Stochastic differential games with inside information

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Abstract

We study stochastic differential games of jump diffusions, where the players have access to inside information. Our approach is based on anticipative stochastic calculus, white noise, Hida-Malliavin calculus, forward integrals and the Donsker delta functional. We obtain a characterization of Nash equilibria of such games in terms of the corresponding Hamiltonians. This is used to study applications to insider games in finance, specifically optimal insider consumption and optimal insider portfolio under model uncertainty.

1 Introduction

In this paper we present a general method for studying *optimal insider games*, i.e. stochastic differential games where the two players have access to some future information about the system. This inside information in the control processes puts the problem outside the context of semimartingale theory, and we therefore apply general *anticipating white noise calculus*, including *forward integrals* and *Hida-Malliavin calculus*. Combining this with the *Donsker delta functional* for the random variable $Y = (Y_1, Y_2)$ which represents the inside information for player number 1 and 2, respectively, we are able to prove both a sufficient and a necessary maximum principle for a Nash equilibrium of such games.

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We now explain this in more detail:

The system we consider, is described by a stochastic differential equation driven by a Brownian motion $B(t)$ and an independent compensated Poisson random measure $\tilde{N}(dt, d\zeta)$, jointly defined on a filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions. We assume that the inside information is of *initial enlargement* type. Specifically, we assume that the two inside filtrations $\mathbb{H}^1, \mathbb{H}^2$ representing the information flows available to player 1 and player 2, respectively, have the form

$$\mathbb{H}^i = \{\mathcal{H}^i_t\}_{t\geq 0}, \text{ where } \mathcal{H}^i_t = \mathcal{F}_t \vee Y_i, \quad i = 1, 2 \tag{1.1}$$ {eq1.1}

for all t, where Y_i is a given $\mathcal{F}_{T_0}$-measurable random variable, for some fixed $T_0 > T > t$. Here and in the following we choose the right-continuous version of $\mathbb{H}^i$, i.e. we put $\mathcal{H}^i_t = \mathcal{H}^i_{t+} = \bigcap_{s>t} \mathcal{H}^i_s$. The insider control process is denoted by $u(t) = (u_1(t), u_2(t))$, where $u_i(t)$ is the control of player i; i=1,2. Thus we assume that the value at time t of our insider control process $u_i(t)$ is allowed to depend on both Y_i and $\mathcal{F}_t; i = 1, 2$. In other words, u_i is assumed to be $\mathbb{H}^i$-adapted for $i = 1, 2$. Therefore they have the form

$$u_i(t, \omega) = \overline{u}_i(t, Y_i, \omega) \tag{1.2}$$ {eq1.2}

for some function $\overline{u}_i : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $\overline{u}_i(t, y_i)$ is $\mathbb{F}$-adapted for each $y_i \in \mathbb{R}$. For simplicity (albeit with some abuse of notation) we will in the following write u_i in stead of $\overline{u}_i; i = 1, 2$.

Consider a controlled stochastic process $X(t) = X^u(t)$ of the form

$$\begin{cases} dX(t) = dX^u(t) = b(t, X(t), u_1(t), u_2(t), Y_1, Y_2)dt + \sigma(t, X(t), u_1(t), u_2(t), Y_1, Y_2)dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t), u_1(t), u_2(t), Y_1, Y_2, \zeta)\tilde{N}(dt, d\zeta); \quad t \geq 0 \\ X(0) = x(Y) \in \mathbb{R}, \end{cases} \tag{1.3}$$ {eq2.1}

where $u_i(t) = u_i(t, y_i)_{y_i = Y_i}$ is the control process of insider $i; i = 1, 2$, and the (anticipating) stochastic integrals are interpreted as *forward integrals*, as introduced in [RV] (Brownian motion case) and in [DMØP1] (Poisson random measure case). A motivation for using forward integrals in the modelling of insider control is given in [BØ]. Note that if Y is a deterministic constant, then the stochastic integrals in (1.3) reduce to classical Itô integrals. Let $\mathcal{A}_i$ denote a given set of admissible $\mathbb{H}^i-$adapted controls u_i of player i, with values in $\mathbf{A}_i \subset \mathbb{R}^d, d \geq 1; i = 1, 2$. Put $\mathbb{U} = \mathbf{A}_1 \times \mathbf{A}_2$. Then $X(t)$ is $\mathbb{F} \vee Y_1 \vee Y_2$-adapted. The *performance functional* $J_i(u); u = (u_1, u_2)$ of player i is defined by

$$J_i(u) = \mathbb{E}[\int_0^T f_i(t, X(t), u_1(t), u_2(t), Y)dt + g_i(X(T), Y)]; \quad i = 1, 2. \tag{1.4}$$ {eq1.4}

A *Nash equilibrium* for the game (1.3)-(3.5) is a pair $\hat{u} = (\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\sup_{u_1 \in \mathcal{A}_1} J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) \tag{1.5}$$ {eq1.5}

and

$$\sup_{u_2 \in \mathcal{A}_2} J_2(\hat{u}_1, u_2) \le J_2(\hat{u}_1, \hat{u}_2). \tag{1.6}$$ {eq1.6}

We use the Donsker delta functional of $Y = (Y_1, Y_2)$ to find a Nash equilibrium for the game (1.4)-(1.6).

Here is an outline of the content of the paper:

- In Section 2 we define the Donsker delta functional.
- In Section 3 we use the Donsker delta functional to rewrite the original insider game (1.5)-(1.6) as a (parametrised) classical stochastic differential game, but with different performance functionals.
- In Sections 4 and 5 we present a sufficient and a necessary maximum principle, respectively, for the insider game problem in Section 3.
- In Section 6 we present the zero-sum game case where we distinguish two approaches: Situation 1: Both players are still maximising their own performance functional; Situation 2: One of the players is maximising and the other is minimising the performance functional.
  We formulate the sufficient and necessary maximum principles corresponding for each approach.
- Then in Section 7 we illustrate our results by applying them to optimal insider consumption and optimal insider portfolio under model uncertainty.

2 The Donsker delta functional

Definition 2.1 *Let $Y_i : \Omega \to \mathbb{R}; i = 1, 2$ be two random variables which also belongs to $(\mathcal{S})^*$. Then a continuous functional*

$$\delta_{Y_1,Y_2}(.) : \mathbb{R} \times \mathbb{R} \to (\mathcal{S})^* \tag{2.1}$$ {donsker}

is called a Donsker delta functional of (Y_1, Y_2) if it has the property that

$$\int_{\mathbb{R}^2} g(y_1, y_2)\delta_{Y_1,Y_2}(y_1, y_2)dy_1 dy_2 = g(Y_1, Y_2) \quad a.s. \tag{2.2}$$ {donsker pr

for all (measurable) $g : \mathbb{R}^2 \to \mathbb{R}$ such that the integral converges.

For example, consider the special case when Y is a first order chaos random variable of the form

$$Y = Y(T_0);\ \text{where } Y(t) = \int_0^t \beta(s)dB(s) + \int_0^t \int_{\mathbb{R}} \psi(s, \zeta)\tilde{N}(ds, d\zeta),\ \text{for } t \in [0, T_0] \tag{2.3}$$ {eq2.5}

for some deterministic functions $\beta \neq 0, \psi$ satisfying

$$\int_0^{T_0} \{\beta^2(t) + \int_{\mathbb{R}} \psi^2(t,\zeta)\nu(d\zeta)\}dt < \infty \text{ a.s.} \tag{2.4}$$

We also assume that for every $\epsilon > 0$ there exists $\rho > 0$ such that

$$\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} e^{\rho\zeta}\nu(d\zeta) < \infty.$$

This condition implies that the polynomials are dense in $L^2(\mu)$, where $d\mu(\zeta) = \zeta^2 d\nu(\zeta)$. It also guarantees that the measure ν integrates all polynomials of degree ≥ 2.
In this case it is well known (see e.g. [MØP], [DØ], Theorem 3.5, and [DØP],[DØ]) that the Donsker delta functional exists in $(\mathcal{S})^*$ and is given by

$$\begin{aligned} \delta_Y(y) &= \frac{1}{2\pi}\int_{\mathbb{R}} \exp^{\diamond}\big[\int_0^{T_0}\int_{\mathbb{R}}(e^{ix\psi(s,\zeta)}-1)\tilde{N}(ds,d\zeta) + \int_0^{T_0} ix\beta(s)dB(s) \\ &+ \int_0^{T_0}\{\int_{\mathbb{R}}(e^{ix\psi(s,\zeta)}-1-ix\psi(s,\zeta))\nu(d\zeta) - \frac{1}{2}x^2\beta^2(s)\}ds - ixy\big]dx. \end{aligned} \tag{2.5}$$ {eq2.7}

Moreover, we have

$$\begin{aligned} &E[\delta_Y(y)|\mathcal{F}_t] \\ =&\frac{1}{2\pi}\int_{\mathbb{R}} \exp\big[\int_0^t\int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t ix\beta(s)dB(s) \quad (2.6)\\ &+ \int_t^{T_0}\int_{\mathbb{R}}(e^{ix\psi(s,\zeta)}-1-ix\psi(s,\zeta))\nu(d\zeta)ds - \int_t^{T_0}\frac{1}{2}x^2\beta^2(s)ds - ixy\big]dx, \quad (2.7) \end{aligned}$$

and

$$\begin{aligned} &E[D_t\delta_Y(y)|\mathcal{F}_t] = \\ &\frac{1}{2\pi}\int_{\mathbb{R}} \exp\big[\int_0^t\int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t ix\beta(s)dB(s) \\ &+ \int_t^{T_0}\int_{\mathbb{R}}(e^{ix\psi(s,\zeta)}-1-ix\psi(s,\zeta))\nu(d\zeta)ds - \int_t^{T_0}\frac{1}{2}x^2\beta^2(s)ds - ixy\big]ix\beta(t)dx \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &E[D_{t,z}\delta_Y(y)|\mathcal{F}_t] = \\ &\frac{1}{2\pi}\int_{\mathbb{R}} \exp\big[\int_0^t\int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t ix\beta(s)dB(s) \\ &+ \int_t^{T_0}\int_{\mathbb{R}}(e^{ix\psi(s,\zeta)}-1-ix\psi(s,\zeta))\nu(d\zeta)ds - \int_t^{T_0}\frac{1}{2}x^2\beta^2(s)ds - ixy\big](e^{ix\psi(t,z)}-1)dx. \end{aligned} \tag{2.9}$$

For more information about the Donsker delta function and some explicit formulas for it, see [DrØ].

3 The general insider optimal control problem for the stochastic differential games

In this section, we formulate and prove a sufficient and a necessary maximum principle for general stochastic differential games (not necessarily zero-sum games) for insiders. The system we consider, is described in (1.1)-(1.6) above. Then $X(t)$ is $\mathbb{F} \vee Y_1 \vee Y_2$-adapted, and hence using the definition of the Donsker delta functional $\delta_{Y_1,Y_2}(y_1, y_2)$ of (Y_1, Y_2) we get

$$X(t) = x(t, Y_1, Y_2) = x(t, y_1, y_2)_{y_1=Y_1, y_2=Y_2} = \int_{\mathbb{R}^2} x(t, y_1, y_2)\delta_{Y_1,Y_2}(y_1, y_2)dy_1dy_2 \tag{3.1}$$

{eq6}

for some y_1, y_2-parametrized process $x(t, y_1, y_2)$ which is $\mathbb{F}$-adapted for each y_1, y_2. Then, again by the definition of the Donsker delta functional and the properties of forward integration ([DrØ]), we can write

$$\begin{aligned}
X(t) &= x + \int_0^t b(s, X(s), u_1(s), u_2(s), Y_1, Y_2)ds + \int_0^t \sigma(s, X(s), u_1(s), u_2(s), Y_1, Y_2)dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, X(s), u_1(s), u_2(s), Y_1, Y_2, \zeta)\tilde{N}(ds, d\zeta) \\
&= x + \int_0^t b(s, x(s, Y_1, Y_2), u_1(s, Y_1), u_2(s, Y_2), Y_1, Y_2)ds \\
&+ \int_0^t \sigma(s, x(s, Y_1, Y_2), u_1(s, Y_1), u_2(s, Y_2), Y_1, Y_2)dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, Y_1, Y_2), u_1(s, Y_1), u_2(s, Y_2), Y_1, Y_2, \zeta)\tilde{N}(ds, d\zeta) \\
&= x + \int_0^t b(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)_{y_1=Y_1, y_2=Y_2} ds \\
&+ \int_0^t \sigma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)_{y_1=Y_1, y_2=Y_2} dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2, \zeta)_{y_1=Y_1, y_2=Y_2} \tilde{N}(ds, d\zeta) \\
&= x + \int_0^t \int_{\mathbb{R}^2} b(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)\delta_{Y_1, Y_2}(y_1, y_2)dy_1 dy_2 ds \\
&+ \int_0^t \int_{\mathbb{R}^2} \sigma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)\delta_{Y_1, Y_2}(y_1, y_2)dy_1 dy_2 dB(s) \\
&+ \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}} \gamma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2, \zeta)\delta_{Y_1, Y_2}(y_1, y_2)dy_1 dy_2 \tilde{N}(ds, d\zeta) \\
&= x + \int_{\mathbb{R}^2} [\int_0^t b(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)ds \\
&+ \int_0^t \sigma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2, \zeta)\tilde{N}(ds, d\zeta)]\delta_{Y_1, Y_2}(y_1, y_2)dy_1 dy_2
\end{aligned} \tag{3.2}$$

{eq7}

Comparing (3.1) and (3.2) we see that (3.1) holds if we choose $x(t, y)$ for each $y = (y_1, y_2)$ as the solution of the classical SDE

$$\begin{aligned}
& dx(t, y_1, y_2) = b(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2)dt \\
& + \sigma(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2)dB(t) \\
& + \int_{\mathbb{R}} \gamma(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2, \zeta)\tilde{N}(dt, d\zeta); \quad t \geq 0
\end{aligned} \tag{3.3}$$

{eq8}

$$x(0, y) = x, \quad x \in \mathbb{R}. \tag{3.4}$$

Using this notation and setting $u = (u_1, u_2), Y = (Y_1, Y_2)$ and $y = (y_1, y_2)$, the *performance functional* $J_i(u); u = (u_1, u_2)$ of player i defined in (1.4) gets the form

$$\begin{aligned} J_i(u) &= \mathbb{E}[\int_0^T f_i(t, X(t), u_1(t), u_2(t))dt + g_i(X(T))] \quad (3.5) \quad \{eq3.5\} \\ &= \mathbb{E}[\int_{\mathbb{R}^2} \{\int_0^T f_i(t, x(t, y_1, y_2), u_1(t, y_1), , u_2(t, y_2), y_1, y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_t]dt \\ &+ g_i(x(T, y_1, y_2), y_1, y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_T]\}dy] \\ &= \int_{\mathbb{R}^2} j_i(u, y)dy; \quad i = 1, 2. \quad (3.6) \end{aligned}$$

where

$$j_i(u, y) = \mathbb{E}[\int_0^T f_i(t, x(t, y), u(t, y), y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt + g_i(x(T, y), y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]] \quad (3.7)$$

As before, a *Nash equilibrium* for the game (1.5)-(1.6) is a pair $\hat{u} = (\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\sup_{u_1 \in \mathcal{A}_1} J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) \quad (3.8) \quad \{eq3.6\}$$

and

$$\sup_{u_2 \in \mathcal{A}_2} J_2(\hat{u}_1, u_2) \leq J_2(\hat{u}_1, \hat{u}_2). \quad (3.9) \quad \{eq3.7\}$$

Or, equivalently,

$$\sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} j_1(u_1(y_1), \hat{u}_2(y_2))dy_2 \Big) dy_1 \leq \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} j_1(\hat{u}_1(y_1), \hat{u}_2(y_2))dy_2 \Big) dy_1 \quad (3.10) \quad \{eq3.6a\}$$

and

$$\sup_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} j_2(\hat{u}_1(y_1), u_2(y_2))dy_1 \Big) dy_2 \leq \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} j_2(\hat{u}_1(y_1), \hat{u}_2(y_2))dy_1 \Big) dy_2. \quad (3.11) \quad \{eq3.7a\}$$

We now solve the maximization problem (3.10) by maximizing the inner integral pointwise for each y_1 and we solve the maximization problem (3.11) by maximizing the integral integral pointwise for each y_2, i.e. we get the following problem:

Problem 3.1 *Find* $(u_1^\star(., y_1), u_2^\star(., y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$ *such that*

$$\sup_{u_1(.,y_1) \in \mathcal{A}_1} \int_{\mathbb{R}} j_1(u_1(., y_1), u_2^*(., y_2))dy_2 \leq \int_{\mathbb{R}} j_1(u_1^*(., y_1), u_2^*(., y_2))dy_2; \text{ for each } y_1 \quad (3.12) \quad \{eq3.6b\}$$

and

$$\sup_{u_2(.,y_2) \in \mathcal{A}_2} \int_{\mathbb{R}} j_2(u_1^*(., y_1), u_2(., y_2))dy_1 \leq \int_{\mathbb{R}} j_2(u_1^*(., y_1), u *_2 (., y_2))dy_1; \text{ for each } y_2. \quad (3.13) \quad \{eq3.7b\}$$

To study this problem we present two maximum principles for the corresponding games. The first is the following:

4 A sufficient maximum principle

The problem (3.12)-(3.13) may be regarded as a stochastic differential game with a standard (albeit parametrized) stochastic differential equation (3.3) for the state process $x(t, y_1, y_2)$, but with a non-standard performance functional given by (3.5). We can solve this problem by a modified maximum principle approach, as follows:

Define the *Hamiltonians* $H_i : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \to \mathbb{R}$ by

$$\begin{aligned}&H_i(t,x,y_1,y_2,u_1,u_2,p,q,r) = H(t,x,y_1,y_2,u_1,u_2,p,q,r,\omega)\\ &= \mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_t] f_i(t,x,u_1,u_2,y_1,y_2) + b(t,x,u_1,u_2,y_1,y_2)p\\ &+ \sigma(t,x,u_1,u_2,y_1,y_2)q + \int_{\mathbb{R}} \gamma(t,x,u_1,u_2,y_1,y_2) r(t,\zeta)\nu(d\zeta); i = 1,2. \end{aligned} \quad (4.1)$$ {eq11}

Here $\mathcal{R}$ denotes the set of all functions $r(.) : \mathbb{R} \to \mathbb{R}$ such that the last integral above converges. For $i = 1,2$ we define the *adjoint* processes $p_i(t,y_1,y_2), q_i(t,y_1,y_2), r_i(t,y_1,y_2,\zeta)$ as the solution of the y_1, y_2-parametrised BSDEs

$$\begin{cases} dp_i(t,y_1,y_2) = -\frac{\partial H_i}{\partial x}(t,y_1,y_2)dt + q_i(t,y_1,y_2)dB(t) + \int_{\mathbb{R}} r_i(t,y_1,y_2,\zeta)\tilde{N}(dt,d\zeta); \quad 0 \le t \le T\\ p_i(T,y_1,y_2) = g_i'(x(T,y_1,y_2),y_1,y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_T] \end{cases} \quad (4.2)$$ {eq12}

To study the problem (3.12)-(3.13) we present two maximum principles for the corresponding games. The first is the following:

Theorem 4.1 *[Sufficient maximum principle]* {sufficient
Let $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ with associated solution $\hat{x}(t,y_1,y_2), \hat{p}_i(t,y_1,y_2), \hat{q}_i(t,y_1,y_2), \hat{r}_i(t,y_1,y_2,\zeta)$ of (3.3) and (4.2); i=1,2. Assume that the following hold:

1. $x \to g_i(x)$ *is concave;* $i = 1,2$
2. *The functions*

$$\hat{\mathcal{H}}_1(x) = \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}} H_1(t,x,y_1,y_2,u_1,\hat{u}_2(t,y_2),\widehat{p}_1(t,y_1,y_2),\widehat{q}_1(t,y_1,y_2),\hat{r}_1(t,y_1,y_2,\cdot))dy_2 \quad (4.3)$$

and

$$\hat{\mathcal{H}}_2(x) = \sup_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} H_2(t,x,y_1,y_2,\hat{u}_1(t,y_1),u_2,\widehat{p}_2(t,y_1,y_2),\widehat{q}_2(t,y_1,y_2),\hat{r}_2(t,y_1,y_2,\cdot))dy_1 \quad (4.4)$$

are concave for all t, y_1, y_2

3.

$$\begin{aligned}&\sup_{u_1 \in \mathbf{A}_1} \int_{\mathbb{R}} H_1\big(t,\widehat{x}(t,y_1,y_2),u_1,\hat{u}_2(t,y_2),\widehat{p}_1(t,y_1,y_2),\widehat{q}_1(t,y_1,y_2),\hat{r}_1(t,y_1,y_2,\cdot)\big)dy_2\\ &= \int_{\mathbb{R}} H_1\big(t,\widehat{x}(t,y_1,y_2),\widehat{u}_1(t,y_1),\hat{u}_2(t,y_2),\widehat{p}_1(t,y_1,y_2),\widehat{q}_1(t,y_1,y_2),\hat{r}_1(t,y_1,y_2,\cdot)\big)dy_2\end{aligned}$$

for all t, y_1. (4.5)

4.

$$\sup_{u_2\in\mathbf{A}_2}\int_{\mathbb{R}} H_2\big(t,\widehat{x}(t,y_1,y_2),\hat{u}_1(t,y_1),u_2,\widehat{p}_2(t,y_1,y_2),\widehat{q}_2(t,y_1,y_2),\hat{r}_2(t,y_1,y_2,\cdot)\big)dy_1$$
$$=\int_{\mathbb{R}} H_2\big(t,\widehat{x}(t,y_1,y_2),\widehat{u}_1(t,y_1),\hat{u}_2(t,y_2),\widehat{p}_2(t,y_1,y_2),\widehat{q}_2(t,y_1,y_2),\hat{r}_2(t,y_1,y_2,\cdot)\big)dy_1$$
$$\text{for all } t,y_2. \tag{4.6}$$

Then $(u_1^*(.,y_1),u_2^*(.,y_2)):=(\widehat{u}_1(.,y_1),\widehat{u}_2(.,y_2))$ *is a Nash equilibrium for Problem 3.1.*

Proof. By considering an increasing sequence of stopping times τ_n converging to T, we may assume that all local integrals appearing in the computations below are martingales and hence have expectation 0. We omit the details in this argument. See [ØS2].
We first prove that

$$\sup_{u_1(.,y_1)\in\mathcal{A}_1}\int_{\mathbb{R}}\int_{\mathbb{R}} j_1(u_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \le \int_{\mathbb{R}}\int_{\mathbb{R}} j_1(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \tag{4.7}$$ {eq2.17}

Choose arbitrary $u_1(.,y_1)\in\mathcal{A}_1$ and let us in the following, for simplicity of notation, put $x(t,y_1,y_2)=x^{u_1,\hat{u}_2}(t,y_1,y_2), \hat{x}(t,y_1,y_2)=x^{\hat{u}_1,\hat{u}_2}(t,y_1,y_2)$,
$b(t,y_1,y_2)=b(t,x(t,y_1,y_2),u_1(t,y_1),u_2(t,y_2),\omega), \hat{b}(t,y_1,y_2)=b(t,\hat{x}(t,y_1,y_2),u_1(t,y_1),\hat{u}_2(t,y_2),\omega)$ and similarly with $\sigma(t,y_1,y_2),\hat{\sigma}(t,y_1,y_2),\gamma(t,y_1,y_2,\zeta),\hat{\gamma}(t,y_1,y_2,\zeta)$ and $\tilde{x}(t,y_1,y_2)=x(t,y_1,y_2)-\hat{x}(t,y_1,y_2)$. Let us also put

$$H_1(t,y_1,y_2)=H_1(t,x(t,y_1,y_2),y_1,y_2,u_1(t,y_1),\hat{u}_2(t,y_2),\widehat{p}_1(t,y_1,y_2),\widehat{q}_1(t,y_1,y_2),\hat{r}_1(t,y_1,y_2,\cdot)) \tag{4.8}$$

and

$$\hat{H}_1(t,y_1,y_2)=H_1(t,\hat{x}(t,y_1,y_2),y_1,y_2,\hat{u}_1(t,y_1),\hat{u}_2(t,y_2),\widehat{p}_1(t,y_1,y_2),\widehat{q}_1(t,y_1,y_2),\hat{r}_1(t,y_1,y_2,\cdot)) \tag{4.9}$$

Consider

$$\int_{\mathbb{R}}\int_{\mathbb{R}}[J_1(u_1(.,y_1),\hat{u}_2(.,y_2))-J_1(\widehat{u}_1(.,y_1),\hat{u}_2(.,y_2))]dy_1dy_2=I_1+I_2,$$

where

$$I_1=\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{E}\Big[\int_0^T\{f_1(t,x(t,y_1,y_2),u_1(t,y_1),\hat{u}_2(t,y_2))-f_1(t,\hat{x}(t,y_1,y_2),\hat{u}_1(t,y_1),\hat{u}_2(t,y_2))\}$$
$$\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_t]dt\Big]dy_1dy_2 \tag{4.10}$$ {I_1}

and

$$I_2=\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{E}\big[\{g_1(x(T,y_1,y_2))-g_1(\widehat{x}(T,y_1,y_2))\}\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_T]\big]dy_1dy_2. \tag{4.11}$$ {I_2}

By the definition of H_1 we have

$$
\begin{aligned}
I_1 \;=\; & \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\int_0^T \{H_1(t,y_1,y_2) - \widehat{H}_1(t,y_1,y_2) - \widehat{p}_1(t,y_1,y_2)\widetilde{b}_1(t,y_1,y_2) - \widehat{q}_1(t,y_1,y_2)\widetilde{\sigma}(t,u_1,u_2,y_1,y_2) \\
- & \int_{\mathbb{R}} \hat{r}_1(t,y_1,y_2,\zeta)\tilde{\gamma}(t,y_1,y_2,\zeta)\nu(d\zeta)\}dt]dy_1dy_2. \qquad (4.12)
\end{aligned}
$$

{I

Since g_1 is concave we have

$$
\begin{aligned}
I_2 \leq & \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[g_1'(\widehat{x}(T,y_1,y_2))\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_T]\widetilde{x}(T,y_1,y_2)]dy_1dy_2 \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\widehat{p}_1(T,y_1,y_2)\widetilde{x}(T,y_1,y_2)]dy_1dy_2 \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\int_0^T \widehat{p}_1(t,y_1,y_2)d\widetilde{x}(t,y_1,y_2) + \int_0^T \widetilde{x}(t,y_1,y_2)d\widehat{p}_1(t,y_1,y_2) + \int_0^T d[\hat{p}_1,\tilde{x}]_t]dy_1dy_2 \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\big[\int_0^T \widehat{p}_1(t,y_1,y_2)(\widetilde{b}(t,y_1,y_2)dt + \widetilde{\sigma}(t,y_1,y_2)dB(t) + \int_{\mathbb{R}} \tilde{\gamma}(t,y_1,y_2,\zeta)\tilde{N}(dt,d\zeta)) \\
- & \int_0^T \frac{\partial \widehat{H}_1}{\partial x}(t,y_1,y_2)\widetilde{x}(t,y_1,y_2)dt + \int_0^T \widehat{q}_1(t,y_1,y_2)\widetilde{x}(t,y_1,y_2)dB(t) \qquad (4.13) \quad \{II\_2\} \\
+ & \int_0^T \int_{\mathbb{R}} \widetilde{x}(t,y_1,y_2)\hat{r}_1(t,y_1,y_2,\zeta)\tilde{N}(dt,d\zeta) + \int_0^T \widetilde{\sigma}(t,y_1,y_2)\widehat{q}_1(t,y_1,y_2)dt \\
+ & \int_0^T \int_{\mathbb{R}} \tilde{\gamma}(t,y_1,y_2,\zeta)\hat{r}_1(t,y_1,y_2,\zeta)\nu(d\zeta)dt + \int_0^T \int_{\mathbb{R}} \tilde{\gamma}(t,y_1,y_2,\zeta)\hat{r}_1(t,y_1,y_2,\zeta)\tilde{N}(dt,d\zeta)\big]dy_1dy_2 \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\big[\int_0^T \widehat{p}_1(t,y_1,y_2)\widetilde{b}(t,y_1,y_2)dt - \int_0^T \frac{\partial \widehat{H}_1}{\partial x}(t,y_1,y_2)\widetilde{x}(t,y_1,y_2)dt + \int_0^T \widetilde{\sigma}(t,y_1,y_2)\widehat{q}_1(t)dt \\
+ & \int_0^T \int_{\mathbb{R}} \tilde{\gamma}(t,y_1,y_2,\zeta)\hat{r}_1(t,y_1,y_2,\zeta)\nu(d\zeta)dt\big]dy_1dy_2.
\end{aligned}
$$

Adding (4.12) - (4.13) we get, by concavity of H_1,

$$
\begin{aligned}
& \int_{\mathbb{R}}\int_{\mathbb{R}} [J_1(u_1(.,y_1),\hat{u}_2(.,y_2)) - J_1(\widehat{u}_1(.,y_1),\hat{u}_2(.,y_2))]dy_1dy_2 \\
\leq & \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\big[\int_0^T \{H_1(t,y_1,y_2) - \widehat{H}_1(t,y_1,y_2) - \frac{\partial \hat{H}_1}{\partial x}(t,y_1,y_2)\widetilde{x}(t,y_1,y_2)\}dt\big]dy_1dy_2 \qquad (4.14) \quad \{eq2.28\}
\end{aligned}
$$

Since $\hat{\mathcal{H}}_1(x)$ is concave, it follows by a standard separating hyperplane argument that there exists a supergradient $a \in \mathbb{R}$ for $\hat{\mathcal{H}}_1(x)$ at $x = \hat{x}(t,y_1,y_2)$ such that if we define

$$\phi(x) = \hat{\mathcal{H}}_1(x) - \hat{\mathcal{H}}_1(\hat{x}(t,y_1,y_2)) - a(x - \hat{x}(t,y_1,y_2)) \qquad (4.15)$$

then

$$\phi(x) \leq 0 \text{ for all } x \qquad (4.16)$$

On the other hand, we clearly have

$$\phi(\hat{x}(t,y_1,y_2)) = 0 \tag{4.17}$$

it follows that

$$\frac{\partial \hat{\mathcal{H}}_1}{\partial x}(\hat{x}(t,y_1,y_2)) = \int_{\mathbb{R}} \frac{\partial \hat{H}_1}{\partial x}(t,\hat{x}(t,y_1,y_2),\hat{u}_1(t,y_1),\hat{u}_2(t,y_2),\hat{p}(t,y_1,y_2),\hat{q}(t,y_1,y_2),\hat{r}(t,y_1,y_2,\zeta)dy_2 = a \tag{4.18}$$

Combining this with (4.14), we get

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}[J_1(u_1(.,y_1),\hat{u}_2(.,y_2)) - J_1(\widehat{u}_1(.,y_1),\hat{u}_2(.,y_2))]dy_1dy_2 \\
&\leq \int_0^T \int_{\mathbb{R}}[\hat{\mathcal{H}}_1(x(t,y_1,y_2)) - \hat{\mathcal{H}}_1(\hat{x}(t,y_1,y_2)) - \frac{\partial \hat{\mathcal{H}}_1}{\partial x}(\hat{x}(t,y_1,y_2))(x(t,y_1,y_2) - \hat{x}(t,y_1,y_2))]dy_1dt \\
&\leq 0 \text{ since } \hat{\mathcal{H}}_1 \text{ is concave.}
\end{aligned} \tag{4.19}$$

Hence

$$\sup_{u_1(.,y_1)\in\mathcal{A}_1} \int_{\mathbb{R}}\int_{\mathbb{R}} J_1(u_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \leq \int_{\mathbb{R}}\int_{\mathbb{R}} J_1(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2. \tag{4.20}$$ {eq2.17}

□

4.1 The case when only one of the players is an insider.

It is useful also to have a formulation in the partly degenerate case when only one of the players, say player number 1, has inside information. Then the control of player 1 is $\mathbb{H}_1$-adapted as before, while player 2 is $\mathbb{F}$-adapted. In this case we define the *Hamiltonians* $H_i : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{U}\times\mathbb{R}\times\mathbb{R}\times\mathcal{R}\times\Omega \to \mathbb{R}$ by

$$\begin{aligned}
&H_i(t,x,y_1,u_1,u_2,p,q,r) = H(t,x,y_1,u_1,u_2,p,q,r,\omega) \\
&= \mathbb{E}[\delta_{Y_1}(y_1)|\mathcal{F}_t]f_i(t,x,u_1,u_2,y_1) + b(t,x,u_1,u_2,y_1)p \\
&+ \sigma(t,x,u_1,u_2,y_1)q + \int_{\mathbb{R}} \gamma(t,x,u_1,u_2,y_1)r(t,\zeta)\nu(d\zeta); i = 1,2.
\end{aligned} \tag{4.21}$$ {eq2.35}

Here, as before, $\mathcal{R}$ denotes the set of all functions $r(.) : \mathbb{R} \to \mathbb{R}$ such that the last integral above converges. For $i = 1,2$ we define the *adjoint* processes $p_i(t,y_1), q_i(t,y_1), r_i(t,y_1,\zeta)$ as the solution of the y_1-parametrised BSDEs

$$\begin{cases} dp_i(t,y_1) = -\frac{\partial H_i}{\partial x}(t,y_1)dt + q_i(t,y_1)dB(t) + \int_{\mathbb{R}} r_i(t,y_1,\zeta)\tilde{N}(dt,d\zeta); \quad 0\leq t\leq T \\ p_i(T,y_1) = g_i'(x(T,y_1),y_1)\mathbb{E}[\delta_{Y_1}(y_1)|\mathcal{F}_T] \end{cases} \tag{4.22}$$ {eq2.36}

Let $j_i(u(.,y_1))$ be defined by

$$j_i(u(.,y_1)) = \mathbb{E}[\int_0^T f_i(t,x(t,y_1),u_1(t,y_1),u_2(t),y_1)\mathbb{E}[\delta_{Y_1}(y_1)|\mathcal{F}_t]dt \\ + g_i(x(T,y_1),y_1)\mathbb{E}[\delta_{Y_1}(y_1)|\mathcal{F}_T]]; \quad i=1,2. \tag{4.23}$$
{eq2.37}

Then, with J_i as in (3.5) we see that

$$J_i(u_1,u_2) = \int_{\mathbb{R}} j_i(u(.,y_1))dy_1; \quad i=1,2. \tag{4.24}$$
{eq2.38}

Then, in analogy with Problem 3.1 we now get the following game problem:

Problem 4.2 *Find* $(u_1^\star(.,y_1),u_2^\star(.)) \in \mathcal{A}_1 \times \mathcal{A}_2$ *such that*

$$\sup_{u_1(.,y_1)\in\mathcal{A}_1} j_1(u_1(.,y_1),u_2^*(.)) \le j_1(u_1^*(.,y_1),u_2^*(.)); \text{ for each } y_1 \tag{4.25}$$
{eq3.6c}

and

$$\sup_{u_2(.)\in\mathcal{A}_2} \int_{\mathbb{R}} j_2(u_1^*(.,y_1),u_2(.)dy_1 \le \int_{\mathbb{R}} j_2(u_1^*(.,y_1),u_2^*(.))dy_1. \tag{4.26}$$
{eq3.7c}

Theorem 4.3 *[Sufficient maximum principle with only one insider]* {sufficient
Suppose $Y_2 = 0$, *i.e. player number 2 has no inside information. Let* $(\hat{u_1},\hat{u_2}) \in \mathcal{A}_1 \times \mathcal{A}_2$ *with associated solution* $\hat{x}(t,y_1),\hat{p}_i(t,y_1),\hat{q}_i(t,y_1),\hat{r}_i(t,y_1,\zeta)$ *of* (3.3) *and* (4.2); *i=1,2. Assume that the following hold:*

1. $x \to g_i(x)$ *is concave;* $i=1,2$
2. *The functions*

$$\hat{\mathcal{H}}_1(x) = \sup_{u_1\in\mathcal{A}_1} H_1(t,x,y_1,u_1,\hat{u_2}(t),\widehat{p}_1(t,y_1),\widehat{q}_1(t,y_1),\hat{r}_1(t,y_1,\cdot)) \tag{4.27}$$

   and

$$\hat{\mathcal{H}}_2(x) = \sup_{u_2\in\mathcal{A}_2} \int_{\mathbb{R}} H_2(t,x,y_1,\hat{u}_1(t,y_1),u_2,\widehat{p}_2(t,y_1),\widehat{q}_2(t,y_1),\hat{r}_2(t,y_1,\cdot))dy_1 \tag{4.28}$$

   are concave for all t.

3.

$$\sup_{u_1\in\mathbf{A}_1} H_1\big(t,\widehat{x}(t,y_1),u_1,\hat{u_2}(t),\widehat{p}_1(t,y_1),\widehat{q}_1(t,y_1),\hat{r}_1(t,y_1,\cdot)\big) \\ = H_1\big(t,\widehat{x}(t,y_1),\widehat{u}_1(t,y_1),\hat{u_2}(t),\widehat{p}_1(t,y_1),\widehat{q}_1(t,y_1),\hat{r}_1(t,y_1,\cdot)\big) \\ \text{for all } t. \tag{4.29}$$

4.

$$\sup_{u_2\in\mathbf{A}_2}\int_{\mathbb{R}} H_2\big(t,\widehat{x}(t,y_1),\hat{u}_1(t,y_1),u_2,\widehat{p}_2(t,y_1),\widehat{q}_2(t,y_1),\hat{r}_2(t,y_1,\cdot)\big)dy_1$$
$$=\int_{\mathbb{R}} H_2\big(t,\widehat{x}(t,y_1),\widehat{u}_1(t,y_1),\hat{u_2}(t),\widehat{p}_2(t,y_1),\widehat{q}_2(t,y_1),\hat{r}_2(t,y_1,\cdot)\big)dy_1$$
$$\text{for all } t. \tag{4.30}$$

Then $(u_1^*(.,y_1),u_2^*(.)):=(\widehat{u}_1(.,y_1),\widehat{u}_2(.))$ *is a Nash equilibrium for Problem 4.2.*

Proof. The proof is similar to the proof of Theorem 4.2 and is omitted. □

5 A necessary maximum principle

We proceed to establish a corresponding *necessary* maximum principle. For this, we do not need concavity conditions, but in stead we need the following assumptions about the set of admissible control values:

- A_1. For all $t_0\in[0,T], y_i\in\mathbb{R}$ and all bounded $\mathcal{F}_{t_0}$-measurable random variables $\alpha_i(y_i,\omega)$, the control $\theta_i(t,y_i,\omega):=\mathbf{1}_{[t_0,T]}(t)\alpha_i(y_i,\omega)$ belongs to $\mathcal{A}_i$ for $i=1,2$.

- A_2. For all $u_i;\beta_0^i\in\mathcal{A}_i$ with $\beta_0^i(t,y_i)\le K<\infty$ for all t,y_i define

$$\delta_i(t,y_i)=\frac{1}{2K}dist((u_i(t,y_i),\partial\mathbb{A}_i)\wedge 1>0 \tag{5.1}$$ {delta}

  and put

$$\beta_i(t,y_i)=\delta_i(t,y_i)\beta_0^i(t,y_i). \tag{5.2}$$ {beta(t,y)}

  Then there exists $\delta>0$ such that the control

$$\widetilde{u}_i(t,y_i)=u_i(t,y_i)+a\beta_i(t,y_i);\quad t\in[0,T]$$

  belongs to $\mathcal{A}_i$ for all $a\in(-\delta,\delta)$ for $i=1,2$.

- $A3$. For all β_i as in (5.2) the derivative processes

$$\chi_1(t,y_1,y_2):=\frac{d}{da}x^{(u_1+a\beta_1,u_2)}(t,y_1,y_2)|_{a=0}$$

  and

$$\chi_2(t,y_1,y_2):=\frac{d}{da}x^{(u_1,u_2+a\beta_2)}(t,y_1,y_2)|_{a=0}$$

exists, and belong to $\mathbf{L}^2(\lambda \times \mathbf{P})$ and

$$\begin{cases} d\chi_1(t,y_1,y_2) = [\frac{\partial b}{\partial x}(t,y_1,y_2)\chi_1(t,y_1,y_2) + \frac{\partial b}{\partial u_1}(t,y)\beta_1(t,y_1)]dt \\ +[\frac{\partial \sigma}{\partial x}(t,y_1,y_2)\chi_1(t,y) + \frac{\partial \sigma}{\partial u_1}(t,y_1,y_2)\beta_1(t,y_1)]dB(t) \\ +\int_{\mathbb{R}}[\frac{\partial \gamma}{\partial x}(t,y_1,y_2,\zeta)\chi_1(t,y_1,y_2) + \frac{\partial \gamma}{\partial u_1}(t,y_1,y_2,\zeta)\beta_1(t,y_1)]\tilde{N}(dt,d\zeta) \\ \chi_1(0,y_1,y_2) = \frac{d}{da}x^{(u_1+a\beta_1,u_2)}(0,y_1,y_2)|_{a=0} = 0. \end{cases} \quad (5.3)$$

{d chi}

and

$$\begin{cases} d\chi_2(t,y) = [\frac{\partial b}{\partial x}(t,y_1,y_2)\chi_2(t,y_1,y_2) + \frac{\partial b}{\partial u_2}(t,y_1,y_2)\beta_2(t,y_2)]dt \\ +[\frac{\partial \sigma}{\partial x}(t,y_1,y_2)\chi_2(t,y_1,y_2) + \frac{\partial \sigma}{\partial u_2}(t,y_1,y_2)\beta_2(t,y_2)]dB(t) \\ +\int_{\mathbb{R}}[\frac{\partial \gamma}{\partial x}(t,y_1,y_2,\zeta)\chi_2(t,y_1,y_2) + \frac{\partial \gamma}{\partial u_2}(t,y_1,y_2,\zeta)\beta_2(t,y_2)]\tilde{N}(dt,d\zeta) \\ \chi(0,y_1,y_2) = \frac{d}{da}x^{(u_1,u_2+a\beta_2)}(0,y_1,y_2)|_{a=0} = 0. \end{cases} \quad (5.4)$$

{d chi2}

Theorem 5.1 *[Necessary maximum principle]* {necessary}
Let $(u_1,u_2) \in \mathcal{A}_1 \times \mathcal{A}_2$. Then the following are equivalent:

1. $\frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} j_1(u_1 + a\beta_1, u_2)|_{a=0}dy_1dy_2 = \frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} j_2(u_1, u_2 + a\beta_2)|_{a=0}dy_1dy_2 = 0$ *for all bounded $\beta_i \in \mathcal{A}_i$ of the form (5.2).*

2. $$\begin{aligned} &[\int_{\mathbb{R}} \frac{\partial H_1}{\partial v_1}(t,x(t,y_1,y_2),v_1,u_2(t,y_2),p_1(t,y_1,y_2),q_1(t,y_1,y_2),r_1(t,y_1,y_2,.))dy_2]_{v_1=u_1(t,y_1)} \\ &= [\int_{\mathbb{R}} \frac{\partial H_2}{\partial v_2}(t,x(t,y_1,y_2),u_1(t,y_1),v_2,p_2(t,y_1,y_2),q_2(t,y_1,y_2),r_2(t,y_1,y_2,.))dy_1]_{v_2=u_2(t,y_2)} \\ &= 0 \quad \forall t \in [0,T]. \end{aligned} \quad (5.5)$$

Proof. By considering an increasing sequence of stopping times τ_n converging to T, we may assume that all local integrals appearing in the computations below are martingales and hence have expectation 0. See [ØS2].
We can write

$$\frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} j_1(u_1 + a\beta_1, u_2)|_{a=0}dy_1dy_2 = I_1 + I_2$$

where

$$\begin{aligned} I_1 = \frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\int_0^T & f_1(t,x^{u_1+a\beta_1}(t,y_1,y_2),u_1(t,y_1)+a\beta_1(t,y_1),u_2(t,y_2),y_1,y_2) \\ & \mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_t]dt]|_{a=0}dy_1dy_2 \end{aligned} \quad (5.6)$$

and

$$I_2 = \frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[g_1(x^{(u_1+a\beta_1,u_2)}(T,y_1,y_2),y_1,y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_T]]|_{a=0}dy_1dy_2.$$

By our assumptions on f_1 and g_1 and by (4.2) we have

$$I_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[\int_0^T \{\frac{\partial f_1}{\partial x}(t, y_1, y_2)\chi_1(t, y_1, y_2) + \frac{\partial f_1}{\partial u_1}(t, y_1, y_2)\beta_1(t, y_1)\}\mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_t]dt]dy_1 dy_2 \tag{5.7}$$

{iii1}

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[g_1'(x(T, y_1, y_2), y_1, y_2)\chi_1(T, y_1, y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1, y_2)|\mathcal{F}_T]]dy_1 dy_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[p_1(T, y_1, y_2)\chi_1(T, y_1, y_2)]dy_1 dy_2 \end{aligned} \tag{5.8}$$

{iii2}

By the Itô formula

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[p_1(T,y_1,y_2)\chi_1(T,y_1,y_2)]dy_1dy_2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\int_0^T p_1(t,y_1,y_2)d\chi_1(t,y_1,y_2) + \int_0^T \chi_1(t,y_1,y_2)dp_1(t,y_1,y_2) + \int_0^T d[\chi_1,p_1](t,y_1,y_2)]dy_1dy_2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\int_0^T p_1(t,y_1,y_2)\{\frac{\partial b}{\partial x}(t,y_1,y_2)\chi_1(t,y_1,y_2) + \frac{\partial b}{\partial u_1}(t,y_1,y_2)\beta_1(t,y_1)\}dt \qquad (5.9)\\
&+ \int_0^T p_1(t,y_1,y_2)\{\frac{\partial \sigma}{\partial x}(t,y_1,y_2)\chi_1(t,y_1,y_2) + \frac{\partial \sigma}{\partial u_1}(t,y_1,y_2)\beta_1(t,y_1)\}dB(t) \\
&+ \int_0^T\int_{\mathbb{R}} p_1(t,y_1,y_2)\{\frac{\partial \gamma}{\partial x}(t,y_1,y_2,\zeta)\chi_1(t,y_1,y_2) + \frac{\partial \gamma}{\partial u_1}(t,y_1,y_2,\zeta)\beta_1(t,y_1)\}\tilde{N}(dt,d\zeta) \\
&- \int_0^T \chi_1(t,y_1,y_2)\frac{\partial H_1}{\partial x}(t,y_1,y_2)dt + \int_0^T \chi_1(t,y_1,y_2)q_1(t,y_1,y_2)dB(t) \\
&+ \int_0^T\int_{\mathbb{R}} \chi_1(t,y_1,y_2)r_1(t,y_1,y_2,\zeta)\tilde{N}(dt,d\zeta) \qquad (5.10)\\
&+ \int_0^T q_1(t,y_1,y_2)\{\frac{\partial \sigma}{\partial x}(t,y)\chi_1(t,y_1,y_2) + \frac{\partial \sigma}{\partial u_1}(t,y_1,y_2)\beta_1(t,y_1)\}dt \\
&+ \int_0^T\int_{\mathbb{R}} \{\frac{\partial \gamma}{\partial x}(t,y_1,y_2,\zeta)\chi_1(t,y_1,y_2) + \frac{\partial \gamma}{\partial u_1}(t,y_1,y_2,\zeta)\beta_1(t,y_1)\}r_1(t,y_1,y_2,\zeta)\nu(\zeta)dt]dy_1dy_2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\int_0^T \chi_1(t,y_1,y_2)\{p_1(t,y_1,y_2)\frac{\partial b}{\partial x}(t,y_1,y_2) + q_1(t,y_1,y_2)\frac{\partial \sigma}{\partial x}(t,y_1,y_2) - \frac{\partial H_1}{\partial x}(t,y_1,y_2) \\
&+ \int_{\mathbb{R}} \frac{\partial \gamma}{\partial x}(t,y_1,y_2,\zeta)r_1(t,y_1,y_2,\zeta)\nu(d\zeta)\}dt \\
&+ \int_0^T \beta_1(t,y_1)\{p_1(t,y_1,y_2)\frac{\partial b}{\partial u_1}(t,y_1,y_2) + q_1(t,y_1,y_2)\frac{\partial \sigma}{\partial u_1}(t,y_1,y_2) \\
&+ \int_{\mathbb{R}} \frac{\partial \gamma}{\partial u_1}(t,y_1,y_2,\zeta)r_1(t,y_1,y_2,\zeta)\nu(d\zeta)\}dt]dy_1dy_2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[-\int_0^T \chi_1(t,y_1,y_2)\frac{\partial f_1}{\partial x}\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_t]dt \\
&+ \int_0^T \{\frac{\partial H_1}{\partial u_1}(t,y_1,y_2) - \frac{\partial f_1}{\partial u_1}(t,y_1,y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_t]\}\beta_1(t,y_1)dt]dy_1dy_2 \\
&= -I_1 + \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\int_0^T \frac{\partial H_1}{\partial u_1}(t,y_1,y_2)\beta_1(t,y_1)dt]dy_1dy_2. \qquad (5.11)
\end{aligned}
$$

Summing (5.7) and (5.9) we get

$$
\frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} j_1(u_1 + a\beta_1, u_2)|_{a=0}dy_1dy_2 = I_1 + I_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\int_0^T \frac{\partial H_1}{\partial u_1}(t,y_1,y_2)\beta_1(t,y_1)dt]dy_1dy_2.
$$

We conclude that

$$\frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} j_1(u_1 + a\beta_1, u_2)|_{a=0} dy_1 dy_2 = 0$$

if and only if

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\int_0^T \frac{\partial H_1}{\partial u_1}(t, y_1, y_2)\beta_1(t, y_1)dt]dy_1 dy_2 = 0$$

for all bounded $\beta_1 \in \mathcal{A}_1$ of the form (5.2).
Changing the order of integration we can write this as follows:

$$\mathbb{E}[\int_0^T \int_{\mathbb{R}} F_1(t, y_1)\beta_1(t, y_1)dtdy_1] = 0, \quad \forall \beta_1 \in \mathcal{A}_1 \tag{5.12}$$

where

$$F_1(t, y_1) := \int_{\mathbb{R}} \frac{\partial H_1}{\partial u_1}(t, y_1, y_2)dy_2. \tag{5.13}$$

In particular, applying this to $\beta_1(t, y_1) = \theta_1(t, y_1)$ as in $A1$, we get that this is again equivalent to

$$\mathbb{E}[F_1(t, y_1)|\mathcal{F}_t] = 0, \quad \forall t, y_1.$$

Since $F_1(t, y_1)$ is already $\mathcal{F}_t$-adapted, we have

$$\mathbb{E}[F_1(t, y_1)|\mathcal{F}_t] = F_1(t, y_1), \quad \forall t, y_1.$$

So we deduce that

$$F_1(t, y_1) = \int_{\mathbb{R}} \frac{\partial H_1}{\partial u_1}(t, y_1, y_2)dy_2 = 0, \quad \forall t, y_1.$$

A similar argument gives that

$$\frac{d}{da}\int_{\mathbb{R}}\int_{\mathbb{R}} j_2(u_1, u_2 + a\beta_2)|_{a=0} dy_1 dy_2 = 0 \text{ for all bounded } \beta_2 \in \mathcal{A}_2 \tag{5.14}$$

is equivalent to

$$\int_{\mathbb{R}} \frac{\partial H_2}{\partial u_2}(t, y_1, y_2)dy_1 = 0, \quad \forall t, y_2, \tag{5.15}$$

where

$$\begin{aligned}&\frac{\partial H_2}{\partial u_2}(t, y_1, y_2)\\ &= \frac{\partial H_2}{\partial v_2}(t, x(t, y_1, y_2), u_1(t, y_1), v_2, p_2(t, y_1, y_2), q_2(t, y_1, y_2), r_2(t, y_1, y_2, .))_{v_2 = u_2(t, y_2)}.\end{aligned} \tag{5.16}$$

□

6 The zero-sum game case

In the zero-sum case we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} j_1(u_1(.,y_1),u_2(.,y_2)) + j_2(u_1(.,y_1),u_2(.,y_2))dy_1dy_2 = 0. \tag{6.1}$$

Then the Nash equilibrium $(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying (3.12)-(3.13) becomes a saddle point for

$$\int_{\mathbb{R}}\int_{\mathbb{R}} j(u_1(.,y_1),u_2(.,y_2))dy_1dy_2 := \int_{\mathbb{R}}\int_{\mathbb{R}} j_1(u_1(.,y_1),u_2(.,y_2))dy_1dy_2. \tag{6.2}$$

To see this, note that (3.12)-(3.13) imply that

$$\begin{aligned}\int_{\mathbb{R}^2} j_1(u_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \le \int_{\mathbb{R}^2} j_1(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 &= -\int_{\mathbb{R}^2} j_2(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \\ &\le -\int_{\mathbb{R}^2} j_2(\hat{u}_1(.,y_1),u_2(.,y_2))dy_1dy_2\end{aligned} \tag{6.3}$$

and hence

$$\int_{\mathbb{R}^2} j(u_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \le \int_{\mathbb{R}^2} j(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \le \int_{\mathbb{R}^2} j(\hat{u}_1(.,y_1),u_2(.,y_2))dy_1dy_2 \tag{6.4}$$

for all u_1, u_2. From this we deduce that

$$\begin{aligned}&\inf_{u_2\in\mathcal{A}_2}\sup_{u_1\in\mathcal{A}_1}\int_{\mathbb{R}^2} j(u_1(.,y_1),u_2(.,y_2))dy_1dy_2 \le \sup_{u_1\in\mathcal{A}_1}\int_{\mathbb{R}^2} j(u_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \\ &\le \int_{\mathbb{R}^2} j(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \le \inf_{u_2\in\mathcal{A}_2}\int_{\mathbb{R}^2} j(\hat{u}_1(.,y_1),u_2(.,y_2))dy_1dy_2 \\ &\le \sup_{u_1\in\mathcal{A}_1}\inf_{u_2\in\mathcal{A}_2}\int_{\mathbb{R}^2} j(u_1(.,y_1),u_2(.,y_2))dy_1dy_2.\end{aligned} \tag{6.5}$$

Since we always have $\inf\sup \ge \sup\inf$, we conclude that

$$\begin{aligned}&\inf_{u_2\in\mathcal{A}_2}\sup_{u_1\in\mathcal{A}_1}\int_{\mathbb{R}^2} j(u_1(.,y_1),u_2(.,y_2))dy_1dy_2 = \sup_{u_1\in\mathcal{A}_1}\int_{\mathbb{R}^2} j(u_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \\ &= \int_{\mathbb{R}^2} j(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 = \inf_{u_2\in\mathcal{A}_2}\int_{\mathbb{R}^2} j(\hat{u}_1(.,y_1),u_2(.,y_2))dy_1dy_2 \\ &= \sup_{u_1\in\mathcal{A}_1}\inf_{u_2\in\mathcal{A}_2}\int_{\mathbb{R}^2} j(u_1(.,y_1),u_2(.,y_2))dy_1dy_2\end{aligned} \tag{6.6}$$

i.e $(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a saddle point for $\int_{\mathbb{R}}\int_{\mathbb{R}} J(u_1(.,y_1),u_2(.,y_2))dy_1dy_2$. Hence we want to find $(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\begin{aligned}\sup_{u_1\in\mathcal{A}_1}\inf_{u_2\in\mathcal{A}_2}\int_{\mathbb{R}^2} j(u_1(.,y_1),u_2(.,y_2))dy_1dy_2 &= \inf_{u_2\in\mathcal{A}_2}\sup_{u_1\in\mathcal{A}_1}\int_{\mathbb{R}^2} j(u_1(.,y_1),u_2(.,y_2))dy_1dy_2 \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} j(\hat{u}_1(.,y_1),\hat{u}_2(.,y_2))dy_1dy_2 \qquad (6.7)\end{aligned}$$

where

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} j(u(.,y_1,y_2))dy_1dy_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\Big[\int_0^T f(t,x(t,y_1,y_2),u_1(t,y_1),u_2(t,y_2),y_1,y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_t]dt \\ + g(x(T,y_1,y_2),y_1,y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_T]\Big]dy_1dy_2. \qquad (6.8)\end{aligned}$$ {J(u)}

We can regard this problem as having one performance functional common to both players, but where one of the players is maximising and the other is minimising it. Then we get just one Hamiltonian and just one BSDE, as follows: In this case the Hamiltonian H is given by:

$$\begin{aligned}&H(t,x,y_1,y_2,u_1,u_2,p,q,r) = H(t,x,y_1,y_2,u_1,u_2,p,q,r,\omega) \\ &= \mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_t]f(t,x,u_1,u_2,y_1,y_2) + b(t,x,u_1,u_2,y_1,y_2)p \\ &+ \sigma(t,x,u_1,u_2,y_1,y_2)q + \int_{\mathbb{R}}\gamma(t,x,u_1,u_2,y_1,y_2)r(t,\zeta)\nu(d\zeta). \qquad (6.9)\end{aligned}$$ {eq11}

Moreover, there is only one triple (p,q,r) of adjoint processes, given by the BSDE

$$\begin{cases} dp(t,y_1,y_2) = -\frac{\partial H}{\partial x}(t,y_1,y_2)dt + q(t,y_1,y_2)dB(t) + \int_{\mathbb{R}} r(t,y_1,y_2,\zeta)\tilde{N}(dt,d\zeta); \quad 0\le t\le T \\ p(T,y) = g'(x(T,y_1,y_2),y_1,y_2)\mathbb{E}[\delta_{Y_1,Y_2}(y_1,y_2)|\mathcal{F}_T]. \end{cases} \qquad (6.10)$$ {eq4.12}

By proceeding as above we obtain the corresponding sufficient maximum principle for the zero-sum game:

Theorem 6.1 *(Sufficient maximum principle for the zero-sum game)*
Let $(\hat{u_1},\hat{u_2}) \in \mathcal{A}_1\times\mathcal{A}_2$ *with associated solution* $\hat{x}(t,y_1,y_2),\hat{p}(t,y_1,y_2),\hat{q}(t,y_1,y_2),\hat{r}(t,y_1,y_2,\zeta)$ *of* (3.3) *and* (6.10)*. Assume that the following holds:*

1. *the function* $x \to g(x)$ *is affine*

2.

$$\begin{aligned}&\sup_{u_1\in\mathbf{A}_1}\int_{\mathbb{R}} H\big(t,\widehat{x}(t,y_1,y_2),u_1,\hat{u_2}(t,y_2),\widehat{p}(t,y_1,y_2),\widehat{q}(t,y_1,y_2),\hat{r}(t,y_1,y_2,\cdot)\big)dy_2 \\ &= \int_{\mathbb{R}} H\big(t,\widehat{x}(t,y_1,y_2),\widehat{u}_1(t,y_1),\hat{u_2}(t,y_2),\widehat{p}(t,y_1,y_2),\widehat{q}(t,y_1,y_2),\hat{r}(t,y_1,y_2,\cdot)\big)dy_2\end{aligned}$$

for all t, y_1. (6.11)

$$\inf_{u_2 \in \mathbf{A}_2} \int_{\mathbb{R}} H\big(t, \widehat{x}(t,y_1,y_2), \hat{u}_1(t,y_1), u_2, \widehat{p}(t,y_1,y_2), \widehat{q}(t,y_1,y_2), \hat{r}(t,y_1,y_2,\cdot)\big) dy_1$$
$$= \int_{\mathbb{R}} H\big(t, \widehat{x}(t,y_1,y_2), \hat{u}_1(t,y_1), \hat{u}_2(t,y_2), \widehat{p}(t,y_1,y_2), \widehat{q}(t,y_1,y_2), \hat{r}(t,y_1,y_2,\cdot)\big) dy_1$$
for all t, y_2. (6.12)

3. *The function*

$$\hat{\mathcal{H}}(x) = \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}} H(t, x, y_1, y_2, u_1, \hat{u_2}(t,y_2), \widehat{p}(t,y_1,y_2), \widehat{q}(t,y_1,y_2), \hat{r}(t,y_1,y_2,\cdot)) dy_2 \tag{6.13}$$

is concave for all t, y_1,
and the function

$$\underline{\mathcal{H}}(x) = \inf_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} H(t, x, y_1, y_2, \hat{u}_1(t,y_1), u_2, \widehat{p}(t,y_1,y_2), \widehat{q}(t,y_1,y_2), \hat{r}(t,y_1,y_2,\cdot)) dy_1 \tag{6.14}$$

is convex for all t, y_2.

Then $\hat{u}(t,y_1,y_2) = (\hat{u}_1(t,y_1), \hat{u}_2(t,y_2))$ *is a saddle point for* $J(u_1,u_2)$.

Similarly we obtain the following necessary maximum principle for the zero sum game problem:

Theorem 6.2 *[Necessary maximum principle for zero-sum games]*
Assume the conditions of Theorem 5.1 hold. Then the following are equivalent:

1. $\frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J(u_1 + a\beta_1, u_2)|_{a=0} dy_1 dy_2 = \frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J(u_1, u_2 + a\beta_2)|_{a=0} dy_1 dy_2 = 0$ *for all bounded* $\beta_i \in \mathcal{A}_i$ *of the form (5.2).*

2. $$[\int_{\mathbb{R}} \frac{\partial H}{\partial v_1}(t, x(t,y_1,y_2), v_1, u_2(t,y_2), p_1(t,y_1,y_2), q_1(t,y_1,y_2), r_1(t,y_1,y_2,.)) dy_2]_{v_1 = u_1(t,y_1)}$$
$$= [\int_{\mathbb{R}} \frac{\partial H}{\partial v_2}(t, x(t,y_1,y_2), u_1(t,y_1), v_2, , p_2(t,y_1,y_2), q_2(t,y_1,y_2), r_2(t,y_1,y_2,.)) dy_1]_{v_2 = u_2(t,y_2)}$$
$$= 0 \quad \forall t \in [0,T], y_1, y_2. \tag{6.15}$$

7 Applications

7.1 Optimal insider consumption under model uncertainty

Suppose we have a cash flow with consumption, modelled by the process $X(t,Y) = X^{c,\mu}(t,Y)$ defined by:

$$\begin{cases} dX(t,Y) = (\alpha(t,Y) + \mu(t,Y_2) - c(t,Y_1))X(t,Y)dt + \beta(t,Y)X(t,Y)dB(t) + \int_{\mathbb{R}} \gamma(t,Y,\zeta)X(t,Y)\tilde{N}(dt,d\zeta) \\ X(0) = x > 0 \end{cases}$$

Here $\alpha(t,Y), \beta(t,Y), \gamma(t,Y)$ are given coefficients, while $c(t,Y_1) > 0$ is the relative consumption rate chosen by the consumer (player number 1) and $\mu(t,Y_2)$ is a perturbation of the drift term, representing the model uncertainty chosen by the environment (player number 2). Define the performance functional by

$$J(c,\mu) = \mathbb{E}[\int_0^T \{\log(c(t)X(t)) + \frac{1}{2}\mu^2(t)\}dt + \theta \log X(T)] \tag{7.1}$$

where $\theta = \theta(\omega) > 0$ is a given $\mathcal{H}_T$-measurable random variable, and $\frac{1}{2}\mu^2(t)$ represents a penalty rate, penalizing μ for being away from 0. We assume that c is $\mathbb{H}^1$-adapted, while μ is $\mathbb{H}^2$-adapted.

We want to find $c^* \in \mathcal{A}_1$ and $\mu^* \in \mathcal{A}_2$ such that

$$\sup_{c\in\mathcal{A}_1} \inf_{\mu\in\mathcal{A}_2} J(c,\mu) = J(c^*,\mu^*). \tag{7.2}$$ {eq0.2}

As before we rewrite this problem as a classical stochastic differential game with two parameters y_1, y_2. Thus we define, for $y = (y_1,y_2) \in \mathbb{R}\times\mathbb{R}$,

$$\begin{cases} dx(t,y) &= (\alpha(t,y) + \mu(t,y_2) - c(t,y_1))x(t,y)dt + \beta(t,y)x(t,y)dB(t) \\ &\quad + \int_{\mathbb{R}} \gamma(t,y,\zeta)x(t,y)\tilde{N}(dt,d\zeta) \\ x(0,y) &= x > 0 \end{cases} \tag{7.3}$$

and

$$\begin{aligned} &j(c(.,y_1),\mu(.,y_2)) \\ &= \mathbb{E}[\int_0^T \{\log(c(t,y_1)x(t,y)) + \frac{1}{2}\mu^2(t,y_2)\}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt + \theta\log x(T,y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]] \end{aligned} \tag{7.4}$$

The Hamiltonian for this problem is

$$\begin{aligned} &H(t,x,y,c,\mu,p,q,r) \\ &= \{\log(cx) + \frac{1}{2}\mu^2\}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + (\alpha(t,y)+\mu-c)xp + \beta(t,y)xq + x\int_{\mathbb{R}}\gamma(t,y,\zeta)r(\zeta)d\nu(\zeta) \end{aligned} \tag{7.5}$$

and the BSDE for the adjoint processes p, q, r is

$$\begin{cases} dp(t,y) &= -[\frac{1}{x(t,y)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + (\alpha(t,y)+\mu(t,y_2)-c(t,y_1))p(t,y) \\ &\quad +\beta(t,y)q(t,y) + \int_{\mathbb{R}}\gamma(t,y,\zeta)r(\zeta)d\nu(\zeta)]dt \\ &\quad +q(t,y)dB(t) + \int_{\mathbb{R}} r(t,y)\tilde{N}(dt,d\zeta); 0\le t\le T \\ p(T,y) &= \frac{\theta}{x(T,y)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \end{cases} \tag{7.6}$$ {eq0.6}

Define

$$h(t,y) = p(t,y)x(t,y). \qquad (7.7)$$ {eq0.7}

Then by the Itô formula we get

$$\begin{aligned}
dh(t,y) &= x(t,y)[-\frac{1}{x(t,y)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] - (\alpha(t,Y) + \mu(t,Y_2) - c(t,Y_1))p(t,y) - \beta(t,y)q(t,y) \\
&- \int_{\mathbb{R}} \gamma(t,y,\zeta)r(t,\zeta)d\nu(\zeta)]dt \\
&+ p(t,y)(\alpha(t,Y) + \mu(t,Y_2) - c(t,Y_1))x(t,y)dt + p(t,y)\beta(t,y)x(t,y)dB(t) + x(t,y)q(t,y)dB(t) \\
&+ q(t,y)\beta(t,y)x(t,y)dt \\
&+ \int_{\mathbb{R}} [(x(t,y) + \gamma(t,y,\zeta)x(t,y))(p(t,y) + r(t,y,\zeta)) - p(t,y)x(t,y) - p(t,y)\gamma(t,y,\zeta)x(t,y) - x(t,y)r(t,y,\zeta]d
\end{aligned} \qquad (7.8)$$ {eq0.8}

$$\begin{aligned}
&+ \int_{\mathbb{R}} [(x(t,y) + \gamma(t,y,\zeta)x(t,y))(p(t,y) + r(t,y,\zeta)) - p(t,y)x(t,y)]\tilde{N}(dt,d\zeta) \\
&= dF(t,y) + h(t,y)\beta(t,y)dB(t) + h(t,y)\int_{\mathbb{R}} \gamma(t,y,\zeta)\tilde{N}(dt,d\zeta)),
\end{aligned} \qquad (7.9)$$

where

$$\begin{aligned}
&dF(t,y) = \\
&- \mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt + x(t,y)q(t,y)dB(t) + x(t,y)\int_{\mathbb{R}} r(t,y,\zeta)(1 + \gamma(t,y,\zeta))\tilde{N}(dt,d\zeta).
\end{aligned} \qquad (7.10)$$

To simplify this, we define the process $k(t,y)$ by the equation

$$dk(t,y) = k(t,y)\Big[b(t,y)dB(t) + \int_{\mathbb{R}} \lambda(t,y,\zeta)\tilde{N}(dt,d\zeta)\Big] \qquad (7.11)$$ {eq0.11}

for suitable processes b, λ (to be determined).

Then again by the Itô formula we get

$$\begin{aligned}
d(h(t,y)k(t,y)) &= h(t,y)k(t,y)\Big[b(t,y)dB(t) + \int_{\mathbb{R}} \lambda(t,y,\zeta)\tilde{N}(dt,d\zeta)\Big] \\
&+ k(t,y)\Big[dF(t,y) + h(t,y)\beta(t,y)dB(t) + h(t,y)\int_{\mathbb{R}} \gamma(t,y,\zeta)\tilde{N}(dt,d\zeta)\Big] \\
&+ (h(t,y)\beta(t,y) + x(t,y)q(t,y))k(t,y)b(t,y)dt \\
&+ \int_{\mathbb{R}} \Big(h(t,y)\gamma(t,y,\zeta) + x(t,y)r(t,y,\zeta)(1 + \gamma(t,y,\zeta)\Big)k(t,y)\lambda(t,y,\zeta)\tilde{N}(dt,d\zeta) \\
&+ \int_{\mathbb{R}} \Big(h(t,y)\gamma(t,y,\zeta) + x(t,y)r(t,y,\zeta)(1 + \gamma(t,y,\zeta)\Big)k(t,y)\lambda(t,y,\zeta)d\nu(\zeta)dt
\end{aligned} \qquad (7.12)$$ {eq0.12}

Define

$$u(t,y) := h(t,y)k(t,y). \tag{7.13}$$ {eq0.13}

Then the equation above can be written

$$\begin{aligned}
du(t,y) = u(t,y)\Big[& \int_{\mathbb{R}} \gamma(t,y,\zeta)\lambda(t,y,\zeta)d\nu(\zeta)dt \\
& + \{\beta(t,y)+b(t,y)\}dB(t) + \beta(t,y)b(t,y)dt + \int_{\mathbb{R}} \{\lambda(t,y,\zeta)+\gamma(t,y,\zeta)+\lambda(t,y,\zeta)\gamma(t,y,\zeta)\}\tilde{N}(dt,d\zeta) \\
& + k(t,y)\Big[dF(t,y) + x(t,y)q(t,y)b(t,y)dt + \int_{\mathbb{R}} x(t,y)r(t,y,\zeta)\lambda(t,y,\zeta)(1+\gamma(t,y,\zeta))d\nu(\zeta)dt \\
& + \int_{\mathbb{R}} x(t,y)r(t,y,\zeta)\lambda(t,y,\zeta)(1+\gamma(t,y,\zeta))\tilde{N}(dt,d\zeta)\Big]
\end{aligned} \tag{7.14}$$ {eq0.14}

Choose

$$\begin{aligned}
b(t,y) &:= -\beta(t,y) \\
\lambda(t,\zeta) &:= -\frac{\gamma(t,y,\zeta)}{1+\gamma(t,y,\zeta)}
\end{aligned} \tag{7.15}$$ {eq0.15}

Then from (7.11) we get

$$\begin{aligned}
k(t,y) = \exp\Big(& \int_0^t -\beta(s,y)dB(s) - \frac{1}{2}\int_0^t \beta^2(s,y)ds - \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,y,\zeta))\tilde{N}(ds,d\zeta) \\
& + \int_0^t \int_{\mathbb{R}} \{\frac{\gamma(s,y,\zeta)}{1+\gamma(t,y,\zeta)} - \ln(1+\gamma(s,y,\zeta))\}\nu(d\zeta)ds\Big),
\end{aligned} \tag{7.16}$$ {eq7.16a}

and (7.14) reduces to

$$\begin{aligned}
du(t,y) = & f(t,y)dt + k(t,y)x(t,y)q(t,y)dB(t) \\
& + \int_{\mathbb{R}} \{x(t,y)r(t,y,\zeta)(1+\gamma(t,y,\zeta))[k(t,y)+k(t,y)\lambda(t,y,\zeta)]\}\tilde{N}(dt,d\zeta),
\end{aligned} \tag{7.17}$$ {eq7.16}

where

$$\begin{aligned}
f(t,y) = & -k(t,y)E[\delta_Y(y)|\mathcal{F}_t] + u(t,y)[\int_{\mathbb{R}} \gamma(t,y,\zeta)\lambda(t,\zeta)d\nu(\zeta) + \beta(t,y)b(t,y)] \\
& + k(t,y)x(t,y)q(t,y)b(t,y) + k(t,y)\int_{\mathbb{R}} x(t,y)r(t,y,\zeta)\lambda(t,y,\zeta)(1+\gamma(t,y,\zeta))d\nu(\zeta)
\end{aligned} \tag{7.18}$$ {eq0.17}

Now define

$$\begin{aligned}
v(t,y) &:= k(t,y)x(t,y)q(t,y) \\
w(t,y) &:= k(t,y)x(t,y)r(t,y,\zeta).
\end{aligned} \tag{7.19}$$ {eq0.18}

Then from (7.12) and (7.15) we get the following BSDE in the unknowns u, v, w:

$$\begin{aligned} du(t,y) =&\Big(- k(t,y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] - u(t,y)[\int_{\mathbb{R}} \frac{\gamma^2(t,y,\zeta)}{1+\gamma(t,y,\zeta)} d\nu(\zeta) + \beta^2(t,y)] \\ &- \beta(t,y)v(t,y) - \int_{\mathbb{R}} \gamma(t,y,\zeta)w(t,y,\zeta)d\nu(\zeta)\Big)dt \\ &+ v(t,y)dB(t) + \int_{\mathbb{R}} w(t,y,\zeta)\tilde{N}(dt,d\zeta); \quad 0 \le t \le T \\ u(T,y) =&\theta k(T,y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \end{aligned} \tag{7.20}$$ {eq0.19}

This is a linear BSDE which has a unique solution $u(t,y) = p(t,y)x(t,y)k(t,y), v(t,y), w(t,y,\zeta)$, where $u(t,y)$ is given explicitly by (see e.g. Theorem 1.7 in [ØS2])

$$u(t,y) = \frac{1}{\Gamma(t,y)} E[\Gamma(T,y)\theta k(T,y)E[\delta_Y(y)|\mathcal{F}_T] + \int_t^T \Gamma(s,y)k(s,y)E[\delta_Y(y)|\mathcal{F}_s]ds|\mathcal{F}_t], \tag{7.21}$$ {eq7.21}

where

$$\begin{aligned} \Gamma(t,y) = \exp\Big(&\int_0^t \beta(s,y)dB(s) + \frac{1}{2}\int_0^t \beta^2(s,y)ds + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,y,\zeta))\tilde{N}(ds,d\zeta) \\ &+ \int_0^t \int_{\mathbb{R}} \{\ln(1+\gamma(s,y,\zeta)) - \frac{\gamma(s,y,\zeta)}{1+\gamma(s,y,\zeta)}\}\nu(d\zeta)ds\Big). \end{aligned} \tag{7.22}$$ {eq7.22}

Combining (7.16) and (7.22) we see that

$$\Gamma(t,y)k(t,y) = 1. \tag{7.23}$$

Substituted into (7.21) this gives

$$u(t,y) = k(t,y)\Big(E[\theta E[\delta_Y(y)|\mathcal{F}_T]|\mathcal{F}_t] + (T-t)E[\delta_Y(y)|\mathcal{F}_t]\Big). \tag{7.24}$$

In particular, we get

$$p(t,y)x(t,y) = \frac{u(t,y)}{k(t,y)} = E[\theta E[\delta_Y(y)|\mathcal{F}_T]|\mathcal{F}_t] + (T-t)E[\delta_Y(y)|\mathcal{F}_t]. \tag{7.25}$$ {eq0.20}

Maximizing $\int_{\mathbb{R}} H dy_2$ with respect to c gives the first order equation

$$\int_{\mathbb{R}} \{\frac{1}{c(t,y_1)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] - x(t,y)p(t,y)\}dy_2 = 0, \tag{7.26}$$

i.e., by (7.25),

$$\begin{aligned} c(t,y_1) = \hat{c}(t,y_1) &= \frac{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dy_2}{\int_{\mathbb{R}} x(t,y)p(t,y)dy_2} \\ &= \frac{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dy_2}{\int_{\mathbb{R}} \Big(E[\theta E[\delta_Y(y)|\mathcal{F}_T]|\mathcal{F}_t] + (T-t)E[\delta_Y(y)|\mathcal{F}_t]\Big)dy_2}. \end{aligned} \tag{7.27}$$ {eq0.22}

Minimizing $\int_{\mathbb{R}} H dy_1$ with respect to μ gives the first order equation

$$\int_{\mathbb{R}} \{\mu(t, y_2)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + x(t, y)p(t, y)\} dy_1 = 0, \tag{7.28}$$

i.e., by (7.25),

$$\mu(t, y_2) = \hat{\mu}(t, y_2) = -\frac{\int_{\mathbb{R}} \Big(E[\theta E[\delta_Y(y)|\mathcal{F}_T]|\mathcal{F}_t] + (T-t)E[\delta_Y(y)|\mathcal{F}_t] \Big) dy_1}{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] dy_1}. \tag{7.29}$$

{eq0.24}

We can now verify that $\hat{c}, \hat{\mu}$ satisfies all the conditions of the sufficient maximum principle, and hence we conclude the following:

Theorem 7.1 (Optimal consumption for an insider under model uncertainty) *The solution (c^*, μ^*) of the stochastic differential game (7.2) is given by*

$$c^*(t, Y_1) = \frac{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] dy_2|_{y_1 = Y_1}}{\int_{\mathbb{R}} \Big(E[\theta E[\delta_Y(y)|\mathcal{F}_T]|\mathcal{F}_t] + (T-t)E[\delta_Y(y)|\mathcal{F}_t] \Big) dy_2|_{y_1 = Y_1}} \tag{7.30}$$

{eq0.25}

and

$$\mu^*(t, Y_2) = -\frac{\int_{\mathbb{R}} \Big(E[\theta E[\delta_Y(y)|\mathcal{F}_T]|\mathcal{F}_t] + (T-t)E[\delta_Y(y)|\mathcal{F}_t] \Big) dy_1|_{y_2 = Y_2}}{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] dy_1|_{y_2 = Y_2}}. \tag{7.31}$$

{eq0.26}

An interesting, and perhaps surprising, consequence of Theorem 7.1 is the following, which is a partial extension to model uncertainty of Theorem 3.1 in [Ø]:

Corollary 7.2 *Suppose θ is a deterministic constant. Then $c^*(t)$ and $\mu^*(t)$ are deterministic also. In fact, we have*

$$c^*(t, Y_1) = c^*(t) = \frac{1}{\theta + T - t} \tag{7.32}$$

and

$$\mu^*(t, Y_2) = -(\theta + T - t). \tag{7.33}$$

Remark 7.3 Note that this last result states that if θ is deterministic, then the two players do not need any information about the system, not even inside information, to find the optimal respective controls.

7.2 Optimal insider portfolio under model uncertainty

Consider a financial market with two investment possibilities:

- (i) A risk free investment possibility with unit price $S_0(t) = 1$ for all $t \in [0, T]$

- (ii) A risky investment, where the unit price $S(t) = S(t, Y)$ is modelled by the (forward) SDE

$$dS(t,Y) = S(t,Y)[(\alpha(t,Y) + \mu(t))dt + \beta(t,Y)dB(t)]; S(0) > 0. \tag{7.34}$$
{eq00.1}

Here $\alpha(t, Y), \beta(t, Y)$ are given $\mathbb{H}$-adapted coefficients, while $\mu(t)$ is a perturbation of the drift term, representing the model uncertainty chosen by the environment (player number 2).

Suppose the wealth process $X(t, Y) = X^{\pi,\mu}(t, Y)$ associated to an insider portfolio $\pi(t, Y)$ (representing the fraction of the wealth invested in the risky asset) is given by:

$$\begin{cases} dX(t,Y) = \pi(t,Y)X(t,Y)[(\alpha(t,Y) + \mu(t))dt + \beta(t,Y)]dB(t) \\ X(0) = x > 0 \end{cases} \tag{7.35}$$
{eq00.2}

Define the performance functional by

$$J(\pi, \mu) = \mathbb{E}[\int_0^T \frac{1}{2}\mu^2(t)dt + \theta \log X(T)], \tag{7.36}$$
{eq00.3}

where $\theta > 0$ is a given (deterministic) constant, and $\frac{1}{2}\mu^2(t)$ represents a penalty rate, penalizing μ for being away from 0. We assume that π is $\mathbb{H}$-adapted, while μ is $\mathbb{F}$-adapted, i.e. has no inside information.

We want to find $\pi^* \in \mathcal{A}_1$ and $\mu^* \in \mathcal{A}_2$ such that

$$\sup_{\pi \in \mathcal{A}_1} \inf_{\mu \in \mathcal{A}_2} J(\pi, \mu) = \inf_{\mu \in \mathcal{A}_2} \sup_{\pi \in \mathcal{A}_1} J(\pi, \mu) = J(\pi^*, \mu^*). \tag{7.37}$$
{eq00.4}

We rewrite this problem as a classical stochastic differential game with one parameter $y_1 = y \in \mathbb{R}$. Thus we define

$$\begin{cases} dx(t,y) & = \pi(t,y)x(t,y)[\{\alpha(t,y) + \mu(t)\}dt + \beta(t,y)dB(t)] \\ x(0,y) & = x(y) > 0 \end{cases} \tag{7.38}$$
{eq00.5}

and

$$J(\pi(.,y), \mu(.)) = \mathbb{E}[\int_0^T \frac{1}{2}\mu^2(t)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt + \theta \log x(T,y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]]. \tag{7.39}$$
{eq00.6}

The Hamiltonian for this problem is

$$H(t,x,y,\pi,\mu,p,q) = \frac{1}{2}\mu^2\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + \pi x(\alpha(t,y) + \mu)p + \pi x\beta(t,y)q \tag{7.40}$$
{eq00.7}

and the BSDE for the adjoint processes p, q is

$$\begin{cases} dp(t,y) = -[\pi(t,y)\{(\alpha(t,y)+\mu(t))p(t,y)+\beta(t,y)q(t,y)\}]dt + q(t,y)dB(t) 0 \leq t \leq T \\ p(T,y) = \frac{\theta\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]}{x(T,y)}. \end{cases} \tag{7.41}$$ {eq00.8}

Maximizing H with respect to π gives the first order equation

$$x(t,y)[(\alpha(t,y)+\mu(t))p(t,y)+\beta(t,y)q(t,y)] = 0. \tag{7.42}$$

Since $x(t,y) > 0$ and $\beta(t,y) \neq 0$, we deduce that

$$(\alpha(t,y)+\mu(t))p(t,y)+\beta(t,y)q(t,y) = 0 \tag{7.43}$$

and

$$q(t,y) = -\frac{\alpha(t,y)+\mu(t)}{\beta(t,y)}p(t,y). \tag{7.44}$$

Hence (7.41) reduces to

$$\begin{cases} dp(t,y) = -\frac{\alpha(t,y)+\mu(t)}{\beta(t,y)}p(t,y)dB(t) \\ p(T,y) = \frac{\theta\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]}{x(T,y)}. \end{cases} \tag{7.45}$$ {eq00.12}

Define

$$h(t,y) = p(t,y)x(t,y). \tag{7.46}$$ {eq00.13}

Then by the Itô formula we get

$$\begin{cases} dh(t,y) &= (\pi(t,y)\beta(t,y) - \frac{\alpha(t,y)+\mu(t)}{\beta(t,y)})h(t,y)dB(t) \\ h(T,y) &= p(T,y)x(T,y) = \theta E[\delta_Y(y)|\mathcal{F}_T]. \end{cases} \tag{7.47}$$ {eq00.14}

This BSDE has the solution

$$h(t,y) = \theta E[\delta_Y(y)|\mathcal{F}_t]. \tag{7.48}$$ {eq00.15}

Moreover, by the generalized Clark-Ocone formula we have

$$(\pi(t,y)\beta(t,y) - \frac{\alpha(t,y)+\mu(t)}{\beta(t,y)})h(t,y) = D_t h(t) = \theta E[D_t\delta_Y(y)|\mathcal{F}_t], \tag{7.49}$$ {eq00.16}

from which we get the following expression for our candidate $\hat{\pi}(t,y)$ for the optimal portfolio

$$\hat{\pi}(t,y) = \frac{\alpha(t,y)+\hat{\mu}(t)}{\beta^2(t,y)} + \frac{E[D_t\delta_Y(y)|\mathcal{F}_t]}{\beta(t,y)E[\delta_Y(y)|\mathcal{F}_t]}, \tag{7.50}$$ {eq00.17}

where $\hat{\mu}(t)$ is the corresponding candidate for the optimal perturbation.

Minimizing $\int_{\mathbb{R}} H dy$ with respect to μ gives the following first order equation for the optimal $\hat{\mu}(t)$:

$$\int_{\mathbb{R}} \{\hat{\mu}(t)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + \hat{\pi}(t,y)\hat{x}(t,y)\hat{p}(t,y)\}dy = 0, \tag{7.51}$$

i.e.,

$$\hat{\mu}(t) = -\frac{\int_{\mathbb{R}} \hat{\pi}(t,y)\hat{x}(t,y)\hat{p}(t,y)dy}{\int_{\mathbb{R}} \hat{x}(t,y)\hat{p}(t,y)dy} = -\int_{\mathbb{R}} \hat{\pi}(t,y)E[\delta_Y(y)|\mathcal{F}_t]dy. \tag{7.52}$$ {eq00.19}

We can now verify that $(\hat{\pi}, \hat{\mu})$ satisfies all the conditions of the sufficient maximum principle, and hence we conclude the following:

Theorem 7.4 (Optimal portfolio for an insider under model uncertainty) *The saddle point $(\pi^*(t,Y), \mu^*(t))$, where $\pi^*(t,Y) = \pi^*(t,y)|_{y=Y}$, of the stochastic differential game (7.35) is given by the solution of the following coupled system of equations*

$$\pi^*(t,y) = \frac{\alpha(t,y) + \mu^*(t)}{\beta^2(t,y)} + \frac{E[D_t\delta_Y(y)|\mathcal{F}_t]}{\beta(t,y)E[\delta_Y(y)|\mathcal{F}_t]}, \tag{7.53}$$ {eq00.20}

and

$$\mu^*(t) = -\int_{\mathbb{R}} \pi^*(t,y)E[\delta_Y(y)|\mathcal{F}_t]dy. \tag{7.54}$$ {eq00.21}

Remark 7.5 This result is an extension to insider trading of a result in [ØS4].

Consider the special case when Y is a Gaussian random variable of the form

$$Y = Y(T_0);\ \text{where } Y(t) = \int_0^t \psi(s)dB(s),\ \text{for } t \in [0, T_0] \tag{7.55}$$ {eqY}

for some deterministic function $\psi \in \mathbf{L}^2[0, T_0]$ such that

$$\|\psi\|^2_{[0,T]} := \int_t^T \psi(s)^2 ds > 0 \text{ for all } t \in [0, T]. \tag{7.56}$$

In this case it is well known that the Donsker delta functional is given by

$$\delta_Y(y) = (2\pi v)^{-\frac{1}{2}} \exp^{\diamond}[-\frac{(Y-y)^{\diamond 2}}{2v}] \tag{7.57}$$

where we have put $v := \|\psi\|^2_{[0,T_0]}$. See e.g. [AaØU], Proposition 3.2. We have

$$\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] = (2\pi\|\psi\|^2_{[t,T_0]})^{-\frac{1}{2}} \exp[-\frac{(Y(t)-y)^2}{2\|\psi\|^2_{[t,T_0]}}]. \tag{7.58}$$

and

$$\mathbb{E}[D_t\delta_Y(y)|\mathcal{F}_t] = -(2\pi\|\psi\|^2_{[t,T_0]})^{-\frac{1}{2}}) \exp[-\frac{(Y(t)-y)^2}{2\|\psi\|^2_{[t,T_0]})}] \frac{Y(t)-y}{\|\psi\|^2_{[t,T_0]}}\psi(t). \tag{7.59}$$

For more details see [DrØ].

Corollary 7.6 *Suppose that Y is Gaussian of the form (7.55). Then the saddle point $(\pi^*(t,Y),\mu^*(t))$, where $\pi^*(t,Y)=\pi^*(t,y)|_{y=Y}$, of the stochastic differential game* (7.35) *is given by the solution of the following coupled system of equations*

$$\pi^*(t,Y)=\frac{\alpha(t,y)+\mu^*(t)}{\beta^2(t,y)}+\frac{Y(T_0)-Y(t)}{\beta(t,y)\|\psi\|^2_{[t,T_0]}}\psi(t), \tag{7.60}$$

{eq00.20}

and

$$\mu^*(t)=-\theta(2\pi\|\psi\|^2_{[t,T_0]})^{-\frac{1}{2}}\int_{\mathbb{R}}\exp[-\frac{(Y(t)-y)^2}{2\|\psi\|^2_{[t,T_0]}}]\pi^*(t,y)dy. \tag{7.61}$$

{eq00.21}

Corollary 7.7 *Suppose that $Y=B(T_0)$ for some $T_0>T$. Then the saddle point $(\pi^*(t,Y),\mu^*(t))$, where $\pi^*(t,Y)=\pi^*(t,y)|_{y=Y}$, of the stochastic differential game* (7.35) *is given by the solution of the following coupled system of equations*

$$\pi^*(t,y)=\frac{\alpha(t,y)+\mu^*(t)}{\beta^2(t,y)}+\frac{B(T_0)-B(t)}{\beta(t,y)(T_0-t)};\quad 0\le t\le T \tag{7.62}$$

{eq00.20}

and

$$\mu^*(t)=-\theta(2\pi(T_0-t))^{-\frac{1}{2}}\int_{\mathbb{R}}\exp[-\frac{(B(t)-y)^2}{2(T_0-t)}]\pi^*(t,y)dy;\quad 0\le t\le T. \tag{7.63}$$

{eq00.21}

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