Housing investment and consumption in an open economy

Kjetil Stiansen

Master of Philosophy in Economics

Department of Economics

University of Oslo

May 10 2016
Housing investment and consumption in an open economy

Kjetil Stiansen
Preface

Acknowledgements

Thanks to supervisor Asbjørn Rødseth for his comments on earlier versions. A big thanks also to Helene Onshuus for valuable discussion and comments along the way. All remaining errors are my own.

Kjetil Stiansen

May 10 2016
Abstract

The article present a three period oig model showing how households can use housing investment to smooth consumption of housing services and consumption goods over the life cycle in the presence of borrowing constraints. Households face a borrowing constraint restricting them to only borrow up to a fraction of the value of their housing investment. Households have an incentive to buy houses when young rather than rent in order to shift some of the cost of housing consumption to the future. When households must buy a minimal house size in order to become homeowners, poor households may be forced to rent rather than buy when young. They then lose access to credit markets, making them unable to smooth consumption between periods. Conditioning access to credit markets on housing investment therefore amplifies the effects of income inequality on consumption inequality.
## Contents

1 Introduction 1

2 Related literature 3

3 Model set up 6

4 Unconstrained model 9
   4.1 Equilibrium ............................................. 11
   4.2 Steady state ............................................. 11

5 Model with borrowing constraint 13
   5.1 Model set up ............................................. 13
   5.2 Behaviour of young households .............................. 14
   5.3 Equilibrium ............................................. 16
   5.4 Steady state ............................................. 17
      5.4.1 Behaviour of prices and debt levels in steady state 19
      5.4.2 Consumption goods consumption of the young ............. 22
   5.5 Simulation of change in $\gamma$ ............................. 22
      5.5.1 Behaviour of prices .................................. 23
      5.5.2 Behaviour of bond holdings ............................. 24
      5.5.3 Behaviour of housing consumption ...................... 26
      5.5.4 Behaviour of consumption goods consumption ......... 26

6 Model with income inequality 29
   6.1 Model set up ............................................. 30
   6.2 Behaviour of poor households .............................. 31
   6.3 Behaviour of rich households .............................. 32
   6.4 Equilibrium ............................................. 34
   6.5 Steady state ............................................. 34
      6.5.1 Housing consumption of poor young in steady state ........ 35
      6.5.2 Consumption goods consumption of rich young in steady state 37
      6.5.3 Prices and debt levels ................................ 39

7 Conclusions 41

8 Appendix 43
   8.1 Households in the unconstrained model do not rent .......... 43
   8.2 Unconstrained households in the models with borrowing constraints do not rent 44
   8.3 Sufficient condition on $\gamma$ for constrained households not wanting to rent 44
8.4 Condition on endowment of young such that they always borrow up to borrowing constraint in steady state ........................................... 46

References ........................................................................... 48
List of Tables

1. Values used in calculation of the constrained model .................................. 18
2. Values used in calculating model with income inequality .............................. 35

List of Figures

1. House prices in steady state in the constrained and unconstrained model ......... 19
2. House purchase demand by young households for different values of $\gamma$ .......... 20
3. Bond holdings for each generation and aggregate bond holdings in steady state for different values of $\gamma$ ................................................................. 21
4. Consumption of consumption goods by the young in steady state in the constrained and unconstrained economy ......................................................... 23
5. The evolution of prices when $\gamma$ is increased from 0.1 to 0.2 ......................... 24
6. The evolution of bond holdings of each generation and aggregate bond holdings when $\gamma$ is increased from 0.1 to 0.2 .................................................... 25
7. The evolution of the housing consumption of different groups when $\gamma$ is increased from 0.1 to 0.2 ................................................................. 27
8. The evolution of the consumption goods consumption of each generation and aggregate consumption when $\gamma$ is increased from 0.1 to 0.2 ................. 28
9. Housing consumption by poor young households in steady state for different values of $\gamma$ ................................................................. 36
10. Consumption level for the rich young for different levels of $\gamma$ with and without rental market ................................................................. 38
11. The purchase price of housing for different levels of $\gamma$ .............................. 39
12. Housing demand by young and interest payments in steady state .................. 40
1 Introduction

House prices have increased substantially in Norway as well as other countries in recent years. In light of this, there are sometimes warnings coming from economists warning households against investing in housing despite the rising prices because prices might drop \(^1\). Furthermore, some argue that prices will drop, pointing to experiences from countries such as the United States and Spain that witnessed similar price increases, followed by massive price reductions \(^2\).

In this article I present a model suggesting that housing investment may be an important way for households to smooth consumption over the life cycle because housing investment can be financed, at least partially, by credit. If other sources of credit are limited, households may therefore have good reasons to buy houses even if they cannot make money from price increases. Particularly, young households expecting higher income in the future may want to borrow against their future income to consume more when young. If access to unsecured consumption loans is limited, housing investment may be the best way to do this.

I analyze this in a three period overlapping generations model of a small open economy where households consume housing services and consumption goods and give bequests to their children. Housing services can be consumed either by buying houses or by renting from other households. Throughout I assume that households have a preference for living in owned as opposed to rented accommodation. This ensures that households that do not face financing constraints choose to buy rather than rent in equilibrium.

In the model, households can borrow to buy houses, subject to a down payment requirement, but cannot borrow to buy consumption goods. Households have low income when young and higher income when middle aged and old. This gives them an incentive to borrow when young and save when middle aged in order to smooth consumption over the life cycle.

By buying houses when young, households in the model are able to transfer some of the cost of housing consumption when young to the future. This allows them to spend more of their income early in life to buy consumption goods rather than on paying for housing services. In this way households are able to smooth consumption of both housing services and consumption goods over the life cycle even though they are only able to borrow for buying houses.

Households in the model optimally overconsume housing relative to consumption goods in the first period of their lives and have an incentive to buy houses rather than rent when young. They overconsume housing because housing consumption can be partially financed by credit while consumption of consumption goods cannot. Similarly, the fact that buying lets households transfer some of the cost of housing consumption to the future gives them an incentive to buy

\(^1\)dn.no: http://www.dn.no/privat/2015/11/12/0615/Boligmarkedet/-folk-tenker-krisen-treffer-ikke-meg-jeg-skal-ha-bolig

\(^2\)dn.no: http://www.dn.no/privat/eiendom/2012/10/17/-forvent-en-ubehagelig-slutt-pa-boligboblen-i-norge
rather than rent. If they rent instead of buying they lose access to credit and must pay the entire cost of housing consumption in the same period.

The model predicts that a sudden change in the down payment requirement affects house prices differently in the short run compared to the long run. In the short run, an increase in the down payment requirement causes a drop in demand for houses by young households because of their reduced access to credit. House prices must then fall in order to induce the middle aged and old to buy the houses freed up by the drop in demand by the young.

While an increase in the down payment requirement causes a fall in house prices in the short run, it causes an increase in the long run. The reason for this is that a higher down payment requirement forces households to borrow less when young. This in turn means that they must spend a smaller share of their income on paying back debt and making interest payments later in life. This frees up resources to be spent on houses as well as consumption goods, pushing up house prices.

Finally, I show how down payment requirements may amplify the effects of income inequality on consumption inequality. If there is a minimal size of houses that households have to buy if they are to become homeowners, some households may be unable to pay the down payment on the smallest house. Households who are unable to pay the down payment requirement on the smallest type of house when young must rent housing instead and are shut out of credit markets. Poor young households with low income are the ones who have the most to gain from getting access to credit, since their income only affords them a very low level of consumption of housing and consumption goods. When they cannot afford the down payment on the smallest kind of house, they become unable to smooth consumption between the first and second period of their lives at all. This means that they must accept a very low level of consumption when young.

When poor young households rent housing, young households with higher income can increase their consumption possibilities by renting out to the poorer group. They do this by borrowing to buy houses and rent out some of it to the poor. In other words, while the rich young cannot borrow for buying consumption goods directly, they can borrow to buy houses, rent these out and spend the rental income on buying more consumption goods. Rich young households therefore benefit from the fact that some groups are shut out of housing markets.

The article will proceed as follows: In section 4 I present a version of the model without borrowing constraints to be used as a baseline. In this section I show the optimal behaviour of households if there are no imperfections in credit markets. Because of households’ preference for owned versus rented housing, all generations buy housing and the rental market drops out of the model.

In section 5 I introduce a borrowing constraint allowing households to borrow to buy houses,
subject to a down payment requirement, but not to buy consumption goods. I show that for sufficiently low levels of the down payment requirement, households strictly prefer buying rather than renting houses in the first period of their lives. When this is the case the rental market drops out of the model as in the model without the borrowing constraint.

In section 6 I introduce income inequality and a minimal size of houses that households have to buy if they are to become homeowners. When the down payment requirement is large enough, the it becomes so costly for a group of poor young households to buy the smallest kind of house that they choose to rent rather than buy houses. When this happens, a rental market for housing exists in equilibrium, enabling rich young households to increase their consumption by buying houses and rent out to poor young households.

The simplicity of the model set up means that I can easily compare model outcomes for different levels of down payment requirements, both in and outside of steady state. The simple set up does however mean that there are a number of interesting topics that I am unable to look at. One of these is how changing down payment requirements changes the optimal timing of households’ first house purchase. To analyze this in a good way, one would need a more complex model with more periods.

Another interesting extension of the model would be to add uncertainty. Uncertainty about future income and prices may influence household behaviour in important ways. Notably, if owning housing enables households to better insure themselves against income shocks, as in Fernandez-Villaverde and Krueger (2011), the motive for housing investment early in life may become even stronger.

Adding production is another way the model could be extended. Notably, the behaviour of prices in the long run would be different if the supply of houses could change over time. I have chosen to look at a pure endowment economy mainly for simplicity and because my main focus of the article is to look at households’ ability to smooth consumption over the life cycle. Adding production to the model, while an interesting extension, would complicate this analysis.

2 Related literature

There is a large literature that studies households’ consumption-savings decisions in the presence of borrowing constraints, as surveyed in Heathcote, Storesletten and Violante (2009). Many of these models focus on how households self insure against idiosyncratic and aggregate income shocks in the presence of borrowing constraints. The relationship between these models and my work is therefore largely in the role of borrowing constraints in restricting households’ ability to smooth consumption between periods. Often, these models do not explicitly consider the role of
housing as a separate good yielding both services for the households as well as acting as collateral for loans.

A paper that does consider the role of housing as collateral for loans explicitly is Fernandez-Villaverde and Krueger (2011). In their model, borrowing is restricted to a fraction of the consumers’ stock of consumer durables. In the presence of idiosyncratic income shocks, households accumulate consumer durables in order to self insure against negative income shocks. A central prediction of the model is that households consume less consumption goods and accumulate less financial assets early in life in order to build up a stock of consumer durables. Unlike in my model, which does not have uncertainty, the main reason why households do this in Fernandez-Villaverde and Krueger (2011) is to be better able to insure themselves against income shocks. Like in my model however, a central premise is that access to credit markets is conditioned on being able to post collateral in form of consumer durables, or in my case houses.

The relationship between house prices, housing investment and consumption is studied by Kiyotaki, Michaelides and Nikolov (2011). They develop a life cycle model to study the interaction between household consumption, production and house prices over the life cycle. In their model borrowing constraints restrict households from consuming as much housing early in life as they would like. Instead they gradually move up the property ladder as they accumulate savings.

Their model predicts that relaxing down payment requirements on houses has a surprisingly small effect on house prices. In their model this is explained by the fact that households that are affected by borrowing constraints only constitute a small share of total housing demand. Changes in fundamentals affecting house prices can however have large distributional effects. Households at an early stage in their life, planning to move up the property ladder lose from an increase in house prices, while older households and retirees planning to scale down their housing stock gain.

Gary-Bobo and Nur (2015) formulates a two period OLG model to study the allocation of housing capital between generations. I base my model set up partly on their article. Their model is a closed economy where households live for two periods, consume housing and consumption goods and leave bequest. The main interest of Gary-Bobo and Nur (2015) is the efficiency of the housing allocation in equilibrium. They find that in a closed economy without outside assets, young consume too little housing compared to what is optimal (and the old too much). They proceed to show how a system of taxes can improve upon the efficiency of the allocation of housing between generations.

Gary-Bobo and Nur (2015) is similar to my work in its focus on the efficiency of the allocation of housing over the life cycle, but differs in important respects. The most important difference is that they do not study the effect of borrowing constraints on household behaviour. Households
are unable to perfectly smooth consumption in their model, but this is mainly a result of the assumption of a closed economy and a bequest motive for old households. Together these features of the model combine to yield a real interest rate that is too high, resulting in too high housing and perishable goods consumption of the old.

Gervais (2001) studies how preferential tax treatment of owner occupied housing causes households to over-invest in and over consume owner occupied housing. The article formulates a more complex overlapping generations model where households choose whether to own or rent housing and how much to invest in business capital. As in many countries, imputed rents of owned housing are not taxed in the model. This causes the return on investing in housing larger compared to business capital for households and leads to overinvestment in housing.

Iacoviello and Pavan (2012) studies the consumption of housing and consumption goods over the life cycle and over the business cycle in an overlapping generations model calibrated to match the US economy. In the model, households have to make a down payment to buy houses and derive higher utility from living in owned versus rented accommodation. Unlike me, Iacoviello and Pavan (2012) include both a labour supply decision and uncertainty in their model in order to study the response of equilibrium variables in the presence of shocks. In their model, households cannot themselves rent out housing to other households. Instead, households can rent housing from the banks.

Iacoviello and Pavan (2012) have a more complex model where households have an uncertain life span. Because of this, they are able to discuss the timing of house purchases and the home ownership rates among different age groups to a greater extent than what I can do in a simple three period set up. The general features of housing consumption and debt over the life cycle in my model are similar to those found in Iacoviello and Pavan (2012): In the presence of borrowing constraints and low income early in life, young households own and consume less housing than older households.

In contrast to the papers discussed above, Attanasio et al (2011) take house prices as given exogenously and asks how different groups respond to shocks to house prices. They develop a realistic life cycle model calibrated to match the UK economy and simulate the response the effects of shocks to house prices and income using both realized and counterfactual shocks.

In Attanasio et al (2011), it is the old that respond the most to exogenous shocks to house prices, while the young respond more to changes in income. The reason for this is that old households in the model plan to sell off housing to finance consumption in retirement. Young households on the other hand plan to increase their housing stock over the life cycle. Because young households have a longer lifespan left, they smooth the impact of changes in house prices over a longer period, meaning they react less in the period of the change.
In my model, it is the young that react the most to a change in down payment requirements causing a drop in prices. Young households are the most affected since they are borrowing constrained and therefore are forced to borrow less for house purchases when down payment requirements increase. Old households in my model actually gain from the drop in house prices caused by increased down payment requirements. The reason for this is that they can afford higher housing consumption early in life. Since they give their houses as bequests to their children, they do not sell their houses in the last period of life to rent. While the old reduce their consumption of consumption goods somewhat in my model, this is mainly because housing has become relatively cheaper, leading them to spend more of their income on housing.

On the empirical side, the relationship between house prices, Mian and Sufi (2010) show that households borrowing against increases in house prices were responsible for much of the build up in household leverage between 2002 and 2006 in the US. They show that this effect is the strongest for young households and households with poor credit scores.

The results of Mian and Sufi (2010) suggest that home-ownership is important in determining households’ access to credit. This is further supported by their finding that there is little variation in growth in credit card debt between areas with large increases in house prices and areas in which house prices did not rise as much. The main driver of credit growth in the US in this period was home-equity based borrowing.

3 Model set up

My model set up is close to that of Gary-Bobo and Nur (2015). Households live for three periods, young middle aged and old. Agents die at the end of the third period of their lives. At the start of the next period, a new generation of young households enter the economy to replace the previous old. Each generation consists of a large number of identical agents summing to one, so the total population size at each time is three.

Households consume two types of goods, housing services and consumption goods. At the beginning of each period households receive an endowment of consumption goods. They get a different endowment in each period. The endowment when young is denoted by $w_y$, the endowment when middle aged by $w_m$ and the endowment when old by $w_o$. Throughout i will assume that household endowment is highest when middle aged and lowest when young:

$$w_m > w_o > w_y$$

In addition to their endowment, when old, agents receive a bequest left by the previous old generation. The bequest can be houses, bonds denominated in units of the consumption good,
or both. Old agents receive this bequest at the beginning of the period, at the same time as they receive the endowment.

Housing services can be consumed by buying housing or by renting it from other households. Households have a preference for living in owned versus rented housing. There is a constant stock of housing in the economy given by \( H \).

Denote consumption of consumption goods of generation \( i \) at time \( t \) by \( c^i_t \), owner occupied housing consumption by \( h^o,i_t \), rented housing consumption by \( h^r,i_t \) and the bequest by \( f_{t+1}^o \). Preferences are represented by the utility function

\[
U = \sum_{j=0}^{2} \beta^j \left[ \log(c^i_{t+j}) + \alpha \log(h^r_{t+j} + \theta h^o_{t+j}) \right] + \beta^2 \delta \log(f_{t+3}^o)
\]  

(1)

\( \beta \) is the discount factor, \( \alpha \) measures the strength of the households’ preference for housing services relative to consumption goods. \( \theta > 1 \) captures households’ preference for living in owned versus rented housing.

\( \delta \) measures the strength of the bequest motive. These are so-called warm glow-bequests: households do not include utility of their children in their utility function directly, but derive utility from leaving some inheritance to their children.

In addition to buying housing to consume themselves, households can purchase buy-to-let housing to rent out to other households. The stock of buy-to-let housing owned by generation \( i \) at time \( t \) is given by \( h^l,i_t \).

One can think of the buy-to-let housing in two ways. Households can buy a separate house to rent out in addition to the one they live in, or they can buy a large house with the intention to live in one part of the house and rent out one or more rooms to other households.

Households can buy a one period bond, \( b^i_{t+1} \), freely available at a world interest rate, \( r \). I will assume throughout that the interest rate satisfies

\[
\beta(1 + r) = 1
\]

so the interest equals the discount rate (\( \beta = 1/(1 + \rho) \) where \( \rho \) is the discount rate).

When the interest rate equals the discount rate households will like to keep consumption constant over the life cycle. Households receive a higher endowment when middle aged and old than when they are young. In addition they receive a bequest in old age. This gives households a desire to borrow when they are young in order to increase consumption in the first period of their lives.

7
Denote the price of buying housing in period $t$ by $p_t$ and the price of renting housing by $q_t$. The bequest given by the old measured in square feet of housing available to the next generation is then given by

$$ f_{t+1}^o = h_t^{o,a} + h_t^{l,o} + \frac{(1 + r)b_{t+1}^o}{p_{t+1}} $$

That is, the bequest is the sum of owner occupied and buy to let housing owned by the old at period $t$ plus the amount of bonds they own including interest. The bond holdings are divided by the price of housing in the next period to give how much housing they buy in the next period.

Households can purchase buy to let housing at a price of $p_t$ and rent it out to receive $q_t$. The net cost of a unit of buy to let housing is therefore $p_t - q_t$. The budget constraint of household $i$ at period $t$ is then given by

$$ c_i^t + p_t h_t^{o,i} + q_t h_t^{l,i} + (p_t - q_t)h_t^{l,i} + b_{t+1}^i = x_i^t $$

(2)

Where $x_i^t$ is the "disposable resources" available for generation $i$ at time $t$. These are the endowment they receive plus the value of any financial savings and housing wealth they carry over from the previous period and bequests they receive if any. The disposable resources for each generation are given by

$$ x_i^y = w^y $$

$$ x_i^m = w^m + p_t (h_{t-1}^{o,y} + h_{t-1}^{l,y}) + (1 + r)b_t^y $$

$$ x_i^o = w^o + p_t (h_{t-1}^{o,m} + h_{t-1}^{l,m}) + (1 + r)b_t^o + p_t f_t^o $$

A household buying a unit of buy to let housing $h_t^l$ at price $p_t$, renting it out at price $q_t$ and selling it next period at price $p_{t+1}$ gets a return of

$$ \frac{p_{t+1}}{p_t - q_t} $$

For the household to be indifferent between buying a one period bond and buying buy to let housing the following no arbitrage condition must hold

$$ (1 + r) = \frac{p_{t+1}}{p_t - q_t} $$

(3)

Assuming $(1 + r) > 1$, rearranging this and iterating forward shows that the price of a house
today is the present discounted value of future rental prices:

\[ p_t = \sum_{j=0}^{\infty} \frac{q_t+j}{(1+r)^j} + \lim_{T \to \infty} \frac{p_{t+T}}{(1+r)^T} = \sum_{j=0}^{\infty} \frac{q_t+j}{(1+r)^j} \]

4 Unconstrained model

I first present the solution of the model without any constraints on borrowing besides the natural constraint that households cannot leave life with negative net assets, where net assets are housing wealth plus financial savings (this is always satisfied due to the log utility of bequests). In other words, households are allowed to die with negative bond holdings as long as the value of their housing wealth exceeds the value of the bond holdings. In the unconstrained model, households keep consumption of both consumption goods and housing services constant over the life cycle. They spend constant shares of their budget on housing and consumption goods. The first results stems from the fact that the interest rate equals the discount rate, while the second from the logarithmic utility specification.

When solving the maximization problem it is useful to define the variable

\[ s_{t+1} = \frac{b_{t+1}}{p_t - q_t} \]

This variable can be thought of as bond holdings measured in terms of buy-to-let housing. Define the housing capital of generation \( i \) at time \( t \) as \( h_{c,i}^t = h_{c,i}^t + h_{l,i}^t \). Using these definitions, the budget constraint of generation \( i \) at time \( t \) can be written as

\[ c_t + q_t h_{c,i}^t + (p_t - q_t)(h_{c,i}^t + s_{t+1}) = x_t \tag{4} \]

Using the no-arbitrage condition, \((1 + r) = \frac{p_{t+1}}{p_t - q_t}\) the bequest of the old can be written as

\[ f_{t+1} = h_{c,o}^t \frac{p_{t+1}}{p_{t+1} + q_{t+1}} + s_{t+1} = h_{c,o}^t + s_{t+1} \]

The period by period constraints can then be written as

\[ c_t + q_t (h_{c,y}^t + h_{l,y}^t) + (p_t - q_t)(h_{c,y}^t + s_{t+1}^y) = w_y^t \]
\[ c_{t+1} + q_{t+1} (h_{c,y}^{t+1} + h_{l,y}^{t+1}) + (p_{t+1} - q_{t+1})(h_{c,y}^{t+1} + s_{t+2}^{t+1}) = w_y^{t+1} + p_{t+1}(h_{c,y}^t + s_{t+1}^y) \]
\[ c_{t+2} + q_{t+2} (h_{c,y}^{t+2} + h_{l,y}^{t+2}) + (p_{t+2} - q_{t+2})f_{t+3} = w_{t+2} + p_{t+2}(h_{c,y}^{t+1} + s_{t+2}^{t+1} + f_{t+2}^{t+1}) \]

Before writing the lifetime budget constraint i use that households do not want to rent in the unconstrained model. I prove this in the appendix. The intuition for this is that households
prefer to live in owner occupied housing and face no financing constraints. Thus they have no incentive to rent versus owning. In a lifetime perspective owning and renting costs the same, since the purchase price is just the discounted rental prices. At the same time, households derive higher utility from consuming owned housing than rented housing. In other words, in the unconstrained model the rental market for housing drops out endogenously.

Drop rented housing consumption from the budget constraints and use that 
\( (1 + r) = \frac{p_{t+j+1}}{p_{t+j} - q_{t+j}} \).

Dividing the second constraint by \((1 + r)\) and the third by \((1 + r)^2\) and adding them together yields the lifetime budget constraint

\[
\frac{c^y_t + q_t h^{o,y}_t}{1 + r} + \frac{c^m_{t+1} + q_{t+1} h^{o,m}_{t+1}}{1 + r} + \frac{c^o_{t+2} + q_{t+2} h^{o,o}_{t+2} + (p_{t+2} - q_{t+2}) f^o_{t+3}}{(1 + r)^2} = Y_t
\]

Where \( Y_t \) is the life time income

\[
Y_t = w^y + \frac{w^m}{1 + r} + \frac{w^o + f^o_{t+2}}{(1 + r)^2}
\]

Households maximize their utility function, (1), subject to the lifetime budget constraint and non-negativity constraints on housing purchases

\[
h^{r,i}_t \geq 0 \\
h^{o,i}_t \geq 0
\]

Using that households do not rent, maximizing the utility function subject to the lifetime budget constraint yields the optimality conditions

\[
\frac{\alpha}{h^{o,i}_{t+j}} = \frac{q_{t+j}}{c^i_{t+j}} \\
c^{k}_{t+j+1} = \beta (1 + r) c^i_{t+j}
\]

The logarithmic utility specification means that households spend constant shares of their income on housing and consumption goods. \( \beta (1 + r) = 1 \) implies that they hold consumption constant.

Inserting for this in the lifetime budget constraint yields the demand functions:

\[
c^y_t = c^m_{t+1} = c^o_{t+2} = \frac{Y_t}{(1 + \alpha)(1 + \beta + \beta^2) + \beta^2 \delta} \\
h^{o,y}_t = \frac{\alpha}{q_t} c^y_t, \ h^{o,m}_t = \frac{\alpha}{q_{t+1}} c^m_{t+1}, \ h^{o,o}_t = \frac{\alpha}{q_{t+2}} c^o_{t+2}
\]
The bequest is given by

\[ f_{t+3}^o = \frac{\delta c_{t+2}}{p_{t+2} - q_{t+2}} \]

### 4.1 Equilibrium

An equilibrium in the unconstrained economy is a set of prices \( \{p_t, q_t\} \) and consumption allocations \( \{c_t^i, h_t^i\} \) and bequests, \( f_{t+3}^o \) such that households solve

\[
\max_{\{c_t^i, h_t^i\}} \sum_{j=0}^{2} \beta^j (\log(c_{t+j}^i) + \alpha \log(h_{t+j}^{y,i}) + \beta h_{t+j}^{o,i}) + \delta \beta^2 \log(f_{t+3}^o)
\]

Subject to

\[
c_{t+1}^y + q_t h_{t+1}^{o,y} + \frac{c_{t+1}^o + q_{t+1} h_{t+1}^{o,m}}{1 + r} + \frac{c_{t+2}^o + q_{t+2} h_{t+2}^{o,o} + (p_{t+2} - q_{t+2}) f_{t+3}^o}{(1 + r)^2} \leq Y_t
\]

\[ h_{t+j}^{o,i} \geq 0 \]

The initial middle aged and old maximize utility and the housing market clears in each period:

\[ H = h_t^{o,y} + h_t^{o,m} + h_t^{o,o}, \forall t \]

The sequence of prices satisfies

\[
(1 + r) = \frac{p_{t+1}}{p_t - q_t}
\]

Where \( r \) is given exogenously.

### 4.2 Steady state

I now look at the steady state version of the unconstrained model where consumption levels and prices are constant. To compute the steady state equilibrium, use the demand functions derived above and insert them in the lifetime budget constraint:

\[
c^y = \frac{w^y + \frac{w^{m}}{(1 + r)} + \frac{w^{o} + p f^{o}}{(1 + r)^2}}{(1 + \alpha)(1 + \beta + \beta^2) + \beta^2 \delta}
\]
Use that

\[ f^\alpha = \frac{\delta c^\alpha}{p - q} = \frac{\delta \beta^2 (1 + r)^2 c^\alpha}{p - q} \]

and

\[ (1 + r) = \frac{p}{p - q} \]

insert for the bequest in the equation for \( c^\alpha \):

\[ c^\alpha = \frac{w^y + w^m + \frac{w^o}{1+r} + \delta \beta^2 (1 + r)c^\alpha}{(1 + \alpha)(1 + \beta + \beta^2) + \beta \delta} \]

And rearrange to get the steady state consumption level of the young (and thus the middle aged and old):

\[ c^y = \frac{w^y + w^m + \frac{w^o}{1+r} - \beta^2 \delta r}{(1 + \alpha)(1 + \beta + \beta^2) - \beta^2 \delta r} \]  \( \text{(6)} \)

Use that the demand for housing relative to consumption in each period is given by

\[ h^{\alpha,i} = \frac{\alpha}{q} \]

Insert for housing consumption for each generation in the market clearing condition for the housing market to get the equation for the steady state rental price of housing, \( q \):

\[ q = \frac{\alpha}{H} (c^y + c^m + c^o) = \frac{3\alpha}{H} c^y \]

Where \( c^y \) is defined above. Using this value for \( q \) in the no arbitrage equation for prices yields the purchase price of housing in steady state.

The solution of the unrestricted model is very simple. Households want to have constant consumption levels over the life cycle. Since they have low income when young they borrow in the first period of their lives and buy both consumption goods and housing. Full access to credit ensures that they smooth consumption perfectly. The preference for living in owned housing makes the rental market for housing drop out of the model and everyone becomes homeowners.
5 Model with borrowing constraint

In this section I introduce a borrowing constraint restricting households’ access to credit markets. The borrowing constraint I consider is that households can borrow for buying houses, but not for buying consumption goods. One can think of this as consumers having access to borrowing for house purchases since this kind of borrowing is secured in the house, but not for unsecured consumption loans.

Furthermore, the borrowing constraint is such that households can only borrow a fraction of the cost of their house purchases. One can interpret this as the down payment they have to pay when buying a house.

As before households have low income when they are young and higher when middle aged and old. This combined with the assumption that the interest rate equals the discount rate means that they would like to borrow when young in order to smooth consumption over the life cycle.

I show that if the down payment requirement is low enough relative to the cost of renting and there is no restriction on the minimal size of houses, households buy houses when young and the rental market for houses drops out of the model. By borrowing to buy houses, households are able to partly smooth consumption of both housing services and consumption goods over the life cycle.

5.1 Model set up

The model with borrowing constraint is identical to the unconstrained model except that households now face a borrowing constraint given by

$$b_{t+1}^i \geq -(1 - \gamma)p_t(h_{t}^{e,1} + h_{t}^{c,1})$$

That is, households are only allowed to borrow up to a fraction \((1 - \gamma)\) of the value of their housing capital. I make no further restrictions on the model for now. The down payment requirement should be interpreted broadly as the fraction of the value of the house that households have to pay within the same period. This can be both a standard down payment that you pay when buying the house, but also payments on the principal and intra-period interest payments that they must make during the first period of their lives.

As before, households have low income when they are young and higher income when they are older. They would therefore like to borrow when young in order to increase consumption of both consumption goods and housing services.

In the model without borrowing constraint, the rental market for housing dropped out of the
model because of households’ preference for living in owned accommodation and the absence of financing constraints. In the model with borrowing constraint, the rental market drops out provided

\[ q_t > \gamma p_t \]  \hspace{1cm} (8)

This condition states that the cost of renting per unit of housing is higher than the down payment households have to make per unit of housing. When this is the case, households must give up less consumption goods per unit of housing when they are young if they buy than if they rent. By buying, households only have to pay the down payment in the first period of their lives, transferring the rest of the cost to future periods. Thus, by buying houses rather than renting, households are able to smooth consumption of both housing and consumption goods between the first and second period of their lives.

Note that households have an incentive to buy rather than rent housing even absent any possibility to make money from increases in house prices. In the model, the price of buying a house is the discounted rental prices, so owning and renting costs the same over the life cycle. What matters for households’ decision whether to buy or rent in the first period of their lives is how what it costs them in terms of consumption goods in the same period.

In the following, I am going to look at values for \( \gamma \) such that condition (8) holds, namely such that the rental cost per unit of housing is higher than the down payment cost. My main reason for doing this is that it is only when this holds than households are able to use house purchases to smooth consumption between periods, and it is this behaviour that I want to study.

If the down payment requirement is large enough, households may want to rent in the first period of their lives rather than buy, since the cost of the down payment becomes too large. When they do this they lose access to credit all together and cannot smooth consumption at all. In section 6 I present a version of the model with income inequality and a minimal size of purchased housing. These elements together create a group of young households who rent when young. I therefore postpone the discussion of how households who rent in the first period of their lives behave until this section.

5.2 Behaviour of young households

I now look at how the borrowing constraint affects the behaviour of young households. The Kuhn-Tucker optimality conditions for young households are given by
\[ c_t^y : \frac{1}{c_t^q} - \lambda_t^y = 0 \]

\[ h_{t+1}^{r,y} : \frac{\alpha}{h_{t+1}^{y} + \theta h_{t+1}^{r,y}} - \lambda_t^y q_t + \eta_t^y = 0 \]

\[ h_{t+1}^{q,y} : \frac{\alpha \theta}{h_{t+1}^{y} + \theta h_{t+1}^{r,y}} - p_t \lambda_t^y + \lambda_{t+1}^m \beta p_{t+1} + \phi_t^y (1 - \gamma) p_t + \mu_t^y = 0 \]

\[ h_{t+1}^{b,y} : -\lambda_t^y (p_t - q_t) + \beta p_{t+1} \lambda_{t+1}^m + \phi_t^y (1 - \gamma) p_t + \xi_t^y = 0 \]

\[ b_{t+1}^{y} - \lambda_t^y + \lambda_{t+1}^m (1 + r) \beta + \phi_t^y = 0 \]

\[ \eta_t^y \geq 0, \eta_t^y = 0, \text{ if } h_{t+1}^{r,y} > 0 \]

\[ \mu_t^y \geq 0, \mu_t^y = 0, \text{ if } h_{t+1}^{q,y} > 0 \]

\[ \xi_t^y \geq 0, \xi_t^y = 0, \text{ if } h_{t+1}^{b,y} > 0 \]

\[ \phi_t^y \geq 0, \phi_t^y = 0, \text{ if } b_{t+1}^{y} > -(1 - \gamma) p_t h_{t+1}^{r,y} \]

Where \( \lambda_t^y, \lambda_{t+1}^m, \eta_t^y, \mu_t^y, \xi_t^y, \phi_t^y \) are the Lagrange multipliers on the budget constraints of young and middle aged, the non-negativity constraints on rented, owned and buy to let housing and the borrowing constraint respectively.

When condition (8) holds, young households become homeowners. Provided their period one endowment is sufficiently low, they borrow up to the borrowing constraint (in the appendix I provide a sufficient condition on the endowment of the young to ensure they borrow up to the borrowing constraint in steady state). Combine the first order conditions for bond purchases and owner occupied housing consumption and use that households do not rent to yield

\[ \frac{\alpha}{h_{t+1}^{y}} = \frac{q_t}{c_t^q} - \phi_t^y (q_t - \gamma p_t) \] (9)

Equation (9) shows how the borrowing constraint distorts the intratemporal allocation of resources between housing and consumption goods for young households. Provided condition (8) holds, the second term on the right hand side is always negative meaning the marginal utility of consumption goods measured in units of housing is larger than the marginal utility of housing. This means that young households overconsume housing relative to consumption goods compare to the model without borrowing constraint.

Young households overconsume housing because they can borrow to buy housing but not to buy consumption goods. In a sense, housing becomes relatively cheaper for borrowing constrained young households since they can borrow to finance this type of consumption but not to finance consumption of consumption goods.

The optimality condition for bond purchases tells a similar story for the intertemporal allocation
of consumption goods:

\[
\frac{1}{c_t^y} = \frac{\beta(1 + r)}{c_{t+1}^m} + \phi_t^y
\]

(10)

The marginal utility of consumption goods while young is larger than the marginal utility of consumption goods when middle aged. Households would like to shift consumption forward, but are unable because of the borrowing constraint.

Because the middle aged and old are unconstrained, the young cannot buy housing to rent out. Thus, the rental market for housing drops out of the model.

5.3 Equilibrium

An equilibrium in the model with borrowing constraints is a set of prices \( \{p_t, q_t\} \) and consumption allocations \( \{c_i^t, h_i^t\} \) and bequests, \( f_t^o \) such that households solve

\[
\max_{\{c_i^t, h_i^t, f_t^o\}} \sum_{j=0}^{2} \beta^j (\log(c_{t+j}^i) + \alpha \log(h_{t+j}^{r,i} + \theta h_{t+j}^{s,i})) + \delta \beta^j \log(f_{t+j}^o)
\]

Subject to

\[
c_t^y + q_t(h_{t}^{o,y} + h_{t+1}^{r,y}) + (p_t - q_t)(h_{t}^{c,y} + s_{t+1}) = w^y
\]
\[
c_{t+1}^m + q_{t+1}(h_{t+1}^{o,m} + h_{t+2}^{r,m}) + (p_{t+1} - q_{t+1})(h_{t+1}^{c,m} + s_{t+2}) = w^m + p_{t+1}(h_{t+1}^{o,y} + s_{t+1})
\]
\[
c_{t+2}^o + q_{t+2}(h_{t+2}^{o,o} + h_{t+1}^{r,o}) + (p_{t+2} - q_{t+2})f_{t+3}^o = w^o + p_{t+2}(h_{t+2}^{c,m} + s_{t+2} + f_{t+2})
\]
\[
b_{t+1}^o \geq -(1 - \gamma)p_t(h_{t}^{o,i} + h_{t+1}^{l,i})
\]
\[
h_{t}^{o,i} \geq 0
\]
\[
h_{t}^{r,i} \geq 0
\]
\[
h_{t}^{l,i} \geq 0
\]

The initial middle aged and old maximize utility, the housing market clears in each period:

\[
H = h_{t}^{o,y} + h_{t}^{o,m} + h_{t}^{o,o}, \forall t
\]

The sequence of prices satisfies

\[
(1 + r) = \frac{p_{t+1}}{p_t - q_t}
\]

(11)

Where \( r \) is given exogenously.
When households become middle aged they have higher income and are no longer borrowing constrained. As in the unconstrained model they therefore do not want to rent housing. To solve the maximization problem for the old and middle aged, solve the maximization problem from the middle aged period forward subject to the "lifetime" budget constraint from the middle aged period forward.

Doing this yields the following expression for middle aged and old consumption:

\[
c_{t+1}^m = \frac{w^m + (p_{t+1} - (1 - \gamma)p_t(1 + r))h_{t+1}^{o,y} + \frac{w^o}{1 + \alpha} + \beta\delta(1 + r)c_{t}^m}{(1 + \alpha)(1 + \beta) + \beta\delta}
\]

(12)

Using the budget constraint of young households and

\[
b_{t+1}^y = -(1 - \gamma)p_t h_{t}^{o,y}
\]

Equations (9) and (10) above can be written as

\[
\frac{q_t}{h_t^{o,y}} = \frac{q_t}{w^y - \gamma p_t h_{t}^{o,y} - \phi^y_t (q_t - \gamma p_t)} - 1
\]

\[
\frac{w^y - \gamma p_t h_{t}^{o,y}}{c_{t+1}^m} + \frac{\beta(1 + r)}{\alpha} c_{t+1}^o + \phi^y_t = 0
\]

(13)  
(14)

Housing market clearing requires

\[
H = h_t^{o,y} + h_t^{o,m} + h_t^{o,o}
\]

Inserting for the housing consumption of middle aged and old households yields

\[
H = h_t^{o,y} + \frac{\alpha}{q_t} (c_{t}^m + c_{t}^o)
\]

(15)

Finally, prices satisfy

\[
(1 + r) = \frac{P_{t+1}}{p_t - q_t}
\]

(16)

5.4 Steady state

As in the unconstrained model I focus on the steady state solution of the model where consumption levels and prices are constant over time. The steady state versions of equations
(12)-(16) above give the solution of the model in steady state:

\[
\begin{align*}
0 &= \frac{\alpha}{h^{\phi,y}} - \frac{q}{w^y - \gamma ph^{\phi,y}} + \phi^y(q - \gamma p) \\
0 &= c^m - \frac{w^m + p(1 - (1 - \gamma)(1 + r))h^{\phi,y} + \frac{w^\phi}{1+r}}{(1 + \alpha)(1 + \beta) - \beta \delta r} \\
0 &= -\frac{1}{w^y - p\gamma h^{\phi,y}} + \frac{\beta(1 + r)}{c^m} + \phi^y \\
0 &= H - h^{\phi,y} - \frac{\alpha}{q}(1 + \beta(1 + r))c^m \\
0 &= p - \frac{1 + r}{\gamma} q
\end{align*}
\]

These are five equations in five unknowns, \(\{p, q, \phi^y, c^m, h^{\phi,y}\}\). I compute the steady state solution of the model for different values of \(\gamma\) using Matlab’s fsolve algorithm. The values used in the calculations are given in Table 1.

<table>
<thead>
<tr>
<th>(w^y)</th>
<th>(w^m)</th>
<th>(w^\phi)</th>
<th>(\theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>0.9</td>
<td>0.3</td>
<td>1.01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(\delta)</th>
<th>(r)</th>
<th>(H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Values used in calculation of the constrained model

The values are chosen to be illustrative and not to match empirical estimates. If one period of the model is 20 years, \(r = 1\) would imply a yearly real interest rate of roughly 3 percent. In steady state, equation (8), which ensures that young, borrowing constrained households want to own housing rather than rent can be written as

\[
\begin{align*}
q &> \gamma p \\
q &> \frac{1 + r}{r} \gamma q \\
\frac{r}{1 + r} &> \gamma
\end{align*}
\]

Where the second line uses the steady state version of the no arbitrage condition for buy to let housing. Thus, as long as this holds, the rental price is higher than what you have to put down as down payment for each value of \(\gamma\). As seen above, when this is the case, the rental market for housing drops out of the model endogenously. I calculate the steady state equilibrium for values of \(\gamma\) such that this holds. In this way I know that young households want to be homeowners in equilibrium and that the rental market drops out.
5.4.1 Behaviour of prices and debt levels in steady state

Figure 1 plots the steady equilibrium house price for different values of the down payment requirement in the constrained and unconstrained model. Prices are higher in the constrained model than in the unconstrained model for all values of $\gamma$ that I consider.

![Steady state purchase price of housing](image)

**Figure 1:** House prices in steady state in the constrained and unconstrained model

The borrowing constraint has two counteracting effects on house prices: on the one hand, when young households have to make a down payment they demand less houses than they do in the unconstrained model since they have reduced ability to borrow for the house. When young households demand less housing, middle aged and old households have to buy more. All else equal this means that the price must fall in order to make middle aged and old willing to buy more.

On the other hand, the borrowing constraint forces households to save more when young. The increased saving when young means that households enter the middle aged period of their lives with lower debt than if they could borrow freely as in the unconstrained model. This means that they have to spend a smaller share of their income on paying back debt and making interest
payments. As such, they have a larger budget to spend on both consumption goods and houses when middle aged. The larger budget means they should demand more housing all else equal, pushing the price up. In the model, this second effect coming through increased savings dominates the first effect coming from lower demand by young households and so the prices rise.

The reduction in demand for houses by young households is largest when the down payment requirement is high. Figure 2 shows the effect on the down payment requirement on demand for houses by young households.

![Steady state housing consumption of young](image)

**Figure 2:** House purchase demand by young households for different values of $\gamma$

The higher is the down payment requirement, the more consumption goods must the young households give up in order to get one more unit of housing. As such, the higher the down payment requirement is, the less housing will young households demand in order to avoid getting too low consumption of consumption goods.

When the down payment requirement is low, households can finance most of the cost of house purchases with credit, and so they demand more houses since these are relatively cheap in terms of period one consumption goods. Notably, when there is no down payment requirement at all
(γ = 0) households demand the same amount of housing when young as they do in the model without the borrowing constraint. They do this because they can fully finance house purchases by borrowing. They then buy the optimal amount of housing from a life time perspective.

At the same time as the down payment requirement causes young households to buy less housing it also means they borrow less. Figure 3 shows the effect on the bond holdings of each generation and the aggregate bond holdings of the economy as γ varies:

![Figure 3: Bond holdings for each generation and aggregate bond holdings in steady state for different values of γ](image)

The higher is the down payment requirement, the less do the young borrow. On the other hand, the middle aged and old borrow more. However, they do not increase their borrowing enough to offset the reduction in borrowing by the young. As such, the aggregate bond holdings of the economy rise with the down payment requirement.

Higher aggregate bond holdings, meaning lower aggregate debt, means a smaller part of the total endowment has to be spent on making interest payments. This has the effect on house prices discussed above: agents enter the second period of their lives with lower debt. They therefore
have to spend a smaller part of their income on making interest payments. This in turn means that they have more available resources to spend on houses and consumption goods. This pushes up the price of houses.

Note that for low levels of the down payment requirement, the aggregate debt level is actually higher than in the model without the borrowing constraint. The reason for this is mainly increased borrowing by the old. The old increase their borrowing in the model with borrowing constraint because house prices are higher meaning they have higher housing wealth when old. They want to use some of this housing wealth to consume more consumption goods and do this by borrowing against the value of their house rather than selling their house and become renters.

5.4.2 Consumption goods consumption of the young

As discussed above, when the down payment requirement is low enough, households can smooth consumption of housing services as well as consumption goods by buying housing and delaying the full payment on the house until later in life. Figure 4 shows the consumption level of young households in steady state for different levels of the down payment requirement.

When the down payment requirement is zero, young households spend all their income on consumption goods and fully finance housing purchases by borrowing. As the down payment requirement increases, young households must spend more of their income on paying the down payment and therefore cut consumption of consumption goods.

As seen by the figure, consumption of consumption goods by young households falls at a decreasing rate as the down payment requirement increases. When the down payment requirement increases, housing becomes relatively more expensive because they must give up more consumption. To avoid consuming too little consumption goods they reduce house purchases and thereby the increase in down payment costs.

5.5 Simulation of change in $\gamma$

In this section I look at how equilibrium variables move from one steady state to another if the economy is in a steady state and $\gamma$ is suddenly changed. The change I consider is a doubling of $\gamma$ from 0.1 to 0.2. This could be because of a change in the law or some other factor making access to credit more strict.
Figure 4: Consumption of consumption goods by the young in steady state in the constrained and unconstrained economy

5.5.1 Behaviour of prices

Figure 5 shows the evolution of prices after the change in $\gamma$. The economy is in the original steady state at time -1 and the change in $\gamma$ happens at time 0. The path from the old steady state to the new illustrates the counteracting effects on the house price when $\gamma$ increases.

In the period of the change in $\gamma$, prices drop. This is because the larger down payment requirement forces young households to buy less housing than the previous generation of young did. The lower demand by the young is not compensated for by the middle aged and old and so the prices drop initially.

Over time, the prices increase. Prices increase because the higher down payment requirement forces young households to save more. Households therefore have to spend a smaller share of their endowment when middle aged on making interest payments and paying back debt. They therefore increase their spending on housing as well as consumption goods, pushing up the price of houses.
Figure 5: The evolution of prices when $\gamma$ is increased from 0.1 to 0.2. The economy is in steady state with $\gamma=0.1$ at period -1. The change happens at period 0. At period 5 the economy it at the new steady state.

5.5.2 Behaviour of bond holdings

Figure 6 shows the evolution of the bond holdings of each group of households and the aggregate bond holdings after the change in $\gamma$.

When the down payment requirement is increased, the current young must borrow less than the previous young did, so the bond holdings of the young immediately increase and move towards the new steady state level.

In the period of the change, the middle aged increase their borrowing compared to the previous middle aged generation. They do this because the drop in the price of houses causes them to demand more housing. Rather than cut their consumption of consumption goods by the full amount required to buy more housing, they partially finance the increase in housing consumption by credit. Bond holdings of middle aged then rise toward their new steady state, which is not much different from the previous one.
Figure 6: The evolution of bond holdings of each generation and aggregate bond holdings when $\gamma$ is increased from 0.1 to 0.2. The economy is in steady state with $\gamma=0.1$ at period -1. The change happens at period 0. At period 5 the economy it at the new steady state.

The generation that is old in the period of the change increase their borrowing compared to the previous old. This is for the same reason as the middle aged: they increase their housing consumption and finance this partially by credit. The old in the next period have higher bond holdings than the old in the period of the change. This can be explained by the fact that this generation borrowed more when middle aged than the previous generation did. They therefore buy less housing and consumption goods and borrow less when old because they have to pay more in interest payments. Bond holdings of the old then fall towards their new steady state level.

Aggregate bond holdings increase from the period of the change in $\gamma$ until they reach the new steady state. The middle aged and old increase their borrowing somewhat, but not enough to offset the forced reduction in borrowing by the young.
5.5.3 Behaviour of housing consumption

Figure 7 shows the evolution of housing consumption of the different groups over time after the increase in $\gamma$. In the period of the change in $\gamma$, young households are forced to buy less housing than the previous young generation did because of the reduced access to credit. This reduction in demand for houses by the young acts to push prices of houses down in equilibrium.

House purchases by the young does not fall immediately to its new steady state level. This is because of the initial drop in prices. As the purchase price of housing rises to its new steady state level, house purchases by the young households fall to the new steady state.

House purchases by the generation that is middle aged in the period of the change in $\gamma$ rise compared to the previous generation of middle aged. The reason for this is that the fall in demand for housing by the young causes the price to drop. When prices drop, middle aged demand more housing.

The next generation of middle aged is the generation that was young in the period of the increase in $\gamma$. As seen by the figure, this group consumes even more housing than the middle aged in the period of the change. This can be explained by the fact that this generation saved more when young than what the generation before them did. At the same time, prices are not yet at their new steady state level, meaning they can afford higher housing consumption.

Like the generation that is middle aged in the period of the change in $\gamma$, the generation that is old in this period increase their housing purchases compared to the previous generation. Again the reason for this is the drop in the price caused by the fall in demand by the young.

The next generation of old is the same generation that was middle aged in the period of the change. This generation have lower housing consumption when old than the old in the period of the change. The reason for this is that the price of housing is higher at the same time as they borrowed more in when middle aged than the previous generation did, meaning they must make higher interest payments when old. In addition house prices are higher than in the previous period. Finally, the generation that was young in the period of the change in $\gamma$ have higher housing consumption when old due to their high savings early in life.

5.5.4 Behaviour of consumption goods consumption

Figure 8 show the evolution of consumption levels of the different groups and aggregate consumption after the change in $\gamma$. Consumption goods consumption of the young immediately falls because they have to cut back on borrowing. They buy less housing when $\gamma$ increases, but also cut consumption since they have to make a larger down payment per unit of housing they still buy.
Figure 7: The evolution of the housing consumption of different groups when $\gamma$ is increased from 0.1 to 0.2. The economy is in steady state with $\gamma=0.1$ at period -1. The change happens at period 0. At period 5 the economy is at the new steady state.

The consumption of the old and middle aged in the period of the change reacts little. These groups are not borrowing constrained and as such are mainly effected through changes in prices. The house price drops in the model due to lower demand by the young, and thus the middle aged and old increase their housing consumption as seen above. They do this mostly by increased borrowing, meaning their consumption of consumption goods are largely unaffected.

The middle aged in the period after the change is the generation that was young in the period of the change. They increase their consumption of consumption goods by a lot compared to the previous generation of middle aged because of their higher savings when young. The same thing happens when this generation of households become old. Because of higher savings early in life, they consume more when old than the previous generations did.

As seen by the final chart in the figure, aggregate consumption falls initially and then rises to its new steady state level. The initial fall is caused by the sharp reduction in consumption by the young as well as the modest fall in consumption of middle aged and old. Over time, the lower
The evolution of the consumption goods consumption of each generation and aggregate consumption when $\gamma$ is increased from 0.1 to 0.2. The economy is in steady state with $\gamma=0.1$ at period -1. The change happens at period 0. At period 5 the economy is at the new steady state.

Figure 8: The evolution of the consumption goods consumption of each generation and aggregate consumption when $\gamma$ is increased from 0.1 to 0.2. The economy is in steady state with $\gamma=0.1$ at period -1. The change happens at period 0. At period 5 the economy is at the new steady state.

borrowing by agents early in life means that they spend a lower share of their life time income on making interest payments. Aggregate consumption therefore rises over time to the new higher steady state level.

It should be noted that although aggregate consumption is higher in the new steady state, agents are not better off. They are not better off because they are less able to smooth consumption between periods of life. Although they consume more housing and consumption goods when middle aged and old in the new steady state than in the old, this is at the cost of lower consumption of both goods when young. Since their consumption level is lower when young, this makes them worse off.

The analysis also suggests that it is the generation that is old in the period of the change in $\gamma$ that gains the most from the increase in the down payment requirement. This generation can afford higher housing consumption than the previous generation of old did because of the drop
in prices. The generation that is middle aged in the period of the change also benefit to some extent, mainly because they can afford higher housing consumption when middle aged than what they would be able to if the change in $\gamma$ had not happened.

6 Model with income inequality

I now introduce income inequality in the model. I split the consumers into two groups, rich and poor. In each period the rich have twice the endowment of the poor.

In addition to income inequality, I introduce a minimal level of owner occupied housing that households must buy if they are to become homeowners as in Gervais (2001). One can think of this as the smallest kind of house or apartment that can be purchased in the economy. I impose no minimal level of how much buy to let housing households can own. The interpretation here is that if households are to become homeowners they have to buy at least the smallest kind of apartment. To become landlords they just have to buy a slightly larger apartment or house than what they plan to consume themselves and rent out one or more rooms to renters.

As in the model without income inequality, i will assume that the down payment requirement is such that the cost of renting is higher than the down payment cost per unit of housing, namely $q_t > \gamma p_t$. As seen above, when this is the case, households would like to buy housing when young. However, for large enough levels of the down payment requirement, the minimal size of house purchases can force the poor into the rental market even when this holds.

When the down payment requirement is low, both rich and poor young households borrow to buy houses in steady state. As the down payment requirement increases, they have to cut back on consumption of consumption goods in order to pay the down payment. Without the minimal size of housing, they simply buy smaller and smaller houses and stay homeowners. However, when there is a lower limit on how much housing they have to buy, at some point they have to sacrifice so much consumption goods that they are better off renting than remaining homeowners despite losing access to credit.

For large enough values of the down payment requirement, the poor young rent rather than buy housing in steady state. When the poor young rent, the rich young can rent out housing to the poor. This enables rich young households to increase the consumption of consumption goods by borrowing to finance additional housing investment and rent out to the poor.

The analysis suggests that when access to credit is tied to house purchases, the combination of down payment requirements and the minimal house size amplify the effects of income inequality on consumption inequality. Poor young households who are unable to afford down payment on the smallest house are shut out of credit markets, meaning they are unable to smooth consumption
between periods. As such they are forced to accept low levels of consumption of housing and consumption goods in the first period of their lives because they are only able to spend their period one endowment.

6.1 Model set up

Households are divided into two groups, rich and poor, of equal size. In each period the rich receive an endowment that is larger than the endowment received by the poor. Both groups have receive a larger endowment when middle aged and old than they do when young. Denote the endowment of the rich of generation $j$ by $w^{jr}$ and the endowment of the poor by $w^{jp}$. For both groups

$$w^{mi} > w^{oi} > w^{yi}, \forall i \in \{r, p\}$$

The rich have a larger endowment than the poor in each period of their lives:

$$w^{jr} > w^{jp}, \forall j \in \{y, m, o\}$$

Both groups of households have the same preferences represented by the same utility function as before. At the end of the third period of their lives, households die. The poor old who die are replaced by a new generation of poor young at the beginning of the next period. The same holds for the rich.

Both poor and rich leave a bequest for the generation directly below them. This bequest is received at the beginning of the old period of life along with the endowment. The poor old get the bequest left by the previous poor old generation and the rich old get the bequest left by the previous rich old generation.

Households remain in the group into which they are born for their entire life. That is, poor young households become poor middle aged and old households as they age.

The minimal house size is given by $h$. The borrowing constraint is the same as before and given by

$$b_{t+1}^i \geq -(1 - \gamma)p_t(h^{o,i}_t + h^{l,i}_t)$$

In addition to the borrowing constraint, households face an additional constraint given by the minimal size of housing. I formulate this as

$$h^{o,i}_t(h^{o,i}_t - h) \geq 0$$

(17)
This constraint is satisfied if households buy an amount of owner occupied housing larger or equal to the minimal house size or if they buy nothing at all.

6.2 Behaviour of poor households

Poor households choose between three modes of housing in the first period of their lives:

- become homeowners by buying more than the minimal house size
- become homeowners by buying the minimal housing size
- become renters

When poor households chose the first mode of housing consumption, they behave in the same way as young households in the model without income inequality. They become homeowners and borrow up to their borrowing constraint. Behaviour then satisfies

\[
\frac{\alpha}{h_{t+1}} = \frac{q_t}{c_t} - \phi_{t+1}^{pp} (q_t - \gamma p_t) \\
- \frac{1}{c_t} + \frac{\beta (1 + r)}{c_{t+1}} + \phi_{t+1}^{pp} = 0
\]

\[
c_t^{yp} = w^{yp} - (q_t - \gamma p_t) h_{t+1}^{yp}
\]

\[
c_{t+1}^{mp} = \frac{w^{mp} + (p_{t+1} - (1 - \gamma) p_t (1 + r)) h_{t+1}^{yp} + \mu_{t+1}^{yp} c_{t+1}}{1 + \alpha (1 + \beta) + \beta \delta}
\]

Thus, the rental market drops out and the poor are homeowners when young. In the same way as in the model without income inequality, poor young households overconsume housing services relative to consumption goods.

When poor young households buy the minimal housing size, behaviour satisfies the same equations as above but with \(h\) replacing \(h^{yp}\) and the first equation being replaced by

\[
\frac{\alpha}{h} = \frac{q_t}{c_t} - \phi_{t}^{yp} (q_t - \gamma p_t) - \mu_{t}^{yp} h
\]

where \(\mu_{t}^{yp}\) is the Lagrange multiplier on the condition restricting house purchases. When households buy exactly \(h\) this must be non-negative.

Equation (18) shows how the constraint on minimal house purchases can distort the intratemporal trade-off additionally compared to if there were no such constraint. When the down payment requirement is large enough, poor households would optimally want to buy a smaller house than
the minimal house size when young. Since they cannot do this, they must either become renters or keep buying the minimal house size. As before, because buying allows households to transfer some of the cost of housing consumption to the future, the poor young have a strong incentive to keep buying the minimal house size rather than become renters.

For large enough values of the down payment requirement, the cost of the down payment on the minimal house size becomes so large that poor households are better off becoming renters rather than buying the minimal house size when young.

When the poor young rent they do not borrow and do not save in bonds. They do not save because they have higher income in the future and do not borrow because they do not own housing. Behaviour then satisfies

\[
\begin{align*}
    c_{t}^{yp} &= \frac{w_{yp}}{1 + \alpha} \\
    h_{t}^{wy} &= \frac{\alpha}{q_{t}} \frac{w_{yp}}{1 + \alpha} \\
    c_{t+1}^{mp} &= \frac{w_{mp} + w_{op}^{t+1} + c_{t}^{mp} \delta (1 + r)}{(1 + \alpha)(1 + \beta) + \beta \delta} \\
    c_{t+2}^{op} &= \beta (1 + r) c_{t+1}^{mp} \\
    f_{t+3}^{op} &= \delta c_{t+2}^{op}
\end{align*}
\]

When renting housing, the poor young split their endowment between renting housing services and buying consumption goods in the optimal way. They no longer have an incentive to over consume housing relative to consumption goods since they no longer have access to credit. They are however not able to bring consumption forward in time at all since they cannot borrow.

6.3 Behaviour of rich households

As long as the poor are homeowners, there is no rental market for housing. The then rich either behave in the same way as in the model without income inequality or they buy the minimal house size, \( \mathbf{h} \).

The rental market for housing emerges when the down payment requirement is so large that the poor young are better off renting housing in than buying the minimal house size. Since the rich receive a larger endowment, the poor young will want to become renters before the rich young do.

When there is a rental market for housing, young rich households may be able to increase their consumption of both consumption goods and housing services by buying housing and renting it out to the poor young.
To see how the possibility of renting out housing can enable the rich to increase consumption of
consumption goods in the first period, suppose first that there is no upper bound on how much
they can rent out. Consider the first order conditions for buy to let and owner occupied housing
for the rich young using that the rich young do not both own and rent at the same time:

\[
\begin{align*}
\frac{\alpha}{h_{t}} - \frac{p_{t}}{c_{y_{t}}^{\text{pr}}} + \beta(1 + r)\frac{p_{t+1}}{c_{y_{t+1}}^{\text{pr}}} + \phi^{\text{byr}}(1 - \gamma)p_{t} + \mu^{\text{byr}}(2h_{t}^{\alpha - \text{yr}} - h_{t}) &= 0 \\
p_{t} - q_{t} - \frac{\beta p_{t+1}}{c_{y_{t+1}}^{\text{pr}}} + \phi^{\text{byr}}(1 - \gamma)p_{t} + \xi^{\text{yr}} &= 0
\end{align*}
\]

If buy to let housing is positive, the Lagrange multiplier, \(\xi^{\text{yr}}\) is zero by the complementary
slackness condition. Suppose also that the rich young buy more than the minimal house size, so
that \(\mu^{\text{byr}} = 0\). Subtracting the second equation from the first then yields

\[
\frac{\alpha}{h_{t}} = \frac{q_{t}}{c_{y_{t}}^{\text{yr}}}
\]

Which is precisely the intratemporal optimality condition of young households when they are
unconstrained. Thus, when young households can rent out as much as they want they can
purchase buy to let housing and rent out, thereby obtaining the optimal level of consumption
goods relative to housing services even though they are only allowed to borrow for buying
houses.

Provided \(q_{t} > \gamma p_{t}\), this also implies that the Lagrange multiplier on the borrowing constraint,
\(\phi^{\text{byr}}\) must be zero.

\[
\phi^{\text{byr}}(q_{t} - \gamma p_{t}) = 0
\]

In other words, when the rich can rent out as much as they want, they can buy housing capital
and rent out, thereby increasing consumption of consumption goods in addition to housing in
the first period of their lives.

In practice, there will be an upper bound on how much the young rich are able to rent out given
by how much the poor young want to rent. The poor young only rent if they are able to meet
the down payment requirement on the smallest amount of housing. The low purchasing power
of the young households also mean that the amount of housing they want to rent in equilibrium
is limited.

The fact that the rich young cannot rent out more than what the poor young want to rent limits
the ability of the rich young to smooth consumption over the life cycle. However, the fact that
they are able to rent out at least some housing to the poor young allows them to consume more
consumption goods than they otherwise could. Suppose the rich young are the only ones renting
out housing to the poor young. In this case buy to let housing by the rich young becomes

$$h_{t}^{l,yr} = \frac{\alpha}{q_{t}^{\gamma}} \frac{w_{t}^{yp}}{1 + \alpha}$$  \hspace{1cm} (21)$$

Inserting this in the budget constraint of the rich young and using

$$b_{t+1}^{yr} = -(1 - \gamma)p_{t}(h_{t}^{o,yr} + h_{t}^{l,yr})$$  \hspace{1cm} (22)$$

Yields the maximal consumption level in period one given the level of housing consumption and buy to let housing:

$$c_{t}^{yr} = w_{t}^{yr} - \gamma p_{t} h_{t}^{o,yr} + (q_{t} - \gamma p_{t})(\frac{\alpha}{q_{t}^{\gamma}}\frac{w_{t}^{yp}}{1 + \alpha})$$  \hspace{1cm} (23)$$

If they cannot rent out at all, their consumption is limited to

$$c_{t}^{yr} = w_{t}^{yr} - \gamma p_{t} h_{t}^{o,yr}$$  \hspace{1cm} (24)$$

Which is lower than when they rent out the maximal amount possible. In other words, even if the young rich households are unable to rent out as much as they like since only the poor young households rent, they can still increase consumption in the first period by renting out to the poor young.

### 6.4 Equilibrium

The definition of equilibrium is similar to the definition given in section 5 and is a sequence of prices and consumption allocations for young and rich households such that both young and rich households maximize utility given endowments and bequests, the market for housing purchases clears and the rental market for housing clears.

### 6.5 Steady state

As before, I focus on the steady state of the model where consumption levels and prices are constant over time. I calculate the equilibrium for different levels of the down payment requirement and show how steady state variables depend on the size of the down payment requirement.

The nature of the steady state equilibrium depends on the level of the down payment requirement, $\gamma$. For low levels of the down payment requirement, the poor young are homeowners buying more than the minimal house size. As the down payment requirement increases, at some
Table 2: Values used in calculating model with income inequality

<table>
<thead>
<tr>
<th>$w^r$</th>
<th>$w^m$</th>
<th>$w^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.6</td>
<td>0.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>1.01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

point the young start buying the minimal housing size. When the down payment requirement becomes large enough, the poor young no longer want to buy and the rental market for housing emerges.

In the equilibria I look at, the rich young borrow up to their borrowing constraint and buy more than the minimal house size for all levels of $\gamma$. When the rental market for housing exists the rich young buy additional housing to rent out to the poor young.

I use the following values for endowments, interest rate, housing stock and preference parameters in the calculations:

The endowment of the rich is twice the endowment of the poor in each period of life. Total endowment in the economy is the same as in the previous versions of the model above. I have deliberately chosen a low level for $\theta$ in order to make sure that the decision of whether to buy or rent when constrained is driven by income and borrowing constraints, not the higher preference for living in owner-occupied housing.

To find the values for $\gamma$ at which the poor young start buying the minimal house size and become renters I use the following approach: I calculate the equilibrium for all three modes of housing and calculate the indirect utility of poor households.

For low levels of $\gamma$ poor young households buy more than the minimal house size. When $\gamma$ becomes sufficiently large, poor young households want to buy less housing than the minimal house size. I then compare the indirect utility of buying the minimal house size to the indirect utility of becoming renters. As long as the former is larger, poor young households keep buying the minimal house size. When $\gamma$ becomes large enough, poor young households are better off renting than buying the minimal house size and the rental market for housing emerges.

6.5.1 Housing consumption of poor young in steady state

Figure 9 shows housing consumption of the poor young in the steady state equilibrium for different values of $\gamma$. The solid line shows the housing consumption in the presence of a minimal house size, while the dotted line shows desired (purchased) housing consumption if households could buy any size of housing they wanted.
Figure 9: Housing consumption by poor young households in steady state for different values of $\gamma$. The dotted line shows optimal house purchase in absence of the minimal house size.

For low levels of the down payment requirement, the poor young would like to buy more than the minimal size of housing in steady state. The dotted and solid lines then overlap. As seen by the figure, poor young households would like to gradually adjust housing consumption down in steady state as the down payment requirement becomes larger. They want to keep buying housing because remaining homeowners lets them finance part of their housing consumption with credit. On the other hand, they must give up more consumption of consumption goods when the down payment requirement is higher, and would therefore like to buy smaller houses.

Because of the minimal size of housing, poor young households cannot adjust housing purchases gradually down as they like. If they want to remain homeowners they have to buy at least the minimal house size. When $\gamma$ increases, the dotted and solid lines of the figure therefore part ways. Households would like to buy a smaller amount of housing than the minimal house size, but cannot do this. To reduce housing consumption they would have to become renters. Becoming renters makes them lose access to credit however, and so they are better off buying the minimal house size than switching to rented accommodation.
When the down payment requirement becomes large enough, the poor young become renters. This is what causes the drop in housing consumption at $\gamma$ roughly equal to 0.25. When $\gamma$ becomes this large, the poor young have to give up so much consumption goods to pay the down payment on the smallest house size that they are better off becoming renters despite losing access to credit. Housing consumption then falls abruptly to the level that is optimal for households when renting. This is lower than the minimal level of housing, but also compared to the amount of housing they would like to buy if there were no minimal house size. Again the reason for this is that rented housing consumption cannot be financed by credit at all.

As discussed above, both poor and rich young households over consume housing when they are borrowing constrained homeowners. The minimal house size causes the poor young to over consume housing to a even larger degree. They do this because only purchased housing consumption can be financed by credit. This gives them a strong incentive to buy the minimal house size despite having to consume very little consumption goods when the down payment requirement is high.

Note that this is not driven by the higher preference for owned versus rented housing consumption. I have deliberately chosen a low value for $\theta$ to make sure of this. The main role of the higher preference of owned versus rented housing consumption in the model is to ensure that households who do not face financing constraints do not want to rent. What drives the emergence of the rental market for housing in the model is the minimal size of housing combined with the borrowing constraint and down payment requirement.

### 6.5.2 Consumption goods consumption of rich young in steady state

Figure 10 shows the consumption of consumption goods by rich young households in steady state for different levels of the down payment requirement. The solid line shows consumption of the rich young in steady state when there is a minimal house size. When there is a minimal house size, a rental market for housing exists for large enough levels of $\gamma$. The dotted line shows consumption good consumption by the rich young in steady state in the absence of the minimal house size.

As seen in figure 9, when there is a minimal size of housing a rental market for housing exists for large enough levels of the down payment requirement. When the poor young rent in equilibrium, the rich can increase their consumption of consumption goods by purchasing buy-to-let housing and rent this out to the poor.

It is the combination of the down payment requirement and the minimal house size that drives this. These two elements together create a market for rental housing by making it so costly for the poor young to become homeowners that they chose to rent. When poor young rent, the rich
Figure 10: Consumption level for the rich young for different levels of $\gamma$ with and without rental market

young can purchase buy to let housing and rent it out to the poor young, thereby increasing their consumption of consumption goods.

In fact, the rich young increase their consumption of both housing and consumption goods when the rental market exists compared to when it does not. They do this for the same reason as before: housing consumption can be financed by credit. They spend the income from renting out housing both on consumption of consumption goods and on getting a larger loan for housing.

When the down payment requirement is large enough, poor young households are shut out of the housing market because the cost of the down payment on the smallest house becomes too large. Young rich households gain from this, as this enables them to rent out housing to the poor young, thereby enabling them to increase their consumption of consumption goods as well as housing.

The down payment requirement combined with the minimal house size amplifies the effects of income inequality on consumption inequality of young households. When the down payment
requirement is so large that poor young households cannot afford to buy, they lose access to credit, making them unable to smooth consumption between the first and second periods of their lives at all. Their consumption of consumption goods and housing services must then be financed by their period one endowment alone. The rich young on the other hand are made better off since they can purchase additional housing and rent this out to the poor young. One should interpret this as the rich buying slightly larger houses and renting out one or more rooms to the poor.

6.5.3 Prices and debt levels

Figure 11 shows the steady state price level of housing for different values of the down payment requirement. When the down payment requirement becomes large enough to make the poor young become renters, prices drop. This does not have to be the case, but depends on the interest rate. There are two counteracting effects on prices when the rental market drops out. The first is the lower demand for housing by young households. This effect pushes prices down.
When the poor young become renters, their demand for purchased housing falls from the minimal house size to zero. The rich young pick up some of this fall in demand by increasing their demand for housing in order to rent out to the poor. However, the increase in demand is insufficient to fully compensate for the drop in demand by the poor young. Total demand for purchased housing by the young therefore drops. The demand is picked up by the middle aged and old. They have a lower marginal utility of housing than the young poor did and so prices drop.

The top chart of figure 12 shows this effect. The top chart shows total demand for buying houses by the young. When the rental market emerges, this demand falls, but by less than the minimal house size (which is the reduction in demand by the poor young) since the rich young increase their purchases.

The second effect acting on the house prices is that aggregate interest payments on debt fall when the young poor stop buying housing. This effect pushes prices up. The lower part of figure 11 illustrates this effect. When the poor young drop out of the market for purchased housing and

![Demand for buying housing by young](image)

![Interest payments in steady state](image)

**Figure 12:** Housing demand by young and interest payments in steady state. Above: Total demand for housing purchases by young households for different values of $\gamma$. Below: total interest payments in steady state for different levels of $\gamma$. 

40
become renters, they stop borrowing completely. This is partially compensated for by increased borrowing on behalf of the rich young and the poor themselves in middle age, but not completely. Lower debt levels means that interest payments are lower in equilibrium. This in turn means that a larger part of national income can be spent on housing rather than paying interest payments, pushing prices up.

In principle, the effect of lower interest payments can dominate the first effect. Prices would then increase when the rental market emerges. The reason is that for a given debt level, the effect of reducing the debt level on interest payments is higher the higher the interest rate is. In the model I have set the interest rate equal to the discount rate, so the desired amount of borrowing is largely unaffected by changing the interest rate (any effect comes through changes in prices and therefore the borrowing ability of constrained households). Thus a higher interest rate would not result in lower debt levels, only higher interest payments. Cutting this debt level would then have a larger effect the higher the initial debt level is.

The main point of this section is not to study the behaviour of prices when the rental market for housing emerges. Rather, it is how the down payment requirement and minimal house size can force poor young households to rent and how this affects their access to credit and consumption levels. The precise effect on house prices when the down payment requirement becomes large enough to make the rental market emerge should therefore be interpreted with caution.

The general features of the effects of the down payment requirement on steady state house prices are the same as in the model without income inequality. A higher down payment requirement causes a higher steady state purchase price of houses. As before this is driven by the fact that a higher down payment requirement forces agents to save more in the first period of their lives. This means that over the life cycle, they spend less of their lifetime income making interest payments, leaving more resources available for house purchases as well as consumption goods, pushing up prices. In the model, this effects dominates the effect of lower demand for houses by rich households caused by the borrowing constraint.

7 Conclusions

In this article I have presented a model of how households can use house purchases to smooth consumption over the life cycle when access to credit is tied to house purchases. The analysis suggests that households may have an incentive to overconsume housing early in life and that they have strong incentives to buy rather than rent housing when they are young.

Notably, households in the model have a strong incentive to buy rather than rent housing early in life even absent opportunities to make money from increases in house prices. As long as it
is easier for households to get access to credit for house purchases than for renting or buying consumption goods, households have a strong incentive to become homeowners early in order to better smooth consumption over the life cycle.

Further, the analysis highlights how the effects of income inequality on consumption inequality may be amplified when access to credit tied to house purchases. Low income households who are unable to afford down payment costs on the smallest types of houses may be shut out of credit markets, making them unable to smooth consumption between periods at all.

Finally, the model predicts that the effects on house prices of a change in down payment requirements are different in the short run compared to the long run. A tightening of borrowing constraints causes a drop in house prices in the short run, due to lower demand by borrowing constrained households. In the long run, however, the model predicts that prices should rise because of lower overall debt levels causing households to spend less of their lifetime income on making interest payments.

A major limitation of this model is that it abstracts from production, both of consumption goods and houses. Particularly, the fact that the housing stock is fixed means that the predictions of the model for house prices should be interpreted with caution. For example, if the housing stock can adjust in the long run, the increase in house prices caused by lower debt levels may be less pronounced.

Despite its simplified set-up, I believe the model sheds light on questions related to how households can smooth consumption over the life cycle. Particularly, I believe the model can be used to better understand the motivations of young households deciding whether to buy or rent housing early in life. The analysis suggests that young households have good reasons both for buying rather than renting as well as spending a large share of their income on housing.
8 Appendix

8.1 Households in the unconstrained model do not rent

The first order conditions when including buy to let housing in the lifetime budget constraint of household are

\[ c^0_t : \frac{1}{c^0_t} - \lambda = 0 \] (25)

\[ h_{t+1}^{r,y} : \frac{\alpha}{h_{t+1}^{r,y} + \theta h_{t+1}^{o,y}} - \lambda q_t + \eta^y_t = 0 \] (26)

\[ h_{t+1}^{o,y} : \frac{\alpha \theta}{h_{t+1}^{r,y} + \theta h_{t+1}^{o,y}} - \lambda q_t + \mu^y_t = 0 \] (27)

\[ c^{0+1}_t : \frac{\beta}{c^{0+1}_t} - \lambda = 0 \] (28)

\[ h_{t+1}^{r,m} : \frac{\beta \alpha}{h_{t+1}^{r,m} + \theta h_{t+1}^{o,m}} - \frac{\lambda q_{t+1}}{(1 + r)} + \eta^m_{t+1} = 0 \] (29)

\[ h_{t+1}^{o,m} : \frac{\beta \alpha \theta}{h_{t+1}^{r,m} + \theta h_{t+1}^{o,m}} - \frac{\lambda q_{t+1}}{(1 + r)} + \mu^m_{t+1} = 0 \] (30)

\[ c^{0+2}_t : \frac{\beta^2}{c^{0+2}_t} - \lambda = 0 \] (31)

\[ h_{t+2}^{r,o} : \frac{\beta^2 \alpha}{h_{t+2}^{r,o} + \theta h_{t+2}^{o}} - \frac{\lambda q_{t+2}}{(1 + r)^2} + \eta^o_{t+2} = 0 \] (32)

\[ h_{t+2}^{o,o} : \frac{\beta^2 \alpha \theta}{h_{t+2}^{r,o} + \theta h_{t+2}^{o}} + \frac{\beta^2 \delta}{h_{t+2}^{r,o} + s^o_{t+2}} - \frac{\lambda p_{t+2}}{(1 + r)^2} + \mu^o_{t+2} = 0 \] (33)

\[ s^{o+3}_t : \frac{\beta^2 \delta}{h_{t+3}^{r,o} + s^o_{t+2}} - \frac{\lambda (p_{t+2} - q_{t+2})}{(1 + r)^2} = 0 \] (34)

These combined with the complementary slackness conditions:

\[ \eta^i_t \geq 0, \eta^i_t = 0, \text{ if } h_{t}^{i} > 0 \] (35)

\[ \mu^i_t \geq 0, \mu^i_t = 0, \text{ if } h_{t}^{o,i} > 0 \] (36)

Are the Kuhn-Tucker optimality conditions of households in the unconstrained model. \( \lambda, \eta^i_t, \mu^i_t \)

are the Lagrange multipliers on the budget constraint and non-negativity constraints for rented and owner occupied housing consumption respectively.

In the unconstrained model, households do not rent housing. To see this combine the first order conditions with respect to rented and owned housing for the young to yield

\[ \frac{\alpha (\theta - 1)}{h_{t}^{r,y} + \theta h_{t}^{o,y}} + \mu^y_t - \eta^y_t = 0 \] (37)
Both \( \eta \) and \( \mu \) cannot be positive at once, since that implies that the household does not consume housing. Suppose \( h^{o,y} > 0 \) meaning \( \mu = 0 \), then

\[
\eta^y_t = \frac{\alpha(\theta - 1)}{h^{o,y}_t + \theta h^{o,y}_t} > 0
\]  
(38)

This is compatible with the complementary slackness condition. \( \eta^y_t > 0, h^{o,y}_t > 0 \) implies that households do not rent when they own.

Suppose that households do rent and see if this is compatible with them not owning. Use \( h^{r,y}_t > 0 \) meaning \( \eta^y_t = 0 \). Then

\[
\mu^y_t = -\frac{\alpha(\theta - 1)}{h^{r,y}_t + \theta h^{r,y}_t} < 0
\]  
(39)

Which contradicts the complementary slackness condition. The same argument holds for the middle aged, so middle aged and young do not rent when they are not borrowing constrained.

For the old, fist combine the first order conditions for \( s^{o}_t + 3 \) and \( h^{o,o}_t + 3 \) to get:

\[
\frac{\beta^2 \delta}{h^{o,o}_t + s^{o}_t + 2} - \frac{\lambda q_{t+2}}{(1+r)^2} = 0
\]  
(40)

Then use the same approach as for the young.

8.2 Unconstrained households in the models with borrowing constraints do not rent

Use the same proof as for the unconstrained model but with the lifetime budget constraint going from the middle aged period forward.

8.3 Sufficient condition on \( \gamma \) for constrained households not wanting to rent

This is a formal presentation of when borrowing constrained young households want to become homeowners rather than rent.
I put the Kuhn-Tucker conditions for optimality again:

\[
\begin{align*}
\ell_t^y : & \quad \frac{1}{\ell_t^y} - \lambda_t^y = 0 \\
h_t^r,y : & \quad \alpha \frac{\ell_t^r}{\ell_t^r + \theta h_t^o,y} - \lambda_t^y q_t + \eta_t^y = 0 \\
h_t^o,y : & \quad \alpha \theta \frac{\ell_t^o}{\ell_t^o + \theta h_t^o} - \lambda_t^o + \lambda_{t+1}^o \beta p_{t+1} + \phi_t^o (1 - \gamma) p_t + \mu_t^o = 0 \\
h_t^l,y : & \quad -\lambda_t^l (p_t - q_t) + \beta p_{t+1} \lambda_{t+1}^l + \phi_t^l (1 - \gamma) p_t + \xi_t^l = 0 \\
b_{t+1}^y : & \quad \alpha \lambda_t^l + \lambda_{t+1}^l (1 + r) + \phi_t^l = 0 \\
\eta_t^l \geq 0, \eta_t^l = 0, \text{ if } h_t^r,y > 0 \\
\mu_t^l \geq 0, \mu_t^l = 0, \text{ if } h_t^r,y > 0 \\
\xi_t^l \geq 0, \xi_t^l = 0, \text{ if } h_t^l,y > 0 \\
\phi_t^l \geq 0, \phi_t^l = 0, \text{ if } b_{t+1}^y > -(1 - \gamma) p_t h_t^r,y
\end{align*}
\]

Insert for $\beta \lambda_{t+1}^o$ from the first order condition for bond purchases in the first order condition for house order purchases:

\[
\alpha \theta \frac{\ell_t^o}{\ell_t^o + \theta h_t^o} - \lambda_t^o q_t + \phi_t^o (q_t - \gamma p_t) + \mu_t^o = 0 \tag{41}
\]

Subtract the first order condition for rented housing from this equation to get

\[
\alpha (\theta - 1) \frac{\ell_t^o}{\ell_t^o + \theta h_t^o} - \eta_t^o + \phi_t^o (q_t - \gamma p_t) + \mu_t^o = 0 \tag{42}
\]

Suppose first that households are renters, $h_t^r,y > 0$ so that $\eta_t^y = 0$. Then

\[
\alpha (\theta - 1) \frac{\ell_t^o}{\ell_t^o + \theta h_t^o} + \phi_t^o (q_t - \gamma p_t) = -\mu_t^o \tag{43}
\]

Then if $\phi_t^o (q_t - \gamma p_t)$ is positive, $\mu_t^o$ is negative, which contradicts the complementary slackness condition. $\phi_t^o (q_t - \gamma p_t)$ is positive if

\[
q_t > \gamma p_t \tag{44}
\]

so if this condition holds, households are not renters. Now see if this condition is compatible with households being homeowners. Suppose $h_t^o,y > 0$ so that $\mu_t^o = 0$. Then equation (41) implies

\[
\alpha (\theta - 1) \frac{\ell_t^o}{\ell_t^o + \theta h_t^o} + \phi_t^o (q_t - \gamma p_t) = \eta_t^o \tag{45}
\]
When equation (43) holds \( \eta_y \) is positive, which is consistent with the complementary slackness condition and zero rented housing. Conclude that if (43) holds, households are only homeowners and not renters when they are borrowing constrained.

This proof also works when households hold the minimal house size in section 4.

8.4 Condition on endowment of young such that they always borrow up to borrowing constraint in steady state

In both the models with and without income inequality I have used that the young borrow up to their borrowing constraint when they become homeowners for all level of \( \gamma \) I consider. In principle, this does not have to be the case. For sufficiently high incomes, the young may not want to shift as much of the cost of house purchases as possible to the future. I now provide a sufficient condition on the endowment on the young such that they always borrow up to their borrowing constraint for all levels of \( \gamma \) in steady state.

Begin by using the first order condition for house purchases in steady state and insert for the consumption of young in steady state from the budget constraint:

\[
\frac{\alpha}{h^{o,y}} - \frac{q}{w^y - \gamma p h^{o,y}} + \phi_y (q - \gamma p) = 0 \tag{46}
\]

\[
\phi_y (1 - \gamma \frac{1 + r}{r}) = \frac{q}{w^y - \gamma p h^{o,y}} - \frac{\alpha}{h^{o,y}} \geq 0 \tag{47}
\]

The second line uses the no arbitrage condition. By the complementary slackness condition, \( \phi_y \) must be non-negative. The term inside brackets on the leftmost side is positive by assumption (this is \( q - \gamma p \)), so the term on the right hand side must be larger or equal to zero. Rearranging this yields a condition on \( h^{o,y} \):

\[
h^{o,y} \geq \frac{\alpha w^y}{q(1 + \alpha \frac{1 + \gamma}{1 + r})} \tag{48}
\]

Now use the optimality condition for bond holdings:

\[
\frac{1}{c^y} = \beta(1 + r) \frac{c^m}{c^{m}} + \phi_y \tag{49}
\]

Similarly, this implies that

\[
\phi_y = \frac{1}{c^y} \beta(1 + r) \frac{c^m}{c^{m}} \geq 0 \tag{50}
\]

\[
\frac{c^m}{c^{m}} \geq \beta(1 + r) \tag{51}
\]
Use the expression for $c^m$ in steady state and $c^y = w^y - \gamma ph^{o,y}$ to get a messy expression:

$$\frac{w^m + \frac{w^o}{1+r}}{(1+\alpha)(1+\beta) - \beta \delta r} + \frac{ph^{o,y}(1 - (1 - \gamma)(1 + r))}{w^y - \gamma ph^{o,y}} \geq \beta(1 + r) \tag{52}$$

Now insert for $h^{o,y}$ from (47) with equality and define $\sigma = (1 + \alpha)(1 + \beta) - \beta \delta r$ for simplicity. Collect terms and rearrange to get:

$$\frac{w^m + \frac{w^o}{1+r}}{\sigma \beta (1 + r) - \frac{\alpha}{1+\alpha \gamma \frac{1+r}{r}}} \geq w^y \tag{53}$$

The strictest this condition can be is if $\gamma \to 0$. Then the expression reduces to

$$\frac{w^m + \frac{w^o}{1+r}}{(1 + r)(\sigma \beta + \alpha)} \geq w^y \tag{54}$$

Throughout I have chosen values of $w^y$ such that this holds. In this way I know that young households are on the borrowing constraint for all values of $\gamma$ that I consider. This simplifies the analysis. That the condition is the strictest when $\gamma = 0$ is fairly intuitive. When $\gamma = 0$, households who borrow up to the borrowing constraint shift the entire cost of the house purchase to future periods. They only want to do this if their income is sufficiently low compared to future periods. The main points of the analysis would not change if this condition did not always holds, but calculations would be more cumbersome. I have therefore chosen to always give young households an endowment such that the condition in (53) holds. In this way I do not have to worry about when they switch from being homeowners borrowing less than the maximal amount to homeowners borrowing the maximal amount.
References


[7] Iacoviello, Matteo and Marina Pavan: *Housing and debt over the life cycle and over the business cycle*, Journal of Monetary Economics 60 (2012) 221-238


48