Rational Quartic Symmetroids

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The study of generic quartic symmetroids in projective 3-space dates back to Cayley, but little is known about the special specimens. This thesis sets out to survey the possibilities in the non-generic case. We prove that the Steiner surface is a symmetroid. We also find quartic symmetroids that are double along a line and have two to eight isolated nodes, symmetroids that are double along two lines and have zero to four isolated nodes and symmetroids that are double along a smooth conic section and have two to four isolated nodes. In addition, we present degenerated symmetroids with fewer isolated singularities. A symmetroid that is singular along a smooth conic section and a line is given. This culminates in a 21-dimensional family of rational quartic symmetroids.
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CHAPTER 1

Introduction

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

Emil Artin

A symmetroid is a hypersurface $V(f)$ in $\mathbb{P}^n$ whose defining equation $f$ can be written as the determinant of a symmetric matrix $A(x)$ of linear forms. In this thesis we will concern ourselves with quartic symmetroids in $\mathbb{C}P^3$, that is,

$$A(x) = A_0 x_0 + A_1 x_1 + A_2 x_2 + A_3 x_3,$$

where the $A_i$ are $(4 \times 4)$-matrices with entries in $\mathbb{C}$. We will mostly omit the argument in the notation and denote $A(x)$ by simply $A$. It should be clear from the context at which point the matrix is evaluated.

The term symmetroid was coined by Cayley in the memoir [Cay69] from 1869, where it is related to the Jacobian surface of four quadric surfaces. It appears later in classical treatises such as [Jes16] and [Cob29]. It is shown in [Cos83] that the $K3$ surfaces that arise as the étale double cover of the generic Reye congruences are quartic symmetroids. Symmetroids are also discussed in [Dol12], which is Dolgachev’s monumental effort to preserve classical algebraic geometry. Nevertheless, it is fair to say that symmetroids have received little attention during their long history. However, they have recently obtained renewed interest with the study of spectrahedra.

A spectrahedron is the intersection of the cone of positive semidefinite matrices with an affine subspace of the space of real symmetric $(n \times n)$-matrices. The motivation for studying spectrahedra comes from semidefinite programming, which is the subfield of convex optimisation concerned with optimising a linear function over a spectrahedron. The algebraic boundary of a spectrahedron in $\mathbb{R}^3$ is a symmetroid in $\mathbb{C}P^3$. If a symmetroid has a representation (1.1) such that each $A_i$ is a real matrix, then its associated spectrahedron is the set

$$\{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid A_0 x_0 + A_1 x_1 + A_2 x_2 + A_3 \text{ is semidefinite}\}.$$

If the $A_i$ are $(n \times n)$-matrices, we say that the spectrahedron has degree $n$. Ottem, Ranestad, Sturmfels and Vinzant gave a new and elementary proof of
1. Introduction

the characterisation of the types of transversal quartic spectrahedra in [Ott+14]. Some of the key tools and ideas used in this thesis stem from that article, such as Algorithm 1.1.4 and the pillow in Example 3.3.1, which inspired Chapter 4.

In this thesis we find several novel examples of quartic symmetroids that are double along a line or a conic section. These are necessarily rational. We describe techniques for finding families of such surfaces. The methods employed also indicate heuristically to what degree the examples are generic or special. This is not a complete classification of rational quartic symmetroids, but it gives a good overview of the land. Hopefully the thesis has laid useful groundwork for future works on symmetroids or spectrahedra. Some existing results about quartic symmetroids are unsatisfactory in that they require the surfaces to be nodal. The explicit examples in this thesis may contribute to discern which statements can be generalised.

1.1 Basic Properties

Having identified the surface with a matrix, we are able to talk about the rank of a point. Namely, a point \( x \) is a rank-\( k \)-point if \( \text{rank} \ A(x) = k \). For a symmetroid of degree \( d \), the rank-(\( d - 2 \))-points are singular on \( \mathcal{V}(f) \), and generically they are nodes [Ott+14, Section 1]. The set of points where the rank is at most \( k \) is called the rank \( k \) locus. The set of points where the rank is at most \( k \) is called the rank \( k \) locus. The rank \( (d - 2) \)-locus is necessarily equal to the singular locus; the inclusion is strict in Example 4.1.7.

Generically, there are \( \binom{d+1}{3} \) nodes on \( \mathcal{V}(f) \). If the symmetroid is nodal, that is, all its singularities are isolated nodes, then it has exactly \( \binom{d+1}{3} \) rank-(\( d - 2 \))-points. The symmetroid is called transversal if it does not have additional nodes. The terminology is motivated by Theorem 1.1.2. Kummer surfaces are examples of nodal symmetroids that are not transversal [Jes16, Article 105].

Our main object of study, the quartic symmetroids, has therefore generically 10 nodes. They have generically no rank-1-points and the rank 2 locus is not contained in a quadric surface.

Moreover, we will concern ourselves with symmetroids that are rational, that is, birationally equivalent to \( \mathbb{P}^2 \). If the only singularities on a quartic surface are finitely many double points, then they can be resolved. Therefore, the quartic has a relatively minimal model with trivial canonical bundle. Hence it is not rational. Thus to find rational symmetroids, we need to look at quartic surfaces that either have nodes along a curve or have a triple point.

Example 1.1.1 (Cayley's symmetroid). Rational cubic symmetroids have been known for a long time. Cayley’s 4-nodal cubic can be realised as the blow-up of \( \mathbb{P}^2 \) in the six intersection points of four general lines. With a suitable choice of coordinates, it can be taken to have

\[
x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0
\]

as its equation. This is the determinant of

\[
\begin{bmatrix}
x_0 + x_3 & x_3 & x_3 \\
x_3 & x_1 + x_3 & x_3 \\
x_3 & x_3 & x_2 + x_3 \\
\end{bmatrix}.
\]

Cubic surfaces with four singularities of type\(^1\) \( A_1 \) that span \( \mathbb{P}^3 \) are projectively

\(^1\)See Appendix C for a note on this notation.
isomorphic, so these are all symmetroids.

1.1. Basic Properties

Genus 3 Sextics

The (3 × 3)-minors of a (3 × 4)-matrix of linear forms in \(\mathbb{C}[x_0, x_1, x_2, x_3]\) define in general a sextic curve in \(\mathbb{C}P^3\) with arithmetic genus 3. Let

\[
A := \begin{bmatrix}
    l_{00} & l_{01} & l_{02} & l_{03} \\
    l_{10} & l_{11} & l_{12} & l_{13} \\
    l_{20} & l_{21} & l_{22} & l_{23} \\
    l_{30} & l_{31} & l_{32} & l_{33}
\end{bmatrix}
\]

be a (4 × 4)-matrix of linear forms in \(\mathbb{C}[x_0, x_1, x_2, x_3]\). Let \(S\) be the quartic in \(\mathbb{C}P^3\) given by \(\det(A) = 0\). The (3 × 3)-minors of any (3 × 4)-submatrix \(A_1\) and any (4 × 3)-submatrix \(A_2\) define such curves, \(C_1\) and \(C_2\), lying on \(S\). Say

\[
A_1 := \begin{bmatrix}
    l_{10} & l_{11} & l_{12} & l_{13} \\
    l_{20} & l_{21} & l_{22} & l_{23} \\
    l_{30} & l_{31} & l_{32} & l_{33}
\end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix}
    l_{00} & l_{01} & l_{03} \\
    l_{10} & l_{11} & l_{13} \\
    l_{20} & l_{21} & l_{23} \\
    l_{30} & l_{31} & l_{33}
\end{bmatrix}
\]

The curves \(C_1\) and \(C_2\) both vanish on the common (3 × 3)-minor of \(A_1\) and \(A_2\). In our example, that is

\[
\begin{vmatrix}
    l_{10} & l_{11} & l_{13} \\
    l_{20} & l_{21} & l_{23} \\
    l_{30} & l_{31} & l_{33}
\end{vmatrix} = 0.
\]

The common minor defines a cubic surface \(S'\) in \(\mathbb{C}P^3\) intersecting \(S\) along \(C_1\) and \(C_2\). The intersection \(S \cap S'\) has degree \(4 \cdot 3 = 12\) by Bézout’s theorem. Since \(C_1, C_2 \subseteq S \cap S'\) both have degree 6, there is nothing else in \(S \cap S'\), except possibly some points.

Now, assume that \(A\) is symmetric and \(A_1, A_2\) are obtained by removing the \(i\)-th row and column, respectively. Then \(C_1 = C_2\). By a continuity argument, or direct computation, one can show that \(S \cap S'\) equals two times \(C_1\), plus some points.

In other words, if \(S \subseteq \mathbb{C}P^3\) is a quartic symmetroid, then there exist cubic surfaces \(S'\) that are tangent to \(S\) along a sextic curve with arithmetic genus 3. This is a rare property among quartic surfaces. In [Ble+12, Section 5], an algorithm is given for producing a representation (1.1) for a 10-nodal quartic with such a curve, using the Hilbert-Burch theorem [Eis95, Theorem 20.15].

Projection From a Node

One of the classic tools for examining a quartic surface \(S := V(f) \subseteq \mathbb{P}^3\) with at least one node \(p\), is to study the projection \(S \setminus \{p\} \to \mathbb{P}^2\) from \(p\). This is a two-to-one map, which extends to a morphism \(\pi_p : \tilde{S} \to \mathbb{P}^2\) on the blow-up \(\tilde{S}\) of \(S\) with centre \(p\). If \(p := [1 : 0 : 0 : 0]\), then \(f\) is on the form \(ax_0^2 + bx_0 + c\) where \(a, b, c \in \mathbb{C}[x_1, x_2, x_3]\). The fibres of \(\pi_p\) are given by the roots of \(f\), so the points where \(f\) has a double root, known as the ramification locus, are described by the discriminant \(R_p := (b^2 - 4ac) \subseteq \mathbb{P}^2\).
A singular point on $R_p$ is the image under $\pi_p$ of either a singularity on $S$ or of a line through $p$ contained in $S$. Conversely, singular points on $S$ and lines on $S$ through $p$ are mapped to singularities on $R_p$.

Cayley gave a characterisation of quartic symmetroids using the sextic curve $R_p$ [Cay69], but we refer to a more modern source for the proofs:

**Theorem 1.1.2** (Cayley, [Ott+14, Theorem 1.2]). If $S$ is a quartic symmetroid in $\mathbb{CP}^3$ and $p \in S$ is a rank-$2$-point, then the ramification locus is the union of two cubic curves $R_p = R_1 \cup R_2$. Moreover, if no line through $p$ is contained in $S$, then $S$ is transversal if and only if $R_1$ and $R_2$ are smooth and intersecting transversally in $9$ points.

Cayley was also able to provide a converse for nodal quartics.

**Theorem 1.1.3** (Cayley, [Ott+14, Theorem 1.2]). Let $p$ be a node on a nodal, quartic surface $S \subset \mathbb{CP}^3$. If the ramification locus $R_p$ splits into two cubics, then $S$ is a symmetroid.

In their proof of [Ott+14, Theorem 1.2], Ottem et al. gave an algorithm for constructing a matrix with the correct determinant from the ramification locus. We omit most of the details in the proof and boil it down to the steps in the algorithm:

**Algorithm 1.1.4.** Suppose that $S := V(f)$ is a nodal quartic surface with a node in $p := [1 : 0 : 0 : 0]$. Then $f = -qx_0^2 + 2gx_0 + \Delta$ for polynomials $q, g, \Delta \in \mathbb{C}[x_1, x_2, x_3]$. Assume also that the ramification locus $R_p$ factors into cubics $V(F_{11})$ and $V(F_{22})$. Up to rescaling of $F_{11}$, we have that

$$F_{11}F_{22} = g^2 + q\Delta$$

in $\mathbb{C}[x_1, x_2, x_3]$. Since $q\Delta$ is a square modulo $F_{11}$, there is a subscheme $Z_\Delta$ of length $6$ in $V(F_{11})$ such that $V(F_{11}, \Delta) = 2Z_\Delta$.

There are no conics containing $Z_\Delta$, thus the space of cubics in $\mathbb{C}[x_1, x_2, x_3]_3$ vanishing on $Z_\Delta$ is $4$-dimensional. Extend $F_{11}$ and $F_{12} := g$ to a basis

$$\{F_{11}, F_{12}, F_{13}, F_{14}\}$$

for that space. For $2 \leq j \leq k \leq 4$, we can find a cubic $F_{jk} \in \mathbb{C}[x_1, x_2, x_3]_3$ and a quadric $q_{jk} \in \mathbb{C}[x_1, x_2, x_3]_2$ satisfying

$$F_{1j}F_{1k} = F_{11}F_{jk} + q_{jk}\Delta.$$

Let

$$F := \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{12} & F_{22} & F_{23} & F_{24} \\ F_{13} & F_{23} & F_{33} & F_{34} \\ F_{14} & F_{24} & F_{34} & F_{44} \end{bmatrix}.$$ 

It follows from the construction that $\Delta^3$ divides $\det(F)$, and since they both have degree $12$, there exists a constant $c \in \mathbb{C}$ such that $\det(F) = c \cdot \Delta^3$. It is a consequence that $c \neq 0$, because $Z_\Delta$ is not contained in a conic.
The entries in the adjoint matrix $F^{\text{adj}}$ are divisible by $\Delta^2$. Define the $(4 \times 4)$-matrix

$$M := \frac{1}{\Delta^2} \cdot F^{\text{adj}}$$

with linear entries $m_{jk}$. A computation shows that

$$A := \begin{bmatrix}
  m_{11} & cx_0 + m_{12} & m_{13} & m_{14} \\
  m_{13} & m_{22} & m_{23} & m_{24} \\
  m_{14} & m_{23} & m_{33} & m_{34} \\
  m_{24} & m_{34} & m_{44}
\end{bmatrix}$$

has determinant $\det(A) = c^3 \cdot f$. Thus $A$ is a determinantal representation of $S$, since $c \neq 0$.

Remark 1.1.5. An important property to note is that the rank 2 locus in the representation obtained from Algorithm 1.1.4 is mapped to $\mathcal{V}(F_{11}) \cap \mathcal{V}(F_{22})$ via the projection $\pi_p$.

Classically, $\mathcal{V}(F_{11})$ is called a contact curve for $\mathcal{V}(\Delta)$, meaning that all the points in their intersection have even multiplicity.

Remark 1.1.6. The first part of Algorithm 1.1.4 is to use Dixon’s method to find a symmetric, determinantal representation of the ternary quartic $\Delta$. In general, it is hard to apply Dixon’s method; the first application was published 36 years after Dixon’s proof [Edge38]. Plaumann, Sturmfels and Vinzant noted in 2012 that it is difficult to construct suitable contact curves to use as input in the algorithm [PSV12]. However, in the particular case of Algorithm 1.1.4, all the input is given.

Remark 1.1.7. We will exclusively use Algorithm 1.1.4 on surfaces which are not nodal. Despite the fact that the algorithm is not proven to always work in that case — we certainly encountered instances where it did not — it has been one of our most valuable tools for producing explicit examples of rational quartic symmetroids.

For a transversal symmetroid, the matrix representation (1.1) is unique up to conjugation by an invertible matrix [Ble+12, Proposition 11]. With Kummer symmetroids, only 10 of the 16 nodes are rank-2-points. However, for each node there exists a representation such that the node has rank 2 [Ott+14]. In [Jcs16, Article 9], there is an 11-nodal quartic symmetroid with a node $p$ that has rank 3 in every representation. The projection from $p$ has a ramification locus that cannot be decomposed into cubics.

1.2 Outline

The rest of the thesis is organised as follows:

Chapter 2 presents the Veronese varieties with particular emphasis on the Veronese surface, which is the rank 1 locus of a symmetroid. We show in Chapter 6 that the general projection of the Veronese surface to $\mathbb{P}^1$ is a symmetroid.
1. Introduction

Chapter 3 develops the theory of quadric hypersurfaces in $\mathbb{P}^3$, employing the moduli space as an abstract tool for analysing symmetroids. This is then applied superficially to two non-generic examples, but we revisit the technique more thoroughly later in Sections 4.3 and 5.1.

Chapter 4 is concerned with symmetroids that are singular along a conic section. These are projections of del Pezzo surfaces and therefore rational. The surfaces are first studied through the projection itself and then via the quadrics.

Chapter 5 serves to bridge the gap between symmetroids having finitely many double points and symmetroids that are double along a curve of degree 2. It uses the same type of analysis of the quadrics as in Chapter 4 on symmetroids that are double along a single line.

Chapter 6 features symmetroids that are degenerations of the surfaces from the previous chapters. It includes symmetroids that are singular along a cubic curve, have a triple point or are cones. Among them is the Steiner surface. The chapter ends with a discussion of our work.

Appendix A is a means to reduce printing costs by collecting colour images that would otherwise be scattered throughout the text.

Appendix B contains the majority of the computational work. It should be read in parallel with the other chapters. Readers who are only interested in the results may skip this section entirely.

Appendix C gives a brief account of the $ADE$ notation used in the text. The content of this appendix is largely disjoint from the rest of the thesis and can be ignored by readers who already have an understanding of rational double points.

Throughout the thesis we assume familiarity with linear systems of divisors and blow-ups. An excellent reference is [Har77, V, Section 4]. We also refer to [Ott+14] for basic facts about symmetroids.
Here we review Veronese varieties, because the image $V$ of a quadratic Veronese map is a quintessential example of a determinantal variety. Its secant variety $\text{Sec}(V)$ is a symmetroid, with $V$ being the rank 1 locus of $\text{Sec}(V)$. We showcase some classical arguments involved in the study of these kinds of varieties.

2.1 Quadratic Forms

Before we plunge into the examination of the Veronese embedding, we allow ourselves a detailed digression. The relationship between quadratic forms and the rank of the corresponding matrix is vital throughout the thesis.

**Proposition 2.1.1.** Let $C$ be the conic in $\mathbb{CP}^2$ given by

$$x^TAx = ax^2 + 2bxy + cy^2 + 2dxyz + 2eyz + fz^2 = 0,$$

where

$$A := \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

Then we have that

(i) $\text{rank } A = 1$ if and only if $C$ is a double line,

(ii) $\text{rank } A = 2$ if and only if $C$ consists of two distinct lines,

(iii) $\text{rank } A = 3$ if and only if $C$ is smooth.
2. Veronese Varieties

Proof. Since $C$ is a conic, $A$ cannot be the zero matrix.

(i) Assume that $C$ is a double line. Then its equation is on the form

$$(ax + by + cz)^2 = a^2x^2 + 2abxy + b^2y^2 + 2acx + 2bcyz + e^2z^2 = 0.$$ 

Thus

$$A = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix},$$

which has rank 1.

Conversely, if rank $A = 1$, then in particular the $(2 \times 2)$-minors

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2 \quad \begin{vmatrix} a & d \\ d & f \end{vmatrix} = af - d^2 \quad \begin{vmatrix} c & e \\ e & f \end{vmatrix} = cf - e^2$$

vanish. Hence $b = \sqrt{ac}$, $d = \sqrt{af}$ and $e = \sqrt{cf}$. Substituting $\tilde{a} := \sqrt{a}$, $\tilde{c} := \sqrt{c}$ and $\tilde{f} := \sqrt{f}$, the equation defining $C$ reduces to

$$\tilde{a}^2x^2 + 2\tilde{a}\tilde{c}xy + \tilde{c}^2y^2 + 2\tilde{a}\tilde{f}xz + 2\tilde{c}\tilde{f}yz + \tilde{f}^2z^2 = (\tilde{a}x + \tilde{c}y + \tilde{f}z)^2 = 0.$$ 

In conclusion, $C$ is a double line.

(iii) Let

$$f(x, y, z) := ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2,$$

then $C$ is nonsingular if and only if the system of equations of partial derivatives,

$$\frac{\partial f}{\partial x} = 2ax + 2by + 2dz = 0,$$

$$\frac{\partial f}{\partial y} = 2bx + 2cy + 2ez = 0,$$

$$\frac{\partial f}{\partial z} = 2dx + 2ey + 2fz = 0,$$

has no nontrivial solution. This is equivalent to the determinant

$$\begin{vmatrix} 2a & 2b & 2d \\ 2b & 2c & 2e \\ 2d & 2e & 2f \end{vmatrix} = \det(2A) \neq 0.$$ 

This occurs precisely when $A$ has full rank.

The case (ii) now follows because we have exhausted the list of possible ranks and conic sections. ■

Remark 2.1.2. Proposition 2.1.1 extends easily to an $(n \times n)$-matrix $A$. The quadratic form is a square if and only if rank $A = 1$, and $x^T A x$ is singular if and only if $A$ is singular. Moreover, since the partial derivatives of $x^T A x$ vanish precisely when $Ax = 0$, the Rank-Nullity theorem implies that the projective dimension of the singular locus of the quadratic form is $n - \text{rank } A - 1$.  ♦
2.2 Veronese Embedding

Recall that the Veronese surface means the image of the map \( \nu_2 : \mathbb{P}^2 \to \mathbb{P}^5 \) given by \([x_0 : x_1 : x_2] \mapsto [x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2]\).

If we let \( \mathbb{P}^5 \) have coordinates \([y_{00} : y_{01} : y_{02} : y_{11} : y_{12} : y_{22}]\), we see that the polynomials
\[
q_1 := y_{00}y_{11} - y_{01}^2 \\
q_2 := y_{00}y_{12} - y_{01}y_{02} \\
q_3 := y_{00}y_{22} - y_{02}^2 \\
q_4 := y_{01}y_{12} - y_{02}y_{11} \\
q_5 := y_{01}y_{22} - y_{02}y_{12} \\
q_6 := y_{11}y_{22} - y_{12}^2
\]
disappear on the Veronese surface. In fact, the Veronese surface is precisely the zero locus of the \(q_i\) \([Har77, I, Exercise 2.12]\). This is equal to the rank 1 locus of the symmetric matrix 
\[
M := \begin{bmatrix}
y_{00} & y_{01} & y_{02} \\
y_{01} & y_{11} & y_{12} \\
y_{02} & y_{12} & y_{22}\end{bmatrix}.
\]

In light of Proposition 2.1.1 this can be interpreted as the set of points
\([y_{00} : \ldots : y_{22}] \in \mathbb{P}^5\)
such that the quadratic form associated to \(M\) defines a double line.

We note two properties about the Veronese surface:

Observation 2.2.1. Let \( V \subseteq \mathbb{P}^5 \) be the Veronese surface. Then

(i) if \( C \) is a curve on \( V \), then there exists a hypersurface \( H \subset \mathbb{P}^5 \) such that \( H \cap V = C \) \([Har77, I, Exercise 2.13]\),

(ii) there are no lines on \( V \) \([Har92, Lecture 19]\).

More generally, a Veronese map of degree \( d \) is a morphism \( \nu_d : \mathbb{P}^n \to \mathbb{P}^N \), where \( N = \binom{n+d}{d} - 1 \), defined by sending a point \([a_0 : \ldots : a_n]\) to
\[
[M_0(a_0, \ldots, a_n) : \ldots : M_N(a_0, \ldots, a_n)].
\]
The \(M_i\) are the different monomials of degree \( d \) in \( x_0, \ldots, x_n \). The map \( \nu_d \) is an isomorphism onto its image, which is called a Veronese variety. The image of a variety \( X \subseteq \mathbb{P}^n \) under \( \nu_d \) is a subvariety of \( \mathbb{P}^N \) \([Har92, Lecture 2]\). The image of the special case \( \nu_d : \mathbb{P}^1 \to \mathbb{P}^d \) is called a rational, normal curve.

Let \( \mathbb{P}^N \) have coordinates \([y_0 : \ldots : y_N]\). Then \( \nu_d(\mathbb{P}^n) \) equals the intersection of the quadratic hypersurfaces \( V(y_iy_j - y_ky_l) \) for multi-indices \((i, j, k, l)\) such that \( M_iM_j = M_kM_l \). In the case of the quadratic Veronese map,

\[
\nu_2 : \mathbb{P}^n \to \mathbb{P}^{\binom{n+1}{2}(n+2)-1},
\]
the image can be expressed as the zero locus of the \((2 \times 2)\)-minors of a symmetric \((n + 1) \times (n + 1)\)-matrix.

Any projective variety is isomorphic to an intersection of a Veronese variety with a linear space \([Har92, Exercise 2.9]\).
2. Veronese Varieties

2.3 The Degree of a Veronese Variety

Before we can compute the degree of a Veronese variety, we need to recollect some well-known results.

Definition 2.3.1. Let \( X \subseteq \mathbb{P}^n \) be an irreducible, \( k \)-dimensional variety. If \( \Omega \) is a general \((n-k)\)-plane, then the \textit{degree} \( \deg(X) \) of \( X \), is the number of points of intersection of \( \Omega \) and \( X \). If \( X \) is reducible and \( X = \bigcup X_i \) is its decomposition into irreducible components, then the definition is extended linearly by \( \deg(X) := \sum \deg(X_i) \).

Definition 2.3.2. Let \( X,Y \subseteq \mathbb{P}^n \) be subvarieties. We say that \( X \) and \( Y \) \textit{intersect transversally} at a point \( P \in X \cap Y \), if they are smooth at \( P \) and their tangent spaces span \( T_P \mathbb{P}^n \). Let \( Z_i \) be the irreducible components of \( X \cap Y \). Then \( X \) and \( Y \) intersect \textit{generically transversally} if they for each \( i \) intersect transversally at a general point \( P_i \in Z_i \).

If \( \dim(X) + \dim(Y) = n \), then saying that \( X \) and \( Y \) intersect generically transversally is the same as saying that they intersect transversally.

Theorem 2.3.3 (Bézout, [Har92, Theorem 18.3]). Let \( X,Y \subseteq \mathbb{P}^n \) be subvarieties of pure dimensions \( k \) and \( l \) with \( k + l \geq n \), and suppose that they intersect generically transversally. Then

\[
\deg(X \cap Y) = \deg(X) \cdot \deg(Y).
\]

In particular, if \( k + l = n \), then \( X \cap Y \) consists of \( \deg(X) \cdot \deg(Y) \) points.

Theorem 2.3.4 (Bertini, [Har92, Theorem 17.16]). Let \( X \) be a quasi-projective variety, \( f : X \to \mathbb{P}^n \) a regular map, \( H \subset \mathbb{P}^n \) a general hyperplane and put \( Y := f^{-1}(H) \). Then \( Y_{\text{sing}} = X_{\text{sing}} \cap Y \).

One can use Bertini’s theorem to show that \( k \) general hypersurfaces in \( \mathbb{P}^n \), for \( k < n \), intersect transversally in a smooth, \((n-k)\)-dimensional variety \( X \subset \mathbb{P}^n \) [Har92, Exercise 17.17].

Now we can turn to the task of computing the degree. Let

\[
P := [a_0 : a_0 : a_1 : a_2 : a_2 : a_2] = [a_0^2 : a_0a_1 : a_0a_2 : a_1^2 : a_1a_2 : a_2^2]
\]

be the point on the Veronese surface \( V \subset \mathbb{P}^5 \) coming from \( P' := [a_0 : a_1 : a_2] \) via the embedding \( \nu_2 : \mathbb{P}^2 \to \mathbb{P}^5 \). Saying that a hyperplane \( H \subset \mathbb{P}^5 \) given by

\[
b_{00}y_{00} + b_{01}y_{01} + b_{02}y_{02} + b_{11}y_{11} + b_{12}y_{12} + b_{22}y_{22} = 0
\]

intersects \( V \) in \( P \), amounts to saying that a certain curve vanish at \( P' \). Indeed,

\[
\sum b_{ij}a_{ij} = 0
\]

if and only if

\[
\left( \sum b_{ij}x_ix_j \right)(a_0, a_1, a_2) = 0.
\]

In other words, there is a bijection between hyperplane sections of the Veronese surface and conic sections in \( \mathbb{P}^2 \).
2.4. Projection of the Veronese Surface to $\mathbb{P}^3$

In precisely the same manner, for a Veronese map $\nu_d: \mathbb{P}^n \rightarrow \mathbb{P}^N$ of degree $d$, the inverse image $\nu_d^{-1}(H)$ of a general hyperplane $H \subset \mathbb{P}^N$ is a general hypersurface of degree $d$ in $\mathbb{P}^n$. The degree of the Veronese variety $\nu_d(\mathbb{P}^n)$ is equal to the number of intersection points of $\nu_d(\mathbb{P}^n)$ and $n$ general hyperplanes $H_i \subset \mathbb{P}^N$. Let $Y_i := \nu_d^{-1}(H_i) \subset \mathbb{P}^n$ be the inverse images. Bézout’s theorem gives that

$$\deg(Y_1 \cap \ldots \cap Y_n) = \deg(Y_1) \cdots \deg(Y_n) = d^n.$$  

Bézout’s theorem implies that the intersection of the $Y_i$ has dimension 0. An $(n - 0)$-plane in $\mathbb{P}^n$ is just $\mathbb{P}^n$ itself, so there are $d^n$ points in

$$(Y_1 \cap \ldots \cap Y_n) \cap \mathbb{P}^n = Y_1 \cap \ldots \cap Y_n.$$  

Subsequently, $\nu_d(\mathbb{P}^n)$ has degree $d^n$ since $\nu_d$ is injective. A similar argument shows that if $Y \subset \mathbb{P}^n$ is a variety of dimension $k$ and degree $e$, then $\nu_d(Y) \subset \mathbb{P}^N$ has degree $d^k e$.

In particular, $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ has degree 4. Its intersection with a general hyperplane is a rational, normal curve in $\mathbb{P}^4$. This is the only incidence where the intersection of a Veronese variety with a general hyperplane is another Veronese variety [Har92, Exercise 18.14].

### 2.4 Projection of the Veronese Surface to $\mathbb{P}^3$

Projection from $\mathbb{P}^5$ to $\mathbb{P}^3$ can be done in two steps: First projecting from a point $p_1 \in \mathbb{P}^5$ to $\mathbb{P}^4$, and then projecting further from another point $p_2 \in \mathbb{P}^4$ to $\mathbb{P}^3$. The composition $\mathbb{P}^5 \rightarrow \mathbb{P}^3$ is said to be projection from the line $L$ between $p_1$ and $p_2$. We shall consider the image of the Veronese surface when $L$ is in two different positions.

#### General Projection

The general projection of the Veronese surface $V$ to $\mathbb{P}^3$ is called a Steiner surface $S$. In this section we will show that a Steiner surface has nodes along three lines which meet in a triple point. First, we need to make a few notes about the secant variety $Sec(V)$ of $V$, starting with a simple definition.

**Definition 2.4.1.** The span $\overline{\Gamma \Phi}$ of two subsets $\Gamma, \Phi \subset \mathbb{P}^n$ is defined as the smallest linear subspace of $\mathbb{P}^n$ containing $\Gamma \cup \Phi$. If $\Gamma := \mathbb{P}(W_1)$ and $\Phi := \mathbb{P}(W_2)$ are projectivisations of subspaces $W_1, W_2 \subset k^{n+1}$, then $\overline{\Gamma \Phi} = \mathbb{P}(W_1 + W_2)$. $\leftarrow$

For linear subspaces $\Lambda_1, \Lambda_2 \subset \mathbb{P}^n$, we have the inequality

$$\dim(\overline{\Lambda_1 \Lambda_2}) \leq \dim(\Lambda_1) + \dim(\Lambda_2) + 1.$$  

Equality holds if and only if $\Lambda_1$ and $\Lambda_2$ are disjoint. In general,

$$\dim(\overline{\Lambda_1 \Lambda_2}) \leq \dim(\Lambda_1) + \dim(\Lambda_2) - \dim(\Lambda_1 \cap \Lambda_2),$$  

with the convention that $\dim(\emptyset) = -1$ [Har92, Example 1.1].

Recall that for an irreducible variety $X \subset \mathbb{P}^n$, the secant line map is a rational map $s: X \times X \rightarrow G(1, n)$, where the Grassmannian $G(1, n)$ is the
2. Veronese Varieties

lines in \( \mathbb{P}^n \). The secant line map is defined on the complement \(( X \times X ) \setminus \Delta \) of the diagonal \( \Delta \) by sending a pair of points \(( p, q )\) to the line \( \overline{pq} \) connecting them. A point \( l \in \text{im} \, s \) is called a \textit{secant line} of \( X \). The \textit{secant variety} of \( X \),

\[
\text{Sec}(X) := \bigcup_{l \in \text{im} \, s} l \subseteq \mathbb{P}^n,
\]

is the union of all secant lines of \( X \).

In general, it is hard to compute secant varieties, but for determinantal varieties we have the following result:

**Proposition 2.4.2** ([Har92, Exercise 11.29]). Let \( M \) be the projective space of \(( m \times n )\)-matrices and let \( M_k \subseteq M \) be the subvariety of matrices of rank at most \( k \). Assume that \( 2k < \min\{ m, n \} \). Then the secant variety \( \text{Sec}(M_k) \) is equal to the subvariety \( M_{2k} \subseteq M \) of matrices of rank at most \( 2k \).

We know that the Veronese surface \( V \) is the rank 1 locus of

\[
M = \begin{bmatrix}
y_{00} & y_{01} & y_{02} \\
y_{01} & y_{11} & y_{12} \\
y_{02} & y_{12} & y_{22}
\end{bmatrix}.
\]

However, \( M \) only parametrises the symmetric \(( 3 \times 3 )\)-matrices, so Proposition 2.4.2 is not directly applicable. Nevertheless, we shall see that the conclusion that \( \text{Sec}(V) \) is the rank 2 locus of \( M \) still holds true.

Any secant line of \( V \) can be written as \( \nu_2(p)\nu_2(q) \) for some points \( p, q \in \mathbb{P}^2 \). The line \( \overline{pq} \) is mapped to a conic \( C \subset V \subset \mathbb{P}^5 \) under \( \nu_2 \). Then \( C \) spans a plane \( \Pi \). A general line in \( \Pi \) intersects \( C \) in two points, hence is a secant line to \( C \) and thus to \( V \). It follows that \( \text{Sec}(V) \) is the union of the planes intersecting \( V \) in a conic.

Explicitly, the planes are parametrised by the column space — or equivalently the row space — of \( M \). For each \([ a : b : c ] \in \mathbb{P}^2 \), the equation

\[
a \begin{bmatrix} y_{00} \\ y_{01} \\ y_{02} \end{bmatrix} + b \begin{bmatrix} y_{01} \\ y_{11} \\ y_{12} \end{bmatrix} + c \begin{bmatrix} y_{02} \\ y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

defines a plane which intersects \( V \) in the conic \( \nu_2(l) \), where \( l \in \mathbb{P}^2 \) is the line given by \( ax_0 + bx_1 + cx_2 = 0 \). This expresses that the columns are linearly dependent, so \( \operatorname{rank}(M) \) is at most 2. Thus the equation for \( \text{Sec}(V) \) is \( \det(M) = 0 \).

Note that \( V \) is the singular locus of \( \text{Sec}(V) \).

As \( \text{Sec}(V) \) is a hypersurface in \( \mathbb{P}^5 \), it has dimension 4. A \( k \)-dimensional variety \( X \subseteq \mathbb{P}^n \) is said to have a \textit{deficient secant variety} if

\[
\dim(\text{Sec}(X)) < \min\{2k + 1, n\}.
\]

The Veronese surface is the only smooth surface with a deficient secant variety [Har92, Lecture 11].

We are now ready to study what happens when \( V \) is projected from a general line \( L \) to \( \mathbb{P}^3 \). Since

\[
\text{Sec}(V) = \mathcal{V}(\det M)
\]
2.4. Projection of the Veronese Surface to \( \mathbb{P}^3 \)

has degree 3, it will meet \( L \) in three distinct points \( P_1, P_2 \) and \( P_3 \). Each \( P_i \) defines a plane \( \Pi_i \subset \text{Sec}(V) \) intersecting \( V \) in a conic \( C_i \). The projection \( \pi_L : \mathbb{P}^5 \to \mathbb{P}^3 \) maps \( \Pi_i \) to a line \( L_i \subset \mathbb{P}^3 \) and surjects \( C_i \) two-to-one onto \( L_i \). This explains the promised double points along three lines. The situation is displayed in Figure 2.1.

Figure 2.1: Under the projection from a general line \( L \) in \( \mathbb{P}^5 \) to \( \mathbb{P}^3 \), three conics on the Veronese surface \( V \) are mapped two-to-one onto lines in a Steiner surface \( S \).

The conics \( C_i \) can be thought of as \( \nu_2(l_i) \) for lines \( l_i \subset \mathbb{P}^2 \). Since the lines \( l_i \) meet pairwise, so must the \( C_i \) and therefore also the \( L_i \). Hence there are two possible configurations, as shown in Figure 2.2 on the next page. Either all the \( L_i \) lie in a plane or they form three axes. Suppose for contradiction that the \( L_i \) lie in a plane. Then this \( L \) together with this plane spans a \( \mathbb{P}^4 \subset \mathbb{P}^5 \). The intersection of this \( \mathbb{P}^4 \) with \( V \) will contain \( C_1 + C_2 + C_3 \), which has degree 6. But \( \text{deg}(\mathbb{P}^4 \cap V) = 4 \). This is impossible, so the lines \( L_i \) meet in a single point with multiplicity 3.

Choose coordinates \([z_0 : z_1 : z_2 : z_3]\) for \( \mathbb{P}^3 \) such that the singular lines of the Steiner surface \( S \) are \( z_1 = z_2 = 0 \), \( z_1 = z_3 = 0 \) and \( z_2 = z_3 = 0 \). Then we may assume by [Dol12, Section 2.1.1] that the equation for \( S \) is

\[
z_0 z_1 z_2 z_3 + z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2 = 0,
\]

after scaling the coordinates. We will henceforth refer to all general projections of the Veronese surface to \( \mathbb{P}^3 \) as the Steiner surface. The affine piece \( z_0 = 1 \) is plotted in Figure 2.3 on the following page. Moreover, an explicit birational map \( \mathbb{P}^2 \dashrightarrow S \) can be given by sending \([x_0 : x_1 : x_2]\) to

\[
[(-x_0 + x_1 + x_2)^2 : (x_0 - x_1 + x_2)^2 : (x_0 + x_1 - x_2)^2 : (x_0 + x_1 + x_2)^2].
\]

This map is surjective [Dol12, Section 2.1.1].

Remark 2.4.3. In Section 6.1, we find a symmetric matrix having (2.1) as its determinant, showing that the Steiner surface is a symmetroid.
2. Veronese Varieties

![Figure 2.2: Pairwise intersection of conics and lines.](image)

Duality and Special Projection

Later, when hunting for rational symmetroids, we shall be interested in quartics that are singular along a curve. As an exercise, we find a line $L \subset \mathbb{P}^5$ such that the dual of the Veronese surface $V$ intersected with the orthogonal subspace $L^\perp$ of $L$, is singular along a curve. We begin by clarifying the terminology.

**Definition 2.4.4.** The dual projective space $(\mathbb{P}^n)^\vee$ consists of all hyperplanes in $\mathbb{P}^n$. That is, the point $[a_0 : \ldots : a_n] \in (\mathbb{P}^n)^\vee$ corresponds to the hyperplane $V(a_0x_0 + \cdots + a_nx_n) \subset \mathbb{P}^n$.

**Definition 2.4.5.** Let $X \subset \mathbb{P}^n$ be an irreducible, projective variety. A hyperplane $H \subset \mathbb{P}^n$ is said to be tangent to $X$ if $H$ contains an embedded tangent space $T_PX$ at some smooth point $P \in X$.

**Definition 2.4.6.** For an irreducible, projective variety $X \subset \mathbb{P}^n$, we define the dual variety $X^* \subset (\mathbb{P}^n)^\vee$ to be the closure of the set of all tangent hyperplanes of $X$.

**Theorem 2.4.7** (Reflexivity Theorem, [Tev03, Theorem 1.7]). Let $X \subset \mathbb{P}^n$ be an irreducible, projective variety. Then

(i) $X^{**} = X$.

(ii) More precisely, if $x \in X$ and $H \in X^*$ are smooth points, then $H$ is tangent to $X$ at $x$ if and only if $x$, regarded as a hyperplane in $(\mathbb{P}^n)^\vee$, is tangent to $X^*$ at $H$. 

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Definition 2.4.8. For a projective subspace $X \subset \mathbb{P}^n$, let the orthogonal subspace $X^\perp \subset (\mathbb{P}^n)^\vee$ be the set of hyperplanes that contain $X$. ♠

To motivate our undertaking, consider a line $L \subset \mathbb{P}^5$. Then $L^\perp \subset (\mathbb{P}^5)^\vee$ is a $\mathbb{P}^3$. Denote the projection $\pi_L(V)$ of $V$ from $L$ by $\overline{V}$. In the irreducible case, projecting from $L$ and then taking the dual is the same as taking the dual first and then intersecting with $L^\perp$, that is $\overline{V}^\ast = V^\ast \cap L^\perp$. Then we could use Theorem 2.4.7 to deduce $\overline{V}$. We will however shortly see that $V^\ast \cap L^\perp$ is reducible in our case.

Let us now turn to the task of finding the dual of the Veronese surface. A hyperplane is tangent to $V$ if and only if its intersection with $V$ is singular. From Section 2.3, we know that a hyperplane section of $V$ is the image $\nu_2(C)$ of a conic $C \subset \mathbb{P}^2$. Thus the only singular hyperplane sections are those that break up into pairs of conics, meaning they are the image of either a line-pair or double line in $\mathbb{P}^2$. Thus Proposition 2.1.1 implies that $V^\ast$ is parametrised by the rank 2 locus of

$$
\begin{bmatrix}
y_{00} & y_{01} & y_{02} \\
y_{10} & y_{11} & y_{12} \\
y_{20} & y_{21} & y_{22}
\end{bmatrix}
$$

This is precisely the same description as the one we gave for $\text{Sec}(V)$. Note that Theorem 2.4.7 implies that $\text{Sec}(V)^\ast \equiv V$.

For simpler notation, let $V_1 := V$, $Y_1 := \text{Sec}(V_1)$, $Y_2 := V_1^\ast$ and $V_2 := Y_1^\ast$, so that $Y_2 = \text{Sec}(V_2)$. Strict notational propriety dictates that $\overline{V}$ should be denoted $\overline{V}_1$, but we shall keep the original symbol in this case.

Since $\text{Sing} Y_2 = V_2$, we have that $\text{Sing}(Y_2 \cap L^\perp) = V_2 \cap L^\perp$. Thus finding a line $L \subset \mathbb{P}^5$ such that $Y_2 \cap L^\perp$ is singular along a curve, amounts to finding a $\mathbb{P}^3 \subset \mathbb{P}^5$ such that $\mathbb{P}^3 \cap V_2$ contains a curve.

What are the possible curves in the intersection of the Veronese surface and a $\mathbb{P}^3$? As noted above, the intersection of $V_2$ and a hyperplane is the image $\nu_2(C)$ of a conic $C \subset \mathbb{P}^2$. Because $\mathbb{P}^3 \subset \mathbb{P}^5$ is contained in a hyperplane, we deduce that $V_2 \cap \mathbb{P}^3 \cong \nu_2(X)$ for some subset $X \subset C$ of a conic $C \subset \mathbb{P}^2$. A $\mathbb{P}^3$ in $\mathbb{P}^5$ is contained in a pencil of hypersurfaces, so $X$ must be the base locus of a pencil of conics. Generically, $X$ will consist of 4 points, but specically it can be a line and a point. See Figure 2.4 on the next page. In the latter case, $\nu_2(X)$ consists of a conic $C$ and a point $P$, where $P$ is either an embedded component in $C$ or $P$ is not in the plane spanned by $C$, since $\nu_2(X)$ span $\mathbb{P}^3$.

Consequently, if $L \subset \mathbb{P}^5$ is such that $V_1^\ast \cap L^\perp$ is singular along a curve, then $V_2 \cap L^\perp$ contains a conic $C$. The conic $C$ spans a plane $\Pi$. Because $\Pi \subset L^\perp$, we have that $L \subset \Pi^\perp$. In addition, $\Pi$ is a plane intersecting $V_2$ in a conic; in Section 2.4 we demonstrated that $\Pi \cap \text{Sec}(V_2) = Y_2 = V_1^\ast$. Thus points in $\Pi$ correspond to hyperplanes that are tangent to $V_1$. It follows that $\Pi^\perp$ is the tangent plane $T_qV_1$ at some point $q \in V_1$. In summary, if $Y_2 \cap L^\perp$ is singular along a curve, then $L$ lies in a tangent plane of $V_1$.

Suppose that $L$ lies in the tangent plane $T_qV_1$ at some point $q \in V_1$. There is a pencil of lines in $T_qV_1$ passing through $q$. This corresponds to conic sections in $V_1$ meeting in $q$ via the embedding $\nu_2: T_qV_1 \rightarrow \mathbb{P}^5$. Any such line $l \subset T_qV_1$ will intersect $L$ in a point $q'$. The projection $\pi_L$ factors through the projection from $q'$. It follows that the conic $\nu_2(l)$ is projected two-to-one onto a line. Consequently, $\pi_L: V_1 \rightarrow \overline{V}$ is a double cover and $\overline{V}$ is a union of concurrent lines; a cone.
2. Veronese Varieties

We now give an account of $V^* \cap L^\perp$. First, note that $Y_2 \cap L^\perp = \text{Sec}(V_2) \cap \mathbb{P}^3$ is a cubic. If $\text{Sing}(Y_2 \cap L^\perp)$ consists of the conic $C$ and the point $P$, then the plane $\Pi$ spanned by $C$ is contained in both $\text{Sec}(V_2)$ and $\mathbb{P}^3$. Hence $Y_2 \cap L^\perp$ is reducible, consisting of a plane and some quadric component. We deduce that the quadric is a cone with base $C$ and apex $P$. There is a depiction in Figure 2.5.

Figure 2.4: The base locus of a pencil of conics consists generically (a) of 4 points, but consist in special cases (b, c) of a line and a point.

Figure 2.5: The projection of the Veronese surface $V$ to $\mathbb{P}^3$ from a line lying in a tangent plane of $V$ (a) and the dual of $V$ intersected with the orthogonal subspace of the line (b).
CHAPTER 3

Space of Quadrics in $\mathbb{P}^3$

I am a great fan of science, but I cannot do a quadratic equation.

Terry Pratchett

Every point $x$ on a symmetroid $V(\det(M))$ is associated with a symmetric matrix $M(x)$. To $M(x)$ we associate the quadratic form $a^T M(x) a$. In order to fully understand symmetroids, we must explore the quadrics. We begin in full generality, describing the moduli space of quadrics in $\mathbb{P}^3$, before we specialise to pencils and webs of quadrics.

3.1 Moduli Space

Consider $\mathbb{P}^9$ as the space of quadratic surfaces in $\mathbb{P}^3$. We stratify this space according to the ranks of the symmetric matrices representing the quadratic forms. More precisely, parametrise $\mathbb{P}^9$ with the matrix

$$
M := \begin{bmatrix}
00 & 01 & 02 & 03 \\
01 & 11 & 12 & 13 \\
02 & 12 & 22 & 23 \\
03 & 13 & 23 & 33
\end{bmatrix}
$$

and let $Q_i$ denote the set of points where $M$ has rank at most $i$.

Then $Q_4 = \mathbb{P}^9$ and $Q_3$ is the discriminant quartic hypersurface $\det(M) = 0$. The rank 2 locus $Q_2$ has dimension 6 and degree 10. Finally, $Q_1$ equals the image of the Veronese map $\nu_2$: $\mathbb{P}^3 \to \mathbb{P}^9$, so it is a threefold of degree 8.

The same argument as in Section 2.4 shows that Sec($Q_1$) is the union of the planes intersecting $Q_1$ in a conic. Furthermore, if the conic is $\nu_2(l)$, for a line $l \subset \mathbb{P}^3$ with ideal

$$(a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3, b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3),$$

then the plane $\Pi \subset \text{Sec}(Q_1)$ intersecting $Q_1$ in $\nu_2(l)$ is given by

$$a_0m_0 + a_1m_1 + a_2m_2 + a_3m_3 = 0,$$

$$b_0m_0 + b_1m_1 + b_2m_2 + b_3m_3 = 0,$$

(3.1)
3. Space of Quadrics in $\mathbb{P}^3$

where $m_i$ are the columns of $M$. This shows that $M$ has nullity at least 2 in $\Pi$, so the rank is at most 2. Ergo $\text{Sec}(Q_1) = Q_2$ and thus it is deficient.

Since any point in $(\mathbb{P}^3 \times \mathbb{P}^3) \setminus \Delta$ yields one of these planes, we have described $Q_2$ as a 6-dimensional family of planes. However, we can also give it a characterisation as a 3-dimensional family of 3-spaces. We begin with some terminology.

**Definition 3.1.1.** Let $X \subset \mathbb{P}^n$ be a smooth variety. Then its **tangential variety** $\text{Tan}(X)$ is defined as the closure of the union of all embedded tangent spaces $T_P X$ for $P \in X$.

**Notation 3.1.2.** If $f$ is the polynomial of a quadric in $\mathbb{P}^3$, let $[f]$ denote its corresponding point.

**Remark 3.1.3.** Unfortunately, Notation 3.1.2 is vague because the same symbol is used for two different, albeit similar, concepts. In this chapter, $[f]$ means the corresponding point in $\mathbb{P}^9$, with the exception of Lemma 3.2.2. In Lemma 3.2.2 and Chapter 4, $[Q]$ is the point with the property that $a^T M([Q]) a = Q$.

Take a point $[l]^2 \in Q_1$, where $l$ is a linear form. We have from [Dol12, Exercise 1.18] that the embedded tangent space $T_{[l]} Q_1$ is the set

$$T_{[l]} Q_1 = \{ l \cdot l' \mid l' \text{ is a linear form} \}.$$  (3.2)

Furthermore, it is a consequence of the Fulton–Hansen connectedness theorem that $\text{Sec}(X)$ is deficient if and only if $\text{Tan}(X) = \text{Sec}(X)$ for a variety $X$ [FH79, Corollary 4]. Together, this proves the claim that $Q_2$ is a 3-dimensional family of 3-spaces.

The purpose of cogitating $\mathbb{P}^9$ is to give us another tool for analysing quartic symmetroids. Given a quartic symmetroid $S \subset \mathbb{P}^3$ with equation

$$\begin{vmatrix}
  l_{00} & l_{01} & l_{02} & l_{03} \\
  l_{10} & l_{11} & l_{12} & l_{13} \\
  l_{20} & l_{21} & l_{22} & l_{23} \\
  l_{30} & l_{31} & l_{32} & l_{33}
\end{vmatrix} = 0,$$

where the $l_{ij}$ are linear forms in the variables $x_0, x_1, x_2$ and $x_3$, then $S$ can be viewed as the intersection of $Q_3$ and the 3-space

$$H := \{ [l_{00}(x_0, x_1, x_2, x_3) : \ldots : l_{33}(x_0, x_1, x_2, x_3)] \mid [x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 \}.$$

Conversely, any $\mathbb{P}^3 \subset \mathbb{P}^9$ gives rise to a quartic symmetroid: It arises as the **discriminant hypersurface** of the web of quadrics. The symmetroid defined by the determinant of a matrix which parametrises a space of quadrics, is often called the **discriminant hypersurface**. The term originates from the following: The quadratic form $ax_0^2 + bx_0x_1 + cx_1^2$ can be written as $x^T A x$, where

$$A := \begin{bmatrix}
  a & \frac{1}{2}b \\
  \frac{1}{2}b & c
\end{bmatrix}.$$  

Then $\det(A)$ is the discriminant of the dehomogenised polynomial $ax_0^2 + bx_0 + c$, up to multiplication with $-4$. 

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Remark 3.1.4. If the symmetroid $S$ is a cone, then $H$ only defines a net of quadrics. It degenerates further if $S$ is defined by a polynomial in only one or two variables. ♦

Bézout’s theorem implies that a $\mathbb{P}^3$ will generically meet $Q_2$ in 10 points and miss $Q_1$. If the 3-space is chosen to be fully contained in $Q_3$ — for instance a tangent space $T_{[2]}Q_1$ — then the polynomial defining the symmetroid will be the zero polynomial, since the matrix is rank deficient at all points. From the description (3.2) of $T_{[2]}Q_1$, it is clear that quadrics in this space have the plane $V(l)$ in common. Observe that the equations (3.1) show that quadrics in a secant plane $P$ are pairs of planes with a fixed line of intersection, since the matrices in $P$ have the same null space.

### 3.2 Web of Quadrics

Let $S \subset \mathbb{P}^3$ be a quartic symmetroid given by the symmetric matrix $M := M(x)$ in the variables $x_0, x_1, x_2$ and $x_3$. Denote the web of quadrics $a^T M a$ by $W$. The union of all the singularities of the quadrics in $W$ form an algebraic set, called the variety of singular loci$^1$ of $W$. It is easy to describe: Recall from Remark 2.1.2 that the singular locus of the quadratic form $a^T M a$ is the solutions of $M a = 0$. Let $A := A(a)$ be the matrix — in the variables $a_0, a_1, a_2$ and $a_3$ — such that $M a = A x$. Then $M a = 0$ if and only if $A x = 0$, so the variety of singular loci is given by $\det(A) = 0$.

Let $S$ be a generic quartic symmetroid and $S'$ the associated variety of singular loci. Then $S$ and $S'$ are birationally equivalent. Indeed, it follows from Remark 2.1.2 that the 10 rank-2-points on $S$ are replaced by lines in $S'$; one can obtain $S'$ by blowing up $S$ in these points.

The connection between the symmetroid and its variety of singular loci might not be as clear in general, but we see some patterns in Chapter 4. There we also compute the quadrics $Q \in W$ that have singularities in the base locus of $W$, so we make a general note about this first.

**Notation 3.2.1.** The base locus of a linear system of divisors $\mathfrak{d}$ is denoted $\operatorname{Bl}(\mathfrak{d})$. ♦

**Lemma 3.2.2.** Let $W$ be the linear space of quadrics in $\mathbb{P}^n$ given by $y^T M y$ for some symmetric $(n+1) \times (n+1)$-matrix $M$ of linear forms in $k \leq \frac{1}{2}(n+1)(n+2)$ variables. Let $D$ be the discriminant hypersurface $\mathcal{V}(\det(M))$. If $[Q_P] \in W$ is a point such that the quadric $Q_P$ is singular at $P \in \operatorname{Bl}(W)$, then $[Q_P] \in \operatorname{Sing} D$.

The proof is due to Kristian Ranestad.

**Proof.** We claim that the tangent space to $D$ in $W$ at a point $[Q_P]$, such that $Q_P$ is singular at a point $P \in \mathbb{P}^n$ — not necessarily in the base locus — is the hyperplane of quadrics that vanish at $P$.

In order to see this, choose coordinates such that $[Q_P] := [1 : 0 : \ldots : 0]$ and such that the hyperplane of quadrics vanishing at $P$ passes through the points $[0 : 1 : 0 : \ldots : 0], \ldots, [0 : \ldots : 0 : 1 : 0]$. Thus if

$$M(x) := M_0 x_0 + \cdots + M_k x_k,$$

$^1$Occasionally singular loci is replaced by singularities or singular points.
then \( Q_P = y^TM_0y \) and \( y^TM_iy \) vanish at \( P \) for \( i = 1, \ldots, k - 1 \). In addition, the hyperplane of quadrics vanishing at \( P \) has equation \( x_k = 0 \).

Moreover, choose coordinates such that \( P := [1 : 0 : \ldots : 0] \). Then \( M_0 \) is on the form

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & m_{11} & \cdots & m_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & m_{1n} & \cdots & m_{nn}
\end{pmatrix}
\]

and the \((0,0)-\)entry in \( M_i \) is 0 for \( i = 1, \ldots, k - 1 \). This implies that there are no \( x_0^ix_i \) terms in \( F(x) := \det(M) \) for \( i = 0, \ldots, k - 1 \). Hence

\[
\frac{\partial F}{\partial x_i}(P) = 0,
\]

for \( i = 0, \ldots, k - 1 \). Thus the tangent space to \( D \) at \([Q_P]\) has equation \( x_k = 0 \), as claimed.

Finally, if \( P \in \text{Bl}(W) \), then all quadrics in \( W \) vanish at \( P \). Hence the tangent space of \( D \) at \([Q_P]\) equals \( W \), so \( D \) is singular at \([Q_P]\). ■

Remark 3.2.3. The converse to Lemma 3.2.2 does not hold. In Example 4.1.7 we see that there are quadrics \( Q \), for \([Q]\) in the singular locus of the symmetroid, that are not singular in any of the base points.

\[\clubsuit\]

3.3 Some Unusual Symmetroids

We give here an exposition of two known symmetroids that are not nodal.

**Example 3.3.1 (Pillow, [Ott+14, Example 5.2])**. The symmetroid defined by the matrix

\[
P := \begin{bmatrix}
x_3 & x_0 & 0 & x_0 \\
x_0 & x_3 & x_1 & 0 \\
0 & x_1 & x_3 & x_2 \\
x_0 & 0 & x_2 & x_3
\end{bmatrix}
\]

is called the pillow for its shape, which is shown in Figure 3.1 on the next page.

It is irreducible and has rank 2 along the two lines \( L_1 := V(x_0,x_3) \) and \( L_2 := V(x_1-x_2,x_3) \), as well as in the four coplanar corners of the pillow. Therefore the surface has the unusual property that the rank 2 locus is contained in a quadric. Consult Appendix B.1 for the Macaulay2 code for the calculations in this section.

Consider

\[
P\alpha = \begin{bmatrix}
x_3 & x_0 & 0 & x_0 \\
x_0 & x_3 & x_1 & 0 \\
0 & x_1 & x_3 & x_2 \\
x_0 & 0 & x_2 & x_3
\end{bmatrix} \begin{bmatrix}
a_0 \\
0 \\
0 \\
a_3
\end{bmatrix} = \begin{bmatrix}
a_1+a_3 & 0 & 0 & a_0 \\
a_0 & a_2 & 0 & a_1 \\
0 & a_1 & a_3 & a_2 \\
a_0 & 0 & a_2 & a_3
\end{bmatrix} \begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3
\end{bmatrix} := A\alpha.
\]

So \( P\alpha = 0 \) is equivalent to \( A\alpha = 0 \). In other words,

\[
\det(A) = a_2(a_1 + a_3)(-a_0^3 + a_1^2 - a_2^2 + a_3^2) = 0
\]
3.3. Some Unusual Symmetroids

describes the variety of the singularities of all the quadrics defined by the pillow. It is the union of two planes and a smooth quadric $Q$.

Each of the quadrics defined by the corners of the pillow are singular along a line on $Q$. The quadratic form of a point $[0 : x_1 : x_2 : 0] \in L_1$ is $2a_2(x_1a_1 + x_2a_3)$. The base locus of the pencil of quadrics parametrised by $L_1$, is the union of the plane $\mathcal{V}(a_2)$ and the line $\mathcal{V}(a_1, a_3)$. Hence the singular loci of the pairs of planes in this pencil can be summarised as all lines in $\mathcal{V}(a_2)$ passing through the intersection point $\mathcal{V}(a_2) \cap \mathcal{V}(a_1, a_3)$. The situation is completely analogous for $L_2$, substituting $\mathcal{V}(a_1 + a_3)$ for $\mathcal{V}(a_2)$ and $\mathcal{V}(a_0, a_2)$ for $\mathcal{V}(a_1, a_3)$.

We can formulate this in light of the pillow being the intersection of $Q_3$ and a $\mathbb{P}^3$ in $\mathbb{P}^9$. Since all the quadrics along $L_i$ share a plane, the $\mathbb{P}^3$ intersects two tangent spaces $T_{[l_i]}Q_1$ in a line each.

![Figure 3.1: The affine piece $x_3 = 1$ of the pillow. The double lines lie in the plane at infinity.](image)

**Example 3.3.2** (Toeplitz, [Ott+14, Example 5.1]). Consider the surface defined by the symmetric Toeplitz matrix

$$
T := \begin{bmatrix}
x_0 & x_1 & x_2 & x_3 \\
x_1 & x_0 & x_1 & x_2 \\
x_2 & x_1 & x_0 & x_1 \\
x_3 & x_2 & x_1 & x_0 \\
\end{bmatrix}.
$$

The determinant $\det(T)$ factors as a product of two irreducible quadratic forms. Each of the components are singular in a single point, wherein the Toeplitz symmetroid has rank $1$. The components intersect in the union of the twisted cubic curve

$$C := \mathcal{V}(x_0^2 - 2x_1^2 + x_0x_2, x_0x_1 - 2x_1x_2 + x_0x_3, x_0^2 - x_0x_2 - 2x_2^2 + 2x_1x_3)$$

and the line $L := \mathcal{V}(x_0 - x_2, x_1 - x_3)$. See **Figure 3.2** on the following page. This intersection forms the rank 2 locus of the symmetroid, with the rank-1-points.
being $C \cap L$. The different rank $k$ loci are computed by the Macaulay2 code in Appendix B.2.

The variety of the singular loci of the quadrics $\mathbf{a}^T \mathbf{a}$ of the Toeplitz surface is given by the asymmetric matrix

$$
A := \begin{bmatrix}
a_0 & a_1 & a_2 & a_3 \\
1 & a_0 + a_2 & a_3 & 0 \\
a_2 & a_1 + a_3 & a_0 & 0 \\
a_3 & a_2 & a_1 & a_0
\end{bmatrix}.
$$

Its determinant is the product

$$
\det(A) = (a_0 + a_1 + a_2 + a_3)(a_0 - a_1 + a_2 - a_3)(a_0^2 - a_0a_2 + a_1a_3 - a_3^2).
$$

It should be noted that quadrics from the rank-1-points are double along the two planes $\Pi_1 := \mathcal{V}(a_0 + a_1 + a_2 + a_3)$ and $\Pi_2 := \mathcal{V}(a_0 - a_1 + a_2 - a_3)$, respectively.

Because $L$ is a secant line of $Q_1$, we know that the quadrics along $L$ make up a pencil of pairs of planes with a common intersection line, $\Pi_1 \cap \Pi_2$. We can also see this by computing the quadrics along $L$. The quadratic form $\mathbf{a}^T \mathbf{a}$ at a point $[x_0 : x_1 : x_0 : x_1] \in L$ is given by

$$
x_0(a_0^2 + a_1^2 + a_2^2 + a_3^2 + 2(a_0a_2 + a_1a_3)) + 2x_1(a_0a_1 + a_0a_3 + a_1a_2 + a_2a_3).
$$

This splits as

$$
((a_0 + a_2)c + (a_1 + a_3)d)((a_0 + a_2)d + (a_1 + a_3)c),
$$

where $c$ and $d$ are complex numbers satisfying

$$
cd = x_0, \\
c^2 + d^2 = x_1.
$$

We see that the base locus is indeed the line $\mathcal{V}(a_0 + a_2, a_1 + a_3) = \Pi_1 \cap \Pi_2$.

![Figure 3.2: The Toeplitz symmetric (a) and its two components (b).](image)

**Pencils of Quadrics with Rank at Most 2**

In both of the above examples there were pencils consisting of quadrics that had rank at most 2, so we make a general note about such pencils.
3.3. Some Unusual Symmetroids

According to Remark 2.1.2, if the symmetric matrix defining a quadratic form has rank 2, then the singular locus of the quadric is a linear space of co-dimension 2. It follows that the quadric is the union of two distinct hyperplanes. Moreover, the hyperplanes coincide if and only if the rank is 1.

Consider a pencil of quadrics \( a_1l_1 + a_2l_4 \), where \([a : b] \in \mathbb{P}^1\) and the \( l_i \) are linear forms, such that each quadric in the pencil has at most rank 2. Then there are two possibilities: Either \( l_3 \) is a scalar multiple of \( l_1 \), or there exist scalars \( c_1, c_2, d_1 \) and \( d_2 \) such that

\[
\begin{align*}
l_3 &= c_1l_1 + c_2l_2 \\
l_4 &= d_1l_1 + d_2l_2
\end{align*}
\]

are linear combinations of \( l_1 \) and \( l_2 \). The pencil would otherwise contain a quadric of higher rank.

In the first case, the hyperplane \( V(l_1) \) and the subspace \( V(l_2, l_4) \) is fixed in the pencil. For us, this is a plane and a line, as in Example 3.3.1. In the second case, the base locus of the pencil is the double linear subspace \( V(l_1, l_2) \). In our case, this is a double line. Substituting (3.3), an element in the pencil becomes

\[
bc_1d_1l_1^2 + (a + bc_1d_2 + bc_2d_1)l_1l_2 + bc_2d_2l_2^2,
\]

which is the square \((\sqrt{c_1d_1l_1} \pm \sqrt{c_2d_2l_2})^2\) for

\[
[\pm 2\sqrt{c_1c_2d_1d_2} - c_1d_2 - c_2d_1 : 1] \in \mathbb{P}^1.
\]

This is two distinct points unless \( c_1c_2d_1d_2 = 0 \). Hence the second kind of pencils has generically two rank-1-points, as in Example 3.3.2. If \( c_1c_2d_1d_2 = 0 \), then the pencil is also of the first kind.
A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.

Paul Halmos

The singular locus of the pillow in Example 3.3.1 has a description similar to that of a projected quartic del Pezzo surfaces, so we are led to pursue these surfaces. A quartic surface in $\mathbb{P}^3$ with a double conic is the projection of a del Pezzo surface. It is well-known that del Pezzo surfaces are rational. Pictures of some of the surfaces in this chapter can be found in Appendix A.

4.1 Mimicking the Pillow

A variety $X \subset \mathbb{P}^n$ is called nondegenerate if it is not contained in a proper, linear subspace. The classical definition of a del Pezzo surface is as follows:

**Definition 4.1.1.** A del Pezzo surface is a nondegenerate, irreducible surface of degree $d$ in $\mathbb{P}^d$ that is not a cone and not isomorphic to a projection of a surface of degree $d$ in $\mathbb{P}^{d+1}$.

It can be showed that surfaces satisfying this definition are also del Pezzo surfaces in the more general definition:

**Definition 4.1.2.** A normal, algebraic surface $S$ is called a del Pezzo surface if its canonical sheaf $\omega_S$ is invertible, $\omega_S^{-1}$ is ample and all its singularities are rational double points.

One of the reasons we are interested in del Pezzo surfaces is the following result:

**Theorem 4.1.3** ([Dol12, Theorem 8.1.15]). Let $X$ be a minimal resolution of a del Pezzo surface. Then, either $X \cong F_0$, $X \cong F_2$ or $X$ is obtained from $\mathbb{P}^2$ by blowing up $N \leq 8$ points.

Here $F_n$ denotes the Hirzebruch surfaces. The importance of Theorem 4.1.3 is that it shows that del Pezzo surfaces are rational. Before we can describe the projection of a quartic del Pezzo surface, we note that they are Segre surfaces:
4. Symmetroids with a Double Conic

**Theorem 4.1.4** ([Dol12, Theorem 8.6.2]). Let $S$ be a del Pezzo surface of degree 4. Then $S$ is a complete intersection of two quadrics in $\mathbb{P}^4$.

The converse also holds [Cas41]. Hence $S$ is the base locus of a pencil of quadrics.

We are now ready to look at the projections of del Pezzo surfaces. Take the projection centre $p$ to be a point in $\mathbb{P}^4$ not on $S$. Let $Q_p$ be the unique quadric in the pencil that contains $p$.

**Theorem 4.1.5** ([Dol12, Theorem 8.6.4]). Assume that the quadric $Q_p$ is nonsingular. Then the projection $X$ of $S$ from $p$ is a quartic surface in $\mathbb{P}^3$ which is singular along a nonsingular conic. Any irreducible quartic surface in $\mathbb{P}^3$ which is singular along a nonsingular conic arise in this way from a Segre surface $S$ in $\mathbb{P}^4$. The surface $S$ is nonsingular if and only if $X$ is nonsingular outside the conic.

Similarly, surfaces with nodes along singular conics can be obtained by taking the projection centre to be a singular or nonsingular point on a singular quadric $Q$ from the pencil. They have been classified by Segre [Seg84]. It is also known that the second statement of the theorem — irreducible quartic surfaces that are singular along a conic are projected del Pezzo surfaces — is true even when the conic is singular. In fact, Dolgachev’s proof of the second statement is still correct if the word *nonsingular* is omitted. We repeat it here with more details.

**Proof.** Let $F := \mathcal{V}(f) \subset \mathbb{P}^3$ be an irreducible quartic surface with a conic section $C \subseteq \text{Sing } F$. Choose coordinates such that $C$ spans the plane $\mathcal{V}(x_0)$. Then the ideal $I_C$ of $C$ is generated by $x_0$ and a quadratic form $Q$. We may assume that $Q$ is a linear combination $L$ of the monomials

$$B_1 := \{x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\}.$$

We take $B_2 := \{x_0^2, x_0x_1, x_0x_2, x_0x_3, Q\}$ as a basis for the vector space of quadrics containing $C$. This gives rise to a rational map $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$, sending a point $p$ to $[x_0^2(p) : x_0x_1(p) : x_0x_2(p) : x_0x_3(p) : Q(p)]$.

Consider the quadric $x_0^2Q$. It is the linear combination $L$ of the monomials in $\{x_0^2s \mid s \in B_1\} = \{q_1q_2 \mid q_1, q_2 \in B_2 \setminus \{x_0, Q\}\}$. Change coordinates such that $B_2$ are the new coordinates in $\mathbb{P}^4$. Then $x_0^2Q - L$ is a quadratic relation in these coordinates. It is satisfied by all points in the image of $\varphi$. Hence $\varphi(\mathbb{P}^3) \subseteq \mathcal{V}(x_0^2Q - L)$ is contained in a quadric.

Since $C \subseteq \text{Sing } F$, we have $f \in I_C^2$, so there exists a quadric $q$, a linear form $L'$ and a constant $a$ such that $f = qx_0^2 + L'x_0Q + aQ^2$. Now $f$ is written as a quadratic form in $B_2$, so it defines a quadric hypersurface in $\mathbb{P}^4$.

Let $S := \mathcal{V}_p(x_0^2Q - L) \cap \mathcal{V}_p(f)$. It is the complete intersection of two quadrics, and thus a quartic del Pezzo surface. Note that $\varphi$ maps $\mathcal{V}_p(x_0)$ to the point $P := [0 : 0 : 0 : 0 : 1]$. The projection from $P$ is the inverse of $\varphi$. It follows that $F$ is the image of $S$ under the projection.

We automatically get that the pillow is a projected del Pezzo surface and therefore rational. We want to see if other projections of del Pezzo surfaces can be realised as symmetroids. The pillow is singular along a conic section, but
4.1. Mimicking the Pillow

it also has four nodes outside of the two lines. To mimic this, we require the original del Pezzo surface in $\mathbb{P}^4$ to have four nodes.

Dolgachev gives a complete list of the possible singularities on quartic del Pezzo surfaces and the configuration of the five points needed to construct them as blow-ups of $\mathbb{P}^2$ [Dol12, Section 8.6.3]. For convenience, this is repeated with a minor change\(^1\) in Table 4.1.

<table>
<thead>
<tr>
<th>Singularities</th>
<th>Position of points</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$P_1, P_2, P_3$ are collinear</td>
</tr>
<tr>
<td>$A_1 + A_2$</td>
<td>$P_3 \succ P_2 \succ P_1$ and $P_5 \succ P_4$</td>
</tr>
<tr>
<td>$A_1 + A_3$</td>
<td>$P_3 \succ P_2 \succ P_1$; $P_5 \succ P_4$ and $P_1, P_4, P_5$ are collinear</td>
</tr>
<tr>
<td>$2A_1$</td>
<td>$P_3, P_2, P_3$ and $P_1, P_4, P_5$ are collinear, or $P_2 \succ P_1$ and $P_3, P_4, P_5$ are collinear</td>
</tr>
<tr>
<td>$2A_1 + A_2$</td>
<td>$P_3 \succ P_2 \succ P_1$; $P_5 \succ P_4$</td>
</tr>
<tr>
<td>$2A_1 + A_3$</td>
<td>$P_3 \succ P_2 \succ P_1$; $P_5 \succ P_4$; $P_1, P_2, P_3$ and $P_1, P_4, P_5$ are collinear</td>
</tr>
<tr>
<td>$3A_1$</td>
<td>$P_2 \succ P_1$; $P_4 \succ P_3$ and $P_1, P_2, P_5$ are collinear</td>
</tr>
<tr>
<td>$4A_1$</td>
<td>$P_2 \succ P_1$; $P_4 \succ P_3$; $P_1, P_2, P_3$ and $P_1, P_4, P_5$ are collinear</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$P_5 \succ P_2 \succ P_1$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$P_4 \succ P_3 \succ P_2 \succ P_1$, or $P_3 \succ P_2 \succ P_1$ and $P_1, P_2, P_4$ are collinear</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1$</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$P_4 \succ P_3 \succ P_2 \succ P_1$ and $P_1, P_2, P_5$ are collinear</td>
</tr>
<tr>
<td>$D_5$</td>
<td>$P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1$ and $P_1, P_2, P_3$ are collinear</td>
</tr>
</tbody>
</table>

Table 4.1: Possible singularities on quartic del Pezzo surfaces and how to obtain them by blowing up $\mathbb{P}^2$ in five points. Here $P_i \succ P_j$ denotes that $P_i$ is a point on the exceptional divisor over $P_j$, and that $P_i, P_j, P_k$ are collinear means that the linear equivalence class $l - e_1 - e_2 - e_3$ is effective.

First we compute the ideal of a 4-nodal del Pezzo surface $S$. The calculations uses the fact that $S$ is the image of the map induced by the linear system of cubics passing through the points $P_i$.

**Example 4.1.6** (4-nodal quartic del Pezzo surface). Following the $4A_1$ entry in Table 4.1, take $P_1, P_3$ and $P_5$ to be distinct points in $\mathbb{P}^2$. For $i = 1, 3$, let $P_{i+1}$ be the points in the exceptional divisor $E_i$ over $P_i$, corresponding to the tangent direction at $P_i$ given by the line $\overline{P_1P_5}$. Then the linear equivalence classes

\[
e_1 - e_2, \quad l - e_1 - e_2 - e_5, \quad e_3 - e_4, \quad l - e_3 - e_4 - e_5
\]

are effective. Here $e_i$ denotes the linear equivalence class of $E_i$ and $l$ is the class of the pullback of a line in $\mathbb{P}^2$. Moreover, the intersection with one of

\(^1\)In later tables, Dolgachev utilises that there are inequivalent linear systems giving rise to quartic del Pezzo surfaces with two $A_1$ singularities. However, the particular table in question lists two equivalent systems.

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4. Symmetroids with a Double Conic

these curves and a general hyperplane section \(3l - \sum_{i=1}^{5} e_i\) is empty, so they are contracted to points.

In Appendix B.3 we use \([1 : 0 : 0], [0 : 1 : 0]\) and \([0 : 0 : 1]\) as \(P_1, P_3\) and \(P_5\), respectively, and

\[
x_0^2 x_1, \quad x_0 x_1^2, \quad x_0 x_1 x_2, \quad x_0 x_2^2, \quad x_1 x_2^2
\]

as a basis for the linear system of cubics passing through the \(P_i\). This linear system induces a rational map \(\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^4\). The image of \(\varphi\) is the del Pezzo surface \(S\) with ideal

\[
\langle x_1 x_3 - x_0 x_4, x_2^2 - x_0 x_4 \rangle.
\]

We see that it is indeed contained in a pencil of quadrics. The surface has nodes in \([1 : 0 : 0 : 0 : 0], [0 : 1 : 0 : 0 : 0], [0 : 0 : 0 : 1 : 0]\) and \([0 : 0 : 0 : 0 : 1]\).

Now that we have an explicit del Pezzo surface, we can study its projections.

Example 4.1.7. We continue with the linear system from Example 4.1.6. The two families of conic sections \(l - e_5\) and \(2l - e_1 - e_2 - e_3 - e_4\) both avoid the singularities. Also, their intersection number is

\[
(l - e_5) \cdot (2l - e_1 - e_2 - e_3 - e_4) = 2.
\]

Hence we can take a curve from each family and they will intersect in two points, which spans a line \(L\). We choose the projection centre \(p\) to be a point on \(L\), but not on the del Pezzo surface \(S\). If the target \(\mathbb{P}^3\) does not contain the two conics, then they are collapsed to double lines under the projection map \(\pi_p: \mathbb{P}^4 \dashrightarrow \mathbb{P}^3\). With a suitable choice of \(p\) and \(\mathbb{P}^3\), the four nodes are not mapped onto the double lines. Also, if \(p\) is contained in the \(\mathbb{P}^3\) spanned by the nodes, their images are coplanar.

In Appendix B.4 we do this for a concrete example and find the quartic \(X\) with equation

\[
f(x_0, x_1, x_2, x_3) = x_0^3 x_2^2 + 2 x_0 x_1 x_2^2 + x_1^2 x_2^2 - x_0^2 x_1 x_3 - x_0 x_2 x_3^2 + 2 x_0 x_2 x_3 + 2 x_1 x_2^2 x_3 - x_0 x_1 x_3^2 - x_1^2 x_3^2 + x_2^2 x_3.
\]

It is singular along two intersecting lines, \(L_1\) and \(L_2\), and in four coplanar points, \(p_1, p_2, p_3\) and \(p_4\). In Appendix B.1 we noted that there are embedded points in the singular locus of the pillow in addition to the intersection between the lines. The same is true for this projected del Pezzo surface, but now we are able to explain them as the branch locus of \(\pi_p\) of the two conics on \(S\).

For Algorithm 1.1.4, we consider the further projection to \(\mathbb{P}^2\) from the node \(p_1 := [1 : 0 : 0 : 0]\). The ramification locus is given by

\[
x_1^2 x_3^2 (x_1 + 2 i x_2 - x_3) (x_1 - 2 i x_2 - x_3) = 0.
\]

In other words, the branch curve consists of two double lines and two reduced lines, \(l_1\) and \(l_2\). Since the projection centre is in the plane spanned by the other nodes on \(X\), their images are collinear in \(\mathbb{P}^2\). One of them, \(p_4\), is mapped to the intersection between the two reduced lines, whereas the other two are points on either of the double lines. The remaining intersection points in the branch locus are the images of the embedded points. This is evinced in Figure 4.1.
4.1. Mimicking the Pillow

There are several ways of dividing the ramification locus into two cubics. If we let \( F_{11} \) in Algorithm 1.1.4 be \( x_3(x_1 + 2ix_2 - x_3)(x_1 - 2ix_2 - x_3) \), where \( \hat{L}_1 := V(x_3) \) is the projected image of \( L_1 \), we find the symmetric matrix

\[
A_1 := \begin{bmatrix}
0 & 4x_0 & 4x_1 & 4x_2 \\
4x_0 & -4x_3 & 2x_1 + 2x_3 & 0 \\
4x_1 & 2x_1 + 2x_3 & -4x_1 & -2x_2 \\
4x_2 & 0 & -2x_2 & -x_3
\end{bmatrix},
\]

whose determinant is \( \det(A_1) = 64f \). Although we constructed \( X \) to imitate the pillow, there is a seemingly remarkable difference between the two. The rank 2 locus of the pillow is the whole singular locus, apart from the embedded points. However, for \( X \) the rank 2 locus is only \( L_1 \), the two embedded points \( q_2, q'_2 \in L_2 \setminus L_1 \) and two of the four nodes, \( p_1 \) and \( p_2 \).

If we instead define \( F_{11} := x_1(x_1 + 2ix_2 - x_3)(x_1 - 2ix_2 - x_3) \), where \( V(x_1) \) is the image of \( L_2 \), we get

\[
A_2 := \begin{bmatrix}
0 & 4x_0 + 8x_3 & -4x_3 & 4x_2 \\
4x_0 + 8x_3 & 4x_1 - 8x_3 & -2x_1 + 6x_3 & -4x_2 \\
-4x_3 & -2x_1 + 6x_3 & -4x_3 & 2x_2 \\
4x_2 & -4x_2 & 2x_2 & -x_1
\end{bmatrix},
\]

Since \( A_1 \) and \( A_2 \) have different eigenvalues at the point \([0 : 1 : 0 : 0]\), they are not conjugate. For \( A_2 \) the situation is opposite of \( A_1 \): It has rank 2 along \( L_2 \), the two embedded points \( q_1, q'_1 \in L_1 \setminus L_2 \) and the two nodes \( p_1 \) and \( p_3 \).
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While the remaining node $p_4$ is a rank-3-point for both $A_1$ and $A_2$, the projection from $p_4$ splits into cubics, so it is possible that it has rank 2 in another representation. We suspect that there might exist a symmetric matrix $A_3$ with $\det(A_3) = f$, where the rank 2 locus equals the singular locus. Unfortunately, since we have to work over $\mathbb{Q}$ with Macaulay2, we have been unable to separate the complex conjugate lines $l_1$ and $l_2$ in the computations, thereby not checking all the ways to divide the branch curve into cubics.

Let $W_1$ denote the web of quadrics $a^T A_1 a$. The matrix

$$M_1 := \begin{bmatrix} 4a_1 & 4a_2 & 4a_3 & 0 \\ 4a_0 & 2a_2 & 0 & -4a_1 + 2a_2 \\ 0 & 4a_0 + 2a_1 - 4a_2 & -2a_3 & 2a_1 \\ 0 & 0 & 4a_0 - 2a_2 & -a_3 \end{bmatrix}$$

is such that $M_1 x = A_1 a$. The variety of singular loci of the quadrics in $W_1$ is given by $\det(M_1) = 0$. It is the union of a plane $\Pi_1$ and a 3-nodal cubic $C_1$. The base locus $\text{Bl}(W_1)$ is five points spanning $\mathbb{P}^3$: The singularities on $C_1$ and two more points in $C_1 \setminus \Pi_1$. One of the nodes on $C_1$ is contained in $C_1 \setminus \Pi_1$ and has multiplicity 2 in $\text{Bl}(W_1)$.

For $P \in \text{Bl}(W_1)$, we look at the quadrics $Q_P$ which are singular at $P$ and consider the corresponding points $[Q_P] \in \mathbb{P}^3$. From Lemma 3.2.2, we know that $[Q_P] \in \text{Sing} X$. For the double point $P \in \text{Bl}(W_1)$, the set of $[Q_P]$ is the line $L_2$. For the other points $P \in \text{Bl}(W_1)$, $[Q_P]$ are the points $p_3, p_4, q_1$ and $q_1'$, respectively.

The situation is completely analogous for $A_2$, albeit reversed. The main aspects are condensed into Table 4.2.

The surface $X$ is shown in Figure A.5a on page 70.

<table>
<thead>
<tr>
<th>$F_{11}$</th>
<th>$F_{22}$</th>
<th>rank 2</th>
<th>$[Q_P]$ for $P \in \text{Bl}(W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{L}_1 + l_1 + l_2$</td>
<td>$\hat{L}_1 + 2\hat{L}_2$</td>
<td>$L_1, p_1, p_2, q_2, q_1'$</td>
<td>$L_2, p_3, p_4, q_1, q_1'$</td>
</tr>
<tr>
<td>$\hat{L}_2 + l_1 + l_2$</td>
<td>$2\hat{L}_1 + \hat{L}_2$</td>
<td>$L_2, p_1, p_3, q_1, q_1'$</td>
<td>$L_1, p_2, p_4, q_2, q_2'$</td>
</tr>
</tbody>
</table>

Table 4.2: Differences resulting from dividing the ramification locus into different cubics $F_{11}$ and $F_{22}$ in Algorithm 1.1.4.

**Remark 4.1.8.** In [Ott+14, Remark 5.4], Kummer symmetroids are interpreted in terms of quadrics, showing that the base locus consists of six distinct points. The base locus in Example 4.1.7 is a non-reduced scheme of length six. In this regard, $X$ can be considered as a degenerated Kummer surface.

**4.2 Menagerie of Projected Del Pezzo Surfaces**

We found some more rational symmetroids using the procedure in Example 4.1.7 by considering different projection centres and del Pezzo surfaces.

By a slight variation of projection centre, we can ensure that the four isolated nodes are not coplanar:
Example 4.2.1. Take the same del Pezzo surface $S$ as in Example 4.1.7 and again let the projection centre $p := [1 : -1 : -2 : -1 : 1]$ be a point on the line $L$, but such that $p$ is not contained in the $\mathbb{P}^3$ spanned by the four nodes.

Projecting from $p$ onto $V(x_4)$ yields a symmetroid $X$, which has two intersecting double lines and four nodes that span $\mathbb{P}^3$. Projecting further from one of the nodes on $X$, the ramification locus has exactly the same description as in Example 4.1.7, except that the nodes are not projected onto a line.

Choosing $F_{11}$ in the two ways as in Example 4.1.7 we obtain the dissimilar matrices

$$A(x) := A_0 x_0 + A_1 x_1 + A_2 x_2 + A_3 x_3,$$

where

$$A_0 := \begin{bmatrix} 0 & 256 & 0 & 0 \\ 256 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 := \begin{bmatrix} -16 & 16 & 0 & 4 \\ 16 & 240 & 128 & -4 \\ 0 & 128 & 0 & 0 \\ 4 & -4 & 0 & -1 \end{bmatrix},$$

$$A_2 := \begin{bmatrix} -128 & 128 & 64 & 0 \\ 128 & -128 & -64 & 0 \\ 64 & -64 & 0 & -16 \\ 0 & 0 & -16 & 8 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 256 & 128 & 0 & 0 \\ 128 & 0 & -128 & 32 \\ 0 & -128 & -256 & 64 \\ 0 & 32 & 64 & -16 \end{bmatrix}.$$

In Example 4.1.7, the rank 2 locus was contained in the union of the planes spanned by the two double lines and the four nodes, respectively. The rank 2 locus here is also contained in a quadric, but it is smooth. Otherwise the description of the rank 2 locus and the base locus of the web of quadrics is just as in Example 4.1.7.

We now consider projected del Pezzo surfaces that are double along a smooth conic section.

Example 4.2.2. Again, start with the del Pezzo surface $S$ from Example 4.1.6, but let the projection centre $p := [1 : 1 : 0 : -1 : 1]$ be a general point in the $\mathbb{P}^3$ spanned by the nodes on $S$. That is, $p$ is not on any line which is the intersection between any planes spanned by two conic sections on $S$.

The projection of $S$ from $p$ is the symmetroid $X$ in Figure A.6a on page 71, which is double along a smooth conic section and has four coplanar nodes. There are four embedded points on the double conic in the singular locus. The projection to $\mathbb{P}^2$ from one of the nodes on $X$ is branched along a double smooth conic and two reduced lines. The configuration of the projected images of the nodes and the embedded points are displayed in Figure 4.2 on the next page.
The dashed line is not part of the ramification locus. The points marked with • are the images of the embedded points and ■ indicates the image of a node.

The two different ways of defining $F_{11}$ in Algorithm 1.1.4 give the dissimilar matrices

$$A_1 := \begin{bmatrix}
8x_1 + 16x_2 + 8x_3 & 4x_0 - 4x_1 - 8x_2 & 2x_1 + 4x_2 + 2x_3 & 2x_3 \\
4x_0 - 4x_1 - 8x_2 & 2x_1 + 4x_2 + 2x_3 & -x_1 - 4x_2 - x_3 & -2x_2 - x_3 \\
2x_1 + 4x_2 + 2x_3 & -x_1 - 4x_2 - x_3 & x_1 + 2x_2 + x_3 & x_2 + x_3 \\
x_2 - x_3 & x_2 + x_3 & x_3 & x_3
\end{bmatrix}$$

and

$$A_2 := \begin{bmatrix}
8x_1 - 16x_2 + 8x_3 & 4x_0 - 4x_1 + 8x_2 & -2x_1 + 4x_2 - 2x_3 & -2x_3 \\
4x_0 - 4x_1 + 8x_2 & 2x_1 - 4x_2 + 2x_3 & x_1 - 4x_2 + x_3 & -2x_2 + x_3 \\
-2x_1 + 4x_2 - 2x_3 & x_1 - 4x_2 + x_3 & x_1 - 2x_2 + x_3 & -x_2 + x_3 \\
x_2 - x_3 & -2x_2 + x_3 & -x_2 + x_3 & x_3
\end{bmatrix}$$

For both representations, the rank 2 locus equals the whole singular locus, sans the embedded points.

The variety of singular loci of the web $W_i$ of quadrics $a^T A_i a$ is the union of two smooth quadrics, $Q_i$ and $Q_i'$. The base locus $\text{Bl}(W_i)$ consists of four points in $Q_i \cap Q_i'$. The points $[Q_P]$ such that the quadric $Q_P \in W_i$ is singular at $P \in \text{Bl}(W_i)$, are the four embedded points in the double conic section on $X$.

Again, the four isolated nodes need not be coplanar.

**Example 4.2.3.** Take that same del Pezzo surface $S$ from Example 4.1.6 as before, and let the projection centre $p := [2 : 1 : 3 : 8 : 2]$ be a generic point as in Example 4.2.2, but such that $p$ is not contained in the $\mathbb{P}^3$ spanned by the nodes on $S$.

Then $S$ is projected onto a symmetroid $X$, which has a double smooth conic section and four nodes that span $\mathbb{P}^3$. The ramification locus for the projection from one of the nodes on $X$ is just as in Example 4.2.2, except the nodes are not mapped onto a line. The dissimilar matrices

$$A(x) := A_0 x_0 + A_1 x_1 + A_2 x_2 + A_3 x_3,$$

$$B(x) := B_0 x_0 + B_1 x_1 + B_2 x_2 + B_3 x_3,$$

where

$$A_0 := \begin{bmatrix}
0 & 36 & 0 & 0 \\
36 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A_1 := \begin{bmatrix}
576 & -432 & 72 & 0 \\
-432 & 144 & 36 & 0 \\
72 & 36 & -36 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$
4.2. Menagerie of Projected Del Pezzo Surfaces

\[ A_2 := \begin{bmatrix} -288 & 216 & -126 & 0 \\ 216 & -144 & 84 & 0 \\ -126 & 84 & -48 & 6 \\ 0 & 0 & 6 & 0 \end{bmatrix}, \quad A_3 := \begin{bmatrix} -279 & 261 & -72 & 18 \\ 261 & -244 & 68 & -17 \\ -72 & 68 & -16 & 4 \\ 18 & -17 & 4 & -1 \end{bmatrix}, \]

and

\[ B_0 := \begin{bmatrix} 0 & 400 & 0 & 0 \\ 400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ B_1 := \begin{bmatrix} -8000 & 16000 & 2000 & 0 \\ 16000 & -32000 & -4000 & 0 \\ 2000 & -4000 & -4000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ B_2 := \begin{bmatrix} -24400 & 36800 & 1780 & 120 \\ 36800 & -54400 & -3040 & -160 \\ 1780 & -3040 & 40 & -20 \\ 120 & -160 & -20 & 0 \end{bmatrix}, \]

\[ B_3 := \begin{bmatrix} -6725 & 13950 & -85 & 85 \\ 13950 & -28400 & 170 & -170 \\ -85 & 170 & -1 & 1 \\ 85 & -170 & 1 & -1 \end{bmatrix}, \]

have the correct determinant.

The rank 2 locus in Example 4.2.2 is contained in the union of the two planes spanned by the nodes and the conic, respectively. The rank 2 locus of this symmetroid is also contained in a quadric, but it is smooth.

Now we turn to symmetroids with fewer isolated nodes. We will use the same technique of projecting a del Pezzo surface. We begin by constructing a del Pezzo surface with three nodes:

Example 4.2.4 (3-nodal quartic del Pezzo surface). We shall find a 3-nodal del Pezzo surface \( S \) by blowing up \( \mathbb{P}^2 \) in five points. Start with distinct points \( P_1, P_3, P_5 \in \mathbb{P}^2 \) and let \( P_2 \) be the point on the exceptional divisor \( E_1 \) that corresponds to the line \( \overline{P_1P_3} \) and let \( P_4 \) be any point in \( E_3 \) that does not correspond to \( \overline{P_3P_5} \). Then the linear equivalence classes

\[ e_1 - e_2, \quad e_3 - e_4, \quad l - e_1 - e_2 - e_5 \]

are effective and contracted to points.

Note that the line \( e_4 \) between the nodes \( e_1 - e_2 \) and \( l - e_1 - e_2 - e_5 \) is contained in the blow-up. The same is true for the line \( l - e_1 - e_3 \) between \( e_1 - e_2 \) and \( e_3 - e_4 \).

In Appendix B.5 we compute the equation of the del Pezzo surface for a concrete example.

We consider the projections of the 3-nodal del Pezzo surface.
**Example 4.2.5.** Let $S$ be the del Pezzo surface given in Appendix B.5. The two families of conic sections in **Example 4.1.7** avoid the three singularities and we construct the line $L$ in the same manner.

Let the projection centre $p := [0 : 3 : 1 : 0 : 0]$ be a point on $L$ and project onto the $\mathbb{P}^3$ given by $x_1 = 0$. The image is the symmetroid $X$ in Figure A.5b on page 70, which is double along two lines $L_1$ and $L_2$, and has three additional nodes $p_1, p_2$ and $p_3$. The singular locus also contains five embedded points: The intersection $L_1 \cap L_2$ and two other points each on $L_i$. Number the nodes such that $p_3$ is the image of $e_1 - e_2$. Then the lines $p_1 p_3$ and $p_2 p_3$ are contained in $X$.

If we project to $\mathbb{P}^2$ from $p_3$, the ramification locus is the union of two double lines, $\hat{L}_1, \hat{L}_2$, and a smooth conic section $C$. If we project from $p_1$ or $p_2$, the ramification locus is the union of two double lines, also called $\hat{L}_1$ and $\hat{L}_2$, and two reduced lines, $l_1$ and $l_2$. In either case, the ramification locus, which is showed in Figure 4.3 on page 36, is decomposable into cubics.

The two different ways of defining $F_{11}$ for the projection from $p_3$, results in the matrices

$$A(x) := A_0 x_0 + A_1 x_1 + A_2 x_2 + A_3 x_3,$$
$$A'(x) := A'_0 x_0 + A'_1 x_1 + A'_2 x_2 + A'_3 x_3,$$

where

$$A_0 := \begin{bmatrix} 0 & 0 & 8 & 0 \\ 0 & 0 & 4 & 0 \\ 8 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 := \begin{bmatrix} -384 & -96 & 72 & 24 \\ -96 & 0 & 12 & 0 \\ 72 & 12 & -12 & -6 \\ 24 & 0 & -6 & 0 \end{bmatrix},$$
$$A_2 := \begin{bmatrix} 192 & 80 & -48 & -8 \\ 80 & 32 & -20 & -4 \\ -48 & -20 & 11 & 3 \\ -8 & -4 & 3 & -1 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 0 & 16 & 0 & 0 \\ 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

and

$$A'_0 := \begin{bmatrix} -256 & 576 & 144 & 16 \\ 576 & 5616 & 1404 & -36 \\ 144 & 1404 & -81 & -9 \\ 16 & -36 & -9 & -1 \end{bmatrix},$$
$$A'_1 := \begin{bmatrix} 3072 & -10368 & -2592 & 96 \\ -10368 & 3456 & 2592 & 0 \\ -2592 & 2592 & 1944 & 0 \\ 96 & 0 & 0 & -24 \end{bmatrix},$$
$$A'_2 := \begin{bmatrix} 256 & 1728 & 1008 & -16 \\ 1728 & -12528 & -5724 & 324 \\ 1008 & -5724 & -2511 & 153 \\ -16 & 324 & 153 & 1 \end{bmatrix}.$$
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\[ A'_3 := \begin{bmatrix} 0 & 2304 & 0 & 0 \\ 2304 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

If for the projection from \( p_1 \) we try to define \( F_{11} \) as \( \hat{L}_2 \) — which does not contain the image of a node — and the two reduced lines, we discover that \( \mathcal{V}(F_{11}, \Delta) \) is set-theoretically seven points. More precisely, five points with multiplicity 2 and two points with multiplicity 1. This is not two times a scheme of length 6 and Algorithm 1.1.4 fails. However, if we take \( \hat{L}_1 \) — which does contain the image of a node instead of \( \hat{L}_2 \), we find the matrix

\[ B(x) := B_0 x_0 + B_1 x_1 + B_2 x_2 + B_3 x_3, \]

where

\[ B_0 := \begin{bmatrix} 0 & 97344 & 0 & 0 \\ 97344 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ B_1 := \begin{bmatrix} -32448 & -405600 & 624 & -4368 \\ -405600 & -324480 & -3432 & -12480 \\ 624 & -3432 & 420 & -132 \\ -4368 & -12480 & -132 & -480 \end{bmatrix}, \]

\[ B_2 := \begin{bmatrix} -67600 & 75712 & 1300 & 1040 \\ 75712 & -400192 & -1456 & -6032 \\ 1300 & -1456 & -25 & -20 \\ 1040 & -6032 & -20 & -16 \end{bmatrix}, \]

\[ B_3 := \begin{bmatrix} 54080 & 189280 & -2912 & 4160 \\ 189280 & -432640 & 9464 & -13520 \\ -2912 & 9464 & -196 & 280 \\ 4160 & -13520 & 280 & -400 \end{bmatrix}. \]

For each of the matrices \( A, A' \) and \( B \), the variety of singular loci associated to the web of quadrics is the union of a plane and a 2-nodal cubic. The base locus of the web is four points, where one of the points has multiplicity 2.

The details of the rank 2 loci and quadrics which are singular at points in the base locus, are summarised in Table 4.3.

<table>
<thead>
<tr>
<th>( F_{11} )</th>
<th>( F_{22} )</th>
<th>rank 2</th>
<th>( [Q_P] ) for ( P \in \text{Bl}(W) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{L}_1 + C )</td>
<td>( \hat{L}_1 + 2\hat{L}_2 )</td>
<td>( L_1, p_1, p_3, q_1, q'_1 )</td>
<td>( L_2, p_2, q_1, q'_1 )</td>
</tr>
<tr>
<td>( \hat{L}_2 + C )</td>
<td>( 2\hat{L}_1 + \hat{L}_2 )</td>
<td>( L_2, p_2, p_3, q_1, q'_1 )</td>
<td>( L_1, p_1, q_2, q'_2 )</td>
</tr>
<tr>
<td>( \hat{L}_1 + l_1 + l_2 )</td>
<td>( \hat{L}_1 + 2\hat{L}_2 )</td>
<td>( L_1, p_1, p_3, q_2, q'_2 )</td>
<td>( L_2, p_2, q_1, q'_1 )</td>
</tr>
</tbody>
</table>

Table 4.3: Division of the ramification locus into different cubics \( F_{11} \) and \( F_{22} \) in Algorithm 1.1.4, with the resulting rank 2 loci and that quadrics that are singular at points in the base loci.
4. Symmetroids with a Double Conic

Figure 4.3: The ramification loci of the surface $X$ in Example 4.2.5 projected from a node. The points marked with $\bullet$ are the images of the embedded points and $\blacksquare$ indicates the image of a node.

Consider now symmetroids with only two isolated singularities. Before we can apply our projection method, we need to construct a del Pezzo surface with two nodes:

Example 4.2.6 (2-nodal quartic del Pezzo surface). Blow up $\mathbb{P}^2$ in five points $P_1, P_2, P_3, P_4, P_5$ that are such that the triples of points $P_1, P_2, P_3$ and $P_1, P_4, P_5$ are collinear. Then $l - e_1 - e_2 - e_3$ and $l - e_1 - e_4 - e_5$ are contracted to nodes. The line $e_1$ between the two singularities is contained in the resulting del Pezzo surface. Readers who are familiar with Segre symbols\(^2\) will recognise this as having Segre symbol $[221]$. We compute the equation of a del Pezzo surface like this in Appendix B.6.

Note that there is an inequivalent linear system of cubics passing through five points that also defines a 2-nodal del Pezzo surface. It has Segre symbol $[(11)111]$. Indeed, take a tangent direction $P_2$ through a point $P_1$, and say that this corresponds to a line $L$ through $P_1$. Let $P_1, P_4$ and $P_5$ be three points lying on a line $L' \neq L$. In this case, the line between the nodes $e_1 - e_2$ and $l - e_3 - e_4 - e_5$ is not contained in the del Pezzo surface.

We use the 2-nodal del Pezzo surface from Example 4.2.6 to find a symmetroid with only two isolated singularities.

Example 4.2.7. We begin with the del Pezzo surface $S$ found in Appendix B.6. Curves in the linear equivalence classes $l - e_1$ and $2l - e_2 - e_3 - e_4 - e_5$ are conics that avoid both singularities on $S$ and meet curves from the other class in two points, which span a line $L$. The point $p := [2 : -2 : 1 : -2 : 2]$ lies on such a line $L$ and is not contained in $S$. We choose $p$ as the projection centre and $\mathcal{V}(x_1)$ as the target hyperplane.

The image of $S$ under the projection is the symmetroid $X$ in Figure A.5c on page 70. Its singular locus is comprised of two double lines $L_1, L_2$, two

\(^2\)We will not need Segre symbols for anything than a shorthand notation for differentiating between the linear systems in this example. Interested readers may consult [Jes16, Article 48] for an introduction to Segre symbols.
nodes $p_1, p_2$, and four embedded points: The intersection $L_1 \cap L_2$, two points $q_1, q'_1 \in L_1$ and one other point $q_2 \in L_2$.

The ramification locus $R_{p_1}$ is the two double lines $\hat{L}_1$ and $\hat{L}_2$ — coming from $L_1$ and $L_2$ — and a smooth, reduced conic section $C$. The line $\hat{L}_2$ is tangent to $C$. The point $p_2$ is mapped onto $\hat{L}_1$, as shown in Figure 4.4.

![Figure 4.4: The ramification locus of the surface $X$ in Example 4.2.7 projected from a node. It consists of two double lines and a reduced conic. One of the lines is tangent to the conic. The points marked with • are the images of the embedded points and ■ indicates the image of a node.](image)

If we let $F_{11} := L_2 + C$, then we find that $V(F_{11}, \Delta)$ is the union of a triple point, a single point and four double points. Consequently, there is no subscheme $Z_\Delta$ of length 6 such that $V(F_{11}, \Delta) = 2Z_\Delta$. Thus Algorithm 1.1.4 does not work. If we on the other hand let $F_{11} := L_1 + C$, then the algorithm produces the matrix

$$A(x) := A_0 x_0 + A_1 x_1 + A_2 x_2 + A_3 x_3,$$

where

$$A_0 := \begin{pmatrix} 0 & 100 & 0 & 0 \\ 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} -9 & 48 & -12 & 3 \\ 48 & -1056 & 264 & -16 \\ -12 & 264 & -16 & 4 \\ 3 & -16 & 4 & -1 \end{pmatrix},$$

$$A_2 := \begin{pmatrix} 24 & 222 & -48 & 2 \\ 222 & 216 & -144 & 6 \\ -48 & -144 & 96 & -4 \\ 2 & 6 & -4 & -4 \end{pmatrix}, \quad A_3 := \begin{pmatrix} 24 & 2 & 12 & 2 \\ 2 & -504 & 276 & 46 \\ 12 & 276 & -144 & -24 \\ 2 & 46 & -24 & -4 \end{pmatrix}.$$

The matrix $A$ has rank 2 in $L_1, p_1, p_2$ and $q_2$. The variety of singular points of the quadrics $a^T A a$ is the union of a plane and a 1-nodal cubic. The base locus of the web is three points, one of which has multiplicity 2. The quadrics that are singular at points in the base locus are the quadrics along $L_2$, as well as the quadrics in $q_1$ and $q'_1$.

There are patterns emerging from the above examples, some of which may be general. In the ramification locus, the intersection between the double and the reduced conic was always the image of embedded points in the singular locus of the surface in $\mathbb{P}^3$. In each example, nodes were mapped either to the double conic or the reduced conic, not to the intersection of the conics.
4. Symmetroids with a Double Conic

In the cases where the singular locus consisted of two double lines and \( n \) additional nodes, the variety of singular points of the quadrics was the union of a plane and an \((n - 1)\)-nodal cubic. The base locus was a scheme of length \( n + 2 \), with support in \( n + 1 \) points.

When we considered a ramification locus \( 2L_1 + 2L_2 + C \) — where \( L_1, L_2 \) are lines and \( C \) a conic — and we defined \( F_{11} \) as \( L_1 + C \), then Algorithm 1.1.4 worked precisely when \( L_1 \) contained the projected image of a node.

4.3 Quadrics Through Four General Points

We now turn to the task of expanding some of what we learnt from the examples into generalities.

In Example 4.2.2 the base locus of the web of quadrics was four points spanning \( \mathbb{P}^3 \). For this reason, we study the quadrics in \( \mathbb{P}^3 \) passing through four given points. We can also motivate this less heuristically. Consider a net \( N \) of quadrics containing a smooth conic \( C \) of rank-2-quadrics. Let \( \mathcal{V}(l_1l_2) \), \( \mathcal{V}(l'_1l'_2) \) be two points on \( C \). These quadrics intersect generically in four lines, say \( L_1, L'_1 \subset \mathcal{V}(l_1) \) and \( L_2, L'_2 \subset \mathcal{V}(l_2) \). Then \( L_1, L'_1, L_2, L'_2 \) are fixed in the pencil spanned by \( \mathcal{V}(l_1l_2) \) and \( \mathcal{V}(l'_1l'_2) \). Letting \( \mathcal{V}(l'_1l'_2) \) run through all the points in \( C \), we obtain a pencil \( P \) of line-pairs \( L_i + L'_i \) in each of the \( \mathcal{V}(l_i) \). Since \( P \) only contains line-pairs and no smooth conic sections, the base locus must contain a line; see Figure 2.4. Suppose that the line is \( L_i \). Then every quadric in \( N \) passes through \( L_1 \) and \( L_2 \). Let \( W \supset N \) be a web of quadrics, generated by \( N \) and a quadric \( Q \). Generically, \( Q \) intersects each line in two points, so \( \text{Bl}(W) \) is generically four general points.

The set of quadrics through four points form a \( \mathbb{P}^5 \) in the \( \mathbb{P}^9 \) of all quadrics. Taking the four points to have coordinates \([1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0] \) and \([0 : 0 : 1 : 1]\), the \( \mathbb{P}^5 \) is parametrised by the matrix

\[
M := \begin{bmatrix}
0 & y_{01} & y_{02} & y_{03} \\
y_{01} & 0 & y_{12} & y_{13} \\
y_{02} & y_{12} & 0 & y_{23} \\
y_{03} & y_{13} & y_{23} & 0
\end{bmatrix}.
\]

As discussed in Chapter 3, intersecting the rank 3 locus of \( M \) with a 3-space yields a symmetroid. Let \( X \) be the rank 2 locus of \( M \). It is a surface of degree 10. See Appendix B.7 for this computation. Recall Theorem 2.3.3: Since the dimensions add up to the dimension of the ambient space, a \( \mathbb{P}^3 \subset \mathbb{P}^5 \) will always intersect \( X \), and generically the intersection consists of 10 points. So even though we have restricted to quadrics passing through four points, the situation is not so special that a general 3-space yields a symmetroid with a non-generic number of rank-2-points.

Furthermore, \( X \) breaks up into three smooth quadric surfaces \( Q_1, Q_2 \) and \( Q_3 \), each spanning a \( \mathbb{P}^3 \), and four planes \( \Pi_1, \Pi_2, \Pi_3 \) and \( \Pi_4 \). The components are configured in the following manner: The \( \Pi_i \) intersect each other in a single point, resulting in six unique points \( P_k \) spanning \( \mathbb{P}^5 \). The \( Q_i \) intersect each other in two points, giving precisely the same six points as the planes. The intersection between \( \Pi_i \) and \( Q_j \) is a line, totalling 12 distinct lines. Four of these lines pass through each \( P_k \). The lines between the points in \( Q_1 \cap Q_2 \) do
not occur as the intersection between either $Q_i$ or $Q_j$ with one of the planes $\Pi_k$. Figure 4.5 illustrates the intersection between the components in $X$.

The singular locus of $\mathcal{V}(\det(M))$ consists of four additional planes. Each of these planes intersects three of the $\Pi_i$ along the same lines as the $\Pi_i$ intersect the $Q_j$. It avoids the fourth $\Pi_i$ completely.

Figure 4.5: Intersection of the components in the rank 2 locus $X$. The points marked with $\bullet$ are the intersection points between two planes $\Pi_i$ and $\Pi_j$, or equivalently between two quadrics $Q_i$ and $Q_j$. The solid lines are the intersections between the planes and the quadrics. The dashed lines are spanned by $Q_i \cap Q_j$.

**Smooth Conic and Four Isolated Nodes**

A symmetroid which is double along a smooth conic section and four additional points inspired us to look at quadrics through four points. The first fruit we want to harvest from this observation is a family of symmetroids having this property. Let $\Pi$ be a plane in the $\mathbb{P}^3$ spanned by $Q_1$. Then $\Pi$ intersects $Q_1$ in a conic section $C$. Assume that $C$ is smooth. Extend $\Pi$ to a linear 3-space $H$. If $H$ meets $Q_1$ outside of $C$, then $Q_1$ is completely contained in $H$, so suppose this is not the case. Generically $H$ meets the planes $\Pi_i$ in a single point each. Since each $\Pi_i$ intersects $Q_1$ along a line, they meet $H$ in a point\(^3\) each on $C$. Similarly, $H$ meets $Q_2$ and $Q_3$ generically in two points each, which we assume to be outside of $C$. Therefore, $H$ contains a conic section and four additional points with rank 2, thus giving rise to a symmetroid with the desired property.

**Smooth Conic and Three Isolated Nodes**

We can reduce the number of rank-2-points in $H$, outside of $C$, by choosing $\Pi$ such that some of the points in $Q_1 \cap Q_2$ or $Q_1 \cap Q_3$ are contained in $\Pi$. For instance, if one of these intersection points are in $\Pi$, we generically obtain a symmetroid which has rank 2 in a smooth conic section and three additional points with rank 2 in the singular locus of the symmetroid.

\(^3\)In the concrete examples we computed, these points corresponded to embedded points in the singular locus of the symmetroid.
points. For example, the matrix
\[
\begin{bmatrix}
0 & x_0 & 3x_1 - 2x_2 - 3x_0 & x_1 \\
x_0 & 0 & x_3 & x_2 \\
3x_1 - 2x_2 - 3x_0 & x_3 & 0 & x_0 \\
x_1 & x_2 & x_0 & 0
\end{bmatrix}
\]
was produced in this fashion. The symmetroid it defines is showed in Figure A.6b on page 71. It has one point embedded in the conic in the rank 2 locus. This is opposed to previous examples, where we have had embedded points in the singular locus, but not in the rank 2 locus. It has two further embedded points in the singular locus. Projecting from one of the isolated nodes, the branch curve is the union of a double conic and two lines meeting on the conic, as shown in Figure 4.6.

![Figure 4.6: The ramification locus of a symmetroid with a double conic and three nodes, projected from an isolated node. It consists of a double smooth conic and two reduced lines. The point marked with ▲ is the image of the point embedded in the rank 2 locus, the points marked with • are the images of the embedded points in the singular locus and ■ indicates the image of a node.](image)

**Smooth Conic and Two Isolated Nodes**

For a symmetroid which has rank 2 along a smooth conic section and two points, let Π be such that it contains \( Q_1 \setminus Q_2 \). Then the 3-space \( H \) will generically only meet \( X \) in two other points. An example of this is the matrix
\[
\begin{bmatrix}
0 & 2x_0 & x_0 + x_1 & x_2 \\
2x_0 & 0 & x_3 & x_1 \\
x_0 + x_1 & x_3 & 0 & x_0 \\
x_2 & x_1 & x_0 & 0
\end{bmatrix}
\]
which defines the symmetroid in Figure A.6c on page 71. It has two points embedded in the conic in the rank 2 locus. The projection from one of the isolated nodes is ramified along a double conic and two lines that are both tangent to the conic, as in Figure 4.7.

If we instead let Π contain one point \( p_1 \) from \( Q_1 \setminus Q_2 \) and another point \( p_2 \) from \( Q_1 \setminus Q_3 \), then \( C \) breaks down into two lines, since the line \( p_1p_2 \) is contained in \( Q_1 \).

**Two Lines**

We can go one step further and let Π be spanned by \( Q_1 \setminus Q_2 \) and a point in \( Q_1 \setminus Q_3 \). Then \( H \) will generically meet \( X \) in two lines and a single point. This
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Figure 4.7: The ramification locus of a symmetroid with a double conic and two nodes, projected from an isolated node. It consists of a double smooth conic and two tangential lines. The points marked with ▲ are the images of the points embedded in the rank 2 locus and ■ indicates the image of the node.

is a type of symmetroid that was missing from our collection. An example of such a symmetroid is displayed in Figure A.5d on page 70. It is given by the matrix

$$
\begin{bmatrix}
0 & x_0 & x_1 & x_2 \\
x_0 & 0 & x_3 & x_0 \\
x_1 & x_3 & 0 & x_0 \\
x_2 & x_0 & x_0 & 0
\end{bmatrix}.
$$

It has three embedded points in the rank 2 locus; one on each line and their intersection. The projection from the isolated node is ramified along two triple lines, as displayed in Figure 4.8.

Figure 4.8: The ramification locus of a symmetroid with two double lines and a node, projected from the isolated node. It consists of two triple lines. The points marked with ▲ are the images of the points embedded in the rank 2 locus.

We can also recover symmetroids whose singular loci consist of two double lines and four additional nodes in this picture. One way to go about is to choose \( \Pi \) such that the conic \( C \) breaks up into two lines that avoid \( Q_2 \) and \( Q_3 \). By the same argument as above, a generic \( \mathbb{P}^3 \) containing \( \Pi \) intersects \( X \) in four more points. Another approach is to let \( \Pi \) be a plane that cuts both \( \Pi_1 \) and \( \Pi_2 \) along a line each, say \( l_1 \) and \( l_2 \). Number the points and the quadrics such that \( P_1 = \Pi_1 \cap \Pi_2 \subset Q_1 \cap Q_2 \). Extend \( \Pi \) to a linear 3-space \( H \). Then \( H \) generically meets \( \Pi_3 \) and \( \Pi_4 \) in a single point each, and the \( Q_i \) in two points each. Since \( P_1 \) is contained in \( \Pi \), \( H \) only meets each of \( Q_1 \) and \( Q_2 \) in one other point. We deduce that there is no further contribution from \( Q_3 \), since \( \Pi_1 \) intersects \( Q_3 \) in a line that meets \( l_1 \), and similarly for \( \Pi_2 \).
4. Symmetroids with a Double Conic

It is possible to reduce the number of rank-2-points outside the two lines in the last case. For instance, if \( l_1 \) is spanned by \( P_1 \) and \( P_2 := \Pi_1 \cap \Pi_3 \) then the number of additional rank-2-points drops to two, since either \( Q_1 \) or \( Q_2 \) pass through \( P_3 \), say \( Q_1 \). If, in addition, \( l_2 \) is spanned by \( P_3 := \Pi_2 \cap \Pi_4 \), the number of rank-2-points only drops to one, since \( Q_1 \) also passes through \( P_3 \), which can be seen from the computations in Appendix B.7.

**Degenerate Cases**

In the above account we utilised that a \( \mathbb{P}^3 \subset \mathbb{P}^5 \) generically meets the quadric \( Q_i \) in two points. It can well happen that these two points coincide, so the \( \mathbb{P}^3 \) intersects \( Q_i \) in one point with multiplicity 2.

For instance, we can start with a plane containing \( Q_1 \cap Q_2 \), and extend this to a 3-space meeting \( Q_3 \) in a single point. Thus the resulting symmetroid has rank 2 along a conic section and in that point. The matrix

\[
\begin{bmatrix}
0 & x_0 & -x_0 - x_1 & x_2 \\
x_0 & 0 & x_3 & -x_0 + x_1 \\
-x_0 - x_1 & x_3 & 0 & x_0 \\
x_2 & -x_0 + x_1 & x_0 & 0
\end{bmatrix}
\]

is an example of this, defining the symmetroid in Figure A.6d on page 71. We can unearth this geometric description by excavating algebra. The saturated ideal of the rank 2 locus is

\[I := \langle x_1^2 + x_2x_3, x_0x_2x_3, x_0x_1x_3, x_0^2x_3, x_0x_1x_2, x_3^2x_2 \rangle.\]

The associated prime ideals are \( \langle x_0, x_1^2 + x_2x_3 \rangle \), corresponding to the conic section; \( \langle x_1, x_2, x_3 \rangle \), corresponding to where the \( \mathbb{P}^3 \) meets \( Q_3 \); the ideals \( \langle x_0, x_1, x_2 \rangle \), \( \langle x_0, x_1, x_3 \rangle \), corresponding to the embedded points \( Q_1 \cap Q_2 \), which is also the points where the planes \( \Pi_1 \) meet the conic section. One possible primary decomposition of \( I \) is

\[\langle x_0, x_1^2 + x_2x_3 \rangle \cap \langle x_1, x_2, x_3 \rangle \cap \langle x_0^2, x_1^2, x_0x_1, x_2 \rangle \cap \langle x_0^2, x_1^2, x_0x_1, x_3 \rangle.\]

Notice that the point corresponding to the intersection with \( Q_3 \) is represented by the ideal \( \langle x_1^2, x_2, x_3 \rangle \) of degree 2 in the decomposition.

Another degenerated example is

\[M := \begin{bmatrix}
0 & x_0 & x_1 & 7x_1 - 3x_2 \\
x_0 & 0 & x_2 & x_3 \\
x_1 & x_2 & 0 & x_0 \\
7x_1 - 3x_2 & x_3 & x_0 & 0
\end{bmatrix}.\]

The matrix \( M \) was found by letting a point \( p \in Q_1 \cap Q_2 \) be contained in \( \Pi \) and then extend \( \Pi \) to a 3-space \( H \) that intersects \( Q_1 \cap Q_2 \) twice in \( p \), rather than in two different points. Hence the rank 2 locus consists of a conic section and two points, instead of three.

### 4.4 Conjectures

Having a accumulated a sizeable collection of symmetroids, we have an idea about what is possible and what is not. We propose two conjectures. The
4.4. Conjectures

attempted proofs are missing the same detail. Namely, we have not been able to exclude the possibility that the ramification locus consists of a double conic and two lines meeting on the conic.

Lemma 4.4.1. All lines on a projected quartic del Pezzo surface $X$ in $\mathbb{P}^3$ meet the double conic.

Proof. In the proof of Theorem 4.1.5, Dolgachev gives the following description of the curve $C \subset S \subset \mathbb{P}^4$ which is mapped onto the double conic: Let $H$ be the tangent hyperplane of $Q_p$ at $p$, then $C = S \cap H$.

Let $l$ be a line on $X$ and $\Pi$ the plane in $\mathbb{P}^4$ spanned by $l$ and $p$. Then $H$ cuts $\Pi$ along a line $L$. And $\Pi$ meets $S$ in some curve $K$, since $l \subset X$. The claim follows because $\emptyset \neq K \cap L \subset S \cap H = C$. $\blacksquare$

Corollary 4.4.2. All singularities on the branch curve in $\mathbb{P}^2$ of the projection from an isolated node on a projected quartic del Pezzo surface $X \subset \mathbb{P}^3$ are the images of singularities on $X$.

Proof. In general, a singularity on the branch curve of the projection from a node on a quartic $Q$ in $\mathbb{P}^3$, is either the image of a singularity on $Q$ or the image of a line on $Q$ passing through the projection centre. Lemma 4.4.1 states that if such a line exits on $X$, its image coincides with the image of a singularity. $\blacksquare$

Conjecture 4.4.3. Let $X$ be an irreducible quartic surface in $\mathbb{P}^3$ which is double along a smooth conic section and has a single isolated node $p$. If $X$ is a symmetroid, then $p$ is a rank-3-point.

Partial proof. If $p$ has rank 2 or less, then Theorem 1.1.2 implies that the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from $p$ is ramified along two cubics. The ramification locus contains a double smooth conic section $C$, which is the image of the double conic on $X$, and another conic section $C'$. For this sextic to split into two cubics, $C'$ must be a pair of lines, hence have a singular point. By Theorem 4.1.5, $X$ is a projected 1-nodal quartic del Pezzo surface. Corollary 4.4.2, combined with the fact that $X$ has no further nodes, implies that $C'$ is either nonsingular or two lines meeting on $C$. We are done if can show the latter is not possible. $\square$

Recall from Section 4.3 that we saw the conic break up into lines when we attempted to reduce the number of isolated nodes to one.

Conjecture 4.4.4. Let $X \subset \mathbb{P}^3$ be an irreducible quartic surface which is double along a smooth conic section and has two additional nodes, $p_1$ and $p_2$. Suppose that $X$ is the projection of a 2-nodal del Pezzo surface $S \subset \mathbb{P}^4$ with Segre symbol $[221]$. If $X$ is a symmetroid, then $p_1$ and $p_2$ are rank-3-points.

Partial proof. Consult Example 4.2.6 for details about del Pezzo surfaces with Segre symbol $[221]$. Just as for Conjecture 4.4.3, the ramification locus for the projection from either $p_1$ or $p_2$ is the union of a smooth double conic $C$ and another conic $C'$. For $p_1$ to be a rank-2-point, $C'$ must be singular.

The line $\overline{p_1p_2}$ is contained in $X$, since the line between the two nodes on $S$ is contained in $S$. By Lemma 4.4.1, $\overline{p_1p_2}$ intersects the double conic. Thus $p_2$ is mapped to $C$ in the projection from $p_1$. Since there are no other singularities on $X$, $C'$ is either smooth or two lines meeting on $C$. $\square$
4. Symmetroids with a Double Conic

One could imagine that one way to complete the proofs is to start with a sextic curve $R$ in the plane, consisting of a double conic and two lines meeting on that conic. Then one constructs a surface $S$ having $R$ as ramification locus when projected from a node. Finally, one shows that $S$ is not one of the surfaces in the conjectures. Alas, this gives no conclusion; several surfaces can have the same ramification locus. For instance, if $l,l_1,l_2 \in \mathbb{C}[x_1,x_2,x_3]_1$ are linear forms, $q \in \mathbb{C}[x_1,x_2,x_3]_2$ is a quadratic form and $k_1,k_2 \in \mathbb{C}$ are such that $k_1k_2 = \frac{1}{4}$, then

$$
\begin{align*}
\mathcal{V}(k_1l_1l_2x_0^2 - k_2q^2) & \quad \mathcal{V}(k_1l_1l_2x_0^2 + l_1l_2lx_0 + k_2(l_1l_2l^2 - q^2)) \\
\mathcal{V}(q(k_1x_0^2 - k_2l_1l_2)) & \quad \mathcal{V}(q(k_1x_0^2 + lx_0 + k_2(l^2 - l_1l_2))) \\
\mathcal{V}(k_1(l^2 - l_1l_2)x_0^2 + qlx_0 + k_2q^2)
\end{align*}
$$

are quartic surfaces in $\mathbb{P}^3$ that all have $\mathcal{V}_{\mathbb{P}^2}(l_1l_2q^2)$ as their ramification locus, when projected to $\mathcal{V}(x_0)$ with projection centre $[1:0:0:0]$. 

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CHAPTER 5

Symmetroids with a Double Line

Art, like morality, consists of drawing the line somewhere.

G. K. Chesterton

Generically, a quartic symmetroid has only finitely many singularities. We have found several symmetroids that are singular along a curve of degree 2. There ought to be intermediate specimens, so we direct our focus at symmetroids that have nodes along a single line, apart from some isolated singularities.

Pictures of some of the surfaces in this chapter can be found in Appendix A.

A First Estimate

Consider a quartic symmetroid \( S \subset \mathbb{P}^3 \). Let \( p \in S \) be an isolated node such that the projection \( \pi_p : S \setminus \{p\} \to \mathbb{P}^2 \) from \( p \) is ramified along two cubics, \( F_{11} \) and \( F_{22} \). Suppose that \( S \) has a determinantal representation obtained from Algorithm 1.1.4. In this representation, the rank 2 locus of \( S \) is mapped onto \( F_{11} \setminus F_{22} \).

Now assume that \( S \) has rank 2 along a line. Then \( F_{11} \) and \( F_{22} \) share a common line. The remaining components of \( F_{11} \) and \( F_{22} \) are two conics that generically intersect in four points. Taking the projection centre into account, we expect, a priori, that a symmetroid which has rank 2 along a line to have generically five other rank-2-points.

Note that the same type of analysis yields that a generic symmetroid which has rank 2 along a double conic, will have two isolated rank-2-points. This is contrary to our experience with four nodes being the least special in Section 4.3. When we computed the ramification curves of symmetroids with more than two isolated nodes, we saw that some of the isolated nodes were mapped to the double conic in the plane.

5.1 Quadrics Through Four Coplanar Points

We are primarily interested in symmetroids which have rank 2 along a line and no rank-1-points. Recall from Section 3.3 that if a pencil of rank-2-quadrics contains no rank-1-quadric, then it fixes a plane and a line not contained in that plane.
5. Symmetroids with a Double Line

We thus start with two quadrics, \( Q_1 \) and \( Q_2 \), having a plane \( \Pi \) and a line \( l \) in common. A symmetroid gives rise to a whole web of quadrics, so we extend this to four independent quadrics, \( Q_1, Q_2, Q_3, Q_4 \). Both \( Q_3 \) and \( Q_4 \) intersect \( \Pi \) in a conic and \( l \) in two points each. Generically, the points on \( l \) do not coincide, but the two conics intersect in four points. Hence the base locus of the web is four coplanar points.

First, observe that if three of the points are collinear, then the symmetroid is a double quadric. Indeed, let \( \Pi := V(x_3) \) and choose the points \([1 : 0 : 0 : 0]\), \([0 : 1 : 0 : 0]\), \([1 : 1 : 0 : 0]\) on the line \( V(x_2, x_3) \), as well as the point \([0 : 0 : 1 : 0]\).

Then the \( \mathbb{P}^5 \) of quadrics passing through these four points is parametrised by

\[
\begin{bmatrix}
0 & 0 & y_{02} & y_{03} \\
0 & 0 & y_{12} & y_{13} \\
y_{02} & y_{12} & 0 & y_{23} \\
y_{03} & y_{13} & y_{23} & y_{33}
\end{bmatrix},
\]

which has determinant \((y_{03}y_{12} - y_{02}y_{13})^2\). Therefore the intersection of a 3-space in \( \mathbb{P}^5 \) with the rank 3 locus is a double quadric.

Assume instead that the four coplanar points are in general position. Let the points have coordinates \([1 : 0 : 0 : 0]\), \([0 : 1 : 0 : 0]\), \([0 : 0 : 1 : 0]\) and \([1 : 1 : 1 : 0]\).

Then the \( \mathbb{P}^5 \) of quadrics passing through these points is parametrised by

\[
M := \begin{bmatrix}
0 & y_{01} & y_{02} & y_{03} \\
y_{01} & 0 & -y_{01} - y_{02} & y_{13} \\
y_{02} & -y_{01} - y_{02} & 0 & y_{23} \\
y_{03} & y_{13} & y_{23} & y_{33}
\end{bmatrix},
\]

We calculate similarly as in Appendix B.7 and find that \( \text{rank} \ M = 1 \) only in the point \( P := [0 : 0 : 0 : 0 : 1] \), which corresponds to the quadric 2\( \Pi \). The rank 2 locus \( X \) consists of a linear 3-space \( T \) and three quadric surfaces \( K_1, K_2 \) and \( K_3 \). All the \( K_i \) meet in \( P \). Each \( K_i \) intersects \( T \) along two lines, \( l_i \) and \( L_i \). The lines are concurrent, meeting in \( P \). It is possible to label the lines in such a way that the all the \( l_i \) are contained in a plane \( \Pi_l \), and the \( L_i \) in another plane \( \Pi_L \). The singular locus \( Y \) of \( V(\det(M)) \) has the same description, but it also includes four embedded planes in \( T \) that intersect each other along the lines \( l_i \) and \( L_i \).

We will now look at the symmetroids we can obtain by choosing different linear 3-spaces in this \( \mathbb{P}^5 \).

Six Isolated Rank-2-Points

A generic \( \mathbb{P}^1 \subset \mathbb{P}^5 \) will cut \( T \) along a line and meet each \( K_i \) in two points. The resulting symmetroid thus has a rank 2 locus consisting of a line and six other points. How does this fit in with our estimate of the number of isolated nodes?

Let us look at an example, such as the symmetroid given by

\[
\begin{bmatrix}
0 & -x_0 & -x_1 - 2x_2 + x_3 & x_1 \\
-x_0 & 0 & x_0 + x_1 + 2x_2 - x_3 & x_2 \\
-x_1 - 2x_2 + x_3 & x_0 + x_1 + 2x_2 - x_3 & 0 & x_3 \\
x_1 & x_2 & x_3 & x_0
\end{bmatrix}.
\]
It is drawn in Figure A.2a on page 68. There are four embedded points in the singular locus and the rank 2 locus is contained in a quadric. The ramification locus is as predicted — two conics and a double line — although the conclusion about the number of rank-2-points is wrong. This is because one of the nodes is mapped onto the double line. The situation is completely analogous to Example 4.2.2, where two nodes were mapped onto the double conic. The branch curve is displayed in Figure 5.1 on the following page.

It can happen that a 3-space $H$ in $\mathbb{P}^5$ intersects the rank 3 locus of $M$ in such a way that the intersection has singularities that are not in $Y$. For instance, the symmetroid in Figure A.1b on page 67, given by

\[
\begin{pmatrix}
0 & x_0 & x_1 & 2x_2 + x_3 \\
x_0 & 0 & -x_0 - x_1 & 2x_3 \\
x_1 & -x_0 - x_1 & 0 & x_2 \\
2x_2 + x_3 & 2x_3 & x_2 & x_3
\end{pmatrix}
\]

is singular along a line and has seven isolated nodes, of which only six are rank-2-points. The rank 2 locus is contained in a quadric surface, but the rank-3-node is not on the quadric. In addition, there are four points embedded in the singular locus.

Projection from one of the rank-2-points gives a ramification locus that consists of a double line, two reduced lines and a smooth conic section. Three of the six points are collinear. If the projection centre is one of these, then the two other nodes on that line are mapped to the same point in $\mathbb{P}^2$. This causes one of the reduced lines to be tangent to the conic. This story is depicted in Figure 5.2 on the following page. Considering the projection from the rank-3-node instead, we get five general lines, with one of them being double, as in Figure 5.3 on the next page.

There is one lesson to take away from these ramification loci: It is impossible to divide them into cubics such that the cubics have all the images of the nodes in common. Thus there exists no matrix representation of this symmetroid such that all the isolated nodes have rank 2.

We can also find symmetroids like this in the context of Section 4.3. Using the notation of Section 4.3, start with a line $l$ in $\Pi_1$ and extend it to a 3-space $H$ containing $l$. Generically, $H$ meets the other $\Pi_i$ in one point each and $Q_i$ in two points each. But since $\Pi_1$ cuts each $Q_i$ along a line, one of these points are on $l$. Hence $H$ contains six isolated rank-2-points. Since there is a plane in the singular locus of $V(\det(M))$ not meeting $\Pi_1$, the symmetroid will have seven isolated nodes.

Likewise, extend a line $l$ on $Q_1$ to a 3-space $H$. A plane through $l$ cuts $Q_1$ in another line $l'$. As noted earlier, the $\Pi_i$ all meet the conic $l + l'$. It is not possible to find a point $p_i \in \Pi_i \cap Q_1$ for each $i$ such that three of them are collinear. One consequence is that exactly two of the $\Pi_i$ meet $l$. The same is true of the four extra planes in the singular locus of $V(\det(M))$. Therefore $H$ meets the rank 2 locus generically in $l$ and six isolated rank-2-points and two rank-3-points. An example is the symmetroid given by

\[
\begin{pmatrix}
0 & x_0 - x_1 + x_2 & x_0 + x_3 - x_1 & x_0 \\
x_0 - x_1 + x_2 & 0 & x_3 & x_1 \\
x_0 + x_3 - x_1 & x_3 & 0 & x_2 \\
x_0 & x_1 & x_2 & 0
\end{pmatrix}
\]
A similar symmetroid with a less pretty matrix, but with only real nodes is shown in Figure A.1a on page 67.

Figure 5.1: The ramification locus of a symmetroid with a double line and six nodes, projected from an isolated node. It consists of two conics and a double line. The points marked with • are the images of the embedded points in the singular locus and ■ indicates the image of a node.

(a) No two nodes lie on the same line through the projection centre.
(b) Two nodes lie on the same line through the projection centre.

Figure 5.2: The ramification locus of a symmetroid with a double line and seven nodes, projected from an isolated rank-2-point. It consists of a conic, two reduced lines and a double line. The point marked with ⊞ is the image of the rank-3-node, the points marked with • are the images of the embedded points in the singular locus and ■ indicates the image of a node.

Figure 5.3: The ramification locus of a symmetroid with a double line and seven nodes, projected from the rank-3-node. It consists of five general lines, one of which is double. The points marked with • are the images of the embedded points in the singular locus and ■ indicates the image of a node.
Five Isolated Rank-2-Points

Let $p$ be a point on one of the lines $l_i$ or $L_j$. A $\mathbb{P}^3$ that contains $p$ will generically intersect $X$ in a line and five points. The symmetroid defined by

$$
\begin{bmatrix}
0 & -x_0 + x_1 & x_2 & x_3 \\
-x_0 + x_1 & 0 & x_0 - x_1 - x_2 & x_0 \\
x_2 & x_0 - x_1 - x_2 & 0 & x_3 \\
x_3 & x_0 & x_3 & x_1
\end{bmatrix}
$$

was found this way. A similar symmetroid with real nodes is plotted in Figure A.3a on page 68. The rank 2 locus is contained in a quadric surface and it contains an embedded point. Two additional points are embedded in the singular locus. The projection from one of the isolated nodes is ramified along two conics and a double line that is tangent to one of the conics, as shown in Figure 5.4.

Figure 5.4: The ramification locus of a symmetroid with a double line and five nodes, projected from an isolated node. It consists of two conics and a double line, which is tangent to one of them. The point marked with ▲ is the image of the point embedded in the rank 2 locus, the points marked with ● are the images of the embedded points in the singular locus and ■ indicates the image of a node.

Again, we can choose the 3-space such that the symmetroid has singularities not contained in $Y$. The determinant of the matrix

$$
\begin{bmatrix}
0 & x_0 & x_1 & x_2 \\
x_0 & 0 & -x_0 - x_1 & x_3 \\
x_1 & -x_0 - x_1 & 0 & x_2 \\
x_2 & x_3 & x_2 & x_3
\end{bmatrix}
$$

defines a symmetroid that is double along a line and has six isolated nodes, of which five are rank-2-points. The rank 2 locus is contained in a quadric which also contains the rank-3-node. One point is embedded in the rank 2 locus and two other points are embedded in the singular locus.

The five isolated rank-2-points are coplanar, and the conic passing through them is a line-pair. The lines intersect in one of the rank-2-points, say $q$, and there are two additional rank-2-points on each line. Hence, no matter which of the nodes we project from, two nodes are mapped to the same point. Projecting from $q$, the other rank-2-points are mapped to two points. The ramification
5. Symmetroids with a Double Line

locus is a conic, two reduced lines and a double line. Projecting from the rank-3-point, the branch curve consists of five lines, one of which is double. The double line and two of the other lines are concurrent. The ramification loci are displayed in Figures 5.5 and 5.6. It is impossible to divide the branch curves into cubics such that they have the image of all the nodes in common. Hence this symmetroid is different from the symmetroid with a double line and six rank-2-nodes.

Figure 5.5: The ramification locus of a symmetroid with a double line, five rank-2-nodes and one rank-3-node, projected from a rank-2-node. It consists of a conic, two reduced lines and a double line. The point marked with ★ is the image of the rank-3-node, the point ▲ is embedded in the rank 2 locus, the points marked with ● are the images of the embedded points in the singular locus and ■ indicates the image of a node.

Figure 5.6: The ramification locus of a symmetroid with a double line, five rank-2-nodes and one rank-3-node, projected from the rank-3-node. It consists of five lines, one of which is double. Three of the lines are concurrent. The point marked with ▲ is embedded in the rank 2 locus, the points marked with ● are the images of the embedded points in the singular locus and ■ indicates the image of a node.

Four Isolated Rank-2-Points

Let $p_1$ be a point on one of the $l_i$, and $p_2$ a point on one of the $L_j$. A 3-space containing both $p_1$ and $p_2$ will generically intersect the rank 2 locus $X$ in the line $p_1p_2$ and in four points. The matrix

$$
\begin{bmatrix}
0 & x_0 & -x_1 + x_2 & x_3 \\
x_0 & 0 & -x_0 + x_1 - x_2 & x_1 \\
-x_1 + x_2 & -x_0 + x_1 - x_2 & 0 & x_1 \\
x_3 & x_1 & x_1 & x_2
\end{bmatrix}
$$
was obtained in this procedure. A symmetroid like this with real singularities is
drawn in Figure A.3b on page 68. It has one point embedded with multiplicity
2 in the rank 2 locus and two points embedded in the singular locus. The rank
2 locus lies on a quadric surface. The projection from one of the isolated nodes
is branched along two smooth conics that are tangent in a point \(q\) and a double
line passing through \(q\). This is illustrated in Figure 5.7.

Figure 5.7: The ramification locus of a symmetroid with a double line and four
nodes, projected from an isolated node. It consists of two tangential conics and
a double line passing through the point of tangency. The point marked with ▲
is the image of the point embedded in the rank 2 locus, the points marked with
• are the images of the embedded points in the singular locus and ■ indicates
the image of a node.

Three Isolated Rank-2-Points

Symmetroids that are singular along a line and have three isolated rank-2-points
come in two flavours.

For the first flavour, let \(p_1 \in l_1\) and \(p_2 \in l_2\). Then the line \(p_1p_2\) meets \(l_3\),
since the \(l_i\) are coplanar. Hence a 3-space containing \(p_1\) and \(p_2\) generically
intersects \(X\) in \(p_1p_2\) and three other points. The matrix

\[
\begin{pmatrix}
0 & x_0 & x_1 & x_2 \\
x_0 & 0 & -x_0 - x_1 & x_3 \\
x_1 & -x_0 - x_1 & 0 & x_3 \\
x_2 & x_3 & x_3 & x_1
\end{pmatrix}
\]

is an example of a symmetroid like this. Another symmetroid with real nodes is
shown in Figure A.4a on page 69. It has no rank-1-points. The rank 2 locus is
contained in a quadric and there are three points embedded in it. The branch
curve has the same description as for the symmetroid with a double line and
four nodes. The images of the embedded points are marked in Figure 5.8.

The second flavour comes about by taking a 3-space \(H\) that contains the
rank-1-point \(P\). Since \(P \in K_1 \cap K_2 \cap K_3\), generically \(H\) intersects each \(K_i\) in
one further point. Hence the resulting symmetroid has a rank-1-point on a line
of rank-2-points and three additional rank-2-points. One such symmetroid is
given by

\[
\begin{pmatrix}
0 & x_0 + x_1 & x_0 & x_2 \\
x_0 + x_1 & 0 & -2x_0 - x_1 & x_1 \\
x_0 & -2x_0 - x_1 & 0 & x_2 \\
x_2 & x_1 & x_2 & x_3
\end{pmatrix}
\]
5. Symmetroids with a Double Line

It is drawn in Figure A.4b on page 69. The rank-1-point is embedded in the rank 2 locus and there is one other point embedded in the singular locus. The rank 2 locus is not contained in a quadric. The branch curve is a conic, a triple line that is tangent to the conic and another reduced line. This is drawn in Figure 5.9.

![Figure 5.8](image)

Figure 5.8: The ramification locus of a symmetroid with a double line, three nodes and no rank-1-point, projected from an isolated node. It consists of two tangential conics and a double line passing through the point of tangency. The points marked with ▲ are the images of the points embedded in the rank 2 locus and ■ indicates the image of a node.

![Figure 5.9](image)

Figure 5.9: Ramification locus for a symmetroid with a double line, three nodes and a rank-1-point, projected from an isolated node. It consists of a triple line, a reduced line and a conic that is tangent to the triple line. The point marked with ▲ is the image of the rank-1-point, the point marked with • is the image of the embedded point in the singular locus and ■ indicates the image of a node.

**Two Isolated Rank-2-Points**

If $H$ contains the rank-1-point $P$ and a point $p \in l_1$, then $l_1$ is contained in $H$. Also, $H$ will generically not intersect $K_1$ further and only meet $K_2$ and $K_3$ in one more point each. In summary, the symmetroid will have one rank-1-point, a line of rank-2-points and two isolated rank-2-points. An example of such a symmetroid is shown in Figure A.4c on page 69. It is given by the matrix

\[
\begin{pmatrix}
0 & 2x_0 + x_1 & x_1 & x_0 \\
2x_0 + x_1 & 0 & -2x_0 - 2x_1 & x_1 \\
x_1 & -2x_0 - 2x_1 & 0 & x_2 \\
x_0 & x_1 & x_2 & x_3
\end{pmatrix}
\]
The rank-1-point is embedded in the rank 2 locus, and there are two further embedded points in the singular locus. There is a quadric surface containing the rank 2 locus. The projection from one of the isolated nodes has a branch curve consisting of a quadruple line and two reduced lines. See Figure 5.10.

Figure 5.10: The ramification locus of a symmetroid with a double line and two nodes, projected from an isolated node. It consists of a quadruple line and two reduced lines. The point marked with ▲ is the image of the rank-1-point, the points marked with • are the images of the embedded points in the singular locus and ■ indicates the image of the node.

If we attempt to reduce the number of nodes further, by letting $H$ contain a point on one of the other lines $l_i$ or $L_j$, then $H$ will cut $T$ along a plane. The symmetroid will thus be reducible.

Remark 5.1.1. Discount the symmetroids that either have a rank-1-point or a singularity that was not originally a singular point on the discriminant hypersurface in the $\mathbb{P}^5$ of quadrics through four coplanar points. Then the number of isolated rank-2-points and points embedded in the rank 2 locus is six, counted with multiplicity for all of the above examples.

In Chapter 4, the corresponding statement is that the number of isolated nodes and points embedded in the rank 2 locus is four. This is not true for Examples 4.2.5 and 4.2.7, but the rank 2 locus is not equal to the singular locus in the matrix representations we have for those symmetroids.

5.2 Linear System

Consider the linear system of quartic curves passing through nine given points $P_1, \ldots, P_9 \in \mathbb{P}^2$, going twice through $P_1$. By [Jes16, Article 79], any quartic surface $S \subseteq \mathbb{P}^3$ with a double line is the image of the map $\varphi: \mathbb{P}^2 \to \mathbb{P}^3$ induced by the linear system. Consequently, all the symmetroids found above are rational.

Let us acquaint ourselves with this linear system. It is the plane cubic passing through the $P_i$ that is contracted to the double line on $S$. Indeed, the plane quartics in the linear system correspond to the hyperplane sections $H := 4l - 2e_1 - \sum_{i=2}^9 e_i$ on $S$. Since these are singular in one point, $S$ must be singular along a curve of degree 1, that is, a line $L$. Intersect $S$ with a plane containing $L$. By Bézout’s theorem, this intersection is $2L + C$ for some conic $C$. The pencil of such planes gives a pencil of conics, which in $\mathbb{P}^2$ corresponds to the pencil $l - e_1$ of lines through $P_1$. Hence $B := H - (l - e_1) = 3l - \sum_{i=1}^9 e_i$ is common for all the hyperplanes containing $L$, which is precisely the linear equivalence class of the cubic passing through $P_1$.

If the $P_i$ constitute the complete intersection of two cubic curves, then $\varphi$ is a two-to-one map and the image is a quadric [Jes16, Article 80].
5. Symmetroids with a Double Line

A quartic surface with a double line can have up to eight isolated nodes, in which case it is called Plücker’s surface. To see this, let $P_{i+4}$ be a point on the exceptional line $E_i$ for $i = 2, 3, 4, 5$ such that $P_i, P_1, P_{i+4}$ are collinear. Then the classes

\[ l - e_1 - e_2 - e_6 \quad e_2 - e_6 \]
\[ l - e_1 - e_3 - e_7 \quad e_3 - e_7 \]
\[ l - e_1 - e_4 - e_8 \quad e_4 - e_8 \]
\[ l - e_1 - e_5 - e_9 \quad e_5 - e_9 \]

are effective and contracted to eight distinct nodes on $S$. One could imagine that it is possible to push this to nine nodes by letting $P_2, P_3, P_4, P_5$ be collinear so that $l - e_2 - e_3 - e_4 - e_5$ is another node. But observe that

\[ (l - e_2 - e_3 - e_4 - e_5) \cdot B = -1. \]

Hence it is contained in the double line. It should be clear how one can let the $P_i$ have a more general position to obtain fewer nodes.

Ideally, we would like to give necessary conditions on the configuration of the nine points for the linear system to give rise to a symmetroid. Recall from Section 1.1 that if $S$ is a symmetroid, then there exist cubic surfaces that are tangent to $S$ along a sextic curve of arithmetic genus 3. We would like to describe these sextics via the linear system. The intersection between $S$ and a cubic surface has linear equivalence class $3H$. The aim is therefore to write $3H$ as $2C + A$, where $C$ has degree 6 and genus 3, and $A$ has degree 0. A specious candidate is $C := 5l - 3e_1 - \sum_{i=2}^{9} e_i$ and $A := 2l - \sum_{i=2}^{9} e_i$. We check that $C$ is as required:

\[
\deg(C) = C \cdot H = 5 \cdot 4 - 3 \cdot 2 - 8 \cdot 1 = 6,
\]
\[
\rho_0(C) = \frac{1}{2} (5 - 1)(5 - 2) - 3 \cdot (3 - 1) - 8 \cdot 1 \cdot (1 - 1) = 3.
\]

The obvious problem is that if $A$ is effective, then the $P_i$ constitute the complete intersection of two cubics and $S$ is a quadric. We therefore settle for describing $A$ and $C$ in examples.

Example 5.2.1. In Appendix B.8 we let the $P_i$ be such that $P_2 \succ P_3, P_3 \succ P_4, P_4 \succ P_5$ and the triples $P_1, P_2, P_3, P_4, P_5$ and $P_1, P_6, P_7$ are collinear. Lastly, we let $P_8$ be $P_9$ be general points. The resulting quartic $S$ is then singular along the double line $B = 3l - \sum_{i=1}^{9} e_i$ and in the six nodes

\[ A_1 := l - e_1 - e_2 - e_3, \quad A_4 := e_2 - e_3, \]
\[ A_2 := l - e_1 - e_4 - e_5, \quad A_5 := e_4 - e_5, \]
\[ A_3 := l - e_1 - e_6 - e_7, \quad A_6 := e_6 - e_7. \]

We project from $A_4$ and find a symmetric matrix $M$ representing $S$, using Algorithm 1.1.4. The rank 2 locus is all of the $A_i$ and $B$.

We consider the curve $C$ defined by the $(3 \times 3)$-minors of the submatrix of $M$ obtained by removing the first column. Having defined the ring map corresponding to $\varphi$ in Macaulay2, we can investigate how $C$ is represented in the linear system. The sextic decomposes into the quartic curve

\[ 3l - e_1 - 2e_2 - e_4 - e_6 - e_7 - e_9, \]
5.2. Linear System

the exceptional line $e_2$ and the associated reduced line of $B$. Hence

$$2C = 2(3l - e_1 - 2e_2 - e_4 - e_6 - e_8 - e_9) + 2e_2 + B$$

$$= 9l - 3e_1 - 3e_2 - 3e_3 - 3e_4 - 3e_5 - 3e_6 - 3e_7 - 3e_8 - 3e_9.$$ 

Note that $3H - 2C = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$.

Lastly, removing different columns of $M$ gives four independent sextic curves of genus $3$. The system

$$|(3l - e_1 - 2e_2 - e_4 - e_6 - e_8 - e_9) + e_2| = |3l - e_1 - e_2 - e_4 - e_6 - e_8 - e_9|$$

has the correct dimension to account for this, since there is a web of plane cubics passing through the six points.

Let $S$ be a symmetroid with matrix $M$. The $(3 \times 3)$-minors of a $(3 \times 4)$-submatrix of $M$ define a sextic curve with arithmetic genus $3$. The rank 2 locus is given by all the $(3 \times 3)$-minors of $M$ and hence is in the base locus of the web generated by these four sextics.

In all the examples we have seen of symmetroids with a double line, the double line has been contained in the rank 2 locus. Therefore it is tempting to conjecture that $B$ must be contained in $2C$, just as in the previous example. We will now present a counterexample to this conjecture.

**Example 5.2.2.** In Appendix B.9 we start with a smooth conic $K$ and let $P_2$, $P_3$ and $P_6$ be three points on $K$. Then for $i = 2, 4, 6$, let $P_{i+1}$ be the point on the exceptional divisor over $P_i$ corresponding to the tangent direction of $K$ at $P_i$. Define $P_1$ as the intersection of the tangent lines of $K$ at $P_2$ and $P_4$. Finally, let $P_5$ and $P_9$ be two points not on $K$, on a line through $P_1$.

Then the image $S$ of $\varphi$ has, in addition to the double line, the nodes

$$A_1 := l - e_1 - e_2 - e_3, \quad A_2 := l - e_1 - e_4 - e_5,$$

$$A_3 := e_2 - e_3, \quad A_4 := e_4 - e_5,$$

$$A_5 := e_6 - e_7.$$ 

The cubic passing through the $P_1$ is the union of $K$ and the line through $P_1$, $P_3$ and $P_6$. Because of this, the singularity $A := l - e_1 - e_8 - e_9$ is a point on the double line $B$.

The projection from $A_1$ is ramified along a singular, irreducible cubic curve, a reduced line and a double line. The two lines meet on the cubic, as in Figure 5.11 on the next page. Hence it is not possible to divide the branch curve into two cubics having a common line, so $B$ is not contained in the rank 2 locus. Using Algorithm 1.1.4 we find a matrix $M$ for $S$. The rank 2 locus is a scheme of length 10. It consists of $A$, with multiplicity 3, and two other points on $B$, with multiplicity 2 each. These three points are embedded in the singular locus. In addition, $A_1$, $A_4$ and $A_5$ are rank-2-points. The rank 2 locus is not contained in a quadric.

Let $C$ be the curve on $S$ determined by the $(3 \times 3)$-minors of the submatrix obtained by removing the first column of $M$. We have not been able to describe $C$ via the linear system, but it is a simple task to verify that the reduced line associated to $B$ is not a component of $C$.
5. Symmetroids with a Double Line

This does not mean that no choice of curve $C$ contains $B$ or its associated reduced line:

**Example 5.2.3.** If we in Example 5.2.2 instead define $C$ by removing the second column of $M$, rather than the first, then $C$ consists of two double lines and a conic. These correspond to $B$, two times the exceptional line $e_3$ and $l - e_4 - e_6$, respectively. Since

$$C = B + 2e_3 + (l - e_4 - e_6)$$

$$= 4l - e_1 - e_2 + e_3 - 2e_4 - e_5 - 2e_6 - e_7 - e_8 - e_9,$$

then

$$3H - 2C = 4l - 4e_1 - e_2 - 5e_3 + e_4 - e_5 + e_6 - e_7 - e_8 - e_9.$$  

This can be written as $A + 3A_1 + 2A_3 + A_4 + A_5$.

![Figure 5.11: Ramification locus for a symmetroid with a double line that is not contained in the rank 2 locus. It consists of a singular cubic, as well as a reduced line and a double line meeting on the cubic. The points marked with • are the images of isolated rank-3-nodes. Both • and □ indicate the images of rank-2-points, but • means that the points are embedded in the singular locus.](image)

5.3 Double Line Not Contained in the Rank 2 Locus

How rare is it that a symmetroid is double along a line, but that line is not contained in the rank 2 locus? It turns out that we are able to find at least as many of these as of symmetroids where the double line is contained in the rank 2 locus.

Consider the web $W$ of quadrics defined by the matrix in Example 5.2.2. Its base locus is a scheme of length 4, consisting of a scheme of length 2 with support in a point $p_1$, as well as two other points $p_2$ and $p_3$. Choosing coordinates such that the points are $p_1 := [1 : 0 : 0 : 0]$, $p_2 := [0 : 1 : 0 : 0]$ and $p_3 := [0 : 0 : 1 : 0]$, we may take $V(xy + xw)$, $V(xz)$, $V(yz)$, $V(yw)$, $V(zw)$ and $V(w^2)$ as a basis for the quadrics passing through $\text{Bl}(W)$. The matrix parametrising this $\mathbb{P}^5$ of quadrics is

$$M := \begin{bmatrix} 0 & y_{01} & y_{02} & y_{03} \\ y_{01} & 0 & y_{12} & y_{13} \\ y_{02} & y_{12} & 0 & y_{23} \\ y_{13} & y_{23} & y_{33} & 0 \end{bmatrix}.$$
With a computation similar to the one in Appendix B.7, we find that Sing $V(\det(M))$ is the union of a linear 3-space $T$, a smooth quadratic surface $Q$ and four planes $\Pi_i$. The quadric $Q$ and two of the planes, $\Pi_1$ and $\Pi_2$, are in the rank 2 locus. In addition, there is a double quadric surface and a double plane in the rank 2 locus which are completely contained in $T$. A general 3-space $H \subset \mathbb{P}^5$ will intersect the singular locus in a line and six points. Also, $H$ meets the rank 2 locus in seven points, three of which are double. One such symmetroid is

$$\begin{pmatrix} 0 & x_0 + x_1 & x_2 + x_3 & x_0 + x_1 \\
 x_0 + x_1 & 0 & x_2 & x_0 \\
 x_2 + x_3 & x_2 & 0 & x_1 \\
 x_0 + x_1 & x_0 & x_1 & x_3 \end{pmatrix},$$

which is displayed in Figure A.2b on page 68.

The 3-space $T$ is the set of quadrics passing through $\text{Bl}(W)$ that are singular at $p_1$. That the discriminant hypersurface in $\mathbb{P}^3$ is singular along $T$ is immediate from Lemma 3.2.2. The quadratic surface of rank 2 quadrics inside $T$ corresponds to surfaces $\Pi_1 \cup \Pi_2$, where $\Pi_1$ is a plane containing $p_1p_2^2$ and $\Pi_2$ is a plane containing $p_1\Pi_3^2$. The plane of rank 2 quadrics in $T$ is the set of quadrics that are the unions of the plane $p_1p_2p_3$ and another plane through $p_1$. We see that both these surfaces contain the rank-1-point corresponding to twice $p_1p_2p_3$.

The components $Q$ and $\Pi_i$ are cut along a line each by $T$. The planes $\Pi_1$ and $\Pi_2$ of rank-2-quadrics meet in a single point. So do the planes $\Pi_3$ and $\Pi_4$; they intersect in the rank-1-point. The two planes $\Pi_1$ and $\Pi_3$ meet in a line on $Q$. Likewise for $\Pi_2$ and $\Pi_4$. The intersections $\Pi_1 \cap \Pi_4$ and $\Pi_2 \cap \Pi_3$ are empty.

**Example 5.3.1.** Let $H$ be a 3-space containing the line $L := \Pi_3 \cap T$. Then $L$ intersects $\Pi_4$ in the rank-1-point. The line $\Pi_1 \cap Q \subset \Pi_3$ cuts $L$ in a point $P$. Generically, $H$ will meet $Q$ in another point and intersect $\Pi_2$ in a single point outside $L$. However, $H$ can in degenerate cases intersect $Q$ with multiplicity 2 in $P$. Thus the resulting symmetroid is double along a line and has one additional rank-2-point. An example of such a symmetroid is defined by the matrix

$$\begin{pmatrix} 0 & x_0 & x_1 & x_0 \\
 x_0 & 0 & x_0 & x_1 \\
 x_1 & x_0 & 0 & x_2 \\
 x_0 & x_1 & x_2 & x_3 \end{pmatrix}.$$ 

It is shown in Figure A.4d on page 69.

### 5.4 Dimensions of Families of Rational Symmetroids

Recall from Section 3.1 that there is a correspondence between the quartic symmetroids in $\mathbb{P}^3$ and linear 3-spaces in $\mathbb{P}^5$. By [Har92, Lecture 11, p. 138], the dimension of the Grassmannian of $k$-dimensional, linear subspaces of $\mathbb{P}^n$ is

$$\dim G(k, n) = (k + 1)(n - k). \quad (5.1)$$

Therefore the symmetroids form a 24-dimensional variety in the $\mathbb{P}^{(3+4)-1} = \mathbb{P}^{34}$ of all quartic surfaces in $\mathbb{P}^3$.

We have looked at webs of quadrics with three different base loci:
5. Symmetroids with a Double Line

(i) four general points in Section 4.3,

(ii) four coplanar points in Section 5.1,

(iii) a scheme of length 4 with support in three points in Section 5.3.

We will now compute the dimensions of the families of rational symmetroids these induce.

Four General Points

We want to compute the dimension of the family of rational symmetroids that are found by considering webs of quadrics with a base locus containing four general points. While we have mostly focused on symmetroids with a double conic, for ease of computation, we restrict to the case where the $\mathbb{P}^3$ cuts one of the $\Pi_\ell$ — or the extra planes in $\text{Sing} \mathcal{V}($det$(M))$ — in a line. These symmetroids are briefly discussed in Section 5.1.

To give four points in $\mathbb{P}^3$ is the same as giving a 4-tuple in $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$. Naturally, there are 4-tuples that do not correspond to points spanning $\mathbb{P}^3$, but removing them do not affect the dimension. Therefore, the set of all choices of four general points has dimension $4 \cdot 3 = 12$.

We have from [Har92, Example 11.42] that the subvariety of $G(k,n)$ consisting of $k$-planes $K$, which intersects a fixed $m$-plane $M$ such that $\dim(K \cap M) \geq l$, has dimension

\[(l + 1)(m - l) + (k - l)(n - k). \tag{5.2}\]

Hence, given four general points we get in this way a 6-dimensional family of symmetroids with a double line. In total, varying the four points yields a family of rational quartic symmetroids of dimension $12 + 6 = 18$.

Four Coplanar Points

The first three points can be chosen freely, and the last point must be in the plane spanned by the other three. Hence giving four coplanar points is the same as giving a 4-tuple in $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^2$, which is 11-dimensional.

We have seen that a general 3-space in the $\mathbb{P}^5$ of quadrics through four coplanar points gives a rational symmetroid. Equation (5.1) gives that the dimension of $G(3,5)$ is 8. So all together with the different choices of four coplanar points, we get a family of rational quartic symmetroids of dimension $11 + 8 = 19$.

Scheme of Length 4 with Support in Three Points

The three points can be chosen freely. For the scheme to be of length 4, one of the points must be doubled in one direction. In $\mathbb{P}^3$ there is a $\mathbb{P}^2$ of directions through a point. Hence the scheme can be represented by a 4-tuple in $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^2$.

Again, we know that a generic $\mathbb{P}^3$ in the space of quadrics through a scheme of length 4 with support in three points gives rise to a rational symmetroid. Just as before, we get in entirety a family of dimension $11 + 8 = 19$ of rational quartic symmetroids.
5.4. Dimensions of Families of Rational Symmetroids

A Larger Family

While we explained why a symmetroid with a line of rank-2-points induces a web of quadrics with a base locus that is generically four coplanar points, we had no such justification for base loci of length 4 with support in three points. We merely studied those after stumbling upon an example with that property. It is in fact a rather special situation.

For a surface to be double along a line it must contain a line. Suppose that $S \coloneqq \mathbb{V}(F)$ is a quartic symmetroid containing the line $L \coloneqq \mathbb{V}(x_0, x_1)$. Then $F$ cannot contain any $x_2^4$, $x_2^3x_3$, $x_2x_3^2$ or $x_3^4$ terms. This is five conditions, but since $\dim \mathbb{G}(1, 3) = 4$, it is only one condition for a quartic to contain an arbitrary line.

Because $S$ is a symmetroid, then $L$ is a pencil of rank-3-quadrics. A pencil not containing a single rank-4-quadric must contain at least three rank-2-quadrics. Hence $S$ has at least three singularities on $L$. We claim that if $S$ has four singularities on $L$, then it must be singular along the whole of $L$. Let again $L$ be $\mathbb{V}(x_0, x_1)$. Then $F$ can be written as

$$F = x_0^2F_{00} + x_0x_1F_{01} + x_1^2F_{11} + x_0F_0 + x_1F_1,$$

where $F_{00}, F_{01}, F_{11} \in \mathbb{C}[x_0, x_1, x_2, x_3]$ and $F_0, F_1 \in \mathbb{C}[x_2, x_3]$. Note that

$$\text{Sing } S \cap L = \mathbb{V}(F_0, F_1) \cap L.$$

Since $\mathbb{V}(F_0)$ and $\mathbb{V}(F_1)$ are cubics, they contain $L$ if $L$ intersects them in more than three points, so the claim follows.

Thus to require a quartic symmetroid to be double along a line is the same as requiring it to contain a line and to have an additional singularity on that line. As we saw, it is one condition to contain a line. Demanding that the quartic is singular at a given point imposes four conditions, but since the symmetroid already passes through the point, it reduces to three extra conditions. Moreover, the extra node can be anywhere on the line, so it amounts to just two conditions. We conclude that the quartic symmetroids with a double line form a family of dimension $24 - 1 - 2 = 21$.

We let Macaulay2 compute the singular locus and the rank 2 locus of the discriminant hypersurface in the $\mathbb{P}^8$ of quadrics through a single given point $p \in \mathbb{P}^3$. The rank 2 locus is a fivefold of degree 10. The singular locus consists in addition of a linear fivefold $X$. By Equation (5.2), there is an 18-dimensional family of linear 3-spaces intersecting $X$ in a line. Since $p$ can be any point in $\mathbb{P}^3$, this totals to a 21-dimensional family of quartic symmetroids that are double along a line.
CHAPTER 6

Degenerated Symmetroids

He who hears the rippling of rivers in these degenerate days will not utterly despair.

Henry David Thoreau

All the symmetroids we have considered thus far are of course degenerate in the sense that they are not generic. However, here we will take a brief look at surfaces that are even more special. This includes symmetroids that are double along a cubic curve, have a triple point or are cones. Lastly, we discuss our work.

6.1 Symmetroids with No Isolated Singularities

We present here three symmetroids that were discovered in the context of Section 5.3. Two of them are singular along reducible cubic curves. We start out with a surface that is singular along two intersecting lines and has a rank-1-point where they meet. It is a degeneration of the surfaces in Chapter 4, having no isolated singularities.

In the notation of Section 5.3, let \( L_1 \) be a line in \( \Pi_3 \) passing through the rank-1-point. The line \( \Pi_1 \cap Q \subset \Pi_3 \) meets \( L_1 \) in a point. Let \( L_2 \) be a line in \( T \) passing through the rank-1-point and meeting \( Q \). Then a 3-space \( H \) containing \( L_1 \) and \( L_2 \) will generically not meet \( \text{Sing} V(\det(M)) \) in any further points. For instance, the symmetroid \( X \) given by

\[
\begin{bmatrix}
0 & x_0 & x_1 & x_0 \\
x_0 & 0 & x_0 & x_2 \\
x_1 & x_0 & 0 & x_1 \\
x_0 & x_2 & x_1 & x_3 \\
\end{bmatrix}
\]

is double along two lines and has no isolated singularities. It is plotted in Figure A.5e on page 70.

Next, let \( L_1 \subset \Pi_3, L_2 \subset \Pi_4, L_3 \subset T \) be lines passing through the rank-1-point. The \( \mathbb{P}^3 \) spanned by the \( L_i \) gives rise to a symmetroid with three double
6. Degenerated Symmetroids

lines. One such symmetroid is given by

$$
\begin{bmatrix}
0 & x_0 & x_1 & x_0 \\
x_0 & 0 & 2x_0 - x_2 & x_2 \\
x_1 & 2x_0 - x_2 & 0 & x_1 \\
x_0 & x_2 & x_1 & x_3
\end{bmatrix}.
$$

By a change of coordinates we obtain

$$
A := \begin{bmatrix}
0 & x_0 & x_1 & x_0 \\
x_0 & 0 & -2x_2 & 4x_0 + 2x_2 \\
x_1 & -2x_2 & 0 & 2x_1 \\
x_0 & 4x_0 + 2x_2 & 2x_1 & 6x_0 + 2x_1 + 2x_2 - x_3
\end{bmatrix},
$$

which has determinant

$$
det(A) = 4(x_0x_1x_2x_3 + x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2).
$$

We recognise this from Equation (2.1) as the Steiner surface.

Finally, let \( \Pi \) be a plane intersecting \( Q \) in a conic. If \( H \) is a 3-space containing \( \Pi \), then it will generically meet \( T \) in a line, so the resulting symmetroid is double along a conic and a line. The symmetroid defined by

$$
\begin{bmatrix}
0 & x_0 & x_1 & x_0 \\
x_0 & 0 & 2x_2 - x_3 & x_2 \\
x_1 & 2x_2 - x_3 & 0 & x_0 \\
x_0 & x_2 & x_0 & x_3
\end{bmatrix}
$$

is an example of this. It is shown in Figure A.7 on page 72.

Base Loci

Both the Steiner surface and \( X \) are uniquely determined by the base loci of their webs of quadrics. For each symmetroid, the base locus is a scheme of length 6. Hence there is a unique web of quadrics passing through it.

The base locus of the Steiner surface consists of three points and a direction through each. The directions correspond to mutually skew lines, none of which are contained in the plane spanned by the three points.

For \( X \) the base locus also consists of three points and a direction through each. Again, the directions correspond to mutually skew lines, but one of them is contained in the plane spanned by the three points.

Cones

As mentioned in Remark 3.1.4, the space of quadrics \( a^TMa \) degenerates to a net when the symmetroid is a cone. The arguments in Section 4.3 and Section 5.1 usually involve a plane \( \Pi \), in some subset of the \( \mathbb{P}^9 \) of quadrics, which is extended to a \( \mathbb{P}^3 \). The discriminant of this \( \mathbb{P}^3 \) is then the sought-after symmetroid. The same type of arguments can be carried out for cones, except that \( \Pi \) is not extended to a \( \mathbb{P}^3 \).
If Π cuts the rank 2 locus in some points, then the resulting cone has rank 2 in the lines through those points and the apex. This symmetroid has no other singularities. For instance, the matrix
\[
\begin{pmatrix}
0 & x_0 & -x_1 + x_2 & x_0 \\
x_0 & 0 & -x_0 + x_1 - x_2 & x_1 \\
-x_1 + x_2 & -x_0 + x_1 - x_2 & 0 & x_1 \\
x_0 & x_1 & x_1 & x_2
\end{pmatrix}
\]
defines a symmetroid that is only singular along a line through the rank-0 point [0 : 0 : 0 : 1]. It is drawn in Figure A.4e on page 69.

Equation (5.1) gives that the Grassmannian $G(2, 9)$ of planes in $\mathbb{P}^9$ has dimension 21.

### 6.2 Quartic Symmetroids with a Triple Point

A quartic surface $S := \mathcal{V}(F)$ in $\mathbb{P}^3$ with a triple point $p$ is automatically rational. Indeed, the projection $\pi_p : S \setminus \{p\} \to \mathbb{P}^2$ from $p$ is a birational map.

If we let $p := [1 : 0 : 0 : 0]$, then $p$ is a triple point for $S$ if $F$ contains no $x_0^4, x_0^3f_1$ or $x_0^2f_2$ terms, where $f_1, f_2 \in \mathbb{C}[x_1, x_2, x_3]$ are polynomials of degree 1 and 2, respectively. Then $F$ can be written as
\[
F = x_0F_3 + F_4,
\]
where $F_3, F_4 \in \mathbb{C}[x_1, x_2, x_3]$ are polynomials of degree 3 and 4, respectively. The surfaces $X_3 := \mathcal{V}(F_3)$ and $X_4 := \mathcal{V}(F_4)$ meet in a curve of degree 12 on $S$. Since both $X_3$ and $X_4$ are cones with apex $p$, the curve breaks up into twelve concurrent lines.

The lines through $p$ are blown down to points $p_i$ via $\pi_p$. These twelve points are the intersection of a cubic and a quartic curve, namely
\[
\{p_1, \ldots, p_{12}\} = \mathcal{V}_{\mathbb{P}^2}(F_3) \cap \mathcal{V}_{\mathbb{P}^2}(F_4).
\]

Now $S$ can be represented as the image of the map induced by the linear system of quartics through the $p_i$. The triple point $p$ has linear equivalence class $3l - \sum_{i=1}^{12} e_i$.

It is a simple task to find quartic symmetroids with triple points. Assume that $l_{ij} \in \mathbb{C}[x_1, x_2, x_3]$ are linear forms for $0 \leq i \leq j \leq 3$. Then the matrix
\[
\begin{pmatrix}
l_{00} + x_0 & l_{01} & l_{02} & l_{03} \\
l_{01} & l_{11} & l_{12} & l_{13} \\
l_{02} & l_{12} & l_{22} & l_{23} \\
l_{03} & l_{13} & l_{23} & l_{33}
\end{pmatrix}
\]
defines a symmetroid which is triple in $[1 : 0 : 0 : 0]$. Clearly $p$ is a rank-1-point.

If $S$ is a symmetroid with $p := [1 : 0 : 0 : 0]$, then
\[
F_3 = \begin{vmatrix}
l_{11} & l_{12} & l_{13} \\
l_{12} & l_{22} & l_{23} \\
l_{13} & l_{23} & l_{33}
\end{vmatrix}
\]
6. Degenerated Symmetroids

As explained in Section 1.1, $X_3$ touches $S$ along a sextic curve of genus 3. Since $X_3$ intersects $S$ in the twelve lines through $p$, these must coincide in such a way that the lines through $p$ have even multiplicity.

Consequently, $V_{p^2}(F_3)$ is a contact curve for $V_{p^2}(F_4)$; their intersection is a scheme which is two times a scheme of length 6. Consider the special case where this scheme has support in six points, $p_i$ for $i = 1, \ldots, 6$. This occurs when two and two lines through $p$ coincide. Then $p_{i+6}$ is a tangent direction through $p_i$ and $S$ have the six nodes $e_i - e_{i+6}$. It is proved in [Jes16, Article 93] that a quartic surface with a triple point and six nodes is always a symmetroid.

Recall from Section 5.4 that the quartics in $\mathbb{P}^3$ form a $\mathbb{P}^{34}$. To require a triple point at a given point imposes ten conditions on the quartic, whereas a double point imposes only four conditions. Allowing the nodes to be any points in $\mathbb{P}^3$, only imposes $4 - 3 = 1$ condition for each. Hence the set of quartics with a given triple point and six nodes has dimension $34 - 10 - 6 = 18$. The dimension is 21 if we allow the triple point to be arbitrary.

6.3 Final Remarks

There are still numerous questions we would like to answer. The long-term goal is a complete classification of rational quartic symmetroids.

How far away are we from achieving that aim? Are there many symmetroids that we have not described? We argued in Section 5.4 that the family of quartic symmetroids that are singular along a line has dimension 21. Under the assumption that the base locus is non-empty, we found a component with the correct dimension. We have not proved that a double line implies a non-empty base locus, so it is unclear if we have missed some symmetroids using this procedure.

We demonstrated that symmetroids with rank 2 along a line have generically four coplanar base points. In addition, symmetroids with rank 2 along a conic section have generically base loci consisting of four general points. We have no results on symmetroids with a curve contained in the rank 2 loci, but with non-generic base loci. Furthermore, we have not examined the possibility of symmetroids with rank-3-nodes along a conic section.

One important step towards a classification is to determine whether all quartic surfaces with the same singular locus, say a double line and six isolated nodes, are symmetroids, or if only a subset of these surfaces has a symmetric, determinantal representation.

There are issues concerning the number of isolated nodes which remains to be addressed. Firstly is the verification or disproving of Conjectures 4.4.3 and 4.4.4. Furthermore, does there exist quartic symmetroids that are singular along a smooth conic section and have only one node — in a nondegenerate way — or no isolated nodes? Similarly, can a symmetroid be singular along a single line and have only one node or no further singularities, in a nondegenerate way?

In our experience, it is more common that a symmetroid with a double line has six nodes than any other number of isolated nodes. Likewise for symmetroids with a double conic and four isolated nodes. Are these heuristic results true?

We have not conducted much work on symmetroids that are double along a cubic curve. The only ones we have seen are the Toeplitz symmetroid in Example 3.3.2, the Steiner surface and the symmetroid with the double line and conic. The Toeplitz symmetroid is singular along a twisted cubic curve and
a line. Its singular locus is the complete intersection of two quadrics, so the surface is reducible. It is not true in general that a quartic surface which is double along a twisted cubic is reducible. Indeed, [CW05, Theorem 2.1] provides
\[256x_0^3x_2 - 256x_0^2x_1^2 - 288x_0x_1x_2x_3 + 256x_1^3x_3 + 27x_2^2x_3^2 = 0\]
as a counterexample. But it is worth investigating if all symmetroids which are singular along a twisted cubic curve are reducible.

We also desire suitable conditions for the rank 2 locus to be contained in a quadric surface. This is not the case for the generic symmetroids, but it is true for the majority of the examples we have given.
Here we showcase some of the symmetroids referred to in the text. The pictures are drawn using the visualising tool *surf*, developed by Stephan Endraß [End10]. All the singularities are real and contained in the affine chart used. For some of the surfaces, in particular Figure A.3a, you may need to squint in order to spot every singular point.

### A.1 Symmetroids Singular along a Line

![Symmetroid](image)

(a) Eight additional nodes.  
(b) Seven additional nodes.

Figure A.1: Rational quartic symmetroids singular along a line.
A. Gallery of Symmetroids

(a) The double line is contained in the rank 2 locus.

(b) The double line is not contained in the rank 2 locus.

Figure A.2: Rational quartic symmetroids singular along a line, with six isolated nodes.

(a) Five additional nodes.

(b) Four additional nodes.

Figure A.3: Rational quartic symmetroids singular along a line.
A.1. Symmetroids Singular along a Line

(a) Three additional nodes, no rank-1-points.
(b) Three additional nodes and a rank-1-point.
(c) Two additional nodes.
(d) One additional node.
(e) No additional nodes.

Figure A.4: Rational quartic symmetroids singular along a line.
A. Gallery of Symmetroids

A.2 Symmetroids Singular along a Conic

(a) Four additional nodes. The lines meet at infinity.

(b) Three additional nodes.

(c) Two additional nodes.

(d) One additional node.

(e) No additional nodes.

Figure A.5: Rational quartic symmetroids singular along two intersecting lines.
A.2. Symmetroids Singular along a Conic

(a) Four additional nodes.
(b) Three additional nodes.
(c) Two additional nodes.
(d) One additional node.

Figure A.6: Rational quartic symmetroids singular along a smooth conic section.
A.3 Symmetroid Singular along a Reducible Cubic

Figure A.7: Quartic symmetroid singular along a smooth conic section and a line.
Worst-case estimates suggested that trying to compute Gröbner bases might be a hopeless approach to solving problems. But from the first prototype, Macaulay was successful surprisingly often.

Computation in Algebraic Geometry with Macaulay2

The purpose of this appendix is to collect all the code necessary for the computations mentioned earlier and the unsightly long polynomial output that otherwise makes for a boring read. The calculations have been performed using Macaulay2, which is a computer algebra system for research in algebraic geometry, developed by Daniel Grayson and Michael Stillman [M2]. Due to the limitations of Macaulay2, the calculations have to be done over \( \mathbb{Q} \), so some of the output is reducible.

**B.1 Computations for Example 3.3.1**

Consider the following snippet:

```plaintext
k = QQ;
R = k[x_0..x_3];

P = matrix{{x_3, x_0, 0, x_0},
           {x_0, x_3, x_1, 0},
           {0, x_1, x_3, x_2},
           {x_0, 0, x_2, x_3}};

pillow = ideal det P
rank2 = saturate minors(3, P)
rank1 = saturate minors(2, P)
sing = saturate ideal singularLocus pillow

degree rank2
codim rank2
```
The equation defining the pillow is
\[
\det(P) = x_0^2 x_1^2 - 2x_0^2 x_1 x_2 + x_0^2 x_2^2 - 2x_0^2 x_3^2 - x_1^2 x_3^2 - x_2^2 x_3^2 + x_3^4 = 0.
\]

There are no rank-1-points on the surface. Its rank 2 locus is the one-dimensional variety of degree 2 given by the ideal

\[
I := \langle (x_1 + x_2) x_3, (2x_2^2 - x_3^2) x_3, (2x_0 - x_3^2) x_3, x_0(x_1 + x_2)(x_1 - x_2), x_0(x_1 x_2 - x_2^2 + x_3^2), x_0^2 x_1 - x_0^2 x_2 + x_2 x_3^2 \rangle,
\]

where the important observation is that the first generator is a quadric. We can describe how the quadric relates to the rank 2 locus: The ideal \(I\) decomposes as the intersection of the ideals

\[
\langle x_0, x_3 \rangle, \quad \langle x_1 - x_2, x_3 \rangle, \quad \langle x_0 - x_2, x_1 + x_2, 2x_2^2 - x_3^2 \rangle, \quad \langle x_0 + x_2, x_1 + x_2, 2x_2^2 - x_3^2 \rangle.
\]

In other words, the rank 2 locus is the union of the two lines \(L_1 := \mathcal{V}(x_0, x_3)\) and \(L_2 := \mathcal{V}(x_1 - x_2, x_3)\) lying in the plane \(x_3 = 0\) and the four points

\[
[1 : 1 : -1 : \pm \sqrt{2}] \quad \text{and} \quad [1 : -1 : 1 : \pm \sqrt{2}]
\]

in the plane \(x_1 + x_2 = 0\).

The singular locus consists of the rank 2 locus and five embedded points: The intersection \(L_1 \cap L_2\), the two points \( [0 : 1 : \pm i : 0] \in L_1 \) and \([1 : \pm i : \pm i : 0] \in L_2\).

Let us now turn to the variety of singular points on quadrics stemming from the pillow. The related claims can be verified easily by hand. Take for instance the corner \([1 : 1 : -1 : \sqrt{2}]\). Then \(Pa = 0\) reduces to

\[
\begin{bmatrix}
a_1 + a_3 + \sqrt{2}a_0 \\
a_0 + a_2 + \sqrt{2}a_1 \\
a_1 - a_3 + \sqrt{2}a_2 \\
a_0 - a_2 + \sqrt{2}a_3
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

Adding the first and the third row, and the second and the last row, we find that the solutions to these equations are described by the ideal

\[
\langle 2a_0 + \sqrt{2}(a_1 + a_3), 2a_1 + \sqrt{2}(a_0 + a_2) \rangle.
\]

We can use Macaulay2 to confirm that the line defined by this ideal does indeed lie on \(Q\).
B.2 Computations for Example 3.3.2

R1 = k[a_0..a_3];

A = matrix{{a_1 + a_3, 0, 0, a_0},
{a_0, a_2, 0, a_1},
{0, a_1, a_3, a_2},
{a_0, 0, a_2, a_3}};

f = factor det A
Q = ideal f#2#0

R2 = R1[s] / (s^2 - 2);
-- s = sqrt(2)
-- Polynomial factorisation is only
-- implemented over rational numbers.

i = map(R2, R1, gens R1); -- Inclusion
Q = i(Q);

L = ideal(2*a_0 + s*(a_1 + a_3),
2*a_1 + s*(a_0 + a_2));

L + Q == L

B.2 Computations for Example 3.3.2

The next lines of code are used to examine the stratification of the Toeplitz symmetroid.

k = QQ;
R = k[x_0..x_3];

T = matrix{{x_0, x_1, x_2, x_3},
{x_1, x_0, x_1, x_2},
{x_2, x_1, x_0, x_1},
{x_3, x_2, x_1, x_0}};

f = factor det T
C1 = ideal f#0#0
C2 = ideal f#1#0

S1 = saturate ideal singularLocus C1
S2 = saturate ideal singularLocus C2

rank2 = saturate minors(3, T)
rank1 = saturate minors(2, T)

MP = minimalPrimes rank2
rank1 == MP_0 + MP_1
rank1 == intersect(S1, S2)

The determinant det $T$ factors as the product of the quadratic forms

\[-x_0^2 - x_0 x_1 + x_1^2 + 2x_1 x_2 + x_2^2 - x_0 x_3 - x_1 x_3\]

and

\[-x_0^2 + x_0 x_1 + x_1^2 - 2x_1 x_2 + x_2^2 + x_0 x_3 - x_1 x_3.\]

Denote the cones these forms define by $C_1$ and $C_2$. The $C_i$ have apices in the points $[1 : -1 : 1 : -1]$ and $[1 : 1 : 1 : 1]$, respectively. These two points constitute the rank 1 locus of the Toeplitz symmetroid.

The rank 2 locus is not contained in any quadric; its ideal is generated by six cubics. It decomposes as the intersection of the ideals

\[
\langle x_0^2 - 2x_1^2 + x_0 x_2, x_0 x_1 - 2x_1 x_2 + x_0 x_3, x_0^2 - x_0 x_2 - 2x_2^2 + 2x_1 x_3 \rangle
\]

and

\[
\langle x_0 - x_2, x_1 - x_3 \rangle.
\]

The twisted cubic and the line defined by these intersect precisely in the two rank-1-points.

### B.3 Computations for Example 4.1.6

Put $P_1 := [1 : 0 : 0]$, $P_3 := [0 : 1 : 0]$ and $P_5 := [0 : 0 : 1]$. The requirement that a cubic $V(f)$ passes through these three points amounts to there being no $x_0^3$, $x_1^3$ or $x_2^3$ terms in $f$. Furthermore, the lines $P_1 P_5$ and $P_3 P_5$ are $V(x_1)$ and $V(x_0)$, respectively. Thus, for $V(f)$ to have these as tangent lines at $P_1$ and $P_3$, we must have

\[
\frac{\partial f}{\partial x_0}(P_1) = \frac{\partial f}{\partial x_2}(P_1) = \frac{\partial f}{\partial x_1}(P_2) = \frac{\partial f}{\partial x_2}(P_2) = 0.
\]

Therefore, there cannot be any $x_0^2 x_2$ or $x_1^2 x_2$ terms in $f$ either. Hence a cubic in this linear system is on the form

\[
f(x_0, x_1, x_2) = a_{210} x_0^2 x_1 + a_{120} x_0 x_1^2 + a_{111} x_0 x_1 x_2 + a_{102} x_0 x_2^2 + a_{012} x_1 x_2^2.
\]

We use the obvious basis $\{x_0^2 x_1, x_0 x_1^2, x_0 x_1 x_2, x_0 x_2^2, x_1 x_2^2\}$. The rest we leave to Macaulay2:

```plaintext
k = QQ;
P2 = k[x_0..x_2];
P4 = k[y_0..y_4];
f = map(P2, P4, {x_0*x_1^2, x_0*x_1*x_2^2, x_0*x_1*x_2, x_0*x_2^2});
```

minPrimes ideal singularLocus DelPezzo
B.4 Computations for Example 4.1.7

We start with the 4-nodal del Pezzo surface from Appendix B.3.

\[
k = \mathbb{Q};
\]
\[
P_2 = \mathbb{Q}[x_0..x_2];
\]
\[
P_3 = \mathbb{Q}[y_0..y_3];
\]
\[
P_4 = \mathbb{Q}[z_0..z_4];
\]
\[
f = \text{map}(P_2, P_4, \{x_0^2*x_1, x_0*x_1^2, x_0*x_1*x_2, x_0*x_2^2, x_1*x_2^2\});
\]
\[
S = \ker f \quad \text{-- Del Pezzo Surface.}
\]
\[
\text{minimalPrimes, ideal, singularLocus } S
\]

Then we compute the strict transform of a line passing through \(P_5\), and the strict transform of a conic that meets all the \(P_i\) except \(P_5\). These strict transforms are two conics, \(C_1\) and \(C_2\), that do not meet any of the nodes on the surface. Also, \(C_1\) and \(C_2\) intersect in two points, which span a line \(L\). In this case, \(L = V(x_3 + x_4, x_1 + x_4, x_0 - x_4)\) and we let the projection centre \(p\) be the point \([1 : -1 : 0 : -1 : 1]\), which we verify is not on the surface.

\[
\text{-- Conics that avoid the singularities:}
\]
\[
C_1 = \text{preimage}(f, \text{ideal}(x_0 + x_1));
\]
\[
C_2 = \text{preimage}(f, \text{ideal}(x_0*x_1 + x_2^2));
\]
\[
\text{-- Find the line spanned by their intersection points.}
\]
\[
L = \text{trim}(C_1 + C_2);
\]
\[
L = \text{ideal}(L_0, L_1, L_2)
\]
\[
\text{-- Take the projection centre } p \text{ to be a point on this line.}
\]
\[
\text{-- } p = [1 : -1 : 0 : -1 : 1]
\]
\[
Ip = \text{ideal}(z_1 + z_0, z_2, z_3 + z_0, z_4 - z_0);
\]
\[
Ip + L == Ip \quad \text{-- } p \text{ is on } L
\]
\[
Ip + S == Ip \quad \text{-- } p \text{ is not on the del Pezzo}
\]

Under the projection, a point \(q\) on the del Pezzo surface \(S\) is mapped to the unique intersection point between the line \(pq\) and the \(P_3\). Consider the union of all lines from \(p\) to a point on \(S\). This is a cone \(C\) over \(S\) with apex in \(p\). The image of the projection is \(C \cap \mathbb{P}^3\). Because the del Pezzo surface is a quartic, \(p\) is a quadruple point on \(C\). Thus, the equation of \(C\) is given by the unique quartic contained in the ideal

\[
I := I^4_p \cap I_{\text{DelPezzo}}.
\]

This explains why we did not use \([0 : 0 : 1 : 0 : 0]\) for \(p\); one of the quadrics in the pencil has apex in \([0 : 0 : 1 : 0 : 0]\), in which case \(I\) contains ten quartics, rather than just one. Not to mention that \([0 : 0 : 1 : 0 : 0]\) is not contained in the \(\mathbb{P}^3\) spanned by the nodes. However, it is of minor importance whether their projected images are coplanar or not.
B. Code and Computations

We use $V(x_4)$ as the linear hyperspace.

```plaintext
C = trim intersect (Ip^4, S); -- Cone with apex in p.
degrees C -- Check number of quartics.

-- Intersect with the hyperplane $V(z_4)$:
i = map(P3, P4, gens P3 | {0});

X = i(ideal(C_0)) -- Projected del Pezzo Surface.
```

First, we make sure that the singular locus of the projected del Pezzo surface does indeed consist of two lines, $L_1$ and $L_2$, and four points.

Next, there are embedded points in the singular locus: The intersection $L_1 \cap L_2$ and two other points on each $L_i$. We pull the latter back to $\mathbb{P}^3$ and find that the preimages of the two embedded points on $L_i$ span a line $L_i'$ in the plane spanned by the conic $C_i$. The line $L_i'$ is tangent to $C_i$.

```plaintext
sing = ideal singularLocus X;
minimalPrimes sing

-- Confirm that the embedded points are
-- ramification points for the projection:
AP = associatedPrimes sing
L1 = preimage(i, AP_3)
L2 = preimage(i, AP_2)
L1 = ideal(L1_0, L1_1, L1_2)
L2 = ideal(L2_0, L2_1, L2_2)
minimalPrimes (L1 + C1)
minimalPrimes (L2 + C2)
```

Since $X$ has a node in $[1 : 0 : 0 : 0]$, its defining equation is of the form $f = ax_0^2 + bx_0 + c$, where $a, b, c \in \mathbb{Q}[x_1, x_2, x_3]$. When projecting from $[1 : 0 : 0 : 0]$ to $V(x_0)$, the branch curve is given by the discriminant $b^2 - 4ac$.

```plaintext
-- Trick in order to use the function 'coefficients':
R1 = k[y_1..y_3][y_0];
X = sub(X, R1);

(Mon, Coeff) = coefficients(X_0,
                         Monomials => {y_0^2, y_0, 1});

X = sub(X, P3);

a = sub(Coeff_(0, 0), P3);
b = sub(Coeff_(1, 0), P3);
c = sub(Coeff_(2, 0), P3);

use P3;

branch = factor(b^2 - 4*a*c)
```

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The built-in method \texttt{adjoint} does not give us what we want, so we copy the correct definition from [Eis+02].

\begin{verbatim}
classicalAdjoint = (G) -> {
    n := degree target G;
    m := degree source G;
    matrix table(n, n, (i, j) -> (-1)^(i + j) * det(
        submatrix(G, {0..j - 1, j + 1..n - 1},
        {0..i - 1, i + 1..m - 1}))));
}
\end{verbatim}

Then we just follow Algorithm 1.1.4 and check that matrix we end up with has the desired determinant.

\begin{verbatim}
F11 = branch#0#0 * branch#2#0 / 4;
F22 = branch#0#0 * branch#1#0^2;
q = -a;
g = b/2;
delta = c;
I = ideal(F11, delta);
RI = radical I;

-- Check that the space of cubics is 4-dimensional:
degrees RI
F11*F22 - g^2 == q*delta

F12 = g;
F13 = RI_0;
F14 = RI_3;

-- Check that the Fij are independent:
degrees trim ideal(F11, F12, F13, F14)
F23 = (F12+F13 // gens I)_(0,0);
F24 = (F12+F14 // gens I)_(0,0);
F33 = (F13+F13 // gens I)_(0,0);
F34 = (F13+F14 // gens I)_(0,0);
F44 = (F14+F14 // gens I)_(0,0);
F = matrix({F11, F12, F13, F14},
{F12, F22, F23, F24},
{F13, F23, F33, F34},
{F14, F24, F34, F44});
M = classicalAdjoint(F) // delta^2;
cc = det F // delta^3;
\end{verbatim}
B. Code and Computations

\[
N = \text{matrix}\{\{0, cc*y_0, 0, 0\}, \{cc*y_0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\};
\]

\[A = M + N\]

\[\text{det } A / X_0\]

Next, we find the variety of the singular loci of the quadrics \(a^T A a\). This is straightforward: Look at the product \(A a\) and create the corresponding matrix \(M\) — called \(B\) in the code, since \(M\) is already in use — of the coefficients to the variables \(a_i\), such that \(A a = M x\). Then \(\text{det}(M) = 0\) is the desired variety. It turns out to be the union of a plane and a 3-nodal cubic.

---

**Trick in order to use the function ‘coefficients’:**

\[R2 = k[w_0..w_3][y_0..y_3];\]

\[A = \text{sub}(A, R2);\]

\[W = \text{matrix}\{\{w_0\}, \{w_1\}, \{w_2\}, \{w_3\}\};\]

\[P = A*W;\]

\[(M0, C0) = \text{coefficients}(P^{0}, \text{Monomials} => \{y_0, y_1, y_2, y_3\});\]

\[(M1, C1) = \text{coefficients}(P^{1}, \text{Monomials} => \{y_0, y_1, y_2, y_3\});\]

\[(M2, C2) = \text{coefficients}(P^{2}, \text{Monomials} => \{y_0, y_1, y_2, y_3\});\]

\[(M3, C3) = \text{coefficients}(P^{3}, \text{Monomials} => \{y_0, y_1, y_2, y_3\});\]

\[B = \text{matrix}\{\{C0_0(0,0), C0_0(1,0), C0_0(2,0), C0_0(3,0)\}, \{C1_0(0,0), C1_0(1,0), C1_0(2,0), C1_0(3,0)\}, \{C2_0(0,0), C2_0(1,0), C2_0(2,0), C2_0(3,0)\}, \{C3_0(0,0), C3_0(1,0), C3_0(2,0), C3_0(3,0)\}\};\]

\[R3 = k[w_0..w_3];\]

\[g = \text{factor sub}(\text{det } B, R3)\]

\[\text{minimalPrimes ideal singularLocus ideal } g\#1\#0\]

The base locus of the web of quadrics \(a^T A a\), equals the common intersection of four generating quadrics. To find four such quadrics, take \(a^T A(P_i) a\) for four points \(P_i\) that span \(P^3\). In this case, the base locus consists of six points, counted with multiplicity. More precisely, it contains five points where one of the points is counted twice.
B.5 Computations for Example 4.2.4

-- General quadratic form:
Q = (transpose(W)*P)_(0, 0)

R4 = k[w_0..w_3,y_0..y_3];

Q = sub(Q, R4)

-- Generators for the base locus:
Q0 = sub(Q, {y_0 => 1, y_1 => 0, y_2 => 0, y_3 => 0});
Q1 = sub(Q, {y_0 => 0, y_1 => 1, y_2 => 0, y_3 => 0});
Q2 = sub(Q, {y_0 => 0, y_1 => 0, y_2 => 1, y_3 => 0});
Q3 = sub(Q, {y_0 => 0, y_1 => 0, y_2 => 0, y_3 => 1});

base = ideal(Q0, Q1, Q2, Q3);
degree base
minimalPrimes base
primaryDecomposition base

For the points $P$ in the base locus, we find the set of points $[Q_P]$ such that the quadrics $Q_P$ are singular at $P$.

-- The function 'singularLocus' differentiates with respect to $y_i$ as well, hence we have to do it manually.
d0 = diff(w_0, Q);
d1 = diff(w_1, Q);
d2 = diff(w_2, Q);
d3 = diff(w_3, Q);

d0 = ideal(d0, d1, d2, d3);
saturate sub(d0, {w_0 => 1, w_1 => 0, w_2 => 0, w_3 => 0})
saturate sub(d0, {w_0 => 1, w_1 => 0, w_2 => 2, w_3 => 0})
saturate sub(d0, {w_0 => 0, w_1 => 1, w_2 => 1, w_3 => 0})
saturate sub(d0, {w_0 => 0, w_2 => 0, w_3 => 2})
saturate sub(d0, {w_0 => 0, w_2 => 0, w_3 => 1})

B.5 Computations for Example 4.2.4

We use the same points as in Appendix B.3, albeit we must swap the names of $P_1$ and $P_3$ in order to be consistent with Example 4.2.4. Again, we can have no terms of the forms $x_0^3$, $x_1^3$, $x_2^3$ or $x_1 x_2^2$, but $x_0^2 x_2$ terms are allowed.

The cubics must have the same tangent at $P_3$, and the tangent line must be different from $P_3 P_5$. We choose the line $V(x_1 - x_2)$ and let

\[
\begin{align*}
x_0^2 x_1 + x_0^2 x_2 + x_0^2 x_1 + x_0^2 x_2 + x_0 x_1 x_2 + x_0 x_1 x_2,
x_0^2 x_1 + x_0^2 x_2 + x_0 x_2^2, x_0^2 x_1 + x_0^2 x_2 + x_0 x_1^2
\end{align*}
\]

be the basis for cubics passing through the $P_i$. Using this, we find the del Pezzo
B. Code and Computations

surface with ideal
\( \langle x_0x_2 + x_1x_2 - 3x_3^2 - x_1x_3 + 2x_2x_3 + x_2x_4 - x_3x_4, \\
x_1^2 - 2x_1x_2 + x_2x_3 + x_2x_4 - x_3x_4 \rangle. \)
It has nodes in \([1 : 0 : 0 : 0 : 0],[0 : 0 : 0 : 1]\) and \([1 : 1 : 1 : 1]\).

\( k = \mathbb{Q}; \)
\( P2 = k[x_0..x_2]; \)
\( P4 = k[y_0..y_4]; \)
\( f = \text{map}(P2, P4, \{x_0^2*x_1 + x_0^2*x_2 + x_1*x_2^2, \}
\text{x}_0^2*x_1 + x_0^2*x_2 + x_0*x_1*x_2, \}
\text{x}_0^2*x_1 + x_0^2*x_2, \}
\text{x}_0^2*x_1 + x_0*x_2^2, \}
\text{x}_0^2*x_1 + x_0*x_1^2); \)
\( \text{DelPezzo} = \ker f \)
\( \text{minimalPrimes ideal singularLocus DelPezzo} \)

B.6 Computations for Example 4.2.6

We choose the points \( P_1 := [0 : 0 : 1], P_2 := [0 : 1 : 0] \) and \( P_3 := [0 : 1 : -1] \)
on the line \( \mathcal{V}(x_0). \) On the line \( \mathcal{V}(x_1) \) through \( P_1, \) we pick \( P_4 := [1 : 0 : 0] \) and \( P_5 := [1 : 0 : -1]. \)
Macaulay2 produces the basis
\( \{x_0^2x_1, x_0x_1^2, x_0x_1x_2, x_1x_2^2, x_0^2x_2 + x_0x_2^2\} \)
of cubics through the \( P_i. \) The resulting del Pezzo surface in \( \mathbb{P}^4 \) has ideal
\( \langle x_1x_3 + x_3^2 - x_2x_4, x_0x_1 - x_2x_3 - x_3^2 \rangle. \)
It has nodes in \([1 : 0 : 0 : 0 : 0] \) and \([0 : 0 : 0 : 0 : 1] \).

\( k = \mathbb{Q}; \)
\( P2 = k[x_0..x_2]; \)
\( P4 = k[y_0..y_4]; \)
\( I1 = \text{ideal}(x_0, x_1); \quad \text{-- } [0 : 0 : 1] \)
\( I2 = \text{ideal}(x_0, x_2); \quad \text{-- } [0 : 1 : 0] \)
\( I3 = \text{ideal}(x_0, x_1 + x_2); \quad \text{-- } [0 : 1 : -1] \)
\( I4 = \text{ideal}(x_1, x_2); \quad \text{-- } [1 : 0 : 0] \)
\( I5 = \text{ideal}(x_1, x_0 + x_2); \quad \text{-- } [1 : 0 : -1] \)
\( I = \text{intersect}(I1, I2, I3, I4, I5) \)
\( \text{-- The generators are listed in this order such that} \)
\( \text{-- one node gets coordinates } [1 : 0 : 0 : 0 : 0]. \)
\( f = \text{map}(P2, P4, \{I_1, I_0*x_0, I_0*x_1, I_0*x_2, I_2}); \)
B.7 Computations for Section 4.3

This is a straightforward factorisation into components and then examination of their intersections.

\[
\begin{align*}
\text{k} & = \mathbb{Q}; \\
\text{R} & = k[x_0..x_5]; \\
\text{M} & = \text{matrix}\{{ \begin{array}{cccc}
0, & x_0, & x_1, & x_2, \\
 x_0, & 0, & x_3, & x_4, \\
x_1, & x_3, & 0, & x_5, \\
x_2, & x_4, & x_5, & 0 \\
\end{array} }\};
\end{align*}
\]

\[
\begin{align*}
\text{rank2} & = \text{saturate minors}(3, \text{M}); \\
\text{sing} & = \text{saturate ideal}\text{ singularLocus}\text{ ideal det}\text{ M};
\end{align*}
\]

\[
\begin{align*}
\text{dim} & \text{ rank2} \\
\text{degree} & \text{ rank2}
\end{align*}
\]

\[
\begin{align*}
\text{dim} & \text{ sing} \\
\text{degree} & \text{ sing}
\end{align*}
\]

\[
\begin{align*}
\text{MP} & = \text{minimalPrimes}\text{ rank2} \\
\text{mp} & = \text{minimalPrimes}\text{ sing}
\end{align*}
\]

\[
\begin{align*}
\text{Pi1} & = \text{MP}_0; \\
\text{Pi2} & = \text{MP}_2; \\
\text{Pi3} & = \text{MP}_3; \\
\text{Pi4} & = \text{MP}_5; \\
\text{Pi5} & = \text{mp}_0; \\
\text{Pi6} & = \text{mp}_4; \\
\text{Pi7} & = \text{mp}_6; \\
\text{Pi8} & = \text{mp}_7; \\
\text{Q1} & = \text{MP}_1; \\
\text{Q2} & = \text{MP}_4; \\
\text{Q3} & = \text{MP}_6;
\end{align*}
\]

\[
\begin{align*}
\text{saturate}(\text{Pi1} + \text{Pi2}) \\
\text{saturate}(\text{Pi1} + \text{Pi3}) \\
\text{saturate}(\text{Pi1} + \text{Pi4}) \\
\text{saturate}(\text{Pi2} + \text{Pi3}) \\
\text{saturate}(\text{Pi2} + \text{Pi4}) \\
\text{saturate}(\text{Pi3} + \text{Pi4})
\end{align*}
\]
B. Code and Computations

\texttt{minimalPrimes(Q1 + Q2)}
\texttt{minimalPrimes(Q1 + Q3)}
\texttt{minimalPrimes(Q2 + Q3)}

\texttt{saturate(Pi1 + Q1)}
\texttt{saturate(Pi1 + Q2)}
\texttt{saturate(Pi1 + Q3)}
\texttt{saturate(Pi2 + Q1)}
\texttt{saturate(Pi2 + Q2)}
\texttt{saturate(Pi2 + Q3)}
\texttt{saturate(Pi3 + Q1)}
\texttt{saturate(Pi3 + Q2)}
\texttt{saturate(Pi3 + Q3)}
\texttt{saturate(Pi4 + Q1)}
\texttt{saturate(Pi4 + Q2)}
\texttt{saturate(Pi4 + Q3)}
\texttt{saturate(Pi5 + Pi1)}
\texttt{saturate(Pi5 + Pi2)}
\texttt{saturate(Pi5 + Pi3)}
\texttt{saturate(Pi5 + Pi4)}
\texttt{saturate(Pi6 + Pi1)}
\texttt{saturate(Pi6 + Pi2)}
\texttt{saturate(Pi6 + Pi3)}
\texttt{saturate(Pi6 + Pi4)}
\texttt{saturate(Pi7 + Pi1)}
\texttt{saturate(Pi7 + Pi2)}
\texttt{saturate(Pi7 + Pi3)}
\texttt{saturate(Pi7 + Pi4)}
\texttt{saturate(Pi8 + Pi1)}
\texttt{saturate(Pi8 + Pi2)}
\texttt{saturate(Pi8 + Pi3)}
\texttt{saturate(Pi8 + Pi4)}

B.8 Computations for Example 5.2.1

We let \(P_1 := [0 : 0 : 1] \), \(P_2 := [1 : 0 : 0] \), \(P_3 := [0 : 1 : 0] \), and \(P_5 := [1 : 1 : 1] \). So we want the quartics to have \(V(x_1), V(x_0) \) and \(V(x_0 - x_1) \) as their tangent lines at \(P_2, P_3 \) and \(P_5 \), respectively. Lastly, we put \(P_8 := [1 : -1 : 2] \) and \(P_9 := [2 : 1 : -1] \). We used MATLAB to find a basis for the quartics that pass through these points with the correct tangent lines and a singularity at \(P_1 \):

\[
\begin{align*}
6x_0x_1^2x_2 - 5x_0^2x_2^2 + 3x_0x_1x_2^2 + 2x_1^2x_2^2 - 6x_0^2x_1x_2, \\
48x_0x_1^2x_2 + 55x_0^2x_2^2 + 3x_0x_1x_2^2 - 82x_1^2x_2^2 - 24x_0x_1^3, \\
48x_0x_1^2x_2 + 73x_0^2x_2^2 - 3x_0x_1x_2^2 - 94x_1^2x_2^2 - 24x_0^2x_1^2, \\
48x_0x_1^2x_2 + 103x_0^2x_2^2 + 3x_0x_1x_2^2 - 130x_1^2x_2^2 - 24x_0x_1^3.
\end{align*}
\]

We use this basis to define the ring map \(f \) associated to the map \(\varphi \) induced by the linear system:

\[84\]
B.8. Computations for Example 5.2.1

\[ k = \mathbb{Q}Q; \]
\[ P_2 = k[x_0..x_2]; \]
\[ P_3 = k[y_0..y_3]; \]
\[ q_1 = 6x_0x_1^2x_2 + 5x_0^2x_2^2 + 3x_0x_1x_2^2 + 2x_1^2x_2^2 - 6x_0^2x_1x_2; \]
\[ q_2 = 48x_0x_1^2x_2 + 55x_0^2x_2^2 + 3x_0x_1x_2^2 - 82x_1^2x_2^2 - 24x_0x_1^3; \]
\[ q_3 = 48x_0x_1^2x_2 + 73x_0^2x_2^2 - 3x_0x_1x_2^2 - 94x_1^2x_2^2 - 24x_0^2x_1^2; \]
\[ q_4 = 48x_0x_1^2x_2 + 103x_0^2x_2^2 + 3x_0x_1x_2^2 - 130x_1^2x_2^2 - 24x_0^3x_1; \]
\[ f = \text{map}(P_2, P_3, \{q_1, q_2, q_3, q_4\}); \]
\[ S = \ker f \]

The result is a surface \( S \) with the equation

\[
41472x_0^4 - 2016x_0^3x_1 - 34848x_0^3x_3 + 31680x_0^3x_1x_2 + 18550x_0^3x_1x_2^2 + 13444x_0^3x_1^2x_2 + 12310x_0^3x_1^2x_2^2 + 6050x_0^3x_2^3 - 1752x_0^3x_1x_2x_3 + 4456x_0^3x_1^2x_2x_3 + 1100x_0^3x_1^3x_2 + 60x_0^3x_1x_2^3 + 215x_0^3x_2^3 + 15x_0^3x_1^2x_2^3 + 60x_0^3x_2^2x_3 + 165x_0^3x_1^2x_2^2x_3 - 165x_0^3x_2^2x_3^2.
\]

We are able to use \( f \) to relate three of the nodes to the linear system right away:

The ideal \( f(node) \) is contained in the ideal of one the lines \( P_1P_2, P_1P_4, P_1P_6 \).

\[ \text{MP} = \text{minimalPrimes ideal singularLocus } S; \]

-- See which singularities we can identify:
\[ \text{minimalPrimes } f(\text{MP}_1) \]
\[ \text{minimalPrimes } f(\text{MP}_2) \]
\[ \text{minimalPrimes } f(\text{MP}_3) \quad \text{-- } l - e_1 - e_6 - e_7 \]
\[ \text{minimalPrimes } f(\text{MP}_4) \quad \text{-- } l - e_1 - e_2 - e_3 \]
\[ \text{minimalPrimes } f(\text{MP}_5) \]
\[ \text{minimalPrimes } f(\text{MP}_6) \quad \text{-- } l - e_1 - e_4 - e_5 \]

We then apply Algorithm 1.1.4 to obtain a matrix representation of \( S \). An implementation of the algorithm is given in Appendix B.4. The output is a symmetric matrix with so large entries that we will not present it here. We remove the first column and consider the curve \( C \) determined by the \((3 \times 3)\)-minors of the remaining submatrix.

\[ C = \text{minors}(3, M_{\{1, 2, 3\}}); \]
\[ \text{mp} = \text{minimalPrimes } C; \]
\[ \text{apply}(\text{mp}, i \rightarrow \text{degree } i) \quad \text{-- } 4, 1, 1 \]

It consists of a quartic and two lines. One of the lines is immediately identified as the reduced line associated to \( B \):

85
-- Check that the reduced line associated to $B$ is a component: 
mp_1 == MP_0 -- True

The quartic is identified with a plane cubic. It is singular at $P_2$. We check which points it passes through.

cubic = (minimalPrimes f(mp_0))_0

saturate ideal singularLocus cubic -- p2

p1 = ideal(x_0, x_1); -- [0 : 0 : 1]
p2 = ideal(x_1, x_2); -- [1 : 0 : 0]
p4 = ideal(x_0, x_2); -- [0 : 1 : 0]
p6 = ideal(x_0 - x_1, x_0 - x_2); -- [1 : 1 : 1]
p8 = ideal(x_0 + x_1, 2*x_0 - x_2); -- [1 : -1 : 2]
p9 = ideal(x_0 - 2*x_1, x_1 + x_2) -- [2 : 1 : -1]

cubic + p1 == p1 -- True
cubic + p2 == p2 -- True
cubic + p4 == p4 -- True
cubic + p6 == p6 -- True
cubic + p8 == p8 -- True
cubic + p9 == p9 -- True

It passes through all of them. We then check if it has some of the tangent lines corresponding to $P_3$, $P_5$ or $P_7$.

dx0 = diff(x_0, cubic_0);
dx1 = diff(x_1, cubic_0);
dx2 = diff(x_2, cubic_0);

sub(dx0, {x_0 => 1, x_1 => 0, x_2 => 0})
sub(dx1, {x_0 => 1, x_1 => 0, x_2 => 0})
sub(dx2, {x_0 => 1, x_1 => 0, x_2 => 0})

sub(dx0, {x_0 => 0, x_1 => 1, x_2 => 0})
sub(dx1, {x_0 => 0, x_1 => 1, x_2 => 0})
sub(dx2, {x_0 => 0, x_1 => 1, x_2 => 0})

sub(dx0, {x_0 => 1, x_1 => 1, x_2 => 1})
sub(dx1, {x_0 => 1, x_1 => 1, x_2 => 1})
sub(dx2, {x_0 => 1, x_1 => 1, x_2 => 1})

It does not, so we conclude that the quartic is $3l - e_1 - 2e_2 - e_4 - e_6 - e_8 - e_9$. This curve is singular at $e_2 - e_3$, so we can relate another node to the linear system.
saturate ideal singularLocus mp_0 -- e_2 - e_3
saturate(mp_2 + mp_0) -- e_2 - e_3
mp_2 + MP_4 == MP_4 -- True
The last component in C passes through both e_2 − e_3 and l = e_1 − e_2 − e_3. We deduce that the line is e_2.

B.9 Computations for Example 5.2.2

We let the conic K be \( V(x_0x_1 - x_2^2) \). Let \( P_2 := [1 : 0 : 0] \), \( P_4 := [0 : 1 : 0] \) and \( P_6 := [1 : 1 : 1] \). The tangent lines to K at \( P_2 \), \( P_4 \) and \( P_6 \) are \( V(x_1) \), \( V(x_0) \) and \( V(x_0 + x_1 - 2x_3) \), respectively. That means that \( P_1 \) is \( [0 : 0 : 1] \). Finally, put \( P_8 := [1 : -1 : 2] \) and \( P_9 := [1 : -1 : 3] \). We found the following basis of quartics passing through the \( P_i \) using MATLAB:

\[
\begin{align*}
19x_0^3x_1 + 5x_0^2x_1^2 - 20x_0x_1x_2^2 - 4x_0^2x_2^2, \\
7x_0^3x_1 - 5x_0^2x_1^2 - 5x_0^2x_1x_2 + 5x_0x_1^2x_2 - 2x_0^2x_2^2, \\
x_0^3x_1 + x_0^2x_1^2 - x_0^2x_2^2 - x_0x_1x_2^2, \\
24x_0^3x_1 - 20x_0^2x_1x_2 - 9x_0^2x_2^2 + 5x_1^2x_2^2.
\end{align*}
\]

We use this to define the ring map induced by the linear system:

\[
\begin{align*}
&k = QQ; \\
&P2 = k[x_0..x_2]; \\
&P3 = k[y_0..y_3]; \\
&Q1 = 19*x_0^3*x_1 - 20*x_0^2*x_1*x_2 - 4*x_0^2*x_2^2 + 5*x_0*x_1^3; \\
&Q2 = 7*x_0^3*x_1 - 5*x_0^2*x_1^2 - 5*x_0^2*x_1*x_2 - 2*x_0^2*x_2^2 + 5*x_0*x_1^2*x_2; \\
&Q3 = x_0^3*x_1 + x_0^2*x_1^2 - x_0^2*x_2^2 - x_0*x_1*x_2^2; \\
&Q4 = 24*x_0^3*x_1 - 20*x_0^2*x_1*x_2 - 9*x_0^2*x_2^2 + 5*x_1^2*x_2^2; \\
f = map(P2, P3, {Q1, Q2, Q3, Q4}); \\
&S = ker f
\end{align*}
\]

We obtain the surface \( S \) with defining polynomial

\[
x_0^2x_1^3 + 10x_0^2x_1x_2 + 25x_0^2x_2^2 - 5x_0^2x_2x_3 + 20x_0x_1^2x_2 - 2x_0x_1^2x_3 - 20x_0x_1x_2^2 \\
- 10x_0x_1x_2x_3 - 25x_0x_2x_3^2 + 5x_0x_2x_3^2 + 300x_1x_2^2 - 20x_1x_2x_3 + x_1^2x_3 \\
+ 100x_1x_3^3 - 80x_1x_3^2x_3 - 400x_1^2 + 100x_1x_3.
\]

Applying Algorithm 1.1.4 as in Appendix B.4, we obtain the matrix

\[
M(x) := M_0x_0 + M_1x_1 + M_2x_2 + M_3x_3,
\]
where

\[
M_0 := \begin{bmatrix}
-16 & 800 & 200 & 100 \\
800 & -60000 & -15000 & -5000 \\
200 & -15000 & -2500 & -1250 \\
100 & -5000 & -1250 & -625
\end{bmatrix},
\]

\[
M_1 := \begin{bmatrix}
-32 & 1600 & 400 & 0 \\
1600 & -120000 & -20000 & 0 \\
400 & -20000 & -5000 & 0 \\
0 & 0 & 0 & 1250
\end{bmatrix},
\]

\[
M_2 := \begin{bmatrix}
-240 & 3200 & 1000 & 500 \\
3200 & 120000 & 10000 & 5000 \\
1000 & 10000 & 12500 & -6250 \\
500 & 5000 & -6250 & 3125
\end{bmatrix},
\]

\[
M_3 := \begin{bmatrix}
0 & 400 & 0 & 0 \\
400 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

We remove the first column of \( M \). Then we verify that the reduced line associated to \( B \) is not a component in the curve \( C \) defined by the \((3 \times 3)\)-minors of this \((4 \times 3)\)-matrix.

\[ C = \text{saturate minors}(3, M_{1, 2, 3}); \]

\[ \text{MP} = \text{minimalPrimes ideal singularLocus S} \]

\[ \text{MP}_2 + C == \text{MP}_2 -- False \]
APPENDIX  C

Rational Double Points

Put thyself into the trick of singularity.

William Shakespeare

Dynkin diagrams crop up in many areas of mathematics. This appendix is dedicated to explaining the connection between simply laced Dynkin diagrams — those with no multiple edges — and rational double points, and thereby making sense of the symbols $A_n$, $D_n$ and $E_n$ used in the thesis.

C.1  ADE Singularities

The collection of simply laced Dynkin diagrams is called the ADE classification. The complete list, comprised of two families and three exceptional diagrams, is given in Table C.1.

<table>
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<tr>
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<td>$E_8$</td>
<td></td>
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Table C.1: ADE classification. The requirement $n ≥ 4$ for $D_n$ is put in to avoid redundancy, as $D_3$ ∼ $A_3$.

Definition C.1.1. An ADE curve is an exceptional divisor $C := \sum C_i$, where each irreducible component $C_i$ is a (-2)-curve. Moreover, $C_i \cdot C_j ≤ 1$ for all $i ≠ j$, that is, two components intersect in at most one point and then transversally.

The ADE curves are named so because their dual graphs — the graphs constructed by representing $C_i$ with a node, and the intersection between $C_i$
C. Rational Double Points

and $C_j$ with an edge connecting their corresponding nodes — are the Dynkin diagrams above. The $A_n$ curves also go under the alias Hirzebruch-Jung strings.

**Definition C.1.2.** Let $X$ and $Y$ be normal surfaces and assume that $X$ is nonsingular. A curve $C$ on $X$ is said to be exceptional if there is a birational map $\pi : X \to Y$ such that $C$ is exceptional for $\pi$. This means that there is a neighbourhood $U \subset X$, a point $y \in Y$ and a neighbourhood $V \subset Y$ of $y$, such that $\pi(C) = y$ and $\pi$ gives an isomorphism of $X \setminus U$ onto $Y \setminus V$. This situation is expressed by saying $C$ contracts to $y$.

**Definition C.1.3.** Let $X$ and $Y$ be normal and irreducible surfaces. Assume further that $X$ is smooth. Suppose that $C \subset X$ is the exceptional curve contracting to the singularity $y \in Y$ under the proper, birational map $\pi : X \to Y$. Then $y$ is called a rational singularity if the first higher direct image $R^1\pi_*\mathcal{O}_X$ vanish.

**Proposition C.1.4 ([Bar+04, III, Proposition 3.4]).** The ADE curves contract to rational singularities.

Suppose that $C := \sum C_i$ is an exceptional curve, and that $U \subset X$ is a neighbourhood of $C$ with $\omega_U = \mathcal{O}_U$. In particular, such a neighbourhood always exists for an ADE curve [Bar+04, III, Proposition 3.5]. An effective divisor $D$ on $U$ is the principal divisor $(f)$ of a rational function $f$ on $U$, if and only if $D \cdot C_i = 0$ for all $i$. Any effective divisor $D$ can be written on the form $D = Z + \sum U_i$, where $Z := \sum r_i C_i$ has support on $C$ and each $U_i$ intersects $C$ in at most finitely many points.

**Proposition C.1.5 ([Bar+04, III, Proposition 3.7]).** The divisors $Z := \sum r_i C_i$, where $r_i \geq 0$, satisfying

$$Z \cdot C_i \leq 0$$

for all $i$, are exactly the parts contained in $C$ of divisors of rational functions defined on some neighbourhood of $C$.

Let $(f), (f')$ be two principal divisors with corresponding parts $Z := \sum r_i C_i$ and $Z' := \sum r'_i C_i$ in $C$. Then for $\alpha, \alpha' \in \mathbb{C} \setminus \{0\}$, the part of $(\alpha f + \alpha' f')$ in $C$ is $\sum \min\{r_i, r'_i\} C_i$. Thus there is a minimal effective divisor satisfying (C.1) for all $i$, which is called the fundamental cycle of $C$ — or of the singularity $C$ contracts to.

Table C.2 on the next page indicates the coefficients for the fundamental cycles of the ADE curves.

An immediate consequence is:

**Proposition C.1.6 ([Bar+04, III, Proposition 3.9]).** For the fundamental cycle $Z$ of an exceptional ADE curve, we have

$$Z^2 = -2.$$  

**Definition C.1.7.** The multiplicity of the maximal ideal $m$ in a local ring $R$ of dimension $d$, is $(d - 1)!$ times the leading coefficient of the Hilbert–Samuel polynomial of $R$. The multiplicity of a variety $X$ at a point $p \in X$ is defined to be the multiplicity of the maximal ideal $m$ in the local ring $\mathcal{O}_{X, p}$. ♠
Artin showed in [Art66] that if $X$ is the spectrum of a two-dimensional local ring and $p \in X$ is a rational singularity with fundamental cycle $Z$, then the multiplicity at $p$ equals $-(Z^2)$. Hence Proposition C.1.6 implies that $ADE$ curves contract to rational double points. This provides a converse to du Val’s result in [Val34], that the resolution of an isolated double point on an embedded surface is an $ADE$ curve.

In light of this, we denominate rational double points according to the $ADE$ classification of their exceptional curves. Rational double points are known in the literature as $du$ Val singularities, simple surface singularities and Kleinian singularities.

**Example C.1.8 ($E_6$ singularity).** We shall construct an $ADE$ curve by sequentially blowing up $\mathbb{P}^2$ in six points. Set $X_0 := \mathbb{P}^2$ and let $P_1$ be a point in $X_{i-1}$. Denote the blow-up of $X_{i-1}$ with centre $P_i$ by $\pi_i : X_i \to X_{i-1}$, and take $E_i$ to be the exceptional line $\pi_i^{-1}(P_i)$. Consider these steps:

(i) Let $L$ be a line in $\mathbb{P}^2$ and $P_1$ any point on $L$.

(ii) Let $P_2$ be the intersection of $E_1$ and $L - E_1$, where $L - E_1$ is the strict transform of $L$ under $\pi_1$. Put $C_1 := E_1 - E_2$.

(iii) Let $P_3$ be the intersection of $E_2$ and $L - E_1 - E_2$. Put $C_2 := E_2 - E_3$ and $C_3 := L - E_1 - E_2 - E_3$.

(iv) Let $P_4$ be any point on $E_3$ that is not on $C_2$ or $C_3$. Put $C_4 := E_3 - E_4$.

(v) Let $P_5$ be any point on $E_4$ that is not on $C_4$. Put $C_5 := E_4 - E_5$.

(vi) Let $P_6$ be any point on $E_5$ that is not on $C_5$. Put $C_6 := E_5 - E_6$.  

---

Table C.2: Fundamental cycles for $ADE$ curves.
Each $C_i$ is irreducible with self-intersection $C_i^2 = -2$. Moreover, they fit together in the following dual graph:

This means that

$$C := C_1 + C_2 + C_3 + C_4 + C_5 + C_6 = L - E_2 - E_3 - E_6$$

is an $E_6$ curve.

We know from Proposition C.1.4 that $C$ contracts to a single point. We can see this explicitly. Let $\pi : X_6 \to \mathbb{P}^2$ be the composition $\pi_1 \circ \cdots \circ \pi_6$ and $\mathfrak{d}$ the linear system of plane cubic curves with assigned base points $P_1, \ldots, P_6$. Suppose $D \in \mathfrak{d}$, then we have a corresponding linear system $\mathfrak{d}'$ on $X_6$ given by

$$\mathfrak{d}' := |\pi^*(D) - E_1 - \cdots - E_6|.$$ 

Note that $\mathfrak{d}'$ is base-point free, so it gives rise to a morphism $X_6 \to X$, where $X$ is a cubic surface in $\mathbb{P}^3$. We claim that this morphism maps $C$ to a point.

Let $l$ denote the linear equivalence class of $\pi^*(L)$ and $e_i$ the classes of $E_i$. By abuse of notation, identify divisors on $X_6$ with their image in $X$. Recall that a hyperplane section $h$ of $X$ is in the linear equivalence class $3l - \sum_{i=1}^{6} e_i$. The intersection number of $C$ and $h$ is then

$$C \cdot h = (l - e_2 - e_3 - e_6) \cdot \left(3l - \sum_{i=1}^{6} e_i\right) = 3l^2 + e_2^2 + e_3^2 + e_6^2 = 3 - 1 - 1 - 1 = 0.$$ 

Since the image of $C$ in $X$ does not meet a general hyperplane section, it must be a point. In other words, it is an $E_6$ singularity. ♥


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