THE INFLUENCE OF THE INTERACTION
BETWEEN THE PARTICLES ON THE VISCOSITY
OF THE SUSPENSION

by

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The Influence of the Interaction between the Particles on the viscosity of the Suspension

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Abstract

The viscosity of suspensions in which interaction between the particles takes place is considered. The Brownian motion, hydrodynamic interaction and effects of coagulation are neglected.

The problems of the interaction between particles and the viscosity of the suspension are very interesting for practical applications. The hydrodynamic interaction between particles on viscosity of the suspension was considered by Happel & Brenner (1973) and Batchelor & Green (1972). But there are many other reasons for the interaction between particles besides hydrodynamic interaction. For example, the interaction by the Londons and the electrical forces take place in a suspension. These forces can be very important for the flow of the suspension.

Let the function U describe the interaction between two particles in a suspension. All the particles are the rigid spheres of radii a. Let Γ be the volume fraction of the spheres and $\Gamma \ll 1$. The particles are immersed in

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incompressible fluid of viscosity η_0. The ambient flow field has velocity $\vec{u}$ which is assumed to be a linear function of position and can therefore be characterized instantaneously by a uniform rate-of-strain tensor

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and a rigid-body rotation with angular velocity

$$\vec{\omega} = \frac{1}{2}\text{rot}\,\vec{u}.$$

Then, the velocity of the fluid is

$$u_i = u_{0i} + \gamma_{ij}x_j + \omega_{ijh}x_j$$

Choose the any two particles in the suspension and denote them by A and B, respectively. The particles are of such small size that the Reynolds number of the fluid motion is small and inertia forces can be neglected. We may write the equations of motion for each sphere without the Brownian motion:

$$
\begin{aligned}
\zeta(\vec{u}_0 - \vec{v}_A) &= \vec{F}_{BA}, \\
\zeta(\vec{u}_0 - \vec{v}_B) &= \vec{F}_{AB},
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
\xi(\vec{\omega}_0 - \vec{\Omega}_A) &= \vec{M}_{BA}, \\
\xi(\vec{\omega}_0 - \vec{\Omega}_B) &= \vec{M}_{AB},
\end{aligned}
\tag{2}
$$

Here,

$$
\begin{aligned}
\zeta &= 6\pi\eta_0 a, \\
\xi &= 8\pi\eta_0 a^3.
\end{aligned}
$$

$\vec{u}_A, \vec{v}_B$ are the translational velocities of the spheres and $\vec{\Omega}_A, \vec{\Omega}_B$ refer to their angular velocities, $\vec{F}_{BA}$ and $\vec{F}_{AB}$ are the forces of the interaction between two spheres, $\vec{M}_{AB}$ and $\vec{M}_{BA}$ are the moments of the forces. Of course, we may write the following equality

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

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Let the position of the centre of the sphere B relative to that of the sphere A be denoted by $\vec{R}$. The relative velocity $\vec{V}$ is

$$\vec{V} = (\vec{u}_0 - \vec{v}_B) + \nabla\vec{u}\cdot\vec{R} - (\vec{u}_0 - \vec{v}_A)$$

We obtain the following expression by equation (1)

$$\vec{V} = \frac{2}{\zeta}\vec{F}_{AB} + \nabla\vec{v}\cdot\vec{R} \tag{3}$$

The force $\vec{F}_{AB}$ may be written

$$\vec{F}_{AB} = -\nabla U \tag{4}$$

If the energy of the interaction depends only on R then the moments of the forces are equal to zero

$$\vec{M}_{AB} = \vec{M}_{BA} = 0$$

We obtain by equation (2)

$$\vec{\Omega}_A = \vec{\Omega}_B = \vec{\omega} \tag{5}$$

Happel and Brenner (1973) obtained the following expression for the viscosity η of the suspension

$$\eta = \eta_0\left(1 + \frac{E^*}{E_0}\right) \tag{6}$$

Here, E_0 is the dissipation of energy in the fluid without the particles and E^* is the dissipation as a result of the presence of suspended spheres. For N particles in the volume V we may write the following expressions

$$E^* = \sum_{\alpha=1}^{N}\left[(\vec{u}_0 - \vec{v}_\alpha)\vec{F}_\alpha + (\vec{\omega} - \vec{\Omega}_\alpha)\vec{T}_\alpha + (\oint_{S_\alpha}\sigma_{ij}\frac{x_j x_k}{a}ds)\gamma_{ik}\right] \tag{7}$$

$$E_0 = \int_V 2\eta_0\gamma_{ik}\gamma_{ij}dV \tag{8}$$

Here

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$$\sigma_{ij} = -p\delta_{ij} + \eta_0 \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

is the stress tensor,

$$F_{\alpha i} = \oint_{S_\alpha} \delta_{ij} \frac{x_j}{a} dS$$

is the hydrodynamic force exerted by the fluid on the αth particle,

$$T_{\alpha i} = e_{ijk} T_{jk} \,,$$

$$T_{jk} = \oint_{S_\alpha} \left(\delta_{jl} \frac{x_l x_k}{a^2} - \delta_{kl} \frac{x_l x_j}{a^2} \right) dS$$

is the hydrodynamic torgue exerted on the αth particle and S_α is the surface of the $\alpha$the particle. For spheres of radius a $\vec{F}_\alpha$ and $\vec{T}_\alpha$ are equal

$$\vec{F}_\alpha = 6\pi\eta_0 a(\vec{u}_0 - \vec{v}_\alpha)\,,$$

$$\vec{T} = 8\pi\eta_0 a^3(\vec{\omega} - \vec{\Omega}_\alpha)\,.$$

For our case we obtain by expression (5)

$$\vec{T}_\alpha = 0.$$

We shall make some use of the solution of the Stokes equatiosn for a single rigid sphere of radius a with centre at $\vec{R}$ in fluid whose velocity in the absence of the sphere is $(\vec{u}_0 - \vec{v}_\alpha)$ with uniform gradient at infinity characterized by γ_{ik} and ω_{ik}. The expressions for the velocity $\vec{v}$ and pressure p in the surrounding fluid are known to be

$$v_i = (u_{0i} - v_{\alpha i}) + \gamma_{ik} x_k + \omega_{ik} x_k + \frac{5}{2} \left(\frac{a^5}{r^7} - \frac{a^3}{r^5} \right) \gamma_{kl} x_l x_k x_i +$$

$$+ \frac{a^5}{r^5} \gamma_{ik} x_k - (u_{0i} - v_{\alpha i}) \left(\frac{3a}{4r} + \frac{a^3}{4r^3} \right) +$$

$$+ \frac{3}{4} \left(\frac{a^3}{r^3} - \frac{a}{r} \right) (u_{0k} - v_{\alpha k}) \frac{x_k x_i}{r^5} \,,$$

$$p = p_0 - \frac{3}{2} \eta_0 \frac{a}{r^3} (u_{0i} - v_{\alpha i}) x_i - 5\eta_0 \frac{a^3}{r^5} \gamma_{ik} x_k x_i$$

These relations may be introduced into expression (7) and we obtain

$$E^* = \sum_{\alpha=1}^{N} \left((\vec{u}_0 - \vec{v}_\alpha)^2 6\pi\eta_0 a + \frac{5}{2} \cdot \frac{4}{3}\pi a^3 \cdot 2\eta_0 \gamma_{ik}\gamma_{ik} \right] \tag{9}$$

Substituting the results (1), (4), (8), (9) into (6) we may write

$$\eta = \eta_0 \left[1 + \frac{5}{2}\Gamma + \frac{9}{4} \cdot \frac{4}{3}\pi a^3 \cdot \frac{1}{V} \sum_{\alpha=1}^{N} \frac{1}{\zeta^2}(\nabla U)^2 \frac{1}{\gamma_{ik}\gamma_{ik}a^2} \right] \tag{10}$$

The last addend in the expression (10) is the function R. Find the mean value of the viscosity by assistane of the function of the probability. Let $q(\vec{R}, t)$ be the probability for the vector $\vec{R}$ separating the centeres of the two particles. The differential equation for the probability function $q(\vec{R}, t)$ is

$$\frac{\partial q}{\partial t} + \vec{V}\nabla q = -q\nabla \cdot \vec{V}, \quad q=0, \quad R<2a$$

we obtain by equations (3), (4)

$$\frac{\partial q}{\partial t} + \left(\nabla_j u_i x_j - \frac{2}{\zeta}\frac{\partial U}{\partial x_i} \right) \frac{\partial q}{\partial x_i} = q\frac{2}{\zeta}\Delta U \tag{11}$$

The solution of the equation (11) contains an arbitrary multiplying constants. We choose the constant so that

$$q(R, t) \rightarrow n \quad \text{as} \quad R \rightarrow \infty \tag{12}$$

Here, n is the number of the particles in a unit volume. If the solution of the equation (11) with condition (12) is known we obtain the following expression for viscosity of the suspension

$$\eta = \eta_0 \left[1 + \frac{5}{2}\Gamma + \frac{9}{4}\Gamma \left(\int_V \frac{1}{\zeta^2}(\nabla U)^2 \frac{1}{\gamma_{ik}\gamma_{ik}a^2} q\,dV \right) \right] \tag{13}$$

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Example

Consider the suspension with magnetic particles in external magnetic field $\vec{H}$. The vector $\vec{H}$ is constant. Magnetic moment of each particle is vector $\vec{m}$. For a uniaxial extensional or compressional flow the rate-of-strain tensor γ_{ij} and angular velocity $\vec{\omega}$ are equal

$$\gamma_{ij} = \begin{bmatrix} \pm\gamma & 0 & 0 \\ 0 & \pm\gamma & 0 \\ 0 & 0 & \pm 2\gamma \end{bmatrix}, \tag{14}$$

$$\vec{\omega} = (0,0,0).$$

The + and − correspond, respectively, to compressional and extensional flow. The orientation of magnetic moment $\vec{m}$ of each particle is defined by external magnetic field and the forces of interaction between particles. Let value of magnetic field so large that directions of the magnetic moment $\vec{m}$ and vector $\vec{H}$ are the same. The function U in this case is written (Landau and Lifshitz)

$$U = \frac{m^2}{R^3}(1 - 3\cos^2\theta), \tag{15}$$

where θ is the angle between vector $\vec{R}$ and vector $\vec{H}$. We obtain for U the identity

$$\Delta U \equiv 0.$$

The solution of equation (11) with boundary condition (12) in this case is

$$q = n_0 \tag{16}$$

Integral in expression for viscosity (13) by assistance of relations (14), (15), (16) may be calculated. We obtain in spherical system of coordinates

$$\frac{9}{4}\Gamma \int_V \frac{1}{\zeta^2}(\nabla U)^2 \frac{q}{\gamma_{ik}\gamma_{ik}a^2}dV =$$

$$\frac{9}{4}\Gamma \frac{n_0 m^2}{6\gamma^2\zeta^2 a^2} \int_{2a}^{\infty}\int_0^{\pi}\int_0^{2\pi} \frac{1}{r^6}(9 + 45\cos^4\theta - 18\cos^2\theta)\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{9}{4}\Gamma \frac{3m^2}{\gamma^2 a^{10}\zeta^2 \cdot 80}\left(\frac{4}{3}\pi a^3 n_0\right) = \frac{3m^2\Gamma^2}{1280\gamma^2 a^{12}\pi^2\eta_0^2}$$

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The viscosity of suspension with magnetic particles without Brownian motion, hydrodynamic interaction and effect of coagulation is

$$\eta = \eta_0 \left(1 + \frac{5}{2}\Gamma + \frac{3m^2\Gamma^2}{1280\gamma^2\pi^2\eta_0^2 a^{12}} \right) .$$

Remark We must remember that the particles can form agregates due to the forces of interaction, and formation of agregates can change the viscosity of the suspension. Therefore, we must consider the kinetic equations of coagulation when the processes of formation and destruction of the agregates take place in the suspension.

References

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