{n-m} \Delta^{m-1}(\eta))\|_{\mu \otimes n}^2. \]

The fourth estimate is similar. We leave the details to the reader. \( \Box \)

Because of the Lemma 5.5.15 and Theorem 5.5.5 there exists a \( C^* \)-tensor category corresponding to the braided commutative Yetter–Drinfeld \( G \)-\( C^* \)-algebra \( \hat{R}(\hat{G}, \mu) \). Denote this \( C^* \)-tensor category by \( \mathcal{D} \). So concretely, \( \mathcal{D} = \mathcal{C}_{\hat{R}(\hat{G}, \mu)} \) is the completion of the category
with objects $\text{Ob}(\text{Rep}(G))$ and morphism sets

$$\text{Hom}_D(U, V) := \{ T \in \tilde{R}(\hat{G}, \mu) \otimes B(\mathcal{H}_U, \mathcal{H}_V) : V_{13}^\ast(\alpha_l \otimes \iota_V)(T)U_{13} = 1 \otimes T \},$$

for $U, V \in \text{Ob}(\text{Rep}(G))$. Recall the C*-tensor category $R(\text{Rep}(G), \mu)$ of $\mu$-regular natural transformations defined in Definition 5.4.2.

**Theorem 5.5.17.** The category $D$ is unitarily monoidally equivalent to $R(\text{Rep}(G), \mu)$ via a monoidal equivalence preserving the canonical functors $\text{Rep}(G) \to D$ and $\text{Rep}(G) \to R(\text{Rep}(G), \mu)$.

**Proof.** For an element $T \in l^\infty(\hat{G}) \otimes B(\mathcal{H}_U, \mathcal{H}_V)$ recall the condition

$$V_{13}^\ast(\alpha_l \otimes \iota_V)(T)U_{13} = 1 \otimes T. \quad (5.5.24)$$

The categories $\text{Rep}(G)_{-1}$ and $C_{-1}({\hat{G}})$ are unitarily monoidally equivalent as $\text{Rep}(G)$-module categories (see Corollary 5.5.9). The correspondence is given by (see Lemma 5.5.8)

$$\text{Hom}_{\text{Rep}(G)_{-1}}(U, V) \overset{\sim}{\leftarrow} \{ T \in l^\infty(\hat{G}) \otimes B(\mathcal{H}_U, \mathcal{H}_V) : V_{13}^\ast(\alpha_l \otimes \iota_V)(T)U_{13} = 1 \otimes T \};$$

$$\eta(T) \leftrightarrow T. \quad (5.5.25)$$

To prove the theorem it therefore suffices to show that for any pair of objects $U, V \in \text{Ob}(\text{Rep}(G))$, this correspondence restricts to an isomorphism

$$\text{Hom}_{R(\text{Rep}(G), \mu)}(U, V) \overset{\sim}{\leftarrow} \{ T \in \tilde{R}(\hat{G}, \mu) \otimes B(\mathcal{H}_U, \mathcal{H}_V) : V_{13}^\ast(\alpha_l \otimes \iota_V)(T)U_{13} = 1 \otimes T \}.$$

Because (5.5.25) is an isomorphism on the level of $\text{Rep}(G)_{-1}$, it is sufficient to show the following two things:

(a) $\eta(T) \in \text{Hom}_{R(\text{Rep}(G), \mu)}(U, V)$ for any $T \in \tilde{R}(\hat{G}, \mu) \otimes B(\mathcal{H}_U, \mathcal{H}_V)$ satisfying condition (5.5.24);

(b) for any $\eta \in \text{Hom}_{R(\text{Rep}(G), \mu)}(U, V)$ there exists a homomorphism $T \in \tilde{R}(\hat{G}, \mu) \otimes B(\mathcal{H}_U, \mathcal{H}_V)$ satisfying condition (5.5.24) such that $\eta = \eta(T)$.

(a) Assume $T \in \tilde{R}(\hat{G}, \mu) \otimes B(\mathcal{H}_U, \mathcal{H}_V)$ satisfies (5.5.24). By Lemma 5.5.8, $\eta(T) \in \text{Hom}_{\text{Rep}(G)_{-1}}(U, V)$. We must show that $\eta(T)$ is $\mu$-regular. Since $B(\mathcal{H}_U, \mathcal{H}_V)$ is finite dimensional we can write $T = \sum_{i=1}^k x_i \otimes T_i$ with $x_i \in \tilde{R}(\hat{G}, \mu)$ and $T_i \in B(\mathcal{H}_U, \mathcal{H}_V)$ such that $\varphi_U(T_i^* T_j) = \varphi_V(T_i T_j^*) = 0$ whenever $i \neq j$ (for instance, take $T_i$ of the form $m_{\xi, \xi'}$ where $\xi$ and $\xi'$ are in a basis of $\mathcal{H}_U$ respectively $\mathcal{H}_V$ in which $\rho$ acts diagonally). Let $n > m$. By Lemma 5.5.10 we obtain

$$\|\hat{\Delta}^{-1}(\eta(T)) - \iota \otimes m \otimes \hat{\Delta}^{m-1}(\eta(T))\|_\mu^2$$

$$= \|\eta((\hat{\Delta}^{-1} \otimes \iota_V)(T) - (1 \otimes m \otimes \hat{\Delta}^{m-1} \otimes \iota_V)(T))\|_\mu^2$$

$$= (\varphi_\mu \otimes \cdots \otimes \varphi_\mu \otimes \varphi_\mu)(((\hat{\Delta}^{-1} \otimes \iota_V)(T) - (1 \otimes m \otimes \hat{\Delta}^{m-1} \otimes \iota_V)(T))^*$$

$$\times ((\hat{\Delta}^{-1} \otimes \iota_V)(T) - (1 \otimes m \otimes \hat{\Delta}^{m-1} \otimes \iota_V)(T))$$
As each $x_i$ is $\mu$-regular, it follows that for $i = 1, \ldots, k$
\[
\|\hat{\Delta}^{-1}(x_i) - 1 \otimes \hat{\Delta}^{-1}(x_i)\|_{\varphi_{\mu}^* \otimes \cdots \otimes \varphi_{\mu}}^2 \to 0 \quad \text{as } n, m \to \infty
\]
and thus (5.5.26) converges to zero as $m, n \to \infty$. A similar calculation shows
\[
\|\hat{\Delta}^{-1}((\eta^*) - e^{-1} \otimes \hat{\Delta}^{-1}(\eta^*))\|_{\varphi_{\mu}^* \otimes \cdots \otimes \varphi_{\mu}}^2 \to 0 \quad \text{as } n, m \to 0.
\]
We conclude that $\eta(T) \in \text{Hom}_{R(\text{Rep}(G),\mu)}(U, V)$.

(b) First assume that $\eta \in \text{Hom}_{R(\text{Rep}(G),\mu)}(1, U_s)$. Pick the unique $T \in C_{-1}(\hat{G}) \otimes B(\mathbb{C}, \mathcal{H}_s)$ satisfying (5.5.24) such that $\eta(T) = \eta$. Given $\xi \in \mathcal{H}_s$, observe that $(\iota \otimes \bar{\xi})(T) = \lambda_1(\bar{\xi} \otimes \eta)$. From Lemma 5.5.16 it immediately follows that $(\iota \otimes \bar{\xi})(T) \in \hat{R}(\hat{G}, \mu)$ and thus $T \in \hat{R}(\hat{G}, \mu) \otimes B(\mathbb{C}, \mathcal{H}_s)$.

The general case of $\text{Hom}_{R(\text{Rep}(G),\mu)}(U, V)$ can be deduced from $\text{Hom}_{R(\text{Rep}(G),\mu)}(1, U_s)$. Indeed, Frobenius reciprocity gives
\[
\text{Hom}_{R(\text{Rep}(G),\mu)}(U, V) \cong \text{Hom}_{R(\text{Rep}(G),\mu)}(1, U \otimes V)
\cong \text{Hom}_{R(\text{Rep}(G),\mu)}(1, \bigoplus_{s \in \text{Irr}(G)} U_s^* V_s)
\cong \bigoplus_{s \in \text{Irr}(G)} m_{U,V}^s \text{Hom}_{R(\text{Rep}(G),\mu)}(1, U_s),
\]
thus (b) follows from the special case.

\begin{theorem}
Given a reduced compact quantum group $G$ and a generating and transient probability measure $\mu$ on $\text{Irr}(G) = \text{Irr}(\text{Rep}(G))$, the random walk defined by $\mu$ on the discrete dual $l^\infty(\hat{G})$ converges to the boundary if and only if the random walk defined by $\mu$ on the C*-tensor category $\text{Rep}(G)$ converges to the boundary.
\end{theorem}

Most of the hard work of the proof of this theorem is already done. We will refer to requirements (i) and (ii) of Definition 5.4.5 as (i)\textsuperscript{cat} respectively (ii)\textsuperscript{cat}, to distinguish them from the corresponding properties in Conjecture 4.3.3.

\begin{proof}[Proof of Theorem 5.5.18]
Note that $\hat{M}(\hat{G}, \mu) \subset C_{-1}(\hat{G})$. Therefore Theorem 5.5.17 implies that $\hat{M}(\hat{G}, \mu) \subset \hat{R}(\hat{G}, \mu)$ if and only if $\hat{M}(\text{Rep}(G), \mu)$ is a C*-tensor subcategory of $\hat{R}(\text{Rep}(G), \mu)$. Thus (i) and (i)\textsuperscript{cat} are equivalent.

Equivalence of (ii) and (ii)\textsuperscript{cat} is proved in the subsequent two lemmas.
\end{proof}

\begin{lemma}
Suppose that $(G, \mu)$ satisfies (ii), then $(\text{Rep}(G), \mu)$ satisfies (ii)\textsuperscript{cat}.
\end{lemma}
Proof. Let \( \nu \in \text{Nat}_\mu(\iota \otimes X, \iota \otimes Y) \) be \( \mu \)-harmonic and \( \eta \in \text{Nat}_0(\iota \otimes U, \iota \otimes V) \). By Lemma 5.5.11, \( \eta \) is of the form \( \eta = \eta(T) \) for a unique \( T \) satisfying (5.5.24). Write \( T \) as

\[
T = \sum_{i=1}^{k} x_i \otimes T_i \in c_{00}(\hat{G}) \otimes B(\mathcal{H}_U, \mathcal{H}_V).
\]

Similarly, by [NY14a, Thm. 4.1] also \( \nu \) is of the form \( \nu = \eta(S) \) for some \( S \), where

\[
S = \sum_{i=1}^{m} y_j \otimes S_j \in H^\infty(\hat{G}, \mu) \otimes B(\mathcal{H}_X, \mathcal{H}_Y).
\]

Note that these sums are finite, because \( B(\mathcal{H}_U, \mathcal{H}_V) \) and \( B(\mathcal{H}_X, \mathcal{H}_Y) \) are finite dimensional. The Poisson boundary is invariant under the right action defined by the comultiplication (see Lemma 3.2.14). Thus if we write \( \hat{T}(y_j) = y_j^{(1)} \otimes y_j^{(2)} \), then \( y_j^{(1)} \in H^\infty(\hat{G}, \mu) \).

Use the description of the tensor product of natural isomorphisms given by (5.5.10) to obtain

\[
\sum_{s \in \text{Irr}(G)} d_s^2 \text{tr}_s(\eta \otimes \nu) = \sum_{s \in \text{Irr}(G)} \sum_{i,j} d_s^2(\varphi_s \otimes \iota_V \otimes \iota_Y)(x_i y_j^{(1)} \otimes T_i \pi_U(y_j^{(2)}) \otimes S_j)
\]

\[
= \sum_{i,j} \left( \sum_{s \in \text{Irr}(G)} d_s^2 \varphi_s(x_i y_j^{(1)}) \right) (T_i \pi_U(y_j^{(2)}) \otimes S_j)
\]

\[
= \sum_{i,j} \varphi_{\mu}^\infty(K_{\mu}(x_i) y_j^{(1)})(T_i \pi_U(y_j^{(2)}) \otimes S_j)
\]

\[
= (\varphi_{\mu}^\infty \otimes \iota_V \otimes \iota_Y) \left( \sum_{i,j} K_{\mu}(x_i) y_j^{(1)} \otimes T_i \pi_U(y_j^{(2)}) \otimes S_j \right)
\]

\[
= \text{tr}_\mu^\infty(K_{\mu}(\eta) \otimes \nu).
\]

In the last step we used Lemma 5.5.10.

\[\square\]

Lemma 5.5.20. If \( (\text{Rep}(G), \mu) \) satisfies (ii)\textsuperscript{cat}, then \( (G, \mu) \) satisfies (ii).

Proof. Let \( \tilde{\xi} \otimes \eta \in \tilde{H}_s \otimes \text{Hom}_{\text{Rep}(G)_{-1}}(1, U_s) \), such that \( \lambda_1(\tilde{\xi} \otimes \eta) \in c_{00}(\hat{G}) \) and let \( \tilde{\zeta} \otimes \nu \in \tilde{H}_t \otimes \text{Hom}_{\text{Rep}(G)_{-1}}(1, U_t) \), such that \( \lambda_1(\tilde{\zeta} \otimes \nu) \in H^\infty(\hat{G}, \mu) \). Recall the tensor product of natural transformations (5.2.3). Using the notation and results of Corollary 5.5.12 we get

\[
\hat{\psi}(\lambda_1(\tilde{\xi} \otimes \eta)\lambda_1(\tilde{\zeta} \otimes \nu)) = \hat{\psi}(\lambda_1((\tilde{\xi} \otimes \eta)(\tilde{\zeta} \otimes \nu)))
\]

\[
= \sum_r d_r^2 \varphi_r(\lambda_1((\tilde{\xi} \otimes \tilde{\zeta}) \otimes ((\eta \otimes \nu)(\iota \otimes \nu))))
\]

\[
= \sum_r d_r^2 \lambda_0 \circ (\iota_{\pi_r \circ \pi_t} \otimes \text{tr}_r)((\tilde{\xi} \otimes \tilde{\zeta}) \otimes (\eta \otimes \nu))
\]

\[
= \sum_r d_r^2 \lambda_0((\tilde{\xi} \otimes \tilde{\zeta}) \otimes \text{tr}_r(\eta \otimes \nu))
\]

\[
= \lambda_0((\tilde{\xi} \otimes \tilde{\zeta}) \otimes \left( \sum_r d_r^2 \text{tr}_r(\eta \otimes \nu) \right)). \tag{5.5.27}
\]
Corollary 5.5.12 gave us that $\lambda_1(\iota \otimes K_{\bar{\mu}}) = K_{\bar{\mu}}\lambda_1$, where the $K_{\bar{\mu}}$ on the left hand side is the Martin kernel acting on $\text{Hom}_{\text{Rep}(G)^{-1}}(1, U_s)$ while on the right hand side acting on $l^\infty(\hat{G})$. By assumption $(\text{Rep}(G), \mu)$ satisfies (ii)$^{\text{cat}}$, so

\[
(5.5.27) = \lambda_0(\xi \otimes \zeta) \otimes \text{tr}_\mu^\infty(K_{\bar{\mu}}(\eta) \otimes \nu)) \\
= \lambda_0 \circ (\iota_{\mathcal{H}_s} \otimes \iota_{\mathcal{H}_t} \otimes \text{tr}_\mu^\infty)((\xi \otimes \zeta) \otimes ((K_{\bar{\mu}}(\eta) \otimes \iota)(\iota \otimes \nu))) \\
= \varphi^\infty_{\mu}(\lambda_1((\xi \otimes \zeta) \otimes ((K_{\bar{\mu}}(\eta) \otimes \iota)(\iota \otimes \nu)))) \\
= \varphi^\infty_{\mu}(K_{\bar{\mu}}(\lambda_1(\xi \otimes \eta))\lambda_1(\zeta \otimes \nu)).
\]

By linearity it follows that for any $x \in \lambda_1(\bigoplus_s \mathcal{H}_s \otimes \text{Nat}_0(1, U_s))$ and any $h = \lambda_1(T)$ where $T \in \bigoplus_s \mathcal{H}_s \otimes \text{Nat}_0(\iota, \iota \otimes U_s)$ with $(\iota \otimes P_{\mu})(T) = T$, it holds that

\[
\hat{\psi}(xh) = \varphi^\infty_{\mu}(K_{\bar{\mu}}(x)h).
\] (5.5.28)

By density and strong*-continuity we obtain that (5.5.28) holds for any $x \in c_0(\hat{G})$ and any $P_{\mu}$-harmonic element $h \in H^\infty(\hat{G}, \mu)$. \[\square\]
An introduction for the layman

“A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.”

— David Hilbert

In the past years many people asked me the question “what are you doing as a mathematician?” Many times I tried to give an idea of what mathematics is about and also to explain in easy terms the kind of problems I am working on. Sometimes I succeeded and sometimes I didn’t. In these last pages of the dissertation I will try to put some these ideas on paper. I hope that after reading this short chapter you got a glimpse of the world I have been wandering around in during the last three years.

Suppose you are lost in a city and don’t know how to find your way back home. Of course, you can ask another person on the street for help, but assume you don’t. Instead, at every intersection you arbitrarily pick a road to continue your path. Will you ever find your way back home like this? This question is kind of hard to answer and depends very much on the city you are in and how you determine which road you select to continue your walk. So let us simplify the situation and assume we are in a city that looks like the figure below.\(^1\) Moreover we assume that we only change directions at intersections (so you cannot turn around halfway in a street) and at every intersection each of the four roads (north, east, south and west)

\(^1\)for example in Manhattan in New York with all its streets and avenues has this layout.

![Figure A: The first 20 steps of a random walk on a grid](image-url)
has an equal chance that we continue in that direction. For example, at each crossing you roll a four-sided die,\(^2\) if you get 1 you go to the north, with 2 to the east, etcetera. Observe that the path you create via this method can intersect itself and sometimes you will walk back along the same street as you just entered. In this specific example, maybe surprising at first sight, there is a theorem saying that you will come back home.\(^3\)

What we described here is what we call in mathematics a “random walk”: you walk randomly from one intersection to the next. There are many sorts of random walks which can have very different behaviours. So let us consider what happens if we modify the situation slightly.

If instead of walking in a two-dimensional grid you change the situation to a three-dimensional space then the behaviour is fundamentally different. Namely, suppose that you are in a three-dimensional grid and at every crossing you go with probability \(\frac{1}{6}\) either up, down, north, south, east or west, then the probability of getting home is strictly less than 1. This is often summarised by saying: “A drunk man will always find his way home, whereas a drunk bird may get lost forever”.\(^4\)

Let us go back to the two-dimensional case and modify the random walk in a different way. If the die is not fair any more (so not every side has probability \(\frac{1}{4}\)) then again you might get lost forever. You somehow drift away to one side of the city. For instance, if the four sided die gives 1 with probability \(\frac{1}{3}\), the number 2 with probability \(\frac{1}{6}\) and 3 and 4 each with probability \(\frac{1}{4}\), then you tend to walk to a north-west direction. A question one can ask is where you expect to end up if you continue this walk for an infinite amount of time. Of course you cannot do this in real life, because cities are limited in size and you don’t live forever. But suppose the grid above (Figure A) extends on every side to infinity, then here this question makes sense. The problem is that if you walk for infinite time and you drift away in some direction, eventually you will get further and further away from your starting point. So you do not end up in a certain area of your space. But what you can say is in which direction you go. In order to do this you can embed the grid in a disc of radius one as indicated by the following picture.\(^5\) If the walk on the grid goes

\[\text{Figure B: Embedding of a grid into a disc}\]

\(^2\)Yes they exist, they look like a pyramid
\(^3\)To be precise the probability of not getting home is 0.
\(^4\)due to Kakutani
\(^5\)A point \((x, y)\) on the grid corresponds to the point \((x/r, y/r)\) on the disc, where \(r = \sqrt{x^2 + y^2 + 1}\).
to infinity, meaning that you walk further and further away from the starting point, then
the same walk on the disc approaches the outer circle of the disc. So a point on the circle
of the disc should be thought of as a point at infinity of the grid. We can now associate
to each interval of the circle the probability that the random walk eventually stays in the
corresponding sector of the circle. For example the sector indicated by the shaded area
in the picture below (see Figure C). In this way we get a nice description of the direction
in which one walks, so where you eventually “end up”.

![Figure C: A sector of the disc](image)

What we have done here can be done in much greater generality. For example it is
not necessary that the grid is two-dimensional. It can be done in any dimension. What
you can do is embed an \( N \)-dimensional grid in an \( N \)-dimensional ball and then the same
can be said about where the random walk is likely to end. But it is even unnecessary to
require that the space of the random walk looks like a grid. A similar construction can
be performed in many cases as long as the random walk is nice enough, meaning that you
can reach every other point in your space from your starting point and eventually you get
further and further away from your starting point. This construction we made above by
associating the circle to the grid is the so-called “Martin boundary”. It is a very useful
tool for understanding random walks.

One of the goals of my PhD project was to study such Martin boundaries. Not for
random walks on grids as described above, because that problem has been solved already
long ago, but for random walks on quantum groups. I am not able to describe in a
nontechnical way what a quantum group is, but I would like to describe the idea behind
quantum groups.\(^6\) The keyword for this is “noncommutativity”. Everyone learned in
primary school that it does not matter in which order you multiply numbers, for example

\[
3 \times 7 = 21 = 7 \times 3.
\]

This is exactly commutativity, the order of multiplication does not matter. But in real
life it is very common that things do not commute. The famous example is “taking your
clothes off” and “taking a shower”. Usually you take off your clothes and then take a
shower. If you do it the other way around, the result is different: you end up with wet

\(^6\)If you do know some mathematics but haven’t heard of quantum groups, take a look at the two-page paper “What is... a Quantum Group” [Maj06].
clothes. For a more mathematical example consider all the rotations of an object in a three dimensional space. These rotations do not commute, for instance take a look at Figure D below. The arrows indicate how the die is rotated. We see that first rotating along the x-axis and then the y-axis gives a different result than first rotating along the y-axis and then the x-axis.\footnote{Consider the die in the upper left corner, the x-axis is the axis passing through the faces with three and four on them, the y-axis is the axis passing through faces indicated by two and five.}

![Figure D: Noncommuting rotations of a die. Recall that the numbers of two opposite faces always sum to seven. So opposite to one is six, the opposite side of two is five and opposite to three is four (nonvisible).]

Before we move to quantum groups, I will first explain what a group is. A group is a space with an operation of composition, an inverse and a neutral element. For example the integers $\{\ldots,-2,-1,0,1,2,\ldots\}$ form a group with composition given by addition, the inverse of a number is equal to its negative and the neutral element is 0. In formulas we have for an integer $n$

\[
n + (-n) = 0 = (-n) + n \quad \text{and} \quad n + 0 = n = 0 + n.
\]

where the first equation shows that the inverse of a number is given by taking the negative of the number and the second equation shows that 0 is the neutral element. Note that $n + m = m + n$ for any two numbers, so the integers from a commutative group.
Also all rotations of an object in a three-dimensional space form a group. The composition of two rotations is given by first applying the first transformation and then the second; the inverse of a transformation is given by rotating in the converse direction and the neutral element is the rotation that does nothing.

The point is now that from a space you can construct in a concrete way a new object, a so-called “C*-algebra”. I will not describe what a C*-algebra is or how this construction works. The only thing you need to know is that from a space you can construct in some concrete way a new object. This correspondence is in such a way, that whenever you have two spaces and the associated C*-algebras are the same, then the spaces you started with must have been the same. Moreover, one can show that a C*-algebra arising from a space is always commutative. Now there is a nice theorem which states that conversely if you have a commutative C*-algebra it always comes from a space via this particular construction. So there is a one-to-one correspondence (or a duality) between spaces and commutative C*-algebras. In other words, both points of view contain the same information. However, sometimes one picture can be more convenient than the other.

Now, also noncommutative C*-algebras make perfect sense. So with this duality in mind it is reasonable to say that these noncommutative C*-algebras come from “noncommutative spaces” or “quantum spaces”. Hopefully a picture clarifies this idea, see Figure E.

![Diagram](image)

**Figure E:** A one-to-one correspondence between spaces and commutative C*-algebras.

Observe that a quantum space does not exist as a space. Because if it was a space then the corresponding C*-algebra would be commutative, which it is not. So for a dot in the outer ring on the right hand side of the picture, there is no corresponding dot on
the left side of the picture. Therefore we only think of noncommutative C*-algebras as “coming” from quantum spaces. This gives a useful intuition when studying these spaces. So studying noncommutative spaces really means studying noncommutative C*-algebras.

As described previously, a group is a space with extra structure. So you can associate to it a C*-algebra. But the additional structure of the group can also be transported to the C*-algebra. So starting with a group you obtain a commutative C*-algebra which contains extra structure from the group. If you now consider noncommutative C*-algebras with such extra structure, you obtain what we call “quantum groups”. The picture thus looks as follows, see Figure F.

An important thing to observe is that these quantum groups are not groups. They should be thought of as objects dual to groups. Even though these “quantum spaces” and “quantum groups” might seem artificial at first sight, they are useful in a lot of places. Originally they were defined to solve some problems in physics. But not much later also from a mathematical point of view, quantum groups became very interesting to study.

In this thesis I worked on problems where these two fields of quantum groups and random walks meet. One can define random walks on quantum groups and study how these behave. Also in the quantum case it is possible to construct Martin boundaries. One of the goals of the thesis was to compute such Martin boundaries in concrete cases. Another goal was to describe how such quantum random walks behave if you let them walk for an infinite amount of time. It is interesting to see how far the theories of random walks on groups and quantum groups agree, but sometimes it is even more interesting to see where they differ and what the reason is for such a difference.
Index of symbols

\( a, 47 \)
\( A_0(F), 159 \)
\( A_{i,j}(r), 99 \)
\( a_j, 113 \)
\( \alpha, 136 \)
\( \alpha_j, 26 \)
\( \alpha_t, 19 \)
\( \alpha_r, 19 \)
\( \alpha^\rho, 33 \)
\( A_n, 37 \)
\( a_{\pm, t}, 141 \)
\( a_+^{\pm, i}, 141 \)
\( B, 152 \)
\( \mathcal{B}, 152, 180 \)
\( \mathcal{B}_{-n}(\hat{G}), 187 \)
\( \mathcal{B}_{-n}(\hat{G}), 187 \)
\( b_j, 116 \)
\( b\text{-lim}, 90 \)
\( B_n, 38 \)
\( \mathcal{X}_{\text{alg}}^\alpha, 153 \)
\( \mathcal{X}^\alpha, 153 \)
\( \mathcal{X}_{\text{red}}^\alpha, 154 \)
\( \bullet, 181 \)
\( c, 91 \)
\( C_{-n}(\hat{G}), 184 \)
\( C(G), 11 \)
\( c_0(\hat{G}), 16 \)
\( c_{00}(\hat{G}), 16 \)
\( C_{-n}, 161 \)
\( \mathbb{C}[G], 13 \)
\( C(G_0), 29 \)
\( \chi_t, 139 \)
\( c_{n,x}(\mu), 77 \)
\( C_+, 26 \)
\( C_q, 138 \)
\( C_q(k,s,t), 105 \)
\( \mathcal{C}_r, 33 \)
\( C(S^2_{q,t}), 139 \)
\( \mathbb{C}[\text{SU}_q(2)], 136 \)
\( \Delta, 9, 11, 14, 26, 136 \)
\( \hat{\Delta}, 16, 162 \)
\( \delta_1, 152 \)
\( \delta_2, 152 \)
\( \delta_C, 58 \)
\( d_i(U), 25 \)
\( d_U, 12 \)
\( E, 137 \)
\( e, 137 \)
\( \mathcal{E}, 161, 166, 172, 173 \)
\( E_t, 28, 97, 120 \)
\( e_i, 36, 98 \)
\( E_n, 37, 174 \)
\( \text{End}(\mathcal{D}), 180 \)
\( \varepsilon_i, 120 \)
\( \varepsilon, 14 \)
\( \eta, 41 \)
\( \eta(T), 184 \)
\( \eta^e, 169 \)
\( \eta^r, 169 \)
\( F, 137 \)
\( f, 137 \)
\( F_t, 28, 97, 120 \)
\( F_n, 38 \)
\( \mathcal{F}, 17 \)
\( f_z, 14 \)
\( \gamma, 136 \)
\( \gamma_C, 51 \)
\( G_{\mu},\ 89 \)
\( g_i,\ 35 \)
\( G_{\mu},\ 78,\ 168 \)
\( g_x,\ 35 \)
\( \text{grad} \tau,\ 90 \)
\( g(x,y),\ 72 \)

\( h,\ 11 \)
\( H^\infty(\hat{G},\mu),\ 79 \)
\( H_\infty(q),\ 36 \)
\( H_n(q),\ 35 \)
\( \text{Hom}(U,V),\ 12 \)
\( \mathcal{H}_s,\ 136 \)

\( (\cdot,\cdot)_\eta,\ 10 \)
\( (\iota \otimes U),\ 160 \)
\( (\iota \otimes \cdot),\ 162 \)
\( \nu_R \otimes F,\ 188 \)

\( J,\ 9,\ 30 \)
\( j,\ 12 \)
\( j^\infty,\ 79,\ 128 \)
\( j_n,\ 79,\ 175 \)

\( K,\ 137 \)
\( k,\ 137 \)
\( \kappa,\ 80 \)
\( \kappa_s,\ 157 \)
\( \kappa^{s,V},\ 164 \)
\( K_{\hat{\mu}},\ 78,\ 170 \)
\( K_{\hat{\mu}}^s,\ 28,\ 97,\ 120 \)
\( k_n,\ 153 \)
\( K_{\omega},\ 97 \)
\( K_{\hat{\phi}}^{s},\ 131 \)
\( k(x,y),\ 72 \)

\( L_{-4\omega_2},\ 120 \)
\( L^2(G),\ 16 \)
\( \lambda,\ 139,\ 182 \)
\( \lambda_n,\ 187 \)
\( \tilde{\lambda},\ 30 \)
\( \tilde{\lambda},\ 139 \)

\( l^\infty(\hat{G}),\ 183 \)
\( l^\infty(\hat{G}'),\ 16 \)
\( L^\infty(G),\ 16 \)
\( L^\infty(SU_q(2)),\ 136 \)

\( \mathcal{M}(\mathcal{C},\mu),\ 173 \)
\( M_{\mathcal{C}},\ 47 \)
\( m_{p},\ 90 \)
\( M(\hat{G},\mu),\ 78 \)
\( M(\hat{G},\mu)_{\text{alg}},\ 190 \)
\( M(\hat{G},\mu)_{\text{alg}},\ 190 \)

\( \mu_s, 13 \)
\( M_{\min}(X,P),\ 74 \)
\( \mathcal{M}(\mathcal{C},\mu),\ 172 \)
\( M(\hat{G},\mu),\ 78 \)

\( \hat{\mu},\ 76 \)
\( \mu_{\mathcal{C}},\ 57 \)
\( m_U^I,\ 13 \)
\( M(X,P),\ 72 \)

\( \mathbb{N},\ 1 \)
\( N_{\mathcal{C}},\ 57 \)
\( [[n]]_q,\ 2 \)
\( [n]_q,\ 2 \)
\( \nu,\ 57 \)
\( \nu^f,\ 73 \)
\( \nu_x,\ 74 \)

\( O(g),\ 2 \)
\( o(g),\ 2 \)
\( \Omega,\ 71 \)
\( \omega,\ 152 \)
\( \omega_{\mathcal{C}},\ 57 \)
\( \omega_j,\ 26 \)

\( P,\ 26,\ 71 \)
\( \mathbb{P}_{\mu},\ 84 \)
\( \mathbb{P}_x,\ 71 \)
\( \mathcal{P}(\mathcal{C},\mu),\ 166 \)

\( \tilde{\varphi},\ 76 \)
\( \phi,\ 17 \)
\( \varphi^\infty,\ 128 \)
\( \varphi^{s},\ 75 \)
\( \varphi^{\infty},\ 141 \)
\( \varphi_{s},\ 75 \)
\( \varphi^{(\infty)},\ 175 \)
\( \pi_{s},\ 15,\ 138 \)
\( \pi_U,\ 15 \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>76, 163</td>
</tr>
<tr>
<td>$p_\mu(s,t)$</td>
<td>76, 164</td>
</tr>
<tr>
<td>$p_c(s,t)$</td>
<td>87</td>
</tr>
<tr>
<td>$P_{\phi}$</td>
<td>75</td>
</tr>
<tr>
<td>$P_-$</td>
<td>139</td>
</tr>
<tr>
<td>$\psi$</td>
<td>17</td>
</tr>
<tr>
<td>$p(x,y)$</td>
<td>71</td>
</tr>
<tr>
<td>$Q_a$</td>
<td>107</td>
</tr>
<tr>
<td>$q_C$</td>
<td>51, 57</td>
</tr>
<tr>
<td>$R$</td>
<td>15, 26</td>
</tr>
<tr>
<td>$(R, \overline{R})$</td>
<td>24</td>
</tr>
<tr>
<td>$r$</td>
<td>98</td>
</tr>
<tr>
<td>$r_{i,j}$</td>
<td>99</td>
</tr>
<tr>
<td>$R(U,V)$</td>
<td>174</td>
</tr>
<tr>
<td>$R_\phi$</td>
<td>128</td>
</tr>
<tr>
<td>$\hat{R}(\hat{G},\mu)$</td>
<td>191</td>
</tr>
<tr>
<td>$S$</td>
<td>14, 15, 41, 120</td>
</tr>
<tr>
<td>$s_\alpha$</td>
<td>26</td>
</tr>
<tr>
<td>$s_i$</td>
<td>26</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>36, 47</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>15</td>
</tr>
<tr>
<td>$\sigma_t^\phi$</td>
<td>17</td>
</tr>
<tr>
<td>$\sigma_t^\psi$</td>
<td>17</td>
</tr>
<tr>
<td>$\sigma_t(x)$</td>
<td>9</td>
</tr>
<tr>
<td>$\sim$</td>
<td>2</td>
</tr>
<tr>
<td>$S_n$</td>
<td>35</td>
</tr>
<tr>
<td>$SU_q(2)$</td>
<td>136</td>
</tr>
<tr>
<td>$\tau$</td>
<td>90</td>
</tr>
<tr>
<td>$\tau_C$</td>
<td>63</td>
</tr>
<tr>
<td>$\mathcal{T}$</td>
<td>90</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\otimes$</td>
<td>7</td>
</tr>
<tr>
<td>$\otimes^\infty_1$</td>
<td>7</td>
</tr>
<tr>
<td>$\theta$</td>
<td>84, 128</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>128</td>
</tr>
<tr>
<td>$\theta_n$</td>
<td>51</td>
</tr>
<tr>
<td>$\text{tr}_C$</td>
<td>52</td>
</tr>
<tr>
<td>$\triangleleft$</td>
<td>119</td>
</tr>
<tr>
<td>$\triangleright$</td>
<td>182</td>
</tr>
<tr>
<td>$\text{tr}_\mu$</td>
<td>162</td>
</tr>
<tr>
<td>$\text{Tr}_U$</td>
<td>25</td>
</tr>
<tr>
<td>$\text{tr}_U$</td>
<td>25</td>
</tr>
<tr>
<td>$\text{Tr}_X$</td>
<td>162</td>
</tr>
<tr>
<td>$U(\hat{G})$</td>
<td>16</td>
</tr>
<tr>
<td>$\hat{U}$</td>
<td>12, 24</td>
</tr>
<tr>
<td>$U^c$</td>
<td>12</td>
</tr>
<tr>
<td>$U_g$</td>
<td>27</td>
</tr>
<tr>
<td>$\hat{U}_q(g)$</td>
<td>27</td>
</tr>
<tr>
<td>$\hat{U}_q(g,S)$</td>
<td>120</td>
</tr>
<tr>
<td>$\hat{U}_q(h)$</td>
<td>28</td>
</tr>
<tr>
<td>$\hat{U}_q(u_N)$</td>
<td>97</td>
</tr>
<tr>
<td>$U_q(su_2)$</td>
<td>137</td>
</tr>
<tr>
<td>$\hat{U}_q(su_2)$</td>
<td>137</td>
</tr>
<tr>
<td>$\hat{U}_q(su_N)$</td>
<td>97</td>
</tr>
<tr>
<td>$V(\lambda)$</td>
<td>28</td>
</tr>
<tr>
<td>$V_\lambda$</td>
<td>29</td>
</tr>
<tr>
<td>$W$</td>
<td>16, 26</td>
</tr>
<tr>
<td>$w_0$</td>
<td>30</td>
</tr>
<tr>
<td>$w_t$</td>
<td>28</td>
</tr>
<tr>
<td>$X$</td>
<td>32, 153</td>
</tr>
<tr>
<td>$X_t$</td>
<td>139</td>
</tr>
<tr>
<td>$\xi^s_1$</td>
<td>13, 138</td>
</tr>
<tr>
<td>$X^-_t$</td>
<td>97</td>
</tr>
<tr>
<td>$X^-_{\infty}$</td>
<td>74</td>
</tr>
<tr>
<td>$X^+_t$</td>
<td>97</td>
</tr>
<tr>
<td>$\hat{X}_t$</td>
<td>139</td>
</tr>
<tr>
<td>$X_{\lambda}$</td>
<td>32</td>
</tr>
<tr>
<td>$X_n$</td>
<td>71</td>
</tr>
<tr>
<td>$X^s$</td>
<td>152</td>
</tr>
<tr>
<td>$\mathbb{Z}_+$</td>
<td>1</td>
</tr>
</tbody>
</table>
INDEX OF SYMBOLS
Index of subjects

action, 18
C*-algebra, 18
adjoint, 19
ergodic, 18
reduced, 154
right adjoint, 119
t von Neumann algebra, 18
analytic element, 9
associativity morphism, 20
big O-notation, 2
braided-commutative, 182
category
rigid, 24
strict, 21
Clebsch–Gordan coefficient, 104
compact quantum group, 11
completion
category, 23
conjugate equations, 24
standard solution, 24
conjugate object, 24
convergence to the boundary, 131, 179
coordinate projection, 71
C*-category, 19
C*-dynamical system, 9
C*-tensor category, 20
discrete Markov chain, 71
discrete quantum group, 16
Drinfeld–Jimbo q-deformation, 29
essentially surjective, 23
extension, 169
fixed point algebra, 18

flip map, 15
Fourier transform, 17
free orthogonal quantum group, 159
fully faithful, 23
fundamental object, 32
fusion semiring, 32

G-C*-algebra, 18
generating, 77, 168
generating object, 180
Green kernel, 72, 78, 168
GT tableau, 98

Haar state, 11
harmonic, 71, 79, 166
minimal, 71
super-, 71, 79
Hecke algebra, 35
highest weight
module, 29
vector, 29
Hopf *-algebra, 14

infinite tensor product, 7
C*-algebras, 7
von Neumann algebras, 8
intertwiner, 12
intrinsic dimension, 25
irreducible, 72

KMS state, 9

link algebra, 152
little o-notation, 2

Markov kernel, 71
Markov operator, 75, 163
Martin
boundary, 72, 78, 173
compactification, 72, 78, 172
kernel, 72, 78, 170
matrix coefficient, 13
matrix unit, 13
mean, 90
minimal tensor product, 7
modular automorphism group, 9
modular conjugation, 9
monoidal category, 20
monoidal equivalence, 22
  $\mu$-regular, 174
multiplicative unitary, 16
multiplicity, 13
natural isomorphism
  monoidal, 22
natural transformation
  compactly supported, 161
  harmonic, 166
  uniformly bounded, 160
  vanishing at infinity, 161
orthogonality relations, 14
path space, 71
pentagon equation, 16
Poisson boundary, 74, 79, 166
$q$-binomial, 2
$q$-factorial, 2
$q$-number, 2
quantized universal enveloping algebra, 27
quantum dimension, 12
quantum homogeneous sphere of Podleś, 139
random walk, 71
  on center, 76
  on torus, 85
regular, 128
regular subalgebra, 180
representation, 12
  admissible, 28
  conjugate, 12, 30
  contragredient, 12
left regular, 16
unitary, 12
restriction, 169
root lattice, 26
spectral subspace, 153
SU($N$)-type category, 32
tensor category, 20
tensor functor, 21
trace
  categorical, 25
  Hecke algebra, 40
  Markov, 40
transient, 72, 77, 168
transition probability, 71
twist, 63
unitarily monoidally equivalent, 22
unitary functor, 22
universal enveloping algebra, 27
weight lattice, 26
Weyl chamber, 26
Weyl group, 26
Woronowicz character, 14
$\mathcal{W}^*$-dynamical system, 9
Yetter–Drinfeld $G$-$C^*$-algebra, 182
Bibliography


